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FROM THE LAPLACIAN WITH VARIABLE MAGNETIC FIELD TO THE ELECTRIC LAPLACIAN IN THE SEMICLASSICAL LIMIT

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We consider a twisted magnetic Laplacian with Neumann condition on a smooth and bounded domain of $\mathbb{R}^2$ in the semiclassical limit $h \to 0$. Under generic assumptions, we prove that the eigenvalues admit a complete asymptotic expansion in powers of $h^{1/4}$.

1. Introduction and main results

Let Ω be an open bounded and simply connected subset of $\mathbb{R}^2$ with smooth boundary. Let us consider a smooth vector potential $\boldsymbol{A}$ such that $\beta = \nabla \times \boldsymbol{A} > 0$ on $\overline{\Omega}$ and a a smooth and positive function on $\overline{\Omega}$. We are interested in estimating the eigenvalues $\lambda_n(h)$ of the operator $P_{h,\boldsymbol{A}} = (ih\nabla + \boldsymbol{A})a(ih\nabla + \boldsymbol{A})$ whose domain is given by

$$\mathrm{Dom}(P_{h,\boldsymbol{A}}) = \left\{\psi \in L^2(\Omega) : (-ih\nabla + \boldsymbol{A})a(-ih\nabla + \boldsymbol{A})\psi \in L^2(\Omega) \text{ and } (-ih\nabla + \boldsymbol{A})\psi \cdot \nu = 0 \text{ on } \partial\Omega\right\}.$$

The corresponding quadratic form, denoted by $Q_{h,\boldsymbol{A}}$, is defined on $H^1(\Omega)$ by

$$Q_{h,\boldsymbol{A}}(\psi) = \int_\Omega a(x)|(-ih\nabla + \boldsymbol{A})\psi|^2\, dx.$$

By gauge invariance, it is standard that the spectrum of $P_{h,\boldsymbol{A}}$ depends on the magnetic field $\beta = \nabla \times \boldsymbol{A}$, but not on the potential $\boldsymbol{A}$ itself.

Motivation and presentation of the problem.

Motivation and context. Before stating our main result, we should briefly describe the context and the motivations of this paper. As much in 2D as in 3D, the magnetic Laplacian, corresponding to the case when $a = 1$, appears in the theory of superconductivity when studying the third critical field H_{C_3} that appears after the linearization of the Ginzburg–Landau functional (see, for instance, [Lu and Pan 1999; 2000; Fournais and Helffer 2010]). It turns out that H_{C_3} can be related to the lowest eigenvalue of the magnetic Laplacian in the regime $h \to 0$.

In fact, the case which is mainly investigated in the literature is the case when the magnetic field is constant. In 2D, the two-terms asymptotics is done in the case of the disk by Bauman, Phillips and Tang in [Bauman et al. 1998] (see also [Bernoff and Sternberg 1998; del Pino et al. 2000]) and is generalized by Helffer and Morame [2001] to smooth and bounded domains. The asymptotic expansion at any order of

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all the lowest eigenvalues is proved by Fournais and Helffer [2006]. In 3D, one can mention the celebrated paper [Helffer and Morame 2004], which gives the two-terms asymptotics of the first eigenvalue.

When the magnetic field is variable (and $a = 1$), fewer results are known. In 2D, [Lu and Pan 1999] provides a one-term asymptotics of the lowest eigenvalue, and [Raymond 2009] gives the two-term asymptotics under generic assumptions (we can also mention [Helffer and Kordyukov 2011], which deals with the case without boundary and provides a full asymptotic expansion of the eigenvalues). In 3D, for the one-term asymptotics, one can mention [Lu and Pan 2000], and for a three-terms asymptotics upper bound, [Raymond 2010] (see also [Raymond 2012], where a complete asymptotics is proved for a toy model).

Here we consider a twist factor $a > 0$. As we will see, the presence of a (which is maybe not the main point of this paper) will not complicate the philosophy of the analysis, even if it will lead us to use generalizations of the Feynman–Hellmann theorems (such generalizations were introduced by physicists to analyze the anisotropic Ginzburg–Landau functional; see [Doria and de Andrade 1996]). In fact, this additional term obliges us to have a more synthetic sight of the structure of the magnetic Laplacian. The motivation to add this term comes from [Chapman et al. 1995], where the authors deal with the anisotropic Ginzburg–Landau functional (which is an effective mass model). We can also refer to [Alama et al. 2010], where closely related problems appear. Moreover, we will see that the quantity to minimize to get the lowest energy is the function $a\beta$, so that this situation recalls what happens in 3D in [Lu and Pan 2000; Raymond 2010] and where the three-terms asymptotics is still not established.

Under generic assumptions, we will prove in this paper that the eigenvalues $\lambda_n(h)$ admit complete asymptotic expansions in powers of $h^{1/4}$.

Heuristics. Let us discuss the heuristics a little bit, to understand the problem. Let us fix a point $x_0 \in \overline{\Omega}$. If $x_0 \in \Omega$ and if we approximate the vector potential $\boldsymbol{A}$ by its linear part, we can locally write the magnetic Laplacian as

$$a(x_0)(h^2 D_x^2 + (hD_y - \beta(x_0)x)^2) + \text{lower-order terms}.$$

The lowest eigenvalue can be computed after a Fourier transform with respect to y and a translation with respect to x (which reduces to a 1D harmonic oscillator); it provides an eigenvalue $a(x_0)\beta(x_0)h$. If $x_0 \in \partial\Omega$, and considering the standard boundary coordinates (s, t) ($t > 0$ being the distance to the boundary and s the curvilinear coordinate), we get the approximation

$$h^2 D_t^2 + (hD_s - \beta(x_0)t)^2 + \text{lower-order terms}.$$

The shape of this formal approximation invites us to recall basic properties of the de Gennes operator.

The de Gennes operator. For $\xi \in \mathbb{R}$, we consider the Neumann realization H_ξ in $L^2(\mathbb{R}_+)$ associated with the operator

$$-\frac{d^2}{dt^2} + (t - \xi)^2, \quad \operatorname{Dom}(H_\xi) = \{u \in B^2(\mathbb{R}_+) : u'(0) = 0\}. \tag{1-1}$$

One knows (see [Dauge and Helffer 1993]) that it has compact resolvent and that its lowest eigenvalue is denoted by $\mu(\xi)$; the associated L^2-normalized and positive eigenstate is denoted by $u_\xi = u(\,\cdot\,, \xi)$ and is in the Schwartz class. The function $\xi \mapsto \mu(\xi)$ admits a unique minimum, say at $\xi = \xi_0$, and we let

$$\Theta_0 = \mu(\xi_0), \quad C_1 = \frac{u_{\xi_0}^2(0)}{3}. \tag{1-2}$$

Let us also recall identities established in [Bernoff and Sternberg 1998]. For $k \in \mathbb{N}$, we let

$$M_k = \int_{t>0} (t-\xi_0)^k |u_{\xi_0}(t)|^2 \, dt,$$

and we have

$$M_0 = 1, \quad M_1 = 0, \quad M_2 = \tfrac{1}{2}\Theta_0, \quad M_3 = \tfrac{1}{2}C_1, \quad \tfrac{1}{2}\mu''(\xi_0) = 3C_1\sqrt{\Theta_0}. \tag{1-3}$$

Main result. Let us introduce the general assumptions under which we will work throughout this paper. As already mentioned, the natural invariant associated with the operator is the function $a\beta$. We will assume that

$$\Theta_0 \min_{\partial\Omega} a(x)\beta(x) < \min_{\Omega} a(x)\beta(x) \tag{1-4}$$

and that

$$x \in \partial\Omega \mapsto a(x)\beta(x) \text{ admits a unique and nondegenerate minimum at } x_0. \tag{1-5}$$

Remark 1.1. Assumption (1-4) is automatically satisfied when the magnetic field is constant (and is sometimes called the surface superconductivity condition), and Assumption (1-5) excludes the case of constant magnetic field. Therefore, our generic assumption deals with a complementary situation analyzed in [Fournais and Helffer 2006], that is, the situation with a generically variable magnetic field.

Let us state our first rough estimate of the n-th eigenvalue $\lambda_n(h)$ of $P_{h,\mathbf{A}}$ that we will prove in this paper:

Proposition 1.2. *Under Assumptions* (1-4) *and* (1-5), *for all* $n \geq 1$, *we have*

$$\lambda_n(h) = \Theta_0 h a(x_0)\beta(x_0) + O(h^{5/4}). \tag{1-6}$$

From this proposition, we see that the asymptotics of $\lambda_n(h)$ is related to local properties of $P_{h,\mathbf{A}}$ near the point of the boundary x_0. That is why we are led to introduce the standard system of local coordinates (s,t) near x_0, where t is the distance to the boundary and s the curvilinear coordinate on the boundary (see (2-1)). We denote by $\Phi : (s,t) \mapsto x$ the corresponding local diffeomorphism. We write the Taylor expansions

$$\tilde{a}(s,t) = a(\Phi(s,t)) = 1 + a_1 s + a_2 t + a_{11}s^2 + a_{12}st + a_{22}t^2 + O(|s|^3 + |t|^3) \tag{1-7}$$

and

$$\tilde{\beta}(s,t) = \beta(\Phi(s,t)) = 1 + b_1 s + b_2 t + b_{11}s^2 + b_{12}st + b_2 t^2 + O(|s|^3 + |t|^3), \tag{1-8}$$

where we have assumed the normalization

$$a(x_0) = \beta(x_0) = 1. \tag{1-9}$$

Let us translate the generic assumptions (1-4) and (1-5). The critical point condition becomes

$$a_1 = -b_1, \tag{1-10}$$

and the nondegeneracy property can be reformulated as

$$b_{11} + a_1 b_1 + a_{11} = a_{11} + b_{11} - a_1^2 = \alpha > 0. \tag{1-11}$$

We can now state the main result of this paper:

Theorem 1.3. *We assume* (1-4) *and* (1-5) *and the normalization condition* (1-9). *For all* $n \geq 1$, *there exist a sequence* $(\gamma_{n,j})_{j\geq 0}$ *and* $h_0 > 0$ *such that for all* $h \in (0, h_0)$, *we have*

$$\lambda_n(h) \underset{h\to 0}{\sim} h \sum_{j\geq 0} \gamma_{n,j} h^{j/4}.$$

Moreover, we have, for all $n \geq 1$,

$$\gamma_{n,0} = \Theta_0, \quad \gamma_{n,1} = 0, \quad \gamma_{n,2} = C(k_0, a_2, b_2) + (2n-1)\left(\frac{\alpha\Theta_0\mu''(\xi_0)}{2}\right)^{1/2},$$

with

$$C(k_0, a_2, b_2) = -C_1 k_0 + \frac{3C_1}{2} a_2 + \left(\frac{C_1}{2} + \xi_0\Theta_0\right) b_2.$$

Comments about the main theorem. Let us first notice that Theorem 1.3 completes the one of Fournais and Helffer [2006, Theorem 1.1] dealing with a constant magnetic field (see also [Fournais and Helffer 2006, Remark 1.2], where the variable magnetic field case is left as an open problem).

It turns out that Theorem 1.3 generalizes [Raymond 2009, Theorem 1.7]. Moreover, as a consequence of the asymptotics of the eigenvalues (which are simple for h small enough), we also get the corresponding asymptotics for the eigenfunctions. These eigenfunctions are approximated (in the L^2 sense) by the power series, which we will use as quasimodes (see (2-10)). In particular, the eigenfunctions are approximated by functions in the form

$$u_{\xi_0}(h^{-1/2}t)g(h^{-1/4}s),$$

where g is a renormalized Hermite function.

As we will see in the proof, the construction of appropriate trial functions can give a hint of the natural scales of the problem ($h^{1/2}$ with respect to t and $h^{1/4}$ with respect to s). Nevertheless, as far as we know, there are no structural explanations in the literature of the double scales phenomena related to the magnetic Laplacian.

In this paper, we will explain how, thanks to conjugations of the magnetic Laplacian (by explicit unitary transforms in the spirit of Egorov's theorem; see [Egorov 1971; Robert 1987; Martinez 2002]), we can reduce the study to an electric Laplacian which is in the Born–Oppenheimer form (see [Combes et al. 1981; Martinez 1989]). The main point of the Born–Oppenheimer approximation is that it naturally involves two different scales (related to the so-called slow and fast variables).

As we recalled at the beginning of the introduction, many papers deal with the two or three first terms of $\lambda_1(h)$ and do not analyze $\lambda_n(h)$ (for $n \geq 2$); see, for instance, [Helffer and Morame 2004; Raymond 2009]. One could think that it is just a technical extension. But, as can be seen in [Fournais and Helffer 2006] (see also [Dombrowski and Raymond 2013]), the difficulty of the extension relies on the microlocalization properties of the operator: The authors have to combine a very fine analysis using pseudodifferential calculus (to catch the a priori behavior of the eigenfunctions with respect to a phase variable) and the

Grušin reduction machinery [1972]. Let us emphasize that these microlocalization properties are one of the deepest features of the magnetic Laplacian and are often found at the core of proofs (see, for instance, [Helffer and Morame 2004, Sections 11.2 and 13.2; Fournais and Helffer 2006, Sections 5 and 6]). We will see how we can avoid the introduction of the pseudodifferential (or abstract functional) calculus. In fact, we will also avoid the Grušin formalism by keeping only the main idea behind it: We can use the true eigenfunctions as quasimodes for the first-order approximation of $P_{h,\mathbf{A}}$ and deduce a tensorial structure for the eigenfunctions.

In our investigation, we will introduce successive changes of variables and unitary transforms, such as changes of gauge and weighted Fourier transforms (which are all associated with canonical transformations of the symbol). By doing this, we will reduce the symbol of the operator (or, equivalently, reduce the quadratic form), thanks to the a priori localization estimates. By gathering all these transforms, one would obtain a Fourier integral operator which transforms (modulo lower-order terms) the magnetic Laplacian into an electric Laplacian in the Born–Oppenheimer form. For this normal form, we can prove Agmon estimates with respect to a phase variable. These estimates involve, for the normal form, strong microlocalization estimates, and spare us, for instance, the multiple commutator estimates needed in [Fournais and Helffer 2006, Section 5].

Scheme of the proof. Let us now describe the scheme of the proof. In Section 2, we perform a construction of quasimodes and quasieigenvalues thanks to a formal expansion in power series of the operator. This analysis relies on generalizations of the Feynman–Hellmann formula and of the virial theorem, which were already introduced in [Raymond 2010], and which are an alternative to the Grušin approach used in [Fournais and Helffer 2006]. Then we use the spectral theorem to infer the existence of a spectrum near each constructed power series. In Section 3, we prove a rough lower bound for the lowest eigenvalues and deduce Agmon estimates with respect to the variable t, which provide a localization of the lowest eigenfunctions in a neighborhood of the boundary of size $h^{1/2}$. In Section 4 , we improve the lower bound of Section 3 and deduce a localization of size $h^{1/4}$ with respect to the tangential coordinate s. In Section 5, we prove a lower bound for $Q_{h,\mathbf{A}}$ thanks to the definition of "magnetic coordinates," and we reduce the study to a model operator (in the Born–Oppenheimer form) for which we are able to estimate the spectral gap between the lowest eigenvalues.

