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L^p ESTIMATES FOR THE HILBERT TRANSFORMS ALONG A ONE-VARIABLE VECTOR FIELD

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Stein conjectured that the Hilbert transform in the direction of a vector field v is bounded on, say, L^2 whenever v is Lipschitz. We establish a wide range of L^p estimates for this operator when v is a measurable, nonvanishing, one-variable vector field in $\mathbb{R}^2$. Aside from an L^2 estimate following from a simple trick with Carleson's theorem, these estimates were unknown previously. This paper is closely related to a recent paper of the first author (*Rev. Mat. Iberoam.* **29**:3 (2013), 1021–1069).

1. Introduction

Given a nonvanishing measurable vector field $v\colon \mathbb{R}^2 \to \mathbb{R}^2$, define for $f : \mathbb{R}^2 \to \mathbb{R}^2$

$$H_v f(x, y) = \text{p.v.} \int \frac{f((x, y) - tv(x, y))}{t}\, dt. \tag{1-1}$$

In this paper we prove:

Theorem 1. *Suppose v is a nonvanishing measurable vector field such that for all $x, y \in \mathbb{R}$,*

$$v(x, y) = v(x, 0),$$

and suppose $p \in \left(\frac{3}{2}, \infty\right)$. Then

$$\|H_v f\|_p \lesssim \|f\|_p.$$

The estimate is understood as an a priori estimate for all f in an appropriate dense subclass of $L^p(\mathbb{R}^2)$, say the Schwartz class, on which the Hilbert transform H_v is initially defined. One can then use the estimate to extend H_v to all of $L^p(\mathbb{R}^2)$.

If the vector field is constant, then this follows from classical estimates for the one-dimensional Hilbert transform by evaluating the L^p norm as an iterated integral, with inner integration in the direction of the vector field. Theorem 1 follows from the special case for vector fields mapping to vectors of unit length, because the Hilbert transforms along v and $v/|v|$ are equal by a simple change of variables in (1-1). To prove the theorem for unit-length vector fields, it suffices to do so for vector fields with nonvanishing first component, because we can apply the result for constant vector fields to the restriction of H_v to the set where v takes the value $(0, 1)$ and the set where it takes the value $(0, -1)$. Dividing v by its first component, we may then assume it is of the form $(1, u(x))$; multiplying v by a negative number merely

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changes the sign of (1-1). We call u the slope of the vector field. The Hilbert transform (1-1) then takes the form

$$H_v f(x, y) = \text{p.v.} \int \frac{f(x-t, y-tu(x))}{t}\, dt. \tag{1-2}$$

1.1. *Remarks and related work.* The case $p = 2$ of Theorem 1 is equivalent to the Carleson–Hunt theorem in L^2. This observation is attributed (without reference) to Coifman in [Lacey and Li 2010] and to Coifman and El Kohen in [Carbery et al. 1999]. We briefly explain how to deduce Theorem 1 for $p = 2$ from the Carleson–Hunt theorem. Denote by $\mathcal{F}_2$ the Fourier transform in the second variable. Then we formally have for (1-2), ignoring principal value notation,

$$\int e^{2\pi i \eta y} \int \mathcal{F}_2 f(x-t, \eta) \frac{e^{-2\pi i u(x)\eta t}}{t}\, dt\, d\eta.$$

As the inner integral is independent of y, it suffices, by Plancherel, to prove

$$\left\| \int \mathcal{F}_2 f(x-t, \eta) \frac{e^{-2\pi i u(x)\eta t}}{t}\, dt \right\|_{L^2(x,\eta)} \lesssim \|\mathcal{F}_2 f\|_2.$$

For each fixed η, applying the Carleson–Hunt theorem in the form

$$\left\| \int g(x-t) \frac{e^{-2\pi i N(x) t}}{t}\, dt \right\|_2 \lesssim \|g\|_2$$

for $g \in L^2(\mathbb{R})$ and measurable function N, proves the desired estimate.

For any regular linear transformation of the plane, we have the identity

$$(H_{T \circ v \circ T^{-1}} f) \circ T = H(f \circ T).$$

The class of vector fields depending on the first variable is invariant under linear transformations that preserve the vertical direction. This symmetry group is generated by the isotropic dilations

$$(x, y) \to (\lambda x, \lambda y),$$

nonisotropic dilations

$$(x, y) \to (x, \lambda y),$$

and shearing transformations

$$(x, y) \to (x, y + \lambda x)$$

for $\lambda \neq 0$. By a simple limiting argument, it suffices to prove Theorem 1 under the assumption that $\|u\|_\infty$ is finite. By the above nonisotropic scaling, the operator norm is independent of $\|u\|_\infty$, and we may therefore assume without loss of generality that

$$\|u\|_\infty \leq 10^{-2}. \tag{1-3}$$

Following general principles of wave packet analysis, it is natural to decompose H_v into wave packet components, where the wave packets are obtained from a generating function ϕ via application of elements of the symmetry group of the operator. These wave packets can be visualized by acting with the same group element on the unit square in the plane. The shapes obtained under the above linear symmetry group of H_v are parallelograms with a pair of vertical edges. All parallelograms in this paper will be of this special type. Under the assumption (1-3), it suffices to consider parallelograms whose nonvertical edges are close to horizontal. Such parallelograms are well approximated by rectangles, which are used in [Bateman 2013b; Lacey and Li 2010].

Theorem 2 [Bateman 2013b]. *Assume $\|u\|_\infty \le 1$ and $1 < p < \infty$. Assume $\hat{f}(\xi, \eta)$ vanishes outside an annulus $A < |(\xi, \eta)| \le 2A$ for some $A > 0$. Then*

$$\|H_v f\|_p \lesssim \|f\|_p.$$

Actually, the theorem is stated in that reference for functions such that $\hat{f}$ vanishes outside a trapezoidal region inside an annulus, but this is inessential, as can be seen from the commentary below. This theorem is weaker than Theorem 1 in the region $p > \frac{3}{2}$, but holds in the full region $1 < p < \infty$. The width of the annulus can be altered by finite superposition of different annuli, at the expense of an implicit constant depending on the conformal width of the annulus. The case $p > 2$ and a weak-type endpoint at $p = 2$ of Theorem 2 are due to Lacey and Li [2006b], and hold for arbitrary measurable vector fields.

We reformulate Theorem 2 in a form invariant under the above linear transformation group. The adjoint linear transformations of this group leave the horizontal direction invariant.

Theorem 3. *Assume $1 < p < \infty$. Assume $\hat{f}(\xi, \eta)$ is supported in a horizontal pair of strips $A < |\eta| < 2A$ for some $A > 0$. Then*

$$\|H_v f\|_p \lesssim \|f\|_p.$$

To deduce Theorem 3 from Theorem 2, we use the nonisotropic dilation $(x, y) \to (\lambda x, y)$ to stretch the annulus in the ξ direction until in the limit it degenerates to a pair of strips $A < |\eta| < 2A$. The restriction $\|u\|_\infty \le \lambda^{-1}$ becomes void in the limit $\lambda \to 0$. This proves Theorem 3. For the converse direction, we use a bounded number of dilated strips to cover the annulus except for two thin annular sectors around the ξ-axis. It remains to prove bounds on functions supported in these sectors. For fixed constant vector v, the operator H_v is given by a Fourier multiplier that is constant on two half-planes separated by a line through the origin perpendicular to v. If $\|u\|_\infty \le 1$, then this line does not intersect the thin annular sectors, and we have, with the constant vector field $(1, 0)$,

$$H_v f(x, y) = H_{(1,0)} f(x, y). \tag{1-4}$$

But $H_{(1,0)}$ is trivially bounded, and this completes the deduction of Theorem 2 from Theorem 3.

Sharpness of the exponent $\frac{3}{2}$ in Theorem 1 is not known. In Remark 9 we mention a potential covering lemma that, when combined with the methods in this paper, would push the exponent down to $\frac{4}{3}$. The truth of this covering lemma is unknown, however. If f is an elementary tensor,

$$f(x, y) = g(x)h(y),$$

then a similar calculation to the above turns $H_v f$ into

$$\int \widehat{h}(\eta) e^{2\pi i \eta y} \int g(x-t) \frac{e^{-2\pi i u(x)\eta t}}{t} \, dt \, d\eta.$$

This expression can be read as a family of Fourier multipliers acting on h. Assuming the norm of h is normalized to $\|h\|_p = 1$, we can estimate the last display by

$$\left\| \left\| \int g(x-t) \frac{e^{-2\pi i u(x)\eta t}}{t} \, dt \right\|_{M^p(\eta)} \right\|_{L^p(x)},$$

where $M_p(\eta)$ denotes the operator norm of the Fourier multiplier acting on L^p. By scaling invariance of the multiplier norm, the factor $u(x)$ in the phase can be ignored. As shown in [Coifman et al. 1988], multiplier norms are controlled by variation norms. Hence we may estimate the last display by

$$\left\| \left\| \int g(x-t) \frac{e^{-2\pi i \eta t}}{t} \, dt \right\|_{V^r(\eta)} \right\|_{L^p(x)},$$

provided $\left|\frac{1}{2} - 1/p\right| \le 1/r$. The bounds on the variation norm Carleson operator in [Oberlin et al. 2012] imply that for $p > \frac{4}{3}$ and $r > p'$, the last display is bounded by a constant times $\|g\|_p$. Hence the exponent in Theorem 1 can be improved to $\frac{4}{3}$ under the additional assumption that the function f is an elementary tensor. The authors learned this argument from Ciprian Demeter. Related multiplier theorems in [Demeter et al. 2008; Demeter 2012] also show a phase transition at this exponent.

