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ANGULAR ENERGY QUANTIZATION FOR LINEAR ELLIPTIC SYSTEMS WITH ANTISYMMETRIC POTENTIALS AND APPLICATIONS

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We establish a quantization result for the angular part of the energy of solutions to elliptic linear systems of Schrödinger type with antisymmetric potentials in two dimensions. This quantization is a consequence of uniform Lorentz–Wente type estimates in degenerating annuli. Moreover this result is optimal in the sense that we exhibit a sequence of functions satisfying our hypothesis whose radial part of the energy is not quantized. We derive from this angular quantization the full energy quantization for general critical points to functionals which are conformally invariant or also for pseudoholomorphic curves on degenerating Riemann surfaces.

Introduction

Conformal invariance is a fundamental property for many problems in physics and geometry. In the last decades it has become an important feature of many questions of nonlinear analysis too. Elliptic conformally invariant Lagrangians for instance share similar analysis behaviors: their Euler–Lagrange equations are critical with respect to the function space naturally given by the Lagrangian and, as a consequence, solutions to these Euler Lagrange equations are subject to *concentration compactness* phenomena. Questions such as the regularity of solutions or energy losses for sequences of solutions cannot be solved by robust general arguments in PDE but require instead a careful study of the interplay between the highest order part of the PDE and its nonlinearity.

For example, in dimension 2, let (Σ, h) be a closed Riemann surface, it has been proved [Rivière 2007, Theorem I.2] that every critical point of a conformally invariant functional, $u : \Sigma \to \mathbb{R}^n$, solves a system

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of the form[1]

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } \Sigma, \tag{1}$$

where $\Omega \in \mathrm{so}(n) \otimes T\Sigma$ and Δ is the negative Laplace–Beltrami operator $\big(1/\sqrt{|h|}\big)\, \partial_i \big(\sqrt{|h|}\, h^{ij} \partial_j\big)$. The fundamental fact here that has been observed in [Rivière 2007] and exploited in this work to obtain the Hölder continuity of $W^{1,2}$-solutions to (1) is the *antisymmetry* of Ω.

The analysis developed in [Rivière 2007] allowed one to extend to general two-dimensional conformally invariant Lagrangians the use of *integrability by compensation theory* as it was introduced originally by H. Wente in the framework of constant mean curvature immersions in $\mathbb{R}^3$ to solve the *CMC system*

$$\Delta u = 2 u_x \wedge u_y \quad \text{on } \Sigma. \tag{2}$$

Solutions to this *CMC system* are in fact critical points to the conformally invariant Lagrangian

$$E(u) = \frac{1}{2} \int_\Sigma |du|_h^2 \, d(\mathrm{vol}_h) + \int_\Sigma u^* \omega,$$

where ω is a 2-form in $\mathbb{R}^3$ satisfying $d\omega = 4\, dx_1 \wedge dx_2 \wedge dx_3$. The natural space to consider (2) is clearly the Sobolev space $W^{1,2}$. The CMC system (2) is critical for $W^{1,2}$ in the following sense: the right-hand side of (2) is *a priori* only in L^1. Classical Calderon Zygmund theory tells us that derivatives of functions in $\Delta^{-1} L^1$ are in the weak L^2 space locally which is "almost" the information we started from. Hence in a sense both the quadratic nonlinearity for the gradient in the right-hand side of the system and the operator in the left-hand side are at the same level from regularity point of view and it requires a more careful analysis in order to decide which one is leading the general dynamic of this system.

H. Wente discovered the special role played by the jacobian in the right-hand side of (2) — see [Hélein 1996] and references therein — and was able to prove that if u satisfies (2) then

$$\|\nabla u\|_2 \le C \|\nabla u\|_2^2, \tag{3}$$

where C is independent on Σ and equals[2] $\sqrt{3/16\pi}$. This inequality implies that if $\sqrt{3/16\pi}\ \|\nabla u\|_2 < 1$ then the solution is constant. This is what we call the *bootstrap test* and it is the key observation for proving Morrey estimates and deduce the Hölder regularity of general solutions to (2) which bootstraps easily in order to establish that solutions to (2) are in fact C^∞.

Another analysis issue for this equation is to understand the behavior of sequences u_k of solutions to the CMC system (2). Inequality (3) tells us again that if the energy does not concentrate at a point then the system will behave locally like a linear system of the form $\Delta u = 0$: the nonlinearity $2u_x \wedge u_y$ in the right-hand side is dominated by the linear highest order term Δu in the left-hand side. As a

[1] In coordinates this system reads

$$-\Delta u_i = \sum_{j=1}^{n} \Omega_i^j \cdot \nabla u_j \quad \text{on } \Sigma \text{ for all } i = 1, \dots, n,$$

where the $\cdot$ operation is the scalar product between the gradient vector fields ∇u_j and the different entries of the vector-valued antisymmetric matrix Ω.

[2] This later fact was discovered later on by Y. Ge [1998]; see also [Hélein 1996].

consequence we deduce that sequences of solutions to (2) with uniformly bounded energy strongly converge in C^p-norm for any $p \in \mathbb{N}$, modulo extraction of a subsequence and possibly away from finitely many points[3] $\{a_\infty^1, \dots, a_\infty^l\}$ in Σ, where the $W^{1,2}$-norm concentrates, towards a smooth limit that solves also (2):

$$u_k \longrightarrow u_\infty \text{ strongly in } C^p_{\mathrm{loc}}\big(\Sigma \setminus \{a_\infty^1, \dots, a_\infty^l\}\big) \text{ for all } p \in \mathbb{N}.$$

The question remains to understand how the convergence at the concentration points a_∞^i fails to be strong, in other words we want to understand how and how much energy has been dissipated at the points a_∞^i. A careful analysis shows that the energy is lost by concentrating solution on $\mathbb{R}^2$ of the CMC system (2), the so-called *bubbles*, that converge to the a_∞^i: there exists points in Σ $a_k^i \to a_\infty^i$ and a family of sequences of radii λ_k^i converging to zero such that

$$u_k(\lambda_k^i x + a_k^i) \longrightarrow \omega^i(x) \text{ strongly in } C^p_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \{\text{finitely many points}\}\big) \text{ for all } p \in \mathbb{N},$$

where ω^i denote the bubbles, solutions on $\mathbb{R}^2$ of the CMC system (2). Because of the nature of the convergence it is clear that the Dirichlet energy lost in the convergences amount at least to the sum of the Dirichlet energies of the *bubbles* ω^i:

$$\liminf_{k\to+\infty} \int_\Sigma |du_k|_h^2 \, d(\mathrm{vol}_h) \geq \int_\Sigma |du_\infty|_h^2 \, d(\mathrm{vol}_h) + \sum_{i=1}^{l} \int_{\mathbb{R}^2} |\nabla \omega^i|^2 \, dx_1 \, dx_2. \tag{4}$$

The question remains to understand if the inequality in (4) is strict or is in fact an equality. This question for general conformally invariant problems is known as the *energy quantization question:* is the loss of energy only concentrated in the forming *bubbles* or is there any additional dissipation in the intermediate regions between the *bubbles* and shrinking at the limiting concentration points a_∞^i in the so-called *neck region*. Since the work of Sacks and Uhlenbeck [1981] where it has been maybe first considered, in the particular framework of minimizing harmonic maps from a Riemann surface into a manifold, this question has generated a special interest, intensive researches and several detailed results have been obtained in the last decades on the subject. We refer to [Rivière 2002] and reference therein for a survey on the *energy quantization results*. Positive results establishing *energy quantization* (that is, the inequality in (4) is in fact an equality) often make use of some special *geometric* objects such as isoperimetric inequality or the Hopf differential, see for instance [Jost 1991] or [Parker 1996]. Rivière, in collaboration with F. H. Lin, introduced [2001; 2002] a more *functional analysis* type technique based on the use of the interpolation Lorentz spaces in order to prove *energy quantization results* in the special cases where the nonlinearity of the conformally invariant PDE can be written as a linear combination of jacobians of $W^{1,2}$-functions. Using this technique we can for instance prove that equality holds in (4): *energy quantization* holds for the CMC system, the whole loss of energy exclusively arises in the bubbles. The main step in the proof consists in using an improvement of Wente inequality (3) which has been obtained by L. Tartar and R. Coifman, P. L. Lions, Y. Meyer and S. Semmes [1993]. This improved *Lorentz–Wente type inequality*

[3] In our notation we can have some a_∞^i that coincide with another.

reads

$$\|\nabla u\|_{L^{2,1}} \leq C \; \|\nabla u\|_2^2, \tag{5}$$

where this time C depends *a priori* on (Σ, h) and where $L^{2,1}$ denotes the Lorentz space "slightly" smaller than L^2 given by the space of measurable function f on Σ satisfying

$$\int_0^{\infty} \left|\left\{x \in \Omega \;\middle|\; |f(x)| \geq \lambda\right\}\right|^{1/2} d\lambda < +\infty.$$

The goal of the present paper is to extend energy quantization results to sequences of critical points to general conformally invariant Lagrangians using functional analysis arguments in the style of [Lin and Rivière 2002].

The constant in the inequality (5) depends *a priori* on the domain, at least on its conformal class since the equation is conformally invariant. But our *neck regions* connecting the *bubbles* are conformally equivalent to degenerating annuli. The first task of the present work is to prove different lemma which give some uniform estimates on the $L^{2,1}$-norm of the gradient for solution to Wente-type equations on degenerating annuli. This is the subject of Section 2.

In the following sections, we use these uniform estimates established in Section 2 for proving various quantization phenomena. In particular we get the quantization of the angular part of the gradient for solution of general elliptic second-order systems with antisymmetric potentials. What we mean here by the angular part is the component of the gradient in the orthogonal of the radial direction with respect to the nearest point of concentration. Precisely the first main result in the present work is the following:

Theorem 1. *Let $\Omega_k \in L^2\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$ and let $u_k \in W^{2,1}(B_1, \mathbb{R}^n)$ be a sequence of solutions of*

$$-\Delta u_k = \Omega_k \cdot \nabla u_k,$$

with bounded energy, that is,

$$\int_{B_1} \left(|\nabla u_k|^2 + |\Omega_k|^2\right) dz \leq M. \tag{6}$$

Then there exists $\Omega_\infty \in L^2\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$ and $u_\infty \in W^{2,1}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u_\infty = \Omega_\infty \cdot \nabla u_\infty$ on B_1, $l \in \mathbb{N}^*$ and

(1) $\omega^1, \ldots, \omega^l$ *a family of solutions to system of the form*

$$-\Delta \omega^i = \Omega^i_\infty \cdot \nabla \omega^i \quad \text{on } \mathbb{R}^2,$$

where $\Omega^i_\infty \in L^2\big(\mathbb{R}^2, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$,

(2) $a^1_k, \ldots, a^l_k$ *a family of converging sequences of points of B_1,*

(3) $\lambda^1_k, \ldots, \lambda^l_k$ *a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$\Omega_k \rightharpoonup \Omega_\infty \text{ in } L^2_{\mathrm{loc}}\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big),$$

$$u_k \to u_\infty \text{ on } W^{1,p}_{\mathrm{loc}}\big(B_1 \setminus \{a^1_\infty, \ldots, a^l_\infty\}\big) \text{ for all } p \geq 1,$$

and

$$\left\|\left\langle\nabla\Big(u_k-u_\infty-\sum_{i=1}^{l}\omega_k^i\Big),\,X_k\right\rangle\right\|_{L^2_{\mathrm{loc}}(B_1)}\to 0,$$

where $\omega_k^i=\omega^i(a_k^i+\lambda_k^i\,\cdot\,)$ *and* $X_k=\nabla^\perp d_k$ *with* $d_k=\min\limits_{1\le i\le l}(\lambda_k^i+d(a_k^i,\cdot\,))$.

Moreover, if we have $\|\Omega_k\|_\infty=O(1)$ *or even just* $\Omega_k=\Lambda(u_k,\nabla u_k)$ *where* $\Lambda(\,\cdot\,,p)=O(|p|)$ *the convergence to the limit solution* u_∞ *is in fact in* $C^{1,\eta}_{\mathrm{loc}}$ *for all* $\eta\in[0,1[$.

This theorem is optimal in the sense that we have also exhibited a sequence of functions satisfying the hypothesis of the theorem whose radial part of the energy is not quantized. Moreover, the loss of energy in the neck region is very rigid. We explain these two facts after the proof of Theorem 1.

The proof of Theorem 1 is established through the iteration of the following result. It says that, if the L^2-norm of the potential Ω is below some threshold on every dyadic sub-annulus of a given annulus, the angular part of the Dirichlet energy of u on a slightly smaller annulus is controlled by the maximal contribution of the Dirichlet energy of u on the dyadic sub-annuli. Precisely we prove the following:

Theorem 2. *There exists* $\delta>0$ *such that for all* $r,R\in\mathbb{R}_+^*$ *with* $4r<R$, *all* $\Omega\in L^2\big(B_R\setminus B_r,\mathrm{so}(n)\otimes\mathbb{R}^2\big)$ *and all* $u\in W^{1,2}\big(B_R\setminus B_r,\mathbb{R}^n\big)$ *satisfying* $-\Delta u=\Omega\cdot\nabla u$, *we have*

$$\sup_{r<\rho<R/2}\int_{B_{2\rho}\setminus B_\rho}|\Omega|^2\,dz\le\delta.$$

Then there exists $C>0$, *independent of* u, r *and* R, *such that*

$$\left\|\frac{1}{\rho}\frac{\partial u}{\partial\theta}\right\|^2_{L^2(B_{R/2}\setminus B_{2r})}\le C\|\nabla u\|_2\left[\sup_{r<\rho<R/2}\int_{B_{2\rho}\setminus B_\rho}|\nabla u|^2\,dz\right]^{1/2}.$$

Thanks to the quantization of the angular part for general elliptic systems with antisymmetric potential, we can derive the *energy quantization* for critical points to an arbitrary continuously conformally invariant elliptic Lagrangian with quadratic growth.

Theorem 3. *Let* N^k *be a* C^2 *submanifold of* $\mathbb{R}^m$ *and* ω *be a* C^1 *2-form on* N^k *such that the* L^∞*-norm of* $d\omega$ *is bounded on* N^k. *Let* u_k *be a sequence of critical points in* $W^{1,2}(B_1,N^k)$ *for the Lagrangian*

$$F(u)=\int_{B_1}\big[|\nabla u|^2+\omega(u)(u_x,u_y)\big]\,dz \tag{7}$$

with uniformly bounded energy, that is,

$$\|\nabla u_k\|_2\le M.$$

Then there exists $\Lambda\in C^0\big(TN\otimes\mathbb{R}^2,\mathrm{so}(n)\otimes\mathbb{R}^2\big)$ *and* $u_\infty\in W^{1,2}(B_1,\mathbb{R}^n)$ *a solution of* $-\Delta u=\Lambda(u,\nabla u)\cdot\nabla u$ *on* B_1, $l\in\mathbb{N}^*$ *and*

(1) $\omega^1,\dots,\omega^l$ *some nonconstant* Λ*-bubbles, that is, nonconstant solutions of*

$$-\Delta\omega=\Lambda(\omega,\nabla\omega)\cdot\nabla\omega\quad\text{on }\mathbb{R}^2,$$

(2) $a_k^1, \dots, a_k^l$ *a family of converging sequences of points of* B_1,

(3) $\lambda_k^1, \dots, \lambda_k^l$ *a family of sequences of positive reals converging all to zero*,

such that, *up to a subsequence*,

$$u_k \to u_\infty \text{ on } C^{1,\eta}_{\mathrm{loc}}\big(B_1 \setminus \{a_\infty^1, \dots, a_\infty^l\}\big) \quad \text{for all} \quad \eta \in [0, 1[$$

and

$$\Big\| \nabla\Big(u_k - u_\infty - \sum_{i=1}^{l} \omega_k^i\Big)\Big\|_{L^2_{\mathrm{loc}}(B_1)} \to 0,$$

where $\omega_k^i = \omega^i(a_k^i + \lambda_k^i\,\cdot)$.

Previous works establishing *energy quantizations* for various conformally invariant elliptic Lagrangian usually require more regularity on the Lagrangian (see for instance [Jost 1991; Parker 1996; Struwe 1985; Ding and Tian 1995; Lin and Wang 1998; Zhu 2010]). For instance in [Parker 1996] or [Lin and Wang 1998] the *energy quantization* for harmonic maps in two dimensions is obtained through the application of the maximum principle to an ordinary differential inequality satisfied by the integration over concentric circles of the angular part of the energy. The application of this procedure required an L^∞ bound on the *derivatives of the second fundamental form* [Lin and Wang 1998, Lemma 2.1]. We insist on the fact that, in comparison to the previously existing energy quantization results, Theorem 3 above requires a C^0 bound on the *second fundamental form* only, which is a weakening of the regularity assumption for the target of a magnitude one with respect to derivation. Another application of Theorem 3 is the *energy quantization* for solutions to the prescribed mean curvature system, see Corollary 17, assuming only a C^0 bound on the mean curvature. Again, previous *energy quantization* results were assuming uniform C^1 bounds on H [Bethuel and Rey 1994; Caldiroli and Musina 2006]. Theorem 3 in the prescribed mean curvature system corresponds again for this problem to weakening of the regularity assumption for the target of a magnitude one with respect to derivation in comparison to previous existing result. These weaker assumptions are the minimal ones required in order that the Lagrangian to be continuously differentiable and this is why it coincides with the original one appearing in the formulation of the Heinz–Hildebrandt regularity conjecture in the 1970s.

In a last part, we present some more applications of the uniform Lorentz–Wente estimates we established in Section 2. The first one, for instance, deals with sequences of pseudoholomorphic immersions of sequences of closed Riemann surfaces whose corresponding conformal class degenerate in the moduli space of the underlying two-dimensional manifold. In particular we give a new proof of the Gromov's compactness theorem in all generality, see Theorem 19. We also give some cohomological condition which guarantees the *energy quantization* for sequences of harmonic maps on degenerating surfaces. Finally we give a very brief introduction to the quantization of the Willmore surface established recently in [Bernard and Rivière 2011], where the uniform Lorentz–Wente estimates of Section 2 play a crucial role.

Notation. In the following, if we consider a norm with out specifying its domain, it is implicitly assumed that its domain of definition is the one of the function. We denote $B_R(p)$ the ball of radius R centered at p and we just denote B_R when $p = 0$.

1. Lorentz spaces and standard Wente's inequalities

Lorentz spaces seem to be the good spaces in order to get precise Wente's inequalities. Here we recall some classical facts about these spaces; see [Stein and Weiss 1971] and [Grafakos 2009] for details.

Definition 4. Let D be a domain of $\mathbb{R}^k$, $p \in]1, +\infty[$ and $q \in [1, +\infty]$. The Lorentz space $L^{p,q}(D)$ is the set of measurable functions $f : D \to \mathbb{R}$ such that

$$\|f\|_{p,q} = \left(\int_0^{+\infty} \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t} \right)^{1/q} < +\infty \quad \text{if } q < +\infty,$$

or

$$\|f\|_{p,\infty} = \sup \left(t^{1/p} f^{**}(t)\right) \quad \text{if } q = +\infty,$$

where $f^{**}(t) = (1/t) \int_0^t f^*(s)\, ds$ and f^* is the decreasing rearrangement of f.

Each $L^{p,q}$ may be seen as a deformation of L^p. For instance, we have the strict inclusions

$$L^{p,1} \subset L^{p,q'} \subset L^{p,q''} \subset L^{p,\infty}$$

if $1 < q' < q''$. Moreover,

$$L^{p,p} = L^p.$$

Furthermore, if $|D|$ is finite, we have that for all q and q',

$$p > p' \Rightarrow L^{p,q} \subset L^{p',q'}.$$

Finally, for $p \in]1, +\infty[$ and $q \in [1, +\infty]$, we have $L^{p,q} = \left(L^{p/(p-1), q/(q-1)}\right)^*$.

In the case $p, q = 2, 1$ we can give an equivalent definition. First we note that the norm $\| \; \|_{p,q}$ is equivalent to

$$\left(\int_0^{+\infty} \left(t^{1/p} f^*(t)\right)^q \frac{dt}{t} \right)^{1/q},$$

which is only a seminorm [Ziemer 1989]. Then, letting $\phi(\lambda) = \left|\left\{t \in [0, |D|] \;\middle|\; f^*(t) \geq \lambda\right\}\right|$, we make the change of variable $t = \phi(\lambda)$ in the definition of the Lorentz-norm, which gives

$$\|f\|_{2,1} \sim 2 \int_{\sup|f|}^0 \phi^{-1/2}(\lambda)\, \lambda \phi'(\lambda)\, d\lambda.$$

Hence integrating by parts, we get

$$\|f\|_{2,1} \sim 4 \int_0^{+\infty} \left|\left\{x \in \Omega \;\middle|\; |f(x)| \geq \lambda\right\}\right|^{1/2} d\lambda. \tag{8}$$

To finish these preliminaries, we quickly present the standard Wente's inequalities for elliptic system in Jacobian form. Indeed if a and b are in $W^{1,2}$ this is clear that $a_x b_y - a_y b_x$ is in L^1 but in fact thanks to its structure, it is subject to compensated phenomena and $a_x b_y - a_y b_x$ is in $\mathcal{H}^1$ the Hardy space which is a strict subspace of L^1 and has better behavior than L^1 with respect to Calderon–Zygmund theory, since the convolution of a function in $\mathcal{H}^1$ and the Green kernel $\log(|z|)$ is in $W^{2,1}$. This improvement of integrability is summarized in the following theorem.

Lemma 5 [Wente 1969; Tartar 1985; Coifman et al. 1993]. *Let $a, b \in W^{1,2}(B_1)$, and let $\phi \in W_0^{1,1}(B_1)$ be the solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1.$$

Then there exists a constant C independent of ϕ such that

$$\|\phi\|_\infty + \|\nabla\phi\|_{2,1} + \|\nabla^2\phi\|_1 \leq C\|\nabla a\|\|\nabla b\|_2. \tag{9}$$

A consequence of the previous theorem was obtained by Bethuel [1992] using a duality argument.

Lemma 6. *Let a and b be two measurable functions such that $\nabla a \in L^{2,\infty}(B_1)$ and $\nabla b \in L^2(B_1)$, and let $\phi \in W_0^{1,1}(B_1)$ be the solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1.$$

Then there exists a constant C independent of ϕ such that

$$\|\nabla\phi\|_2 \leq C\|\nabla a\|_{2,\infty}\|\nabla b\|_2. \tag{10}$$

2. Wente-type lemmas

In this section we are going to prove some uniform Wente's estimates on annuli whose conformal class is *a priori* not bounded. In fact those estimate were already known for the L^∞-norm and the L^2-norm of the gradient, since it has been proved that the constant is in fact independent of the domain considered, see [Topping 1997] and [Ge 1998]. But this fact is to our knowledge new for the $L^{2,1}$-norm of the gradient.

Lemma 7. *Let $a, b \in W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{2}$, and let $\phi \in W_0^{1,1}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1 \setminus B_\varepsilon.$$

Then $\nabla\phi \in L^{2,1}(B_1 \setminus B_\varepsilon)$, and for each $\lambda > 1$ there exists a positive constant $C(\lambda)$, independent of ε and ϕ, such that

$$\|\nabla\phi\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq C(\lambda)\|\nabla a\|_2\,\|\nabla b\|_2.$$

Proof. First we consider a solution of our equation on the whole disk, that is to say $\varphi \in W_0^{1,1}(B_1)$ which satisfies

$$\Delta\varphi = a_x b_y - a_y b_x \quad \text{on } B_1.$$

Then thanks to the classical Wente's inequality (9), we have

$$\|\varphi\|_\infty + \|\nabla\varphi\|_{2,1} \leq C\|\nabla a\|_2\,\|\nabla b\|_2, \tag{11}$$

where C is a positive constant independent of φ.

Then we set $\psi = \phi - \varphi$, which satisfies

$$\begin{cases} \Delta\psi = 0 & \text{on } B_1 \setminus B_\varepsilon, \\ \psi = 0 & \text{on } \partial B_1, \\ \psi = -\varphi & \text{on } \partial B_\varepsilon. \end{cases}$$

Hence $\widetilde{\psi} = \psi - \left(\int_{\partial B_\varepsilon} \psi \, d\sigma\right) \log(|z|)/(2\pi\varepsilon\log(\varepsilon))$ satisfies the hypothesis of Lemma A.1, then

$$\|\nabla\widetilde{\psi}\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le C(\lambda)\|\nabla\widetilde{\psi}\|_2 \quad \text{for all } \lambda > 1.$$

Hence, computing the L^2-norm of the gradient of the logarithm on $B_1 \setminus B_{\lambda\varepsilon}$, we get that

$$\|\nabla\widetilde{\psi}\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le C(\lambda)\left(\|\nabla\psi\|_2 + \frac{1}{\varepsilon\sqrt{\log(1/\varepsilon)}} \int_{\partial B_\varepsilon} |\psi| \, d\sigma \right). \tag{12}$$

But ψ is the harmonic on $B_1 \setminus B_\varepsilon$ and is equal to $-\varphi$ on the boundary, then

$$\|\nabla\psi\|_2 \le \|\nabla\varphi\|_2 \quad \text{and} \quad \|\psi\|_\infty \le \|\varphi\|_\infty. \tag{13}$$

Hence we get that

$$\int_{\partial B_\varepsilon} |\psi| \, d\sigma \le \varepsilon C(\lambda)\|\nabla a\|_2 \, \|\nabla b\|_2, \tag{14}$$

which gives, using (12) and (13), that

$$\|\nabla\widetilde{\psi}\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le C(\lambda)\|\nabla a\|_2 \, \|\nabla b\|_2. \tag{15}$$

Finally, computing the $L^{2,1}$-norm of the gradient of the logarithm on $B_1 \setminus B_{\lambda\varepsilon}$, we get that

$$\|\nabla \log r\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} = 4\sqrt{\pi}\log(1/\lambda\varepsilon). \tag{16}$$

Hence, thanks to (14), (15) and (16), we get that

$$\|\nabla\psi\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le C(\lambda)\|\nabla a\|_2 \, \|\nabla b\|_2. \tag{17}$$

Then, thanks to (11) and (17), we get the desired estimate. □

Lemma 8. *Let $a, b \in W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, and let $\phi \in W^{1,1}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1 \setminus B_\varepsilon \tag{18}$$

such that

$$\int_{\partial B_\varepsilon} \phi \, d\sigma = 0 \quad \text{and} \quad \left| \int_{\partial B_1} \phi \, d\sigma \right| \le K,$$

where K is a constant independent of ε. Then for each $0 < \lambda < 1$ there exists a positive constant $C(\lambda)$, independent of ε, such that

$$\|\nabla\phi\|_{L^{2,1}(B_\lambda\setminus B_{\varepsilon/\lambda})} \le C(\lambda)(\|\nabla a\|_2 \, \|\nabla b\|_2 + \|\nabla\phi\|_2 + 1).$$

Proof. Let $u \in W^{1,1}(B_1 \setminus B_\varepsilon)$ be the solution of

$$\begin{cases} \Delta u = 0 & \text{on } B_1 \setminus B_\varepsilon, \\ u = \phi & \text{on } \partial B_1 \cup \partial B_\varepsilon. \end{cases}$$

Hence $\|\nabla u\|_2 \le \|\nabla \phi\|_2$. Moreover, thanks to Lemmas A.2 and 7 we have $\nabla u \in L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})$ and $\nabla(u - \phi) \in L^{2,1}(B_1 \setminus B_{\varepsilon/\lambda})$, with

$$\|\nabla u\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\,(\|\nabla\phi\|_2 + 1) \quad \text{and} \quad \|\nabla(u-\phi)\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\|\nabla a\|_2 \|\nabla b\|_2,$$

which proves Lemma 8. □

Remark. As in Lemma A.2 we cannot control the $L^{2,1}$-norm of $\nabla\phi$ by its L^2-norm, as it is shown by the following example:

$$z \mapsto \frac{\log(|z|/\varepsilon)}{\log(1/\varepsilon)}.$$

Lemma 9. *Let $a, b \in W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, and let $\phi \in W^{1,2}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1 \setminus B_\varepsilon.$$

Moreover, assume that

$$\|\phi\|_\infty < +\infty. \tag{19}$$

Then for each $0 < \lambda < 1$ there exists a positive constant $C(\lambda)$, independent of ε and ϕ, such that

$$\|\nabla\phi\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\big(\|\nabla a\|_2\,\|\nabla b\|_2 + \|\phi\|_\infty\big). \tag{20}$$

Proof. We introduce first $\varphi \in W_0^{1,2}(B_1 \setminus B_\varepsilon)$ to be the unique solution to

$$\begin{cases} \Delta\varphi = a_x b_y - a_y b_x & \text{on } B_1 \setminus B_\varepsilon, \\ \varphi = 0 & \text{on } \partial B_1 \cup \partial B_\varepsilon. \end{cases}$$

Then thanks to Lemma 7, we have

$$\|\nabla\varphi\|_{L^{2,1}(B_1 \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\|\nabla a\|_2\,\|\nabla b\|_2,$$

where $C(\lambda)$ is a positive constant depending on λ but not on ϕ and ε.

Then we set $\psi = \phi - \varphi$, which is harmonic. Thanks to standard estimates on harmonic functions [Han and Lin 2011], there exists a positive constant $C(\lambda)$ independent of ψ and ε such that

$$\|\psi\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\|\psi\|_{L^\infty(\partial B_1 \cup \partial B_\varepsilon)} \le C(\lambda)\|\phi\|_{L^\infty}.$$

This proves the desired inequality, and Lemma 9 is proved. □

Lemma 10. *Let $a, b \in L^2(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, assume that $\nabla a \in L^{2,\infty}(B_1)$ and $\nabla b \in L^2(B_1)$, and let $\phi \in W^{1,(2,\infty)}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1 \setminus B_\varepsilon.$$

For each $\varepsilon \le r \le 1$, *set* $\phi_0(r) = (1/2\pi r) \int_{\partial B_r(0)} \phi \, d\sigma$, *and assume that*

$$\int_\varepsilon^1 \left|\dot{\phi_0}\right|^2 r \, dr < +\infty. \tag{21}$$

Then for each $0 \le \lambda < 1$ *there exists a positive constant* $C(\lambda) > 0$, *independent of* ε *and* ϕ, *such that*

$$\|\nabla\phi\|_{L^2(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\big(\|\nabla a\|_{2,\infty} \|\nabla b\|_2 + \|\nabla\phi_0\|_{L^2(B_1\setminus B_\varepsilon)} + \|\nabla\phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}\big). \tag{22}$$

Proof. First we consider $\varphi \in W_0^{1,2}(B_1)$ to be the solution of

$$\begin{cases} \Delta\varphi = a_x b_y - a_y b_x & \text{on } B_1, \\ \varphi = 0 & \text{on } \partial B_1. \end{cases}$$

Then thanks to the generalized Wente's inequality, see (10), we have

$$\|\nabla\varphi\|_2 \le C \, \|\nabla a\|_{2,\infty} \|\nabla b\|_2. \tag{23}$$

Consider the difference $v := \phi - \varphi - (\phi_0 - \varphi_0)$; it is a harmonic function on $B_1 \setminus B_\varepsilon$ which does not have 0-frequency Fourier modes:

$$v = \sum_{n\in\mathbb{Z}^*} \big(c_n \rho^n + d_n \rho^{-n}\big) \, e^{in\theta},$$

which implies in particular that

$$\int_{\partial B_\rho} \frac{\partial v}{\partial \nu} \, d\sigma = 0 \quad \text{for all } \varepsilon < \rho < 1. \tag{24}$$

Moreover, due to the assumption (21) and due to (23) we have

$$\begin{aligned} \|\nabla v\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)} &\le 2\|\nabla\varphi\|_2 + \|\nabla\phi_0\|_2 + \|\nabla\phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)} \\ &\le C\big(\|\nabla a\|_{2,\infty} \|\nabla b\|_2 + \|\nabla\phi_0\|_2 + \|\nabla\phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}\big). \end{aligned} \tag{25}$$

Here we used the fact that $L^{2,\infty}$-norm is controlled by the L^2-norm on a set of finite measure [Ziemer 1989]. Let $\lambda \in]0, 1[$; then standard elliptic estimates on harmonic functions give that for all $\rho \in (\varepsilon/\lambda, \lambda)$,

$$\begin{aligned} \|\nabla v\|_{L^\infty(\partial B_\rho)} &\le C(\lambda)\rho^{-1} \, \|\nabla v\|_{L^{2,\infty}(B_{\rho/\lambda}\setminus B_{\lambda\rho})} \\ &\le C(\lambda)\rho^{-1} \, \big(\|\nabla a\|_{2,\infty} \|\nabla b\|_2 + \|\nabla\phi_0\|_2 + \|\nabla\phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}\big). \end{aligned} \tag{26}$$

Denote $\Omega_\varepsilon := B_\lambda \setminus B_{\varepsilon/\lambda}$. We have that

$$\|\nabla v\|_{L^2(\Omega_\varepsilon)} = \sup_{\|X\|_{L^2(\Omega_\varepsilon)}\le 1} \int_{\Omega_\varepsilon} \nabla v \cdot X \, dz. \tag{27}$$

For such an $X \in L^2(\Omega_\varepsilon)$, we denote by $\widetilde{X}$ its extension by 0 in the complement of Ω_ε in B_1. Let g be the solution of

$$\begin{cases} \Delta g = -\operatorname{div} \widetilde{X}^\perp & \text{in } B_1, \\ g = 0 & \text{on } \partial B_1, \end{cases}$$

where $\widetilde{X}^\perp = (-\widetilde{X}_2, \widetilde{X}_1)$. We easily see that

$$\|\nabla g\|_{L^2(B_1)} \leq C\ \|\widetilde{X}\|_{L^2(B_1)} \leq C. \tag{28}$$

The Poincaré lemma gives the existence of $f \in W^{1,2}(B_1)$ such that

$$\widetilde{X} = \nabla f + \nabla^\perp g,$$

and we have

$$\|\nabla f\|_{L^2(B_1)} \leq \|\nabla g\|_{L^2(B_1)} + \|\widetilde{X}\|_{L^2(B_1)} \leq C + 1. \tag{29}$$

We have

$$\int_{\Omega_\varepsilon} \nabla v \cdot X\, dz = \int_{\Omega_\varepsilon} \nabla v \cdot \nabla f\, dz + \int_{\Omega_\varepsilon} \nabla v \cdot \nabla^\perp g\, dz.$$

We write

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla v \cdot \nabla^\perp g\, dz &= \int_{\partial B_\lambda} \partial_\tau v\, g\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\tau v\, g\, d\sigma \\ &= \int_{\partial B_\lambda} \partial_\tau v\, (g - g_\lambda)\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\tau v\, (g - g_{\varepsilon/\lambda})\, d\sigma, \end{aligned} \tag{30}$$

where ∂_τ is the tangential derivative along the circles ∂B_λ and $\partial B_{\varepsilon/\lambda}$, and g_λ and $g_{\varepsilon/\lambda}$ denote the averages of g on ∂B_λ and $\partial B_{\varepsilon/\lambda}$, respectively.

We have for any $\rho \in (0, 1)$

$$\frac{1}{\rho} \int_{\partial B_\rho} |g - g_\rho|\, d\sigma \leq C \|g\|_{H^{1/2}(\partial B_\rho)} \leq C \|\nabla g\|_2 \leq C, \tag{31}$$

where C is independent of ρ. Combining (26), (31) and (30) gives on one hand

$$\left| \int_{\Omega_\varepsilon} \nabla v \cdot \nabla^\perp g\, dz \right| \leq C(\lambda) \|\nabla v\|_{L^{2,\infty}(B_1 \setminus B_\varepsilon)}. \tag{32}$$

On the other hand, using the fact that v is harmonic and satisfies (24) we have

$$\begin{aligned} \int_{\Omega_\varepsilon} \nabla v \cdot \nabla f\, dz &= \int_{\partial B_\lambda} \partial_\nu v\, f\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\nu v\, f\, d\sigma \\ &= \int_{\partial B_\lambda} \partial_\nu v\, (f - f_\lambda)\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\nu v\, (f - f_{\varepsilon/\lambda})\, d\sigma. \end{aligned} \tag{33}$$

We have for any $\rho \in (0, 1)$

$$\frac{1}{\rho} \int_{\partial B_\rho} |f - f_\rho|\, d\sigma \leq C \|f\|_{H^{1/2}(\partial B_\rho)} \leq C \|\nabla f\|_2 \leq C. \tag{34}$$

Combining now (26), (33), and (34) we obtain

$$\left| \int_{\Omega_\varepsilon} \nabla v \cdot \nabla f\, dz \right| \leq C(\lambda) \|\nabla v\|_{L^{2,\infty}(B_1 \setminus B_\varepsilon)}. \tag{35}$$

Combining (32), (35), and (27) gives

$$\|\nabla v\|_{L^2(\Omega_\varepsilon)} \le C(\lambda)\|\nabla v\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}. \tag{36}$$

This inequality, together with (21) and (23), gives (22), and the lemma is proved. □

3. Angular energy quantization for solutions to elliptic systems with antisymmetric potential

The aim of this section is to prove that the angular part of the energy of a bounded sequence of solutions of an elliptic system with antisymmetric potential is always quantized. But before starting the proof of the quantization, we remind the reader of some facts about elliptic systems with antisymmetric potential which have intensively studied by the second author [Rivière 2007].

Let $\Omega \in L^2(B_1, \mathrm{so}(n)\otimes\mathbb{R}^2)$. We consider $u \in W^{1,2}(B_1, \mathbb{R}^n)$ a solution of the equation

$$-\Delta u = \Omega\cdot\nabla u \quad \text{on } B_1.$$

One of the fundamental facts about this system is the discovery a conservation law using a Coulomb gauge for Ω when its L^2-norm is small enough which is the aim of the following theorem.

Theorem 11 [Rivière 2007, Theorem I.4]. *There exists $\varepsilon_0 > 0$ such that for all $\Omega \in L^2(B_1, \mathrm{so}(n)\otimes\mathbb{R}^2)$ satisfying*

$$\int_{B_1} |\Omega|^2\,dz \le \varepsilon_0,$$

there exists $A \in W^{1,2}\cap L^\infty(B_1, \mathrm{Gl}_n(\mathbb{R}))$ such that

$$\mathrm{div}(\nabla A - A\Omega) = 0$$

and

$$\int_{B_1}\big(|\nabla A|^2 + |\nabla A^{-1}|^2\big)\,dz + \mathrm{dist}\big(\{A, A^{-1}\}, \mathrm{SO}(n)\big) \le C\int_{B_1}|\Omega|^2\,dz,$$

where C is a constant independent of Ω.

Then, using this theorem and Poincaré's lemma, we get the existence of $B \in W^{1,2}(B_1, M_n(\mathbb{R}))$ such that

$$\mathrm{div}(A\,\nabla u) = \nabla^\perp B\cdot\nabla u$$

and

$$\int_{B_1}|\nabla B|^2\,dz \le C\int_{B_1}|\Omega|^2\,dz.$$

Hence the system is rewritten in Jacobian form and we can use standard Wente's estimates. In particular, this permits one to prove three fundamental properties of the solutions of this equation which are the ε-regularity, the energy gap for solutions defined on the whole plane and the passage to the weak limit in the equation. These properties are summarized in the following theorem.

Theorem 12 [Rivière 2007; 2010]. *There exists $\varepsilon_0 > 0$ and $C_q > 0$, depending only on $q \in \mathbb{N}^*$, such that if $\Omega \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$ (respectively, $L^2(\mathbb{R}^2, \mathrm{so}(n) \otimes \mathbb{R}^2)$) satisfies $\|\Omega\|_2^2 \leq \varepsilon_0$, then:*

(1) (ε-regularity) *If $u \in W^{1,2}(B_1, \mathbb{R}^n)$ satisfies*

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } B_1,$$

then we have

$$\|\nabla u\|_{L^q(B_{1/4})} \leq C_q \|\nabla u\|_2 \quad \text{for all } q \in \mathbb{N}^*.$$

(2) (energy gap) *If $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^n)$ satisfies*

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } \mathbb{R}^2,$$

then it is constant.

(3) (weak limit property) *Let $\Omega_k \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$ be such that Ω_k weakly converges in L^2 to Ω, and let u_k be a bounded sequence in $W^{1,2}(B_1, \mathbb{R}^n)$ which satisfies*

$$-\Delta u_k = \Omega_k \cdot \nabla u_k \quad \text{on } B_1.$$

Then, there exists a subsequence of u_k which weakly converge in $W^{1,2}(B_1, \mathbb{R}^n)$ to a solution of

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } B_1.$$

For the convenience of the reader we recall the arguments developed in [Rivière 2007] and [Rivière 2010] to prove Theorem 12.