2. Accurate construction of quasimodes

This section is devoted to the proof of the following theorem:

Theorem 2.1. *For all $n \geq 1$, there exists a sequence $(\gamma_{n,j})_{j\geq 0}$ such that, for all $J \geq 0$, there exist $h_0 > 0$, $C > 0$ such that*

$$d\left(h\sum_{j=0}^{J}\gamma_{n,j}h^{j/4}, \sigma(P_{h,\mathbf{A}})\right) \leq Ch^{(J+1)/4}.$$

Moreover, we have, for all $n \geq 1$:

$$\gamma_{n,0} = \Theta_0, \quad \gamma_{n,1} = 0, \quad \gamma_{n,2} = C(k_0, a_2, b_2) + (2n-1)\left(\frac{(a_{11}+b_{11}-a_1^2)\Theta_0\mu''(\xi_0)}{2}\right)^{1/2}.$$

The proof of Theorem 2.1 is based on a construction of quasimodes for $P_{h,\boldsymbol{A}}$ localized near x_0.

Local coordinates (s,t). We use the local coordinates (s,t) near $x_0=(0,0)$, where $t(x)=d(x,\partial\Omega)$ and $s(x)$ is the tangential coordinate of x. We choose a parametrization of the boundary:

$$\gamma:\mathbb{R}/(|\partial\Omega|\mathbb{Z})\to\partial\Omega.$$

Let $\nu(s)$ be the unit vector normal to the boundary, pointing inward at the point $\gamma(s)$. We choose the orientation of the parametrization γ to be counterclockwise, so that

$$\det(\gamma'(s),\nu(s))=1.$$

The curvature $k(s)$ at the point $\gamma(s)$ is given in this parametrization by

$$\gamma''(s)=k(s)\nu(s).$$

The map Φ defined by

$$\Phi:\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times\,]0,t_0[\,\to\Omega,\quad (s,t)\mapsto\gamma(s)+t\nu(s) \tag{2-1}$$

is clearly a diffeomorphism, when t_0 is sufficiently small, with image

$$\Phi\big(\mathbb{R}/(|\partial\Omega|\mathbb{Z})\times\,]0,t_0[\big)=\{x\in\Omega\mid d(x,\partial\Omega)<t_0\}=\Omega_{t_0}.$$

We let

$$\tilde{A}_1(s,t)=(1-tk(s))\boldsymbol{A}(\Phi(s,t))\cdot\gamma'(s),\quad \tilde{A}_2(s,t)=\boldsymbol{A}(\Phi(s,t))\cdot\nu(s),\quad \tilde{\beta}(s,t)=\beta(\Phi(s,t)),$$

and we get

$$\partial_s\tilde{A}_2-\partial_t\tilde{A}_1=(1-tk(s))\tilde{\beta}(s,t).$$

The quadratic form becomes

$$Q_{h,\boldsymbol{A}}(\psi)=\int\tilde{a}(1-tk(s))\big|(-ih\partial_t+\tilde{A}_2)\psi\big|^2+\tilde{a}(1-tk(s))^{-1}\big|(-ih\partial_s+\tilde{A}_1)\psi\big|^2\,ds\,dt.$$

In a (simply connected) neighborhood of (0, 0), we can choose a gauge such that

$$\tilde{A}_1(s,t)=-\int_{t_1}^{t}(1-t'k(s))\tilde{\beta}(s,t')\,dt',\quad \tilde{A}_2=0. \tag{2-2}$$

The operator in the coordinates (s,t). Near x_0 and using a suitable gauge (see (2-2)), we are led to construct quasimodes for the operator

$$\mathscr{L}(s,-ih\partial_s;t,-ih\partial_t)=\\ -h^2(1-tk(s))^{-1}\partial_t(1-tk(s))\tilde{a}\partial_t+(1-tk(s))^{-1}(-ih\partial_s+\tilde{A})(1-tk(s))^{-1}\tilde{a}(-ih\partial_s+\tilde{A}),$$

where (see (1-8))

$$\tilde{A}(s,t)=(t-\xi_0h^{1/2})+b_1s(t-\xi_0h^{1/2})+(b_2-k_0)\frac{t^2}{2}+b_{11}s^2(t-\xi_0h^{1/2})+O(|t|^3+|st^2|).$$

Let us now perform the scaling

$$s=h^{1/4}\sigma\quad\text{and}\quad t=h^{1/2}\tau.$$

The operator becomes

$$\mathscr{L}(h) = \mathscr{L}\big(h^{1/4}\sigma, -ih^{3/4}\partial_\sigma; h^{1/2}\tau, -ih^{1/2}\partial_\tau\big).$$

We can formally write $\mathscr{L}(h)$ as a power series:

$$\mathscr{L}(h) \sim h \sum_{j\geq 0} \mathscr{L}_j h^{j/4},$$

where

$$\mathscr{L}_0 = -\partial_\tau^2 + (\tau - \xi_0)^2, \tag{2-3}$$

$$\begin{aligned}\mathscr{L}_1 &= -a_1\sigma\partial_\tau^2 - 2i\partial_\sigma(\tau-\xi_0) + a_1(\tau-\xi_0)^2\sigma + 2b_1\sigma\tau(\tau-\xi_0)\\ &= a_1\sigma H_{\xi_0} - 2i\partial_\sigma(\tau-\xi_0) + 2b_1\sigma(\tau-\xi_0)^2,\end{aligned} \tag{2-4}$$

$$\begin{aligned}\mathscr{L}_2 = {} & -a_2\tau\partial_\tau^2 - a_2\partial_\tau + k_0\partial_\tau + 2k_0\tau(\tau-\xi_0)^2 + a_2\tau(\tau-\xi_0)^2\\ & + (b_2-k_0)\tau^2(\tau-\xi_0) - ia_1(\tau-\xi_0)\\ & + \sigma^2\big(a_{11}H_{\xi_0} - a_1^2(\tau-\xi_0)^2 + 2b_{11}(\tau-\xi_0)^2\big)\\ & - \partial_\sigma^2 - 2ia_1(\tau-\xi_0)\sigma\partial_\sigma + ia_1(\tau-\xi_0)\partial_\sigma\sigma.\end{aligned} \tag{2-5}$$

The aim is now to define good quasimodes for $\mathscr{L}(h)$. Before starting the construction, we shall recall in the next subsection a few formulas coming from perturbation theory.

Feynman–Hellmann and virial formulas. For $\rho > 0$ and $\xi \in \mathbb{R}$, let us introduce the Neumann realization on $\mathbb{R}_+$ of

$$H_{\rho,\xi} = -\rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2.$$

By scaling, we observe that $H_{\rho,\xi}$ is unitarily equivalent to H_ξ and that $H_{1,\xi} = H_\xi$ (the corresponding eigenfunction is $u_{1,\xi} = u_\xi$). The form domain of $H_{\rho,\xi}$ is $B^1(\mathbb{R}_+)$ and is independent from ρ and ξ so that the family $(H_{\rho,\xi})_{\rho>0,\xi\in\mathbb{R}}$ is a holomorphic family of type (B) (see [Kato 1966, p. 395]). The lowest eigenvalue of $H_{\rho,\xi}$ is $\mu(\xi)$ and we will denote by $u_{\rho,\xi}$ the corresponding normalized eigenfunction:

$$u_{\rho,\xi}(\tau) = \rho^{1/4} u_\xi(\rho^{1/2}\tau).$$

Since u_ξ satisfies the Neumann condition, we observe that $\partial_\rho^m\partial_\xi^n u_{\rho,\xi}$ also satisfies it. In order to lighten the notation, when it is not ambiguous we will write H for $H_{\rho,\xi}$, u for $u_{\rho,\xi}$, and μ for $\mu(\xi)$.

The main idea is now to take derivatives of

$$Hu = \mu u \tag{2-6}$$

with respect to ρ and ξ. Taking the derivative with respect to ρ and ξ, we get the following proposition:

Proposition 2.2. *We have*

$$(H-\mu)\partial_\xi u = 2(\rho^{1/2}\tau - \xi)u + \mu'(\xi)u \tag{2-7}$$

and

$$(H-\mu)\partial_\rho u = -\rho^{-2}\partial_\tau^2 - \xi\rho^{-1}(\rho^{1/2}\tau-\xi) - \rho^{-1}\tau(\rho^{1/2}\tau-\xi)^2. \tag{2-8}$$

Moreover, we get

$$(H-\mu)(Su) = Xu, \tag{2-9}$$

where

$$X = -\frac{\xi}{2}\mu'(\xi) + \rho^{-1}\partial_\tau^2 + (\rho^{1/2}\tau - \xi)^2$$

and

$$S = -\frac{\xi}{2}\partial_\xi - \rho\partial_\rho.$$

Proof. Taking the derivatives with respect to ξ and ρ of (2-6), we get

$$(H-\mu)\partial_\xi u = \mu'(\xi)u - \partial_\xi Hu$$

and

$$(H-\mu)\partial_\rho u = -\partial_\rho H.$$

We have $\partial_\xi H = -2(\rho^{1/2}\tau - \xi)$ and $\partial_\rho H = \rho^{-2}\partial_\rho^2 + \rho^{-1/2}\tau(\rho^{1/2}\tau - \xi)$. □

Taking $\rho = 1$ and $\xi = \xi_0$ in (2-7), we deduce, with the Fredholm alternative:

Corollary 2.3. *We have*

$$(H_{\xi_0} - \mu(\xi_0))v_{\xi_0} = 2(t-\xi_0)u_{\xi_0},$$

with

$$v_{\xi_0} = (\partial_\xi u_\xi)_{|\xi=\xi_0}.$$

Moreover, we have

$$\int_{\tau>0} (\tau - \xi_0)u_{\xi_0}^2 \, d\tau = 0.$$

Corollary 2.4. *We have, for all $\rho > 0$,*

$$\int_{\tau>0} (\rho^{1/2}\tau - \xi_0)u_{\rho,\xi_0}^2 \, d\tau = 0$$

and

$$\int_{\tau>0} (\tau - \xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0} u \, d\tau = -\frac{\xi_0}{4}.$$

Corollary 2.5. *We have*

$$(H_{\xi_0} - \mu(\xi_0))S_0 u = (\partial_\tau^2 + (\tau-\xi_0)^2)u_{\xi_0},$$

where

$$S_0 u = -(\partial_\rho u_{\rho,\xi})_{|\rho=1,\xi=\xi_0} - \frac{\xi_0}{2}v_{\xi_0}.$$

Moreover, we have

$$\|\partial_\tau u_{\xi_0}\|^2 = \|(\tau-\xi_0)u_{\xi_0}\|^2 = \frac{\Theta_0}{2}.$$

The next three propositions deal with the second derivatives of (2-6) with respect to ξ and ρ.

Proposition 2.6. *We have*

$$(H_\xi - \mu(\xi))w_{\xi_0} = 4(\tau - \xi_0)v_{\xi_0} + (\mu''(\xi_0) - 2)u_{\xi_0},$$

with

$$w_{\xi_0} = (\partial_\xi^2 u_\xi)_{|\xi=\xi_0}.$$

Moreover, we have

$$\int_{\tau>0} (\tau - \xi_0)v_{\xi_0}u_{\xi_0}\, d\tau = \frac{2 - \mu''(\xi_0)}{4}.$$

Proof. Taking the derivative of (2-7) with respect to ξ (with $\rho = 1$), we get

$$(H_\xi - \mu(\xi))\partial_\xi^2 u_\xi = 2\mu'(\xi)\partial_\xi u_\xi + 4(\tau - \xi)\partial_\xi u_\xi + (\mu''(\xi) - 2)u_\xi.$$

It remains to take $\xi = \xi_0$ and to write the Fredholm alternative. □

Proposition 2.7. *We have*

$$(H-\mu)(\partial_\rho^2 u)_{\rho=1,\xi=\xi_0} = -2(\partial_\tau^2 + (\tau-\xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0} - 2\xi_0(\tau-\xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0} + \left(2\partial_\tau^2 - \frac{\xi_0\tau}{2}\right)u_{\xi_0}$$

and

$$\left\langle(\partial_\tau^2 + (\tau-\xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0}, u_{\xi_0}\right\rangle = -\frac{\Theta_0}{2}.$$

Proof. We just have to take the derivative of (2-8) with respect to ρ and $\rho = 1$, $\xi = \xi_0$. To get the second identity, we use the Fredholm alternative, Corollaries 2.4 and 2.5. □

Taking the derivative of (2-9) with respect to ρ, we find:

Lemma 2.8. *We have*

$$(H-\mu)(\partial_\rho Su)_{\rho=1,\xi=\xi_0} = (-\partial_\tau^2 + \tau(\tau-\xi_0))u_{\xi_0} - (\partial_\rho H)_{\rho=1,\xi=\xi_0}(S_0u) + (\partial_\tau^2 + (\tau-\xi_0)^2)(\partial_\rho u)_{\rho=1,\xi=\xi_0}$$

and

$$\left\langle(\partial_\rho H)_{\rho=1,\xi=\xi_0}(S_0u), u\right\rangle = \frac{\Theta_0}{2}.$$

Lemma 2.9. *We have*

$$\langle(\tau-\xi_0)S_0u, u_{\xi_0}\rangle = \frac{\xi_0}{8}\mu''(\xi_0).$$

Proof. We have

$$\mu'(\xi) = -2\int_{\tau>0}(\rho^{1/2}\tau - \xi)u_{\rho,\xi}^2\, d\tau$$

and

$$S_0\mu' = -2\int_{\tau>0} S_0(\rho^{1/2}\tau - \xi)u_{\xi_0}^2\, d\tau - 4\int_{\tau>0}(\tau-\xi_0)S_0u\, u_{\xi_0}\, d\tau.$$ □

Combining Lemmas 2.8 and 2.9, we deduce:

Proposition 2.10. *We have*

$$\left\langle(-\partial_\tau^2 - (\tau-\xi_0)^2)S_0u, u_{\xi_0}\right\rangle = -\frac{\Theta_0}{2} + \frac{\Theta_0}{8}\mu''(\xi_0).$$

Proposition 2.11. *We have*

$$\big\langle(\partial_\tau^2+(\tau-\xi_0)^2)v_{\xi_0},u_{\xi_0}\big\rangle=\frac{\xi_0\mu''(\xi_0)}{4}.$$

Proof. We take the derivative of (2-7) with respect to ρ (after having fixed $\xi=\xi_0$):

$$(H-\mu)(\partial_\xi u)_{\xi=\xi_0}=2(\rho^{1/2}\tau-\xi_0)u_{\rho,\xi_0}.$$

We deduce

$$(H-\mu)(\partial_\rho\partial_\xi u)_{\rho=1,\xi=\xi_0}=-(\partial_\rho H)_{\rho=1,\xi=\xi_0}v_{\xi_0}+\tau u_{\xi_0}+2(\tau-\xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0}.$$

The Fredholm alternative provides

$$\big\langle(\partial_\tau^2+\tau(\tau-\xi_0))v_{\xi_0},u_{\xi_0}\big\rangle=\xi_0+2\big\langle(\tau-\xi_0)(\partial_\rho u)_{\rho=1,\xi=\xi_0},u_{\xi_0}\big\rangle=\frac{\xi_0}{2},$$

where we have used Corollary 2.4. □

We have now the elements to perform an accurate construction of quasimodes.

Construction. We look for quasimodes expressed as power series,

$$\psi\sim\sum_{j\geq 0}\psi_j h^{j/4},$$

and eigenvalues,

$$\lambda\sim h\sum_{j\geq 0}\lambda_j h^{j/4},$$

so that, in the sense of formal series,

$$\mathscr{L}(h)\psi\sim\lambda\psi.$$

Term in h. We consider the equation

$$(\mathscr{L}_0-\lambda_0)\psi_0=0.$$

We are led to take $\lambda_0=\Theta_0$ and $\psi_0(\sigma,\tau)=f_0(\sigma)u_{\xi_0}(\tau)$.