The Hilbert transform along a one-variable vector field was studied by Carbery, Seeger, Wainger, and Wright in [Carbery et al. 1999]. There, boundedness in L^p for $1 < p$ is proved under additional conditions on the vector field.

In a different direction, Stein conjectured that a truncation of H_v is bounded on L^2 under the assumption that the two-variable vector field v is Lipschitz with sufficiently small Lipschitz constant depending on the truncation. Stein's conjecture is related to a well-known conjecture of Zygmund on the differentiation of Lipschitz vector fields. Define

$$M_v f(x,y) = \sup_{0<L<1} \frac{1}{2L} \int_{-L}^{L} f\big((x,y) - v(x,y)t\big) \, dt.$$

Zygmund conjectured that M_v is (say) weak-type $(2,2)$ if $\|v\|_\infty$ is bounded and the Lipschitz norm $\|\nabla v\|_\infty$ is small enough. Proving a weak-type estimate on this operator would yield corresponding differentiation results analogous to the Lebesgue differentiation theorem, except the averaging takes place over line segments instead of balls. Estimates on M_v are unknown on any L^p space, except for the trivial $p = \infty$ case, unless more stringent requirements are placed on v; for example, Bourgain [1989] proved M_v is bounded on L^p, $p > 1$, when v is real-analytic and the operator is restricted to a bounded domain. The corresponding result for the Hilbert transform was announced in [Stein and Street 2011], although the $p = 2$ case follows from work of Lacey and Li [2010]. Previously the Hilbert transform case in such a range of exponents was only known under the additional assumption that no integral curve of the vector field forms a straight line [Christ et al. 1999].

There is some history of using singular integral and time-frequency methods to control positive maximal operators. See Lacey's bilinear maximal theorem [2000] or the extension of Bourgain's return times theorem by Demeter, Lacey, Tao, and Thiele [Demeter et al. 2008].

This paper is structured as follows: Section 2 contains the main approach, a separation of frequency space into horizontal dyadic strips and application of Littlewood–Paley theory in the second variable to reduce to some vector-valued inequality; this step uses the one-variable property of the vector field to ensure that the strips are invariant under H_v. This fact was brought to our attention by Ciprian Demeter. The vector-valued inequality is proved by restricted weak-type interpolation, a tool that allows us to localize the operator to some benign sets G and H and prove strong L^2 bounds on these sets.

Section 3 gives the crucial construction of the sets G and H, relying on two covering lemmas. One is essentially an argument by Cordoba and R. Fefferman [1975], while the other is essentially an argument by Lacey and Li [2006a].

Section 4 outlines the proof of the L^2 bounds on the sets G and H, using time-frequency analysis as in [Bateman 2013b]. The operator that we estimate at this point is a refinement of the operator in that paper. We refer to the decomposition of this operator there without recalling details. The terms in this decomposition satisfy Estimates 16 through 20, which are also taken from the same paper. To complete the proof of Theorem 1, we need the additional Estimates 21 and 22, which depend on the sets G and H. These additional estimates are proved in Section 5, again with much reference to [Bateman 2013b].

Throughout the paper, we write $x \lesssim y$ to mean there is a universal constant C such that $x \le Cy$. We write $x \sim y$ to mean $x \lesssim y$ and $y \lesssim x$. We write $\mathbf{1}_E$ to denote the characteristic function of a set E.

2. Reduction to estimates for a single frequency band

We fix the vector field v with the normalization (1-2) and assume bounded slope as in (1-3). Let P_c be the Fourier restriction operator to a double cone:

$$\widehat{P_c f}(\xi,\eta) = \mathbf{1}_{10|\xi|\le|\eta|}\hat{f}(\xi,\eta).$$

It suffices to estimate $H_v P_c$ in place of H_v because, similarly to (1-4),

$$H_v(\mathbf{1}-P_c)f(\xi,\eta) = H_{(1,0)}(\mathbf{1}-P_c)f(\xi,\eta),$$

due to the restriction on the slope of v. Define the horizontal pair of bands

$$B_k := \{(\xi,\eta)\in\mathbb{R}^2 : |\eta| \in [2^k, 2^{k+1/100})\},$$

and define the corresponding Fourier restriction operator $\widehat{P_k f} = \mathbf{1}_{B_k}\hat{f}$. Since the Hilbert transform in a constant direction is given by a Fourier multiplier, and the vector field v is constant on vertical lines, we can formally write, for a family of multipliers parametrized by x,

$$H_v f(x,y) = \iint m_x(\xi,\eta)\hat{f}(\xi,\eta)e^{2\pi i(x\xi+y\eta)}\,d\xi\,d\eta.$$

Then it is clear that

$$H_v(P_k f)(x,y) = \int 1_{[2^k, 2^{k+1/100})}(\eta) e^{2\pi i y\eta} \Big[\int m_x(\xi,\eta) \hat{f}(\xi,\eta) e^{2\pi i x\xi}\, d\xi \Big] d\eta = P_k(H_v f)(x,y).$$

Define

$$H_k := P_k H_v P_c = P_k H_v P_c P_k.$$

Littlewood–Paley theory implies

$$\|H_v P_c f\|_p \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}/100} |H_k f|^2 \Big)^{1/2} \Big\|_p ,$$

where the summation is over integer multiples of $\frac{1}{100}$. Using Littlewood–Paley theory once more, it suffices to prove

$$\Big\| \Big(\sum_{k\in\mathbb{Z}/100} |H_k(P_k f)|^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}/100} |P_k f|^2 \Big)^{1/2} \Big\|_p ,$$

which follows from the more general estimate

$$\Big\| \Big(\sum_{k\in\mathbb{Z}/100} |H_k f_k|^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_{k\in\mathbb{Z}/100} |f_k|^2 \Big)^{1/2} \Big\|_p$$

for any sequence of functions $f_k \in L^p$. By a limiting argument, it suffices to prove, for all $k_0 > 0$,

$$\Big\| \Big(\sum_{|k|\le k_0} |H_k f_k|^2 \Big)^{1/2} \Big\|_p \lesssim \Big\| \Big(\sum_{|k|\le k_0} |f_k|^2 \Big)^{1/2} \Big\|_p , \tag{2-1}$$

with implicit constant independent of k_0, where it is understood that k runs through elements of $\mathbb{Z}/100$. Compare this inequality with a vector-valued Carleson inequality as in [Grafakos et al. 2005].

Theorem 3 implies that H_k is bounded in L^p for $1 < p < \infty$ for each k. In particular, (2-1) is true for $p = 2$ by interchanging the order of square summation and L^2 norm.

Note that H_k is defined a priori on all of L^p (by Theorem 3), and we may drop the assumption that f is in the Schwartz class. By Marcinkiewicz interpolation for l^2 vector valued functions, it suffices to prove, for $G, H \subseteq \mathbb{R}^2$ and $\sum_k |f_k|^2 \le \mathbf{1}_H$,

$$\Big| \Big\langle \Big(\sum_{|k|\le k_0} |H_k f_k|^2 \Big)^{1/2}, \mathbf{1}_G \Big\rangle \Big| \lesssim |H|^{1/p} |G|^{1-1/p}. \tag{2-2}$$

By Lebesgue's monotone convergence theorem, it suffices to prove this under the assumption that G is supported on a large square $[-N', N']^2$ as long as the implicit constant does not depend on N'. By another limiting argument using crude estimates in case the sets G and H have large distance, it suffices to prove this under the assumption that H is supported in a much larger square $[-N, N]$, again with bounds independent of N. Generalizing, we will only assume both G and H are supported on the larger square.

Since we already have (2-2) for $p = 2$, we immediately obtain this estimate for $p > 2$ provided $|H| \lesssim |G|$ and for $p < 2$ provided $|G| \lesssim |H|$. By a standard induction on the ratio of $|H|$ and $|G|$, it then suffices to prove the following lemma.