Proof. In order to prove the ε-regularity, let us prove that it suffices to show, for $\alpha > 0$, that we have

$$\sup_{\substack{p \in B_{1/2} \\ 0<\rho<1/2}} \frac{1}{\rho^\alpha} \int_{B_\rho(p)} |\Delta u| \, dz \leq C \|\nabla u\|_{L^2(B_1)}. \tag{37}$$

Indeed, a classical estimate on Riesz potentials gives

$$|\nabla u|(p) \leq C \frac{1}{|x|} * \chi_{B_{1/2}} |\Delta u| + C \|\nabla u\|_{L^2(B_1)} \quad \text{for all } p \in B_{1/4},$$

where $\chi_{B_{1/2}}$ is the characteristic function of the ball $B_{1/2}$. Together with injections proved by Adams [1975] (see also [Grafakos 2009, 6.1.6]), the latter shows that

$$\|\nabla u\|_{L^r(B_{1/4})} \leq C \|\nabla u\|_{L^2(B_1)},$$

for some $r > 1$. Then bootstrapping this estimate (see [Rivière 2010, Lemma IV.1] or [Sharp and Topping 2013, Theorem 1.1]), we get

$$\|\nabla u\|_{L^q(B_{1/4})} \leq C_q \|\nabla u\|_{L^2(B_1)} \quad \text{for all } q \in \mathbb{N}^*,$$

which will prove the ε-regularity.

In order to prove (37), we assume that ε_0 is small enough to apply Theorem 11. Hence there exists $A \in W^{1,2} \cap L^\infty(B_1, \mathrm{Gl}_n(\mathbb{R}))$ and $B \in W^{1,2} \cap L^\infty(B_1, M_n(\mathbb{R}))$ such that

$$\int_{B_1} \big(|\nabla A|^2 + |\nabla B|^2\big)\, dz + \mathrm{dist}\big(\{A, A^{-1}\}, \mathrm{SO}(n)\big) \le C \int_{B_1} |\Omega|^2\, dz,$$

$\mathrm{div}(A\,\nabla u) = \nabla^\perp B \cdot \nabla u$, and $\mathrm{curl}(A\,\nabla u) = \nabla^\perp A \cdot \nabla u$. Let $p \in B_{1/2}$ and $0 < \rho < \frac{1}{2}$; we proceed by introducing on $B_\rho(p)$ the linear Hodge decomposition in L^2 of $A\nabla u$. Namely, there exist two functions C and D, unique up to additive constants, elements of $W_0^{1,2}(B_\rho(p))$ and $W^{1,2}(B_\rho(p))$ respectively, and such that

$$A\,\nabla u = \nabla C + \nabla^\perp D, \tag{38}$$

with

$$\Delta C = \mathrm{div}(A\,\nabla u) = \nabla^\perp B \cdot \nabla u \quad \text{and} \quad \Delta D = -\nabla A \cdot \nabla^\perp u.$$

Wente's Lemma 5 guarantees that C lies in $W^{1,2}$, and moreover

$$\int_{B_\rho(p)} |\nabla C|^2\, dz \le C\Big(\int_{B_\rho(p)} |\nabla B|^2\, dz\Big)\Big(\int_{B_\rho(p)} |\nabla u|^2\, dz\Big). \tag{39}$$

Then, we introduce the decomposition $D = \phi + v$, with ϕ satisfying

$$\begin{cases} \Delta\phi = -\nabla A \cdot \nabla^\perp u & \text{in } B_\rho(p), \\ \qquad \phi = 0 & \text{on } \partial B_\rho(p), \end{cases} \tag{40}$$

and with v being harmonic. Once again, Wente's Lemma 5 gives us the estimate

$$\int_{B_\rho(p)} |\nabla\phi|^2\, dz \le C\Big(\int_{B_\rho(p)} |\nabla A|^2\, dz\Big)\Big(\int_{B_\rho(p)} |\nabla u|^2\, dz\Big).$$

Since $\rho \mapsto (1/\rho^2)\int_{B_\rho(p)} |\nabla v|^2\, dz$ is increasing for any harmonic function [Rivière 2010, Lemma II.1], we get, for any $0 \le \delta \le 1$, that

$$\int_{B_{\delta\rho}(p)} |\nabla v|^2\, dz \le \delta^2 \int_{B_\rho(p)} |\nabla v|^2\, dz.$$

Finally, we have

$$\int_{B_{\delta\rho}(p)} |\nabla D|^2\, dz \le 2\delta^2 \int_{B_\rho(p)} |\nabla D|^2\, dz \; + \; 2\int_{B_\rho(p)} |\nabla\phi|^2\, dz. \tag{41}$$

Bringing together (38), (39), and (41) produces

$$\int_{B_{\delta\rho}(p)} |A\,\nabla u|^2\, dz \le 2\delta^2 \int_{B_\rho(p)} |A\,\nabla u|^2\, dz \; + \; C\varepsilon_0 \int_{B_\rho(p)} |\nabla u|^2\, dz. \tag{42}$$

Using the hypotheses that A and A^{-1} are bounded in L^∞, it follows from (42) that for all $0 < \delta < 1$,

$$\int_{B_{\delta\rho}(p)} |\nabla u|^2\, dz \le 2\|A^{-1}\|_\infty \|A\|_\infty \delta^2 \int_{B_\rho(p)} |\nabla u|^2\, dz + C\,\|A^{-1}\|_\infty \varepsilon_0 \int_{B_\rho(p)} |\nabla u|^2\, dz. \tag{43}$$

Next, we choose ε_0 and δ strictly positive, independent of ρ and p, and such that

$$2\|A^{-1}\|_\infty \|A\|_\infty \delta^2 + C\|A^{-1}\|_\infty \varepsilon_0 = \frac{1}{2}.$$

For this particular choice of δ, we have thus obtained the inequality

$$\int_{B_{\delta\rho}(p)} |\nabla u|^2\, dz \le \frac{1}{2}\int_{B_\rho(p)} |\nabla u|^2\, dz.$$

Classical results then yield the existence of some constant $\alpha > 0$ for which

$$\sup_{\substack{p\in B_{1/2}(0)\\ 0<\rho<1/2}} \frac{1}{\rho^\alpha}\int_{B_\rho(p)} |\nabla u|^2\, dz < +\infty,$$

which proves the ε-regularity as already remarked above.

Then, the energy gap follows easily remarking that, thanks to the conformal invariance, for all $R > 0$ and some $q > 2$, we have

$$\|\nabla u\|_{L^q(B_{R/4})} \le \frac{C_q}{R^{(q-2)/q}}\|\nabla u\|_{L^2(B_R)}.$$

Finally, the weak limit property is a just a special case of [Rivière 2007, Theorem I.5] which is one of the many consequences of Theorem 11. ☐

We will be in position to prove Theorem 2 which is the main result of this section once we will have established the following lemma.

Lemma 13. *There exists $\delta > 0$ such that for all $r, R \in \mathbb{R}_+^*$ satisfying $2r < R$, all $\Omega \in L^2(B_R \setminus B_r, \mathrm{so}(n)\otimes\mathbb{R}^2)$, and all $u \in W^{1,2}(B_R \setminus B_r, \mathbb{R}^n)$ satisfying*

$$-\Delta u = \Omega\cdot\nabla u \quad \textit{and} \quad \sup_{r<\rho<R/2}\int_{B_{2\rho}\setminus B_\rho} |\Omega|^2\, dz \le \delta,$$

there exists $C > 0$, independent of u, r and R, such that

$$\|\nabla u\|_{L^{2,\infty}(B_R\setminus B_r)} \le C\left[\sup_{r<\rho<R/2}\int_{B_{2\rho}\setminus B_\rho} |\nabla u|^2\, dz\right]^{1/2}. \tag{44}$$

Proof. Let

$$\varepsilon := \sup_{r<\rho<R/2}\int_{B_{2\rho}\setminus B_\rho} |\nabla u|^2\, dz.$$

We assume δ to be smaller than ε_0 in the ε-regularity result Theorem 12 in such a way that for any $2r < \rho < R/4$ one has

$$\left[\frac{1}{\rho^2}\int_{B_{2\rho}\setminus B_\rho} |\nabla u|^4\, dz\right]^{1/4} \le C\frac{\sqrt{\varepsilon}}{\rho}. \tag{45}$$

Let $\lambda > 0$. Let $f(x) := |\nabla u|$ in $B_{R/2}\setminus B_{2r}$ and $f = 0$ otherwise; then

$$\int_{B_{2\rho}\setminus B_\rho} f^4\, dz \le C\,\frac{\varepsilon^2}{\rho^2} \quad \text{for all } \rho > 0. \tag{46}$$

For any $\rho > 0$ denote

$$U(\lambda, \rho) := \left\{ z \in B_{2\rho} \setminus B_\rho \;\middle|\; f(z) > \lambda \right\}.$$

Let $j \in \mathbb{Z}$ such that $2^j/\rho \le \lambda < 2^{j+1}/\rho$. For any j, using (46), one has that

$$\lambda^4 \, |U(\lambda, \rho)| \le C \frac{\varepsilon^2}{\rho^2}.$$

Let $k \in \mathbb{Z}$. By summing over $j \ge k$ one obtains

$$\lambda^2 \left| \left\{ z \in \mathbb{R}^2 \setminus B_{2^k \lambda^{-1}} \;\middle|\; f(x) > \lambda \right\} \right| \le C \sum_{j=k}^{\infty} 2^{-2j} \varepsilon^2 \le C 2^{-2k} \varepsilon^2.$$

So we deduce that for any $k \in \mathbb{Z}$

$$\lambda^2 \left| \left\{ z \in \mathbb{R}^2 \;\middle|\; f(z) > \lambda \right\} \right| \le C 2^{-2k} \varepsilon^2 + \pi 2^{2k}. \tag{47}$$

Taking $2^{2k} \simeq \varepsilon$ we obtain

$$\|\nabla u\|_{L^{2,\infty}(B_{R/2} \setminus B_{2r})} \le C \left[\sup_{r<\rho<R/2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u|^2 \, dx \right]^{1/2}. \tag{48}$$

Using now the triangle inequality for the $L^{2,\infty}$-norm and the fact that the $L^{2,\infty}$-norm of ∇u is controlled by the L^2-norm of ∇u over $B_R \setminus B_{R/2}$ and $B_{2r} \setminus B_r$, (48) implies (44) and Lemma 13 is proved. □

3.1. *Proofs of Theorems 1 and 2.*

Proof of Theorem 2. Let $\varepsilon_0 > 0$ be as in Theorem 11.

Step 1: We reduce the problem to an $L^{2,1}$ estimate. Indeed, we use the duality $L^{2,1}$–$L^{2,\infty}$ to infer that

$$\int_{B_{R/2} \setminus B_{2r}} \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2 dx \le \left\| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{R/2} \setminus B_{2r})} \left\| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right\|_{L^{2,\infty}(B_{R/2} \setminus B_{2r})}.$$

Combining this inequality with (44) we obtain

$$\int_{B_{R/2} \setminus B_{2r}} \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2 dx \le C \left\| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{R/2} \setminus B_{2r})} \left[\sup_{r<\rho<R/2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u|^2 \, dx \right]^{1/2}. \tag{49}$$

Hence, thanks to duality, it suffices to control the $L^{2,1}$-norm of $(1/\rho)(\partial u/\partial \theta)$ by the L^2-norm of ∇u in the annulus in order to prove the theorem.

Step 2: We prove the theorem assuming that $\int_{B_R \setminus B_r} |\Omega|^2 \, dz < \varepsilon_0$. We start by extending Ω, setting

$$\widetilde{\Omega} = \begin{cases} \Omega & \text{on } B_R \setminus B_r, \\ 0 & \text{on } B_r. \end{cases}$$

Hence, thanks to Theorem 11, there exists $\widetilde{A} \in W^{1,2}(B_R, \mathrm{Gl}_n(\mathbb{R})) \cap L^\infty(B_R, \mathrm{Gl}_n(\mathbb{R}))$ such that

$$\operatorname{div}(\nabla \widetilde{A} - \widetilde{A}\, \widetilde{\Omega}) = 0$$

and

$$\int_{B_R}\big(|\nabla\widetilde{A}|^2+|\nabla\widetilde{A}^{-1}|^2\big)\,dz+\operatorname{dist}\big(\{\widetilde{A},\,\widetilde{A}^{-1}\},\mathrm{SO}(n)\big)\le C\int_{B_R}|\widetilde{\Omega}|^2\,dz. \tag{50}$$

Then, thanks to Poincaré's lemma, there exists $\widetilde{B}\in W^{1,2}(B_R(0),M_n(\mathbb{R}))$ such that

$$\nabla\widetilde{A}-\widetilde{A}\,\widetilde{\Omega}=\nabla^{\perp}\widetilde{B}, \tag{51}$$

and, thanks to (50) and (51), we get

$$\|\nabla\widetilde{B}\|_{L^2(B_R)}\le C\|\Omega\|_{L^2(B_R\setminus B_r)},$$

where C is a constant independent of Ω. Hence, u satisfies

$$\operatorname{div}(\widetilde{A}\,\nabla u)=\nabla^{\perp}\widetilde{B}\cdot\nabla u\quad\text{ on } B_R\setminus B_r.$$

We extend u to B_R by $\widetilde{u}$ using Whitney's extension theorem (see [Adams and Fournier 2003] or [Stein 1970] for instance); then we get $\widetilde{u}\in W^{1,2}(B_R)$ such that

$$\int_{B_R}|\nabla\widetilde{u}|^2\,dz\le C\int_{B_R\setminus B_r}|\nabla u|^2\,dz. \tag{52}$$

We consider the Hodge decomposition of $\widetilde{A}\nabla\widetilde{u}$ on B_R, that is, $C\in W_0^{1,2}(B_R)$ and $D\in W^{1,2}(B_R)$ such that

$$\widetilde{A}\,\nabla\widetilde{u}=\nabla C+\nabla^{\perp}D. \tag{53}$$

Moreover, thanks to (52), we get

$$\int_{B_R}|\nabla C|^2\,dz+\int_{B_R}|\nabla D|^2\,dz=\int_{B_R}|\widetilde{A}\,\nabla\widetilde{u}|^2\,dz\le C\int_{B_R\setminus B_r}|\nabla u|^2\,dz.$$

Here we use the fact that C vanishes on the boundary to get that

$$\int_{B_R}\nabla C\cdot\nabla^{\perp}D\,dz=0.$$

Then, on $B_R\setminus B_r$, C satisfies

$$\Delta C=\nabla^{\perp}\widetilde{B}\cdot\nabla u.$$

As usual, we write $C=v+\phi$, where $\phi\in W_0^{1,2}(B_R\setminus B_r)$ and $v\in W^{1,2}(B_R\setminus B_r)$ satisfy

$$\Delta\phi=\nabla^{\perp}\widetilde{B}\cdot\nabla u\quad\text{ and }\quad\Delta v=0.$$

On the one hand, thanks to Lemma 7 we get, for $0<\lambda<1$, that

$$\|\nabla\phi\|_{L^{2,1}(B_R\setminus B_{r/\lambda})}\le C(\lambda)\|\nabla\widetilde{B}\|_2\,\|\nabla u\|_2.$$

On the other hand, we decompose v as a Fourier series:

$$v=c_0+d_0\log(\rho)+\sum_{n\in\mathbb{Z}^*}(c_n\rho^n+d_n\rho^{-n})e^{in\theta}.$$

Since $(1/\rho)(\partial v/\partial\theta)$ has no logarithm part, we conclude as in Lemma A.2 that for any $0<\lambda<1$ we have

$$\left\|\frac{1}{\rho}\frac{\partial v}{\partial\theta}\right\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\le C(\lambda)\|\nabla v\|_2.$$

The Dirichlet principle implies that

$$\|\nabla v\|_2\le\|\nabla C\|_2,$$

then we get

$$\left\|\frac{1}{\rho}\frac{\partial C}{\partial\theta}\right\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\le C(\lambda)\|\nabla u\|_{L^2(B_R\setminus B_r)}. \tag{54}$$

Now we estimate D, which satisfies the equation

$$\Delta D=\nabla\widetilde{A}\cdot\nabla^{\perp}\widetilde{u}\quad\text{on } B_R.$$

Then, we also decompose D as $D=v+\phi$, where $\phi\in W_0^{1,2}(B_R)$ and $v\in W^{1,2}(B_R)$ satisfy

$$\Delta\phi=\nabla\widetilde{A}\cdot\nabla^{\perp}\widetilde{u}\quad\text{and}\quad\Delta v=0.$$

On the one hand, thanks to Lemma 5, we have

$$\|\nabla\phi\|_2\le\|\nabla\phi\|_{L^{2,1}(B_R)}\le C\|\nabla\widetilde{A}\|_2\,\|\nabla\widetilde{u}\|_2\le C\|\nabla u\|_{L^2(B_R\setminus B_r)}.$$

On the other hand, since v is harmonic, for any $0<\lambda<1$ we have

$$\|\nabla v\|_{L^{2,1}(B_{\lambda R})}\le C(\lambda)\|\nabla v\|_{L^2(B_R)}\le C(\lambda)\|\nabla D\|_{L^2(B_R)}\le C(\lambda)\|\nabla u\|_2.$$

Finally,

$$\|\nabla D\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\le C(\lambda)\|\nabla u\|_2. \tag{55}$$

Combining (53), (54) and (55), we get

$$\left\|\widetilde{A}\,\frac{1}{r}\frac{\partial\widetilde{u}}{\partial\theta}\right\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\le C(\lambda)\|\nabla u\|_2.$$

Finally, using (50), we get that

$$\left\|\frac{1}{\rho}\frac{\partial\widetilde{u}}{\partial\theta}\right\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\le C(\lambda)\|\nabla u\|_2, \tag{56}$$

which proves, as remarked at the end of Step 1, the theorem under the extra assumption.

Step 3: We prove the general case. We construct two sequences of radii r_i and R_i such that

$$r=r_0<r_1=R_0<\cdots<r_{i+1}=R_i<\cdots<R_N=R,$$

with

$$\int_{B_{R_i}\setminus B_{r_i}}|\Omega|^2\,dz\le\varepsilon_0\quad\text{and}\quad N\le\frac{1}{\varepsilon_0}\int_{B_R\setminus B_r}|\Omega|^2\,dz.$$

First, applying (56) of Step 2, we get that

$$\left\| \frac{1}{\rho}\frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{\lambda R_i}\setminus B_{r_i/\lambda})} \leq C(\lambda)\|\nabla u\|_{L^2(B_{R_i}\setminus B_{r_i})}. \tag{57}$$

We choose δ such that $\delta < \varepsilon_0/4$; hence for all i we have

$$\int_{B_{4r_i}\setminus B_{r_i/4}} |\Omega|^2\,dz < 4\delta < \varepsilon_0.$$

Let $S_i = \min(R, 4r_i)$ and $s_i = \max(r, r_i/4)$, then we apply again (56) of Step 2 on $B_{S_i}\setminus B_{s_i}$, which gives

$$\left\| \frac{1}{\rho}\frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{\lambda S_i}\setminus B_{s_i/\lambda})} \leq C(\lambda)\|\nabla u\|_{L^2(B_{S_i}\setminus B_{s_i})}. \tag{58}$$

Finally, summing (57) and (58), for $i = 0$ to N, we get

$$\left\| \frac{1}{\rho}\frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})} \leq C(\lambda)\|\nabla u\|_2,$$

which achieves the proof of Theorem 2. □

We shall now make use of the Theorem 2 in order to prove the quantization of the angular part of the energy for solutions to antisymmetric elliptic systems.

We will call a *bubble* a solution $u \in W^{2,1}(\mathbb{R}^2, \mathbb{R}^n)$ of the equation

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } \mathbb{R}^2,$$

where $\Omega \in L^2(\mathbb{R}^2, \mathrm{so}(n) \otimes \mathbb{R}^2)$.

Proof of Theorem 1. First we are going to separate B_1 into three parts: one where u_k converges to a limit solution; some neighborhoods where the energy concentrates and you blow some bubbles; and some neck regions which join the first two parts. This "bubble-tree" decomposition is by now classical (see [Parker 1996] for instance); hence we just sketch briefly how to proceed.

Step 1: Find the point of concentration. Let ε_0 be the one of Theorem 12 and δ the one of Theorem 2. Then, thanks to (6), we easily prove that there exist finitely many points $a^1, \dots, a^n$, where

$$\liminf_k \int_{B(a_i,r)} |\Omega_k|^2\,dz \geq \varepsilon_0 \quad \text{for all } r > 0. \tag{59}$$

Moreover, using Theorem 12, we prove that there exists $\Omega_\infty \in L^2(B_1, \mathrm{so}(n)\otimes\mathbb{R}^2)$ and $u_\infty \in W^{2,1}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u = \Omega_\infty \cdot \nabla u$ on B_1, such that, up to a subsequence,

$$\Omega_k \rightharpoonup \Omega_\infty \quad \text{in } L^2_{\mathrm{loc}}(B_1, \mathrm{so}(n)\otimes\mathbb{R}^2),$$

and

$$u_k \to u_\infty \quad \text{in } W^{1,p}_{\mathrm{loc}}(B_1 \setminus \{a^1, \dots, a^n\}) \text{ for all } p \geq 1.$$

Of course, if $\|\Omega_k\|_\infty = O(1)$ or $\Omega_k = \Lambda(u_k, \nabla u_k)$ where $\Lambda(\,\cdot\,, p) = O(|p|)$, then u_k is bounded in $W^{2,\infty}$ which gives the convergence in $C^{1,\eta}_{\mathrm{loc}}$ for all $\eta \in [0, 1[$.

Step 2: Blow-up around a^i. We choose $r_i > 0$ such that

$$\int_{B(a^i,r^i)} |\Omega_\infty|^2 \, dz \leq \frac{\varepsilon_0}{4}.$$

Then, we define a center of mass of $B(a^i, r^i)$ with respect to Ω_k in the following way:

$$a_k^i = \left(\frac{\displaystyle\int_{B(a^i,r^i)} x^\alpha |\Omega_k|^2 \, dz}{\displaystyle\int_{B(a^i,r^i)} |\Omega_k|^2 \, dz} \right)_{\alpha=1,2}.$$

Let λ_k^i be a positive real such that

$$\int_{B(a_k^i,r^i)\setminus B(a_k^i,\lambda_k^i)} |\Omega_k|^2 \, dz = \min\left(\delta, \frac{\varepsilon_0}{2}\right).$$

If $\lambda_k^i \neq o(1)$, then we restart the process replacing r^i by $\liminf \lambda_k^i$ until $\lambda_k^i = o(1)$. Then we set

$$\widetilde{u}_k(z) = u_k(a_k^i + \lambda_k^i z), \quad \widetilde{\Omega}_k(z) = \lambda_k^i \, \Omega_k(a_k^i + \lambda_k^i z), \quad \text{and} \quad N_k^i = B(a_k^i, r^i) \setminus B(a_k^i, \lambda_k^i).$$

Observe that the scaling we chose for defining $\widetilde{\Omega}_k(z)$ guarantees that

$$\int_{B(0,r^i/\lambda_k^i)} \left(|\widetilde{\Omega}_k|^2 + |\nabla \widetilde{u}_k^i|^2\right) dz = \int_{B(a_k^i,r^i)} \left(|\Omega_k|^2 + |\nabla u_k^i|^2\right) \, dx \leq C < +\infty;$$

moreover, we have

$$-\Delta \widetilde{u}_k^i = \widetilde{\Omega}_k^i \cdot \nabla \widetilde{u}_k^i.$$

Modulo extraction of a subsequence, we can assume that for each i

$$\nabla \widetilde{u}_k^i \rightharpoonup \nabla \widetilde{u}_\infty^i \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^n) \quad \text{and} \quad \widetilde{\Omega}_k^i \rightharpoonup \widetilde{\Omega}_\infty^i \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^2, \text{so}(n) \otimes \mathbb{R}^2).$$

The *weak limit property* of Theorem 12 implies that $\widetilde{u}_\infty$ and $\widetilde{\Omega}_\infty$ satisfy what we call a *bubble* equation

$$-\Delta \widetilde{u}_\infty^i = \widetilde{\Omega}_\infty^i \cdot \nabla \widetilde{u}_\infty^i.$$

In fact the convergence of u_k^i to u_∞^i is in $W^{1,p}_{\text{loc}}\left(\mathbb{R}^2 \setminus \{a_i^1, \dots, a_i^n\}\right)$ for all $p \geq 1$, where the a_i^j are possible points of concentration of $\widetilde{\Omega}_k^i$ where

$$\liminf_k \int_{B(a_i^j,r)} |\widetilde{\Omega}_k^i|^2 \, dz \geq \varepsilon_0 \quad \text{for all } r > 0, \tag{60}$$

which are necessarily finite in number and in B_1.

Step 3: Iteration. Two cases have to be considered separately: either $\widetilde{\Omega}_k$ is subject to some concentration phenomena as (59), and then we find some new points of concentration, in such a case we apply Step 2 to our new concentration points; or $\widetilde{u}_k$ converges in $W^{1,p}_{\text{loc}}(\mathbb{R}^2)$ to a (possibly trivial) bubble.

Of course this process has to stop, since we are assuming a uniform bound on $\|\Omega_k\|_2$ and each step is consuming at least $\min(\delta, \varepsilon_0/2)$ of energy of Ω_k. This process is sketched in Figure 1.

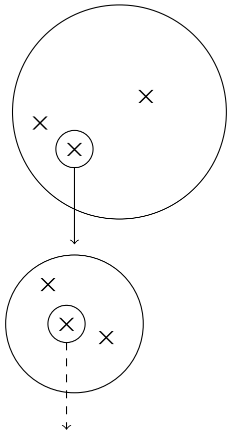

Figure 1. Decomposition of B_1.

Analysis of a neck region. A neck region is a finite union of annuli $N_k^i = B(a_k^i, \mu_k^i) \setminus B(a_k^i, \lambda_k^i)$ such that

$$\lim_{k\to+\infty} \frac{\lambda_k^i}{\mu_k^i} = 0, \qquad X_k = \nabla^\perp d(a_k^i, \cdot),$$

and

$$\int_{N_k^i} |\Omega_k|^2 \, dz \le \min\left(\delta, \frac{\varepsilon_0}{2}\right). \tag{61}$$

In order to prove Theorem 1, we start by proving a weak estimate on the energy of gradient in the region N_k^i. First we remark that, for each $\varepsilon > 0$, there exists $r > 0$ such that for all $\rho > 0$ such that

$$B_{2\rho}(a_k^i) \setminus B_\rho(a_k^i) \subset N_k^i(r),$$

where $N_k^i(r) = B(a_k^i, r\mu_k^i) \setminus B(a_k^i, \lambda_k^i/r)$, we have

$$\int_{B_{2\rho}(a_k^i)\setminus B_\rho(a_k^i)} |\nabla u|^2 \, dz \le \varepsilon. \tag{62}$$

If this were not the case there would exist a sequence $\rho_k^i \to 0$ such that, up to a subsequence, $\hat{u}_k = u_k(a_k^i + \rho_k^i z)$ converges with respect to every $W^{1,p}$-norm to a nontrivial solution of

$$-\Delta \hat{u} = \widehat{\Omega}_\infty \cdot \nabla \hat{u} \quad \text{on } \mathbb{R}^2 \setminus \{0\},$$

where $\widehat{\Omega}_\infty$ is a weak limit, up to a subsequence, of $\widehat{\Omega}_k$. Using the fact that the $W^{1,2}$-norm of $\hat{u}_k$ is bounded, we deduce using Schwartz lemma that it has to be in fact a solution on the whole plane. Using this time the second part of Theorem 12 we deduce that $\hat{\Omega}_\infty$ have energy at least ε_0, which contradicts (61).

Finally, using Theorem 2 on each $N_k^i(r)$, we obtain

$$\lim_{r\to 0} \lim_{k\to+\infty} \left\| \langle \nabla u_k, X_k \rangle \right\|_{L^2(N_k^i(r))} \le C \lim_{r\to 0} \lim_{k\to+\infty} \left(\sup_\rho \int_{B_{2\rho}(a_k^i)\setminus B_\rho(a_k^i)} |\nabla u|^2 \, dz \right) = 0,$$

which achieves the proof of Theorem 1. $\square$

This phenomena of quantization of the angular part of the gradient seems to be quite general for systems with antisymmetric potentials. In a forthcoming paper [Laurain and Rivière 2011] we investigate the quantization for some fourth-order elliptic systems in four dimensions.

3.2. ***Description of the function in the neck regions.*** In this subsection we give a precise description of the behavior of ∇u_k in the neck regions when the radial part is not quantized. In particular we prove that the loss of quantization is due to pure radial part to the form $a(r)/r$ with a uniformly bounded.

Proving Theorem 2, we have proved, see (53) and what follows, that if the L^2-norm of Ω is smaller than a positive constant δ_0 on an annulus $B_R \setminus B_r$, then there exists $A \in W^{1,2} \cap L^\infty(B_1, \mathrm{Gl}_n(\mathbb{R}))$, $h \in L^2(B_1, \mathbb{R}^2 \otimes \mathbb{R}^n)$ and $C \in \mathbb{R}^2 \otimes \mathbb{R}^n$ such that

$$A\nabla u = \frac{C}{r} + h,$$

where C is a constant and $\|h\|_{L^{2,1}(B_{R/2}\setminus B_{2r})}$ is uniformly bounded by the L^2-norm of ∇u, independently of the conformal class of the annulus. Moreover, up to a choice of δ_0 small enough, we can assume that A is very closed to $\mathrm{SO}(n)$. Then using this fact and the fact we can decompose a neck region into a finite number of such regions, we are going to prove that, in the whole neck region,

$$\nabla u = C\frac{a(r)}{r} + h + g, \tag{63}$$

where C is a constant, $a \in L^\infty(B_1, M_n(\mathbb{R}))$ is uniformly bounded by the L^2-norm of ∇u and *radial*, and $\|h\|_{L^{2,1}(B_{R/2}\setminus B_{2r})}$ is uniformly bounded by the L^2-norm of ∇u and $\|g\|_{L^2(B_{R/2}\setminus B_{2r})}$ as the $\|\nabla u\|_{L^{2,\infty}}$ goes to zero.

Indeed, a neck region is an annular region of the form $B_{R_k} \setminus B_{r_k}$. Since the L^2-norm of Ω_k is uniformly bounded we can divide the annulus into a finite number of annuli where the L^2-norm of Ω_k is smaller than $\varepsilon_0/2$. Let $\left(B_{r_k^{i+1}} \setminus B_{r_k^i}\right)_{1\le i\le N}$ be the different annuli, where $r_k^1 = r_k$ and $r_k^{N+1} = R_k$.

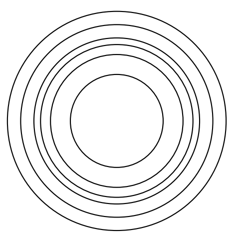

Figure 2. Decomposition of the neck region.

On $B_{r_k^i} \setminus B_{r_k^{i+1}}$ the L^2-norm of Ω_k is smaller than δ_0, so there exist $A_k^i \in W^{1,2} \cap L^\infty\left(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathrm{Gl}_n(\mathbb{R})\right)$, $h_k^i \in L^2\left(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathbb{R}^2 \otimes \mathbb{R}^n\right)$ and $C_k^i \in \mathbb{R}^2 \otimes \mathbb{R}^n$ such that

$$A_k^i \nabla u_k = \frac{C_k^i}{r} + h_k^i \quad \text{on } B_{r_k^{i+2}} \setminus B_{r_k^i},$$

where $\|h_k^i\|_{L^{2,1}}$ is uniformly bounded by the L^2-norm of ∇u_k. Hence we have

$$\nabla u_k = \frac{D_k^i(r)}{r} C_k^i + \tilde{h}_k^i + \tilde{g}_k^i \quad \text{on } B_{r_k^{i+2}} \setminus B_{r_k^i}, \tag{64}$$

where $D_k^i \in L^\infty\big(B_{r_k^{i+2}} \setminus B_{r_k^i}, M_n(\mathbb{R})\big)$ is uniformly bounded by the L^2-norm of ∇u_k and *radial*, $\|\tilde{h}_k^i\|_{L^{2,1}}$ is uniformly bounded, and $\tilde{g}_k^i \in L^2\big(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathbb{R}^2 \otimes \mathbb{R}^n\big)$ with $\|\tilde{g}_k^i\|_{L^2} = o(1)$. Indeed, we have

$$\frac{(A_k^i)^{-1}}{r} = \frac{\overline{(A_k^i)^{-1}}(r)}{r} + \frac{(A_k^i)^{-1} - \overline{(A_k^i)^{-1}}}{r},$$

where $\overline{(A_k^i)^{-1}}$ is the mean value of $(A_k^i)^{-1}$ on each circle. Since $(A_k^i)^{-1}$ is uniformly bounded in $W^{1,2} \cap L^\infty\big(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathrm{Gl}_n(\mathbb{R})\big)$, we have

$$\begin{aligned}
\left\| \frac{(A_k^i)^{-1} - \overline{(A_k^i)^{-1}}}{r} \right\|^2_{L^2\left(B_{r_k^{i+1}} \setminus B_{r_k^i}\right)} &= \int_{r_k^i}^{r_k^{i+1}} \frac{1}{r} \int_0^{2\pi} \left|(A_k^i)^{-1} - \overline{(A_k^i)^{-1}}\right|^2 d\theta\, dr \\
&\leq \int_{r_k^i}^{r_k^{i+1}} \frac{1}{r} \int_0^{2\pi} \left| \frac{\partial (A_k^i)^{-1}}{\partial \theta} \right|^2 d\theta\, dr \\
&\leq \left\| \nabla (A_k^i)^{-1} \right\|_2^2,
\end{aligned}$$

here we use the Poincaré inequality. Finally, we conclude using the fact that $\|\nabla u_k\|_2$ is bounded, which implies

$$\|C_k^i\| = O\Big(\big(\log\big(r_k^{i+1}/r_k^i\big)\big)^{-1/2}\Big) = o(1),$$

since $\tilde{g}_k^i = \frac{1}{r}\big((A_k^i)^{-1} - \overline{(A_k^i)^{-1}}\big) C_k^i$, this proves (64). Then we glue all the functions to get the whole decomposition.

Hence we have the following theorem:

Theorem 14 (see Theorem 1). *Let $\Omega_k \in L^2\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$ and let $u_k \in W^{2,1}(B_1, \mathbb{R}^n)$ be a sequence of solutions of*

$$-\Delta u_k = \Omega_k \cdot \nabla u_k \tag{65}$$

with bounded energy; *that is*,

$$\int_{B_1} \left(|\nabla u_k|^2 + |\Omega_k|^2\right) dz \leq M.$$

Then there exist $u_\infty \in W^{1,2}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u_\infty = \Lambda(u_\infty, \nabla u_\infty) \cdot \nabla u_\infty$ on B_1, $l \in \mathbb{N}^*$, and

(1) *$\omega^1, \dots, \omega^l$ a family of solutions to system*

$$-\Delta \omega^i = \Omega_i \cdot \nabla \omega^i \quad \text{on } \mathbb{R}^2,$$

where $\Omega_i \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$,

(2) *$a_k^1, \dots, a_k^l$ a family of converging sequences of points of B_1,*

(3) $\lambda_k^1, \dots, \lambda_k^l$ *a family of sequences of positive reals converging all to zero,*

(4) $C_k^1, \dots, C_k^l$ *a family of sequences of vectors converging all to zero,*

(5) $A_k^1, \dots, A_k^l$ *a family of sequences of uniformly bounded and radial functions from* $\mathbb{R}^2$ *to* $M_n(\mathbb{R})$,

such that, up to a subsequence,

$$u_k \to u_\infty \quad \text{on } C^{1,\eta}_{\text{loc}}\big(B_1 \setminus \{a^1_\infty, \dots, a^l_\infty\}\big) \text{ for all } \eta \in [0,1[$$

and

$$\left\| \nabla\Big(u_k - u_\infty - \sum_{i=1}^{l} \omega_k^i\Big) + \sum_{i=1}^{l} \frac{A_k^i(d(a_k^i,\cdot))}{d(a_k^i,\cdot)} C_k^i \right\|_{L^2_{\text{loc}}(B_1)} \to 0,$$

where $\omega_k^i = \omega^i(a_k^i + \lambda_k^i\,\cdot)$.

3.3. *Counterexample to the quantization of the radial part of the gradient.* Thanks to the previous subsection, we know that the failure of quantization is given in the neck region by a function of the form $c_k \log(r)$. Hence we look for $u_k : B_1 \to \mathbb{R}^3$ whose third component behaves as $c_k \log(r)$. For this we define the following smooth functions:

$$U_k^3(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1/2, \\ \dfrac{\log(r)}{\log(k)^{1/2}} & \text{if } r \ge 2, \end{cases}$$

such that $|(U_k^3)'(r)| \le \log(k)^{-1/2}$ on $[1/2, 2]$; and

$$\phi(r) = \begin{cases} 2r & \text{if } 0 \le r \le \frac{1}{4}, \\ 1 & \text{if } \frac{1}{2} \le r \le 2, \\ 2/r & \text{if } r \ge 4, \end{cases}$$

such that $|\phi'(r)| \le 4$ on $[1/4, 1/2] \cup [2, 4]$. We set $\psi = r(r\phi')'/\phi - 1$, and we easily see that ψ is a smooth function with compact support in $[1/4, 4]$. Finally we set

$$u_k(r,\theta) = \begin{pmatrix} \cos(\theta)\phi(kr) \\ \sin(\theta)\phi(kr) \\ U_k^3(kr) \end{pmatrix}$$

and

$$\Omega_k^\theta(r,\theta) = \begin{pmatrix} 0 & \psi(kr)/r & \sin(\theta) r \Delta u_k^3 \\ -\psi(kr)/r & 0 & -\cos(\theta) r \Delta u_k^3 \\ -\sin(\theta) r \Delta u_k^3 & \cos(\theta) r \Delta u_k^3 & 0 \end{pmatrix}.$$

We easily verify that $\Delta u_k = \Omega_k \cdot \nabla u_k$ where $\Omega_k = \Omega_k^\theta r\, d\theta$ and that the L^2-norms of ∇u_k and Ω_k are bounded on B_1. We have a bubble which blows up at radius $1/k$, and

$$\lim_{R\to+\infty} \lim_{k\to+\infty} \int_{B_{1/R} \setminus B_{R/k}} |\Omega_k|^2 \, dz = 0,$$

but

$$\lim_{R\to+\infty}\lim_{k\to+\infty}\int_{B_{1/R}\setminus B_{R/k}}|\nabla u_k|^2\,dz=1,$$

which is a failure of energy quantization and proves the optimality of the conclusion of Theorem 1.

4. Energy quantization for critical points to conformally invariant Lagrangians.

In the present section we are going to use Theorem 1 in order to prove Theorem 3.

In his proof of the Heinz–Hildebrandt's regularity conjecture, the second author prove that the Euler Lagrange equations to general conformally invariant Lagrangians which are coercive and of quadratic growth can be written in the form of an elliptic system with an antisymmetric potential. Precisely we have:

Theorem 15 [Rivière 2007, Theorem I.2]. *Let N^k be a C^2 submanifold of $\mathbb{R}^m$ and ω be a C^1 2-form on N^k such that the L^∞-norm of $d\omega$ is bounded on N^k. Then every critical point in $W^{1,2}(B_1,N^k)$ of the Lagrangian*

$$F(u)=\int_{B_1}\left[|\nabla u|^2+u^*\omega\right]\,dz \tag{66}$$

satisfies

$$-\Delta u=\Omega\cdot\nabla u,$$

with

$$\Omega^i_j=\left[A^i(u)_{j,l}-A^j(u)_{i,l}\right]\nabla u^l+\frac{1}{4}\left[H^i(u)_{j,l}-H^j(u)_{i,l}\right]\nabla^\perp u^l, \tag{67}$$

where A and H are in $C^0(N,M_m(\mathbb{R})\otimes\wedge^1\mathbb{R}^2)$ and satisfy

$$\sum_{j=1}^m A^j_{i,l}\nabla u^j=0$$

and $H^i_{j,l}:=d(\pi^*\omega)(\varepsilon_i,\varepsilon_j,\varepsilon_l)$ where, in a neighborhood of N^k, π is the orthogonal projection onto N^k and $(\varepsilon_i)_{i=1,\dots,m}$ is the canonical basis of $\mathbb{R}^m$.