Term in $h^{5/4}$. We want to solve the equation

$$(\mathscr{L}_0-\Theta_0)\psi_1=\lambda_1\psi_0-\mathscr{L}_1\psi_0.$$

We have, using that $b_1=-a_1$ and by Proposition 2.2,

$$(\mathscr{L}_0-\Theta_0)\big(\psi_1-if_0'(\sigma)v_{\xi_0}-a_1\sigma f_0(\sigma)S_0u\big)=\lambda_1 u_{\xi_0}.$$

This implies that $\lambda_1=0$, and we take

$$\psi_1(\sigma,\tau)=if_0'(\sigma)v_{\xi_0}+a_1\sigma f_0(\sigma)S_0u+f_1(\sigma)u_{\xi_0}(\tau),$$

f_0 and f_1 being to determine.

Term in $h^{3/2}$. We consider the equation

$$(\mathscr{L}_0-\Theta_0)\psi_2=\lambda_2\psi_0-\mathscr{L}_1\psi_1-\mathscr{L}_2\psi_0.$$

Let us rewrite this equation by using the expression of ψ_1:

$$(\mathscr{L}_0-\Theta_0)\psi_2=\lambda_2\psi_0-\mathscr{L}_1\big(if_0'(\sigma)v_{\xi_0}+a_1\sigma f_0(\sigma)S_0u\big)-\mathscr{L}_1(f_1(\sigma)u_{\xi_0})-\mathscr{L}_2\psi_0.$$

With Proposition 2.2, we deduce

$$(\mathscr{L}_0-\Theta_0)\big(\psi_2-if_1'(\sigma)v_{\xi_0}-a_1\sigma f_1(\sigma)S_0u\big)=\lambda_2\psi_0-\mathscr{L}_1\big(if_0'(\sigma)v_{\xi_0}+a_1\sigma f_0(\sigma)S_0u\big)-\mathscr{L}_2\psi_0.$$

We take the partial scalar product (with respect to τ) of the right-hand side with u_{ξ_0}, and we get the equation

$$\big\langle\mathscr{L}_1\big(if_0'(\sigma)v_{\xi_0}+a_1\sigma f_0(\sigma)S_0u\big)+\mathscr{L}_2\psi_0,u_{\xi_0}\big\rangle_\tau=\lambda_2 f_0.$$

This equation can be written in the form

$$\big(AD_\sigma^2+B_1\sigma D_\sigma+B_2D_\sigma\sigma+C\sigma^2+D\big)f_0=\lambda_2 f_0.$$

Terms in D_σ^2. Let us first analyze $\langle\mathscr{L}_2u_{\xi_0},u_{\xi_0}\rangle$. It is easy to see that this term is 1. Let us then analyze $\langle\mathscr{L}_1\psi_1,u_{\xi_0}\rangle$. With Proposition 2.6, we deduce that this term is $-2\langle(\tau-\xi_0)v_{\xi_0}u_{\xi_0}\rangle=(\mu''(\xi_0)/2)-1$. We get $A=\mu''(\xi_0)/2>0$.

Terms in σ^2. Let us collect the terms of $\langle\mathscr{L}_2u_{\xi_0},u_{\xi_0}\rangle$. We get

$$\Theta_0a_{11}+2b_{11}\langle(\tau-\xi_0)^2u_{\xi_0},u_{\xi_0}\rangle-a_1^2\langle(\tau-\xi_0)^2u_{\xi_0},u_{\xi_0}\rangle.$$

With Corollary 2.5, this term is equal to

$$\Theta_0a_{11}+\Theta_0b_{11}-\frac{\Theta_0}{2}a_1^2.$$

Let us analyze the terms coming from $\langle\mathscr{L}_1\psi_1,u_{\xi_0}\rangle$. We obtain the term

$$a_1^2\big\langle(-\partial_\tau^2-(\tau-\xi_0)^2)S_0u,u_{\xi_0}\big\rangle=-\frac{\Theta_0}{2}a_1^2+\Theta_0\frac{\mu''(\xi_0)}{8}a_1^2,$$

where we have used Proposition 2.10. Thus, we have

$$C=\Theta_0a_{11}+\Theta_0b_{11}-\Theta_0a_1^2+\frac{\Theta_0}{8}\mu''(\xi_0)a_1^2>0.$$

Terms in σD_σ. This term only comes from $\langle\mathscr{L}_1\psi_1,u_{\xi_0}\rangle$. It is equal to

$$a_1\big\langle(\partial_\tau^2+(\tau-\xi_0)^2)v_{\xi_0},u_{\xi_0}\big\rangle=a_1\frac{\xi_0\mu''(\xi_0)}{4},$$

where we have used Proposition 2.11.

Terms in $D_\sigma\sigma$. This term is

$$2a_1\langle(\tau-\xi_0)S_0u,u_{\xi_0}\rangle=a_1\frac{\xi_0\mu''(\xi_0)}{4},$$

where we have applied Lemma 2.9.

Value of D. We have:

$$D = \big\langle\big(-a_2\tau\partial_\tau^2 - a_2\partial_\tau + k_0\partial_\tau + 2k_0\tau(\tau-\xi_0)^2 + a_2\tau(\tau-\xi_0)^2\big)u_{\xi_0}, u_{\xi_0}\big\rangle + \big\langle\big((b_2-k_0)\tau^2(\tau-\xi_0) - ia_1(\tau-\xi_0)\big)u_{\xi_0}, u_{\xi_0}\big\rangle.$$

Using the relations (1-3) and the definition of C_1 given in (1-2), we get

$$D = C(k_0, a_2, b_2).$$

Let us introduce the quadratic form, which is fundamental in the analysis. We let

$$\mathfrak{Q}(\sigma,\eta) = \frac{\mu''(\xi_0)}{2}\eta^2 + a_1\frac{\xi_0\mu''(\xi_0)}{4}\eta\sigma + a_1\frac{\xi_0\mu''(\xi_0)}{4}\sigma\eta + \Theta_0\Big(a_{11}+b_{11}-a_1^2+a_1^2\frac{\mu''(\xi_0)}{8}\Big)\sigma^2.$$

Lemma 2.12. *$\mathfrak{Q}$ is definite and positive.*

Proof. We notice that $\mu''(\xi_0) > 0$ and $a_{11}+b_{11}-a_1^2+a_1^2(\mu''(\xi_0)/8) > 0$. The determinant is given by

$$\Theta_0\frac{\mu''(\xi_0)}{2}\Big(a_{11}+b_{11}-a_1^2+a_1^2\frac{\mu''(\xi_0)}{8}\Big) - a_1^2\frac{\Theta_0\mu''(\xi_0)^2}{16} = \frac{\Theta_0\mu''(\xi_0)}{2}(a_{11}+b_{11}-a_1^2) > 0. \qquad \square$$

We immediately deduce that $\mathfrak{Q}(\sigma, -i\partial_\sigma)$ is unitarily equivalent to a harmonic oscillator and that the increasing sequence of its eigenvalues is given by

$$\left\{(2n+1)\Big(\frac{\Theta_0\mu''(\xi_0)}{2}(a_{11}+b_{11}-a_1^2)\Big)^{1/2}\right\}_{n\in\mathbb{N}}.$$

The compatibility equation becomes

$$\mathfrak{Q}(\sigma, D_\sigma)f_0 = (\lambda_2 - D)f_0.$$

Thus, we choose λ_2 such that $\lambda_2 - D$ is in the spectrum of $\mathfrak{Q}(\sigma, D_\sigma)$ and we take for f_0 the corresponding normalized eigenfunction (which is in the Schwartz class). For that choice of f_0, we can consider the unique solution $\psi_2^\perp$ (which is in the Schwartz class) of

$$(\mathcal{L}_0 - \Theta_0)\psi_2^\perp = \lambda_2\psi_0 - \mathcal{L}_1\big(if_0'(\sigma)v_{\xi_0} + a_1\sigma f_0(\sigma)S_0u\big) - \mathcal{L}_2\psi_0$$

satisfying $\langle\psi_2^\perp, u_{\xi_0}\rangle = 0$. It follows that ψ_2 is in the form

$$\psi_2 = \psi_2^\perp(\sigma,\tau) + if_1'(\sigma)v_{\xi_0} + a_1\sigma f_1(\sigma)S_0u + f_2(\sigma)u_{\xi_0},$$

where f_1 and f_2 are still to be determined.

Higher-order terms. Let $N \geq 2$. Let us assume that, for $0 \leq j \leq N-2$, the functions ψ_j are determined and belong to the Schwartz class. Moreover, let us also assume that, for $j = N-1, N$, we can write

$$\psi_j(\sigma,\tau) = \psi_j^\perp(\sigma,\tau) + if_{j-1}'(\sigma)v_{\xi_0} + a_1\sigma f_{j-1}(\sigma)S_0u + f_j(\sigma)u_{\xi_0},$$

where the $(\psi_j^\perp)_{j=N-1,N}$ and f_{N-2} are determined functions in the Schwartz class and the $(f_j)_{j=N-1,N}$ are not determined. Finally, we also assume that the $(\lambda_j)_{0\leq j\leq N}$ are determined. We notice that this recursion assumption is satisfied for $N=2$. Let us write the equation of order $N+1$:

$$(\mathscr{L}_0-\Theta_0)\psi_{N+1}=\lambda_{N+1}\psi_0-\mathscr{L}_1\psi_N+(\lambda_2-\mathscr{L}_2)\psi_{N-1}-\mathscr{L}_{N+1}\psi_0+\sum_{j=1}^{N-2}(\lambda_{N+1-j}-\mathscr{L}_{N+1-j})\psi_j.$$

This equation takes the form

$$(\mathscr{L}_0-\Theta_0)\psi_{N+1}=\lambda_{N+1}\psi_0-\mathscr{L}_1\psi_N+(\lambda_2-\mathscr{L}_2)\,\psi_{N-1}+F_N(\sigma,\tau),$$

where F_N is a determined function in the Schwartz class by the recursion assumption. By Proposition 2.2, we can rewrite

$$\begin{aligned}(\mathscr{L}_0-\Theta_0)\big(\psi_{N+1}-if_N'(\sigma)v_{\xi_0}-a_1\sigma f_N(\sigma)S_0u\big)&\\ =\lambda_{N+1}\psi_0-\mathscr{L}_1\big(\psi_N^{\perp}(\sigma,\tau)+if_{N-1}'(\sigma)v_{\xi_0}+a_1\sigma f_{N-1}(\sigma)S_0u\big)+(\lambda_2-\mathscr{L}_2)\psi_{N-1}+F_N(\sigma,\tau)&\\ =\lambda_{N+1}\psi_0-\mathscr{L}_1\big(if_{N-1}'(\sigma)v_{\xi_0}+a_1\sigma f_{N-1}(\sigma)S_0u\big)+(\lambda_2-\mathscr{L}_2)(f_{N-1}u_{\xi_0})+G_N(\sigma,\tau),&\end{aligned}$$

where G_N is a determined function of the Schwartz class. We now write the Fredholm condition. The same computation as previously leads to an equation in the form

$$\mathfrak{Q}(\sigma,-i\partial_\sigma)f_{N-1}=\big(\lambda_2-C(a_2,b_2,k_0)\big)f_{N-1}+\lambda_{N+1}f_0+g_N(\sigma),$$

with $g_N=\langle G_N,u_{\xi_0}\rangle_\tau$. This can be rewritten as

$$\big(\mathfrak{Q}(\sigma,-i\partial_\sigma)-(\lambda_2-C(a_2,b_2,k_0))\big)f_{N-1}=g_N(\sigma)+\lambda_{N+1}f_0.$$

The Fredholm condition applied to this equation provides $\lambda_{N+1}=-\langle g_N,f_0\rangle_\sigma$ and a unique solution f_{N-1} in the Schwartz class such that $\langle f_{N-1},f_0\rangle_\sigma=0$. For this choice of f_{N-1} and λ_{N+1}, we can consider the unique solution $\psi_{N+1}^{\perp}$ (in the Schwartz class) such that

$$\begin{aligned}(\mathscr{L}_0-\Theta_0)\psi_{N+1}^{\perp}&\\ =\lambda_{N+1}\psi_0-\mathscr{L}_1\big(\psi_N^{\perp}(\sigma,\tau)+if_{N-1}'(\sigma)v_{\xi_0}+a_1\sigma f_{N-1}(\sigma)S_0u\big)+(\lambda_2-\mathscr{L}_2)\psi_{N-1}+F_N(\sigma,\tau).&\end{aligned}$$

This leads us to take

$$\psi_{N+1}=\psi_{N+1}^{\perp}+if_N'(\sigma)v_{\xi_0}+a_1\sigma f_N(\sigma)S_0u+f_{N+1}u_{\xi_0}.$$

This ends the proof of the recursion. Thus, we have constructed two sequences $(\lambda_j)_j$ and $(\psi_j)_j$ which depend on n (through the choice of f_0). Let us write $\lambda_{n,j}$ for λ_j and $\psi_{n,j}$ for ψ_j to emphasize this dependence.

Conclusion: proof of Theorem 2.1. Let us consider a smooth cutoff function χ_0 near x_0. For $n\geq 1$ and $J\geq 0$, we let

$$\psi_h^{[n,J]}(x)=\chi_0(x)\sum_{j=0}^{J}\psi_{n,j}\big(h^{-1/4}s(x),h^{-1/2}t(x)\big)h^{j/4}\tag{2-10}$$

and

$$\lambda_h^{[n,J]}=\sum_{j=0}^{J}\lambda_{n,j}h^{j/4}.$$

Using the fact that the ψ_j are in the Schwartz class, we get

$$\big\|\big(P_{h,\boldsymbol{A}}-\lambda_h^{[n,J]}\big)\psi_h^{[n,J]}\big\|\le C(n,J)h^{(J+1)/4}\|\psi_h^{[n,J]}\|.$$

Thanks to the spectral theorem, we deduce Theorem 2.1.

3. Rough lower bound and consequence

This section is devoted to establishing a rough lower bound for $\lambda_n(h)$. In particular, we give the first term of the asymptotics and deduce the so-called normal Agmon estimates, which are rather standard (see, for instance, [Helffer and Morame 2001; Fournais and Helffer 2006; Raymond 2009]).