Lemma 4. *Let $G', H' \subset [-N, N]^2$ be measurable and let $\frac{3}{2} < p < \infty$.*

If $p > 2$ and $10|G'| < |H'|$, then there exists a subset $H \subset H'$ depending only on p, G', and H' with $|H| \geq |H'|/2$ such that (2-2) *holds with $G = G'$ and any sequence of functions f_k with $\sum_{|k|\leq k_0} |f_k|^2 \leq \mathbf{1}_H$.*

If $p < 2$ and $10|H'| < |G'|$, then there exists a subset $G \subset G'$ depending only on p, G', and H' with $|G| \geq |G'|/2$ such that (2-2) *holds with $H = H'$ and any sequence of functions f_k with $\sum_{|k|\leq k_0} |f_k|^2 \leq \mathbf{1}_H$.*

For example, in case $p > 2$ and $10|G'| < |H'|$, we split H' into H and $H' \setminus H$ and apply the triangle inequality. On $H' \setminus H$ we apply the induction hypothesis, which yields an estimate better than the desired one by a factor of $2^{-1/p}$ because of the size estimate for $H' \setminus H$. On H we use the conclusion of the lemma, which we may assume (by choosing the induction statement properly) to provide a bound no more than $1 - 2^{-1/p}$ times the desired bound.

By Cauchy–Schwarz, (2-2) follows from

$$\int \sum_{|k|\leq k_0} |H_k f_k|^2 \mathbf{1}_G \lesssim |H|^{2/p} |G|^{1-2/p}.$$

This in turn follows from

$$\int \sum_{|k|\leq k_0} |H_k f_k|^2 \mathbf{1}_G \lesssim \left(\frac{|G|}{|H|}\right)^{1-2/p} \int \sum_k |f_k|^2 \tag{2-3}$$

by the assumption on the sequence f_k. Now define the operator $H_{k,G,H}$ by

$$H_{k,G,H} f = \mathbf{1}_G H_k(\mathbf{1}_H f).$$

Then (2-3) follows from the estimate

$$\|H_{k,G,H} f\|_2 \lesssim \left(\frac{|G|}{|H|}\right)^{1/2-1/p} \|f\|_2$$

for any $f \in L^2$, and $|k| \leq k_0$, assuming the implicit constant does not depend on k or k_0. We will prove this L^2 estimate again by Marcinkiewicz interpolation between weak-type estimates. More precisely, we will prove:

Theorem 5. *Let p be as in Theorem 1 and let $G', H' \subseteq \mathbb{R}^2$ be as in Lemma 4. Then there are sets G, H as in Lemma 4 such that for any measurable sets $E, F \subset \mathbb{R}^2$ and each $|k| \leq k_0$, we have*

$$|\langle H_{k,G,H} \mathbf{1}_F, \mathbf{1}_E \rangle| \lesssim \left(\frac{|G|}{|H|}\right)^{1/2-1/p} |F|^{1/2} |E|^{1/2}. \tag{2-4}$$

Again, [Bateman 2013b] proves

$$|\langle H_{k,G,H} \mathbf{1}_F, \mathbf{1}_E \rangle| \lesssim |F|^{1/q} |E|^{1-1/q} \tag{2-5}$$

for all $1 < q < \infty$. The refinement we need here is the localization to G and H, with corresponding improvement in the estimate. The parameter k is irrelevant in proving (2-4), but it is crucial that the sets H and G be constructed independent of k. By interpolating Theorem 5 with (2-5) for q near 1 and ∞, we obtain strong-type estimates

$$|\langle H_{k,G,H} f, e\rangle| \lesssim \left(\frac{|G|}{|H|}\right)^{1/2-1/r} \|f\|_q \|e\|_{q'},$$

where r is as close to p as we wish and q is in a small punctured neighborhood of 2 whose size depends on r. Another interpolation allows q to be 2 as well, and we obtain (2-3) with power r instead of p, which is no harm since we seek an open range of exponents. We have thus reduced Theorem 1 to Theorem 5.

3. Construction of the sets G and H

In this section we present the sets G and H of Lemma 4 and prove the size estimates $|G| \geq |G'|/2$ and $|H| \geq |H'|/2$. Inequality (2-4) will be proved in subsequent sections.

We work with two shifted dyadic grids on the real line:

$$\mathcal{I}_1 = \left\{\left[2^k\left(n + \frac{(-1)^k}{3}\right), 2^k\left(n+1+\frac{(-1)^k}{3}\right)\right) : k, n \in \mathbb{Z}\right\},$$
$$\mathcal{I}_2 = \left\{\left[2^k\left(n - \frac{(-1)^k}{3}\right), 2^k\left(n+1-\frac{(-1)^k}{3}\right)\right) : k, n \in \mathbb{Z}\right\}.$$

The exceptional sets will be the union of two sets:

$$H' \setminus H = H_1 \cup H_2,$$
$$G' \setminus G = G_1 \cup G_2.$$

Fix $i \in \{1, 2\}$. The sets H_i and G_i will be constructed using the grid $\mathcal{I}_i$, and we will prove $4|H_i| \leq |H'|$ and $4|G_i| \leq |G'|$.

Given a parallelogram with two vertical edges, we define the height $H(R)$ of the parallelogram to be the common length of the two vertical edges. We define the shadow $I(R)$ to be the projection of R onto the x axis. The central line segment of R is the line segment that connects the midpoints of the two vertical edges. If a line segment can be written

$$\{(x, y) : x \in I(R) : y = ux + b\},$$

then we call u the slope of the line segment. For each parallelogram R, let $U(R)$ be the set of slopes of lines that intersect both vertical edges. Maximal and minimal slopes in $U(R)$ are attained by the diagonals of the parallelogram. Hence $U(R)$ is an interval of length $2H(R)/|I(R)|$ centered at the slope of the central line of R.

For an interval U and a positive number C, define CU to be the interval with the same center but length $C|U|$. If R is a parallelogram, define CR to be the parallelogram with the same central line segment as R but height $CH(R)$ (this definition of CR is used in Section 3 only). Note that $CU(R) = U(CR)$. For an interval $I \subset I(R)$, define

$$R_I = R \cap (I \times \mathbb{R}).$$

Given N and k_0 as in Lemma 4, we consider a finite set $\mathcal{R}_i$ of parallelograms R as follows: the projection of both vertical edges of R onto the y-axis are in $\mathcal{I}_1 \cup \mathcal{I}_2$, and $I(R) \in \mathcal{I}_i$. Further, the parallelogram is contained in the square $[-10^2 N, 10^2 N]^2$, the height is at least 2^{-k_0}, and the slope is at most 10^{-1}. These assumptions imply also that $|I(R)|$ is at least 2^{-k_0}.

We will use the following simple geometric observation:

Lemma 6. *Let R, R' be two parallelograms and assume $I(R) = I(R')$, $U(R) \cap U(R') \neq \varnothing$, $R \cap R' \neq \varnothing$, and without loss of generality $H(R) \leq H(R')$. Then we have $R \subseteq 7R'$. Moreover, if $7H(R) \leq H(R')$, then $7R \subseteq 7R'$.*

Proof. Since $U(R) \cap U(R') \neq \varnothing$, there exist two parallel lines, one intersecting both vertical edges of R and the other intersecting both vertical edges of R'. Since $R \cap R' \neq \varnothing$, the vertical displacement of these lines is less than $H(R) + H(R')$. If $H(R) \leq H(R')$, then the vertical edges of R have distance at most $2H(R')$ from the respective vertical edges of R' and are contained in the vertical edges of $7R'$. This proves the first statement of the lemma. The second statement follows similarly. □

Let M_V denote the Hardy–Littlewood maximal operator in the vertical direction:

$$M_V f(x, y) = \sup_{y \in J} \frac{1}{|J|} \int_J |f(x, z)|\, dz,$$

where the supremum is taken over all intervals J containing y. For a measurable function $u : \mathbb{R} \to \mathbb{R}$ (which will be the slope function associated with the given vector field), define

$$E(R) := \{(x, y) \in R : u(x) \in U(R)\}.$$

3.1. ***Construction of the set H.*** With the sets G', H' as in Lemma 4, we define

$$H_i = \bigcup \{R \in \mathcal{R}_i : |E(R) \cap G'| \geq \delta |R|\},$$

with

$$\delta = C_\alpha \left(\frac{|G'|}{|H'|} \right)^{1-\alpha}$$

for some small α to be determined later through application of Estimate 22 and some constant C_α large enough that the desired estimate $4|H_i| \leq |H'|$ follows from the following lemma, applied with $G = G'$, $q = 1/(1-\alpha)$. We are essentially eliminating all rectangles R with large density parameter, where density has the meaning from [Bateman 2013b]. This will be used in the proof of Estimate 22 later in the paper. Essentially, trees with density $\geq \delta$ will have extremely small size, and will therefore be mostly negligible.