From (67) we observe that for critical points to a conformally invariant C^1-Lagrangian, there exists

$$\Lambda\in C^0(TN\otimes\mathbb{R}^2,\mathrm{so}(n)\otimes\mathbb{R}^2) \tag{68}$$

such that

$$\Lambda(v)=O(|v|); \tag{69}$$

moreover we remark that $\Lambda(u,\nabla u)\cdot\nabla u$ is always orthogonal to ∇u in the following sense:

$$\left\langle\frac{\partial u}{\partial x_k},\Lambda(u,\nabla u)\cdot\nabla u\right\rangle=0\quad\text{for } k=1,2. \tag{70}$$

For $\Lambda\in C^0(TN\otimes\mathbb{R}^2,\mathrm{so}(n)\otimes\mathbb{R}^2)$, we call a Λ-*bubble* a solution $\omega\in W^{2,1}(\mathbb{R}^2,\mathbb{R}^n)$ of the equation

$$-\Delta\omega=\Lambda(\omega,\nabla\omega)\cdot\nabla\omega\quad\text{on }\mathbb{R}^2.$$

Theorem 16. *Let $u_k \in W^{1,2}(B_1, \mathbb{R}^n)$ be a sequence of critical points of a functional which is conformally invariant, which satisfies*

$$-\Delta u_k = \Lambda(u_k, \nabla u_k) \cdot \nabla u_k, \tag{71}$$

where Λ satisfies (68), (69) *and* (70). *Moreover, assume that u_k has a bounded energy, that is,*

$$\|\nabla u_k\|_2 \leq M.$$

Then there exists $u_\infty \in W^{1,2}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u_\infty = \Lambda(u_\infty, \nabla u_\infty) \cdot \nabla u_\infty$ on B_1, $l \in \mathbb{N}^*$ and

(1) *$\omega^1, \ldots, \omega^l$ some nonconstant Λ-bubbles,*

(2) *$a_k^1, \ldots, a_k^l$ a family of converging sequences of points of B_1,*

(3) *$\lambda_k^1, \ldots, \lambda_k^l$ a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$u_k \to u^\infty \quad \text{on } C^{1,\eta}_{\text{loc}}\big(B_1 \setminus \{a_\infty^1, \ldots, a_\infty^l\}\big) \quad \text{for all } \eta \in [0, 1[$$

and

$$\Big\|\nabla\Big(u_k - u_\infty - \sum_{i=1}^{l} \omega_k^i\Big)\Big\|_{L^2_{\text{loc}}(B_1)} \to 0,$$

where $\omega_k^i = \omega(a_k^i + \lambda_k^i z)$.

Since (70) holds for any system issued from a Lagrangian of the form (66), it is clear that Theorem 3 is a consequence of Theorem 16.

Proof. From the previous section, we have the quantization of the angular part of the gradient. To prove Theorem 16 it suffices then to prove the energy quantization for the radial part of the energy. Since u_k satisfies (71) then $u_k \in W^{2,p}(B_{\mu_k^i}(a_k^i))$ for all $p < \infty$ (see [Rivière 2010, Theorem IV.3] or [Sharp and Topping 2013, Lemma 7.1]); hence we can multiply (71) by $\rho(\partial u_k/\partial\rho)$ and integrate. Using (70) we have, for any $r \in [0, \mu_k^i]$,

$$0 = \int_{B_r} \Big\langle \rho \frac{\partial u_k}{\partial \rho}, \Omega \cdot \nabla u_k \Big\rangle \, dz = \int_{B_r} \Big\langle \rho \frac{\partial u_k}{\partial \rho}, \Delta u_k \Big\rangle \, dz.$$

Using Pohozaev identity, we get for all $r \in [0, \mu_k^i]$

$$\int_{\partial B_r} \Big|\frac{\partial u_k}{\partial \rho}\Big|^2 d\sigma = \int_{\partial B_r} \Big|\frac{1}{\rho}\frac{\partial u_k}{\partial \theta}\Big|^2 d\sigma.$$

Finally, we have

$$\lim_{r \to 0} \lim_{k \to +\infty} \|\nabla u_k\|_{L^2(N_k^i(r))} = 0,$$

which concludes the proof of the theorem. □

In particular we get the quantization for the solution of the problem of prescribed mean curvature. Indeed, an immersion of a Riemann surface Σ into $\mathbb{R}^3$ with prescribed mean curvature $H \in C^0(\mathbb{R}^3, \mathbb{R})$ satisfies the H-system

$$\Delta u = 2H(u)\, u_x \wedge u_y, \tag{72}$$

where $z = x + iy$ are some local conformal coordinates on Σ.

In order to state precisely our theorem, we define the notion of H-bubble as being a map $\omega \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^3)$ satisfying

$$\Delta\omega = 2H(\omega)\, \omega_x \wedge \omega_y \quad \text{on } \mathbb{R}^2.$$

We shall also rescale the Riemann surface around a point. To that aim we will introduce some conformal chart. Precisely there exists $\delta > 0$ such that for any $a \in \Sigma$ and $0 < \lambda < \delta$ there exists a map $\Phi_{a,\lambda} : B(a, \delta) \to \mathbb{R}^2$ which is a conformal-diffeomorphism, sends a to 0 and $B(a, \lambda)$ to $B(0, 1)$. We also associate to each point a cut-off function $\chi_a \in C^\infty(\Sigma)$ which satisfies

$$\begin{cases} \chi_a \equiv 1 & \text{on } B(a, \delta/2), \\ \chi_a \equiv 0 & \text{on } \Sigma \setminus B(a, \delta). \end{cases}$$

Corollary 17. *Let Σ be a closed Riemann surface, $H \in C^0(\mathbb{R}^3, \mathbb{R})$ and $u_k \in W^{2,1}(\Sigma, \mathbb{R}^3)$ a sequence of nonconstant solution of* (72) *on Σ then there exists, $u_\infty \in W^{2,1}(\Sigma, \mathbb{R}^3)$ a solution of* (72), *$k \in \mathbb{N}^*$ and*

(1) *$\omega^1, \dots, \omega^l$ a family of H-bubbles,*

(2) *$a_k^1, \dots, a_k^l$ a family of converging sequences of point of Σ,*

(3) *$\lambda_k^1, \dots, \lambda_k^l$ a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$u_k \to u^\infty \quad \text{on } C^{1,\eta}_{\mathrm{loc}}\big(\Sigma \setminus \{a_1^\infty, \dots, a_k^\infty\}\big) \quad \text{for all } \eta \in [0, 1[$$

and moreover

$$\left\| \nabla\Big(u_k - u_\infty - \sum_{i=1}^{l} \chi_{a_k^i}\big(\omega^i \circ \Phi_{a_k^i, \lambda_k^i}\big)\Big) \right\|_2 \to 0.$$

We end up this section by mentioning recent work by Da Lio [2011] in which energy quantization results for fractional harmonic maps (which are also conformally invariant in some dimension) are established using also Lorentz space uniform estimates.

5. Other applications to pseudoholomorphic curves, harmonic maps and Willmore surfaces

In this section we give some more applications of the uniform Lorentz–Wente estimates of Section 2 to problems where the conformal invariance play again a central role. We are interested in Wente's type estimate for first-order system of the form

$$\nabla\phi = \sum_{i=1}^{n} a_i \, \nabla^{\perp} b_i. \tag{73}$$

Taking the divergence of this system gives the classical second-order Wente system

$$\Delta\phi = \sum_{i=1}^{n} \nabla a_i \cdot \nabla^{\perp} b_i. \tag{74}$$

The gain of information provided by a first-order system of the form (73) in comparison to classical second-order system (74) is illustrated by the fact that, in the first-order case, no assumption on the behavior of the solution ϕ at the boundary of the annulus is needed in order to obtain the Lorentz–Wente-type estimates of Section 2. This is proved in Lemma 18. This fact can be applied to geometrically interesting situations that we will describe at the end of the present section.

5.1. *Lorentz–Wente-type estimates for first-order Wente-type equations.* The goal of this subsection is to prove the following lemma.

Lemma 18. *Let $n \in \mathbb{N}^*$, let $(a_i)_{1\le i\le n}$ and $(b_i)_{1\le i\le n}$ be two families of maps in $W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, and assume that $\phi \in W^{1,2}(B_1 \setminus B_\varepsilon)$ satisfies*

$$\nabla\phi = \sum_{i=1}^{n} a_i \, \nabla^{\perp} b_i. \tag{75}$$

Then for each $0 < \lambda < 1$ there exists a positive constant $C(\lambda)$, independent of ϕ, a_i, and b_i, such that

$$\|\nabla\phi\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\Big(\sum_{i=1}^{n} \|\nabla a_i\|_2 \, \|\nabla b_i\|_2 + \|\nabla\phi\|_2\Big).$$

Proof. Taking the divergence of (75) gives

$$\Delta\phi = \sum_{i=1}^{n} \nabla a_i \cdot \nabla^{\perp} b_i.$$

Hence, as in the previous lemma, we start by considering a solution of this equation on the whole disk and equal to zero on the boundary. Let $\varphi \in W_0^{1,1}(B_1)$ be the solution of

$$\Delta\varphi = \sum_{i=1}^{n} \nabla a_i \cdot \nabla^{\perp} b_i.$$

Then, thanks to the improved Wente's inequality (9), we have

$$\|\nabla\varphi\|_{L^{2,1}(B_1)} \le C \sum_{i=1}^{n} \|\nabla a_i\|_2 \, \|\nabla b_i\|_2. \tag{76}$$

We now consider the difference $v = \phi - \varphi$, which is a harmonic function on $B_1 \setminus B_\varepsilon$. Following the proof of Lemma A.2, it suffices to control the logarithmic part of the decomposition in Fourier series. To that aim we set

$$\overline{\phi}(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\rho,\theta)\, d\theta.$$

We have

$$\frac{d\overline{\phi}}{d\rho} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial\phi}{\partial\rho}(\rho,\theta)\, d\theta = \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{2\pi} a_i \frac{\partial b_i}{\partial\theta}\frac{d\theta}{\rho} = \frac{1}{2\pi}\sum_{i=1}^{n}\int_0^{2\pi} (a_i - \overline{a}_i)\frac{\partial b_i}{\partial\theta}\frac{d\theta}{\rho}.$$

Hence

$$\left|\frac{d\overline{\phi}}{d\rho}\right| \leq \frac{1}{2\pi}\sum_{i=1}\left(\int_0^{2\pi} \left|a_i - \overline{a}_i\right|^2 d\theta\right)^{1/2}\left(\int_0^{2\pi}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial\theta}\right|^2 d\theta\right)^{1/2}.$$

Which gives, thanks to Poincaré's inequality on the circle,

$$\left|\frac{d\overline{\phi}}{d\rho}\right| \leq C\sum_{i=1}\left(\int_0^{2\pi} \left|\frac{\partial a_i}{\partial\theta}\right|^2 d\theta\right)^{1/2}\left(\int_0^{2\pi}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial\theta}\right|^2 d\theta\right)^{1/2},$$

where C is a constant independent of ϕ.

Then integrating over $[1,\varepsilon]$, we get

$$\begin{aligned}
\int_\varepsilon^1 \left|\frac{d\overline{\phi}}{d\rho}\right| d\rho &\leq C\sum_{i=1}^{n}\int_\varepsilon^1\left(\int_0^{2\pi} \left|\frac{\partial a_i}{\partial\theta}\right|^2 d\theta\right)^{1/2}\left(\int_0^{2\pi}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial\theta}\right|^2 d\theta\right)^{1/2} d\rho \\
&\leq C\sum_{i=1}^{n}\left(\int_{\mathrm{D}(0,1)\setminus B_\varepsilon} \left|\frac{1}{\rho}\frac{\partial a_i}{\partial\theta}\right|^2 \rho\, d\rho\, d\theta\right)^{1/2}\left(\int_{\mathrm{D}(0,1)\setminus B_\varepsilon}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial\theta}\right|^2 \rho\, d\rho\, d\theta\right)^{1/2} \qquad (77)\\
&\leq C\sum_{i=1}^{n}\|\nabla a_i\|_2\,\|\nabla b_i\|_2.
\end{aligned}$$

Moreover, by duality, we obtain

$$\int_\varepsilon^1 \left|\frac{d\overline{\varphi}}{d\rho}\right| d\rho \leq \left\|\nabla\varphi\frac{1}{\rho}\right\|_1 \leq \|\nabla\varphi\|_{L^{2,1}}\left\|\frac{1}{\rho}\right\|_{L^{2,\infty}} \leq C\|\nabla\varphi\|_{L^{2,1}}. \qquad (78)$$

The combination of (76), (77) and (78) gives then

$$\int_\varepsilon^1 \left|\frac{d\overline{v}}{d\rho}\right| d\rho \leq C\sum_{i=1}^{n}\|\nabla a_i\|_2\|\nabla b_i\|_2. \qquad (79)$$

Following the approaches we used in the proofs of the various lemmas in Section 2, we decompose v as a Fourier series, which gives

$$v(\rho,\theta) = c_0 + d_0\log(\rho) + \sum_{n\in\mathbb{Z}^*}(c_n\rho^n + d_n\rho^{-n})e^{in\theta}.$$

We have

$$\bar{v}(\rho) = c_0 + d_0 \log(\rho).$$

Thanks to (79), we get that

$$|d_0| \ \log \frac{1}{\varepsilon} \leq C \sum_{i=1}^{n} \|\nabla a_i\|_2 \ \|\nabla b_i\|_2. \tag{80}$$

We have moreover

$$\begin{aligned} \|\nabla \bar{v}\|_{L^{2,1}(B_1 \setminus B_\varepsilon)} &\simeq |d_0| \int_0^\infty \left|\left\{x \in B_1 \setminus B_\varepsilon \;\middle|\; |x|^{-1} > t\right\}\right|^{1/2} dt \\ &= |d_0| \int_0^\infty \left|(B_1 \setminus B_\varepsilon) \cap B_{1/t}\right|^{1/2} dt \\ &\leq \pi \, |d_0| \int_0^{1/\varepsilon} \frac{dt}{\max\{t, 1\}} = \pi \ |d_0| \left[1 + \log \frac{1}{\varepsilon}\right]. \end{aligned} \tag{81}$$

Thus combining (80) and (81) we have on one hand

$$\|\nabla \bar{v}\|_{L^{2,1}(B_1 \setminus B_\varepsilon)} \leq C \sum_{i=1}^{n} \|\nabla a_i\|_2 \ \|\nabla b_i\|_2; \tag{82}$$

on the other hand, as in Lemma A.2, we have

$$\Big\| \sum_{n \in \mathbb{Z}^*} (c_n \rho^n + d_n \rho^{-n}) e^{in\theta} \Big\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq C(\lambda) \|\nabla v\|_2 \leq C(\lambda) \|\nabla \phi\|_2. \tag{83}$$

Combining (82), (83) we have for any $\lambda \in (0, 1)$ the existence of a positive constant $C(\lambda) > 0$ such that

$$\|\nabla v\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq C(\lambda) \Big(\sum_{i=1}^{n} \|\nabla a_i\|_2 \ \|\nabla b_i\|_2 + \|\nabla \phi\|_2 \Big). \tag{84}$$

Finally summing (76) and (84) gives the desired inequality and Lemma 18 is proved. □

5.2. ***Quantization of pseudoholomorphic curves on degenerating Riemann surfaces.*** We consider a closed Riemann surface (Σ, h), where Σ is a smooth compact surface without boundary, and is h a metric on Σ. Since we are only interested in the conformal structure of Σ, we can assume, thanks to the uniformization theorem [Hubbard 2006] that h has constant scalar curvature. We consider (N, J) to be a smooth almost-complex manifold and we look at pseudoholomorphic curves between (Σ, h) and (N, J); in other words we consider applications $u \in W^{1,2}(\Sigma, N)$ satisfying

$$\frac{\partial u}{\partial x} = J(u) \frac{\partial u}{\partial y}, \tag{85}$$

where $z = x + iy$ are some local conformal coordinates on Σ. These objects are fundamental in symplectic geometry [McDuff and Salamon 2004]. In the study of the *moduli space* of pseudoholomorphic curves in an almost complex manifold, the compactification question comes naturally. In other words it is of first importance to understand and describe how sequences of pseudoholomorphic curves with possibly degenerating conformal class behave at the limit.

The so-called Gromov's compactness theorem [Gromov 1985] (see also [Parker and Wolfson 1993; Sikorav 1994; Hummel 1997]) provides an answer to this question.

Theorem 19. *Let (N, J) be a compact almost complex manifold, Σ a closed surface and (j_n) a sequence of complex structures on Σ. Assume $u_n : (\Sigma, j_n) \to (N, J)$ is a sequence of pseudoholomorphic curves of bounded area with respect to an arbitrary metric on N. Then u_n converges weakly to some cusp curve*[4] *$\bar{u} : \overline{\Sigma} \to (N, J)$ and there exist finitely many bubbles, holomorphic maps $(\omega^i)_{i=1,\dots,l}$ from S^2 into (N, J), such that, modulo extraction of a subsequence,*

$$\lim_{n\to+\infty} E(u_n) = E(\bar{u}) + \sum_{i=1}^{l} E(\omega^i).$$

In fact the bound on the energy is not necessary assuming that the target manifold is symplectic, that is, if there is ω a closed 2-form on N compatible with J. Indeed, in that case (see [McDuff and Salamon 2004, Chapter 2] for instance), all $u : \Sigma \to N(J, \omega)$, regular enough, satisfies

$$A(u) = \int_\Sigma d(\mathrm{vol}_{u^*g}) \geq \int_\Sigma u^*\omega,$$

where $g = \omega(\,\cdot\,, J)$, with equality if and only if u is pseudoholomorphic. Hence, for symplectic manifolds, pseudoholomorphic curves are area-minimizing in their homology class. In particular, they are minimal surfaces, that is, conformal and harmonic, and we can use the general theory of harmonic maps; see [Zhu 2010, Remark 4.2].

We propose below a proof of Theorem 19 that follows the main lines of the most classical one (that is, we shall decompose our curves into thin and thick parts at the limit) but the argument we provide in order to prove that there is no energy in the neck and collar regions is new. We don't make use of the standard isoperimetric machinery but we simply apply the first-order Wente's estimate on annuli given by Lemma 18 which fits in an optimal way the particular structure of the pseudoholomorphic equation (85).

Proof of Theorem 19. The proof consists in splitting the surface in several pieces where the sequence converges either strongly to a nonconstant limiting map or weakly to a constant. Then in a second step, we prove that there is in fact no energy in the pieces where the converge is weak. Note that in contrast to the previous section, in the present case the complex structure of the surface is not fixed and is *a priori* free to degenerate.

Our aim is to show how Lemma 18 can be used in this context and therefore we shall be more brief on the classical parts such as the limiting Deligne–Mumford *thin-thick* decomposition which is described for instance in [Hummel 1997] or in [Zhu 2010]. Observe that due to the structure of the equation the ε-regularity theorem for pseudoholomorphic curves is a consequence of Theorem 12.

For simplicity, we will also assume that we have a surface of genus g greater or equal to 2. Hence let h_n be the hyperbolic metric of volume 1 associated to the complex structure j_n.

[4] We refer to [Hummel 1997, Chapter 5] for precise definitions.

According to the Deligne–Mumford compactification of Riemann surfaces [Hummel 1997, Chapter 4], modulo extraction of a subsequence, (Σ, h_n) converges to a hyperbolic Riemannian (Σ, h) surface by collapsing p $(0 \leq p \leq 3g-3)$ pairwise disjoint simple closed geodesics (γ_n^i).

Far from the collapsing geodesics, the metric uniformly converges, and we have a classical "bubble-tree" decomposition, that is to say u_n converges to a pseudoholomorphic curves of the (Σ, h) except possibly at finitely many points where, as in the previous section, u_n is forming bubbles (pseudoholomorphic curves from $\mathbb{C}$ to N) which are "connected" to each other by some *neck regions* $N_n^i = B(a_n^i, \mu_n^i) \setminus B(a_n^i, \lambda_n^i)$ where the weak L^2 energy goes to zero,

$$\lim_{r\to 0} \lim_{n\to+\infty} \|\nabla u_n\|_{L^{2,\infty}(N_n^i(r))} = 0,$$

where $N_n^i(r) = B(a_n^i, r\mu_n^i) \setminus B(a_n^i, \lambda_n^i/r)$. This can be established by combining the fact that, on such annular regions, the maximal L^2 energy of ∇u_n on dyadic annuli has to vanish (otherwise we would have another bubble) and the fact that Lemmas 13 and 18 apply to this situation.

Near the collapsing geodesics, our surface becomes asymptotically isometric to a hyperbolic cylinder of the form

$$A_l = \left\{ z = re^{i\phi} \in \mathbb{H} \;\middle|\; 1 \leq r \leq e^l, \arctan(\sinh(l/2)) < \phi < \pi - \arctan(\sinh(l/2)) \right\},$$

where the geodesic corresponds to $\{re^{i\pi/2} \in \mathbb{H} \mid 1 \leq r \leq e^l\}$, and the lines $\{r = 1\}$ and $\{r = e^l\}$ are identified via $z \mapsto e^l z$. This is the *collar region*. It is sometimes easier to consider the following cylindrical parametrization:

$$P_l = \left\{ (t, \theta) \;\middle|\; \frac{2\pi}{l} \arctan(\sinh(l/2)) < t < \frac{2\pi}{l}\big(\pi - \arctan(\sinh(l/2))\big), 0 \leq \theta \leq 2\pi \right\}.$$

In this parametrization the constant scalar curvature metric reads

$$ds^2 = \left(\frac{l}{2\pi \sin(lt/2\pi)} \right)^2 (dt^2 + d\theta^2),$$

where the geodesic corresponds to $\{t = \pi^2/l\}$, and the lines $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ are identified.

Then, as the length l_n of the degenerating geodesic goes to zero, $P_{l_n} = [0, T_n] \times S^1$ up to translation, which can be decomposed as follows [Zhu 2010, Proposition 3.1]. For each such a thin part, one can extract a subsequence such that the following decomposition holds. There $p \in \mathbb{N}$ and $2p$ sequences (a_n^1), (b_n^1), (a_n^2), (b_n^2),..., (a_n^p), (b_n^p) of positive numbers between 0 and T_n such that

$$\lim_{n\to+\infty} \frac{b_n^i - a_n^i}{T_n} = 0$$

and up to rescaling and identifying $]-\infty, +\infty[\times S^1$ with $\mathbb{C} \setminus \{0\}$, there exists a bubble ω^i (that is, a pseudoholomorphic curve from $\mathbb{C}$ to N) such that

$$u^n\left(\frac{a_n^i + b_n^i}{2} + \frac{t}{b_n^i - a_n^i}, \theta \right) \to \omega^i \qquad \text{on } C^2_{\text{loc}}(\mathbb{C} \setminus \{0\}).$$

Moreover, for any $\varepsilon > 0$, there exists $r > 0$ such that for any $T \in [b_n^i + r^{-1}, a_n^{i+1} - r^{-1}]$,

$$\int_{[T,T+1]\times S^1} |\nabla u_n|^2 \le \varepsilon. \tag{86}$$

Denoting

$$J_n^i = [a_n^i, b_n^i] \times S^1, \quad I_n^i = [b_n^i, a_n^{i+1}] \times S^1, \quad I_n^0 = [0, a_n^1] \times S^1,$$
$$I_n^p = [b_n^p, T_n] \times S^1, \quad \text{and} \quad I_n^i(r) = [b_n^i + r^{-1}, a_n^{i+1} - r^{-1}],$$

equation (86) combined with Lemma 13 implies that

$$\lim_{r\to 0} \lim_{n\to+\infty} \|\nabla u_n\|_{L^{2,\infty}(I_n^i(r))} = 0. \tag{87}$$

This decomposition is illustrated by Figure 3.

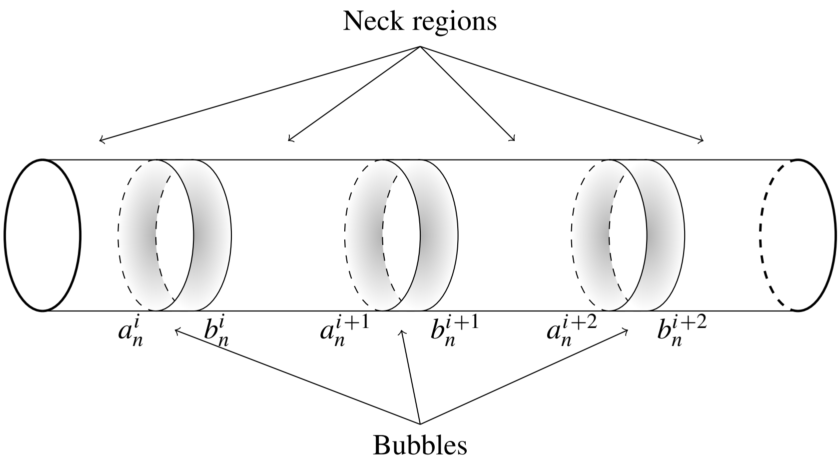

Figure 3. Decomposition into necks and bubbles.

As in the previous section, in order to prove that there is no energy at the limit in the *neck regions* of the thin parts, we combine the vanishing of the $L^{2,\infty}$-norm given by (87) with a uniform estimate on the $L^{2,1}$-norm of $|\nabla u^n|$ on each $I_n^i(r)$, which is a direct consequence of Lemma 18 applied to the pseudoholomorphic equation

$$\nabla u_n = J(u_n)\nabla^\perp u_n.$$

This concludes the proof of Theorem 19. □

Remark 20. Here again, in addition to the fact that our argument is not specific to J-holomorphic curves, our proof, in comparison with previous ones such as the one given in [Zhu 2010], has the advantage to require less regularity on the target manifold N. In fact, following the approach of [Parker 1996] or [Lin and Wang 1998], in order to establish the angular energy quantization, M. Zhu goes through a lower estimate of the second derivative

$$\frac{d^2}{d\theta^2}\int_{S^1\times\{t\}} |u_\theta|^2 \, d\theta.$$

Such an estimate requires for the metric of N to be at least C^2. In the alternative proof we are providing, in order to apply Lemma 18, we only require the almost complex structure and the compatible metric to be C^1 which corresponds to a weakening of the assumption of magnitude 1 in the derivative.

5.3. ***Quantification for harmonic maps on a degenerating surface, a cohomological condition.*** The aim of this section is to shed a new light on the quantization for harmonic maps on a degenerating surfaces, which has been fully described by M. Zhu in [2010].

The main result in the present subsection is the following result, which connects energy quantization for harmonic maps into spheres with a cohomological condition.

Theorem 21. *Let (Σ, h_n) be a sequence of closed Riemann surfaces equipped with their constant scalar curvature metric with volume* 1. *Let u_n be a sequence of harmonic maps from (Σ, h_n) into the unit sphere S^{m-1} of the euclidean space $\mathbb{R}^m$. Assume that*

$$\limsup_{n\to+\infty} E(u_n) < +\infty,$$

and assume that the closed forms

$$\star(u_n^i\, du_n^j - u_n^j\, du_n^i)$$

are exact for all $i, j = 1, \ldots, m$. Then the energy quantization holds: modulo extraction of a subsequence, on each component of the limiting thick part, u_n converges strongly, away from the punctures, to some limiting harmonic map u and there exists finitely many bubbles, holomorphic maps $(\omega^i)_{i=1,\ldots,l}$ from S^2 into S^{m-1} —forming possibly both on the thick and the thin parts — such that, modulo extraction of a subsequence

$$\lim_{n\to+\infty} E(u_n) = E(u) + \sum_{i=1}^{l} E(\omega^i). \tag{88}$$

Proof. In fact, assuming that our sequence of harmonic maps u_n get valued into S^{m-1} the equation simply written

$$\Delta u_n^i = \left(u_n^i \nabla (u_n)_j - (u_n)_j \nabla u_n^i\right) \nabla u_n^j.$$

But div $\left(u_n^i \nabla (u_n)_j - (u_n)_j \nabla u_n^i\right) = 0 = d(*u_n \wedge du_n)$. Hence *assuming that the closed $\wedge^2 \mathbb{R}^m$-valued* 1*-form $\star(u_n \wedge du_n)$ is exact* , there exists $b_n \in W^{1,2}$ such that

$$\star(u_n \wedge du_n) = db_n \quad \text{and} \quad \|b_n\|_{W^{1,2}} = O(\|u_n\|_{W^{1,2}}).$$

Then we have

$$\operatorname{div}(\nabla u_n - \nabla^{\perp} b_n\, u_n) = 0.$$

If we are on a neck region such as $B_1 \setminus D(0, \varepsilon_n)$, it can be integrated as

$$\nabla u_n = \nabla^{\perp} b_n\, u_n + \nabla^{\perp} c_n + d_n \nabla \log(\rho), \tag{89}$$

where $c_n \in W^{1,2}(B_1)$ and $d_n \in \mathbb{R}$. Then we try to control the gradient of the logarithmic part, remarking that

$$\frac{d}{d\rho}\int_0^{2\pi} u_n\, d\theta = \int_0^{2\pi} \frac{1}{\rho}\frac{\partial b_n}{\partial\theta}\, u_n\, d\theta + 2\pi\, \frac{d_n}{\rho} = \int_0^{2\pi} \frac{1}{\rho}\frac{\partial b_n}{\partial\theta}\big(u_n - \bar{u}_n^{\rho}\big)\, d\theta + 2\pi\, \frac{d_n}{\rho},$$

where $\bar{u}_n^{\rho}$ is the mean value of u_n over ∂B_ρ. Integrating the previous identity from ε_n to an arbitrary ρ gives

$$2\pi\big(\bar{u}_n^{\rho} - \bar{u}_n^{\varepsilon_n}\big) = \int_{\varepsilon_n}^{\rho}\int_0^{2\pi} \frac{1}{t}\frac{\partial b_n}{\partial\theta}\big(u_n - \bar{u}_n^{t}\big)\, d\theta\, dt + 2\pi d_n\, \log\left(\rho/\varepsilon_n\right). \tag{90}$$

And, thanks to Poincaré's inequality, we get

$$\left|\int_{\varepsilon_n}^{\rho}\int_0^{\pi} \frac{1}{t}\frac{\partial b_n}{\partial\theta}\big(u_n - \bar{u}_n^{t}\big)\, d\theta\, dt\right| \le C\|\nabla b_n\|_2\|\nabla u_n\|_2. \tag{91}$$

Then, combining (90) and (91), we finally obtain that

$$d_n = O\left(\frac{1}{\log\left(1/\varepsilon_n\right)}\right).$$

Which implies, as in the proof of Lemma 18, that the $L^{2,1}$-norm of $d_n\nabla\log(\rho)$ in $B_1\setminus D(0,\varepsilon_n)$ is uniformly bounded. By Lemma 18 and thanks to (89), we see that he $L^{2,1}$-norm of $\nabla(u_n - d_n\log(\rho))$ is also uniformly bounded and these two uniform bounds imply the uniform $L^{2,1}$ bound of ∇u_n in neck regions. Combining the uniform $L^{2,1}$ bound of ∇u_n in neck regions together with the Lemma 13 gives the desired energy quantization (88) and Theorem 21 is proved. □

More generally we can raise the following question: *Considering a sequence of harmonic maps from a degenerating surface to a general target manifolds, is there is a simple cohomological condition similar to the one in Theorem 21 ensuring the quantization of the energy in collar region?*

5.4. ***Energy Quantization for Willmore Surfaces.*** Finally we would like to recall a last application of Lemma 18 that has been used in a recent work by Y. Bernard and T. Rivière in [2011] for proving energy quantization for sequences of Willmore surfaces with uniformly bounded energy and nondegenerating conformal classes. The problem can be described as follows: for a sufficiently smooth immersion $u:\Sigma\to\mathbb{R}^m$, where Σ is a closed two-dimensional Riemannian surface, we can define its mean curvature vector $\vec{H}$ and we consider the functional

$$W(u) = \int_\Sigma |\vec{H}|^2\, u^*(dy),$$

where $u^*(dy)$ denotes the metric induced on Σ by the immersion u. This functional is called, the Willmore functional and is known to be conformally invariant [Rivière 2010]. Critical points to the functional W are called *Willmore immersions* or *Willmore surfaces*. Hence as for harmonic maps or pseudoholomorphic curves the question of the quantization of sequences of Willmore surfaces arise naturally. The second author has developed appropriate tools to study weak critical points to W in [Rivière 2008] and [Rivière] and proved the ε-regularity for these weak critical points. Using in particular Lemma 18 the following energy quantization has been established:

Theorem 22 [Bernard and Rivière 2011]. *Let u_n be a sequence of Willmore immersions of a closed surface Σ. Assume that*

$$\limsup_{n\to+\infty} W(u_n) < +\infty,$$

and that the conformal class of $u_n^*(\xi_{\mathbb{R}^m})$ remains within a compact subdomain of the moduli space of Σ. Then, modulo extraction of a subsequence, the following energy identity holds:

$$\lim_{n\to+\infty} W(u_n) = W(u_\infty) + \sum_{l=1}^{L} W(\omega_l) + \sum_{k=1}^{K} (W(\Omega_k) - 4\pi\theta_k),$$

where u_∞ is a possibly branched smooth Willmore immersion of Σ. The maps ω_l and Ω_k are smooth, possibly branched, Willmore immersions of S^2 and θ_k is the integer density of the current $(\Omega_k)_*(S^2)$ at some point $p_k \in \Omega_k(S^2)$, namely

$$\theta_k = \lim_{\rho\to 0} \frac{\mathcal{H}^2\big(B_\rho(p_k)\cap\Omega_k(S^2)\big)}{\pi\rho^2}.$$

Appendix A. Lorentz estimates on harmonic functions.

Here we prove two lemmas on harmonic functions which insure that we can control the $L^{2,1}$-norm by the L^2-norm on a smaller domain up to some appropriate boundary condition.

Lemma A.1. *Let $0 < \varepsilon < \frac{1}{2}$ and let $f : B_1 \setminus B_\varepsilon \to \mathbb{R}$ be a harmonic function which satisfies*

$$f = 0 \text{ on } \partial B_1 \quad \text{and} \quad \int_{\partial B_\varepsilon} f\, d\sigma = 0. \tag{92}$$

Then for each $\lambda > 1$ there exists positive a constant $C(\lambda)$, independent of ε and f, such that

$$\|\nabla f\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le C(\lambda)\|\nabla f\|_2.$$

Proof. We start by decomposing f as a Fourier series, which gives

$$f(\rho,\theta) = c_0 + d_0\log(\rho) + \sum_{n\in\mathbb{Z}^*}(c_n\rho^n + d_n\rho^{-n})e^{in\theta}.$$

Hence, using (92), we easily prove that $c_0 = d_0 = c_n + d_n = 0$; then we get

$$f(\rho,\theta) = \sum_{n\in\mathbb{Z}^*} c_n(\rho^n - \rho^{-n})e^{in\theta}.$$

Then we estimate the gradient as follows:

$$|\nabla f(\rho,\theta)| \le 2\sum_{n\in\mathbb{Z}^*} |nc_n|(\rho^{n-1} + \rho^{-n-1}).$$

Then, we estimate the $L^{2,1}$-norm of the $f_m(z) = |z|^m$ on $B_1 \setminus B_{\lambda\varepsilon}$, for $m \in \mathbb{Z}\setminus\{-1\}$ and $\lambda \in]1,2]$, which gives

$$\|f_m\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le \sqrt{\pi}\int_0^{(\lambda\varepsilon)^m} t^{1/m}\, dt \le 2\sqrt{\pi}(\lambda\varepsilon)^{m+1} \quad \text{for } m < -1,$$

and $\|f_m\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le \sqrt{\pi}$ for $m \ge 0$. Here we use the characterization of the $L^{2,1}$ norm given in (8). Hence we get

$$\|\nabla f\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le 4\sqrt{\pi}\left(\sum_{n>0}|n\,c_n|\,\big((\lambda\varepsilon)^{-n}+1\big)+\sum_{n<0}|n\,c_n|\,\big((\lambda\varepsilon)^{n}+1\big)\right).$$

Hence, thanks to the Cauchy–Schwarz and the fact that $\lambda > 1$, we get

$$\|\nabla f\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le 8\sqrt{\pi}\left(\sum_{n\neq 0}|n|\lambda^{-2|n|}\right)\left(\sum_{n\neq 0}|n|\,|c_n|^2\varepsilon^{-2|n|}\right)^{1/2}.$$

Finally we compute the L^2-norm of ∇f:

$$\|\nabla f\|_2 = \left(2\pi\int_\varepsilon^1\sum_{n\neq 0}|n\,c_n|^2\big(\rho^{2n-2}+\rho^{-2n-2}\big)\rho\,d\rho\right)^{1/2} \ge \sqrt{\frac{\pi}{2}}\left(\sum_{n\neq 0}|n|\,|c_n|^2\varepsilon^{-2|n|}\right)^{1/2},$$

which achieves the proof of Lemma A.1. □

Lemma A.2. *Let $0 < \varepsilon < \frac{1}{4}$ and let $f : B_1 \setminus B_\varepsilon \to \mathbb{R}$ be a harmonic function which satisfies*

$$\int_{\partial B_\varepsilon} f\,d\sigma = 0 \quad \text{and} \quad \left|\int_{\partial B_1} f\,d\sigma\right| \le K, \tag{93}$$

where K is a constant independent of ε. Then for each $0 < \lambda < 1$ there exists positive constant $C(\lambda)$, independent of ε and f, such that

$$\|\nabla f\|_{L^{2,1}(B_\lambda\setminus B_{\varepsilon/\lambda})} \le C(\lambda)(\|\nabla f\|_2 + 1).$$

Proof. We start by decomposing f as a Fourier series, which gives

$$f(\rho,\theta) = c_0 + d_0\log(\rho) + \sum_{n\in\mathbb{Z}^*}(c_n\rho^n + d_n\rho^{-n})e^{in\theta}.$$

Hence, using (93), we easily prove that $c_0 + d_0\log(\varepsilon) = 0$ and $|c_0| = O(1)$. Hence

$$d_0 = O\left(-\frac{1}{\log(\varepsilon)}\right). \tag{94}$$

Then we estimate the gradient as follows:

$$|\nabla f(\rho,\theta)| \le |d_0|\,\frac{1}{\rho} + \sum_{n\in\mathbb{Z}^*}|n\,c_n|\rho^{n-1} + |n\,d_n|\rho^{-n-1}.$$

Then we estimate the $L^{2,1}$-norm of $f_m(z) = |z|^m$ on $B_\lambda \setminus B_{\varepsilon/\lambda}$ for $m \in \mathbb{Z}\setminus\{-1\}$ and $0 < \lambda < 1$, which gives

$$\|f_m\|_{2,1} \le \sqrt{\pi}\int_0^{(\varepsilon/\lambda)^m} t^{1/m}\,dt \le 2\sqrt{\pi}(\varepsilon/\lambda)^{m+1} \quad \text{for } m < -1,$$

$\|f_m\|_{2,1} \leq \sqrt{\pi}\lambda^m$ for $m \geq 0$, and $\|f_{-1}\|_{2,1} = O(-\log(\varepsilon))$. Here we use the following characterization (8). Thanks to (94) and the above, we get

$$\|\nabla f\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq 2\sqrt{\pi}\left(\sum_{n>0}\left(|n\,c_n|\lambda^n + |n\,d_n|(\varepsilon/\lambda)^{-n}\right) + \sum_{n<0}\left(|n\,c_n|(\varepsilon/\lambda)^n + |n\,d_n|\lambda^{-n}\right)\right) + O(1).$$

Hence, thanks to Cauchy–Schwarz and the fact that $0 < \lambda < 1$, we get

$$\|\nabla f\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq 4\sqrt{\pi}\left(\sum_{n\neq 0}|n|\lambda^{2|n|}\right)\left(\sum_{n<0}|n|\,\frac{|c_n|^2 + |d_{-n}|^2}{\varepsilon^{2|n|}} + \sum_{n>0}|n|\,\frac{|c_n|^2 + |d_{-n}|^2}{2^n}\right)^{1/2} + O(1).$$

Finally we compute the L^2-norm of ∇f:

$$\begin{aligned}\|\nabla f\|_2 &= |d_0|\left(\int_\varepsilon^1 \frac{1}{\rho}\,d\rho\right)^{1/2} + \left(2\pi\int_\varepsilon^1 \sum_{n\neq 0}\left(|n\,c_n|^2\rho^{2n-2} + |n\,d_n|^2\rho^{-2n-2}\right)\rho\,d\rho\right)^{1/2} \\ &\geq \sqrt{\frac{\pi}{2}}\left(\sum_{n<0}|n|\,\frac{|c_n|^2 + |d_{-n}|^2}{\varepsilon^{2|n|}} + \sum_{n>0}|n|\,\frac{|c_n|^2 + |d_{-n}|^2}{2^n}\right)^{1/2},\end{aligned}$$

which achieves the proof of Lemma A.2. □

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GLOBAL WELL-POSEDNESS OF SLIGHTLY SUPERCRITICAL ACTIVE SCALAR EQUATIONS

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The paper is devoted to the study of slightly supercritical active scalars with nonlocal diffusion. We prove global regularity for the surface quasigeostrophic (SQG) and Burgers equations, when the diffusion term is supercritical by a symbol with roughly logarithmic behavior at infinity. We show that the result is sharp for the Burgers equation. We also prove global regularity for a slightly supercritical two-dimensional Euler equation. Our main tool is a nonlocal maximum principle which controls a certain modulus of continuity of the solutions.