A first lower bound. We now aim at proving a lower bound:

Proposition 3.1. *We have*

$$\lambda_n(h)\ge\Theta_0 h a(x_0)\beta(x_0)-Ch^{5/4}.$$

Proof. We use a partition of unity with balls D_j of size $r=h^\rho$, satisfying

$$\sum_j\chi_j^2=1\quad\text{and}\quad\sum_j\|\nabla\chi_j\|^2\le Cr^{-2}=Ch^{-2\rho}.\tag{3-1}$$

The so-called IMS formula (see [Cycon et al. 1987]) provides

$$Q_{h,\boldsymbol{A}}(\psi)=\sum_j Q_{h,\boldsymbol{A}}(\chi_j\psi)-h^2\sum_j\int_\Omega a\|\nabla\chi_j\|^2|\psi|^2\,dx,$$

and thus

$$Q_{h,\boldsymbol{A}}(\psi)\ge\sum_j Q_{h,\boldsymbol{A}}(\chi_j\psi)-Ch^{2-2\rho}\|\psi\|^2.$$

In each ball, we approximate a by a constant:

$$Q_{h,\boldsymbol{A}}(\chi_j\psi)\ge(a(x_j)-Ch^\rho)\|(-ih\nabla+\boldsymbol{A})(\chi_j\psi)\|^2.$$

If D_j does not intersect the boundary, then

$$\|(-ih\nabla+\boldsymbol{A})(\chi_j\psi)\|^2\ge h\int_\Omega\beta(x)|\chi_j\psi|^2\,dx.$$

We deduce

$$Q_{h,\boldsymbol{A}}(\chi_j\psi)\ge\big(a(x_j)\beta(x_j)h-Ch^{1+\rho}\big)\|\chi_j\psi\|^2.$$

If D_j intersects the boundary, we can assume that its center is on the boundary, and we write in the local coordinates (up to a change of gauge):

$$Q_{h,\boldsymbol{A}}(\chi_j\psi)\ge(1-Ch^\rho)\int\tilde a\big(h^2|\partial_t(\chi_j\psi)|^2+|(-ih\partial_s+\tilde A_1)(\chi_j\psi)|^2\big)\,ds\,dt.$$

We deduce

$$Q_{h,\boldsymbol{A}}(\chi_j\psi)\ge(1-Ch^\rho)(a(x_j)-Ch^\rho)\int h^2|\partial_t(\chi_j\psi)|^2+|(-ih\partial_s+\tilde A_1)(\chi_j\psi)|^2\,ds\,dt.$$

We approximate A_1 by its linear approximation A_1^{lin}, and we have

$$\int h^2|\partial_t(\chi_j\psi)|^2+|(-ih\partial_s+\tilde{A}_1)(\chi_j\psi)|^2\,ds\,dt$$
$$\geq (1-\varepsilon)\int h^2|\partial_t(\chi_j\psi)|^2+|(-ih\partial_s+\tilde{A}_1^{\text{lin}})(\chi_j\psi)|^2\,ds\,dt-C\varepsilon^{-1}\int |x-x_j|^4|\chi_j\psi|^2\,dx$$
$$\geq \big((1-\varepsilon)\Theta_0\beta(x_j)h-C\varepsilon^{-1}h^{4\rho}\big)\|\chi_j\psi\|^2.$$

To optimize the remainder, we choose $\varepsilon=h^{2\rho-1/2}$. Then we take $\rho=\frac{3}{8}$, and the conclusion follows. □

Normal Agmon estimates: localization in t. We now prove the following (weighted) localization estimates:

Proposition 3.2. *Let us consider a smooth cutoff function χ supported in a fixed neighborhood of the boundary. Let $(\lambda_n(h),\psi_h)$ be an eigenpair of $P_{h,\boldsymbol{A}}$. For all $\delta\geq 0$, there exist ε_0, $C\geq 0$ and h_0 such that, for $h\in(0,h_0)$,*

$$\|e^{\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\|^2\leq C\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\|^2,$$
$$Q_{h,\boldsymbol{A}}\big(e^{\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big)\leq Ch\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\|^2.$$

Proof. The proof is based on a technique of Agmon (see, for instance, [Agmon 1982; 1985; Helffer 1988]). Let us recall the IMS formula; we have, for an eigenpair $(\lambda_n(h),\psi_h)$,

$$Q_{h,\boldsymbol{A}}(e^{\Phi}\psi_h)=\lambda_n(h)\|e^{\Phi}\psi_h\|^2+h^2\|a^{1/2}\nabla\Phi e^{\Phi}\psi_h\|^2.$$

We take

$$\Phi=\varepsilon_0 t(x)h^{-1/2}+\delta\chi(x)|s(x)|h^{-1/4},\tag{3-2}$$

where χ is a smooth cutoff function supported near the boundary and where $s:\partial\Omega\mapsto(-|\partial\Omega|/2,|\partial\Omega|/2)$ is the curvilinear coordinate such that $s(x_0)=0$. We use a partition of unity χ_j as in (3-1), but with balls of radius $Rh^{1/2}$ with R large enough (the x_j denote the centers), and we get

$$\sum_j\big(Q_{h,\boldsymbol{A}}(\chi_je^{\Phi}\psi_h)-\lambda_n(h)\|\chi_je^{\Phi}\psi_h\|^2-CR^{-2}h-h^2\|\chi_ja^{1/2}\nabla\Phi e^{\Phi}\psi_h\|^2\big)\leq 0.$$

We now distinguish between the balls intersecting the boundary (bnd) and the others (int). For the interior balls, we have the lower bound, for $\eta>0$ and h small enough,

$$Q_{h,\boldsymbol{A}}(\chi_je^{\Phi}\psi_h)\geq\big(a(x_j)\beta(x_j)h-Ch^{3/2}\big)\|\chi_je^{\Phi}\psi_h\|^2.$$

For the boundary balls, we have

$$Q_{h,\boldsymbol{A}}(\chi_je^{\Phi}\psi_h)\geq\big(\Theta_0a(x_j)\beta(x_j)h-Ch^{3/2}\big)\|\chi_je^{\Phi}\psi_h\|^2.$$

Let us now split the sum:

$$\sum_{j\ \text{int}}\int\big(a(x_j)\beta(x_j)h-\Theta_0a(x_0)\beta(x_0)h-Ch^{3/2}-CR^{-2}h-Ch^2\|\nabla\Phi\|^2\big)|\chi_je^{\Phi}\psi_h|^2\,dx$$
$$\leq Ch\sum_{j\ \text{bnd}}\|\chi_je^{\Phi}\psi_h\|^2.$$

With (3-2), we can notice that

$$\|\nabla\Phi\|^2 \le C(\varepsilon_0^2 h^{-1} + \delta^2 h^{-1/2}).$$

Taking R large enough and ε_0 and h small enough and using (1-5), we get the existence of $c > 0$ such that

$$a(x_j)\beta(x_j)h - \Theta_0 a(x_0)\beta(x_0)h - Ch^{3/2} - CR^{-2}h - Ch^2\|\nabla\Phi\|^2 \ge ch.$$

We deduce

$$c\sum_{j \text{ int}} \|\chi_j e^{\Phi}\psi_h\|^2 \le C \sum_{j \text{ bnd}} \|\chi_j e^{\Phi}\psi_h\|^2.$$

Due to support considerations, we can write

$$C\sum_{j \text{ bnd}} \|\chi_j e^{\Phi}\psi_h\|^2 \le \tilde{C} \sum_{j \text{ bnd}} \big\|\chi_j e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

Thus, we infer

$$\|e^{\Phi}\psi_h\|^2 \le \tilde{C}\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

We deduce that

$$\sum_j Q_{h,\boldsymbol{A}}(\chi_j e^{\Phi}\psi_h) \le Ch\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2,$$

and thus

$$Q_{h,\boldsymbol{A}}(e^{\Phi}\psi_h) \le Ch\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2. \qquad \square$$

Corollary 3.3. *Let $\eta \in \big(0, \frac{1}{2}\big]$. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,\boldsymbol{A}}$. For all $\delta \ge 0$, there exist $\varepsilon_0, C \ge 0$ and h_0 such that, for $h \in (0, h_0)$,*

$$\big\|\chi_{h,\eta} e^{\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 \le C\big\|\chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2,$$

$$Q_{h,\boldsymbol{A}}\big(\chi_{h,\eta} e^{\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big) \le Ch\big\|\chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2,$$

where $\chi_{h,\eta}(x) = \hat{\chi}(t(x)h^{-1/2+\eta})$, and with $\hat{\chi}$ a smooth cutoff function being 1 near 0.

Proof. With Proposition 3.2, we have

$$\big\|\chi_{h,\eta} e^{\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 \le C\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

We can write

$$\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 = \big\|\chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 + \big\|\sqrt{1-\chi_{h,\eta}}\, e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.$$

Using Proposition 3.2, we have the estimate

$$\begin{aligned}\big\|\sqrt{1-\chi_{h,\eta}}\, e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 &= \big\|\sqrt{1-\chi_{h,\eta}}\, e^{-\varepsilon_0 t(x)h^{-1/2}} e^{\chi(x)\varepsilon_0 t(x)h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2 \\ &= O(h^{\infty})\big\|e^{\delta\chi(x)|s(x)|h^{-1/4}}\psi_h\big\|^2.\end{aligned}$$

The IMS formula provides

$$Q_{h,\boldsymbol{A}}(e^{\Phi}\psi_h) = Q_{h,\boldsymbol{A}}(\chi_{h,\eta} e^{\Phi}\psi_h) + Q_{h,\boldsymbol{A}}\big(\sqrt{1-\chi_{h,\eta}}\, e^{\Phi}\psi_h\big) + O(h^{1+2\eta})\|e^{\Phi}\psi_h\|^2. \qquad \square$$

Corollary 3.4. *Let $\eta \in \left(0, \frac{1}{2}\right]$. Let $(\lambda_n(h), \psi_h)$ be an eigenpair of $P_{h,\mathbf{A}}$. For all $\delta \geq 0$, there exist ε_0, $C \geq 0$ and h_0 such that, for $h \in (0, h_0)$,*

$$\left\| \chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} (-ih\partial_s + \tilde{A}_1)\psi_h \right\|^2 \leq Ch \left\| \chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h \right\|^2,$$
$$\left\| \chi_{h,\eta} e^{\varepsilon_0 t(x) h^{-1/2} + \delta\chi(x)|s(x)|h^{-1/4}} (-ih\partial_t + \tilde{A}_2)\psi_h \right\|^2 \leq Ch \left\| \chi_{h,\eta} e^{\delta\chi(x)|s(x)|h^{-1/4}} \psi_h \right\|^2.$$

4. Order of the second term: localization in *s*

It is well-known that the order of the second term in the asymptotics of $\lambda_n(h)$ is closely related to localization properties of the corresponding eigenfunctions. The aim of this section is to establish such properties. Let us mention that similar estimates were proved in [Raymond 2009] through a technical analysis. Here we give a less technical proof using a very rough functional calculus.

Proposition 4.1. *Under the generic assumptions, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$,*

$$\lambda_n(h) \geq \Theta_0 a(x_0)\beta(x_0)h - Ch^{3/2}.$$

Moreover, for all $\delta \geq 0$, there exist $C > 0$ and $h_0 > 0$ such that for $h \in (0, h_0)$,

$$\int e^{2\delta\chi(x)|s|h^{-1/4}} |\psi|^2 \, ds \, dt \leq C\|\psi\|^2.$$

Proof. Let us recall the so-called IMS formula (see, for instance, [Cycon et al. 1987]); we have, for an eigenpair $(\lambda_n(h), \psi)$,

$$Q_{h,\mathbf{A}}(e^{\Phi}\psi) - \lambda_n(h)\|e^{\Phi}\psi\|^2 - h^2\|a^{1/2}\nabla\Phi e^{\Phi}\psi\|^2 = 0.$$

We take

$$\Phi = \delta\chi(x)|s(x)|h^{-1/4}, \quad \text{with } \delta \geq 0. \tag{4-1}$$

The idea is now to prove a suitable lower bound for $Q_{h,\mathbf{A}}$. We use a partition of unity (χ_j) (see (3-1)) with balls of radius $h^{1/4}$ and centers (s_j, t_j). We get the lower bound

$$Q_{h,\mathbf{A}}(e^{\Phi}\psi) \geq \sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2}\|e^{\Phi}\psi\|^2,$$

where

$$\psi_j = \chi_j e^{\Phi}\psi,$$

and we deduce

$$\sum_j Q_{h,\mathbf{A}}(\psi_j) - Ch^{3/2}\|\psi_j\|^2 - \lambda_n(h)\|\psi_j\|^2 \leq 0, \tag{4-2}$$

since we have, thanks to (4-1), $\|\nabla\Phi\|^2 \leq Ch^{-1/2}$.

Interior balls. Considering the balls not intersecting the boundary, we get (see the proof of Proposition 3.1):

$$\sum_{j \text{ int}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ int}} \left(a(x_j)\beta(x_j)h - Ch^{5/4}\right)\|\psi_j\|^2.$$

Using Assumption (1-4), we deduce

$$\sum_{j \text{ int}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ int}} \left(\Theta_0 a(x_0)\beta(x_0)h - Ch^{5/4}\right)\|\psi_j\|^2. \tag{4-3}$$

Boundary balls. Let us consider the j such that D_j intersects the boundary (we can assume that its center is $(s_j, 0)$). Using first the normal Agmon estimates, we have the lower bound

$$\sum_{j \text{ bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ bnd}} \int \tilde{a}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt - Ch^{3/2}\|e^{\Phi}\psi\|^2,$$

where we have used the IMS formula to get

$$\sum_{j \text{ bnd}} \int t\tilde{a}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt$$
$$\leq C \int_{0<t<t_0} t\tilde{a}\big(|(-ih\partial_t + \tilde{A}_2)e^{\Phi}\psi|^2 + |(ih\partial_s + \tilde{A}_1)e^{\Phi}\psi|^2\big)\, ds\, dt + Ch^{3/2}\|e^{\Phi}\psi\|^2.$$

Using again the normal estimates (see Corollaries 3.3 and 3.4) and also the size of the balls, we get

$$\sum_{j \text{ bnd}} Q_{h,\mathbf{A}}(\psi_j) \geq \sum_{j \text{ bnd}} \int \tilde{a}_j^{\text{lin}}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt - Ch^{3/2}\|e^{\Phi}\psi\|^2, \quad (4\text{-}4)$$

where

$$\tilde{a}_j^{\text{lin}} = a_j + (s - s_j)\partial_s\tilde{a}(x_j).$$

Let us fix $S_0 > 0$ to distinguish between the balls whose centers are close to $x_0 = (0, 0)$ and the others.

Case $|s_j| \geq S_0$. Let us consider the boundary balls such that $|s_j| \geq S_0$. Using the size of the balls, we get the lower bound

$$\int \tilde{a}_j^{\text{lin}}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt \geq \big(\Theta_0 a(x_j)\beta(x_j)h - Ch^{5/4}\big)\|\psi_j\|^2$$
$$\geq \Theta_0(1+\varepsilon)a(x_0)\beta(x_0)h\|\psi_j\|^2, \quad (4\text{-}5)$$

where $\varepsilon > 0$ only depends on S_0, β, a and Ω.

Case $|s_j| \leq S_0$. Let us consider the boundary balls such that $|s_j| \leq S_0$. In each ball, we can use a new gauge so that

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq S_0}} \int \tilde{a}_j^{\text{lin}}\big(|(-ih\partial_t + \tilde{A}_2)\psi_j|^2 + |(ih\partial_s + \tilde{A}_1)\psi_j|^2\big)\, ds\, dt$$
$$= \sum_{\substack{j \text{ bnd} \\ |s_j| \leq S_0}} \int \tilde{a}_j^{\text{lin}}\big(|h\partial_t\psi_j|^2 + |(ih\partial_s + \tilde{A}_1^{\text{new}})\psi_j|^2\big)\, ds\, dt,$$

where $\tilde{A}_1^{\text{new}}$ (we omit the dependence on j) satisfies

$$|\tilde{A}_1^{\text{new}} - t\tilde{\beta}_j^{\text{lin}}| \leq C(t|s - s_j|^2 + t^2),$$

with

$$\tilde{\beta}_j^{\text{lin}} = \tilde{\beta}_j + \partial_s\tilde{\beta}(x_j)(s - s_j).$$

We obtain, thanks to the (weighted) estimates of Agmon,

$$\sum_{\substack{j \text{ bnd} \\ |s_j|\le S_0}} \int \tilde{a}_j^{\text{lin}}\big(|h\partial_t\psi_j|^2 + |(ih\partial_s + \tilde{A}_1^{\text{new}})\psi_j|^2\big)\,ds\,dt \\ \geq (1-h^{1/2}) \sum_{\substack{j \text{ bnd} \\ |s_j|\le S_0}} \int \tilde{a}_j^{\text{lin}}\big(h^2|\partial_t\psi_j|^2 + |(ih\partial_s + t\tilde{\beta}_j^{\text{lin}})\psi_j|^2\big)\,ds\,dt - Ch^{3/2}\|e^{\Phi}\psi\|^2. \tag{4-6}$$

In each ball, we use the change of variables (which is a scaling with respect to τ depending on σ)

$$\sigma = s \quad \text{and} \quad \tau = \{\tilde{\beta}_j^{\text{lin}}\}^{1/2} t.$$

We can write

$$\partial_t = \{\tilde{\beta}_j^{\text{lin}}\}^{1/2}\partial_\tau \quad \text{and} \quad \partial_s = \partial_\sigma + \partial_s(\{\tilde{\beta}_j^{\text{lin}}\}^{1/2})\partial_\tau$$

and

$$ds\,dt = \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau.$$

We obtain

$$\int \tilde{a}_j^{\text{lin}}\big(h^2|\partial_t\psi_j|^2 + |(ih\partial_s + t\tilde{\beta}_j^{\text{lin}})\psi_j|^2\big)\,ds\,dt \\ \geq (1-h^{1/2})\int \tilde{a}_j^{\text{lin}}\tilde{\beta}_j^{\text{lin}}\big(h^2|\partial_t\hat{\psi}_j|^2 + |\big(ih\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\partial_\sigma + \tau\big)\hat{\psi}_j|^2\big)\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ - Ch^{3/2}\int |\tau\partial_\tau\hat{\psi}_j|^2\,d\sigma\,d\tau, \tag{4-7}$$

where $\hat{\psi}_j$ denotes ψ_j in the coordinates (σ, τ). With the normal Agmon estimates (see Corollaries 3.3 and 3.4), we have

$$\sum_{\substack{j \text{ bnd} \\ |s_j|\le S_0}} \int |\tau\partial_\tau\hat{\psi}_j|^2\,d\sigma\,d\tau \le C\|e^{\Phi}\psi\|^2.$$

We must now obtain an appropriate lower bound for

$$\int \tilde{a}_j^{\text{lin}}\tilde{\beta}_j^{\text{lin}}\big(h^2|\partial_t\hat{\psi}_j|^2 + |\big(ih\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\partial_\sigma + \tau\big)\hat{\psi}_j|^2\big)\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau.$$

This is the end of the following lemma.