Lemma 7. *Let $\delta > 0$ and $q > 1$ and let $G \subset \mathbb{R}^2$ be a measurable set and $u : \mathbb{R} \to \mathbb{R}$ be a measurable function. Let $\mathcal{R}$ be a finite collection of parallelograms with vertical edges and dyadic shadow such that*

$$|E(R) \cap G| \geq \delta |R|$$

for each $R \in \mathcal{R}$. Then

$$\Big|\bigcup_{R\in\mathcal{R}} R\Big| \lesssim \delta^{-q}|G|.$$

Proof. We will find a subset $\mathcal{G} \subset \mathcal{R}$ such that

$$\Big|\bigcup_{R\in\mathcal{R}} R\Big| \lesssim \sum_{R\in\mathcal{G}} |R|, \tag{3-1}$$

$$\int \Big(\sum_{R\in\mathcal{G}} \mathbf{1}_{E(R)}\Big)^{q'} \lesssim \sum_{R\in\mathcal{G}} |R|. \tag{3-2}$$

Inequality (3-1) will complete the proof of Lemma 7, provided

$$\sum_{R\in\mathcal{G}} |R| \lesssim \delta^{-q}|G|. \tag{3-3}$$

But with the density assumption for the parallelograms in $\mathcal{R}$, we have

$$\sum_{R\in\mathcal{G}} |R| \le \sum_{R\in\mathcal{G}} \frac{1}{\delta}|E(R)\cap G| = \frac{1}{\delta}\Big\|\sum_{R\in\mathcal{G}} \mathbf{1}_{E(R)}\mathbf{1}_G\Big\|_1 \lesssim \frac{1}{\delta}\Big(\sum_{R\in\mathcal{G}} |R|\Big)^{1/q'} |G|^{1/q},$$

where in the last line we have used Hölder's inequality and (3-2). After division by the middle factor of the right hand side, we obtain (3-3).

The following argument is essentially the one used in [Cordoba and Fefferman 1975] to prove endpoint estimates for the strong maximal operator. We select parallelograms according to the following iterative procedure. Initialize

$$\text{STOCK} \leftarrow \mathcal{R},$$
$$\mathcal{G} \leftarrow \varnothing,$$
$$\mathcal{B} \leftarrow \varnothing.$$

While $\text{STOCK} \neq \varnothing$, choose an $R \in \text{STOCK}$ with maximal $|I(R)|$. If

$$\sum_{R'\in\mathcal{G}:\ E(R)\cap E(R')\neq\varnothing} |7R \cap 7R'| \ge 10^{-2}|R|, \tag{3-4}$$

then update

$$\text{STOCK} \leftarrow \text{STOCK} \setminus R,$$
$$\mathcal{G} \leftarrow \mathcal{G},$$
$$\mathcal{B} \leftarrow \mathcal{B} \cup \{R\}.$$

Otherwise update

$$\text{STOCK} \leftarrow \text{STOCK} \setminus R,$$
$$\mathcal{G} \leftarrow \mathcal{G} \cup \{R\},$$
$$\mathcal{B} \leftarrow \mathcal{B}.$$

It is clear that this procedure yields a partition $\mathcal{R} = \mathcal{G} \sqcup \mathcal{B}$.

To prove (3-1), let $R \in \mathcal{B}$ and let R' be in the set $\mathcal{G}(R)$ of all elements in $\mathcal{G}$ that are chosen prior to R and satisfy $E(R) \cap E(R') \neq \varnothing$. The last property implies $U(R) \cap U(R') \neq \varnothing$ and $R \cap R' \neq \varnothing$. Also, $I(R) \subset I(R')$. By Lemma 6 applied to R and $R'_{I(R)}$, we have, for every vertical line L through the interval $I(R)$,

$$|L \cap 7R \cap 7R'| \geq \min(H(R), H(R')) \geq \frac{|7R \cap 7R'|}{7|I(R)|}.$$

Comparing for $(x, y) \in R$ and corresponding vertical line L the maximal function M_V with an average over the segment $L \cap 7R$, we obtain

$$M_V\Big(\sum_{R' \in \mathcal{G}(R)} \mathbf{1}_{7R'}\Big)(x, y) \geq 7^{-1} H(R)^{-1} \sum_{R' \in \mathcal{G}(R)} |L \cap 7R \cap 7R'| \geq 49^{-1} |R|^{-1} \sum_{R' \in \mathcal{G}(R)} |7R \cap 7R'| \geq 10^{-4},$$

where the last estimate follows from (3-4). Hence

$$\Big|\bigcup_{R \in \mathcal{B}} R\Big| \leq \Big|\Big\{x : M_V\Big(\sum_{r \in \mathcal{G}} \mathbf{1}_R\Big)(x) \geq 10^{-4}\Big\}\Big| \lesssim \sum_{R \in \mathcal{G}} |R|,$$

by the weak (1, 1) inequality for M_V. This proves (3-1), because the corresponding estimate for the union of elements in $\mathcal{G}$ is trivial.

To prove (3-2), consider $R', R \in \mathcal{G}$ with $E(R) \cap E(R') \neq \varnothing$. If R' was selected first, then $H(R) > 7H(R')$, for otherwise we can use Lemma 6 as above to conclude, for $(x, y) \in R$,

$$M_V(\mathbf{1}_{7R'})(x, y) \geq 7^{-1} |H(R)|^{-1} \sum_{R' \in \mathcal{G}(R)} |L \cap 7R \cap 7R'| \geq 49^{-1},$$

and hence R would have been put into $\mathcal{B}$. Hence we have, by Lemma 6,

$$7R'_I \subset 7R_I \tag{3-5}$$

for every $I \subset I(R)$. Hence

$$\sum_{R' \in \mathcal{G}(R)} |7R'_I \cap 7R_I| = \sum_{R' \in \mathcal{G}(R)} |7R'_I|$$

is proportional to $|I|$ for $I \subset I(R)$. Hence we have, for all such I,

$$\sum_{R' \in \mathcal{G}(R)} |7R'_I \cap 7R_I| \lesssim |R_I|, \tag{3-6}$$

since for $I = I(R)$, this holds when condition (3-4) fails.

Let's say an n-tuple $(R^1, R^2, \ldots, R^n)$ of elements in $\mathcal{G}$ is *admissible* if R^j is selected after R^{j+1} for each j and $E(R^j) \cap E(R^{j+1}) \neq \varnothing$. Then we have

$$\begin{aligned}\int\Big(\sum_{R\in\mathcal{G}}\mathbf{1}_{E(R)}\Big)^n &\lesssim \sum_{R^1,\dots,R_n}\big|E(R^1)\cap E(R^2)\cap\cdots\cap E(R^n)\big|\\ &\lesssim \sum_{(R^1,R^2,\dots,R^n)\text{ adm.}}\big|E(R^1)\cap E(R^2)\cap\cdots\cap E(R^n)\big|\\ &\lesssim \sum_{(R^1,R^2,\dots,R^n)\text{ adm.}}\big|7R^1\cap 7R^2\cap\cdots\cap 7R^n\big|\\ &\lesssim \sum_{(R^1,R^2,\dots,R^n)\text{ adm.}}\big|7R^1\cap 7R^2_{I(R_1)}\cap\cdots\cap 7R^n_{I(R_1)}\big|.\end{aligned}$$

Using (3-5), which implies that the sets $7R^j_{I(R_1)}$ are nested, and the estimate (3-6) for the last pair of sets, we can estimate the last display by

$$\lesssim \sum_{(R^1,R^2,\dots,R^{n-1})\text{ adm.}}\big|7R^1\cap 7R^2_{I(R_1)}\cap\cdots\cap 7R^{n-1}_{I(R_1)}\big|. \tag{3-7}$$

Iterating the argument allows us to conclude (3-2) for q' an integer, which is clearly not a restriction, as the estimate is harder for larger q'. This completes the proof of Lemma 7. □

3.2. *Construction of the set G.* Let G', H', u be as in Lemma 4 and define

$$G_i=\bigcup_{k\in\mathbb{Z},k<0}\left\{R\in\mathcal{R}_i:\frac{|E(R)|}{|R|}\ge 2^k\text{ and }\frac{|H'\cap R|}{|R|}\ge C_\epsilon 2^{-(1/2+\epsilon)k}\left(\frac{|H'|}{|G'|}\right)^{1/2}\right\}$$

for some small $\epsilon>0$, to be determined later through application of Estimate 21, and some constant C_ϵ large enough that we obtain, with Theorem 8 below,

$$|G_i|\le\sum_{k\in\mathbb{Z},k<0}C2^{-k}\left(C_\epsilon 2^{-(1/2+\epsilon)k}\left(\frac{|H'|}{|G'|}\right)^{1/2}\right)^{-2}|H'|\le\frac{|G'|}{4}.$$

This construction essentially allows us to ignore trees with size and density both too large. This will be used in the proof of Estimate 21.