1. Introduction

Active scalars play an important role in fluid mechanics. An active scalar equation is given by

$$\partial_t \theta + (u \cdot \nabla)\theta + \mathscr{L}\theta = 0, \quad \theta(x, 0) = \theta_0(x), \tag{1-1}$$

where $\mathscr{L}$ is typically some dissipative operator, such as fractional dissipation, and u is the flow velocity that is determined by θ. A common setting is either on $\mathbb{R}^d$ or $\mathbb{T}^d$. Active scalar equations are nonlinear, and most active scalars of interest are nonlocal. This makes the analysis of these equations challenging. The best known active scalar equations are the two-dimensional Euler equation in vorticity form (for which $u = \nabla^{\perp}(-\Delta)^{-1}\theta$), the surface quasigeostrophic (SQG) equation ($d = 2$, $u = \nabla^{\perp}(-\Delta)^{-1/2}\theta$), and the one-dimensional Burgers equation ($u = \theta$). The two-dimensional Euler and Burgers equations are classical in fluid mechanics, while the SQG equation was first considered in the mathematical literature by Constantin, Majda, and Tabak [Constantin et al. 1994], and since then has attracted significant attention, in part due to certain similarities with three-dimensional Euler and Navier–Stokes equations.

Observe that for the SQG and Burgers equations the drift velocity u and the advected scalar θ are of the same order of regularity, while for the two-dimensional Euler equation u is more regular by a derivative. The two-dimensional Euler equation has global regular solutions, and can be thought of as a critical case. For the Burgers and SQG equations, fractional dissipation $\mathscr{L} = \Lambda^{\alpha}$, where $\Lambda = (-\Delta)^{1/2}$ is the Zygmund operator, have often been considered. Both of these equations possess the L^{∞} maximum

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principle [Resnick 1995; Córdoba and Córdoba 2004], and this makes $\alpha = 1$ critical with respect to the natural scaling of the equations. It has been known for a while that in the subcritical case $\alpha > 1$, global regular solutions exist for sufficiently smooth initial data (see [Resnick 1995] for the SQG equation; the analysis for the Burgers equation is very similar; see, for example, [Kiselev et al. 2008]). The critical case $\alpha = 1$ has been especially interesting for the SQG equation since it is well motivated physically, with the $\Lambda\theta$ term modeling so called the Ekman boundary layer pumping effect; see, for example, [Pedlosky 1987]. The global regularity in the critical case has been settled independently by Kiselev, Nazarov, and Volberg [Kiselev et al. 2007] and Caffarelli and Vasseur [2010]. A third proof of the same result was provided by Kiselev and Nazarov [2009], and a fourth by Constantin and Vicol [2012]. All these proofs are quite different. The method of [Caffarelli and Vasseur 2010] is inspired by DeGiorgi iterative estimates, while the duality approach of [Kiselev and Nazarov 2009] uses an appropriate set of test functions and estimates on their evolution. The proof in [Constantin and Vicol 2012] takes advantage of a new nonlinear maximum principle, which gives a nonlinear bound on a linear operator. The method of [Kiselev et al. 2007], on the other hand, is based on a technique which may be called a nonlocal maximum principle. The idea is to prove that the evolution (1-1) preserves a certain modulus of continuity ω of the solution. In the critical SQG case, the control is strong enough to give a uniform bound on $\|\nabla\theta\|_{L^\infty}$, which is sufficient for global regularity.

In the supercritical case, until recently the only results available (for large initial data) have been on conditional regularity and finite time regularization of solutions. It was shown by Constantin and Wu [2008] that if the solution is C^δ with $\delta > 1 - \alpha$, it is smooth (see also [Silvestre 2011] for drift velocity that is not divergence free). Dong and Pavlovic [2009] improved this result to $\delta = 1 - \alpha$. Finite time regularization has been proved by Silvestre [2010] for α sufficiently close to 1, and for the whole dissipation range $0 < \alpha < 1$ by Dabkowski [2011] (with an alternative proof of the latter result given in [Kiselev 2011]). The issue of global regularity in the case $\alpha \in (0, 1)$ remains an outstanding open problem. A small advance into the supercritical regime was made in [Dabkowski et al. 2012], where the SQG equation with velocity given by

$$u = \nabla^\perp \Lambda^{-1} m(\Lambda)\theta$$

was considered. Here m is a Fourier multiplier which may grow at infinity at any rate slower than double logarithm. The method of [Dabkowski et al. 2012] was based on the technique of [Kiselev et al. 2007]. The main issue is that even with very slow growth of m, the equation loses scaling, which plays an important role in every proof of regularity for the critical case. [Dabkowski et al. 2012] was partly inspired by the slightly supercritical Navier–Stokes regularity result of Tao [2009], and partly by recent work on generalized Euler and SQG models [Chae et al. 2011; Chae et al. 2010].

In this paper, we analyze a slightly supercritical SQG equation and the Burgers equation equation. As opposed to [Dabkowski et al. 2012], we keep the velocity definition the same as for classical SQG and Burgers equations, and instead treat supercritical diffusion. We also consider nonlocal diffusion terms more general than the fractional Laplacian, including cases where the L^∞ maximum principle does not hold. We show, roughly, that symbols supercritical by a logarithm or less lead to global regular solutions for both equations. Our main technique is the control of an appropriate family of moduli of continuity of the solutions. For the Burgers equation, when the conditions we impose on the diffusion in order to obtain global regularity are not satisfied, we prove that some smooth initial data leads to finite time blow-up; see also [Alibaud et al. 2007; Dong et al. 2009; Kiselev et al. 2008]. In this respect, our well-posedness result is sharp. For the SQG equation, the global regularity proof is more sophisticated than for the Burgers equation. The upgrade from the double logarithmic supercriticality of [Dabkowski et al. 2012] to the logarithmic one is made possible by exploiting the structure of nonlinearity, in particular the $\nabla^{\perp}$ in $u = \nabla^{\perp}\Lambda^{-1}\theta$. This idea is based on [Kiselev 2011], where this structure was exploited to prove finite time regularization for power supercritical SQG equations. We note that Xue and Zheng [2012] observed a similar improvement from $\log\log$ to $\log$ in the context of supercritical velocity.

We also consider the slightly supercritical two-dimensional Euler equation, and generalize the results of [Chae et al. 2011] on global regularity of solutions.

General diffusion of integral type arises from probabilistic models which involve discontinuous Lévy processes. Indeed, the classical Lévy–Khintchine representation formula shows that very general integral diffusion arises as the generator of Lévy processes. This type of diffusion has many applications in the physical sciences; see, for example, [Klafter and Sokolov 2005] and the references therein.

Below, we state main results proved in the paper. In Section 2, we provide some basic background results on the nonlocal maximum principles. Section 3 is devoted to proving global regularity for the slightly supercritical SQG equation with nonlocal diffusions. The Burgers case is handled in Section 4. In Section 5, we consider the case of dissipation given by Fourier multipliers. Some natural multipliers can lead to nonpositive convolution kernels for the corresponding nonlocal diffusion, and we generalize our technique to this case. Section 6 is devoted to the slightly supercritical two-dimensional Euler equation.

To state our main results, we need to introduce some notation.

1A. ***Assumptions on m.*** Let $m\colon (0,\infty) \to [0,\infty)$ be a nonincreasing smooth function which is singular at the origin, that is, $\lim_{r\to 0} m(r) = \infty$, and satisfies the following conditions:

(i) There exists a sufficiently large positive constant $C_0 > 0$ such that

$$r m(r) \le C_0 \quad \text{for all } r \in (0, r_0) \tag{1-2}$$

for some $r_0 > 0$. This condition is natural in the present context, since otherwise the dissipative operator defined below is subcritical, which is not the purpose of this paper.

(ii) There exists some $\alpha > 0$ such that

$$r^{\alpha} m(r) \text{ is nonincreasing.} \tag{1-3}$$

This assumption is slightly stronger than just having $m(r)$ be nonincreasing.

Throughout this paper we also denote by m the radially symmetric function $m\colon \mathbb{R}^d \setminus \{0\} \to [0,\infty)$ such that $m(y) = m(|y|)$ for each $y \in \mathbb{R}^d \setminus \{0\}$. Note that the above conditions allow for m to be identically zero on the complement of a ball.

The examples of functions m which are relevant to this paper are those that are less singular than r^{-1} near $r = 0$. These functions yield dissipative nonlocal operators (cf. (1-5) below) that are less smoothing than Λ, which makes the corresponding SQG and Burgers equations *supercritical*. The main examples we have in mind are

$$m(r) = \frac{1}{r^a} \quad \text{and} \quad m(r) = \frac{1}{r(\log(2/r))^a} \tag{1-4}$$

with $0 < a \leq 1$ and $0 < r \leq 1$, coupled with enough regularity and decay for $r > 1$. The first class corresponds to power supercriticality. The second class, supercritical by a logarithm, is relevant for the global well-posedness results we prove. It is not hard to verify that the functions in (1-4) verify (1-2)–(1-3) on $(0, 1]$, and that they can be suitably extended on $[1, \infty)$.

1B. ***Dissipative nonlocal operators.*** Associated to any such function m, we consider the nonlocal operator

$$\mathscr{L}\theta(x) = \int_{\mathbb{R}^d} (\theta(x) - \theta(x+y)) \frac{m(y)}{|y|^d}\, dy. \tag{1-5}$$

Above and throughout the rest of the paper the integral is meant in principal value sense, but we omit the P.V. in front of the integral. For example, when $m(r) = r^{-\alpha} C_{d,\alpha}$ for a suitable normalizing constant, $C_{d,\alpha}$, $\mathscr{L} = \Lambda^\alpha$. The nonlocal operators $\mathscr{L}$ we consider here are dissipative because m is singular at the origin: due to (1-3), we have that $m(r) \geq m(1) r^{-\alpha}$ for some $\alpha > 0$ when $r \leq 1$, so that $\mathscr{L}$ is at least as dissipative as Λ^α. It is now evident that when $\lim_{r\to 0} rm(r) = 0$, the corresponding nonlocal operator $\mathscr{L}$ is less smoothing than Λ. We emphasize that, for $\theta \in C^\infty(\mathbb{T}^d)$, the P.V. integral in (1-5) converges only if m is subquadratic near $r = 0$, that is,

$$\int_0^1 rm(r)\, dr < \infty$$

holds. In our case, the above condition is satisfied in view of assumption (1-2). Convergence near infinity is not an issue due to assumption (1-3).

All results in this paper can be generalized to a more general class of dissipative operators. Namely, for each function m that satisfies (1-2)–(1-3), consider the class of smooth radially symmetric kernels $K : \mathbb{R}^d \setminus \{0\} \to (0, \infty)$ which satisfy

$$\frac{m(y)}{C|y|^d} \leq K(y) \leq \frac{Cm(y)}{|y|^d} \tag{1-6}$$

for some constant $C > 0$ and all $y \neq 0$. Associated to each such kernel K we may consider the dissipative nonlocal operator

$$\mathscr{L}\theta(x) = \int_{\mathbb{R}^d} (\theta(x) - \theta(x+y)) K(y)\, dy, \tag{1-7}$$

which generalizes the definition in (1-5). As we will see, such generalization will be useful when working with dissipative operators generated by Fourier multipliers. Moreover, as we will see later, conditions on K can be relaxed further.

1C. ***Main results.*** The generalized dissipative SQG equation reads

$$\partial_t\theta + u\cdot\nabla\theta + \mathscr{L}\theta = 0, \tag{1-8}$$

$$u = \nabla^{\perp}\Lambda^{-1}\theta, \tag{1-9}$$

where $\mathscr{L}$ is as defined in (1-5) and m is as described above. The main result of this paper with respect to the dissipative SQG equation is the following.

Theorem 1.1 (global regularity for slightly supercritical SQG). *Assume that θ_0 is smooth and periodic, and m satisfies an additional assumption*

$$\lim_{\varepsilon\to 0+}\int_{\varepsilon}^{1} m(r)\,dr = \infty. \tag{1-10}$$

Then there exists a unique, global in time, C^{∞} smooth solution θ of the initial value problem associated to (1-8)–(1-9).

In analogy, one may consider the generalized dissipative Burgers equation

$$\partial_t\theta - \theta\theta_x + \mathscr{L}\theta = 0, \tag{1-11}$$

where $\mathscr{L}$ and m are as before, and $d = 1$. Then we prove

Theorem 1.2 (global regularity for fractal Burgers). *Assume that θ_0 is smooth and periodic, and m is such that* (1-2)–(1-3) *hold and*

$$\lim_{\varepsilon\to 0+}\int_{\varepsilon}^{1} m(r)\,dr = \infty. \tag{1-12}$$

Then there exists a unique, global in time, C^{∞} smooth solution θ of the initial value problem associated to (1-11).

In addition, in the case of the Burgers equation we prove that condition (1-12) is sharp.

Theorem 1.3 (finite time blow-up for fractal Burgers). *Assume that m is such that* (1-2)–(1-3) *hold, and in addition we have*

$$r|m'(r)| \le Cm(r) \tag{1-13}$$

for $r > 0$ and some constant $C \ge 1$. Furthermore, suppose that

$$\lim_{\varepsilon\to 0+}\int_{\varepsilon}^{1} m(r)\,dr < \infty \tag{1-14}$$

holds. Then there exists an initial datum $\theta_0 \in C^{\infty}(\mathbb{T})$, and $T > 0$ such that $\lim_{t\to T}\|\theta_x(t)\|_{L^\infty} = \infty$, that is, we have finite time blow-up arising from smooth initial data.

A natural class of dissipation terms is associated with Fourier multiplier operators. This representation is closely related to the form (1-7). As noted above, when $m(r) = r^{-\alpha}C_{d,\alpha}$ for a suitable constant $C_{d,\alpha}$, $\mathscr{L} = \Lambda^{\alpha}$, corresponding to the Fourier multiplier with symbol $P(\zeta) = |\zeta|^{\alpha}$. One may generalize this statement as follows. Let $P(\zeta)$ be a sufficiently nice Fourier multiplier (see Lemmas 5.1 and 5.2 for precise assumptions on P), and let $K(y)$ be the convolution kernel associated to the multiplier P, that is, $\widehat{\mathscr{L}\theta}(\zeta) = P(\zeta)\hat{\theta}(\zeta)$, where $\mathscr{L}$ is the operator defined in (1-7). Then there exists a positive constant C that depends only on P, such that (1-6) holds for all sufficiently small y, with $m(y) = P(1/\zeta)$. This turns out to be sufficient to prove an analog of Theorem 1.1 (and Theorem 1.2).

Theorem 1.4 (global regularity for slightly supercritical SQG). *Let P be a radially symmetric Fourier multiplier that is smooth away from zero, nondecreasing, satisfies $P(0) = 0$, $P(\zeta) \to \infty$ as $|\zeta| \to \infty$, as well as conditions* (5-3)–(5-4), *and* (5-9). *Suppose also*

$$P(|\zeta|) \le C|\zeta| \tag{1-15}$$

for all $|\zeta|$ sufficiently large,

$$|\zeta|^{-\alpha} P(|\zeta|) \text{ is nondecreasing} \tag{1-16}$$

for some $\alpha > 0$, and

$$\lim_{\epsilon \to 0} \int_{\epsilon}^{1} P(|\zeta|^{-1})\, d|\zeta| = \infty. \tag{1-17}$$

Then, for any θ_0 that is smooth and periodic, the Cauchy problem for the dissipative SQG equation (5-1)–(5-2) *has a unique global in time smooth solution.*

In particular, Theorem 1.4 proves global regularity of solutions for dissipative terms given by multipliers with behavior $P(\zeta) \sim |\zeta|(\log|\zeta|)^{-a}$ for large ζ, where $0 \le a \le 1$. The details of the assumptions on P and more discussion can be found in Section 5 below.

2. Pointwise moduli of continuity

Definition 2.1 (modulus of continuity). *We call a function $\omega\colon [0,\infty) \to [0,\infty)$ a modulus of continuity if $\omega(0) = 0$, ω is nondecreasing, continuous, concave, piecewise C^2 with one sided derivatives, and additionally satisfies $\omega'(0+) < \infty$ and $\omega''(0+) = -\infty$. We say that a smooth function f* obeys *the modulus of continuity ω if $|f(x) - f(y)| < \omega(|x-y|)$ for all $x \neq y$.*

We recall that if $f \in C^\infty(\mathbb{T}^2)$ obeys the modulus ω, then $\|\nabla f\|_{L^\infty} < \omega'(0+)$ [Kiselev et al. 2007]. In addition, observe that a function $f \in C^\infty(\mathbb{T}^2)$ automatically obeys any modulus of continuity $\omega(\xi)$ that lies above the function $\min\{\xi\|\nabla f\|_{L^\infty}, 2\|f\|_{L^\infty}\}$.

The following lemma gives the modulus of continuity of the Riesz transform of a given function.

Lemma 2.2 (modulus of continuity under a Riesz transform). *Assume that θ obeys the modulus of continuity ω, and that the drift velocity is given by the constitutive law $u = \nabla^\perp \Lambda^{-1}\theta$. Then u obeys the modulus of continuity Ω defined as*

$$\Omega(\xi) = A\left(\int_0^\xi \frac{\omega(\eta)}{\eta}\, d\eta + \xi \int_\xi^\infty \frac{\omega(\eta)}{\eta^2}\, d\eta\right) \tag{2-1}$$

for some positive universal constant $A > 0$.

Moreover, for any two points x, y with $|x-y| = \xi > 0$, we have

$$\left|(u(x)-u(y))\cdot\frac{x-y}{|x-y|}\right| \le \widetilde{\Omega}(\xi)+\Omega^{\perp}(\xi), \tag{2-2}$$

where

$$\widetilde{\Omega}(\xi) = A\left(\omega(\xi)+\xi\int_{\xi}^{\infty}\frac{\omega(\eta)}{\eta^2}\,d\eta\right) \tag{2-3}$$

and

$$\Omega^{\perp}(\xi) = A\int_0^{\xi/4}\int_{\xi/4}^{3\xi/4}(\theta(\eta,\nu)-\theta(-\eta,\nu)-\theta(\eta,-\nu)+\theta(-\eta,-\nu))\frac{\nu}{((\xi/2-\eta)^2+\nu^2)^{3/2}}\,d\eta\,d\nu, \tag{2-4}$$

where A is a universal constant.

The proof of (2-1) may be found in [Kiselev et al. 2007, Appendix], while (2-2) was obtained in [Kiselev 2011, Lemma 5.2], to which we refer for further details.

Lemma 2.3 (dissipation control). *Let $\mathcal{L}$ be defined as in* (1-7), *with K satisfying* (1-6). *Assume that $\theta \in C^{\infty}(\mathbb{T}^2)$ obeys a concave modulus of continuity ω. Suppose that there exist two points x, y with $|x-y| = \xi > 0$ such that $\theta(x)-\theta(y) = \omega(\xi)$. Then we have*

$$\mathcal{L}\theta(x)-\mathcal{L}\theta(y) \ge \mathfrak{D}(\xi)+\mathfrak{D}^{\perp}(\xi) \tag{2-5}$$

where

$$\mathfrak{D}(\xi) = \frac{1}{A}\int_0^{\xi/2}(2\omega(\xi)-\omega(\xi+2\eta)-\omega(\xi-2\eta))\frac{m(2\eta)}{\eta}\,d\eta \\ + \frac{1}{A}\int_{\xi/2}^{\infty}(2\omega(\xi)-\omega(\xi+2\eta)+\omega(2\eta-\xi))\frac{m(2\eta)}{\eta}\,d\eta \tag{2-6}$$

and

$$\mathfrak{D}^{\perp}(\xi) = \frac{1}{A}\int_0^{\xi/4}\int_{\xi/4}^{3\xi/4}(2\omega(2\eta)-\theta(\eta,\nu)+\theta(-\eta,\nu)-\theta(\eta,-\nu)+\theta(-\eta,-\nu)) \\ \times\frac{m(\sqrt{(\xi/2-\eta)^2+\nu^2})}{(\xi/2-\eta)^2+\nu^2}\,d\eta\,d\nu \tag{2-7}$$

with some sufficiently large universal constant $A > 0$ (that we can, for simplicity of presentation, choose to be the same as in Lemma 2.2). The corresponding lower bound in one dimension includes only the $\mathfrak{D}$ term.

Note that $\mathfrak{D} \ge 0$ due to the concavity of ω, while $\mathfrak{D}^{\perp} \ge 0$ since θ obeys the modulus of continuity ω. The above lemma may be obtained along the lines of [Kiselev 2011, Section 5], where it was obtained for $\mathcal{L} = \Lambda$. However, some modifications are necessary for more general diffusions we consider, and we provide a proof in the Appendix below. We conclude this section by establishing a bound for $\Omega^{\perp}(\xi)$ in terms of $\mathfrak{D}^{\perp}(\xi)$.

Lemma 2.4 (connection between $\Omega^\perp$ and $\mathscr{D}^\perp$). *Let m be as in Section 1A, and assume θ obeys the modulus of continuity ω. For $\Omega^\perp(\xi)$ and $\mathscr{D}^\perp(\xi)$ defined via* (2-4) *and* (2-7), *respectively, we have*

$$m(\xi)\Omega^\perp(\xi) \le A^2 \mathscr{D}^\perp(\xi) \tag{2-8}$$

for all $\xi > 0$.

Proof of Lemma 2.4. To prove (2-8), first observe that since θ obeys ω, we have $\theta(\eta,\nu)-\theta(-\eta,\nu)\le\omega(2\eta)$ and also $\theta(\eta,-\nu)-\theta(-\eta,-\nu)\le\omega(2\eta)$. Therefore we have that

$$\begin{aligned}|\theta(\eta,\nu)-\theta(-\eta,\nu)-\theta(\eta,-\nu)+\theta(-\eta,-\nu)|&\\ \le 2\omega(2\eta)-\theta(\eta,\nu)+\theta(-\eta,\nu)-\theta(\eta,-\nu)+\theta(-\eta,-\nu)&\end{aligned} \tag{2-9}$$

holds, for any $(\eta,\nu)\in\mathbb{R}^2$.

Next, we claim that, for any $0<\nu\le\xi/4$ and any $|\eta-\xi/2|\le\xi/4$, we have

$$\frac{\nu\, m(\xi)}{((\xi/2-\eta)^2+\nu^2)^{3/2}} \le \frac{m(\sqrt{(\xi/2-\eta)^2+\nu^2})}{(\xi/2-\eta)^2+\nu^2}. \tag{2-10}$$

To prove (2-10), we observe that in this range for (η,ν) we have $\sqrt{(\xi/2-\eta)^2+\nu^2}\le\xi$, and due to the monotonicity of m, it follows that $m(\xi)\le m(\sqrt{(\xi/2-\eta)^2+\nu^2})$. Since $\nu\le\sqrt{(\xi/2-\eta)^2+\nu^2}$, (2-10) holds. Recalling the definitions of $\Omega^\perp$ and $\mathscr{D}^\perp$, it is clear that (2-8) follows directly from (2-9) and (2-10). concluding the proof of the lemma. □

3. Global regularity for slightly supercritical SQG

Proof of Theorem 1.1. The local well-posedness for smooth solutions to SQG-type equations is by now standard. In particular, we have.

Proposition 3.1 (local existence of a smooth solution). *Given a periodic $\theta_0\in C^\infty$, there exists $T>0$ and a periodic solution $\theta(\cdot,t)\in C^\infty$ of* (1-8)–(1-9). *Moreover, the smooth solution may be continued beyond T as long as $\|\nabla\theta\|_{L^1(0,T;L^\infty)}<\infty$.*

The local in time propagation of C^∞ regularity may in fact even be obtained in the absence of dissipation. Since in (1-8)–(1-9) we have a dissipative term, one may actually show local C^∞ regularization of sufficiently regular initial data. The proof may be obtained in analogy to the usual supercritical SQG equation [Dong 2010], since, in view of (1-3), $\mathscr{L}$ is smoothing more than Λ^α for some $\alpha>0$. The presence of the general dissipation $\mathscr{L}$ instead of the usual Λ^α does not introduce substantial difficulties.

The main difficulty in proving Theorem 1.1 is the supercriticality of the dissipation in (1-8)–(1-9). Thus, as opposed to the critical case [Kiselev et al. 2007], here we *cannot* construct a single modulus of continuity $\omega(\xi)$ preserved by the equation, and then use the scaling $\omega_B(\xi)=\omega(B\xi)$ to obtain a family of moduli of continuity such that each initial data obeys a modulus in this family. Instead, we will separately construct a modulus of continuity $\omega_B(\xi)$ for each initial data, and each such modulus will be preserved by the equation for all times; see also [Dabkowski et al. 2012].

Before constructing the aforementioned family of moduli, let us recall the breakthrough scenario of [Kiselev et al. 2007].

Lemma 3.2 (breakthrough scenario). *Assume ω is a modulus of continuity such that $\omega(0+) = 0$ and $\omega''(0+) = -\infty$. Suppose that the initial data θ_0 obeys ω. If the solution $\theta(x, t)$ violates ω at some positive time, there must exist $t_1 > 0$ and $x \neq y \in \mathbb{T}^2$ such that*

$$\theta(x, t_1) - \theta(y, t_1) = \omega(|x - y|),$$

and $\theta(x, t)$ obeys ω for every $0 \le t < t_1$.

Let us consider the breakthrough scenario for a modulus of continuity ω. A simple computation [Kiselev 2011] combined with Lemma 2.2 and Lemma 2.3 yields

$$\begin{aligned}\partial_t(\theta(x,t) - \theta(y,t))|_{t=t_1} &= u \cdot \nabla\theta(y, t_1) - u \cdot \nabla\theta(x, t_1) + \mathscr{L}\theta(y, t_1) - \mathscr{L}\theta(x, t_1) \\ &\le \left|(u(x, t_1) - u(y, t_1)) \cdot \frac{x - y}{|x - y|}\right| \omega'(\xi) + \mathscr{L}\theta(y, t_1) - \mathscr{L}\theta(x, t_1) \\ &\le \min\{\Omega(\xi), \widetilde{\Omega}(\xi) + \Omega^{\perp}(\xi)\}\omega'(\xi) - (\mathscr{D}(\xi) + \mathscr{D}^{\perp}(\xi)),\end{aligned} \tag{3-1}$$

where Ω, $\widetilde{\Omega}$, $\Omega^{\perp}$, $\mathscr{D}$, and $\mathscr{D}^{\perp}$ are given in (2-1), (2-3), (2-4), (2-6), and (2-7),, respectively. If we can show that the expression on the right side of (3-1) must be strictly negative, we obtain a contradiction: ω cannot be broken, and hence it is preserved by the evolution (1-8).

3A. ***Construction of the family of moduli of continuity.*** We now construct a family of moduli of continuity ω_B, such that, given any periodic C^∞ function θ_0, there exists $B \ge 1$ such that θ_0 obeys ω_B.

Fix a sufficiently small positive constant $\kappa > 0$, to be chosen precisely later in terms of the constant A of (2-3) and the function m. For any $B \ge 1$, we define $\delta(B)$ to be the unique solution of

$$m(\delta(B)) = \frac{B}{\kappa}. \tag{3-2}$$

Since m is continuous, nonincreasing, $m(r) \to +\infty$ as $r \to 0+$, and (1-3) holds, such a solution $\delta(B)$ exists for any $B \ge 1$ (if κ is small enough). For convenience we can ensure that $\delta(B) \le r_0/4$ for any $B \ge 1$ by using (1-2) and choosing $\kappa < r_0/(4C_0)$. Note that $\delta(B)$ is a nonincreasing function of B.

We let $\omega_B(\xi)$ be the continuous function with $\omega_B(0) = 0$ and

$$\omega_B'(\xi) = B - \frac{B^2}{2C_\alpha\kappa}\int_0^\xi \frac{3 + \ln(\delta(B)/\eta)}{\eta m(\eta)}\, d\eta \quad \text{for } 0 < \xi < \delta(B), \tag{3-3}$$

$$\omega_B'(\xi) = \gamma m(2\xi) \quad \text{for } \xi > \delta(B), \tag{3-4}$$

where $C_\alpha = (1 + 3\alpha)/\alpha^2$ and $\gamma > 0$ is a constant to be chosen later in terms of κ, A, and the function m (through C_0, α, r_0 of assumptions (1-2)–(1-3)). We emphasize that neither κ nor γ will depend on B.

Let us now verify that the above defined function ω_B is indeed a modulus of continuity in the sense of Definition 2.1. First notice that by construction $\omega_B'(0+) = B$ and $\omega_B(\xi) \le B\xi$ for all $0 < \xi \le \delta(B)$. To verify that ω_B is nondecreasing, since m is nonnegative, we only need to check that $\omega_B' > 0$ for $\xi < \delta(B)$.

This is equivalent to verifying that $\omega_B'(\delta(B)-) > 0$. Using (1-3) and the change of variables $\delta(B)/\eta \mapsto \xi$, we may estimate

$$\int_0^{\delta(B)} \frac{3+\ln(\delta(B)/\eta)}{\eta m(\eta)}\, d\eta \le \int_0^{\delta(B)} \frac{3+\ln(\delta(B)/\eta)}{\eta^{1-\alpha}\delta(B)^\alpha m(\delta(B))}\, d\eta = \frac{1}{m(\delta(B))}\int_1^\infty \frac{3+\ln\xi}{\xi^{1+\alpha}} d\xi = \frac{C_\alpha}{m(\delta(B))},$$

where $C_\alpha = (1+3\alpha)/\alpha^2$ may be computed explicitly. The above estimate and (3-2)–(3-3) imply that

$$\omega_B'(\delta(B)-) \ge B - \frac{C_\alpha B^2}{2C_\alpha \kappa m(\delta(B))} = \frac{B}{2}, \tag{3-5}$$

which concludes the proof that $\omega_B' > 0$.

In order to verify that $\omega_B''(0+) = -\infty$, we use (1-2) and (3-3) to obtain

$$\omega_B''(\xi) = -\frac{B^2}{2C_\alpha\kappa\xi m(\xi)}\left(3+\ln\frac{\delta(B)}{\xi}\right) \le -\frac{B^2}{2C_\alpha\kappa C_0}\left(3+\ln\frac{\delta(B)}{\xi}\right),$$

which is strictly negative for $0 < \xi < \delta(B)$, and also converges to $-\infty$ as $\xi \to 0+$.

Since m is nonincreasing, the concavity may only fail at $\xi = \delta(B)$. By (3-2) and (3-5) we have

$$\omega_B'(\delta(B)+) = \gamma m(2\delta(B)) \le \gamma m(\delta(B)) = \frac{\gamma B}{\kappa} \le \frac{B}{2} \le \omega_B'(\delta(B)-)$$

provided that $2\gamma \le \kappa$, and therefore ω_B is concave on $(0,\infty)$. It will also be useful to observe that, due to the concavity of ω_B and the mean value theorem, we have

$$\omega_B(\delta(B)) \ge \delta(B)\omega_B'(\delta(B)-) \ge \frac{\delta(B)B}{2}. \tag{3-6}$$

3B. ***Each initial data obeys a modulus.*** In order to show that given any $\theta_0 \in C^\infty(\mathbb{T}^2)$ there exits $B \ge 1$ such that θ_0 obeys $\omega_B(\xi)$, it is enough to find a B such that $\omega_B(\xi) > \min\{\xi\|\nabla\theta_0\|_{L^\infty}, 2\|\theta_0\|_{L^\infty}\}$ for all $\xi > 0$. Letting $a = 2\|\theta_0\|_{L^\infty}/\|\nabla\theta_0\|_{L^\infty}$, due to the concavity of ω_B, it is sufficient to find $B \ge 1$ such that $\omega_B(a) > 2\|\theta_0\|_{L^\infty}$. First, by choosing B large enough, we can ensure that $a > \delta(B)$. Then we have that

$$\omega_B(a) = \omega_B(\delta(B)) + \int_{\delta(B)}^a \omega_B'(\eta)\, d\eta \ge \gamma\int_{\delta(B)}^a m(2\eta)\, d\eta \to \infty \quad \text{as } \delta(B) \to 0 \tag{3-7}$$

due to the assumption (1-10). Therefore each initial data obeys a modulus from the family $\{\omega_B\}_{B\ge1}$.

3C. ***The moduli are preserved by the evolution.*** It is left to verify that the above constructed family of moduli of continuity satisfy

$$\min\{\Omega_B(\xi), \widetilde{\Omega}_B(\xi) + \Omega_B^\perp(\xi)\}\omega_B'(\xi) - (\mathscr{D}_B(\xi) + \mathscr{D}_B^\perp(\xi)) < 0 \tag{3-8}$$

for any $\xi > 0$ and $B \ge 1$. Here Ω_B and others are defined just as Ω and others, but with ω replaced by ω_B.

The case $\xi \geq \delta(B)$. First we observe that, by Lemma 2.4 and the fact that m is nonincreasing, we have

$$\omega_B'(\xi)\Omega_B^{\perp}(\xi) = \gamma m(2\xi)\Omega_B^{\perp}(\xi) \leq \gamma m(\xi)\Omega_B^{\perp}(\xi) \leq \gamma A^2 \mathfrak{D}_B^{\perp}(\xi) \leq \mathfrak{D}_B^{\perp}(\xi)$$

for all $\xi \geq \delta(B)$ if we choose $\gamma \leq 1/A^2$. In view of (3-8), it is left to prove that

$$\widetilde{\Omega}_B(\xi)\omega_B'(\xi) - \mathfrak{D}_B(\xi) < 0$$

for all $\xi \geq \delta(B)$. In order to do this we claim that, for all $\xi > \delta(B)$, we have

$$\omega_B(2\xi) \leq c_\alpha \omega_B(\xi), \tag{3-9}$$

where $c_\alpha = 1 + (3/2)^{-\alpha}$ and $\alpha > 0$ is as in assumption (1-3). Note that, by definition, $1 < c_\alpha < 2$. We postpone the proof of (3-9) to the end of this subsection. Using Lemma 2.3, (3-9), and the concavity and the monotonicity of ω_B, we may bound $-\mathfrak{D}_B$ as

$$\begin{aligned} -\mathfrak{D}_B(\xi) &\leq \frac{1}{A}\int_{\xi/2}^{\infty} (\omega_B(\xi+2\eta) - \omega_B(2\eta-\xi) - \omega_B(2\xi) - (2-c_\alpha)\omega_B(\xi))\frac{m(2\eta)}{\eta}\,d\eta \\ &\leq -\frac{2-c_\alpha}{A}\omega_B(\xi)\int_{\xi/2}^{\xi} \frac{m(2\eta)}{\eta}\,d\eta \leq -\frac{2-c_\alpha}{A}\omega_B(\xi)m(2\xi). \end{aligned} \tag{3-10}$$

We emphasize that, for the upper bound (3-10), only the contribution from $\eta \in (\xi/2, \xi)$ was used.

On the other hand, integrating by parts, the contribution from $\widetilde{\Omega}_B$ may be rewritten as

$$\frac{\widetilde{\Omega}_B(\xi)}{A} = \omega_B(\xi) + \xi\int_\xi^\infty \frac{\omega_B(\eta)}{\eta^2}\,d\eta = 2\omega_B(\xi) + \gamma\xi\int_\xi^\infty \frac{m(2\eta)}{\eta}.$$

Using (1-3), we may bound

$$\int_\xi^\infty \frac{m(2\eta)}{\eta}\,d\eta \leq \xi^\alpha m(2\xi)\int_\xi^\infty \frac{1}{\eta^{1+\alpha}}\,d\eta \leq \frac{m(2\xi)}{\alpha},$$

where $\alpha > 0$ is given. Hence we obtain

$$\frac{\widetilde{\Omega}_B(\xi)}{A} \leq 2\omega_B(\xi) + \frac{\gamma\xi m(2\xi)}{\alpha}. \tag{3-11}$$

Now, for $\delta(B) \leq \xi \leq 2\delta(B)$, by (1-3) we have

$$\frac{\gamma\xi m(2\xi)}{\alpha} \leq \frac{\gamma}{\alpha}(2\delta(B))^{1-\alpha}\delta(B)^\alpha m(\delta(B)) \leq \frac{2\gamma}{\alpha}\delta(B)\frac{B}{\kappa} \leq \frac{B\delta(B)}{2} \leq \omega_B(\delta(B)) \leq \omega_B(\xi) \tag{3-12}$$

by (3-6), if γ is small. On the other hand, for $\xi > 2\delta(B)$ we have $\xi - \delta(B) \geq \xi/2$ and therefore

$$\omega_B(\xi) \geq \gamma\int_{\delta(B)}^{\xi} m(2\eta)\,d\eta \geq \gamma m(2\xi)(\xi - \delta(B)) \geq \frac{\gamma\xi m(2\xi)}{2}.$$

Combining the above estimates with (3-11) leads to a bound

$$\widetilde{\Omega}_B(\xi) \leq A\left(2 + \frac{2}{\alpha}\right)\omega_B(\xi). \tag{3-13}$$

From (3-4) and the bounds (3-10) and (3-13), we hence obtain

$$\widetilde{\Omega}_B(\xi)\omega_B'(\xi) - \mathfrak{D}_B(\xi) \le \left(A\gamma\frac{2\alpha+2}{\alpha} - \frac{2-c_\alpha}{A}\right)\omega_B(\xi)m(2\xi) < 0$$

for all $\xi \ge \delta(B)$, if we set γ small enough, depending only on A, C, α, and c_α.

Proof of estimate (3-9). To verify (3-9) for $\delta(B) \le \xi \le 2\delta(B)$ is straightforward, since, by the mean value theorem and (1-3), similarly to (3-12), we obtain

$$\begin{aligned}\omega_B(2\xi) \le \omega_B(\xi) + \xi\omega_B'(\xi) &= \omega_B(\xi) + \gamma\xi m(2\xi)\\ &\le \omega_B(\xi) + \frac{2\gamma}{\kappa}B\delta(B) \le \omega_B(\xi) + (c_\alpha - 1)\omega_B(\delta(B)),\end{aligned}$$

by choosing γ small enough.