Lemma 4.2. *We have*

$$\int \tilde{a}_j^{\text{lin}}\tilde{\beta}_j^{\text{lin}}\big(h^2|\partial_t\hat{\psi}_j|^2 + \big|\big(ih\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\partial_\sigma + \tau\big)\hat{\psi}_j\big|^2\big)\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ \geq h\Theta_0\int \Big(a(x_0)\beta(x_0) + \frac{\alpha}{4}\sigma^2\Big)|\hat{\psi}_j|^2\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}\,d\sigma\,d\tau - Ch^2\|\hat{\psi}_j\|^2.$$

Proof. We can notice that the Dirichlet realization on $(-\tilde{S}_0, \tilde{S}_0)$ of $D_\sigma\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}$ is self-adjoint on $L^2(\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}d\sigma)$. Thus, we shall commute D_σ and $\{\tilde{\beta}_j^{\text{lin}}\}^{-1/2}$ and control the remainder due to the commutator.

Notation 4.3. Henceforth, $\partial_\sigma(f)$ will denote the derivative of the function f, whereas $\partial_\sigma f$ will denote the composition of the differentiation ∂_σ with the multiplication by f.

We can write

$$\begin{aligned}\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\partial_\sigma+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ =\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}h^2|\partial_t\hat{\psi}_j|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ +\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}-ih\partial_\sigma(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2})+\tau\big)\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau.\end{aligned}$$

We can estimate the double product:

$$\begin{aligned}2h\Re\Big(\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j i\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\big)\overline{\hat{\psi}_j}\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau\Big) \\ =-2h^2\Re\Big(\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\big)\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\overline{\hat{\psi}_j}\,d\sigma\,d\tau\Big) \\ =-h^2\int a_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\big)\partial_\sigma\Big(\big|\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big|^2\Big)\,d\sigma\,d\tau=O(h^2)\|\psi_j\|^2,\end{aligned}$$

where we have used an integration by parts for the last estimate. We deduce

$$\begin{aligned}\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\partial_\sigma+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ \geq\int \tilde{a}_j^{\mathrm{lin}}\tilde{\beta}_j^{\mathrm{lin}}\Big(h^2|\partial_t\hat{\psi}_j|^2+\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j\big|^2\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau-Ch^2\|\psi_j\|^2. \quad (4\text{-}8)\end{aligned}$$

For S_0 small enough, we have, using the nondegeneracy, for s such that $|s|\leq\tilde{S}_0$ (with $\tilde{S}_0$ slightly bigger than S_0),

$$\tilde{a}_j^{\mathrm{lin}}(s)\tilde{\beta}_j^{\mathrm{lin}}(s)\geq a(x_0)\beta(x_0)+\frac{\alpha}{4}|s|^2.$$

Let us analyze the integral:

$$\begin{aligned}\int \big|\sigma\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ =\int\big|\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\sigma\hat{\psi}_j-ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau.\end{aligned}$$

We must estimate the double product:

$$\begin{aligned}2\Re\int\Big(\big(ih\partial_\sigma\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}+\tau\big)\sigma\hat{\psi}_j ih\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\overline{\hat{\psi}_j}\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ =-2h^2\Re\int\Big(\partial_\sigma\big(\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\sigma\hat{\psi}_j\big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\overline{\hat{\psi}_j}\Big)\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau \\ =-h^2\int\partial_\sigma\big|\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\hat{\psi}_j\big|^2\{\tilde{\beta}_j^{\mathrm{lin}}\}^{-1/2}\,d\sigma\,d\tau+O(h^2)\|\hat{\psi}_j\|^2=O(h^2)\|\hat{\psi}_j\|^2.\end{aligned}$$

We infer:

$$\int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \Big(h^2 |\partial_t \hat{\psi}_j|^2 + \big| \big(ih\partial_\sigma \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} + \tau \big) \hat{\psi}_j \big|^2 \Big) \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} \, d\sigma \, d\tau$$
$$\geq a(x_0)\beta(x_0) \int \Big(h^2 |\partial_t \hat{\psi}_j|^2 + \big| \big(ih\partial_\sigma \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} + \tau \big) \hat{\psi}_j \big|^2 \Big) \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} \, d\sigma \, d\tau$$
$$+ \frac{\alpha}{4} \int \Big(h^2 |\partial_t (\sigma \hat{\psi}_j)|^2 + \big| \big(ih\partial_\sigma \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} + \tau \big) \sigma \hat{\psi}_j \big|^2 \Big) \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} \, d\sigma \, d\tau - Ch^2 \|\hat{\psi}_j\|^2.$$

We recall that, for all $\xi \in \mathbb{R}$,

$$\int \Big(h^2 |\partial_t \phi|^2 + \big| (\tau - h\xi - \xi_0 h^{1/2}) \phi \big|^2 \Big) \, d\tau \geq h\mu(\xi_0 + h^{1/2}\xi) \|\phi\|^2 \geq \Theta_0 h \|\phi\|^2.$$

We infer with the functional calculus:

$$\int \tilde{a}_j^{\text{lin}} \tilde{\beta}_j^{\text{lin}} \Big(h^2 |\partial_t \hat{\psi}_j|^2 + \big| \big(ih\partial_\sigma \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} + \tau - \xi_0 h^{1/2} \big) \hat{\psi}_j \big|^2 \Big) \{\hat{\beta}_j^{\text{lin}}\}^{-1/2} \, d\sigma \, d\tau$$
$$\geq h\Theta_0 \int \Big(a(x_0)\beta(x_0) + \frac{\alpha}{4}\sigma^2 \Big) |\hat{\psi}_j|^2 \{\tilde{\beta}_j^{\text{lin}}\}^{-1/2} \, d\sigma \, d\tau - Ch^2 \|\hat{\psi}_j\|^2. \quad \text{(4-9)}$$

This concludes the proof. ☐

Lower bound for $\lambda_n(h)$. If we take $\delta = 0$, we deduce, with (4-2)–(4-7) and Lemma 4.2,

$$\lambda_n(h) \|\psi\|^2 \geq \sum_j \Theta_0 h a(x_0) \beta(x_0) \int |\psi_j|^2 \, dx - Ch^{3/2} \|\psi\|^2.$$

Tangential Agmon estimate. Gathering the estimates (4-3), (4-5), (4-7) and Lemma 4.2, we deduce the existence of $c > 0$ such that

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq S_0}} \Big(\Theta_0 h \int \Big(a(x_0)\beta(x_0) + \frac{\alpha}{4} s^2 \Big) |\psi_j|^2 \, ds \, dt - \Theta_0 h \|\psi_j\|^2 - Ch^{3/2} \|\psi_j\|^2 \Big)$$
$$+ \sum_{\substack{j \text{ bnd} \\ |s_j| \geq S_0}} ch \|\psi_j\|^2 + \sum_{j \text{int}} ch \|\psi_j\|^2 \leq 0$$

and

$$\sum_{\substack{j \text{ bnd} \\ 2C_0 h^{1/4} \leq |s_j| \leq s_0}} \Big(\Theta_0 h \int \frac{\alpha}{4} s^2 |\psi_j|^2 \, ds \, dt - Ch^{3/2} \|\psi_j\|^2 \Big) \leq Ch^{3/2} \|\psi_0\|^2 \leq Ch^{3/2} \|\psi\|^2.$$

Taking C_0 large enough, we infer

$$\sum_{\substack{j \text{ bnd} \\ 2C_0 h^{1/4} \leq |s_j| \leq s_0}} \|\psi_j\|^2 \leq C \|\psi\|^2,$$

so that

$$\sum_{\substack{j \text{ bnd} \\ |s_j| \leq s_0}} \|\psi_j\|^2 \leq C \|\psi\|^2 \quad \text{and} \quad \sum_j \|\psi_j\|^2 = \|e^{\Phi} \psi\|^2 \leq C \|\psi\|^2.$$ ☐

Let us write an immediate corollary (see Corollaries 3.3 and 3.4).

Corollary 4.4. *Let* $(\eta_1,\eta_2)\in\left(0,\frac{1}{2}\right]\times\left(0,\frac{1}{4}\right]$. *Let* $(\lambda_n(h),\psi_h)$ *be an eigenpair of* $P_{h,\mathbf{A}}$. *For all* $(k,l)\in\mathbb{N}$, *there exist* $C\geq 0$ *and* $h_0>0$ *such that, for* $h\in(0,h_0)$,

$$\left\|\chi_{h,\eta_1,\eta_2}s^k t^l\psi_h\right\|^2\leq Ch^{k/2}h^l\|\psi_h\|^2,$$
$$\left\|\chi_{h,\eta_1,\eta_2}s^k t^l(-ih\partial_s+\tilde{A}_1)\psi_h\right\|^2\leq Chh^{k/2}h^l\|\psi_h\|^2,$$
$$\left\|\chi_{h,\eta_1,\eta_2}s^k t^l(-ih\partial_t+\tilde{A}_2)\psi_h\right\|^2\leq Chh^{k/2}h^l\|\psi_h\|^2,$$

where $\chi_{h,\eta_1,\eta_2}(x)=\hat{\chi}(t(x)h^{-1/2+\eta_1})\hat{\chi}(s(x)h^{-1/4+\eta_2})$. *Moreover, we have*

$$\left\|(1-\chi_{h,\eta_1,\eta_2})s^k t^l\psi_h\right\|^2=O(h^\infty)\|\psi_h\|^2,$$
$$\left\|(1-\chi_{h,\eta_1,\eta_2})s^k t^l(-ih\partial_s+\tilde{A}_1)\psi_h\right\|^2=O(h^\infty)\|\psi_h\|^2,$$
$$\left\|(1-\chi_{h,\eta_1,\eta_2})s^k t^l(-ih\partial_t+\tilde{A}_2)\psi_h\right\|^2=O(h^\infty)\|\psi_h\|^2.$$

Remark 4.5. In the following, each reference to the "estimates of Agmon" will be a reference to this last corollary. Moreover, at some point, the localization ideas behind Section 3 and 4, which are summarized in the last corollary, follow from the general philosophy developed in the last decade (an improvement of the approximation of the eigenvalues provides an improvement of localization and conversely). In the next section, we will strongly use these a priori estimates.

5. Unitary transforms and the Born–Oppenheimer approximation

We use a cutoff function χ_h near x_0 with support or order $h^{1/4-\tilde{\eta}}$ with $\tilde{\eta}>0$. For all $N\geq 1$, let us consider L^2-normalized eigenpairs $(\lambda_n(h),\psi_{n,h})_{1\leq n\leq N}$ such that $\langle\psi_{n,h},\psi_{m,h}\rangle=0$ when $n\neq m$. We consider the N dimensional space defined by

$$\mathfrak{E}_N(h)=\underset{1\leq n\leq N}{\operatorname{span}}\,\tilde{\psi}_{n,h},\quad\text{where }\tilde{\psi}_{n,h}=\chi_h\psi_{n,h}.$$

Remark 5.1. The estimates of Agmon of Corollary 4.4 are satisfied by all the elements of $\mathfrak{E}_N(h)$.

We can notice that, with the estimates of Agmon, for all $\tilde{\psi}\in\mathfrak{E}_N(h)$,

$$Q_{h,\mathbf{A}}(\tilde{\psi})\leq\lambda_N(h)\|\tilde{\psi}\|^2+O(h^\infty)\|\tilde{\psi}\|^2. \tag{5-1}$$

In the following subsection, we provide a lower bound for $Q_{h,\mathbf{A}}$ on $\mathfrak{E}_N(h)$.

Remark 5.2. Let us underline the main spirit of this section. We are going to use successive canonical transformations of the symbol of our operator (change of variable, change of gauge, weighted Fourier transform) or, equivalently, of the associated quadratic form. In the spirit of Egorov's theorem, all these transformations will give rise to different remainders which can be treated thanks to the a priori localization estimates. Then, after conjugations by these successive unitary transforms, we will reduce the analysis to one of an electric Laplacian in the Born–Oppenheimer form.

Choice of gauge and new coordinates: a first lower bound. On the support of χ_h, we use a gauge such that $\tilde{A}_2 = 0$ and

$$|\tilde{A}_1 - \tilde{A}_1^{\mathrm{app}}| \le C(t^3 + |s|t^2 + |s|^2 t),$$

where

$$\tilde{A}_1^{\mathrm{app}} = t(1 + b_1 s + b_{11} s^2) - \xi_0 \hat{b}(s)^{1/2} h^{1/2} + \frac{\hat{b}_2}{2} t^2 = t\hat{b}(s) - \xi_0 \hat{b}(s)^{1/2} h^{1/2} + \frac{\hat{b}_2}{2} t^2,$$

where $\hat{b}_2 = b_2 - k_0$. We also let

$$\tilde{a}^{\mathrm{app}}(s, t) = 1 + a_1 s + a_{11} s^2 + a_2 t = \hat{a}(s) + a_2 t.$$

Moreover, in this neighborhood of (0, 0), we introduce new coordinates:

$$\tau = t(\hat{b}(s))^{1/2}, \quad \sigma = s. \tag{5-2}$$

In particular, we get

$$\partial_t = (\hat{b}(\sigma))^{1/2} \partial_\tau, \quad \partial_s = \partial_\sigma + \tfrac{1}{2} \hat{b}^{-1} \partial_s \hat{b} \tau \partial_\tau$$

and

$$ds\, dt = \hat{b}^{-1/2}\, d\sigma\, d\tau.$$

To simplify the notation, we let $p = \hat{b}^{-1/2}$. We will also use the change of variable

$$\check{\sigma} = \int_0^\sigma \frac{1}{p(u)}\, du = f(\sigma)$$

so that $L^2(p\, d\sigma)$ becomes $L^2(\check{p}^2\, d\check{\sigma})$.