The following theorem is a variant of the result in [Lacey and Li 2006a]. The theorem there is valid for arbitrary Lipschitz vector fields. As stated here, the theorem is valid for vector fields depending on one variable. In fact, the theorem holds for vector fields that are Lipschitz in the vertical direction only. We recreate the proof given in [Lacey and Li 2006a] below in the one-variable case. The only use of the one-variable property comes in the proof of Lemma 12 below.

Theorem 8. *Let $0\le\delta,\sigma\le 1$, let H be a measurable set, and let $\mathcal{R}$ be a finite collection of parallelograms with vertical edges and dyadic shadow such that for each $R\in\mathcal{R}$, we have*

$$|E(R)|\ge\delta|R|,\qquad |H\cap R|\ge\sigma|R|.$$

Then

$$\Big|\bigcup_{R\in\mathcal{R}}R\Big|\lesssim\delta^{-1}\sigma^{-2}|H|.$$

Remark 9. It is of interest whether a result like Theorem 8 holds with σ-power less than 2. In the single height case, optimal results are already known with power all the way to $1+\epsilon$; see [Bateman 2009; Bateman 2013a]. However the important point is that the parallelograms in Theorem 8 can have arbitrary height, which is necessary for creating the exceptional sets needed in the current paper.

Proof. It is enough to find a subset $\mathcal{G} \subset \mathcal{R}$ such that

$$\Big|\bigcup_{R\in\mathcal{R}} R\Big| \lesssim \sum_{R\in\mathcal{G}} |R|, \tag{3-8}$$

$$\int \Big(\sum_{R\in\mathcal{G}} 1_R\Big)^2 \lesssim \delta^{-1} \sum_{R\in\mathcal{G}} |R|. \tag{3-9}$$

Namely, with (3-9) we have

$$\begin{aligned}\sum_{R\in\mathcal{G}} |R| \le \sigma^{-1} \int \sum_{R\in\mathcal{G}} \mathbf{1}_R(x)\mathbf{1}_H \, dx &\le \sigma^{-1}\|H\|^{1/2}\bigg(\int\Big(\sum_{R\in\mathcal{G}} \mathbf{1}_R(x)\Big)^2 dx\bigg)^{1/2} \\ &\lesssim \sigma^{-1}\delta^{-1/2}|H|^{1/2}\Big(\sum_{R\in\mathcal{G}} |R|\Big)^{1/2},\end{aligned}$$

and the desired estimate follows from (3-8).

We define the set $\mathcal{G}$ by a recursive procedure. Initialize

$$\begin{aligned}\mathcal{G} &\leftarrow \varnothing, \\ \text{STOCK} &\leftarrow \mathcal{R}.\end{aligned}$$

While STOCK is not empty, select $R \in \text{STOCK}$ such that $|I(R)|$ is maximal. Update

$$\begin{aligned}\mathcal{G} &\leftarrow \mathcal{G} \cup \{R\}, \\ \mathcal{B} &\leftarrow \bigg\{R' \in \text{STOCK} : R' \subset \bigg\{x : M_V\Big(\sum_{R\in\mathcal{G}} 1_R\Big)(x) \ge 10^{-3}\bigg\}\bigg\}, \\ \text{STOCK} &\leftarrow \text{STOCK} \setminus \mathcal{B}.\end{aligned}$$

This loop will terminate, because the collection $\mathcal{R}$ is finite and we remove at each step at least the selected R from STOCK.

By the Hardy–Littlewood maximal bound, it is clear that (3-8) holds and it remains to show (3-9). By expanding the square in (3-9) and using symmetry, it suffices to show

$$\sum_{(R,R')\in\mathcal{P}} |R \cap R'| \lesssim \delta^{-1} \sum_{R\in\mathcal{G}} |R|,$$

where $\mathcal{P}$ is the set of all pairs $(R, R') \in \mathcal{G} \times \mathcal{G}$ with $R \cap R' \neq \varnothing$, and R is chosen prior to R'. We partition $\mathcal{P}$ into

$$\mathcal{P}' = \{(R, R') \in \mathcal{P} : U(R) \not\subset 10^2 U(R')\} \quad \text{and} \quad \mathcal{P}'' = \{(R, R') \in \mathcal{P} : U(R) \subset 10^2 U(R')\}.$$

Theorem 8 is reduced to the following two lemmas:

Lemma 10. *For fixed $R' \in \mathcal{G}$, we have*

$$\sum_{\substack{R \in \mathcal{R} \\ (R,R') \in \mathcal{P}''}} |R \cap R'| \lesssim |R'|.$$

Lemma 11. *For fixed $R \in \mathcal{G}$, we have*

$$\sum_{\substack{R' \in \mathcal{R} \\ (R,R') \in \mathcal{P}'}} |R \cap R'| \lesssim \delta^{-1} |R|.$$

Proof of Lemma 10. We first argue by contradiction that $\mathcal{P}''$ does not contain a pair (R, R') with $H(R') < H(R)$. By definition of $\mathcal{P}''$, we have $U(R) \cap U(100R') \neq \varnothing$. By Lemma 6 applied to $100R_{I(R')}$ and $100R'$, we conclude that R' is contained in $700R$. But then

$$R' \subset \left\{ M_V 1_R > \tfrac{1}{700} \right\},$$

which contradicts the selection of R' and completes the proof that we have $H(R) \leq H(R')$ for all $(R, R') \in \mathcal{P}''$.

Now we use Lemma 6 again to conclude that for each $(R, R') \in \mathcal{P}''$, we have $R_{I(R')} \subset 700R'$. Hence we have, for some point (x, y) in R',

$$10^{-3} \geq M_V \Bigg(\sum_{R \in \mathcal{G} : (R,R') \in \mathcal{P}''} 1_R \Bigg)(x, y) \geq \frac{1}{700 H(R')} \sum_{R : (R,R') \in \mathcal{P}''} H(R) \geq \frac{1}{700} \sum_{R : (R,R') \in \mathcal{P}''} |R \cap R'| / |R'|.$$

This proves Lemma 10. □

There remains to give the proof of Lemma 11, which will occupy us through the end of the section. Fix $R \in \mathcal{G}$. We decompose $\{R' : (R, R') \in \mathcal{P}'\}$ by the following iterative procedure: Initialize

$$\text{STOCK} \leftarrow \{R' : (R, R') \in \mathcal{P}'\},$$
$$\mathcal{G}' \leftarrow \varnothing.$$

While STOCK is nonempty, select $R' \in \text{STOCK}$ with maximal $|I_{R'}|$. Update

$$\mathcal{G}' \leftarrow \mathcal{G}' \cup \{R'\},$$
$$\mathcal{B}(R') \leftarrow \{R'' \in \text{STOCK} : \Pi E(R'') \cap \Pi E(R') \neq \varnothing\},$$
$$\text{STOCK} \leftarrow \text{STOCK} \setminus \mathcal{B}(R'),$$

where Π denotes the projection onto the x axis. By construction, the sets $\Pi E(R')$ with $R' \in \mathcal{G}'$ are disjoint and we have

$$\sum_{R' \in \mathcal{G}'} |I_{R'}| \leq \delta^{-1} \sum_{R' \in \mathcal{G}'} |\Pi E(R')| \leq \delta^{-1} |I(R)|.$$

As the sets $\mathcal{B}(R')$ with $R' \in \mathcal{G}'$ partition the summation set of the left side of Lemma 11, it suffices to show that, for each $R' \in \mathcal{G}'$,

$$\sum_{R'' \in \mathcal{B}(R')} |R'' \cap R| \lesssim |R_{I(R')}|.$$

In what follows we fix $R' \in \mathcal{G}'$.

Lemma 12. *There is an interval U of slopes (depending on R and R') with*

$$5|U(R)| \le |U|, \tag{3-10}$$

$$U(R) \cap 5U = \varnothing, \tag{3-11}$$

$$U(R) \subset 6U, \tag{3-12}$$

$$U(R'') \subset U \tag{3-13}$$

for all $R'' \subset \mathcal{B}(R')$.

Proof. We distinguish two cases: $|U(R)| \le |U(R')|$ and $|U(R)| > |U(R')|$.

<u>First case</u>: $|U(R)| \le |U(R')|$. In the first case we use the definition of $\mathcal{P}'$ to conclude

$$U(R) \cap 25U(R') = \varnothing.$$

We then define $U = KU(R')$, where $K \ge 5$ is the largest number (or very close to that) such that $U(R) \cap 5KU(R') = \varnothing$. Then we have immediately (3-10), (3-11) and (3-12). To see (3-13), assume $U(R'') \not\subset U$ to get a contradiction.