Now, for $\xi > 2\delta(B)$, by (3-4) and (3-6) we have

$$\begin{aligned}&c_\alpha\omega_B(\xi) - \omega_B(2\xi)\\ &\quad = (c_\alpha - 1)\omega_B(\delta(B)) + (c_\alpha - 1)\gamma\int_{\delta(B)}^{\xi} m(2\eta)\,d\eta - \gamma\int_{\xi}^{2\xi} m(2\eta)\,d\eta\\ &\quad \ge (c_\alpha - 1)\frac{B\delta(B)}{2} - \gamma\int_{2\xi-\delta(B)}^{2\xi} m(2\eta)\,d\eta + \gamma\int_{\delta(B)}^{\xi}((c_\alpha - 1)m(2\eta) - m(2\eta + 2\xi - 2\delta(B)))\,d\eta.\end{aligned}$$

We next note that for $\xi \ge 2\delta(B)$, due to the monotonicity of m, we have

$$\gamma\int_{2\xi-\delta(B)}^{2\xi} m(2\eta)\,d\eta \le \gamma\delta(B)m(\delta(B)) = \frac{\gamma}{\kappa}B\delta(B) \le (c_\alpha - 1)\frac{B\delta(B)}{2}$$

by letting γ be small enough. We next verify that

$$(c_\alpha - 1)m(2\eta) \ge m(2\eta + 2\xi - 2\delta(B))$$

holds for all $\eta \in (\delta(B), \xi)$. Using (1-3) and recalling that $c_\alpha = 1 + (3/2)^{-\alpha}$, the above inequality follows once we check that

$$(3/2)^{-\alpha}(2\eta + 2\xi - 2\delta(B))^\alpha \ge (2\eta)^\alpha$$

holds for all $\eta \in (\delta(B), \xi)$. But since $\xi > 2\delta(B)$, we have

$$\frac{\eta + \xi - \delta(B)}{\eta} \ge 1 + \frac{\xi - \delta(B)}{\xi} \ge \tfrac{3}{2}. \qquad \square$$

The case $0 < \xi \le \delta(B)$. For small values of ξ, we prefer to bound the contribution from the advective term using Ω_B instead of $\widetilde{\Omega}_B + \Omega_B^\perp$. It is sufficient to prove that

$$\Omega_B(\xi)\omega_B'(\xi) - \mathfrak{D}_B(\xi) < 0.$$

Using the concavity of ω_B and the mean value theorem, we may estimate

$$-\mathfrak{D}_B(\xi) \le \frac{1}{A}\int_0^{\xi/2}(\omega_B(\xi+2\eta) + \omega_B(\xi-2\eta) - 2\omega_B(\xi))\frac{m(2\eta)}{\eta}\,d\eta \le \frac{C}{A}\omega_B''(\xi)\int_0^{\xi/2}\eta m(2\eta)\,d\eta.$$

From (1-3) we obtain $\eta^\alpha m(2\eta) \geq (\xi/2)^\alpha m(\xi)$ for $\eta \in (0, \xi/2)$. Since $\omega_B''(\xi) < 0$, we may further bound

$$-\mathcal{D}_B(\xi) \leq \frac{C}{A}\omega_B''(\xi)\xi^\alpha m(\xi)\int_0^{\xi/2}\eta^{1-\alpha}\,d\eta \leq \frac{C}{A}\omega_B''(\xi)\xi^2 m(\xi). \tag{3-14}$$

The contribution from the advecting velocity is bounded as

$$\begin{aligned}\frac{\Omega_B(\xi)}{A} &= \int_0^\xi \frac{\omega_B(\eta)}{\eta}\,d\eta + \xi\int_\xi^{\delta(B)}\frac{\omega_B(\eta)}{\eta^2}\,d\eta + \xi\int_{\delta(B)}^\infty \frac{\omega_B(\eta)}{\eta^2}\,d\eta \\ &\leq B\xi + B\xi\ln\frac{\delta(B)}{\xi} + \xi\left(\frac{\omega_B(\delta(B))}{\delta(B)} + \gamma\int_{\delta(B)}^\infty \frac{m(2\eta)}{\eta}\,d\eta\right).\end{aligned} \tag{3-15}$$

Here we used that $\omega_B(\eta) \leq B\eta$ for $\eta \in (0, \delta(B))$. Using (1-2)–(1-3) and (3-2), we bound

$$\int_{\delta(B)}^\infty \frac{m(2\eta)}{\eta}\,d\eta \leq \frac{m(2\delta(B))}{\alpha} \leq \frac{B}{\alpha\kappa} \leq \frac{B}{\gamma}$$

for $\gamma \leq \alpha\kappa$. Therefore, (3-15) gives

$$\Omega_B(\xi) \leq AB\xi\left(3 + \log\frac{\delta(B)}{\xi}\right). \tag{3-16}$$

From (3-3) and the bounds (3-14) and (3-16), we obtain

$$\begin{aligned}\Omega_B(\xi)\omega_B'(\xi) - \mathcal{D}_B(\xi) &\leq AB^2\xi\left(3 + \log\frac{\delta(B)}{\xi}\right) + \frac{C}{A}\xi^2 m(\xi)\omega_B''(\xi) \\ &\leq AB^2\xi\left(3 + \log\frac{\delta(B)}{\xi}\right)\left(1 - \frac{C}{2A^2C_\alpha\kappa}\right) < 0\end{aligned} \tag{3-17}$$

for any $\xi \in (0, \delta(B))$, if we choose κ small enough. Here we used the explicit expression of ω_B'' for small ξ. Note that the choice of κ is independent of γ and B, which is essential in order to avoid a circular argument. This concludes the proof of Theorem 1.1. □

4. Global regularity vs. finite time blow-up for slightly supercritical Burgers

In this section we prove Theorems 1.2 (global regularity) and 1.3 (finite time blow-up).

Proof of Theorem 1.2. Due to evident similarities to the SQG proof given in Section 3 above, we only sketch the necessary modifications. See also [Kiselev et al. 2008] for more details.

First, we note that a modulus of continuity ω_B is preserved by (1-11) if

$$\omega_B(\xi)\omega_B'(\xi) - \mathcal{D}_B(\xi) < 0, \tag{4-1}$$

where $\mathcal{D}_B$ is defined as by (2-6), with ω replaced by ω_B. We will consider exactly the same family of moduli of continuity ω_B as in the SQG case, defined via (3-3)–(3-4).

We need to verify that (4-1) holds for any $\xi > 0$. In the case $\xi \geq \delta(B)$, by using (3-10), we have

$$\omega_B(\xi)\omega_B'(\xi) - \mathcal{D}_B(\xi) \leq \omega_B(\xi)\gamma m(2\xi) - \frac{C}{4}\omega_B(\xi)m(2\xi) < 0 \tag{4-2}$$

if $\gamma \le C/8$. On the other hand, for $\xi \in (0, \delta(B))$, we use (3-14) and obtain

$$\begin{aligned} \omega_B(\xi)\omega_B'(\xi) - \mathcal{D}_B(\xi) &\le B\xi\omega_B'(\xi) + C\xi^2 m(\xi)\omega_B''(\xi) \\ &\le B^2\xi - \frac{CB^2\xi}{2C_\alpha\kappa}\left(3 + \ln\frac{\delta(B)}{\xi}\right) \le B^2\xi\left(1 - \frac{3C}{2C_\alpha\kappa}\right) < 0 \end{aligned} \tag{4-3}$$

if $\kappa \le 3C/(2C_\alpha)$. This concludes the proof of Theorem 1.2. ☐

Proof of Theorem 1.3. The proof will use ideas from [Dong et al. 2009], which builds an appropriate Lyapunov functional to show blow-up. Throughout this section we assume that (1-14) holds, that is,

$$\int_0^1 m(r)\,dr = A < \infty. \tag{4-4}$$

Let $\theta_0 \in C^\infty$ be periodic and odd, with $\theta_0(0) = 0$. For simplicity we may take θ_0 to be $\mathbb{T} = [-\pi, \pi]$ periodic, and consider that $r_0 = 1$ in (1-2). It is clear that the proof carries over for any period length and for any value of $r_0 > 0$, with obvious modifications. Assume the solution $\theta(x, t)$ of (1-11) corresponding to this initial data lies in $C(0, T; W^{1,\infty})$ for some $T > 0$, and is hence C^∞ smooth on $[0, T]$. The Burgers equation preserves oddness of a smooth solution so that we have $\theta(0, t) = 0$ for $t \in [0, T]$, and also $\theta(x, t) = -\theta(-x, t)$ for all $x \in \mathbb{T}$ and $t \in [0, T]$.

Let $w(x)$ be defined as the *odd* function with

$$w(x) = 1 - x \quad \text{for } x \in (0, 1), \tag{4-5}$$

$$w(x) = 0 \quad \text{for } x \ge 1. \tag{4-6}$$

Associated to this function w we define the Lyapunov functional

$$L(t) = \int_0^\infty \theta(x, t)w(x)\,dx = \int_0^1 \theta(x, t)w(x)\,dx. \tag{4-7}$$

Then, due to the maximum principle $\|\theta(\,\cdot\,, t)\|_{L^\infty} \le \|\theta_0\|_{L^\infty}$, which holds on $[0, T]$, and the definition of $w(x)$, we have

$$L(t) \le \|\theta_0\|_{L^\infty} \tag{4-8}$$

for all $t \in [0, T]$. We will next show, using our assumption that $\theta \in C(0, T; W^{1,\infty})$, that if T is sufficiently large, the bound (4-8) is violated. This shows that our assumption has been wrong, and θ has finite time blow-up in the $W^{1,\infty}$ norm, concluding the proof of Theorem 1.3.

To proceed, we first need the following lemma.

Lemma 4.1. *If $\mathcal{L}$ is a diffusive operator defined by* (1-5) *with m satisfying* (4-4) *in addition to our usual assumptions and w is given by* (4-5),

$$\int_{\mathbb{R}} |\mathcal{L}w(x)|\,dx < \infty.$$

Proof. It is sufficient to estimate the integral over positive x, since $\mathcal{L}w(x)$ is odd.

The case $x \geq 1$. Here we have $w(x) = 0$ and w is odd, hence

$$\begin{aligned}\mathcal{L}w(x) = \int_{\mathbb{R}} (w(x) - w(y)) \frac{m(x-y)}{|x-y|}\, dy &= -\int_0^1 w(y) \frac{m(x-y)}{|x-y|}\, dy - \int_{-1}^0 w(y) \frac{m(x-y)}{|x-y|}\, dy \\ &= -\int_0^1 w(y) \frac{m(x-y)}{|x-y|}\, dy + \int_0^1 w(y) \frac{m(x+y)}{|x+y|}\, dy \\ &= -\int_0^1 (1-y) \left(\frac{m(x-y)}{|x-y|} - \frac{m(x+y)}{|x+y|} \right) dy. \end{aligned} \tag{4-9}$$

Using the mean value theorem and the monotonicity of m, we estimate

$$\left| \frac{m(x-y)}{|x-y|} - \frac{m(x+y)}{|x+y|} \right| \leq 2y \sup_{r \in [x-y, x+y]} \frac{r|m'(r)| + m(r)}{r^2} \leq 4Cy \frac{m(x-y)}{|x-y|^2}.$$

But the above bound is only convenient when $x \geq 2$, and in this range we obtain

$$\begin{aligned}\int_2^\infty |\mathcal{L}w(x)|\, dx &\leq 4C \int_2^\infty \int_0^1 (1-y) y \frac{m(x-y)}{|x-y|^2}\, dy\, dx \\ &\leq 4C \int_2^\infty \int_0^1 (1-y) y \frac{m(1)}{|x-1|^2}\, dy\, dx \leq Cm(1). \end{aligned} \tag{4-10}$$

On the other hand, when $x \in [1, 2]$, it is convenient to work with (4-9) directly. By the monotonicity of m, we have that

$$|\mathcal{L}w(x)| \leq \int_0^1 (1-y) \left(\frac{m(x-y)}{|x-y|} + \frac{m(x+y)}{|x+y|} \right) dy \leq 2 \int_0^1 m(1-y)\, dy = 2A \tag{4-11}$$

for any $x \in [1, 2)$, by using (4-4). Therefore, $\int_1^2 |\mathcal{L}w(x)|\, dx \leq 2A$, and by using (4-10), we obtain that

$$\int_1^\infty |\mathcal{L}w(x)|\, dx \leq 2A + CC_0 =: C_1. \tag{4-12}$$

The case $0 < x < 1$. Here we have $w(x) = 1 - x$, and therefore

$$\begin{aligned}&\mathcal{L}w(x) \\ &= \int_{1-x}^\infty (1-x) \frac{m(y)}{y}\, dy + \int_{-x}^{1-x} y \frac{m(y)}{|y|}\, dy + \int_{-x-1}^{-x} (2+y) \frac{m(y)}{|y|}\, dy + \int_{-\infty}^{-x-1} (1-x) \frac{m(y)}{|y|}\, dy \\ &=: T_1(x) + T_2(x) + T_3(x) + T_4(x). \end{aligned} \tag{4-13}$$

Using condition (1-3), we may easily bound T_1 and T_4. More precisely, using a change of variables $y \to -y$ in T_4, we may write

$$\begin{aligned}|T_1(x) + T_4(x)| &= (1-x) \int_{1-x}^{1+x} \frac{m(y)}{y}\, dy + 2(1-x) \int_{1+x}^\infty \frac{y^\alpha m(y)}{y^{1+\alpha}}\, dy \\ &\leq (1-x) \int_{1-x}^{1+x} \frac{m(1-x)}{1-x}\, dy + 2(1-x)(1+x)^\alpha m(1+x) \int_{1+x}^\infty \frac{1}{y^{1+\alpha}}\, dy \\ &\leq 2x\, m(1-x) + \frac{2(1-x)m(1)}{\alpha}, \end{aligned}$$

and therefore

$$\int_0^1 |T_1(x)+T_4(x)|\,dx \le 2\int_0^1 xm(1-x)\,dx + \frac{2C_0}{\alpha}\int_0^1 (1-x)\,dx \le 2A+\frac{C_0}{\alpha}. \tag{4-14}$$

To bound T_2, we recall that m is even and hence $T_2(x)=\int_x^{1-x} m(y)\,dy$, which in turn implies

$$\int_0^1 |T_2(x)|\,dx \le \int_0^{1/2}\int_x^{1-x} m(y)\,dy\,dx + \int_{1/2}^1\int_{1-x}^x m(y)\,dy\,dx \le \int_0^1 m(y)\,dy = A. \tag{4-15}$$

Lastly, due to the monotonicity of m, we have that

$$\begin{aligned}|T_3(x)| &\le 2\int_x^1 \frac{m(y)}{y}\,dy + 2\int_1^{x+1}\frac{m(y)}{y}\,dy + \int_x^{x+1} m(y)\,dy\\ &\le 2\int_x^1 \frac{m(y)}{y}\,dy + 2\int_1^2 \frac{m(1)}{y}\,dy + \int_x^1 m(y)\,dy + \int_1^2 m(y)\,dy\\ &\le 2\int_x^1 \frac{m(y)}{y}\,dy + 2m(1)\log 2 + A + m(1),\end{aligned}$$

and therefore

$$\begin{aligned}\int_0^1 |T_3(x)|\,dx &\le 2\int_0^1\int_x^1 \frac{m(y)}{y}\,dy\,dx + 3m(1) + A\\ &\le 2\int_0^1\int_0^y \frac{m(y)}{y}\,dx\,dy + 3C_0 + A \le 3(C_0+A). \end{aligned}\tag{4-16}$$

Summarizing (4-14), (4-15), and (4-16), we obtain that

$$\int_0^1 |\mathscr{L}w(x)|\,dx \le 6A + 3C_0 + \frac{C_0}{\alpha} =: C_2. \tag{4-17}$$

This concludes the proof. □

Coming back to our Lyapunov functional $L(t)$, using the evolution (1-11) and integrating by parts, we obtain

$$\begin{aligned}\frac{d}{dt}L(t) = \int_0^\infty \theta_t(x,t)w(x)\,dx &= \int_0^\infty \left(\partial_x \frac{\theta(x,t)^2}{2} - \mathscr{L}\theta(x,t)\right) w(x)\,dx\\ &= -\frac{1}{2}\int_0^1 \theta(x,t)^2 w_x(x)\,dx - \int_0^\infty \theta(x,t)\mathscr{L}w(x)\,dx. \end{aligned}\tag{4-18}$$

Here we employed the identity $\int_0^\infty \mathscr{L}\theta(\cdot,t)w = \int_0^\infty \theta(\cdot,t)\mathscr{L}w$. This equality can be derived by using the oddness of *both* θ and w, the evenness of m, and Lemma 4.1, ensuring $\mathscr{L}w(x)\in L^1$ (see [Dong et al. 2009, (2.8)] for more details). Also, the integration by parts in the first term of (4-18) is justified, since, by our assumption, θ vanishes as $C|x|$ when $x\to 0$, with $C=\sup_{[0,T]}\|\nabla\theta(\cdot,t)\|_{L^\infty}$.

Now, since $w_x = -1$ for $0 < x < 1$, and using the Cauchy–Schwartz inequality, we obtain

$$L(t)^2 = \left(\int_0^1 \theta(x,t)w(x)\,dx\right)^2 \le \int_0^1 \theta(x,t)^2\,dx \int_0^1 w(x)^2\,dx$$
$$= \frac{1}{3}\int_0^1 \theta(x,t)^2\,dx = -\frac{1}{3}\int_0^1 \theta(x,t)^2 w_x(x)\,dx.$$

Therefore, by (4-18) on $[0, T]$ we have

$$\frac{d}{dt}L(t) \ge \frac{3}{2}L(t)^2 - \int_0^\infty |\theta(x,t)||\mathscr{L}w(x)|\,dx \ge L(t)^2 - \|\theta_0\|_{L^\infty}\int_0^\infty |\mathscr{L}w(x)|\,dx. \tag{4-19}$$

By Lemma 4.1, we then have

$$\frac{d}{dt}L(t) \ge L(t)^2 - (C_1 + C_2)\|\theta_0\|_{L^\infty}. \tag{4-20}$$

But (4-20) implies that $L(t)$ blows up in finite time provided that

$$0 < L(0)^2 - (C_1 + C_2)\|\theta_0\|_{L^\infty} = \left(\int_0^1 (1-x)\theta_0(x)\,dx\right)^2 - (C_1 + C_2)\|\theta_0\|_{L^\infty}.$$

It is easy to design initial data satisfying this condition, and thus leading to finite time blow-up. This completes the proof of Theorem 1.3. □

5. Global regularity with dissipative Fourier multiplier

In this section we establish a connection between the global regularity results obtained for (1-8)–(1-9) when the dissipative nonlocal operators $\mathscr{L}$ are replaced by dissipative Fourier multiplier operators, an approach that has been more standard in fluid dynamics. More precisely, we will replace $\mathscr{L}\theta(x)$ by

$$(P(\zeta)\hat{\theta}(\zeta))^\vee(x)$$

for a nice enough radially symmetric Fourier multiplier symbol P, and consider the global regularity for the slightly supercritical SQG equation

$$\partial_t\theta + u\cdot\nabla\theta + (P\hat{\theta})^\vee = 0, \tag{5-1}$$

$$u = \nabla^\perp\Lambda^{-1}\theta. \tag{5-2}$$

The setting can be either $\mathbb{T}^2$ or $\mathbb{R}^2$ with decaying initial data. In the latter case, an additional argument is needed for Lemma 3.2 to remain valid due to lack of compactness; see [Dong and Du 2008]. We will focus on the periodic case. Note that working on $\mathbb{T}^d$ is equivalent to working on $\mathbb{R}^d$ with $\theta(x,t)$ extended periodically. We will henceforth pursue this strategy, thinking of the Fourier multiplier P and its corresponding convolution kernel K in $\mathbb{R}^d$.

Intuitively, the Fourier multiplier corresponds to a nonlocal operator $\mathscr{L}$ as defined in (1-5), with $m(y)$ that is comparable to $P(1/|y|)$. We make this connection more precise in the following two lemmas.

Lemma 5.1 (dissipative operator associated to Fourier multiplier — upper bound). *Let $P(\zeta) = P(|\zeta|)$ be a radially symmetric function which is smooth away from zero, nonnegative, nondecreasing, with $P(0) = 0$ and $P(\zeta) \to \infty$ as $|\zeta| \to \infty$. In addition assume the following*:

(i) *P satisfies the doubling condition*:

$$P(2|\zeta|) \le c_D P(|\zeta|) \tag{5-3}$$

for some doubling constant $c_D \ge 1$.

(ii) *P satisfies the Hörmander–Mikhlin condition*:

$$|\partial_\zeta^k P(\zeta)|\,|\zeta|^{|k|} \le c_H P(\zeta) \tag{5-4}$$

for some constant $c_H \ge 1$, and for all multi-indices $k \in \mathbb{Z}^d$ with $|k| \le N$, with N depending only on the dimension d and on the doubling constant c_D.

(iii) *P has subquadratic growth at ∞, that is,*

$$\int_0^1 P(|\zeta|^{-1})|\zeta|\,d|\zeta| < \infty. \tag{5-5}$$

Then the Fourier multiplier operator with symbol $P(\zeta)$ is given as a nonlocal operator defined as the principal value of

$$(P(\,\cdot\,)\hat{\theta}(\,\cdot\,))^\vee(x) = \int_{\mathbb{R}^d} (\theta(x) - \theta(x+y))K(y)\,dy \tag{5-6}$$

and the radially symmetric kernel K satisfying

$$|K(y)| \le C|y|^{-d} P(|y|^{-1}) \tag{5-7}$$

for all $y \ne 0$, for some positive constant $C > 0$. Similarly $|\nabla K(y)| \le C|y|^{-d-1}P(|y|^{-1})$ for $y \ne 0$.

Proof. As in Littlewood–Paley theory, consider smooth, radially symmetric functions φ, supported on $1/2 \le |\zeta| \le 2$, such that

$$1 = \sum_{j\in\mathbb{Z}} \varphi(2^{-j}\zeta) \tag{5-8}$$

holds for $\zeta \in \mathbb{R}^d \setminus \{0\}$. We write $\varphi_j(\zeta) = \varphi(2^{-j}\zeta)$ and note that $P(\zeta)\varphi_j(\zeta)$ is smooth and compactly supported with $P(0)\varphi_j(0) = 0$. Hence

$$K_j(y) = -\int_{\mathbb{R}^d} P(\zeta)\varphi_j(\zeta)e^{iy\cdot\zeta}\,d\zeta$$

is a family of L^1 kernels, which are smooth at the origin, radially symmetric, and have zero mean on $\mathbb{R}^d$. Thus we may write (in order to avoid principal value integrals we use double differences)

$$(P(\,\cdot\,)\varphi_j(\,\cdot\,)\hat{\theta}(\,\cdot\,))^\vee(x) = \int_{\mathbb{R}^d} K_j(y)\big(2\theta(x) - \theta(x-y) - \theta(x+y)\big)\,dy.$$

For $y \neq 0$, let $j_0 = [\log_2 |y|^{-1}]$ and fix $N > d + \log_2 c_D$ to be an even integer. By (5-3)–(5-4) we have

$$
\begin{aligned}
\sum_j |K_j(y)| &= \sum_{j<j_0} \|K_j\|_{L^\infty} + \sum_{j\geq j_0} \left| \int_{\mathbb{R}^d} P(\zeta)\varphi_j(\zeta) e^{i\zeta\cdot y}\, d\zeta \right| \\
&\leq \sum_{j<j_0} \|\hat{K}_j\|_{L^1} + \sum_{j\geq j_0} |y|^{-N} \left| \int_{\mathbb{R}^d} (-\Delta)^{N/2}(P(\zeta)\varphi_j(\zeta)) e^{i\zeta\cdot y}\, d\zeta \right| \\
&\leq \sum_{j<j_0} \int_{\mathbb{R}^d} P(\zeta)\varphi(2^{-j}\zeta)\, d\zeta + C|y|^{-N} \sum_{j\geq j_0} 2^{-jN} \int_{2^{j-1}\leq|\zeta|\leq 2^{j+1}} P(\zeta)\, d\zeta \\
&\leq P(2^{j_0}) \sum_{j<j_0} \int_{\mathbb{R}^d} \varphi(2^{-j}\zeta)\, d\zeta + C|y|^{-N} \sum_{j\geq j_0} 2^{-j(N-d)} P(2^{j+1}) \\
&\leq C P(2^{j_0}) \sum_{j<j_0} 2^{jd} + C|y|^{-N} 2^{-j_0(N-d)} \sum_{j\geq j_0} 2^{-(j-j_0)(N-d)} P(2^{j_0}) c_D^{j-j_0} \\
&\leq C P(|y|^{-1})|y|^{-d} + C P(|y|^{-1})|y|^{-d} \sum_{j\geq j_0} 2^{-(j-j_0)(N-d-\log_2 c_D)},
\end{aligned}
$$

which shows that the sum $K(y) = \sum_j K_j(y)$ converges absolutely for all $y \neq 0$, and proves (5-7). The purpose of condition (5-5) is now evident. For a smooth function θ (say at least of class C^2), in order to make sense of the integral

$$
\int_{|y|\leq 1} K(y)(\theta(x-y) + \theta(x+y) - 2\theta(x))\, dy,
$$

in view of (5-7), we need to assume that $\int_{|y|\leq 1} P(|y|^{-1})|y|^{-d+2}\, dy < +\infty$, which is equivalent to (5-5). The bound for $|\nabla K|(y)$ is analogous, and we omit further details. □

Lemma 5.2 (dissipative operator associated to Fourier multiplier — lower bound). *Let the Fourier multiplier P and its associated kernel K be as in Lemma 5.1. Assume additionally that*

(v) *P satisfies*

$$
(-\Delta)^{(d+2)/2} P(\zeta) \geq c_H^{-1} P(\zeta)|\zeta|^{-d-2} \tag{5-9}
$$

for all $|\zeta|$ sufficiently large (*say larger than $c_0 > 0$*), *for some constant $c_H \geq 1$.*

Then the kernel K corresponding to P (*see* (5-6)) *may be bounded from below as*

$$
K(y) \geq C^{-1}|y|^{-d} P(|y|^{-1}) \tag{5-10}
$$

for all sufficiently small $|y|$, for some sufficiently large constant $C > 0$.

Proof. From our assumptions, the symbol P is a C^{d+2} smooth function except perhaps at the origin. Without loss of generality we can assume P to be smooth at the origin as well. Otherwise, we can write $P = \widetilde{P} + R$ where $\widetilde{P}$ is a C^{d+2} function everywhere with $\widetilde{P}(\zeta) = P(\zeta)$ for all $|\zeta| > c_0$, $\widetilde{P}(0) = 0$, and $(-\Delta)^{N/2}\widetilde{P}$ is bounded in $\mathbb{R}^d$. The remainder R is a bounded compactly supported function with $R(0) = 0$. Therefore, the Fourier multiplier operator with symbol P is the sum of the operators with multipliers $\widetilde{P}$

and R. For $\widetilde{P}$ we apply the proof below and obtain a kernel satisfying (5-10), and for the remainder R we have $(R\hat{\theta})^\vee = R^\vee * \theta$ and $R^\vee$ is a bounded, mean zero L^1 kernel. Thus, adding $R^\vee$ will not destroy the estimate (5-10) for small enough y.

If P is smooth near $\zeta = 0$, we have that $Q(\zeta) = (-\Delta)^{(d+2)/2} P(\zeta) \in L^1(\mathbb{R}^d)$. Indeed, $\int_{|\zeta|\le 1} |Q(\zeta)|\, d\zeta$ is finite since P is smooth, while by (5-4) and (5-5) we have

$$\int_{|\zeta|\ge 1} |Q(\zeta)|\, d\zeta \le c_H \int_{|\zeta|\ge 1} |\zeta|^{-(d+2)} P(\zeta)\, d\zeta = c_H \int_1^\infty |\zeta|^{-3} P(|\zeta|) d|\zeta| = c_H \int_0^1 r P(r^{-1})\, dr < \infty.$$

We may hence define the function M, the inverse Fourier transform of $-Q$, as

$$M(y) = -\int_{\mathbb{R}^d} Q(\zeta) e^{i\zeta\cdot y}\, d\zeta = -\int_{\mathbb{R}^d} Q(\zeta)\cos(y\cdot\zeta)\, d\zeta,$$

where we have used the fact that Q is radially symmetric and real. Moreover, note that Q has zero mean, since in view of Lemma 5.1 we have the bound $|Q^\vee(x)| \le |x|^{d+2}|P^\vee(x)| \le C|x|^2 P(|x|^{-1}) \to 0$ as $|x| \to 0$, since P is subquadratic at infinity; cf. (5-5). Thus we may rewrite $M(y)$ as

$$M(y) = \int_{\mathbb{R}^d} Q(\zeta)(1-\cos(y\cdot\zeta))\, d\zeta = \int_{\mathbb{R}^d} Q(\zeta)(1-\cos(\zeta_1|y|))\, d\zeta \tag{5-11}$$

by using that Q is radially symmetric. In order to appeal to (5-9), we further split

$$M(y) = \int_{|\zeta|\le c_0} Q(\zeta)(1-\cos(\zeta_1|y|))\, d\zeta + \int_{|\zeta|>c_0} Q(\zeta)(1-\cos(\zeta_1|y|))\, d\zeta. \tag{5-12}$$

For all $|y| \le c_0^{-1}$, the first integral in (5-12) can be estimated from below by $-C_Q|y|^2$, where $C_Q = \int_{|\zeta|\le c_0} |Q(\zeta)|\, d\zeta$. Then, using (5-9), for $|y| \le c_0^{-1}$ we obtain

$$\begin{aligned} M(y) &\ge -C_Q|y|^2 + c_H^{-1}\int_{|\zeta|\ge c_0} |\zeta|^{-(d+2)} P(\zeta)(1-\cos(\zeta_1|y|))\, d\zeta \\ &\ge -C_Q|y|^2 + c_H^{-1}|y|^2 \int_{|z|\ge c_0|y|} |z|^{-(d+2)} P(z|y|^{-1})(1-\cos(z_1))dz \\ &\ge -C_Q|y|^2 + c_H^{-1}|y|^2 \int_{2\ge|z|\ge 1} |z|^{-(d+2)} P(z|y|^{-1})(1-\cos(z_1))dz \\ &\ge -C_Q|y|^2 + c_H^{-1} 2^{-(d+2)}|y|^2 P(|y|^{-1}) \int_{2\ge|z|\ge 1} (1-\cos z_1)dz \\ &\ge -C_Q|y|^2 + 2C^{-1}|y|^2 P(|y|^{-1}) \end{aligned} \tag{5-13}$$

for some sufficiently large constant $C > 0$ that depends only on c_H and d. The assumption that $P(\zeta) \to \infty$ as $|\zeta| \to \infty$ combined with (5-13) shows that

$$M(y) \ge C^{-1}|y|^2 P(|y|^{-1})$$

holds for all sufficiently small $|y|$.

To conclude, we note that since $\widehat{M} = -Q = -(-\Delta)^{(d+2)/2}P$, we have that $K(y) = -P^{\vee}(y) = |y|^{-(d+2)}M(y)$ in the sense of tempered distributions, and hence we obtain that, for sufficiently small $|y|$, the bound $K(y) \geq C^{-1}|y|^{-d}P(|y|^{-1})$ holds, concluding the proof of the lemma. □

Remark 5.3 (examples of symbols P). The conditions (5-3)–(5-5) that were assumed on the symbol P in order to obtain the upper bound for the associated kernel are fairly common assumptions in Fourier analysis. For all symbols of interest to us in this paper, condition (5-9) also naturally holds. The dimension relevant to the SQG equation is $d = 2$. When $P(\zeta) = |\zeta|(\log(|\zeta|))^{-a}$ for sufficiently large $|\zeta|$ and $0 < a \leq 1$, corresponding to (1-4), one may verify that $(-\Delta)^2 P(\zeta)|\zeta|^4/P(\zeta) \to 1$ as $|\zeta| \to \infty$, so that we may take $c_H = 2$ in (5-9) if c_0 is sufficiently large. Thus condition (5-9) is not restrictive for the class of symbols we have in mind.

The proof of Theorem 1.4 combines the estimates in Lemmas 5.1 and 5.2 above with the argument given in Section 3. One complication arises due to the fact that (5-10) only holds for small enough $|y|$. In fact, for the class of multipliers P that we consider, positivity of the kernel K is not assured. Because of that, the L^∞ maximum principle is no longer available. However, there is an easy substitute which is sufficiently strong for our purpose.

Lemma 5.4. *Assume that a smooth function $\theta(x,t)$ solves* (5-2). *Suppose that the kernel $K(y)$ corresponding to the multiplier P via* (5-6) *satisfies $|K(y)| \leq C|y|^{-d}P(|y|^{-1})$ for all y and $K(y) \geq C^{-1}|y|^{-d}P(|y|^{-1})$ for all $|y| \leq 2\sigma$, where σ, C are positive constants. Then there exists $M_* = M_*(P,\theta_0)$ such that $\|\theta(x,t)\|_{L^\infty} \leq M$ for all $t \geq 0$.*

Proof. Letting $M(t) = \|\theta(\cdot\,,t)\|_{L^\infty}$, we prove that there exists $M_* \geq M(0)$, sufficiently large, such that $M(t) \leq M_*$ for all $t \geq 0$. If not, then, for any fixed M_*, there exists a $t_* > 0$ such that $M(t_*) = M_*$ and $\partial_t M(t_*) \geq 0$. For this fixed t_* let $\bar{x}$ be a point of maximum for $\theta(\cdot\,,t_*)$. We have

$$
\begin{aligned}
\mathscr{L}\theta(\bar{x}) &\geq \int_{\sigma\leq|y|\leq\infty} (\theta(\bar{x}) - \theta(\bar{x}+y))K(y)\,dy \\
&\geq cM_* \int_{\sigma\leq|y|\leq 2\sigma} \frac{P(|y|^{-1})}{|y|^d}\,dy - C\|\theta\|_{L^2(\mathbb{T}^d)} \biggl(\int_{\sigma\leq|y|} \frac{P(|y|^{-1})^2}{|y|^{2d}}\,dy\biggr)^{1/2} \\
&\geq cM_* P((2\sigma)^{-1}) - C\|\theta_0\|_{L^2(\mathbb{T}^d)} P(\sigma^{-1})\sigma^{-d/2}.
\end{aligned}
\tag{5-14}
$$

We used that $P \geq 0$ implies $\|\theta(\cdot\,,t)\|_{L^2(\mathbb{T}^d)} \leq \|\theta_0\|_{L^2(\mathbb{T}^d)}$ in the above calculation. The estimate (5-14) proves that $\partial_t M(t_*)$ must be negative if M_* is large enough, depending only on P (through σ and other constants) and θ_0. It follows that $M(t)$ will never exceed the larger of these bound or $\|\theta_0\|_{L^\infty}$. □

Proof of Theorem 1.4. The first two lemmas of this section show that, for the multiplier P satisfying (5-3)–(5-5) and (5-9), we have that $(P\hat{\theta})^{\vee}(x) = \int(\theta(x) - \theta(x+y))K(y)\,dy$, with K being radial and smooth away from zero. Moreover, K satisfies $|K(y)| \leq C|y|^{-d}P(|y|^{-1)}$ for all y and $K(y) \geq c|y|^{-d}P(|y|^{-1})$ for all $|y| \leq 2\sigma$, where C, c, σ are positive constants depending only on P.

Consider a smooth radially decreasing function $\varphi_0(y)$ that is identically 1 on $|y| \le \sigma$ and vanishes identically on $|y| \ge 2\sigma$. We decompose

$$K(y) = K(y)\varphi_0(y) + K(y)(1-\varphi_0(y)) =: K_1(y) + K_2(y),$$

so that

$$(P\hat{\theta})^{\vee}(x) = \int_{\mathbb{R}^d} (\theta(x)-\theta(x+y))K_1(y) + \int_{\mathbb{R}^d} (\theta(x)-\theta(x+y))K_2(y) =: \mathscr{L}_1\theta(x) + \mathscr{L}_2\theta(x).$$

The nonlocal operator $\mathscr{L}_1$ is of type (1-7), since by letting

$$m(r) = C^{-1} P(r^{-1})\varphi_0(r), \tag{5-15}$$

we have that

$$K_1(y) \ge m(|y|)|y|^{-d}$$

for all y and some $C > 0$. It is clear that the above defined m satisfies properties (1-2)–(1-3) and (1-10) in view of assumptions (1-15)–(1-17) imposed on P. Therefore, for $\mathscr{L}_1$, we will be able to directly use the estimate in Lemma 2.3, which relies only on lower bounds for the kernel associated to $\mathscr{L}_1$.

On the other hand, we observe that $K_2 \in L^1(\mathbb{R}^d)$ since $K_2(y) = 0$ for $y \le \sigma$, and we have

$$|K_2(y)| \le |K(y)| \le C P(|y|^{-1})|y|^{-d} \le \sigma^{\alpha} P(\sigma^{-1})|y|^{-d-\alpha}$$

for any $|y| \ge \sigma$, by using (1-16). Let us fix the constant $C_2 = \|K_2\|_{L^1(\mathbb{R}^d)}$. Then if $\theta(\cdot, t)$ obeys the modulus of continuity $\omega(\xi)$, it is clear that

$$|\mathscr{L}_2\theta(x,t) - \mathscr{L}_2\theta(y,t)| \le 2C_2 \min\{\omega(\xi), M_*\}, \tag{5-16}$$

where M_* is the L^∞ norm bound from Lemma 5.4, holds for all $x, y \in \mathbb{R}^d$, where $|x-y| = \xi$.

Now the argument of Section 3 goes through with minor changes. We provide an outline of the argument to verify this. First, similarly to (3-7) we may prove that for B large enough (now depending on σ as well) we have $\omega_B(\sigma) \ge 3M_* \ge 3\|\theta(\cdot,t)\|_{L^\infty}$, so that the modulus of continuity can only be broken at values of $\xi \in (0,\sigma)$. Let $\mathscr{D}_B$ and $\mathscr{D}_B^{\perp}$ be the bounds obtained from the dissipative operator $\mathscr{L}_1$ via Lemma 2.3. Note that the only contribution from the integral defining $\mathscr{D}_B$ that is used in the estimates is for $\eta \in (0,\xi)$ (see (3-10) and (3-14)), and for us $\xi < \sigma$ so all the bounds on the dissipation given in the proof of Theorem 1.1 require no modification. Therefore, provided κ and γ are chosen sufficiently small, we have

$$\min\{\Omega_B(\xi), \widetilde{\Omega}_B(\xi) + \Omega_B^{\perp}(\xi)\}\omega_B'(\xi) - (\tfrac{1}{2}\mathscr{D}_B(\xi) + \mathscr{D}_B^{\perp}(\xi)) < 0$$

for any $B \ge 1$ and $\xi \in (0,\sigma)$, exactly as in the proof of Theorem 1.1. The proof is hence completed once we show that the contribution of $\mathscr{L}_2$ is controlled:

$$2C_2 \min\{\omega_B(\xi), 2\|\theta\|_{L^\infty}\} \le \tfrac{1}{2}\mathscr{D}_B(\xi) \tag{5-17}$$

for any $\xi \in (0, \sigma)$ and any $B \geq 1$. The range $\xi \in (0, \delta(B))$ is clear, since here $\omega_B(\xi) \leq B\xi$ and by (3-14) we have

$$\mathfrak{D}_B(\xi) \geq -\frac{C}{A}\xi^2 m(\xi)\omega_B''(\xi) = \frac{CB^2}{2\kappa C_\alpha A}\xi\left(3+\ln\frac{\delta(B)}{\xi}\right) \geq \frac{3C}{2\kappa C_\alpha A}B\xi \geq 4C_2 B\xi \geq 4C_2\omega_B(\xi)$$

by letting κ be small enough (independent of $B \geq 1$).

We next consider the range $\xi \in (\delta(B), \sigma)$. In view of (3-10), we have $\mathfrak{D}_B(\xi) \geq C\omega_B(\xi)m(2\xi)$, where $C = (2-c_\alpha)/A$. Since $P(\zeta) \to \infty$ as $|\zeta| \to \infty$, we have that $Cm(2\xi) \geq 4C_2$, for all $\xi \in (\delta(B), \kappa)$, for some $\kappa > 0$. If $\kappa \geq \sigma$, the proof is completed, but this cannot be guaranteed, so we have to also consider the case $\kappa < \sigma$. For $\xi \in (\kappa, \sigma)$, we have

$$\mathfrak{D}_B(\xi) \geq C\omega_B(\xi)m(2\xi) \geq C\omega_B(\kappa)m(\sigma) \geq Cm(\sigma)\gamma\int_{\delta(B)}^{\kappa} m(2\eta)\,d\eta. \tag{5-18}$$

By making B large enough, we can ensure that the right hand side of (5-18) is larger than $2M_*$. □

6. Global well-posedness for a two-dimensional Euler-type equation with more singular velocity

In this section we address the issue of global regularity for the *inviscid* active scalar equation

$$\partial_t\theta - u\cdot\nabla\theta = 0, \tag{6-1}$$

$$u = \nabla^\perp\Lambda^{-2}P(\Lambda)\theta, \tag{6-2}$$

where the multiplier $P(\zeta) = P(|\zeta|)$ is a radially symmetric function which is smooth, nondecreasing, with $P(0) = 0$ and $P(\zeta) \to \infty$ as $|\zeta| \to \infty$. In addition, we assume that P satisfies a doubling property

$$P(2|\zeta|) \leq c_D P(|\zeta|) \tag{6-3}$$

for some doubling constant $c_D \geq 1$,

$$|\zeta|^{-\alpha}P(|\zeta|) \text{ is nonincreasing} \tag{6-4}$$

for some $\alpha \in (0, 1)$, and a Hörmander–Mikhlin type condition

$$|\partial_\zeta^k P(\zeta)|\,|\zeta|^{|k|} \leq c_H P(\zeta) \tag{6-5}$$

holds for some constant $c_H \geq 1$ and for all multi-indices $k \in \mathbb{Z}^d$ with $|k| \leq N$, where N depends only on the dimension d and on the doubling constant c_D. Condition (6-4) is quite natural in view of (6-7) below, while conditions (6-3) and (6-5) are standard in Fourier analysis. We remark that, while finalizing this paper, we learned of [Elgindi 2014], which proves a result very similar to the one proved in this section under slightly less restrictive assumptions on P.