This subsection is devoted to the proof of the following lower bound of $Q_{h,\boldsymbol{A}}$ on $\mathfrak{E}_N(h)$.

Proposition 5.3. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$,*

$$Q_{h,\boldsymbol{A}}(\tilde{\psi}) \ge \check{Q}_{h,\mathrm{app}}(\check{\psi}) - C h^{3/2+1/4} \|\tilde{\psi}\|^2, \tag{5-3}$$

where

$$\begin{aligned}\check{Q}_{h,\mathrm{app}}(\check{\psi}) = &\int (1 + a_2\tau)(1 - \tau k_0) |h \partial_\tau \check{\psi}|^2 \check{p}^2\, d\check{\sigma}\, d\tau \\ &+ \int (1 + a_2 \tau)(1 - \tau k_0)^{-1} \Big| \Big(i h \check{p}^{-1} \partial_{\check{\sigma}} \check{p} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} \tau^2 - h \frac{b_1}{2} \tau D_\tau \Big) \check{\psi} \Big|^2 \check{p}^2\, d\check{\sigma}\, d\tau \\ &+ h\alpha\Theta_0 \int \check{\sigma}^2 |\check{\psi}|^2 \check{p}^2\, d\check{\sigma}\, d\tau,\end{aligned}$$

where $\check{\psi}$ denotes $\tilde{\psi}$ in the coordinates $(\check{\sigma}, \tau)$.

In order to prove Proposition 5.3, we will need this lemma:

Lemma 5.4. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$,*

$$Q_{h,\boldsymbol{A}}(\tilde{\psi}) \ge \hat{Q}_{h,\mathrm{app}}(\hat{\psi}) - C h^{3/2+1/4} \|\tilde{\psi}\|^2,$$

where

$$\hat{Q}_{h,\mathrm{app}}(\hat{\psi}) = \int m_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau + \int m_1(\sigma,\tau)\Big|\Big(h\Xi+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau,$$

with

$$\Xi = i\partial_\sigma\hat{b}^{-1/2},\quad m_1(\sigma,\tau)=(1+\alpha\sigma^2)(1+a_2\tau)(1-\tau k_0)^{-1},\quad m_2(\sigma,\tau)=(1+\alpha\sigma^2)(1+a_2\tau)(1-\tau k_0),$$

and where $\hat{\psi}$ denotes $\tilde{\psi}$ in the coordinates (σ,τ).

Proof. We have

$$Q_{h,\boldsymbol{A}}(\tilde{\psi}) = \int \tilde{a}\big(1-tk(s)\big)\big|(-ih\partial_t+\tilde{A}_2)\tilde{\psi}\big|^2+\tilde{a}\big(1-tk(s)\big)^{-1}\big|(ih\partial_s+\tilde{A}_1)\tilde{\psi}\big|^2\,ds\,dt.$$

Thanks to the normal and tangential Agmon estimates, we get

$$Q_{h,\boldsymbol{A}}(\tilde{\psi}) \geq \int \tilde{a}(1-tk_0)h^2|\partial_t\tilde{\psi}|^2+\tilde{a}(1-tk_0)^{-1}\big|(ih\partial_s+\tilde{A}_1)\tilde{\psi}\big|^2\,ds\,dt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

The Agmon estimates imply

$$Q_{h,\boldsymbol{A}}(\tilde{\psi}) \geq \int \tilde{a}^{\mathrm{app}}(1-tk_0)h^2|\partial_t\tilde{\psi}|^2+\tilde{a}^{\mathrm{app}}(1-tk_0)^{-1}\big|(ih\partial_s+\tilde{A}_1^{\mathrm{app}})\tilde{\psi}\big|^2\,ds\,dt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

We get

$$Q_{h,\boldsymbol{A}}(\tilde{\psi}) \geq \int \hat{a}(1+a_2t)\Big((1-tk_0)h^2|\partial_t\tilde{\psi}|^2+(1-tk_0)^{-1}\big|\big(ih\partial_s+\tilde{A}_1^{\mathrm{app}}\big)\tilde{\psi}\big|^2\Big)\,ds\,dt - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

With the coordinates (σ,τ), we obtain

$$\int \hat{a}(1+a_2t)(1-tk_0)h^2|\partial_t\tilde{\psi}|^2+(1+a_2t)(1-tk_0)^{-1}\big|(ih\partial_s+\tilde{A}_1^{\mathrm{app}})\tilde{\psi}\big|^2\,ds\,dt \geq \hat{Q}_h(\hat{\psi})-Ch^{3/2+1/4}\|\tilde{\psi}\|^2,$$

where

$$\hat{Q}_h(\hat{\psi}) = \int \tilde{m}_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau + \int \tilde{m}_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\hat{b}^{-1/2}-h\frac{\partial_\sigma\hat{b}}{2\hat{b}^{3/2}}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau,$$

where

$$\tilde{m}_1(\sigma,\tau)=\hat{a}\hat{b}(1+a_2\tau)(1-\tau k_0)^{-1},\qquad \tilde{m}_2(\sigma,\tau)=\hat{a}\hat{b}(1+a_2\tau)(1-\tau k_0).$$

With the estimates of Agmon, we can simplify the quadratic form modulo lower-order terms:

$$\begin{aligned}\hat{Q}_h(\hat{\psi}) \geq & \int \tilde{m}_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau \\ & + \int \tilde{m}_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau \\ & - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

We recall that $\hat{a}\hat{b} = 1 + \alpha\sigma^2 + O(|\sigma|^3)$, so that with the estimates of Agmon we infer

$$\begin{aligned}\hat{Q}_h(\hat{\psi}) \geq & \int m_2(\sigma,\tau)|h\partial_\tau\hat{\psi}|^2\hat{b}^{-1/2}\,d\sigma\,d\tau \\ & + \int m_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau \\ & - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

We now want to replace $\hat{b}^{-1/2}i\partial_\sigma$ by $i\partial_\sigma\hat{b}^{-1/2}$, which is self-adjoint on $L^2(\hat{b}^{-1/2}d\sigma d\tau)$. Writing a commutator, we get

$$\begin{aligned}& \int m_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}d\sigma d\tau \\ & \quad = \int m_1(\sigma,\tau)\Big|\Big(hi\partial_\sigma\hat{b}^{-1/2} - ih(\partial_\sigma\hat{b}^{-1/2}) + \tau - \xi_0 h^{1/2} + \frac{b_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau.\end{aligned}$$

Let us consider the double product

$$\begin{aligned}2h\Re & \Big(\int m_1(\sigma,\tau)\Big(hi\partial_\sigma\hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\, i(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}d\sigma\,d\tau\Big) \\ & = 2h\Re\Big(\int m_1(\sigma,\tau)\Big(hi\partial_\sigma\hat{b}^{-1/2} - h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\, i(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}\,d\sigma\,d\tau\Big) \\ & = -2h^2\Re\int m_1(\sigma,\tau)\Big(\partial_\sigma(\hat{b}^{-1/2}\hat{\psi})(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}\,d\sigma\,d\tau\Big) + O(h^2)\|\hat{\psi}\|^2,\end{aligned}$$

where we have used the normal Agmon estimates. We deduce that

$$\begin{aligned}& 2\Re\Big(\int m_1(\sigma,\tau)\Big(hi\partial_\sigma\hat{b}^{-1/2} + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2}\tau^2 - h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\, i(\partial_\sigma\hat{b}^{-1/2})\overline{\hat{\psi}}\hat{b}^{-1/2}\,d\sigma\,d\tau\Big) \\ & \qquad = -h^2\int m_1(\sigma,\tau)(\partial_\sigma\hat{b}^{-1/2})\partial_\sigma|\hat{b}^{-1/2}\hat{\psi}|^2\,d\sigma\,d\tau + O(h^2)\|\hat{\psi}\|^2 \\ & \qquad = O(h^2)\|\hat{\psi}\|^2.\end{aligned}$$

This implies

$$\int m_1(\sigma,\tau)\Big|\Big(h\hat{b}^{-1/2}i\partial_\sigma+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau$$
$$\geq\int m_1(\sigma,\tau)\Big|\Big(hi\partial_\sigma\hat{b}^{-1/2}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\hat{\psi}\Big|^2\hat{b}^{-1/2}\,d\sigma\,d\tau-Ch^2\|\hat{\psi}\|^2.\qquad\square$$

Proof of Proposition 5.3. We use Lemma 5.4. In the coordinates $(\check{\sigma},\tau)$, we have

$$\hat{Q}_{h,\text{app}}(\hat{\psi})=\int m_2(f^{-1}(\check{\sigma}),\tau)|h\partial_\tau\check{\psi}|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int m_1(f^{-1}(\check{\sigma}),\tau)\Big|\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau,$$

where

$$m_1(f^{-1}(\check{\sigma}),\tau)=\big(1+\alpha f^{-1}(\check{\sigma})^2\big)(1+a_2\tau)(1-\tau k_0)^{-1},$$
$$m_2(f^{-1}(\check{\sigma}),\tau)=\big(1+\alpha f^{-1}(\check{\sigma})^2\big)(1+a_2\tau)(1-\tau k_0).$$

We notice that $f^{-1}(\check{\sigma})=\check{\sigma}+O(|\check{\sigma}|^2)$, so we can use the estimates of Agmon to get

$$\hat{Q}_{h,\text{app}}(\hat{\psi})\geq\int m_2(\check{\sigma},\tau)|h\partial_\tau\check{\psi}|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int m_1(\check{\sigma},\tau)\Big|\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau-Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

This inequality can be rewritten as

$$\hat{Q}_{h,\text{app}}(\hat{\psi})\geq\check{Q}_{h,\text{app},1}(\check{\psi})+\check{Q}_{h,\text{app},2}(\check{\psi})-Ch^{3/2+1/4}\|\tilde{\psi}\|^2,$$

where

$$\check{Q}_{h,\text{app},1}(\check{\psi})=\int(1+a_2\tau)(1-\tau k_0)|h\partial_\tau\check{\psi}|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int(1+a_2\tau)(1-\tau k_0)^{-1}\Big|\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau$$

and

$$\check{Q}_{h,\text{app},2}(\check{\psi})=\int(1+a_2\tau)(1-\tau k_0)|h\partial_\tau(\check{\sigma}\check{\psi})|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int(1+a_2\tau)(1-\tau k_0)^{-1}\Big|\check{\sigma}\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau.$$

Reduction of $\check{Q}_{h,\text{app},2}(\check{\psi})$. By the estimates of Agmon, we have

$$\check{Q}_{h,\text{app},2}(\check{\psi})\geq\int|h\partial_\tau(\check{\sigma}\check{\psi})|^2\check{p}^2\,d\check{\sigma}\,d\tau$$
$$+\int\Big|\check{\sigma}\Big(ih\check{p}^{-1}\partial_{\check{\sigma}}\check{p}+\tau-\xi_0h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check{\psi}\Big|^2\check{p}^2\,d\check{\sigma}\,d\tau-Ch^{3/2+1/4}\|\tilde{\psi}\|^2.$$

Moreover, we get

$$\int \Big|\check\sigma\Big(ih\check p^{-1}\partial_{\check\sigma}\check p+\tau-\xi_0 h^{1/2}+\frac{\hat b_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\Big)\check\psi\Big|^2\check p^2 d\check\sigma\, d\tau \\ \geq \int \big|\check\sigma\big(ih\check p^{-1}\partial_{\check\sigma}\check p+\tau-\xi_0 h^{1/2}\big)\check\psi\big|^2\check p^2 d\check\sigma\, d\tau - Ch^{3/2+1/4}\|\tilde\psi\|^2.$$

Let us analyze $\int |\check\sigma(ih\check p^{-1}\partial_{\check\sigma}\check p+\tau-\xi_0h^{1/2})\check\psi|^2\check p^2\, d\check\sigma\, d\tau$. We have

$$\int \big|\check\sigma\big(ih\check p^{-1}\partial_{\check\sigma}\check p+\tau-\xi_0h^{1/2}\big)\check\psi\big|^2\check p^2\, d\check\sigma\, d\tau=\int\big|\big(ih\check p^{-1}\partial_{\check\sigma}\check p+\tau-\xi_0h^{1/2}\big)\check\sigma\check\psi-ih\check\psi\big|^2\check p^2\, d\check\sigma\, d\tau.$$

The double product is

$$2\Re\Big(\int\big(ih\check p^{-1}\partial_{\check\sigma}\check p+\tau-\xi_0h^{1/2}\big)\check\sigma\check\psi\, ih\overline{\check\psi}\,\check p^2\, d\check\sigma\, d\tau\Big)=-2h^2\Re\Big(\int(\check p^{-1}\partial_{\check\sigma}\check p)\check\sigma\check\psi\overline{\check\psi}\,\check p^2\, d\check\sigma\, d\tau\Big).$$

But we have

$$2\Re\Big(\int\partial_{\check\sigma}(\check\sigma\check p\check\psi)\overline{\check p\check\psi}\, d\check\sigma\, d\tau\Big)=2\Re\Big(\int\check p\check\psi\,\overline{\check p\check\psi}\, d\check\sigma\, d\tau\Big)+\int\check\sigma\partial_{\check\sigma}|\check p\check\psi|^2\, d\check\sigma\, d\tau$$

and

$$\int\check\sigma\partial_{\check\sigma}|\check p\check\psi|^2\, d\check\sigma\, d\tau=-\int|\check p\check\psi|^2\, d\check\sigma\, d\tau.$$

Gathering the estimates, we obtain the lower bound:

$$\hat Q_{h,\mathrm{app}}(\hat\psi)\geq\check Q_{h,\mathrm{app}}(\check\psi)-Ch^{3/2+1/4}\|\tilde\psi\|^2.$$

A weighted Fourier transform: toward a model operator. We now define the unitary transform which diagonalizes the self-adjoint operator $\check p^{-1}D_{\check\sigma}\check p$ (for completeness, one should extend $\check p$ by 1 away from a neighborhood of 0). As we will see, with the coordinate $\check\sigma$, this transform admits a nice expression.

Weighted Fourier transform. Let us now introduce the weighted Fourier transform $\mathscr{F}_{\check p}$:

$$(\mathscr{F}_{\check p}\psi)(\lambda)=\int_{\mathbb{R}}e^{-i\lambda\check\sigma}\psi(\check\sigma)\check p(\check\sigma)\, d\check\sigma=\mathscr{F}(\check p\psi).$$

We observe that $\mathscr{F}_{\check p}:L^2(\mathbb{R},\check p^2d\check\sigma)\to L^2(\mathbb{R},d\lambda)$ is unitary. Standard computations provide

$$\mathscr{F}_{\check p}((\check p^{-1}D_{\check\sigma}\check p)\psi)=\lambda\mathscr{F}_{\check p}(\psi)\quad\text{and}\quad\mathscr{F}_{\check p}(\check\sigma\psi)=-D_\lambda\mathscr{F}_{\check p}(\psi).$$

Proposition 5.5. *There exist $h_0>0$ and $C>0$ such that for $h\in(0,h_0)$ and all $\tilde\psi\in\mathfrak{E}_N(h)$,*

$$\check Q_{h,\mathrm{app}}(\check\psi)\geq\int(1+a_2\tau)(1-\tau k_0)|hD_\tau\check\phi|^2\, d\lambda\, d\tau \\ +\int(1+a_2\tau)(1-\tau k_0)^{-1}\Big|\Big(-h\lambda+\tau-\xi_0h^{1/2}+\frac{\hat b_2}{2}\tau^2\Big)\check\phi\Big|^2\, d\lambda\, d\tau \\ +h\alpha\Theta_0\int|D_\lambda\check\phi|^2\, d\lambda\, d\tau-Ch^{3/2+1/4}\|\tilde\psi\|^2,$$

where $\check{\phi} = e^{-ib_1/2h(-h\lambda\tau^2/2+\tau^3/3-\xi_0 h^{1/2}\tau^2/2+(b_2/8)\tau^4)}\mathscr{F}_{\check{p}}(\check{\psi})$.