By the construction of $\mathcal{B}(R')$, we know that $\Pi(E(R''))$ and $\Pi(E(R'))$ intersect, which implies that $U(R'') \cap U(R') \neq \varnothing$, since the underlying vector field v is constant along vertical lines. Since $U(R')$ is contained in the middle fifth of the interval U, we conclude $|U| \le 3|U(R'')|$ and $U \subset 7U(R'')$. But then $U(R) \subset 10^2 U(R'')$, a contradiction to $(R, R'') \in \mathcal{P}'$.

<u>Second case</u>: $|U(R)| > |U(R')|$. Then $H(R) > H(R')$ because $|I(R')| \le |I(R)|$. Since R' is not contained in the set $\{M_V 1_R > 10^{-3}\}$ and thus not in $10^3 R$, we conclude that $U(R')$ contains an element not in $400U(R)$. Hence

$$25 \frac{|U(R)|}{|U(R')|} U(R')$$

does not intersect $U(R)$. From there we may proceed as before, with $U(R')$ replaced by this bigger interval. This completes the proof of Lemma 12. □

Lemma 13. *Let I be a dyadic interval contained in $I_{R'}$. Then for all $R'' \in \mathcal{B}(R')$ with $H(R'') \le 20|U|\,|I|$, we have that*

$$R_I \cap R'' \neq \varnothing \implies R''_I \subset 50\big(1 + |U|\,|I| H(R)^{-1}\big) R \tag{3-14}$$

and

$$|R_I \cap R''| \le 10|U|^{-1} H(R'') H(R). \tag{3-15}$$

Proof. By a shearing transformation and translation we may assume that the central line segment of R is on the x axis.

Statement (3-14) follows immediately from the central slope of R'' being less than $10|U|$ and $H(R'') \le 20|U|\,|I|$, and hence the vertical distance of any point in R'' from R is at most $50|U|\,|I|$. To see the second statement, note that the central slope u_0 of R'' is at least $2|U|$. Hence (3-15) follows, because $R \cap R''$ is contained in a parallelogram of height $H(R)$ and base $H(R'')u_0^{-1}$. This proves Lemma 13. □

Lemma 14. *Let I be a dyadic interval contained in $I_{R'}$. If*

$$\sum_{\substack{R''\in\mathcal{B}(R')\\ \text{s.t. } I\subset I_{R''}}} |R_I\cap R''| > 10^{-1}|R_I|,$$

then there does not exist $R'''\in\mathcal{B}(R')$ with $I_{R'''}\subset I$, $I_{R'''}\neq I$.

Proof. For every $R'''\in\mathcal{B}(R')$, we have $U(R''')\subset U$, and thus

$$H(R''')\le 10U|I_{R'''}|.$$

Hence if $I_{R'''}\subset I$, then $H(R''')\le 20U|I|$. The parallelogram R''' has been selected for $\mathcal{G}$ after the parallelogram R and the parallelograms $R''\in\mathcal{B}(R')$ with $I\subset I_{R''}$. By Lemma 13, it suffices to show that the maximal function

$$M_V\Bigg(1_R+\sum_{\substack{R''\in\mathcal{B}(R')\\ \text{s.t. } I\subset I_{R''}}} 1_{R''}\Bigg)$$

is larger than 10^{-3} on the parallelogram

$$\tilde{R}:=50\big(1+|U|\,|I|H(R)^{-1}\big)R.$$

First assume there exists $R''\in\mathcal{B}(R')$ with $I\subset I_{R''}$ and $R_I\cap R''\neq\varnothing$ and $H(R'')\ge 20\,|U|\,|I|$. Note that $U(R'')$ and $U(\tilde{R})$ have nonempty intersection because $U(R'')\subset U\subset U(\tilde{R})$. Applying Lemma 6 to the rectangles R''_I and $\tilde{R}_I$, we obtain similarly as before

$$M_V(1_{R''}+1_R)\ge 7^{-1}H(\tilde{R})^{-1}\big(\min(H(R''),H(\tilde{R}))+H(R)\big)>10^{-3}$$

on $\tilde{R}_I$, which proves Lemma 14 in the given case.

Hence we may assume

$$H(R'')\le 20\,|U|\,|I|$$

for every $R''\in\mathcal{B}(R')$ with $I\subset I_{R''}$ and $R_I\cap R''\neq\varnothing$. We then have on $\tilde{R}_I$, by Lemma 13,

$$\begin{aligned}
M_V\Bigg(1_R+\sum_{\substack{R''\in\mathcal{B}(R')\\ \text{s.t. } I\subset I_{R''}}} \mathbf{1}_{R''}\Bigg) &\ge H(\tilde{R})^{-1}\Bigg(H(R)+\sum_{\substack{R''\in\mathcal{B}(R')\\ \text{s.t. } I\subset I_{R''}}} H(R'')\Bigg)\\
&\ge H(\tilde{R})^{-1}\Bigg(H(R)+\sum_{\substack{R''\in\mathcal{B}(R')\\ \text{s.t. } I\subset I_{R''}}} |R_I\cap R''||U|H(R)^{-1}\Bigg)\\
&\ge H(\tilde{R})\big(H(R)+|U|H(R)^{-1}10^{-1}|R_I|\big)\ge 500^{-1}.
\end{aligned}$$

This completes the proof of Lemma 14. □

We have used the hypothesis $I_{R'''}\neq I$ of Lemma 14 only to conclude that R''' has been selected last to $\mathcal{G}$. Consider the collection of all $R''\in\mathcal{B}(R')$ with $I=I_{R''}$ and let R''' be the parallelogram chosen last in this collection. Since $|R_I\cap R'''|\le|R_I|$, the proof of the previous lemma also gives:

Lemma 15. *For every $I \subset I_{R'}$,*

$$\sum_{R'' \in \mathcal{B}(R') : I = I_{R''}} |R_I \cap R''| \le 2|R_I|.$$

Now let $\mathcal{I}$ be the set of maximal dyadic intervals contained in $I_{R'}$ such that

$$\sum_{\substack{R'' \in \mathcal{B}(R') \\ \text{s.t. } I \subset I_{R''}}} |R_I \cap R''| > 2|R_I|.$$

By Lemma 15, we have $I_{R'} \notin \mathcal{I}$. Let $I \in \mathcal{I}$ and denote the parent of I by $\tilde{I}$. By Lemma 14 and by maximality of I and Lemma 15, we have

$$\sum_{R'' \in \mathcal{B}(R')} |R_I \cap R''| = \sum_{R'' \in \mathcal{B}(R') : \tilde{I} \subset I_{R''}} |R_I \cap R''| + \sum_{R'' \in \mathcal{B}(R') : I = I_{R''}} |R_I \cap R''| \le 2|R_{\tilde{I}}| + 2|R_I| \le 6|R_I|.$$

By adding over all $I \in \mathcal{I}$, we obtain

$$\sum_{I \in \mathcal{I}'} \sum_{R'' \in \mathcal{B}(R')} |R_I \cap R''| \le 6|R_{I(R')}|. \tag{3-16}$$

Now let $\mathcal{I}'$ be the set of maximal dyadic intervals that are contained in $I_{R'}$, disjoint from any interval in $\mathcal{I}$, and do not contain any $I(R'')$ with $R'' \in \mathcal{R}(R')$. By construction of $\mathcal{I}$, we have for each $I \in \mathcal{I}'$

$$\sum_{R'' \in \mathcal{R}(R')} |R_I \cap R''| = \sum_{R'' \in \mathcal{R}(R') : I \subset I_{R''}} |R_I \cap R''| \le 2|R_I|.$$

Summing over all intervals in $\mathcal{I}'$ gives

$$\sum_{I \in \mathcal{I}'} \sum_{R'' \in \mathcal{R}(R')} |R_I \cap R''| \le 2|R_{I(R')}|. \tag{3-17}$$

Together with (3-16) this completes the proof of Lemma 11, because $\mathcal{I}$ and $\mathcal{I}'$ form a partition of $I(R')$. □

4. Outline of the proof of Theorem 5

Recall that we need to prove, for each $|k| \le k_0$, the inequality

$$|\langle H_{k,G,H} \mathbf{1}_F, \mathbf{1}_E \rangle| \lesssim \left(\frac{|G|}{|H|} \right)^{1/2 - 1/p} |F|^{1/2} |E|^{1/2}. \tag{4-1}$$

We assume without loss of generality that $E \subset G$ and $F \subset H$. Recall also that Theorem 2 implies, for $1 < q < \infty$,

$$|\langle H_k \mathbf{1}_F, \mathbf{1}_E \rangle| \lesssim \left(\frac{|E|}{|F|} \right)^{1/2 - 1/q} |F|^{1/2} |E|^{1/2}. \tag{4-2}$$

The left sides of (4-1) and (4-2) are identical. Hence our task is to strengthen the proof of Theorem 2 in [Bateman 2013b] in case the factor involving G and H in (4-1) is less than the corresponding factor involving E and F in (4-2).