Using the technique of Lemma 5.1, one may show using (6-3) and (6-5) that the convolution kernel K corresponding to the operator $\nabla^\perp\Lambda^{-2}P(\Lambda)$, that is, to the Fourier multiplier $i\zeta^\perp|\zeta|^{-2}P(|\zeta|)$, satisfies the estimates

$$|K(x)| \leq C|x|^{-d+1}P(|x|^{-1}), \quad |\nabla K(x)| \leq C|x|^{-d}P(|x|^{-1}), \quad |\nabla\Delta K(x)| \leq C|x|^{-d-2}P(|x|^{-1}) \tag{6-6}$$

for all $x \neq 0$. Moreover, we note that K integrates to 0 around the unit sphere, and hence convolution with K annihilates constants.

The study of Euler equations with more singular velocities, (6-1)–(6-2), was recently initiated by Chae, Constantin, and Wu [Chae et al. 2011]. They prove the global regularity for the loglog-Euler equation; namely, they prove global regularity in the case that arises when $P(\zeta) = [\ln(1 + \ln(1 + |\zeta|^2))]^\gamma$, for $\gamma \in [0, 1]$. Their approach relies on estimates for the Fourier localized gradient of the velocity for a particular class of symbols. Our aim here is to provide a proof of global regularity for a slightly more general class of symbols P, via the modulus of continuity method. The main result of this section is the following.

Theorem 6.1 (global regularity for the P-Euler equation). *Let P be a smooth radially symmetric function which is smooth and nondecreasing with $P(0) = 0$ and $P(\zeta) \to \infty$ as $|\zeta| \to \infty$ and satisfies assumptions* (6-3)–(6-5). *If θ_0 is periodic and smooth and we assume that*

$$\int_1^M \frac{dr}{r \ln(2r) P(r)} \to \infty, \quad \text{as } M \to \infty, \tag{6-7}$$

the $P-$Euler equation (6-1)–(6-2) *has a global in time smooth solution.*

Remark 6.2 (integral formulation). In fact, our proof provides a stronger result if we state the constitutive law relating u and θ in terms of an integrodifferential operator instead of a Fourier multiplier

$$u(x) = \int_{\mathbb{R}^d} \theta(x + y) K(y)\, dy,$$

where K is any kernel which satisfies the hypothesis (6-6) for any function P for which (6-3), (6-4), and (6-7) hold, but not necessarily (6-5).

In the previous sections, we constructed autonomous families of moduli of continuity preserved by the dynamics of the respective equations. In the inviscid case, we will construct a single modulus of continuity and then scale it autonomously. The following lemma makes the above observation precise.

Lemma 6.3 (modulus of continuity under pure transport). *Let u be a Lipschitz vector field and let θ solve the transport equation*

$$\partial_t \theta + u \cdot \nabla \theta = 0. \tag{6-8}$$

If $\theta_0 = \theta(\,\cdot\,, 0)$ has some modulus of continuity $\omega(\xi)$, then $\theta(\,\cdot\,, t)$ has the modulus of continuity $\omega(B(t)\xi)$, where $B(t)$ is given by

$$B(t) = \exp\left(\int_0^t \|\nabla u(\,\cdot\,, s)\|_{L^\infty} \right).$$

Equivalently, $B(t)$ solves $B(0) = 1$ and $\dot{B}(t) = \|\nabla u(\,\cdot\,, t)\|_{L^\infty} B(t)$.

Proof. The solution to the transport equation can be obtained by following the flow of the vector field backwards. Indeed, $\theta(x, t) = \theta_0(X(t))$ where X solves the ordinary differential equation

$$\dot{X}(s) = u(X(s), t - s), \quad X(0) = x.$$

If $X(t)$ and $Y(t)$ are two such trajectories starting at x and y, respectively, from Grönwall's inequality

$$|X(t)-Y(t)| \le \exp\biggl(\int_0^t \|\nabla u(\,\cdot\,,s)\|_{L^\infty}\biggr)|x-y| = B(t)|x-y|.$$

Therefore,

$$|\theta(x,t)-\theta(y,t)| \le |\theta_0(X(t))-\theta_0(Y(t))| \le \omega(B(t)|x-y|), \qquad \square$$

which concludes the proof of the lemma.

Proof of Theorem 6.1. Let us consider an initial data θ_0 whose Lipschitz L^∞ and L^2 norms are bounded by an arbitrary constant A. Applying Lemma 6.3 with $\omega(\xi) = A\xi$, we obtain that $\theta(\,\cdot\,,t)$ obeys the modulus of continuity $AB(t)\xi$, that is, it is Lipschitz continuous with Lipschitz constant

$$\|\nabla\theta(\,\cdot\,,t)\|_{L^\infty} \le A\,B(t), \tag{6-9}$$

where $B(0)=1$ and $\dot{B} = \|\nabla u(\,\cdot\,,t)\|_{L^\infty}\,B(t)$.

By the maximum principle, $\|\theta(\,\cdot\,,t)\|_{L^\infty} \le \|\theta_0\| \le A$ for any time t. Moreover, since u is divergence-free, $\|\theta(\,\cdot\,,t)\|_{L^2} \le \|\theta_0\|_{L^2} \le A$ for any time t. In order to combine the last two estimates, we have to estimate the Lipschitz norm of u at time t. Let $\varphi(y)$ be a radially nonincreasing nonnegative function that is constant 1 on $|y| \le 1/2$ and vanishes for $|y| \ge 1$. For some $r \in (0,1)$ to be chosen later, we split the integral defining ∇u into three pieces to estimate

$$\begin{aligned}
|\nabla u(x)| &= \left|\int_{\mathbb{R}^d} \nabla K(y)\theta(x+y)\,dy\right| \\
&\le \int_{\mathbb{R}^d} |\nabla(\varphi(y/r)K(y))|\,|\theta(x+y)-\theta(x)|\,dy + \int_{\mathbb{R}^d} |\nabla((1-\varphi(y/r))\varphi(y)K(y))|\,|\theta(x+y)|\,dy \\
&\qquad + \left|\int_{\mathbb{R}^d} \nabla((1-\varphi(y))K(y))\theta(x+y)\,dy\right|.
\end{aligned}$$

Using the bounds on K and its derivatives obtained in (6-6) and the fact that θ is Lipschitz with constant given by (6-9), we may further bound

$$\begin{aligned}
|\nabla u(x)| &\le C\int_{|y|\le r} \frac{P(|y|^{-1})}{|y|^d}|\theta(x+y)-\theta(x)|\,dy + C\int_{r/2\le|y|\le 1} \frac{P(|y|^{-1})}{|y|^d}|\theta(x+y)|\,dy \\
&\qquad + \int_{\mathbb{R}^d} |(-\Delta)\nabla((1-\varphi(y))K(y))|\,|(-\Delta)^{-1}\theta(x+y)|\,dy \qquad (6\text{-}10) \\
&\le CAB(t)\int_0^r P(\rho^{-1})\,d\rho + C\|\theta_0\|_{L^\infty}\int_{r/2}^1 \frac{P(\rho^{-1})}{\rho}\,d\rho \\
&\qquad + C\|(-\Delta)^{-1}\theta\|_{L^\infty(\mathbb{R}^d)}\int_{|y|\ge 1/2} \frac{P(|y|^{-1})}{|y|^{d+2}}\,dy \\
&\le CAB(t)rP(r^{-1}) + CAP(r^{-1})\ln\frac{2}{r} + CAP(2). \qquad (6\text{-}11)
\end{aligned}$$

In the last inequality above, we have additionally used two facts: first, that by (6-4) we have

$$\int_0^r P(\rho^{-1})\,d\rho \le CrP(r^{-1});$$

and second, that since θ is periodic and has zero mean on the torus, we can use the Sobolev inequality and estimate

$$\|(-\Delta)^{-1}\theta\|_{L^\infty(\mathbb{R}^d)} = \|(-\Delta)^{-1}\theta\|_{L^\infty(\mathbb{T}^d)} \le C\|\theta\|_{L^2(\mathbb{T}^d)} \le CA.$$

By choosing $r = B(t)^{-1}$ in (6-11), which is allowed since $B(0) = 1$ and $\dot{B} \ge 0$, we arrive at

$$\|\nabla u(\,\cdot\,, t)\|_{L^\infty} \le CA(1 + P(B(t))(1 + \ln 2B(t))).$$

Finally we rewrite the differential equation for $B(t)$ as

$$\dot{B}(t) = \|\nabla u(\,\cdot\,, t)\|_{L^\infty} B(t) \le CA(1 + P(B(t))(1 + \ln 2B(t)))B(t).$$

Clearly this ODE has a global in time solution *if and only if*

$$\int_1^\infty \frac{1}{r\ln(2r)P(r)}\,dr = \infty$$

holds, which finishes the proof. □

Appendix: Estimate on the dissipative operator at points of modulus breakdown

Here we prove Lemma 2.3. Our argument parallels that of [Kiselev 2011], but is slightly simpler and more general, as we use the integral representation of the diffusion generator $\mathscr{L}$ instead of generalized Poisson kernels employed in [Kiselev 2011]. We remark that a more general argument that allows one to also handle the Cordoba–Cordoba–Fontelos model has recently been given by Tam Do [2013].

Proof of Lemma 2.3. Due to translation invariance and radial symmetry, we may assume without loss of generality that

$$x = (\xi/2, 0) \quad \text{and} \quad y = (-\xi/2, 0).$$

For a point $(\eta, \nu) \in \mathbb{R}^2$, we write $K(\eta, \nu)$ for the dissipation kernel corresponding to $\mathscr{L}$. Then we have

$$\begin{aligned} &\mathscr{L}\theta\Big(\frac{\xi}{2}, 0\Big) - \mathscr{L}\theta\Big(-\frac{\xi}{2}, 0\Big) \\ &\qquad = \int_{\mathbb{R}}\int_{\mathbb{R}} \Big(\theta\Big(\frac{\xi}{2}, 0\Big) - \theta\Big(-\frac{\xi}{2}, 0\Big) - \theta\Big(\frac{\xi}{2} + \eta, \nu\Big) + \theta\Big(-\frac{\xi}{2} + \eta, \nu\Big)\Big) K(\eta, \nu)\,d\eta\,d\nu. \qquad \text{(A-1)} \end{aligned}$$

Note that since θ obeys the modulus of continuity ω, one may bound $\mathscr{L}\theta(\xi/2, 0) - \mathscr{L}\theta(-\xi/2, 0)$ from below by the expression on the right side of (A-1), with $K(\eta, \nu)$ replaced by $m(\sqrt{\eta^2 + \nu^2})(\eta^2 + \nu^2)^{-1}$. We will henceforth write K as a shortcut for the latter expression, and assume without loss of generality that K is radially nonincreasing and nonnegative, since so is m in (1-6).

$$
\begin{aligned}
&\mathscr{L}\theta\Big(\frac{\xi}{2},0\Big)-\mathscr{L}\theta\Big(-\frac{\xi}{2},0\Big)\\
&\quad=\int_{\mathbb{R}}\int_{\mathbb{R}}\Big(\omega(\xi)-\theta\Big(\frac{\xi}{2}+\eta,\nu\Big)+\theta(-\frac{\xi}{2}-\eta,\nu)\Big)K(\eta,\nu)\,d\eta\,d\nu\\
&\quad=\int_{\mathbb{R}}\int_{-\xi/2}^{\infty}\Big(\omega(\xi)-\theta\Big(\frac{\xi}{2}+\eta,\nu\Big)+\theta\Big(-\frac{\xi}{2}-\eta,\nu\Big)\Big)K(\eta,\nu)\,d\eta\,d\nu\\
&\qquad\qquad+\int_{\mathbb{R}}\int_{-\xi/2}^{\infty}\Big(\omega(\xi)-\theta\Big(-\frac{\xi}{2}-\eta,\nu\Big)+\theta\Big(\frac{\xi}{2}+\eta,\nu\Big)\Big)K(-\xi-\eta,\nu)\,d\eta\,d\nu\\
&\quad=\int_{\mathbb{R}}\int_{-\xi/2}^{\infty}\omega(\xi)(K(\eta,\nu)+K(-\xi-\eta,\nu))\\
&\qquad\qquad-\Big(\theta\Big(\frac{\xi}{2}+\eta,\nu\Big)-\theta\Big(-\frac{\xi}{2}-\eta,\nu\Big)\Big)(K(\eta,\nu)-K(\xi+\eta,\nu))\,d\eta\,d\nu\\
&\quad=\int_{\mathbb{R}}\int_{-\xi/2}^{\infty}\omega(\xi)(K(\eta,\nu)+K(-\xi-\eta,\nu))-\omega(\xi+2\eta)(K(\eta,\nu)-K(\xi+\eta,\nu))\,d\eta\,d\nu\\
&\qquad\qquad+\int_{\mathbb{R}}\int_{-\xi/2}^{\infty}\omega(\xi)(K(\eta,\nu)+K(-\xi-\eta,\nu))\\
&\qquad\qquad+\Big(\omega(\xi+2\eta)-\theta\Big(\frac{\xi}{2}+\eta,\nu\Big)+\theta\Big(-\frac{\xi}{2}-\eta,\nu\Big)\Big)(K(\eta,\nu)-K(\xi+\eta,\nu))\,d\eta\,d\nu\\
&\quad=: T^{\parallel}+T^{\perp}.
\end{aligned}
$$

Note that

$$K(\eta,\nu)-K(\xi+\eta,\nu)\geq 0$$

for $\eta\geq-\xi/2$ due to the monotonicity of K (or that of its lower bound). Hence, using that θ obeys the modulus of continuity ω, we see that $T^{\perp}\geq 0$. To obtain a useful lower bound for $T^{\perp}$, we only retain the singular piece centered about $\eta=0$. Changing variables $\eta+\xi/2\mapsto\eta$, we have

$$T^{\perp}=\int_{\mathbb{R}}\int_{0}^{\infty}(\omega(2\eta)-\theta(\eta,\nu)+\theta(-\eta,\nu))\Big(K\Big(\eta-\frac{\xi}{2},\nu\Big)-K\Big(\eta+\frac{\xi}{2},\nu\Big)\Big)\,d\eta\,d\nu. \tag{A-2}$$

When $|\nu|\leq\xi/4$ and $|\eta-\xi/2|\leq\xi/4$, using that m is radially nonincreasing, we have that

$$
\begin{aligned}
K(\eta-\xi/2,\nu)-K(\eta+\xi/2,\nu)&=\frac{m(\sqrt{(\eta-\xi/2)^2+\nu^2})}{(\eta-\xi/2)^2+\nu^2}-\frac{m(\sqrt{(\eta+\xi/2)^2+\nu^2})}{(\eta+\xi/2)^2+\nu^2}\\
&\geq m\Big(\sqrt{(\eta-\xi/2)^2+\nu^2}\Big)\Big(\frac{1}{(\eta-\xi/2)^2+\nu^2}-\frac{1}{(\eta+\xi/2)^2+\nu^2}\Big)\\
&\geq m\Big(\sqrt{(\eta-\xi/2)^2+\nu^2}\Big)\frac{1}{2((\eta-\xi/2)^2+\nu^2)}=\frac{K(\eta-\xi/2,\nu)}{2}.
\end{aligned}\tag{A-3}
$$

Inserting estimate (A-3) into expression (A-2) and recalling that θ obeys ω, we obtain

$$\begin{aligned}T^{\perp} &\geq \frac{1}{2}\int_{-\xi/4}^{\xi/4}\int_{\xi/4}^{3\xi/4}(\omega(2\eta)-\theta(\eta,\nu)+\theta(-\eta,\nu))K\left(\eta-\frac{\xi}{2},\nu\right)d\eta\,d\nu\\ &=\frac{1}{2}\int_{0}^{\xi/4}\int_{\xi/4}^{3\xi/4}(2\omega(2\eta)-\theta(\eta,\nu)+\theta(-\eta,\nu)-\theta(\eta,-\nu)+\theta(-\eta,-\nu))K\left(\eta-\frac{\xi}{2},\nu\right)d\eta\,d\nu=\frac{\mathcal{D}^{\perp}}{2}.\end{aligned}$$

On the other hand, the dissipation contribution from the direction parallel to $x-y$ may be rewritten as

$$\begin{aligned}T^{\parallel} &= \int_{\mathbb{R}}\int_{-\xi/2}^{\infty}\omega(\xi)(K(\eta,\nu)+K(-\xi-\eta,\nu))-\omega(\xi+2\eta)(K(\eta,\nu)-K(\xi+\eta,\nu))\,d\eta\,d\nu\\ &= \int_{\mathbb{R}}\int_{-\infty}^{-\xi/2}(\omega(\xi)+\omega(-\xi-2\eta))K(\eta,\nu)\,d\eta\,d\nu+\int_{\mathbb{R}}\int_{-\xi/2}^{\infty}(\omega(\xi)-\omega(\xi+2\eta))K(\eta,\nu)\,d\eta\,d\nu\\ &= \int_{-\infty}^{-\xi/2}(\omega(\xi)+\omega(-\xi-2\eta))\widetilde{K}(\eta)\,d\eta+\int_{-\xi/2}^{\infty}(\omega(\xi)-\omega(\xi+2\eta))\widetilde{K}(\eta)\,d\eta\\ &= \int_{0}^{\xi/2}(2\omega(\xi)-\omega(\xi+2\eta)-\omega(\xi-2\eta))\widetilde{K}(\eta)\,d\eta+\int_{\xi/2}^{\infty}(2\omega(\xi)-\omega(\xi+2\eta)+\omega(2\eta-\xi))\widetilde{K}(\eta)\,d\eta,\end{aligned}$$

where we have denoted

$$\widetilde{K}(\eta)=\int_{\mathbb{R}}K(\eta,\nu)\,d\nu.$$

Since ω is concave, the proof of the lemma is concluded once we establish the existence of a positive constant C such that

$$\widetilde{K}(\eta)\geq\frac{Cm(2\eta)}{\eta}$$

for all $\eta>0$. But this is immediate since m is nonincreasing, and hence

$$\int_{\mathbb{R}}K(\eta,\nu)\,d\nu\geq\int_{-\eta}^{\eta}K(\eta,\nu)\,d\nu\geq Cm(2\eta)\int_{-\eta}^{\eta}\frac{d\nu}{\eta^2+\nu^2}\geq\frac{Cm(2\eta)}{\eta}. \qquad \square$$

Remark A.1 (one-dimensional version). It is clear that this proof also holds in the one-dimensional case relevant for the Burgers equation. In fact this case is simpler since there is no need to introduce $\widetilde{K}$.

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THE NONLINEAR SCHRÖDINGER EQUATION GROUND STATES ON PRODUCT SPACES

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We study the nature of the nonlinear Schrödinger equation ground states on the product spaces $\mathbb{R}^n \times M^k$, where M^k is a compact Riemannian manifold. We prove that for small L^2 masses the ground states coincide with the corresponding $\mathbb{R}^n$ ground states. We also prove that above a critical mass the ground states have nontrivial M^k dependence. Finally, we address the Cauchy problem issue, which transforms the variational analysis into dynamical stability results.

1. Introduction

Our goal here is to study the nature of the nonlinear Schrödinger equation ground states when the problem is posed on the product spaces $\mathbb{R}^n \times M^k$, where M^k is a compact Riemannian manifold. We thus consider the Cauchy problems

$$\begin{cases} i\partial_t u - \Delta_{x,y} u - u|u|^\alpha = 0, & (t, x, y) \in \mathbb{R} \times \mathbb{R}^n_x \times M^k_y, \\ u(0, x, y) = \varphi(x, y), \end{cases} \tag{1-1}$$

where

$$\Delta_{x,y} = \sum_{j=1}^{n} \partial^2_{x_j} + \Delta_y$$

and Δ_y is the Laplace–Beltrami operator on M^k_y. Recall that the Laplace–Beltrami operator is defined in local coordinates by

$$\frac{1}{\sqrt{\det(g_{i,j}(y))}} \partial_{y_i} \sqrt{\det(g_{i,j}(y))}\, g^{i,j}(y) \partial_{y_j},$$

where $g^{i,j}(y) = (g_{i,j}(y))^{-1}$ and $g_{i,j}(y)$ is the metric tensor.

We assume that $0 < \alpha < 4/(n+k)$, which corresponds to L^2 subcritical nonlinearity. In this paper, we shall study the following two questions:

- the existence and stability of solitary waves for (1-1);
- the global well-posedness of the Cauchy problem associated to (1-1).

Equation (1-1) has two (at least formal) conservation laws: the energy

$$\mathscr{E}_{n,M^k,\alpha}(u) = \int_{M^k_y} \int_{\mathbb{R}^d_x} \left(\frac{1}{2} |\nabla_{x,y} u|^2 - \frac{1}{2+\alpha} |u|^{2+\alpha} \right) dx\, d\mathrm{vol}_{M^k_y}, \tag{1-2}$$

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and the L^2 mass,

$$\|u\|^2_{L^2(\mathbb{R}^n\times M^k)} = \int_{M^k_y}\int_{\mathbb{R}^n_x} |u|^2\,dx\,d\mathrm{vol}_{M^k_y}. \tag{1-3}$$

Here we denote by $d\mathrm{vol}_{M^k_y}$ the volume form on M^k. Recall that in local coordinates it can be written as $\sqrt{\det(g_{i,j}(y))}\,dy$. Moreover, the i-th component (in local coordinates) of the gradient $(\nabla_y u(y))$ is

$$g^{i,j}(y)\partial_{y_j}u.$$

One has the classical Gagliardo–Nirenberg inequality

$$\|u\|^{2+\alpha}_{L^{2+\alpha}(\mathbb{R}^n\times M^k)} \le C\|u\|^{\theta(\alpha)}_{H^1(\mathbb{R}^n\times M^k)}\|u\|^{2+\alpha-\theta(\alpha)}_{L^2(\mathbb{R}^n\times M^k)}, \tag{1-4}$$

where $\theta(\alpha) = (n+k)\alpha/2$. Thus $\theta(\alpha) < 2$ under our assumption $0<\alpha<4/(n+k)$. This implies that the conservation laws (1-2) and (1-3) imply a control on the H^1 norm which excludes an L^2 self-focusing blow-up, and thus one expects that (1-1) has well defined global dynamics. This problem seems quite delicate for a general M^k. However, if we replace M^k with $\mathbb{R}^k$, it is well known (see [Tsutsumi 1987; Cazenave 2003] and the references therein) that (1-1) has a global strong solution for every $L^2(\mathbb{R}^{n+k})$ initial data.

Our argument to construct stable solutions to (1-1) follows the one proposed in [Cazenave and Lions 1982]. Hence we shall look at the following minimization problems:

$$K^\rho_{n,M^k,\alpha} = \inf_{\substack{u\in H^1(\mathbb{R}^n\times M^k)\\ \|u\|_{L^2(\mathbb{R}^n\times M^k)}=\rho}} \mathscr{E}_{n,M^k,\alpha}(u) \tag{1-5}$$

and $\mathscr{E}_{n,M^k,\alpha}(u)$ is defined in (1-2). In the following we shall use the notation

$$\mathcal{M}^\rho_{n,M^k,\alpha} = \{v\in H^1(\mathbb{R}^n\times M^k) : \|v\|_{L^2(\mathbb{R}^n\times M^k)}=\rho \text{ and } \mathscr{E}_{n,M^k,\alpha}(v) = K^\rho_{n,M^k,\alpha}\}. \tag{1-6}$$

The first result we state concerns the compactness of minimizing sequences to (1-5).

Theorem 1.1. *Let M^k be a compact manifold and $0<\alpha<4/(n+k)$. Then*

$$K^\rho_{n,M^k,\alpha} > -\infty \quad \textit{and} \quad \mathcal{M}^\rho_{n,M^k,\alpha} \neq \varnothing \quad \textit{for all } \rho>0. \tag{1-7}$$

Also, for any sequence $u_j\in H^1(\mathbb{R}^n\times M^k)$ such that $\|u_j\|_{L^2(\mathbb{R}^n\times M^k)}=\rho$ and $\lim_{j\to\infty}\mathscr{E}_{n,M^k,\alpha}(u_j) = K^\rho_{n,M^k,\alpha}$, there exists a subsequence u_{j_l} and $\tau_l\in\mathbb{R}^n_x$ such that

$$u_{j_l}(x+\tau_l, y) \textit{ converges in } H^1(\mathbb{R}^n\times M^k). \tag{1-8}$$

The proof of Theorem 1.1 is based on the concentration compactness principle which will be given in the Appendix. Also, the following stability theorem follows from a standard argument, hence its classical proof will be recalled in the Appendix.

Theorem 1.2. *Let $\rho>0$ be fixed and n, M^k, α as in Theorem 1.1. Assume moreover that*

$$\textit{the Cauchy problem (1-1) is globally well posed for any data } \varphi\in\mathcal{U}, \tag{1-9}$$

where $\mathcal{U}$ *is an* $H^1(\mathbb{R}^n \times M^k)$*-neighborhood of* $\mathcal{M}^\rho_{n,M^k,\alpha}$. *Then the set* $\mathcal{M}^\rho_{n,M^k,\alpha}$ *is orbitally stable*; *that is, for all* $\epsilon > 0$, *there exists* $\delta = \delta(\epsilon) > 0$ *such that*, *for any* $\varphi \in \mathcal{U}$ *with* $\inf_{v \in \mathcal{M}^\rho_{n,M^k,\alpha}} \|\varphi - v\|_{H^1(\mathbb{R}^n \times M^k)} < \delta(\epsilon)$, *we have*

$$\sup_{t\in\mathbb{R}} \inf_{v\in\mathcal{M}^\rho_{n,M^k,\alpha}} \|u_\varphi(t) - v\|_{H^1(\mathbb{R}^n\times M^k)} < \epsilon,$$

where $u_\varphi(t, x, y)$ *is the unique global solution to* (1-1).

Let us emphasize that the stability result stated in Theorem 1.2 has two major defaults: the first one is that we don't have an explicit description of the minimizers $\mathcal{M}^\rho_{n,M^k,\alpha}$; the second one is that it is subordinated to (1-9), that is, the global well-posedness of the Cauchy problem (1-1). The main contributions of this paper concern a partial understanding of the aforementioned questions.

Notice that [Cazenave 2003] a special family of solutions to (1-1) is given by

$$u(t, x, y) = e^{-i\omega t} u_{n,\omega,\alpha}(x),$$

where $\omega > 0$ and $u_{n,\omega,\alpha}(x)$ is defined as the unique radial solution to

$$-\Delta_x u_{n,\omega,\alpha} + \omega u_{n,\omega,\alpha} = u_{n,\omega,\alpha}|u_{n,\omega,\alpha}|^\alpha, \quad u_{n,\omega,\alpha} \in H^1(\mathbb{R}^n_x),\ u_{n,\omega,\alpha}(x) > 0,\ x \in \mathbb{R}^n_x. \tag{1-10}$$

Next, we set

$$\mathcal{N}_{n,\omega,\alpha} = \{e^{i\theta} u_{n,\omega,\alpha}(x+\tau) : \tau \in \mathbb{R}^n, \theta \in \mathbb{R}\}. \tag{1-11}$$

Notice that there is a natural embedding $H^1(\mathbb{R}^n_x) \subset H^1(\mathbb{R}^n_x \times M^k_y)$. In fact, every function in $H^1(\mathbb{R}^n_x)$ can be extended in a trivial way with respect to the y variable on $\mathbb{R}^n_x \times M^k_y$, and this extension will belong to $H^1(\mathbb{R}^n \times M^k)$. In particular, since now the set $\mathcal{N}_{n,\omega,\alpha}$ defined in (1-11) will be considered without any further comment both as a subset of $H^1(\mathbb{R}^n_x)$ and as a subset of $H^1(\mathbb{R}^n_x \times M^k_y)$, by a rescaling argument, one can prove that the function

$$(0, \infty) \ni \omega \to \|u_{n,\omega,\alpha}\|^2_{L^2(\mathbb{R}^n_x)} \in (0, \infty)$$

is strictly increasing for any $0 < \alpha < 4/n$ and

$$\lim_{\omega\to\infty} \|u_{n,\omega,\alpha}\|_{L^2(\mathbb{R}^n_x)} = \infty \quad \text{and} \quad \lim_{\omega\to 0} \|u_{n,\omega,\alpha}\|_{L^2(\mathbb{R}^n_x)} = 0.$$

As a consequence, for any fixed $0 < \alpha < 4/n$, we have

$$\text{for all } \rho > 0 \text{ there exists a unique } \omega(\rho) > 0 \text{ such that } \|u_{n,\omega(\rho),\alpha}\|_{L^2(\mathbb{R}^n_x)} = \rho. \tag{1-12}$$

In the next theorem, the set $\mathcal{N}_{n,\omega,\alpha}$ is the one defined in (1-11) and $\mathcal{M}^\rho_{n,M^k,\alpha}$ is defined in (1-6).

Theorem 1.3. *Let* n, M^k, α *be as in Theorem 1.2. There exists* $\rho^* \in (0, \infty)$ *such that*

$$\mathcal{M}^\rho_{n,M^k,\alpha} = \mathcal{N}_{n,\omega(\rho/\sqrt{\mathrm{vol}(M^k)}),\alpha} \text{ for all } \rho < \rho^* \tag{1-13}$$

and

$$\mathcal{M}^\rho_{n,M^k,\alpha} \cap \mathcal{N}_{n,\omega(\rho/\sqrt{\mathrm{vol}(M^k)}),\alpha} = \varnothing \text{ for all } \rho > \rho^*, \tag{1-14}$$

where $\omega\big(\rho/\sqrt{\mathrm{vol}(M^k)}\big)$ *is uniquely defined in* (1-12). *In particular for* $\rho > \rho^*$ *the elements of* $\mathcal{M}^{\rho}_{n,M^k,\alpha}$ *depend in a nontrivial way on the* M^k *variable.*

By the approach of Weinstein [1986] one may expect that $\mathcal{N}_{n,\omega,\alpha}$ is stable under (1-1) for $\alpha < 4/n$ and ω small enough; see [Rousset and Tzvetkov 2012] for a recent related work. It should however be pointed out that in such a stability result one would not get the variational description of $\mathcal{N}_{n,\omega,\alpha}$ as is the case in Theorem 1.3 ($\alpha < 4/(n+k)$). We underline that, by combining Theorem 1.2 and Theorem 1.3, we get a stable set for large values of the mass ρ, and in general it is independent of the solitary waves associated to the nonlinear Schrödinger equation in $\mathbb{R}^n$.

Next we shall focus on the question of the global well-posedness of the Cauchy problem associated to (1-1) in the particular case $n \geq 1$, $k = 1$. For every $n > 1$ we fix the numbers

$$p := p(n,\alpha) = \frac{4(2+\alpha)}{n\alpha} \quad \text{and} \quad q := q(n,\alpha) = 2+\alpha,$$

and for every $T > 0$ we define the localized norms

$$\|u(t,x,y)\|_{X_T} \equiv \|u(t,x,y)\|_{L^p((-T,T);L^q(\mathbb{R}^n_x;H^1(M^1_y))} \tag{1-15}$$

and

$$\|u(t,x,y)\|_{Y_T} \equiv \|\nabla_x u\|_{L^p((-T,T);L^q(\mathbb{R}^n_x;L^2(M^1_y))}. \tag{1-16}$$

Theorem 1.4. *Let* $n \geq 1$ *be fixed and* $\alpha < 4/(n+1)$. *Then, for every initial data* $\varphi \in H^1(\mathbb{R}^n \times M^1)$, *the Cauchy problem* (1-1) *has a unique global solution* $u(t,x,y)$ *satisfying*

$$u(t,x,y) \in \mathscr{C}((-T,T); H^1(\mathbb{R}^n \times M^1)) \cap X_T \cap Y_T \quad \textit{for all } T > 0.$$

Remark 1.5. The main difficulty in the analysis of the Cauchy problem (1-1) (compared with the Cauchy problem in the euclidean space) is related to the fact that the propagator $e^{-it\Delta_{x,y}}$ on $\mathbb{R}^n \times M^1_y$ does not satisfy the Strichartz estimates which are available for the propagator $e^{-it\Delta_{\mathbb{R}^{n+k}}}$ on the euclidean space $\mathbb{R}^{n+k}$.

Let us now describe some other known cases when (1-1) is well posed in $H^1(\mathbb{R}^n \times M^k)$ under the assumption $\alpha < 4/(n+k)$. Using the analysis of [Burq et al. 2004; Burq et al. 2003], one may prove such a well-posedness result in the case $\mathbb{R} \times M^2$, that is, $n = 1$ and $k = 2$. Moreover, using the analysis of [Herr et al. 2010; Ionescu and Pausader 2012], one may also prove such a well-posedness result in the cases $\mathbb{R}^2 \times \mathbb{T}^2$ and $\mathbb{R} \times \mathbb{T}^3$, respectively.

Notation. Next we fix some notations. We denote by L^p_x and H^s_x the spaces $L^p(\mathbb{R}^n_x)$ and $H^s(\mathbb{R}^n_x)$, respectively. We also use the notation $L^p_{x,y} = L^p(\mathbb{R}^n_x \times M^k_y)$ and $L^p_x L^q_y = L^p(\mathbb{R}^n_x; L^q(M^k_y))$. If $v(t)$ is a time dependent function defined on $\mathbb{R}_t$ and valued in a Banach space X, we define

$$\|v\|^p_{L^p_t(X)} = \int_{\mathbb{R}} \|v(t)\|^p_X \, dt.$$

For every $p \in [1,\infty]$ we denote by $p' \in [1,\infty]$ its conjugate Hölder exponent. We denote by $e^{-it\Delta_{x,y}}$ the free propagator associated to the Schrödinger equation on $\mathbb{R}^n_x \times M^k_y$.

2. Some useful results on the euclidean space $\mathbb{R}^n_x$ with $n \geq 1$

In this section we recall some well-known facts (see [Cazenave 2003]) related to the following minimization problem on $\mathbb{R}^n_x$:

$$I^\rho_{n,\alpha} = \inf_{\substack{u \in H^1_x \\ \|u\|_{L^2_x} = \rho}} \mathscr{E}_{n,\alpha}(u), \tag{2-1}$$

where, for $\alpha < 4/n$,

$$\mathscr{E}_{n,\alpha}(u) = \frac{1}{2} \int_{\mathbb{R}^n_x} |\nabla_x u|^2 \, dx - \frac{1}{2+\alpha} \int_{\mathbb{R}^n_x} |u|^{2+\alpha} \, dx. \tag{2-2}$$

By an elementary rescaling argument we have

$$I^\rho_{n,\alpha} = \rho^{(8+4\alpha-2\alpha n)/(4-\alpha n)} I^1_{n,\alpha}. \tag{2-3}$$

It is well known that

$$-\infty < I^\rho_{n,\alpha} < 0, \quad \text{for all } \rho > 0, \tag{2-4}$$

and

$$\mathscr{M}^\rho_{n,\alpha} = \mathscr{N}_{n,\omega(\rho),\alpha}, \tag{2-5}$$

where $\mathscr{N}_{n,\omega,\alpha}$ is defined in (1-11),

$$\mathscr{M}^\rho_{n,\alpha} = \{u \in H^1_x \,|\, \|u\|_{L^2_x} = \rho \text{ and } \mathscr{E}_{n,\alpha}(u) = I^\rho_{n,\alpha}\} \tag{2-6}$$

and $\omega(\rho)$ is defined uniquely (see (1-12)) by the relation

$$\|u_{n,\omega(\rho),\alpha}\|_{L^2_x} = \rho.$$

We also recall that the functions $u_{n,\omega,\alpha}$ (defined as the unique radially symmetric and positive solution to (1-10)) satisfy the following Pohozaev type identity (for a proof of (2-7) see the proof of (3-21) in the next section):

$$\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 \, dx = \frac{\alpha n}{2(\alpha+2)} \int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha} \, dx. \tag{2-7}$$

On the other hand, if we multiply (1-10) by $u_{n,\omega,\alpha}$ and integrate by parts, we get

$$\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 \, dx + \omega \|u_{n,\omega,\alpha}\|^2_{L^2_x} = \int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha} \, dx,$$

which, in conjunction with (2-7), gives

$$\begin{aligned} \omega \|u_{n,\omega,\alpha}\|^2_2 = \frac{2\alpha+4-\alpha n}{\alpha n} & \int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 \, dx \\ &= \frac{4\alpha+8-2\alpha n}{\alpha n - 4} \left(\frac{1}{2} \int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2 \, dx - \frac{1}{2+\alpha} \int_{\mathbb{R}^n_x} |u_{n,\omega,\alpha}|^{2+\alpha} \, dx \right) \\ &= \frac{4\alpha+8-2\alpha n}{\alpha n - 4} I^{\|u_{n,\omega,\alpha}\|_{L^2_x}}_{n,\alpha} \end{aligned} \tag{2-8}$$

(in the last step we have used the fact that due to (2-5) we have that $u_{n,\omega,\alpha}$ is a minimizer for $\mathscr{E}_{n,\alpha}$ on its associated constrained).

Finally notice that by (2-7) we deduce

$$I_{n,\alpha}^{\|u_{n,\omega,\alpha}\|_{L^2_x}} = \mathscr{E}_{n,\alpha}(u_{n,\omega,\alpha}) = \frac{\alpha n-4}{2\alpha n}\int_{\mathbb{R}^n_x} |\nabla_x u_{n,\omega,\alpha}|^2\,dx. \tag{2-9}$$

3. An auxiliary problem

In this section we study the minimizers of the minimization problems

$$J_{n,M^k,\alpha,\lambda} = \inf_{\substack{u\in H^1(\mathbb{R}^n\times M^k)\\ \|u\|_{L^2_{x,y}}=1}} \mathscr{E}_{n,M^k,\alpha,\lambda}(u), \tag{3-1}$$

where

$$\mathscr{E}_{n,M^k,\alpha,\lambda}(u) = \int_{M^k_y}\int_{\mathbb{R}^n_x}\left(\frac{\lambda}{2}|\nabla_y u|^2 + \tfrac{1}{2}|\nabla_x u|^2 - \frac{1}{2+\alpha}|u|^{2+\alpha}\right)dx\,d\mathrm{vol}_{M^k_y}.$$

We also introduce the sets

$$\mathscr{M}_{n,M^k,\alpha,\lambda} = \{w\in H^1(\mathbb{R}^n\times M^k) : \|w\|_{L^2_{x,y}} = 1 \text{ and } \mathscr{E}_{n,M^k,\alpha,\lambda}(w) = J_{n,M^k,\alpha,\lambda}\}.$$

Theorem 3.1. *Let n, M^k, and $0<\alpha<4/(n+k)$ be given. There exists $\lambda^*\in(0,\infty)$ such that*

$$\mathscr{M}_{n,M^k,\alpha,\lambda} = \mathscr{N}_{n,\bar\omega,\alpha} \quad \textit{for all } \lambda>\lambda^* \tag{3-2}$$

and

$$\mathscr{M}_{n,M^k,\alpha,\lambda}\cap\mathscr{N}_{n,\bar\omega,\alpha} = \varnothing \quad \textit{for all } \lambda<\lambda^*, \tag{3-3}$$

where $\bar\omega$ is defined by the condition

$$\mathrm{vol}(M^k)\|u_{n,\bar\omega,\alpha}\|^2_{L^2_x} = 1.$$

We fix a sequence $\lambda_j\to\infty$ and a corresponding sequence of functions $u_{\lambda_j}\in\mathscr{M}_{n,M^k,\alpha,\lambda_j}$. In the sequel we shall assume that

$$u_{\lambda_j}(x,y)\ge 0 \quad \text{for all } (x,y)\in\mathbb{R}^n_x\times M^k_y. \tag{3-4}$$

Indeed, it is well known that if u_{λ_j} is a minimizer, $|u_{\lambda_j}|$ is also a minimizer. In particular there exists at least one minimizer which satisfies (3-4).

Notice that the functions u_{λ_j} depend in principle on the full set of variables (x,y). Our aim is to prove that, for j large and up to subsequence, the functions u_{λ_j} will not depend explicitly on the variable y.

First we prove some a priori bounds satisfied by $u_{\lambda_j}(x,y)$. Recall that the quantities $I^\rho_{n,\alpha}$ are defined in (2-1).