Proof. We have

$$\check{Q}_{h,\mathrm{app}}(\check{\psi}) = \int (1+a_2\tau)(1-\tau k_0)|h\partial_\tau\check{\varphi}|^2\,d\lambda\,d\tau + \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\right)\check{\varphi}\right|^2 d\lambda\,d\tau + h\alpha\Theta_0\int |D_\lambda\check{\varphi}|^2\,d\lambda\,d\tau,$$

where $\check{\varphi} = \mathscr{F}_{\check{p}}(\check{\psi})$. With the normal estimates, we can write

$$\begin{aligned}\int (1+a_2\tau)(1-\tau k_0)^{-1}&\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2-h\frac{b_1}{2}\tau D_\tau\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&\geq \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&\quad - b_1\Re\left(\int (1+a_2\tau)(1-\tau k_0)^{-1}\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\overline{\tau h D_\tau\check{\varphi}}\,d\lambda\,d\tau\right)\\
&\geq \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&\quad - b_1\Re\left(\int \left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\overline{\tau h D_\tau\check{\varphi}}\,d\lambda\,d\tau\right) - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

Completing a square and using the normal Agmon estimates to control the additional terms, we get

$$\begin{aligned}\check{Q}_{h,\mathrm{app}}(\check{\psi}) &\geq \int (1+a_2\tau)(1-\tau k_0)\left|\left(hD_\tau-\frac{b_1}{2}\tau\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\right)\check{\varphi}\right|^2 d\lambda\,d\tau\\
&+\int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\varphi}\right|^2 d\lambda\,d\tau + h\alpha\Theta_0\int |D_\lambda\check{\varphi}|^2\,d\lambda\,d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

We now change the gauge by letting

$$\check{\varphi} = e^{ib_1/2h(-h\lambda\tau^2/2+\tau^3/3-\xi_0 h^{1/2}\tau^2/2+(b_2/8)\tau^4)}\check{\phi}.$$

We deduce

$$\begin{aligned}\check{Q}_{h,\mathrm{app}}(\check{\psi}) &\geq \int (1+a_2\tau)(1-\tau k_0)|hD_\tau\check{\phi}|^2\,d\lambda\,d\tau\\
&\quad + \int (1+a_2\tau)(1-\tau k_0)^{-1}\left|\left(-h\lambda+\tau-\xi_0 h^{1/2}+\frac{\hat{b}_2}{2}\tau^2\right)\check{\phi}\right|^2 d\lambda\,d\tau\\
&\quad + h\alpha\Theta_0\int \left|D_\lambda\left(e^{-i\lambda b_1\tau^2/4}\check{\phi}\right)\right|^2 d\lambda\,d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2.\end{aligned}$$

Finally we write

$$\begin{aligned}\int \left|D_\lambda\left(e^{-i\lambda b_1\tau^2/4}\check{\phi}\right)\right|^2 d\lambda\, d\tau &= \int \left|D_\lambda\check{\phi} - \frac{b_1}{4}\tau^2\check{\phi}\right|^2 d\lambda\, d\tau \\ &\geq \int |D_\lambda\check{\phi}|^2\, d\lambda\, d\tau - C\|\tau^2\check{\phi}\|\|D_\lambda\check{\phi}\| \\ &\geq \int |D_\lambda\check{\phi}|^2\, d\lambda\, d\tau - C\|\tau^2\check{\psi}\|\|D_\lambda\check{\phi}\|.\end{aligned}$$

In addition, we notice that

$$\|D_\lambda\check{\phi}\| \leq C\left(\|\check{\sigma}\check{\psi}\| + \|\tau^2\check{\psi}\|\right) \leq Ch^{1/4}\|\tilde{\psi}\|. \qquad \square$$

In order to get a good model operator, we shall add a cutoff function with respect to τ. Let $\eta \in \left(0, \frac{1}{100}\right)$. Let χ be a cutoff function such that

$$\chi(t) = 1 \ \text{ for } |t| \leq 1, \qquad 0 \leq \chi \leq 1, \qquad \operatorname{supp}\chi \subset [-2, 2].$$

We define

$$l(x) = x\chi(h^\eta x).$$

Applying the normal Agmon estimates, we have:

Proposition 5.6. *There exist $h_0 > 0$ and $C > 0$ such that for $h \in (0, h_0)$ and all $\tilde{\psi} \in \mathfrak{E}_N(h)$,*

$$\begin{aligned}\check{Q}_{h,\mathrm{app}}(\check{\psi}) \geq &\int \left(1 + a_2 h^{1/2} l(h^{-1/2}\tau)\right)\left(1 - h^{1/2} l(h^{-1/2}\tau)k_0\right)|hD_\tau\check{\phi}|^2\, d\lambda\, d\tau \\ &+ \int \left(1 + a_2 h^{1/2} l(h^{-1/2}\tau)\right)\left(1 - h^{1/2} l(h^{-1/2}\tau)k_0\right)^{-1}\left|\left(-h\lambda + \tau - \xi_0 h^{1/2} + \frac{\hat{b}_2}{2} hl(h^{-1/2}\tau)^2\right)\check{\phi}\right|^2 d\lambda\, d\tau \\ &+ h\alpha\Theta_0 \int |D_\lambda\check{\phi}|^2\, d\lambda\, d\tau - Ch^{3/2+1/4}\|\tilde{\psi}\|^2,\end{aligned}$$

where $\check{\phi} = e^{-ib_1/2h\left(-h\lambda\tau^2/2 + \tau^3/3 - \xi_0 h^{1/2}\tau^2/2 + (b_2/8)\tau^4\right)}\mathcal{F}_{\check{p}}(\check{\psi})$.

Remark 5.7. In particular, we have reduced the analysis to an electric Laplacian (with curvature terms), which has essentially the Born–Oppenheimer form (see our recent work [Bonnaillie-Noël et al. 2012], where a similar and simpler model appears). To see this more precisely, let us adopt a heuristical point of view. If we forget the different terms due to curvature, the operator which appears is in the form

$$h\alpha\Theta_0 D_\lambda^2 + h^2 D_\tau^2 + (-h\lambda + \tau - \xi_0 h^{1/2})^2.$$

After the rescaling $\lambda = h^{-1/4}\tilde{\lambda}$, $\tau = h^{1/2}x$, we get

$$h\left(h^{1/2}\alpha\Theta_0 D_{\tilde{\lambda}}^2 + D_x^2 + (-h^{1/4}\tilde{\lambda} - x - \xi_0)^2\right).$$

Therefore we are led to analyze a problem which is semiclassical with respect to just one variable. At some point (that we will justify at the end of this section), we can reduce the study to

$$h\big(h^{1/2}\alpha\Theta_0 D_{\tilde{\lambda}}^2+\mu(\xi_0+h^{1/4}\tilde{\lambda})\big),$$

and then (Taylor expansion)

$$h\left(h^{1/2}\alpha\Theta_0 D_{\tilde{\lambda}}^2+\Theta_0+\frac{\mu''(\xi_0)}{2}h^{1/2}\tilde{\lambda}^2\right).$$

Finally we recognize the harmonic oscillator, whose spectrum is well-known.

A simpler model in the Born–Oppenheimer spirit. We introduce the rescaled quadratic form:

$$\begin{aligned}Q_{\eta,h}(\varphi)=&\int\big(1+a_2h^{1/2}l(x)\big)\big(1-l(x)k_0h^{1/2}\big)|\partial_x\varphi|^2\,d\lambda\,dx\\&+\int\big(1+a_2l(x)h^{1/2}\big)\big(1-l(x)k_0h^{1/2}\big)^{-1}\left|\left(x-\xi_0+h^{1/2}\lambda+\frac{\hat{b}_2}{2}l(x)^2h^{1/2}\right)\varphi\right|^2 d\lambda\,dx\\&+\alpha\Theta_0\int|D_\lambda\varphi|^2\,d\lambda\,dx.\end{aligned}$$

We recall that $\hat{b}_2=b_2-k_0$. We will denote by $H_{\eta,h}$ its corresponding Friedrichs extension. We will denote by $\nu_n(Q_{\eta,h})$ the sequence of its Rayleigh quotients. For each λ, we will need to consider the quadratic form

$$\begin{aligned}q_{\lambda,\eta,h}(\varphi)=&\int\big(1+a_2h^{1/2}l(x)\big)\big(1-l(x)k_0h^{1/2}\big)\,|\partial_x\varphi|^2\,dx\\&+\int\big(1+a_2l(x)h^{1/2}\big)\big(1-l(x)k_0h^{1/2}\big)^{-1}\left|\left(x-\xi_0+h^{1/2}\lambda+\frac{\hat{b}_2}{2}l(x)^2h^{1/2}\right)\varphi\right|^2 dx,\end{aligned}$$

whose domain is $B^1(\mathbb{R}^+)$. We denote by $\nu_j(q_{\lambda,\eta,h})$ the increasing sequence of the eigenvalues of the associated operator. The main proposition of this subsection is the following:

Proposition 5.8. *For all $n\geq 1$, there exist $h_0>0$ and $C>0$ such that, for $h\in(0,h_0)$:*

$$\nu_n(Q_{\eta,h})\geq\Theta_0+\left(C(k_0,a_2,b_2)+(2n-1)\sqrt{\frac{\alpha\mu''(\xi_0)\Theta_0}{2}}\right)h^{1/2}-Ch^{1/2+1/8}.$$

With Propositions 5.6 and 5.3, inequality (5-1), and the min-max principle, we first deduce the size of the spectral gap between the lowest eigenvalues of $P_{h,\boldsymbol{A}}$. Then, with Theorem 2.1, we deduce Theorem 1.3.

Elementary properties of the spectrum. This subsection is devoted to basic properties of the spectrum of $Q_{\eta,h}$. The following proposition provides a lower bound for $\nu_1(q_{\lambda,\eta,h})$.

Proposition 5.9. *There exist positive constants C, c_0, M and h_0 such that if $h\in(0,h_0)$, then:*

(1) *If $|\lambda|\geq Mh^{-1/4-\eta}$, then*

$$\nu_1(q_{\lambda,\eta,h})\geq\Theta_0+c_0\min(1,\lambda^2h).$$

(2) *If* $|\lambda| \leq Mh^{-1/4-\eta}$, *then*

$$\nu_1(q_{\lambda,\eta,h}) \geq \Theta_0 + C(k_0, a_2, b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h - Ch^{3/4-3\eta},$$

where $C(k_0, a_2, b_2)$ *is given in Theorem 1.3.*

Proof. The proof is left to the reader as an adaptation of [Fournais and Helffer 2010, Proposition 5.2.1]. □

Let us now prove a lower bound for the essential spectrum of $H_{\eta,h}$.

Proposition 5.10. *There exist* $h_0 > 0$ *and* $\tilde{c}_0 > 0$ *such that, if* $h \in (0, h_0)$, *then*

$$\inf \sigma_{\mathrm{ess}}(Q_{\eta,h}) \geq \Theta_0 + \tilde{c}_0.$$

Proof. Let $\phi \in \mathrm{Dom}(Q_{\eta,h})$ such that $\mathrm{supp}(\phi) \subset \mathbb{R}^2_+ \setminus [-\tilde{R}, \tilde{R}]^2$. Let us use a partition of unity $\chi^2_{1,R} + \chi^2_{2,R} = 1$ such that $\chi_{1,R}(x) = \chi_1(R^{-1}x)$ and where χ_1 is a smooth cutoff function being 1 near 0. We have

$$Q_{\eta,h}(\phi) \geq Q_{\eta,h}(\chi_{1,R}\phi) + Q_{\eta,h}(\chi_{2,R}\phi) - CR^{-2}\|\phi\|^2.$$

For $R \geq 2h^{-\eta}$, we have (the metrics becomes flat and we can compare with a problem in $\mathbb{R}^2$)

$$Q_{\eta,h}(\chi_{2,R}\phi) \geq \|\chi_{2,R}\phi\|^2.$$

We have

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|\chi_{1,R}\phi|^2 + \alpha\Theta_0|D_\lambda(\chi_{1,R}\phi)|^2 \, dx \, d\lambda.$$

Taking $h \in (0, h_0)$ (where h_0 is given by Proposition 5.9) and $\tilde{R} \geq h^{-1/2}$, we infer

$$Q_{\eta,h}(\chi_{1,R}\phi) \geq \int_{\mathbb{R}^2_+} (\Theta_0 + c_0)|\chi_{1,R}\phi|^2 \, dx \, d\lambda.$$

This implies that

$$Q_{\eta,h}(\phi) \geq \big(\min(1, \Theta_0 + c_0) - Ch^{2\eta}\big)\|\phi\|^2.$$

The conclusion follows from a Persson's lemma-like argument (see [Persson 1960; Fournais and Helffer 2010, Appendix B.3]). □

The following proposition provides an upper bound for the lowest eigenvalues of $H_{\eta,h}$.

Proposition 5.11. *For all* $M \geq 1$, *there exist* $h_0 > 0$, $C > 0$ *such that for all* $1 \leq n \leq M$:

$$\nu_n(Q_{\eta,h}) \leq h^{-1}\lambda_n(h) + O(h^\infty).$$

Proof. This is a consequence of (5-1) together with the lower bounds of Propositions 5.3 and 5.6 and the min-max principle (see for instance [Reed and Simon 1978]). □

Remark 5.12. For h small enough, we deduce that there are at least M eigenvalues below $\Theta_0 + \tilde{c}_0$. Let us consider the first M eigenvalues $\nu_n(Q_{\eta,h})$ below $\Theta_0 + \tilde{c}_0$. With Theorem 2.1, we deduce that, for all $M \geq 1$, there exist $h_0 > 0$ and $C(M) > 0$ such that, for $1 \leq n \leq M$,

$$0 \leq \nu_n(Q_{\eta,h}) - \Theta_0 \leq C(M)h^{1/2}.$$

For $1 \leq n \leq M$, let us consider a normalized eigenfunction $f_{n,\eta,h}$ associated to $\nu_n(Q_{\eta,h})$ so that $f_{n,\eta,h}$ and $f_{m,\eta,h}$ are orthogonal if $n \neq m$. Let us introduce:

$$\mathfrak{F}_M(h) = \operatorname{span}_{1\leq j\leq M}(f_{j,\eta,h}).$$

Agmon estimates. First, let us state Agmon estimates with respect to x.