We recall some details about the proof in [Bateman 2013b]. The form $\langle H_k \mathbf{1}_F, \mathbf{1}_E\rangle$ is written as a linear combination of a bounded number of model forms

$$\sum_{s\in\mathcal{U}_k} \langle \mathbf{C}_{s,k}\mathbf{1}_F, \mathbf{1}_E\rangle,$$

where the index set $\mathcal{U}_k$ is a set of parallelograms with vertical edges and constant height (depending on k). The paper proves the bound analogous to (4-2) for the absolute sum

$$\sum_{s\in\mathcal{U}'_k} |\langle \mathbf{C}_{s,k}\mathbf{1}_F, \mathbf{1}_E\rangle|, \tag{4-3}$$

where $\mathcal{U}'_k$ is an arbitrary finite subset of $\mathcal{U}_k$ and the bound is independent of the choice of subset, which may be assumed to only account for nonzero summands.

To estimate (4-3), one first proves estimates for the sum over certain subsets of $\mathcal{U}'_k$ called trees. Each tree T is assigned a parallelogram $\mathbf{top}(T)$. It is also assigned a density $\delta(T)$, which measures the contribution of E to the tree, and a size $\sigma(T)$, which measures the contribution of F to the tree. One obtains, for each tree T,

$$\sum_{s\in T} |\langle \mathbf{C}_s\mathbf{1}_F, \mathbf{1}_E\rangle| \lesssim \delta(T)\sigma(T)|\mathbf{top}(T)|.$$

The collection $\mathcal{U}'_k$ is then written as a disjoint union of subcollections $\mathcal{U}_{\delta,\sigma}$, where δ and σ run through the set of integer powers of two. Each $\mathcal{U}_{\delta,\sigma}$ is written as a disjoint union of a collection $\mathcal{T}_{\delta,\sigma}$ of trees with density at most δ and size at most σ. With the above tree estimate, it remains to estimate $\sum_{\delta,\sigma} S_{\delta,\sigma}$ with

$$S_{\delta,\sigma} := \sum_{T\in\mathcal{T}_{\delta,\sigma}} \delta\sigma|\mathbf{top}(T)|.$$

We list the estimates on $S_{\delta,\sigma}$ used in [Bateman 2013b]; we include an additional factor of $\delta\sigma$ relative to the corresponding expressions in [Bateman 2013b].

Estimate 16 (orthogonality). $S_{\delta,\sigma} \lesssim |F|\delta\sigma^{-1}$.

Estimate 17 (density). $S_{\delta,\sigma} \lesssim |E|\sigma$.

Estimate 18 (maximal). *For any* $\epsilon > 0$, $S_{\delta,\sigma} \lesssim |F|^{1-\epsilon}|E|^{\epsilon}\sigma^{-\epsilon}$.

Estimate 19 (trivial density restriction). *If* $\delta > 1$, *then* $S_{\delta,\sigma} = 0$.

Estimate 20 (trivial size restriction). *There is a universal* σ_0 *such that if* $\sigma > \sigma_0$, *then* $S_{\delta,\sigma} = 0$.

Our improvement comes through two additional estimates depending on G and H that will be proved in Section 5.

Estimate 21 (second maximal). *If* $p < 2$ *and* G, H *are as in Theorem 5, then for every* $\epsilon > 0$,

$$S_{\delta,\sigma} \lesssim |E|\left(\frac{|H|}{|G|}\right)^{1/2} \sigma^{-\epsilon}\delta^{-1/2-\epsilon}.$$

Estimate 22 (size restriction). *Let $p > 2$ and let G, H be as in Theorem 5. Let $n > 2$ be a large integer and $\alpha = 1/n$ and C_α be some constant. Then there is a constant σ_1 such that if*

$$\sigma \geq \sigma_1 \left(\frac{\tilde{\delta}}{\delta} \right)^n$$

with

$$\tilde{\delta} = C_\alpha \left(\frac{|G|}{|H|} \right)^{1-\alpha},$$

then we have $S_{\delta,\sigma} = 0$.

To obtain summability for small σ, it is convenient to take weighted geometric averages of Estimates 16, 18, and 21 with Estimate 17 to obtain positive powers of σ. We record these modified estimates, where we simplify exponents using that we may assume universal upper bounds on δ and σ. We have, for any $\epsilon > 0$:

Estimate 23 (modified orthogonality). $S_{\delta,\sigma} \lesssim |E|^{1/2+\epsilon} |F|^{1/2-\epsilon} \delta^{1/2-\epsilon} \sigma^{2\epsilon}$.

Estimate 24 (modified maximal). $S_{\delta,\sigma} \lesssim |F|^{1-4\epsilon} |E|^{4\epsilon} \sigma^{\epsilon}$.

Estimate 25 (modified second maximal). *Under the assumptions of Estimate 21*,

$$S_{\delta,\sigma} \lesssim |E| \left(\frac{|H|}{|G|} \right)^{1/2-\epsilon} \sigma^{\epsilon} \delta^{-1/2}.$$

In the rest of this section we show how these estimates are used to estimate $\sum_{\delta,\sigma} S_{\delta,\sigma}$, and thereby complete the proof of Theorem 5.

4.1. ***Case $p < 2$ and $|H| \leq |G|$.*** Inequality (4-1) for $\frac{3}{2} < p < 2$ follows from inequality (4-2) for $1 < q < 2$ unless

$$\left(\frac{|H|}{|G|} \right)^{1/3} \leq \frac{|F|}{|E|}, \tag{4-4}$$

which we shall therefore assume.

Pick $\epsilon > 0$ small compared to the distance of p to $\frac{3}{2}$. We split the sum over δ at

$$\delta_0 = \left(\frac{|H|}{|G|} \frac{|E|}{|F|} \right)^{1/2}.$$

For $\delta \leq \delta_0$, we use Estimate 23 together with Estimate 20 to obtain

$$\sum_{\delta \leq \delta_0} \sum_{\sigma} S_{\delta,\sigma} \lesssim \delta_0^{1/2-\epsilon} |E|^{1/2+\epsilon} |F|^{1/2-\epsilon} = |E|^{3/4+\epsilon/2} |F|^{1/4-\epsilon/2} \left(\frac{|H|}{|G|} \right)^{1/4-\epsilon/2}.$$

For $\delta \geq \delta_0$ we use Estimate 25 together with Estimate 20 to obtain

$$\sum_{\delta \geq \delta_0} \sum_{\sigma} S_{\delta,\sigma} \lesssim \delta_0^{-1/2} |E| \left(\frac{|H|}{|G|} \right)^{1/2-\epsilon} = |E|^{3/4} |F|^{1/4} \left(\frac{|H|}{|G|} \right)^{1/4-\epsilon}.$$

Using (4-4) and $|H| \le |G|$, we may estimate both partial sums by

$$\lesssim |E|^{1/2}|F|^{1/2}\left(\frac{|H|}{|G|}\right)^{1/6-3\epsilon}.$$

This completes the proof of (4-1) in case $p < 2$.

4.2. ***Case $p > 2$ and $|G| \le |H|$.*** Pick ϵ very small compared to $1/p$. Inequality (4-1) for $2 < p < \infty$ follows from inequality (4-2) unless

$$\frac{|G|}{|H|} \le \left(\frac{|E|}{|F|}\right)^{1+\epsilon}, \tag{4-5}$$

which we shall therefore assume. Let α and $1/n$ be very small compared to ϵ, let C_α be as in the construction of the set H, and let $\tilde{\delta}$ be as in Estimate 22. We split the sum over δ at

$$\delta_1 := \tilde{\delta}\left(\frac{1}{\tilde{\delta}}\frac{|E|}{|F|}\right)^{1/n}.$$

For $\delta \le \delta_1$ we use a weighted geometric mean of Estimates 23 and 24 together with Estimate 20 to obtain

$$\sum_{\delta\le\delta_1}\sum_{\sigma} S_{\delta,\sigma} \lesssim \delta_1^{1/2-4\epsilon}|E|^{1/2-\epsilon}|F|^{1/2+\epsilon} \lesssim \tilde{\delta}^{(1-1/n)(1/2-4\epsilon)}|E|^{1/2}|F|^{1/2}\left(\frac{|G|}{|H|}\right)^{-2\epsilon},$$

where in the last line we have used (4-5) and $|G| \le |H|$. Using the definition of $\tilde{\delta}$ in Estimate 22, we may estimate the last display by

$$\lesssim |E|^{1/2}|F|^{1/2}\left(\frac{|G|}{|H|}\right)^{1/2-10\epsilon}. \tag{4-6}$$

For $\delta \ge \delta_1$ we use Estimate 17 together with Estimate 22 to obtain

$$\sum_{\delta\ge\delta_1}\sum_{\sigma} S_{\delta,\sigma} \lesssim \sum_{\delta\ge\delta_1}(\tilde{\delta}/\delta)^n|E| \lesssim (\tilde{\delta}/\delta_1)^n|E| \lesssim \tilde{\delta}|F| \lesssim |F|^{1/2}|E|^{1/2}\left(\frac{|G|}{|H|}\right)^{1/2-10\epsilon},$$

where in the last line we have used (4-5) and $|G| \le |H|$. This completes the proof of (4-1) in case $p > 2$.