Lemma 3.2. *Make the same assumptions as in Theorem 3.1. Then we have*

$$\lim_{j\to\infty} J_{n,M^k,\alpha,\lambda_j} = \mathrm{vol}(M^k)\,I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}} \tag{3-5}$$

and

$$\lim_{j\to\infty} \lambda_j \int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} = 0. \tag{3-6}$$

Proof. First notice that

$$J_{n,M^k,\alpha,\lambda_j} \le \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}. \tag{3-7}$$

In fact, let $w(x) \in H_x^1$ be such that $\|w\|_{L_x^2} = 1/\sqrt{\mathrm{vol}(M^k)}$ and $\mathscr{E}_{n,\alpha}(w) = I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$. Then we easily get

$$\begin{aligned} J_{n,M^k,\alpha,\lambda_j} \le \mathscr{E}_{n,M^k,\alpha,\lambda_j}(w(x)) &= \mathrm{vol}(M^k)\biggl(\frac{1}{2}\int_{\mathbb{R}_x^n} |\nabla_x w|^2 \, dx - \frac{1}{2+\alpha}\int_{\mathbb{R}_x^n} |w|^{2+\alpha} \, dx\biggr) \\ &= \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}, \end{aligned}$$

which concludes the proof of (3-7).

Next we claim that

$$\lim_{j\to\infty} \int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} = 0. \tag{3-8}$$

Assume for a contradiction that this is false. Then there exists a subsequence of λ_j (that we still denote by λ_j) such that

$$\lim_{j\to\infty} \lambda_j = \infty \quad \text{and} \quad \int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} \ge \epsilon_0 > 0,$$

and, in particular,

$$\lim_{j\to\infty} (\lambda_j - 1) \int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} = \infty. \tag{3-9}$$

On the other hand, by the classical Gagliardo–Nirenberg inequality (see (1-4)) we deduce the existence of $0 < \mu < 2$ such that

$$\begin{aligned} &\frac{1}{2}\int_{M_y^k}\int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) \, dx \, d\mathrm{vol}_{M_y^k} - \frac{1}{2+\alpha}\int_{M_y^k}\int_{\mathbb{R}_x^n} |v|^{2+\alpha} \, dx \, d\mathrm{vol}_{M_y^k} \\ &\quad \ge \frac{1}{2}\int_{M_y^k}\int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) \, dx \, d\mathrm{vol}_{M_y^k} - C\biggl[\int_{M_y^k}\int_{\mathbb{R}_x^n} (|\nabla_y v|^2 + |\nabla_x v|^2 + |v|^2) \, dx \, d\mathrm{vol}_{M_y^k}\biggr]^\mu \\ &\quad \ge \inf_{t>0}(1/2t^2 - Ct^\mu) = C(\mu) > -\infty \end{aligned}$$

for all $v \in H^1(\mathbb{R}^n \times M^k)$ such that $\|v\|_{L_{x,y}^2} = 1$. By the previous inequality we get

$$\mathscr{E}_{n,M^k,\alpha,\lambda_j}(v) - \tfrac{1}{2}(\lambda_j - 1)\int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_y v|^2 \ge -\tfrac{1}{2} + C(\mu)$$

for all $v \in H^1(\mathbb{R}^n \times M^k)$ such that $\|v\|_{L_{x,y}^2} = 1$. In particular, if we choose $v = u_{\lambda_j}$, we get

$$J_{n,M^k,\alpha,\lambda_j} = \mathscr{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \ge \tfrac{1}{2}(\lambda_j - 1)\int_{M_y^k}\int_{\mathbb{R}_x^n} |\nabla_y u_{\lambda_j}|^2 \, dx \, d\mathrm{vol}_{M_y^k} - \tfrac{1}{2} + C(\mu).$$

By (3-9) this implies $\lim_{n\to\infty} J_{n,M^k,\alpha,\lambda_j} = \infty$, which is in contradiction with (3-7). Hence (3-8) is proved.

Next we introduce the functions

$$w_j(y) = \|u_{\lambda_j}(x, y)\|^2_{L^2_x}.$$

Notice that

$$\|w_j(y)\|_{L^1_y} = 1 \tag{3-10}$$

and, moreover,

$$\begin{aligned}\int_{M^k_y} |\nabla_y w_j(y)|\, d\mathrm{vol}_{M^k_y} &\le C \int_{M^k_y} \int_{\mathbb{R}^n_x} |u_{\lambda_j}(x, y)|\,|\nabla_y u_{\lambda_j}(x, y)|\, dx\, d\mathrm{vol}_{M^k_y} \\ &\le C \|u_{\lambda_j}\|_{L^2_{x,y}} \|\nabla_y u_{\lambda_j}\|_{L^2_{x,y}}.\end{aligned}$$

Hence, due to (3-8), we get

$$\lim_{j\to\infty} \|\nabla_y w_j\|_{L^1_y} = 0. \tag{3-11}$$

By combining (3-10) and (3-11) with the Rellich compactness theorem and with the Sobolev embedding $W^{1,1}(M^1) \subset L^\infty(M^1)$ and $W^{1,1}(M^2) \subset L^2(M^2)$, we deduce in the cases $k = 1$ and $k = 2$ that (up to a subsequence)

$$\lim_{j\to\infty} \|w_j(y) - 1/\operatorname{vol}(M^1)\|_{L^r_y} = 0 \quad \text{for all } 1 \le r < \infty \tag{3-12}$$

and

$$\lim_{j\to\infty} \|w_j(y) - 1/\operatorname{vol}(M^2)\|_{L^r_y} = 0 \quad \text{for all } 1 \le r < 2, \tag{3-13}$$

respectively. For $k > 2$ we use the Sobolev embedding $H^1(M^k) \subset L^{2k/(k-2)}(M^k)$ and we get

$$\sup_j \|u_{\lambda_j}\|_{L^2_x L^{2k/(k-2)}_y} \le C \sup_j \|u_{\lambda_j}\|_{L^2_x H^1(M^k_y)} < \infty$$

(where in the last step we have used the fact that $\sup_j(\|u_{\lambda_j}\|_{L^2_{x,y}} + \|\nabla_y u_{\lambda_j}\|_{L^2_{x,y}}) < \infty$). By the Minkowski inequality the bound above implies $\sup_j \|u_{\lambda_j}\|_{L^{2k/(k-2)}_y L^2_x}$, which is equivalent to the condition

$$\sup_j \|w_j(y)\|_{L^{k/(k-2)}_y} < \infty \quad \text{for } k > 2. \tag{3-14}$$

By combining (3-10) and (3-11) with the Rellich compactness theorem, we deduce that up to a subsequence

$$\|w_j(y) - 1/\operatorname{vol}(M^k)\|_{L^1_y} = 0 \quad \text{for } k > 2,$$

and hence, by interpolation with (3-14), we get

$$\|w_j(y) - 1/\operatorname{vol}(M^k)\|_{L^r_y} = 0 \quad \text{for } k > 2,\ 1 \le r < k/(k-2). \tag{3-15}$$

By the definition of $I^\rho_{n,\alpha}$ (see (2-1)) and (2-3) we get

$$\begin{aligned}&\frac{1}{2}\int_{\mathbb{R}^n_x} |\nabla_x u_{\lambda_j}(x, y)|^2\, dx - \frac{1}{2+\alpha}\int_{\mathbb{R}^n_x} |u_{\lambda_j}(x, y)|^{2+\alpha}\, dx \\ &\qquad \ge I^{\|u_{\lambda_j}(\cdot,y)\|_{L^2_x}}_{n,\alpha} = I^1_{n,\alpha} \|u_{\lambda_j}(\cdot, y)\|^{(8+4\alpha-2\alpha n)/(4-\alpha n)}_{L^2_x} = I^1_{n,\alpha} w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)}\end{aligned} \tag{3-16}$$

for all $y \in M^k$ and all $j \in \mathbb{N}$. Next notice that, by definition,

$$\begin{aligned} J_{n,M^k,\alpha,\lambda_j} &= \mathscr{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j}) \\ &= \frac{1}{2}\int_{M_y^k}\int_{\mathbb{R}_x^n}(\lambda_j|\nabla_y u_{\lambda_j}|^2+|\nabla_x u_{\lambda_j}|^2)\,dx\,dy - \frac{1}{2+\alpha}\int_{M_y^k}\int_{\mathbb{R}_x^n}|u|^{2+\alpha}\,dx\,d\mathrm{vol}_{M_y^k}, \end{aligned} \tag{3-17}$$

and we can continue

$$\begin{aligned} \cdots &\geq \int_{M_y^k}\left(\frac{1}{2}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}(x,y)|^2\,dx - \frac{1}{2+\alpha}\int_{\mathbb{R}_x^n}|u_{\lambda_j}(x,y)|^{2+\alpha}\,dx\right)d\mathrm{vol}_{M_y^k} \\ &\geq I_{n,\alpha}^1\int_{M_y^k} w_j(y)^{(4+2\alpha-\alpha n)/(4-\alpha n)}\,d\mathrm{vol}_{M_y^k} \\ &= I_{n,\alpha}^1\,\mathrm{vol}(M^k)\,\mathrm{vol}(M^k)^{-(4+2\alpha-\alpha n)/(4-\alpha n)}+o(1), \end{aligned} \tag{3-18}$$

where $o(1)\to 0$ as $j\to\infty$ and in the last step we have combined (3-12), (3-13), and (3-15) for $k=1$, $k=2$, and $k>2$, respectively, and we used our assumption on α. By combining this fact with (2-3), we have

$$\liminf_{j\to\infty} J_{n,M^k,\alpha,\lambda_j} \geq \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}. \tag{3-19}$$

Hence (3-5) follows by combining (3-7) with (3-19).

Next we prove (3-6). For that purpose, it suffices to keep the term $\lambda_j|\nabla_y u_{\lambda_j}|^2$ in the previous analysis. Namely, by combining (3-5) with (3-17) and (3-18), we get

$$\mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}} + g(j) \geq \tfrac{1}{2}\lambda_j\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k} + h(j), \tag{3-20}$$

where

$$\lim_{j\to\infty} g(j) = 0 \quad\text{and}\quad \liminf_{j\to\infty} h(j) \geq \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}.$$

Hence (3-6) follows by (3-20). □

Lemma 3.3. *We have the identity*

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k} = \frac{\alpha n}{2(2+\alpha)}\int_{M_y^k}\int_{\mathbb{R}_x^n}|u_{\lambda_j}|^{2+\alpha}\,dx\,d\mathrm{vol}_{M_y^k}. \tag{3-21}$$

Moreover, there exist $J \in \mathbb{N}$ such that for all $j > J$ there exists $\omega(\lambda_j) > 0$ such that

$$-\lambda_j\Delta_y u_{\lambda_j} - \Delta_x u_{\lambda_j} + \omega(\lambda_j)u_{\lambda_j} = u_{\lambda_j}|u_{\lambda_j}|^\alpha, \tag{3-22}$$

and the following limit exists:

$$\lim_{j\to\infty}\omega(\lambda_j) = \bar{\omega} \in (0,\infty). \tag{3-23}$$

Proof. Since u_{λ_j} is a constrained minimizer for $\mathscr{E}_{n,M^k,\alpha,\lambda_j}$ on the ball of size 1 in $L^2(\mathbb{R}^n\times M^k)$, we get

$$\frac{d}{d\epsilon}[\mathscr{E}_{n,M^k,\alpha,\lambda_j}(\epsilon^{n/2}u_{\lambda_j}(\epsilon x, y)]_{\epsilon=1} = 0,$$

which is equivalent to

$$\frac{d}{d\epsilon}\left[\tfrac{1}{2}\lambda_j\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}+\tfrac{1}{2}\epsilon^2\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}-\frac{1}{2+\alpha}\epsilon^{\alpha n/2}\|u_{\lambda_j}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right]_{\epsilon=1}=0.$$

By computing explicitly the derivative (in ϵ), we deduce (3-21).

Next notice that by using the Lagrange multiplier technique we get (3-22) for a suitable $\omega(\lambda_j)\in\mathbb{R}$. On the other hand, by (3-22), we get

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}(\lambda_j|\nabla_y u_{\lambda_j}|^2+|\nabla_x u_{\lambda_j}|^2)\,dx\,d\mathrm{vol}_{M_y^k}+\omega(\lambda_j)\|u_{\lambda_j}\|^2_{L^2_{x,y}}=\int_{M_y^k}\int_{\mathbb{R}_x^n}|u_{\lambda_j}|^{2+\alpha}\,dx\,d\mathrm{vol}_{M_y^k},$$

which, by (3-21), gives

$$\omega(\lambda_j)=\frac{-\alpha n+4+2\alpha}{\alpha n}\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}-\lambda_j\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k},$$

and hence, by (3-6), we get

$$\omega(\lambda_j)=\frac{-\alpha n+4+2\alpha}{\alpha n}\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}+o(1), \tag{3-24}$$

where $\lim_{j\to\infty}o(1)=0$.

On the other hand, notice that, by (3-21), we get

$$J_{n,M^k,\alpha,\lambda_j}=\mathscr{E}_{n,M^k,\alpha,\lambda_j}(u_{\lambda_j})=\frac{\alpha n-4}{2\alpha n}\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}+\frac{1}{2}\int_{M_y^k}\int_{\mathbb{R}_x^n}\lambda_j|\nabla_y u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k},$$

and by (3-6)

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}=\frac{2\alpha n}{\alpha n-4}J_{n,M^k,\alpha,\lambda_j}+o(1). \tag{3-25}$$

By (3-5) this implies

$$\int_{M_y^k}\int_{\mathbb{R}_x^n}|\nabla_x u_{\lambda_j}|^2\,dx\,d\mathrm{vol}_{M_y^k}=\frac{2\alpha n}{\alpha n-4}\,\mathrm{vol}(M^k)\,I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}+o(1), \tag{3-26}$$

which, in conjunction with (3-24) and (2-4), implies $\omega(\lambda_j)>0$ for j large enough. Moreover, (3-23) follows by (3-24) and (3-26). □

Next recall that the sets $\mathcal{M}^{\rho}_{n,\alpha}$ are the ones defined in (2-6).

Lemma 3.4. *Let $\bar{\omega}$ be as in* (3-23) *and let $v(x)\in\mathcal{M}_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$ be such that $v(x)>0$. Then*

$$-\Delta_x v+\bar{\omega}v=v|v|^{\alpha}.$$

Proof. It is well known that

$$-\Delta_x v+\omega_1 v=v|v|^{\alpha}$$

for a suitable $\omega_1 > 0$. More precisely, we can assume that up to translation $v = u_{n,\omega_1,\alpha}$. Our aim is to prove that $\omega_1 = \bar{\omega}$. Notice that, by (2-8),

$$\omega_1 \frac{1}{\mathrm{vol}(M^k)} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{\|v\|_{L_x^2}} = \frac{4\alpha + 8 - 2\alpha n}{\alpha n - 4} I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}. \tag{3-27}$$

On the other hand, by (3-24) and (3-26), we get

$$\omega(\lambda_j) = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}} + o(1),$$

and hence, passing to the limit in j, we get

$$\bar{\omega} = \frac{-2\alpha n + 8 + 4\alpha}{\alpha n - 4} \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}. \tag{3-28}$$

By combining (3-27) and (3-28), we get $\bar{\omega} = \omega_1$. □

Lemma 3.5. *There exist a subsequence of λ_j (that we shall denote still by λ_j) and a sequence $\tau_j \in \mathbb{R}_x^n$ such that*

$$\lim_{j \to \infty} \|u_{\lambda_j}(x + \tau_j, y) - u_{\bar{\omega}}\|_{H^1(\mathbb{R}^n \times M^k)} = 0,$$

where $u_{\bar{\omega}} \in \mathcal{N}_{n,\bar{\omega},\alpha}$, $u_{\bar{\omega}} > 0$ and $\bar{\omega}$ is defined in (3-23).

Proof. By combining (3-6) and (3-26), and since $\|u_{\lambda_j}\|_{L_{x,y}^2} = 1$, we deduce that u_{λ_j} is bounded in $H^1(\mathbb{R}^n \times M^k)$. Moreover, by combining (3-5) with the fact that $I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}} < 0$ (see (2-4)), we get

$$\inf_j \|u_{\lambda_j}\|_{L_{x,y}^{2+\alpha}} > 0.$$

By using the localized version of the Gagliardo–Nirenberg inequality (A-5) (in the same spirit as in the Appendix), we get the existence (up to subsequence) of $\tau_j \in \mathbb{R}_x^n$ such that

$$u_{\lambda_j}(x + \tau_j, y) \rightharpoonup w \neq 0 \quad \text{in } H^1(\mathbb{R}^n \times M^k).$$

Moreover, due to (3-4), we can assume that

$$w(x, y) \geq 0 \quad \text{a.e. in } (x, y) \in \mathbb{R}_x^n \times M_y^k,$$

and by (3-6) we get $\nabla_y w = 0$. In particular w is y-independent.

By combining (3-6) and (3-23), we pass to the limit in (3-22) in the distribution sense, and we get

$$-\Delta_x w + \bar{\omega} w = w|w|^\alpha \text{ in } \mathbb{R}_x^n, \quad w(x) \geq 0, \ w \neq 0. \tag{3-29}$$

We claim that

$$\|w\|_{L_x^2} = \frac{1}{\sqrt{\mathrm{vol}(M^k)}}. \tag{3-30}$$

If not, we can assume $\|w\|_{L_x^2} = \beta < 1/\sqrt{\mathrm{vol}(M^k)}$, and since w solves (3-29) by (2-5), we get

$$w \in \mathcal{M}_{n,\alpha}^{\beta}. \tag{3-31}$$

On the other hand, by Lemma 3.4, (3-29) is satisfied by any $v \in \mathcal{M}_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$. Hence, again by (2-5) and by the injectivity of the map $\rho \to \omega(\rho)$ (see (1-12)), we deduce that, necessarily, $\beta = 1/\sqrt{\mathrm{vol}(M^k)}$.

In particular, by (3-30), we deduce

$$\lim_{j\to\infty} \|u_{\lambda_j}(x+\tau_j, y) - w\|_{L^2_{x,y}} = 0.$$

Next notice that, by (3-6) and since we have already proved that $\nabla_y w = 0$, we can deduce that

$$\lim_{j\to\infty} \|\nabla_y u_{\lambda_j}(x+\tau_j, y)\|_{L^2_{x,y}} = 0 = \|\nabla_y w\|_{L^2_{x,y}}.$$

Hence, in order to conclude that $u_{\lambda_j}(x+\tau_j, y)$ converges strongly to w in $H^1(\mathbb{R}^n \times M^k)$, it is sufficient to prove that

$$\lim_{j\to\infty} \|\nabla_x u_{\lambda_j}(x+\tau_j, y)\|_{L^2_{x,y}} = \sqrt{\mathrm{vol}(M^k)}\|\nabla_x w\|_{L^2_x} = \|\nabla_x w\|_{L^2_{x,y}}.$$

This last fact follows by combining (2-9) (where we use the fact that $w \in \mathcal{N}_{n,\bar\omega,\alpha}$ by (3-29) and $\|w\|_{L^2_x} = 1/\sqrt{\mathrm{vol}(M^k)}$ by (3-30)) and (3-26). □

Lemma 3.6. *There exists $j_0 > 0$ such that*

$$\nabla_y u_{\lambda_j} = 0 \quad \textit{for all } j > j_0.$$

Proof. By Lemma 3.5 we can assume that

$$u_{\lambda_j} \to u_{\bar\omega} \quad \text{in } H^1(\mathbb{R}^n \times M^k). \tag{3-32}$$

We introduce $w_j = \sqrt{-\Delta_y} u_{\lambda_j}$. Notice that due to (3-22) the functions w_j satisfy

$$-\lambda_j \Delta_y w_j - \Delta_x w_j + \omega(\lambda_j) w_j = \sqrt{-\Delta_y}(u_{\lambda_j}|u_{\lambda_j}|^\alpha), \tag{3-33}$$

which, after multiplication by w_j, implies

$$\int_{M_y^k}\int_{\mathbb{R}_x^n} [\lambda_j |\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \omega(\lambda_j)|w_j|^2 - \sqrt{-\Delta_y}(u_{\lambda_j}|u_{\lambda_j}|^\alpha) w_j]\, dx\, d\mathrm{vol}_{M_y^k} = 0. \tag{3-34}$$

In turn this gives

$$\begin{aligned}
0 = &\int_{M_y^k}\int_{\mathbb{R}_x^n} (\lambda_j - 1)|\nabla_y w_j|^2 - (\alpha+1)\sqrt{-\Delta_y}(u_{\lambda_j}|u_{\bar\omega}|^\alpha) w_j \, dx\, d\mathrm{vol}_{M_y^k} \\
&+ \int_{M_y^k}\int_{\mathbb{R}_x^n} (|\nabla_y w_j|^2 + |\nabla_x w_j|^2 + \bar\omega |w_j|^2 + \sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha)) w_j \, dx\, d\mathrm{vol}_{M_y^k} \\
&+ \int_{M_y^k}\int_{\mathbb{R}_x^n} (\omega(\lambda_j) - \bar\omega)|w_j|^2 \, dx\, dy \equiv I_j + II_j + III_j.
\end{aligned} \tag{3-35}$$

Next we fix an orthonormal basis of eigenfunctions for $-\Delta_y$, that is, $-\Delta_y \varphi_k = \mu_k \varphi_k$ and $\varphi_0 = \text{const}$.

We can write the following development:

$$w_j(x,y)=\sum_{k\in\mathbb{N}\setminus\{0\}} a_{j,k}(x)\varphi_k(y) \tag{3-36}$$

(where the eigenfunction φ_0 does not enter in the development). By using the representation in (3-36), we get

$$I_j \geq \sum_{k\neq 0}(\lambda_j-1)|\mu_k|^2\int_{\mathbb{R}^n_x}|a_{j,k}(x)|^2\,dx-(\alpha+1)\sum_{k\neq 0}\int_{\mathbb{R}^n_x}|u_{\bar\omega}(x)|^\alpha|a_{j,k}(x)|^2\,dx, \tag{3-37}$$

and by (3-23) we get

$$III_j=o(1)\|w_j\|^2_{L^2_{x,y}}. \tag{3-38}$$

By combining (3-37) with (3-38), we get

$$I_j+III_j\geq 0 \tag{3-39}$$

for j large enough. In order to estimate II_j, notice that, by the Cauchy–Schwartz inequality, we get

$$\begin{aligned}\left|\int_{M^k_y}\int_{\mathbb{R}^n_x}\sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha))w_j\,dx\,d\mathrm{vol}_{M^k_y}\right|&\\ \leq\|\sqrt{-\Delta_y}(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha))\|_{L_x^{2(n+k)/(n+k+2)}L_y^{2(n+k)/(n+k+2)}}\|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}}&\\ \leq C\|\nabla_y(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha))\|_{L_x^{2(n+k)/(n+k+2)}L_y^{2(n+k)/(n+k+2)}}\|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}},&\end{aligned} \tag{3-40}$$

where in the last step we have used the following estimate: for all $p\in(1,\infty)$ there exist $c(p), C(p)>0$ such that

$$c(p)\|\sqrt{-\Delta_y}f\|_{L^p_y}\leq\|\nabla_y f\|_{L^p_y}\leq C(p)\|\sqrt{-\Delta_y}f\|_{L^p_y}. \tag{3-41}$$

Indeed, using [Sogge 1993, Theorem 3.3.1], we have that $\sqrt{-\Delta_y}$ is a first-order classical pseudodifferential operator on M with a principal symbol $(g^{i,j}(y)\xi_i\xi_j)^{1/2}$. Observe that

$$C_1\sum_{i,j}g^{i,j}(y)\xi_i\xi_j\leq\sum_i\left|\sum_j g^{i,j}(y)\xi_j\right|^2\leq C_2|\xi|^2\leq C_3\sum_{i,j}g^{i,j}(y)\xi_i\xi_j.$$

Moreover, one can assume that in (3-41) f has no zero frequency. Then one can deduce (3-41) by working in local coordinates, introducing a classical angular partition of unity according to the index $l\in[1,\dots,k]$ such that

$$\sum_{i,j}g^{i,j}(y)\xi_i\xi_j\leq c\left|\sum_j g^{l,j}(y)\xi_j\right|^2,$$

and, most importantly, using the L^p boundedness of zero-order pseudodifferential operators on $\mathbb{R}^k$ (for the proof of this fact we refer to [Sogge 1993, Theorem 3.1.6]).

Next, by the chain rule, we get

$$\nabla_y\big(u_{\lambda_j}((\alpha+1)|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha)\big)=(\alpha+1)\nabla_y u_{\lambda_j}(|u_{\bar\omega}|^\alpha-|u_{\lambda_j}|^\alpha),$$

and by the Hölder inequality we can continue the estimate (3-40):

$$\cdots \leq C \Big\| \, \|\nabla_y u_{\lambda_j}\|_{L^q_y} \| |u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha \|_{L^r_y} \Big\|_{L_x^{2(n+k)/(n+k+2)}} \|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}},$$

where

$$\frac{1}{q}+\frac{1}{r}=\frac{n+k+2}{2(n+k)},$$

and, again by the Hölder inequality in the x-variable, we can continue

$$\cdots \leq C \|\nabla_y u_{\lambda_j}\|_{L^q_{x,y}} \| |u_{\bar\omega}|^\alpha - |u_{\lambda_j}|^\alpha \|_{L^r_{x,y}} \|w_j\|_{L_{x,y}^{2(n+k)/(n+k-2)}}.$$

Notice that if we fix

$$q=\frac{2(n+k)}{n+k-2} \quad \text{and} \quad r=\frac{n+k}{2},$$

then, by combining the Sobolev embedding

$$H^1_{x,y} \subset L_{x,y}^{2(n+k)/(n+k-2)} \tag{3-42}$$

with (3-32) and (3-41), we can continue the estimate:

$$\cdots \leq o(1)\|\sqrt{-\Delta_y}\, u_{\lambda_j}\|_{L^q_{x,y}} \|w_j\|_{H^1_{x,y}} = o(1)\|w_j\|^2_{H^1_{x,y}},$$

where $\lim_{j\to\infty} o(1)=0$. By combining this information in conjunction with the structure of II_j, we get

$$II_j \geq \|w_j\|^2_{H^1_{x,y}}(1-o(1)) \geq 0 \quad \text{for } j > j_0. \tag{3-43}$$

By combining (3-35), (3-39), and (3-43), we deduce $w_j = 0$ for j large enough. □

Proof of Theorem 3.1. By using the diamagnetic inequality, we deduce that (up to a remodulation factor $e^{i\theta}$) we can assume that $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ is real valued. Moreover, if $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$, then also $|v| \in \mathcal{M}_{n,M^k,\alpha,\lambda}$. By a standard application of the strong maximum principle, we finally deduce that it is not restrictive to assume that $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ and $v(x,y)>0$ for all $(x,y)\in \mathbb{R}^n_x \times M^k_y$.

First step: *there exists $\tilde\lambda>0$ such that for all $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$, $v(x,y)>0$ we have $\nabla_y v = 0$ for all $\lambda > \tilde\lambda$.* Assume that the conclusion is false. Then there exists $\lambda_j \to \infty$ such that $u_{\lambda_j}(x,y) \in M_{n,M^k,\alpha,\lambda_j}$, $u_{\lambda_j}(x,y) > 0$ and $\nabla_y u_{\lambda_j} \neq 0$. This is absurd due to Lemma 3.6.

Second step: *conclusion.* We define

$$\lambda^* = \inf_\lambda \{\lambda > 0 : \nabla_y v = 0 \text{ for all } v \in \mathcal{M}_{n,M^k,\alpha,\lambda}\}.$$

By the first step, $\lambda^* < \infty$. Moreover, it is easy to deduce that if $\lambda > \lambda^*$, the minimizers of the problem $J_{n,M^k,\alpha,\lambda}$ are precisely the same minimizers as those of the problem $I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$, which in turn are characterized in Section 2 (hence we get (3-2)).

Next we prove that $\lambda^* > 0$. It is sufficient to show that

$$\lim_{\lambda\to 0} J_{n,M^k,\alpha,\lambda} < \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}} \tag{3-44}$$

(see (2-1) and (3-1) for a definition of the quantities involved in the inequality above). Let us fix $\rho(y) \in C^\infty(M^k)$ such that

$$\int_{M^k} |\rho|^2 \, d\mathrm{vol}_{M^k_y} = 1$$

and $\rho^2(y_0) \neq 1/\mathrm{vol}(M^k)$ for some $y_0 \in M^k$ (that is, $\rho(y)$ is not identically constant). Then we introduce the functions

$$\psi(x, y) = \rho(y)^{4/(4-\alpha n)} Q(\rho(y)^{\frac{2\alpha}{4-\alpha n}} x),$$

where $Q(x)$ is the unique radially symmetric minimizer for $I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$. Then we get

$$\|\psi(x, y)\|^2_{L^2_x} = (\rho(y))^2 \quad \text{and} \quad \mathscr{E}_{n,\alpha}(\psi(x, y)) = I^1_{n,\alpha}(\rho(y))^{\frac{8+4\alpha-2\alpha n}{(4-\alpha n)}},$$

and, as a consequence, we deduce

$$\begin{aligned}
\int_{M^k_y} \int_{\mathbb{R}^n_x} \left(\frac{1}{2} |\nabla_x \psi(x, y)|^2 - \frac{1}{2+\alpha} |\psi(x, y)|^{2+\alpha} \right) dx \, d\mathrm{vol}_{M^k_y} \\
&= I^1_{n,\alpha} \int_{M^k_y} (\rho(y))^{\frac{8+4\alpha-2\alpha n}{4-\alpha n}} \, d\,\mathrm{vol}_{M^k_y} \\
&< I^1_{n,\alpha} \left(\int_{M^k} (\rho(y))^2 \, d\mathrm{vol}_{M^k_y} \right)^{\frac{4-\alpha n+2\alpha}{4-\alpha n}} \mathrm{vol}(M^k)^{-\frac{2\alpha}{4-\alpha n}} = I^1_{n,\alpha} \, \mathrm{vol}(M^k)^{-\frac{2\alpha}{4-\alpha n}},
\end{aligned}$$

where in the last inequality we have used the fact that $I^1_{n,\alpha} < 0$ in conjunction with the Hölder inequality (moreover, we get the inequality $<$ since by hypothesis $\rho(y)$ is not identically constant). As a byproduct we get

$$\lim_{\lambda \to 0} \mathscr{E}_{n,M^k,\alpha,\lambda}(\psi(x, y)) < I^1_{n,\alpha} \, \mathrm{vol}(M^k)^{-2\alpha/(4-\alpha n)} = \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}$$

(where we have used (2-3)), which in turn implies (3-44).

Let us finally prove (3-3). It is sufficient to show that if $v \in \mathcal{M}_{n,M^k,\alpha,\lambda}$ for $\lambda < \lambda^*$, then $\nabla_y v \neq 0$. Assume for a contradiction that this is false. Then we get $\lambda_1 < \lambda^*$ and $v_1 \in \mathcal{M}_{n,M^k,\alpha,\lambda_1}$ such that $\nabla_y v_1 = 0$. Arguing as above implies that

$$J_{n,M^k,\alpha,\lambda_1} = \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}}. \tag{3-45}$$

On the other hand, by the definition of λ^*, there exists $\lambda_2 \in (\lambda_1, \lambda^*]$ and $v_2 \in \mathcal{M}_{n,M^k,\alpha,\lambda_2}$ such that $\nabla_y v_2 \neq 0$. As a consequence, we deduce that

$$J_{n,M^k,\alpha,\lambda_1} < \mathscr{E}_{n,M^k,\alpha,\lambda_2}(v_2) = J_{n,M^k,\alpha,\lambda_2} \leq \mathrm{vol}(M^k) I_{n,\alpha}^{1/\sqrt{\mathrm{vol}(M^k)}},$$

where in the last step we have used (3-7). Hence we get a contradiction with (3-45). □

4. Proof of Theorem 1.3

The homogeneity of the euclidean space $\mathbb{R}^n$ will play a key role in the sequel. Due to this property we shall be able to reduce the proof of Theorem 1.3 to the problem studied in the previous section.

In view of Section 2 it is sufficient to prove that there exists $\rho^* > 0$ such that

$$v \in \mathcal{M}^{\rho}_{n,M^k,\alpha} \quad \text{implies} \quad \nabla_y v = 0 \text{ for } \rho < \rho^* \tag{4-1}$$

and

$$v \in \mathcal{M}^{\rho}_{n,M^k,\alpha} \quad \text{implies} \quad \nabla_y v \neq 0 \text{ for } \rho > \rho^*. \tag{4-2}$$

By an elementary computation, we have that the map

$$S_1 \ni u \to \rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y) \in S_\rho,$$

where

$$S_\lambda = \{v \in H^1(\mathbb{R}^n \times M^k) : \|v\|_{L^2_{x,y}} = \lambda\}$$

is a bijection. Moreover, we have

$$\begin{aligned}
&\mathscr{E}_{n,M_k,\alpha}\big(\rho^{4/(4-\alpha n)} u(\rho^{2\alpha/(4-\alpha n)} x, y)\big) \\
&\quad = \rho^{(8-2\alpha n)/(4-\alpha n)} \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y u|^2 \,dx\, d\mathrm{vol}_{M^k_y} + \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x u|^2 \,dx\, d\mathrm{vol}_{M^k_y} \\
&\qquad\qquad - \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \frac{1}{2+\alpha} \int_{M^k_y}\int_{\mathbb{R}^n_x} |u|^{2+\alpha} \,dx\, d\mathrm{vol}_{M^k_y} \\
&\quad = \rho^{(8-2\alpha n+4\alpha)/(4-\alpha n)} \bigg(\frac{1}{2}\rho^{-4\alpha/(4-\alpha n)} \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y u|^2 \,dx\, d\mathrm{vol}_{M^k_y} \\
&\qquad\qquad + \frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x u|^2 - \frac{1}{2+\alpha}|u|^{2+4/d} \,dx\, d\mathrm{vol}_{M^k_y} \bigg).
\end{aligned}$$

In particular, (4-1) and (4-2) are satisfied provided that there exists $\rho^* > 0$ such that

$$v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \quad \text{implies} \quad \nabla_y v = 0 \text{ for } \rho < \rho^* \tag{4-3}$$

and

$$v \in \mathcal{M}_{n,M^k,\alpha,\rho^{-4\alpha/(4-\alpha n)}} \quad \text{implies} \quad \nabla_y v \neq 0 \text{ for } \rho > \rho^*, \tag{4-4}$$

which in turn follow by Theorem 3.1.

5. Proof of Theorem 1.4

The main tool we use is the following Strichartz type estimate (whose proof follows by [Tzvetkov and Visciglia 2012]).

Proposition 5.1. *For every manifold M^k_y, $n \geq 1$ and $p, q \in [2, \infty]$ such that*

$$\frac{2}{p} + \frac{n}{q} = \frac{n}{2}, \quad (p, n) \neq (2, 2),$$

there exists $C>0$ such that

$$\|e^{-it\Delta_{x,y}}f\|_{L_t^pL_x^qH_y^1}+\left\|\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L_t^pL_x^qH_y^1}\le C(\|f\|_{L_x^2H_y^1}+\|F\|_{L_t^{p'}L_x^{q'}H_y^1}), \tag{5-1}$$

$$\|\nabla_x e^{-it\Delta_{x,y}}f\|_{L_t^pL_x^qL_y^2}+\left\|\nabla_x\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L_t^pL_x^qL_y^2}\le C(\|\nabla_x f\|_{L_x^2L_y^2}+\|\nabla_x F\|_{L_t^{p'}L_x^{q'}L_y^2}), \tag{5-2}$$

and

$$\left\|\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L_t^pL_x^qL_y^2}\le C\|F\|_{L_t^{p'}L_x^{q'}L_y^2}. \tag{5-3}$$

Moreover,

$$\|e^{-it\Delta_{x,y}}f\|_{L_t^\infty L_x^2H_y^1}+\left\|\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L_t^\infty L_x^2H_y^1}\le C(\|f\|_{L_x^2H_y^1}+\|F\|_{L_t^{p'}L_x^{q'}H_y^1}) \tag{5-4}$$

and

$$\|\nabla_x e^{-it\Delta_{x,y}}f\|_{L_t^\infty L_x^2L_y^2}+\left\|\nabla_x\int_0^t e^{-i(t-s)\Delta_{x,y}}F(s)\,ds\right\|_{L_t^\infty L_x^2L_y^2}\le C(\|\nabla_x f\|_{L_x^2L_y^2}+\|\nabla_x F\|_{L_t^{p'}L_x^{q'}L_y^2}). \tag{5-5}$$

Next we shall use the norms $\|\cdot\|_{X_T}$ and $\|\cdot\|_{Y_T}$ introduced in (1-15) and (1-16) for time dependent functions. We also introduce the space Z_T whose norm is defined by

$$\|v\|_{Z_T}\equiv\|v\|_{X_T}+\|v\|_{Y_T}$$

and the nonlinear operator associated to the Cauchy problem (1-1):

$$\mathcal{T}_\varphi(u)\equiv e^{-it\Delta_{x,y}}\varphi+\int_0^t e^{-i(t-s)\Delta_{x,y}}u(s)|u(s)|^\alpha\,ds.$$

We split the proof of Theorem 1.4 in several steps.

5A. ***Local well-posedness.*** We devote this subsection to proving the following: for all $\varphi\in H^1(\mathbb{R}^n\times M^1)$ there exists a $T=T(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)})>0$ and there exists a unique $v(t,x)\in Z_T\cap\mathscr{C}((-T,T);H^1(\mathbb{R}^n\times M^1))$ such that $\mathcal{T}_\varphi v(t)=v(t)$ for all $t\in(-T,T)$

First step: *for all $\varphi\in H^1(\mathbb{R}^n\times M^1)$ there exist $T=T(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)})>0$, $R=R(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)})>0$ such that $\mathcal{T}_\varphi(B_{Z_{\widetilde{T}}}(0,R))\subset B_{Z_{\widetilde{T}}}(0,R)$ for all $\widetilde{T}<T$.* First we estimate the nonlinear term:

$$\|u|u|^\alpha\|_{L_t^{p'}L_x^{q'}H_y^1}\le\big\|\|u^\alpha(t,x,\cdot)\|_{L_y^\infty}\|u(t,x,\cdot)\|_{H_y^1}\big\|_{L_t^{p'}L_x^{q'}}$$

(where (p,q) is the couple in (1-15) and (1-16)). After applying the Hölder inequality in (t,x), we get

$$\cdots\le\|u\|_{L_t^pL_x^qH_y^1}\|u\|^\alpha_{L_t^{\alpha\tilde p}L_x^{\alpha\tilde q}L_y^\infty}\le C\|u\|_{L_t^pL_x^qH_y^1}\|u\|^\alpha_{L_t^{\alpha\tilde p}L_x^{\alpha\tilde q}H_y^1},$$

where we have used the embedding $H_y^1\subset L_y^\infty$ and we have chosen

$$\frac{1}{\tilde p}+\frac{1}{p}=1-\frac{1}{p}\quad\text{and}\quad\frac{1}{\tilde q}+\frac{1}{q}=1-\frac{1}{q}.$$

By direct computation we have

$$\alpha\tilde{q} = q \quad \text{and} \quad \alpha\tilde{p} < p. \tag{5-6}$$

By combining the nonlinear estimate above with (5-1), (5-6), and the Hölder inequality (in the time variable), we get

$$\|\mathcal{T}_\varphi u\|_{X_T} \le C(\|\varphi\|_{L^2_x H^1_y} + T^{a(d)}\|u\|^{1+\alpha}_{X_T}) \tag{5-7}$$

with $a(d) > 0$.

Arguing as above, we get

$$\|\nabla_x(u|u|^\alpha)\|_{L^{p'}_t L^{q'}_x L^2_y} \le C\|\nabla_x u\|_{L^p_t L^q_x L^2_y}\|u^\alpha\|_{L^{\tilde{p}}_t L^{\tilde{q}}_x L^\infty_y} \le C\|u\|_{Y_T}\|u\|^\alpha_{L^{\alpha\tilde{p}}_t L^{\alpha\tilde{q}}_x H^1_y},$$

where $\tilde{p}$ and $\tilde{q}$ are as above and we have used the embedding $H^1_y \subset L^\infty_y$. As a consequence of this estimate and (5-2), we get

$$\|\mathcal{T}_\varphi u\|_{Y_T} \le C(\|\nabla_x\varphi\|_{L^2_{x,y}} + T^{a(d)}\|u\|_{Y_T}\|u\|^\alpha_{X_T}) \tag{5-8}$$

with $a(d) > 0$.

By combining (5-7) with (5-8), we get

$$\|\mathcal{T}_\varphi u\|_{Z_T} \le C(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)} + T^{a(d)}\|u\|_{Z_T}\|u\|^\alpha_{Z_T}).$$

The proof follows by a standard continuity argument.

Next we introduce the norm

$$\|w(t,x,y)\|_{\widetilde{Z}_T} \equiv \|w(t,x,y)\|_{L^p((-T,T);L^q_x L^2_y)}.$$

and we shall prove the following.