Proposition 5.13. *There exist $h_0 > 0$, $\varepsilon_0 > 0$, $C > 0$ such that, for all $f \in \mathfrak{F}_M(h)$,*

$$\int_{\mathbb{R}^2_+} e^{\varepsilon_0 x}|f|^2\,dx\,d\lambda \leq C\|f\|^2.$$

Proof. Let us use a partition of unity, $\chi_{1,R}^2 + \chi_{2,R}^2 = 1$, with $R \geq h^{-\eta}$. We take $\Phi = \varepsilon_0\chi(x/r)|x|$. This IMS formula implies (with $f = f_{n,\eta,h}$)

$$Q_{\eta,h}(\chi_{1,R}e^{\Phi}f) + Q_{\eta,h}(\chi_{2,R}e^{\Phi}f) - C\varepsilon_0^2\|e^{\Phi}f\|^2 - \nu_n(Q_{\eta,h})\|e^{\Phi}f\|^2 \leq 0.$$

We recall that

$$Q_{\eta,h}(\chi_{2,R}e^{\Phi}f) \geq \|\chi_{2,R}e^{\Phi}f\|^2$$

and that

$$Q_{\eta,h}(\chi_{1,R}e^{\Phi}f) \geq \int \nu_1(q_{\lambda,\eta,h})|\chi_{1,R}e^{\Phi}f|^2\,dx\,d\lambda.$$

On the one hand, we have

$$Q_{\eta,h}(\chi_{2,R}e^{\Phi}f) - C\varepsilon_0^2\|\chi_{2,R}e^{\Phi}f\|^2 - (\Theta_0 + Ch^{1/2})\|\chi_{2,R}e^{\Phi}f\|^2 \geq (1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2})\|\chi_{2,R}e^{\Phi}f\|^2.$$

On the other hand, we get

$$\begin{aligned} Q_{\eta,h}(\chi_{1,R}e^{\Phi}f) - C\varepsilon_0^2\|\chi_{1,R}e^{\Phi}f\|^2 - (\Theta_0 + Ch^{1/2})\|\chi_{1,R}e^{\Phi}f\|^2 \\ \geq \int \big(\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2}\big)|\chi_{1,R}e^{\Phi}f|^2\,dx\,d\lambda. \end{aligned}$$

When $|\lambda| \geq Mh^{-1/4-\eta}$, we have

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - Ch^{1/2}.$$

When $|\lambda| \leq Mh^{-1/4}$, we have

$$\nu_1(q_{\eta,\lambda,h}) - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2} \geq -C\varepsilon_0^2 - \tilde{C}h^{1/2}.$$

If h and ε_0 are small enough, we deduce that

$$(1 - C\varepsilon_0^2 - \Theta_0 - Ch^{1/2})\|\chi_{2,R}e^{\Phi}f\|^2 \leq C\|\chi_{1,R}e^{\Phi}f\|^2,$$

so that

$$\|\chi_{2,R}e^{\Phi}f\|^2 \leq \tilde{C}\|f\|^2 \quad \text{and} \quad \|e^{\Phi}f\|^2 \leq \hat{C}\|f\|^2,$$

where $\tilde{C}$ and $\hat{C}$ are independent from r. It remains to make $r \to +\infty$ and apply the Fatou lemma. Finally, it is easy to extend the inequality to $f \in \mathfrak{F}_M(h)$. ☐

Then, we will need Agmon estimates with respect to λ:

Proposition 5.14. *There exist $h_0 > 0$, $C > 0$ such that, for all $f \in \mathfrak{F}_M(h)$,*

$$\int_{\mathbb{R}^2_+} e^{2h^{1/4}|\lambda|}|f|^2\,dx\,d\lambda \le C\|f\|^2 \tag{5-4}$$

and

$$\int_{\mathbb{R}^2_+} e^{2h^{1/4}|\lambda|}|D_\lambda f|^2\,dx\,d\lambda \le Ch^{1/2}\|f\|^2. \tag{5-5}$$

Remark 5.15. Heuristically, these estimates with respect to λ correspond to the phase space localization of [Fournais and Helffer 2006, Section 5].

Proof. We take $f = f_{j,\eta,h}$ and use the IMS formula (with $\Phi = h^{1/4}\chi(r^{-1}|\lambda|)|\lambda|$) to get

$$Q_{\eta,h}(e^\Phi f) \le \nu_j(Q_{\eta,h})\|e^\Phi f\|^2 + C\|\nabla\Phi e^\Phi f\|^2 \le \big(\Theta_0 + C(M)h^{1/2} + Ch^{1/2}\big)\|e^\Phi f\|^2.$$

We recall that

$$Q_{\eta,h}(e^\Phi f) \ge \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2 + \alpha\Theta_0|D_\lambda(e^\Phi f)|^2\,dx\,d\lambda \ge \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda.$$

We have, for all $D > 0$,

$$\int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda = \int_{|\lambda|\le Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda + \int_{|\lambda|\ge Dh^{-1/4}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda.$$

Moreover, we get

$$\int_{|\lambda|\ge Mh^{-1/4-\eta}} \nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda \ge \int_{|\lambda|\ge Mh^{-1/4-\eta}} \big(\Theta_0 + c_0\min(1,h\lambda^2)\big)|e^\Phi f|^2\,dx\,d\lambda$$

and

$$\begin{aligned}\int_{Dh^{-1/4}\le|\lambda|\le Mh^{-1/4-\eta}} &\nu_1(q_{\lambda,\eta,h})|e^\Phi f|^2\,dx\,d\lambda \\ &\ge \int_{Dh^{-1/4}\le|\lambda|\le Mh^{-1/4-\eta}} \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h - Ch^{3/4-3\eta}\right)|e^\Phi f|^2\,dx\,d\lambda.\end{aligned}$$

This leads to

$$\int_{|\lambda|\ge Dh^{-1/4}} \big(c_1\min(1,h\lambda^2) - \tilde{C}h^{1/2} - C\alpha^2 h^{1/2}\big)|e^\Phi f|^2\,dx\,d\lambda \le \tilde{C}h^{1/2}\int_{|\lambda\le Dh^{-1/4}} |f|^2\,d\lambda\,dx.$$

It remains to take D large enough, and we get (5-4). Then we have

$$\int_{\mathbb{R}^2_+} \big(\nu_1(q_{\lambda,\eta,h}) - \Theta_0\big)|e^\Phi f|^2 + \alpha\Theta_0|D_\lambda(e^\Phi f)|^2\,dx\,d\lambda \le Ch^{1/2}\|f\|^2.$$

But we notice that

$$\int_{\mathbb{R}^2_+} \big(\nu_1(q_{\lambda,\eta,h}) - \Theta_0\big)|e^{\Phi} f|^2 \, dx \, d\lambda$$
$$\geq \int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - C h^{3/4-3\eta} \right) |e^{\Phi} f|^2 \, dx \, d\lambda$$
$$+ \int_{|\lambda \leq Dh^{-1/4}} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - C h^{3/4-3\eta} \right) |e^{\Phi} f|^2 \, d\lambda \, dx.$$

Taking D larger, we get

$$\int_{Dh^{-1/4} \leq |\lambda| \leq Mh^{-1/4-\eta}} \left(\frac{\mu''(\xi_0)}{2} \lambda^2 h - C h^{1/2} - C h^{3/4-3\eta} \right) |e^{\Phi} f|^2 \, dx \, d\lambda \geq 0.$$

Moreover, we have

$$\left| \int_{|\lambda \leq Dh^{-1/4}} \left(C(k_0, a_2, b_2) h^{1/2} + \frac{\mu''(\xi_0)}{2} \lambda^2 h - C h^{3/4-3\eta} \right) |e^{\Phi} f|^2 \, d\lambda \, dx \right| \leq C h^{1/2} \|f\|^2. \qquad \square$$

Approximations of eigenvectors by tensor products. Let us define the quadratic form q_0 with domain $B^1(\mathbb{R}_+) \otimes L^2(\mathbb{R})$:

$$q_0(\varphi) = Q_0(\varphi) - \Theta_0 \|\varphi\|^2 = \int_{\mathbb{R}^2_+} |\partial_x \varphi|^2 + |(x - \xi_0)\varphi|^2 - \Theta_0 |\varphi|^2 \, dx \, d\lambda.$$

The Friedrichs extension of q_0 is the operator $H_{\xi_0} \otimes \mathrm{Id}_{L^2(\mathbb{R})}$. We also define the Feshbach–Grušin projection on the kernel of $H_{\xi_0} \otimes \mathrm{Id}_{L^2(\mathbb{R})}$:

$$\Pi_0 \varphi = \langle \varphi, u_{\xi_0} \rangle_x u_{\xi_0}(x).$$

The next proposition states an approximation result for the elements of $\mathfrak{F}_M(h)$ (which behave as tensor products):

Proposition 5.16. *For all $M \geq 1$, there exist $h_0 > 0$ and $C > 0$ such that we have, for all $f \in \mathfrak{F}_M(h)$,*

$$\|f - \Pi_0 f\|_{L^2} + \|\partial_x (f - \Pi_0 f)\|_{L^2} + \|x(f - \Pi_0 f)\|_{L^2} \leq C h^{1/8} \|f\|, \tag{5-6}$$
$$\|(\lambda f - \Pi_0 \lambda f\|_{L^2} + \|\partial_x (\lambda f - \Pi_0 \lambda f)\|_{L^2} + \|x(\lambda f - \Pi_0 \lambda f)\|_{L^2} \leq C h^{-1/8} \|f\|, \tag{5-7}$$
$$\|(\partial_\lambda f - \Pi_0 \partial_\lambda f\|_{L^2} + \|\partial_x (\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} + \|x(\partial_\lambda f - \Pi_0 \partial_\lambda f)\|_{L^2} \leq C h^{3/8} \|f\|. \tag{5-8}$$

In particular, Π_0 *is an isomorphism from* $\mathfrak{F}_M(h)$ *onto its range.*

Proof. We take $f = f_{j,\eta,h}$. By definition, we have

$$H_{\eta,h} f = \nu_j(Q_{\eta,h}) f. \tag{5-9}$$

Approximation of f. We deduce

$$Q_{\eta,h}(f) = \nu_j(Q_{\eta,h})\|f\|^2 \leq (\Theta_0 + Ch^{1/2})\|f\|^2.$$

We have

$$Q_{\eta,h}(f) \geq (1 - Ch^{1/2-\eta}) \int_{\mathbb{R}^2_+} |\partial_x f|^2 + \left|\left(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{\hat{b}_2}{2}l(x)^2\right)f\right|^2 dx\,d\lambda.$$

Moreover, we get (using the estimates of Agmon), for all $\varepsilon \in (0, 1)$:

$$\int_{\mathbb{R}^2_+} |\partial_x f|^2 + \left|\left(x - \xi_0 + h^{1/2}\lambda + h^{1/2}\frac{b_2}{2}l(x)^2\right)f\right|^2 dx\,d\lambda \geq (1-\varepsilon)Q_0(f) - C\varepsilon^{-1}h^{1/2}\|f\|^2.$$

Taking $\varepsilon = h^{1/4}$, we deduce

$$q_0(f) \leq Ch^{1/4}\|f\|^2.$$

We deduce (5-6).

Approximation of λf. We multiply (5-9) by λ and take the scalar product with λf:

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + \left|\langle [H_{\eta,h}, \lambda]f, \lambda f\rangle\right|.$$

Thus, it follows that

$$Q_{\eta,h}(\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\lambda f\|^2 + \alpha\Theta_0|\langle D_\lambda f, \lambda f\rangle| \leq \Theta_0\|\lambda f\|^2 + C\|f\|^2.$$

We get

$$Q_{\eta,h}(\lambda f) \geq (1 - Ch^{1/2-\eta})\big((1-\varepsilon)Q_0(\lambda f) - C\varepsilon^{-1}\|f\|^2\big).$$

We take $\varepsilon = h^{1/4}$ to deduce

$$q_0(\lambda f) \leq Ch^{-1/4}\|f\|^2.$$

We infer (5-7).

Approximation of $D_\lambda f$. We take the derivative of (5-9) with respect to λ and take the scalar product with $\partial_\lambda f$:

$$Q_{\eta,h}(\partial_\lambda f) \leq (\Theta_0 + Ch^{1/2})\|\partial_\lambda f\|^2 + \left|\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f\rangle\right|.$$

The estimates of Agmon give

$$\left|\langle [H_{\eta,h}, \partial_\lambda]f, \partial_\lambda f\rangle\right| \leq Ch^{3/4}\|f\|^2.$$

We have

$$Q_{\eta,h}(\partial_\lambda f) \geq (1 - Ch^{1/2-\eta})\big((1-\varepsilon)Q_0(\partial_\lambda f) - C\varepsilon^{-1}h\|f\|^2\big).$$

We take $\varepsilon = h^{1/4}$ and deduce

$$q_0(\partial_\lambda f) \leq Ch^{3/4}\|f\|^2.$$

We infer (5-8). □

Conclusion: proof of Proposition 5.8. For all $f \in \mathfrak{F}_M(h)$, we have the lower bound

$$
\begin{aligned}
Q_{\eta,h}(f) &\geq \int_{\mathbb{R}^2_+} \nu_1(q_{\lambda,\eta,h})|f|^2 + \alpha\Theta_0|D_\lambda f|^2\,dx\,d\lambda \\
&\geq \int_{\mathbb{R}^2_+}\left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)\right)|f|^2\,dx\,d\lambda \\
&\quad + \int_{\mathbb{R}^2_+}\left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)|f|^2\,dx\,d\lambda + \alpha\Theta_0|D_\lambda f|^2\,dx\,d\lambda.
\end{aligned}
$$

We now estimate

$$
\begin{aligned}
&\int_{\mathbb{R}^2_+}\left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)\right)|f|^2\,dx\,d\lambda \\
&\quad = \int_{|\lambda|\geq Mh^{-1/4-\eta}}\left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)\right)|f|^2\,dx\,d\lambda \\
&\qquad + \int_{|\lambda|\leq Mh^{-1/4-\eta}}\left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)\right)|f|^2\,dx\,d\lambda.
\end{aligned}
$$

Moreover, we get

$$
\begin{aligned}
&\int_{|\lambda|\geq Mh^{-1/4-\eta}}\left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)\right)|f|^2\,dx\,d\lambda \\
&\quad \geq \int_{|\lambda|\geq Mh^{-1/4-\eta}} -\left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)|f|^2\,dx\,d\lambda = O(h^\infty)\|f\|^2,
\end{aligned}
$$

where the last estimate is a consequence of the estimates of Agmon. Then we get

$$
\int_{|\lambda|\leq Mh^{-1/4-\eta}}\left(\nu_1(q_{\lambda,\eta,h}) - \left(\Theta_0 + C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)\right)|f|^2\,dx\,d\lambda \geq -Ch^{3/4-3\eta}\|f\|^2.
$$

We deduce

$$
\begin{aligned}
Q_{\eta,h}(f) \geq \int_{\mathbb{R}^2_+}\left(C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)|f|^2\,dx\,d\lambda + \alpha\Theta_0|D_\lambda f|^2\,dx\,d\lambda \\
+ \Theta_0\|f\|^2 - Ch^{3/4-3\eta}\|f\|^2.
\end{aligned}
$$

We now use Proposition 5.16 to get

$$
\begin{aligned}
Q_{\eta,h}(f) \geq \int_{\mathbb{R}^2_+}\left(C(k_0,a_2,b_2)h^{1/2} + \frac{\mu''(\xi_0)}{2}\lambda^2 h\right)|\Pi_0 f|^2\,dx\,d\lambda + \alpha\Theta_0|D_\lambda \Pi_0 f|^2\,dx\,d\lambda \\
+ \Theta_0\|f\|^2 - Ch^{1/2+1/8}\|\Pi_0 f\|^2.
\end{aligned}
$$

But we notice that for all $f \in \mathfrak{F}_M(h)$,

$$
Q_{\eta,h}(f) \leq \nu_M(Q_{\eta,h})\|f\|^2,
$$

and thus:

$$\int_{\mathbb{R}^2_+}\left(C(k_0,a_2,b_2)h^{1/2}+\frac{\mu''(\xi_0)}{2}\lambda^2 h\right)|\Pi_0 f|^2\,dx\,d\lambda+\alpha\Theta_0|D_\lambda\Pi_0 f|^2\,dx\,d\lambda \le \big(\nu_M(Q_{\eta,h})-\Theta_0\big)\|f\|^2+Ch^{1/2+1/8}\|\Pi_0 f\|^2 \le \big(\nu_M(Q_{\eta,h})-\Theta_0\big)\|\Pi_0 f\|^2+\tilde{C}h^{1/2+1/8}\|\Pi_0 f\|^2.$$

The conclusion follows from the min-max principle.

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