5. Proof of the additional Estimates 21 and 22

In this section we deviate from the notation in Section 3 as follows: for a parallelogram R we denote by CR the isotropically scaled parallelogram with the same center and slope as R but with height $H(CR) = CH(R)$ and shadow $I(CR) = CI(R)$.

We say that a set is approximated by a parallelogram R if it is contained in the parallelogram and the parallelogram has at most one hundred times the area of the set. Any parallelogram R can be approximated by a parallelogram R' with $I(R') \in \mathcal{I}_1 \cup \mathcal{I}_2$ and both vertical edges of R' in $\mathcal{I}_1 \cup \mathcal{I}_2$. To see this, first identify an interval I in $\mathcal{I}_1 \cup \mathcal{I}_2$ that contains $I(R)$ and has at most three times the length; this interval I will be the shadow of R'. Consider the extension of R that has same central line and height as R but shadow I. Then find two intervals in $\mathcal{I}_1 \cup \mathcal{I}_2$ that have mutually equal length at most three times the height of R and that contain the respective vertical edges of the extended parallelogram. These intervals define the vertical edges of R'.

We recall some details of the proof of Estimate 17 in [Bateman 2013b]. Given δ, σ, one constructs a collection $\mathcal{R}_{\delta,\sigma}$ of parallelograms of the same height as the parallelograms in $\mathcal{U}'_k$ such that each tree T in $T_{\delta,\sigma}$ is assigned a parallelogram R in $\mathcal{R}_{\delta,\sigma}$ with $\mathbf{top}(T) \subset C_0 R$ and $\mathbf{top}(T') \subset C_0 R$ for every subtree T' of T, for some constant C_0. If $\mathcal{T}(R)$ denotes the trees in $\mathcal{T}_{\delta,\sigma}$ that are assigned a given parallelogram $R \in \mathcal{R}_{\delta,\sigma}$, then we have

$$\sum_{T \in \mathcal{T}(R)} |\mathbf{top}(T)| \le C_1 |R|$$

for some constant C_1. Estimate 17 is then deduced from the inequality

$$\sum_{R \in \mathcal{R}_{\sigma,\delta}} |R| \lesssim |E| \delta^{-1}, \tag{5-1}$$

which follows essentially from pairwise incomparability of the parallelograms in $\mathcal{R}_{\delta,\sigma}$. (In other words, if two parallelograms P_1, P_2 overlap, then they are pointed in different directions, resulting in disjointness of the sets $E(P_1)$ and $E(P_2)$.) All parallelograms in $\mathcal{R}_{\delta,\sigma}$ have height at least 2^{-k_0}, length of shadow at least 2^{-k_0}, and slope at most 10^{-1}.

Let $Q = [-N, N]^2$ be the large square with N as in Lemma 4. We claim that every set $Q \cap 2^k R$ with $R \in \mathcal{R}_{\delta,\sigma}$ and $k \ge 0$ can be approximated by a parallelogram in $\mathcal{R}_1 \cup \mathcal{R}_2$. If $Q \cap 2^k R$ is a parallelogram, then this is clear by the remarks above. If $Q \cap 2^k R$ is not a parallelogram, then we first extend it to the minimal parallelogram containing it, which thanks to the bounded slope of R is not much larger than $Q \cap 2^k R$, and then approximate the extension by a parallelogram in $\mathcal{R}_1 \cup \mathcal{R}_2$.

5.1. ***Proof of Estimate 21.*** We partition $\mathcal{R}_{\delta,\sigma}$ into subset $\mathcal{R}_{\delta,\sigma,j}$ consisting of all parallelograms in $\mathcal{R}_{\delta,\sigma}$ such that

$$C_1 2^{-j-1} |R| \le \sum_{T \in \mathcal{T}(R)} |\mathbf{top}(T)| < C_1 2^{-j} |R|.$$

We claim that $\mathcal{R}_{\delta,\sigma,j}$ is empty unless j satisfies (5-3) below. Specifically, the number j_0 used in the following display is implicitly defined in (5-3); our present claim justifies that the summation immediately below should only be over $j \gtrsim j_0$. This claim together with (5-1) will prove Estimate 21:

$$S_{\delta,\sigma} \lesssim \delta\sigma \sum_{j_0 \lesssim j} \sum_{\mathcal{R}_{\delta,\sigma,j}} 2^{-j} |R| \lesssim \sum_{j_0 \lesssim j} 2^{-j} |E| \sigma \lesssim |E| \sigma^{-\epsilon} \delta^{-1/2\epsilon} \left(\frac{|H|}{|G|} \right)^{1/2}.$$

It remains to prove the claim. Suppose there is a parallelogram R in $R \in \mathcal{R}_{\sigma,\delta,j}$. It has large density as defined and discussed in [Bateman 2013b], which implies that there is a $k \ge 0$ with

$$|E(2^k R) \cap G| \ge 2^{20k} \delta |2^k R|.$$

Since G is contained in Q, we may approximate $Q \cap 2^k R$ by a parallelogram R' of $\mathcal{R}_1 \cup \mathcal{R}_2$ and obtain

$$|E(R')| \ge |E(R') \cap G| \gtrsim 2^{20k} \delta |R'|. \tag{5-2}$$

Now suppose first that $2^k \geq \sigma^{-\epsilon}$. By Claim 18 in [Bateman 2013b], and using $F \subset Q$, we obtain

$$\frac{|F \cap H \cap R'|}{|R'|} \gtrsim \frac{|F \cap H \cap 2^k R|}{|2^k R|} \gtrsim 2^{-2k} 2^{-j} \sigma^{1+\epsilon}.$$

On the other hand, (5-2) implies in particular $R' \cap G \neq \varnothing$, which by construction of G (see Section 3) implies, using $k \geq 1$,

$$2^{-2k} 2^{-j} \sigma^{1+\epsilon} \lesssim (2^{20k}\delta)^{-(1/2+\epsilon)} \left(\frac{|H|}{|G|}\right)^{1/2},$$

$$2^{-j} \lesssim 2^{-j_0} := \sigma^{-1-\epsilon} \delta^{-1/2-\epsilon} \left(\frac{|H|}{|G|}\right)^{1/2}. \tag{5-3}$$

If $2^k \leq \sigma^{-\epsilon}$, we use the variant

$$\frac{|F \cap H \cap \sigma^{-\epsilon} R|}{|\sigma^{-\epsilon} R|} \geq 2^{-j} \sigma^{1+3\epsilon}$$

of Claim 18 in [Bateman 2013b] to obtain the same conclusion.

5.2. ***Proof of Estimate 22.*** By Estimates 19 and 20, we may assume $C_0 \tilde{\delta} \leq \delta$ with C_0 as above. Suppose $T_{\delta,\sigma}$ is nonempty. Consider a tree T in $\mathcal{T}_{\delta,\sigma}$ and let $R \in \mathcal{R}_{\delta,\sigma}$ be the associated parallelogram as above. As above, for some $k \geq 0$ we have

$$|E(2^k R) \cap G| \geq 2^{20k} \delta |2^k R|.$$

Define m so that δ is within a factor of two of $C_0^2 2^m \tilde{\delta}$ and note that $m \geq 0$. Let $R' \in \mathcal{R}_1 \cup \mathcal{R}_2$ be an approximation of $Q \cap \max(2^k, C_0 2^m) R$. We then have

$$|E(R') \cap G| \geq \tilde{\delta} |R'|.$$

By construction, R' is disjoint from H. Since $\mathbf{top}(T)$ is contained in $C_0 R$, we have that $2^m \mathbf{top}(T)$ is contained in $R' \cup Q^c$, and the same holds with T replaced by any subtree T' of T.

But by Lemma 29 of [Bateman 2013b] with $f = \mathbf{1}_{F \cap H}$, we obtain, with the notation in that lemma for every subtree T' of T,

$$\sum_{s \in T'} |\langle f, \phi_s \rangle|^2 = \sum_{m' \geq m} \sum_{s \in T'} \left| \left\langle f \mathbf{1}_{2^{m'+1}\mathbf{top}(T') \setminus 2^{m'}\mathbf{top}(T')}, \phi_s \right\rangle \right|^2$$
$$\lesssim \sum_{m' \geq m} 2^{-4nm'} \| f 1_{2^{m'+1}\mathbf{top}(T')} \|_2^2 \lesssim 2^{-2nm} |\mathbf{top}(T')|.$$

By the definition of $\sigma(T)$, this implies

$$\sigma(T) \leq 2^{-nm},$$

which in turn implies Estimate 22.

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