Second step: *Let $T, R > 0$ as in the previous step. Then there exists $T' = T'(\|\varphi\|_{H^1(\mathbb{R}^n\times M^1)}) < T$ such that $\mathcal{T}_\varphi$ is a contraction on $B_{Z_{T'}}(0,R)$ endowed with the norm $\|\cdot\|_{\widetilde{Z}_{T'}}$.* It is sufficient to prove

$$\|\mathcal{T}_\varphi v_1 - \mathcal{T}_\varphi v_2\|_{\widetilde{Z}_T} \le CT^{a(d)}\|v_1 - v_2\|_{\widetilde{Z}_T}\sup_{i=1,2}\{\|v_i\|_{Z_T}\}^\alpha \tag{5-9}$$

with $a(d) > 0$. Notice that we have

$$\begin{aligned}\|v_1|v_1|^\alpha - v_2|v_2|^\alpha\|_{L^{p'}((-T,T);L^{q'}_x L^2_y)} &\le C\big\|\|v_1 - v_2\|_{L^2_y}(\|v_1\|_{L^\infty_y} + \|v_2\|_{L^\infty_y})^\alpha\big\|_{L^{p'}((-T,T);L^{q'}_x)}\\ &\le CT^{a(d)}\|v_1 - v_2\|_{\widetilde{Z}_T}\sup_{i=1,2}\{\|v_i\|_{Z_T}\}^\alpha,\end{aligned}$$

where we have used the Sobolev embedding $H^1_y \subset L^\infty_y$ and the Hölder inequality in the same spirit as in the proof of (5-7) and (5-8). We conclude by combining the estimate above with the Strichartz estimate (5-3).

Third step: *existence and uniqueness of the solution in $Z_{T'}$, where T' is as in the previous step.* We apply the contraction principle to the map $\mathcal{T}_\varphi$ defined on the complete space $B_{Z_{T'}}(0,R)$ endowed with the topology induced by $\|\cdot\|_{\widetilde{Z}_{T'}}$. It is well known that this space is complete.

Fourth step: *regularity of the solution*. By combining the previous steps with the fixed point argument, we get the existence of a solution $v \in Z_{T'}$. In order to get the regularity $v \in \mathscr{C}((-T', T'); H^1(\mathbb{R}^n \times M^1))$, it is sufficient to argue as in the first step (to estimate the nonlinearity) in conjugation with the Strichartz estimates (5-4) and (5-5).

5B. ***Global well-posedness.*** Next we prove that the local solution (whose existence has been proved above) cannot blow up in finite time. The argument is standard and follows from the conservation laws

$$\|u(t)\|_{L^2_{x,y}} \equiv \|\varphi\|_{L^2_{x,y}}, \tag{5-10}$$

$$\mathscr{E}_{n,M^1,\alpha}(u(t)) + \tfrac{1}{2}\|u(t)\|^2_{L^2_{x,y}} \equiv \mathscr{E}_{n,M^1,\alpha}(\varphi) + \tfrac{1}{2}\|\varphi\|^2_{L^2_{x,y}}, \tag{5-11}$$

where $\mathscr{E}_{n,M^1,\alpha}$ is defined in (1-2). By the Gagliardo–Nirenberg inequality we deduce

$$\mathscr{E}_{n,M^1,\alpha}(u(t)) + \tfrac{1}{2}\|u(t)\|^2_{L^2_{x,y}} \geq \tfrac{1}{2}\|u(t)\|^2_{H^1(\mathbb{R}^n\times M^1)} - C\|u(t)\|^{2+\alpha-\mu}_{L^2_{x,y}}\|u(t)\|^{\mu}_{H^1(\mathbb{R}^n\times M^1)}$$

for a suitable $\mu \in (0, 2)$. By combining the estimate above with (5-10) and (5-11), we get

$$\tfrac{1}{2}\|u(t)\|^2_{H^1(\mathbb{R}^n\times M^1)} - C\|\varphi\|^{2+\alpha-\mu}_{L^2_{x,y}}\|u(t)\|^{\mu}_{H^1(\mathbb{R}^n\times M^1)} \leq \mathscr{E}_{n,M^1,\alpha}(\varphi) + \tfrac{1}{2}\|\varphi\|^2_{L^2_{x,y}}.$$

Since $\mu \in (0, 2)$, it implies that $\|u(t)\|_{H^1(\mathbb{R}^n\times M^1)}$ cannot blow up in finite time.

Appendix

For the sake of completeness we prove in this appendix Theorems 1.1 and 1.2. Our argument is heavily inspired by [Cazenave and Lions 1982] even if, in our opinion, the following presentation of Theorem 1.1 is simpler compared with the original one.

Proof of Theorem 1.1. For any given $\rho > 0$ we shall denote by $u_{j,\rho} \in H^1(\mathbb{R}^n \times M^k)$ any constrained minimizing sequence, that is,

$$\|u_{j,\rho}\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}) = K^{\rho}_{n,M^k,\alpha}. \tag{A-1}$$

Next we split the proof into many steps.

First step: $K^{\rho}_{n,M^k,\alpha} > -\infty$ *and* $\sup_j \|u_{j,\rho}\|_{H^1_{x,y}} < \infty$ *for all* $\rho > 0$. By the classical Gagliardo–Nirenberg inequality (see (1-4)) we get the existence of $\mu \in (0, 2)$ such that

$$\begin{aligned}\mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}) + \tfrac{1}{2}\rho^2 &\geq \frac{1}{2}\int_{M^k_y}\int_{\mathbb{R}^n}(|\nabla_{x,y}u_{j,\rho}|^2 + |u_{j,\rho}|^2)\,dx\,d\mathrm{vol}M^k_y - C(\rho)\|u_{j,\rho}\|^{\mu}_{H^1(\mathbb{R}^m\times M^k)}\\ &\geq \inf_{t>0}(1/2t^2 - C(\rho)t^{\mu}) > -\infty.\end{aligned}$$

The conclusion follows by a standard argument.

Second step: *the map* $(0,\infty)\ni\rho\to K^{\rho}_{n,M^k,\alpha}$ *is continuous*. Fix $\rho\in(0,\infty)$ and let $\rho_j\to\rho$. Then we have

$$
\begin{aligned}
K^{\rho_j}_{n,M^k,\rho} &\le \mathscr{E}_{n,M^k,\alpha}\left(\frac{\rho_j}{\rho}u_{j,\rho}\right)\\
&=\left(\frac{\rho_j}{\rho}\right)^2\left(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\left(\frac{\rho_j}{\rho}\right)^{\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right)\\
&=\left(\frac{\rho_j}{\rho}\right)^2\left(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right)+\frac{1}{2+\alpha}\left(\frac{\rho_j}{\rho}\right)^2\left(1-\left(\frac{\rho_j}{\rho}\right)^{\alpha}\right)\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\\
&=\left(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right)+\left(\left(\frac{\rho_j}{\rho}\right)^2-1\right)\left(\frac{1}{2}\|\nabla_{x,y}u_{j,\rho}\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right)\\
&\qquad+\frac{1}{2+\alpha}\left(\frac{\rho_j}{\rho}\right)^2\left(1-\left(\frac{\rho_j}{\rho}\right)^{\alpha}\right)\|u_{j,\rho}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}.
\end{aligned}
$$

Since we are assuming that $\rho_j\to\rho$ and $\sup_n\|u_{j,\rho}\|_{H^1(\mathbb{R}^n\times M^k)}<\infty$ (see the first step), we get

$$\limsup_{j\to\infty}K^{\rho_j}_{n,M^k,\alpha}\le K^{\rho}_{n,M^k,\alpha}.$$

To prove the opposite inequality, let us fix $u_j\in H^1(\mathbb{R}^n\times M^k)$ such that

$$\|u_j\|_{L^2_{x,y}}=\rho_j\quad\text{and}\quad\mathscr{E}_{n,M^k,\alpha}(u_j)<K^{\rho_j}_{n,M^k,\alpha}+\frac{1}{j}. \tag{A-2}$$

By looking at the proof of the first step, we also deduce that u_j can be chosen in such a way that

$$\sup_j\|u_j\|_{H^1(\mathbb{R}^n\times M^k)}<\infty. \tag{A-3}$$

Then we can argue as above and we get

$$
\begin{aligned}
K^{\rho}_{n,M^k,\alpha}\le\mathscr{E}_{n,M^k,\alpha}\left(\frac{\rho}{\rho_j}u_j\right)&=\left(\frac{1}{2}\|\nabla_{x,y}u_j\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right)\\
&+\left(\left(\frac{\rho}{\rho_j}\right)^2-1\right)\left(\frac{1}{2}\|\nabla_{x,y}u_j\|^2_{L^2_{x,y}}-\frac{1}{2+\alpha}\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right)+\frac{1}{2+\alpha}\left(\frac{\rho}{\rho_j}\right)^2\left(1-\left(\frac{\rho}{\rho_j}\right)^{\alpha}\right)\|u_j\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}.
\end{aligned}
$$

By using (A-2), (A-3), and the assumption $\rho_j\to\rho$, we get

$$K^{\rho}_{n,M^k,\alpha}\le\liminf_{j\to\infty}K^{\rho_j}_{n,M^k,\alpha}.$$

Third step: *for every* $\rho>0$ *we have (up to subsequence)* $\inf_j\|u_{j,\rho}\|_{L^{2+\alpha}_{x,y}}>0$. It is sufficient to prove that $K^{\rho}_{n,M^k,\alpha}<0$. In fact, we have

$$K^{\rho}_{n,M^k,\alpha}\le\mathrm{vol}(M^k)\mathscr{E}_{n,\alpha}(u_{n,\omega,\alpha})=\mathrm{vol}(M^k)I^{\rho/\sqrt{\mathrm{vol}(M^k)}}_{n,\alpha}<0, \tag{A-4}$$

where $\mathscr{E}_{n,\alpha}$ is the energy defined in (2-2) and ω is chosen in such a way that $\|u_{n,\omega,\alpha}\|_{L^2_x}=\rho/\sqrt{\mathrm{vol}(M^k)}$. Notice that in (A-4) we have used (2-4) and (2-5).

Fourth step: *for any minimizing sequence $u_{j,\rho}$, there exists $\tau_j \in \mathbb{R}^n$ such that (up to subsequence) $u_{j,\rho}(x+\tau_j, y)$ has a weak limit $\bar{u} \neq 0$.* We have the localized Gagliardo–Nirenberg inequality:

$$\|v\|_{L^{2+4/(n+k)}_{x,y}} \le C \sup_{x\in\mathbb{R}^n} (\|v\|_{L^2_{Q^n_x \times M^k}})^{2/(n+k+2)} \|v\|^{(n+k)/(n+k+2)}_{H^1(\mathbb{R}^n \times M^k)}, \tag{A-5}$$

where

$$Q^n_x = x + [0,1]^n \quad \text{for all } x \in \mathbb{R}^n.$$

The estimate above can be proved as follows (see [Lions 1984] for a similar argument on the flat space $\mathbb{R}^{d+k}$). We fix $x_h \in \mathbb{R}^n$ in such a way that $\bigcup_h Q^n_{x_h} = \mathbb{R}^n$ and $\operatorname{meas}_n(Q^n_{x_i} \cap Q^n_{x_j}) = 0$ for $i \neq j$, where meas_n denotes the Lebesgue measure in $\mathbb{R}^n$. By the classical Gagliardo–Nirenberg inequality we get

$$\|v\|^{2+4/(n+k)}_{L^{2+4/(n+k)}_{Q^n_{x_h}\times M^k}} \le C \|v\|^{4/(n+k)}_{L^2_{Q^n_{x_h}\times M^k}} \|v\|^2_{H^1(Q^n_{x_h}\times M^k)}.$$

The proof of (A-5) follows by taking the sum of the previous estimates on $h \in \mathbb{N}$.

Due to the boundedness of $u_{j,\rho}$ in $H^1(\mathbb{R}^m \times M^k)$ (see the first step), we deduce by (A-5) that

$$0 < \epsilon_0 = \inf_j \|u_{j,\rho}\|_{L^{2+4/(n+k)}_{x,y}} \le C \sup_{x\in\mathbb{R}^n} \|u_{j,\rho}\|^{2/(n+k+2)}_{L^2_{Q^n_x\times M^k}} \tag{A-6}$$

(the left side above follows by combining the Hölder inequality with the third step). The proof can be concluded by the Rellich compactness theorem once we choose a sequence $\tau_j \in \mathbb{R}^n_x$ in such a way that

$$\inf_j \|u_{j,\rho}\|_{L^2_{Q^n_{\tau_j}\times M^k}} > 0$$

(the existence of such a sequence τ_j follows by (A-6)).

Fifth step: *the map $(0,\bar{\rho}) \ni \rho \to \rho^{-2} K^{\rho}_{n,M^k,\alpha}$ is strictly decreasing*. Let us fix $\rho_1 < \rho_2$ and u_{j,ρ_1} a minimizing sequence for $K^{\rho_1}_{n,M^k,\alpha}$. Then we have

$$\begin{aligned}
K^{\rho_2}_{n,M^k,\alpha} &\le \mathscr{E}_{n,M^k,\alpha}\left(\frac{\rho_2}{\rho_1} u_{j,\rho_1}\right) \\
&= \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2}\|\nabla_{x,y} u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\left(\frac{\rho_2}{\rho_1}\right)^{\alpha} \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right) \\
&= \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2}\|\nabla_{x,y} u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right) + \frac{1}{2+\alpha}\left(\frac{\rho_2}{\rho_1}\right)^2 \left(1 - \left(\frac{\rho_2}{\rho_1}\right)^{\alpha}\right) \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} \\
&\le \left(\frac{\rho_2}{\rho_1}\right)^2 \left(\frac{1}{2}\|\nabla_{x,y} u_{j,\rho_1}\|^2_{L^2_{x,y}} - \frac{1}{2+\alpha}\|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}\right) + \frac{1}{2+\alpha}\left(\frac{\rho_2}{\rho_1}\right)^2 \left(1 - \left(\frac{\rho_2}{\rho_1}\right)^{\alpha}\right) \inf_j \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}}.
\end{aligned}$$

By recalling (see the third step) that $\inf_j \|u_{j,\rho_1}\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} > 0$, we get

$$K^{\rho_2}_{n,M^k,\alpha} < \left(\frac{\rho_2}{\rho_1}\right)^2 K^{\rho_1}_{n,M^k,\alpha}.$$

Sixth step: *Let $\bar{u}$ be as in the fourth step. Then* $\|\bar{u}\|_{L^2_{x,y}} = \rho$. Up to a subsequence we get

$$u_{j,\rho}(x+\tau_j, y) \to \bar{u}(x, y) \neq 0 \quad \text{a.e. in } (x, y) \in \mathbb{R}^n_x \times M^k_y,$$

and hence, by the Brezis–Lieb lemma [1983], we get

$$\|u_{j,\rho}(x+\tau_j, y) - \bar{u}(x, y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} = \|u_{j,\rho}(x+\tau_j, y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} - \|\bar{u}(x, y)\|^{2+\alpha}_{L^{2+\alpha}_{x,y}} + o(1). \tag{A-7}$$

Assume that $\|\bar{u}\|_{L^2_{x,y}} = \theta$. Our aim is to prove $\theta = \rho$. Since $\bar{u} \neq 0$, necessarily $\theta > 0$. Notice that since $L^2_{x,y}$ is a Hilbert space, we have

$$\rho^2 = \|u_{j,\rho}(x+\tau_j, y)\|^2_{L^2_{x,y}} = \|u_{j,\rho}(x+\tau_j, y) - \bar{u}(x, y)\|^2_{L^2_{x,y}} + \|\bar{u}(x, y)\|^2_{L^2_{x,y}} + o(1), \tag{A-8}$$

and hence

$$\|u_{j,\rho}(x+\tau_j, y) - \bar{u}(x, y)\|^2_{L^2_{x,y}} = \rho^2 - \theta^2 + o(1). \tag{A-9}$$

By a similar argument,

$$\begin{aligned}\int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_x(u_{j,\rho}(x+\tau_j, y)) - \nabla_x\bar{u}(x, y)|^2\, dx\, dy \\ + \int_{M^k_y}\int_{\mathbb{R}^n_x} |\nabla_y(u_{j,\rho}(x+\tau_j, y)) - \nabla_y\bar{u}(x, y)|^2\, dx\, d\mathrm{vol}_{M^k_y} + \int_{M^k_y}\int_{\mathbb{R}^n_x} (|\nabla_x\bar{u}(x, y)|^2 + |\nabla_y\bar{u}(x, y)|^2)\, dx\, d\mathrm{vol}_{M^k_y} \\ = \int_{M^k_y}\int_{\mathbb{R}^n_x} (|\nabla_x(u_{j,\rho}(x+\tau_j, y)|^2 + |\nabla_y u_{j,\rho}(x+\tau_j, y)|^2)\, dx\, d\mathrm{vol}_{M^k_y} + o(1).\end{aligned} \tag{A-10}$$

By combining (A-10) with (A-7), we get

$$K^\rho_{n,M^k,\alpha} = \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}(x+\tau_j, y)) = \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{j,\rho}(x+\tau_j, y) - \bar{u}(x, y)) + \mathscr{E}_{n,M^k,\alpha}(\bar{u}), \tag{A-11}$$

and we can continue the estimate as follows:

$$\cdots \geq K^{\sqrt{\rho^2-\theta^2}+o(1)}_{n,M^k,\alpha} + K^\theta_{n,M^k,\alpha},$$

where we have used (A-9). Hence, by using the second step, we get

$$K^\rho_{n,M^k,\alpha} \geq K^{\sqrt{\rho^2-\theta^2}}_{n,M^k,\alpha} + K^\theta_{n,M^k,\alpha}.$$

Assume that $\theta < \rho$. Then, by using the monotonicity proved in the fifth step, we get

$$K^\rho_{n,M^k,\alpha} > \frac{\rho^2-\theta^2}{\rho^2} K^\rho_{n,M^k,\alpha} + \frac{\theta^2}{\rho^2} K^\rho_{n,M^k,\alpha} = K^\rho_{n,M^k,\alpha},$$

and we have an absurdity. □

Proof of Theorem 1.2. Assume for a contradiction that the conclusion is false. Then there exists ρ and two sequences $\varphi_j \in H^1(\mathbb{R}^n \times M^k)$ and $t_j \in \mathbb{R}$ such that

$$\lim_{j\to\infty} \mathrm{dist}_{H^1(\mathbb{R}^n\times M^k)}(\varphi_j, \mathcal{M}^\rho_{n,M^k,\alpha}) = 0 \tag{A-12}$$

and

$$\liminf_{j\to\infty} \operatorname{dist}_{H^1(\mathbb{R}^n\times M^k)}(u_{\varphi_j}(t_j), \mathcal{M}^{\rho}_{n,M^k,\alpha}) > 0, \tag{A-13}$$

where u_{φ_j} is the solution to (1-1) with Cauchy data φ_j. By (A-12) we deduce the following information:

$$\lim_{j\to\infty} \|\varphi_j\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(\varphi_j) = K^{\rho}_{n,M^k,\alpha},$$

and hence, due to the conservation laws satisfied by solutions to (1-1), we get

$$\lim_{j\to\infty} \|u_{\varphi_j}(t_j)\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(u_{\varphi_j}(t_j)) = K^{\rho}_{n,M^k,\alpha}.$$

In turn, by an elementary computation, we get

$$\|\tilde{u}_j\|_{L^2_{x,y}} = \rho \quad \text{and} \quad \lim_{j\to\infty} \mathscr{E}_{n,M^k,\alpha}(\tilde{u}_j) = K^{\rho}_{n,M^k,\alpha}$$

(more precisely $\tilde{u}_j$ is a constrained minimizing sequence for $K^{\rho}_{n,M^k,\alpha}$), where

$$\tilde{u}_j = \rho \frac{u_{\varphi_j}(t_j)}{\|u_{\varphi_j}(t_j)\|_{L^2_{x,y}}}.$$

Moreover, by (A-13), it is easy to deduce

$$\liminf_{j\to\infty} \operatorname{dist}_{H^1(\mathbb{R}^n\times M^k)}(\tilde{u}_j, \mathcal{M}^{\rho}_{n,M^k,\alpha}) > 0,$$

which is in contradiction with the compactness of minimizing sequences for $K^{\rho}_{n,M^k,\alpha}$ from Theorem 1.1. □

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ORTHONORMAL SYSTEMS IN LINEAR SPANS

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We show that any N-dimensional linear subspace of $L^2(\mathbb{T})$ admits an orthonormal system such that the L^2 norm of the square variation operator V^2 is as small as possible. When applied to the span of the trigonometric system, we obtain an orthonormal system of trigonometric polynomials with a V^2 operator that is considerably smaller than the associated operator for the trigonometric system itself.

1. Introduction

Let $(\mathbb{T}, \mathcal{B}, \mu)$ denote a probability space and $\Phi := \{\phi_n\}_{n=1}^N$ an orthonormal system (ONS) of (μ-measurable) functions from $\mathbb{T}$ to $\mathbb{R}$. Motivated by questions regarding almost everywhere convergence, one is often interested in the behavior of the maximal function

$$\mathcal{M}f := \max_{\ell \le N} \left| \sum_{n=1}^{\ell} a_n \phi_n \right|.$$

Here we let $f := \sum_{n=2}^{N} a_n \phi_n$. For an arbitrary ONS, the Rademacher–Menshov theorem states that $\|\mathcal{M}f\|_{L^2} \ll \log(N) \|f\|_{L^2}$, where the $\log(N)$ factor is known to be sharp. However, one can do much better for many classical systems; for instance one can replace $\log(N)$ with an absolute constant in the case of the trigonometric system (the Carleson–Hunt inequality). More recently, there has been interest in variational refinements of these maximal results. Define the r-th variation operator by

$$\mathcal{V}^r f := \left(\max_{\pi \in \mathcal{P}_N} \sum_{I \in \pi} \left| \sum_{n \in I} a_n \phi_n \right|^r \right)^{1/r},$$

where $\mathcal{P}_N$ denotes the set of partitions of $[N]$ into subintervals. Clearly, $|\mathcal{M}f| \le |\mathcal{V}^r f|$ for all $r < \infty$. In the case of the trigonometric system, strengthening the Carleson–Hunt theorem, Oberlin, Seeger, Tao, Thiele, and Wright [Oberlin et al. 2012] have shown that $\|\mathcal{V}^r f\|_{L^2} \ll \|f\|_{L^2}$ for $r > 2$. When $r = 2$, it has been shown that $\|\mathcal{V}^2 f\|_{L^2} \ll \sqrt{\log(N)} \|f\|_{L^2}$ [Lewko and Lewko 2012a], where the factor $\sqrt{\log(N)}$ is optimal. This later inequality has some applications to sieve theory [Lewko and Lewko 2012c]. The factor $\sqrt{\log(n)}$ is rather unfortunate, leading to inefficiencies in these applications. It is likely that this factor can be improved for the functions arising in the applications, for instance, if the Fourier support of f is contained in certain arithmetic sets. This is a potential route towards improving the estimates

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in [Lewko and Lewko 2012c]. Some results in this direction can be found in section 7 of [Lewko and Lewko 2012a].

In a different direction, it seems that the $\sqrt{\log(n)}$ factor might also be an eccentricity of the standard ordering of the trigonometric system. In [Lewko and Lewko 2012a] the following problem was posed.

Problem 1. Is there a permutation $\sigma : [N] \to [N]$ such that the reordering of the trigonometric system $\Phi := \{\phi_n = e(\sigma(n)x)\}$ (where $e(x) := e^{2\pi i x}$) satisfies

$$\|\mathcal{V}^2 f\|_{L^2} \ll o(\sqrt{\log(N)})\|f\|_{L^2}$$

for all f in the span of the system[1]?

This problem can be thought of as a variational variant of Garsia's conjecture. A longstanding problem in the theory of orthonormal systems, often called Kolmogorov's rearrangement problem, asks if every (infinite) ONS can be reordered such that the expansion of every L^2 function converges almost everywhere. Garsia's conjecture is the stronger assertion (see [Garsia 1970] for a proof of this implication) that any finite ONS can be reordered to satisfy $\|\mathcal{M}f\|_{L^2} \ll \|f\|_{L^2}$ where the implicit constant is absolute. Towards Garsia's conjecture, Bourgain [1989] proved that one can rearrange a uniformly bounded ONS such that $\|\mathcal{M}f\|_{L^2} \ll \log\log(N)\|f\|_{L^2}$. His proof proceeds by showing that this holds for a uniformly randomly selected permutation with high probability. Unfortunately this is the best estimate one can obtain from a purely probabilistic approach. Bourgain showed that if one is allowed to select a new ONS with the same span as Φ (which allows more freedom than just reordering the system), one can obtain $\|\mathcal{M}f\|_{L^2} \ll \|f\|_{L^2}$ for the new system with the same span.

In this paper, we will study the analogous linear span version of Problem 1. Given an ONS $\Phi := \{\phi_n(x)\}_{n=1}^N$ and an $N \times N$ orthogonal matrix $O = \{o_{i,n}\}_{1\le i,n\le N}$, we define a new ONS, $\Psi := \{\psi_n(x)\}_{n=1}^N$, by

$$\psi_n(x) := \sum_{i=1}^N o_{i,n}\phi_i(x).$$

This new system will span the same space as the original system. Conversely, every such ONS can be obtained from some element of the orthogonal group, $\mathbb{O}(N)$. Let us write $\Phi(O) := \Psi$. Furthermore, in what follows Q will denote a measurable subset of $\mathbb{O}(N)$ and $\mathbb{P}[Q]$ will denote the Haar measure of Q.

Theorem 2. *Given an N-dimensional subspace of $L^2(\mathbb{T})$, there exists an ONS Ψ that satisfies*

$$\|\mathcal{V}^2 f\|_{L^2} \ll \sqrt{\log\log(N)}\|f\|_{L^2} \tag{1}$$

for all f in the span. In fact, if we take an arbitrary basis Φ for F, the conclusion holds for the ONS $\Phi(O)$ for all $O \in Q$ for some $Q \subset \mathbb{O}(N)$ with $\mathbb{P}[Q] \ge 1 - Ce^{-cN^{2/5}}$ (for some absolute positive constants C, c).

[1]We have recently proved [Lewko and Lewko 2012b] that there exists a rearrangement such that $\|\mathcal{V}^2 f\|_{L^2} \ll_\epsilon \log^{9/22+\epsilon}(N)\|f\|_{L^2}$ for $\epsilon > 0$ (for all uniformly bounded ONS). This is likely far from best possible.

If we take $\Phi := \{e(nx)\}_{n=1}^N$ (on the circle with the Lebesgue measure), this produces an ONS of trigonometric polynomials (spanning the same space as the first N elements of the trigonometric system) with much smaller square variation than the trigonometric system. Strictly speaking, Theorem 2 is stated for real valued ONS, but the result for the trigonometric system can be obtained by splitting into real and imaginary parts and noting that the corresponding result holds on each with large probability. We note that Problem 1 asks for a similar conclusion where O is restricted to be a permutation matrix instead of just an orthogonal matrix.

Theorem 2 is sharp. Consider an ONS of independent, mean zero, variance on Gaussians, $\{g_i\}_{i=1}^N$. Notice that applying an orthogonal transformation to this system leaves it metrically unchanged. On the other hand, we have (almost surely) that $\max_{\pi \in \mathcal{P}_N} \sum_{I \in \pi} |\sum_{n \in I} g_n|^2 \sim 2N \log\log(N)$ from the variational law of the iterated logarithm [Lewko and Lewko 2011].

Let us briefly outline the key idea in the proof of Theorem 2. In [Lewko and Lewko 2012a], we proved an estimate of the form (1) for systems of bounded independent random variables; see Theorem 9. The key ingredient in that case is that for every f in the span of the system we have the subgaussian tail estimate $\|f\|_{\mathscr{G}} \ll \|f\|_{L^2}$ (where $\|\cdot\|_{\mathscr{G}}$ is the Orlicz space norm associated to $e^{x^2}-1$). This clearly cannot hold in the setting of Theorem 2, since any L^2 function can be in the span of the system. However, we will show that a function f in the span of a generic basis $\Phi(O)$ can be split as $f = G + E$, where G satisfies a subgaussian tail inequality and E has small L^2 norm (decreasing with the size of the Fourier support of f). More precisely, we will prove the following (note that we abuse the notation c below to denote multiple distinct constants):

Proposition 3. *For N fixed, let $\Phi = \{\phi_n(x)\}_{n=1}^N$ be an ONS such that $\sum_{n=1}^N |\phi_n(x)|^2 \le N$ holds (pointwise). There exists $Q \subset \mathbb{O}(N)$ with $\mathbb{P}[Q] \ge 1 - Ce^{-cN^{2/5}}$ such that for $O \in Q$, we have that the associated ONS $\Phi(O) = \{\psi_n\}_{n=1}^N$ satisfies the following property. For any $f = \sum a_n \psi_n$, letting m denote* support$(\{a_n\})$ *(the number of nonzero a_i values), we have that the function defined by*

$$f := \sum a_n \psi_n(x)$$

can be decomposed as $f := G + E$ where $\|G\|_{\mathscr{G}} \ll \|f\|_{L^2}$ and $\|E\|_{L^2} \ll (m/N)^c \|f\|_{L^2}$ for some universal constant $c > 0$.

See Proposition 15 below, which gives a stronger maximal form of this statement. The condition $\sum_{n=1}^N |\phi_n(x)|^2 \le N$ can usually be removed in applications (such as Theorem 2) by a change of measure argument (see Lemma 6). It seems likely that this decomposition has other applications.

2. Preliminaries

We fix the probability space $(\mathbb{T}, \mathscr{B}, \mu)$. We define several different norms on the space of functions from $\mathbb{T}$ to $\mathbb{R}$. First, for a positive constant c, let $\|\cdot\|_{\mathscr{G}(c)}$ denote the norm of the Orlicz space associated to the convex function $e^{cx^2}-1$. That is,

$$\|f\|_{\mathscr{G}(c)} := \inf_{\lambda \in \mathbb{R}^+} \left\{ \int_{\mathbb{T}} e^{c|f/\lambda|^2} \, d\mu - 1 \le 1 \right\}.$$

When we write $\|\cdot\|_{\mathcal{G}}$ with the specification of c omitted, we mean $c = 1$.

We next define the convex function

$$\Gamma_K(t) := \begin{cases} e^{t^2} - 1, & |t| \leq K, \\ e^{K^2} t^2 + e^{K^2}(1 - K^2) - 1, & |t| \geq K \end{cases}$$

and denote the associated Orlitz norm by $\|\cdot\|_{\Gamma_K}$.

Lemma 4. *When $K \geq 1$, for all t, we have that*

$$\Gamma_K(t) \leq e^{t^2} - 1, \quad \Gamma_K(t) \leq e^{K^2} t^2.$$

It follows that for $f : \mathbb{T} \to \mathbb{R}$ we have $\|f\|_{\Gamma_K} \leq \|f\|_{\mathcal{G}}$ and $\|f\|_{\Gamma_K} \leq e^{K^2/2}\|f\|_{L^2}$.

Proof. We first prove $\Gamma_K(t) \leq e^{t^2} - 1$ for all t. For t such that $|t| \leq K$, this is clear since $\Gamma_K(t) = e^{t^2} - 1$. We consider t such that $|t| \geq K$. Then $\Gamma_K(t) = e^{K^2} t^2 + e^{K^2}(1 - K^2) - 1$, so we must show that $e^{K^2} t^2 + e^{K^2}(1 - K^2) \leq e^{t^2}$. We note that for all real $x \geq 0$, $1 + x \leq e^x$. Applying this to the quantity $t^2 - K^2 + 1 > 0$, we have

$$e^{K^2} t^2 + e^{K^2}(1 - K^2) = e^{K^2}(t^2 - K^2 + 1) \leq e^{K^2} e^{t^2 - K^2} = e^{t^2},$$

as required.

We let f be a function from $\mathbb{T}$ to $\mathbb{R}$. For any fixed positive real number λ such that $\int_{\mathbb{T}} e^{|f/\lambda|^2}\, d\mu - 1 \leq 1$ (that is, $\lambda \geq \|f\|_{\mathcal{G}}$), we have

$$\int_{\mathbb{T}} \Gamma_K(f/\lambda)\, d\mu \leq \int_{\mathbb{T}} e^{|f/\lambda|^2}\, d\mu - 1 \leq 1,$$

since $\Gamma_K(t) \leq e^{t^2} - 1$ for all t. This shows that $\lambda \geq \|f\|_{\Gamma_K}$. Hence $\|f\|_{\Gamma_K} \leq \|f\|_{\mathcal{G}}$.

Next we prove $\Gamma_K(t) \leq e^{K^2} t^2$. We first consider t such that $|t| \geq K$. $\Gamma_K(t) = e^{K^2} t^2 + e^{K^2}(1 - K^2) - 1$ in this case. Since $K \geq 1$, we see that $e^{K^2}(1 - K^2) < 0$, so $\Gamma_K(t) \leq e^{K^2} t^2$ follows. For t such that $|t| \leq K$, we have $\Gamma_K(t) = e^{t^2} - 1$, so we must show that $e^{t^2} - 1 \leq e^{K^2} t^2$ for $|t| \leq K$.

We consider $(e^{t^2} - 1)/t^2$ as a function of t for $t \geq 0$. Its derivative is

$$2(t^{-1} e^{t^2} - t^{-3} e^{t^2} + t^{-3}).$$

We observe that this is always nonnegative. To see this, consider multiplying the quantity by t^3 to obtain $2(t^2 e^{t^2} - e^{t^2} + 1)$. Nonnegativity then follows from the inequality $1 + x e^x \geq e^x$ for all real $x \geq 0$. (This inequality can be proved by noting that $x e^x \geq \int_0^x e^u\, du$.) Hence $(e^{t^2} - 1)/t^2$ is a nondecreasing function of t in the range $0 \leq t \leq K$. So it suffices to consider the value at $t = K$, which is $K^{-2}(e^{K^2} - 1)$. Since $K \geq 1$, this is $< e^{K^2}$, as required.

For $f : \mathbb{T} \to \mathbb{R}$, we consider $\lambda := e^{K^2/2}\|f\|_{L^2}$. Then

$$\int_{\mathbb{T}} \Gamma_K(f/\lambda)\, d\mu \leq \int_{\mathbb{T}} e^{K^2} \frac{f^2}{\lambda^2}\, d\mu = \frac{e^{K^2}}{\lambda^2}\|f\|_{L^2}^2 = 1,$$

since $\Gamma_K(t) \leq e^{K^2} t^2$. Thus, $\|f\|_{\Gamma_K} \leq e^{K^2/2}\|f\|_{L^2}$. ∎

Lemma 5. *For any (measurable) $f : \mathbb{T} \to \mathbb{R}$, we can decompose $f = f_1 + f_2$ such that*

$$\|f_1\|_{\mathscr{G}} \ll \|f\|_{\Gamma_K} \quad \text{and} \quad \|f_2\|_{L^2} \ll e^{-cK^2}\|f\|_{\Gamma_K}$$

for some universal constant $c > 0$.

Proof. Given f, we define $\gamma := 2\|f\|_{\Gamma_K}$ to simplify our notation. We then set

$$f_1 := f \cdot \mathbb{1}_{|f/\gamma| \le K} \quad \text{and} \quad f_2 := f \cdot \mathbb{1}_{|f/\gamma| \ge K},$$

where $\mathbb{1}_S$ for a set $S \subset \mathbb{T}$ denotes the indicator function for that set. By definition of $\gamma = 2\|f\|_{\Gamma_K} > \|f\|_{\Gamma_K}$, we have that

$$\int_{\mathbb{T}} \Gamma_K(f/\gamma)\, d\mu = \int_{\mathbb{T}} (e^{|f/\gamma|^2} - 1) \cdot \mathbb{1}_{|f/\gamma| \le K}\, d\mu + \int_{\mathbb{T}} (e^{K^2} f^2/\gamma^2 + e^{K^2}(1-K^2) - 1) \cdot \mathbb{1}_{|f/\gamma| \ge K}\, d\mu \le 1. \tag{2}$$

Since this is a sum of two nonnegative quantities, this implies

$$\int_{\mathbb{T}} (e^{|f/\gamma|^2} - 1) \cdot \mathbb{1}_{|f/\gamma| \le K}\, d\mu \le 1.$$

This is equivalent to

$$\int_{\mathbb{T}} e^{|f_1/\gamma|^2}\, d\mu - 1 \le 1,$$

and so $\|f_1\|_{\mathscr{G}} \le \gamma \ll \|f\|_{\Gamma_K}$.

Again considering (2), we also have

$$\int_{\mathbb{T}} (e^{K^2} f^2/\gamma^2 + e^{K^2}(1-K^2) - 1) \cdot \mathbb{1}_{|f/\gamma| \ge K}\, d\mu \le 1.$$

We let $\mu(|f/\gamma| \ge K)$ denote the measure of the set in $\mathbb{T}$ on which $|f/\gamma| \ge K$. We can then rewrite the above as

$$\mu\left(\left|\frac{f}{\gamma}\right| \ge K\right)(e^{K^2}(1-K^2) - 1) + \int_{\mathbb{T}} e^{K^2} f_2^2/\gamma^2\, d\mu \le 1. \tag{3}$$

Now, since $\int_{\mathbb{T}} \Gamma_K(f/\gamma)\, d\mu \le 1$ and $\Gamma_K(f/\gamma) \ge e^{K^2} - 1$ whenever $|f/\gamma| \ge K$, we must have

$$\mu\left(\left|\frac{f}{\gamma}\right| \ge K\right)(e^{K^2} - 1) \le 1.$$

Thus, $\mu(|f/\gamma| \ge K) \le 1/(e^{K^2} - 1)$. Combining this with (3), we have

$$\int_{\mathbb{T}} e^{K^2} f_2^2/\gamma^2\, d\mu \le 1 + \mu\left(\left|\frac{f}{\gamma}\right| \ge K\right)(e^{K^2}(K^2 - 1) + 1) \ll K^2,$$

and hence

$$\|f_2\|_{L^2}^2 \ll K^2 e^{-K^2}\gamma^2,$$

implying that $\|f_2\|_{L^2} \ll e^{-cK^2}\|f\|_{\Gamma_K}$ for some universal constant $c > 0$. □

Finally, we note the following.

Lemma 6. *It suffices to prove* (*the second formulation of*) *Theorem 2 with the restriction* $\sum_{n=1}^{N} |\phi_n|^2 \leq N$.

Proof. Consider an arbitrary ONS $\Phi := \{\phi_n\}_{n=1}^{N}$ and define $\nu(x) = N^{-1} \sum_{n=1}^{N} |\phi_n(x)|^2$. Fix $O \in \mathbb{O}(N)$. Define $\widetilde{\Phi} := \Phi(O)$. Furthermore, consider the ONS Ψ defined on $\mathbb{T}$ (with the measure induced by integration against $\nu(x)\,d\mu$) by $\psi_n(x) := \nu^{-1/2}(x)\phi_n(x)$. Furthermore, define $\widetilde{\Psi} = \Psi(O)$. We have the trivial identity

$$\int_{\mathbb{T}} \max_{\pi \in \mathcal{P}_N} \sum_{I \in \pi} \Big| \sum_{n \in I} a_n \tilde{\phi}_n(x) \Big|^2 d\mu = \int_{\mathbb{T}} \max_{\pi \in \mathcal{P}_N} \sum_{I \in \pi} \Big| \sum_{n \in I} a_n \tilde{\psi}_n(x) \Big|^2 \nu(x)\,d\mu.$$

Thus, the conclusion of Theorem 2 holds for Φ if and only if it holds for Ψ. However, $\sum_{n=1}^{N} |\psi_n|^2 \leq N$ by construction. □

3. Probabilistic Methods

In this section we establish the following result.

Proposition 7. *For N fixed, let $\{\phi_n(x)\}_{n=1}^{N}$ be an ONS such that $\sum_{n=1}^{N} |\phi_n(x)|^2 \leq N$. Define for each $1 \leq m \leq N$ the function*

$$\Gamma_* := \Gamma_{\sqrt{(2/5)\log((N/m)\log(N/m+1))}}$$

(*the dependence on m is implicit in this notation*). *There exists a subset $Q \subset \mathbb{O}(N)$ with $\mathbb{P}[Q] \geq 1 - C(e^{-cN^{2/5}})$ such that for all $O = \{o_{i,n}\}_{1 \leq i,n \leq N} \in Q$ the corresponding base change of $\{\phi_n\}_{n=1}^{N}$, that is*

$$\psi_n(x) := \sum_{i=1}^{N} o_{i,n}\phi_i(x),$$

satisfies the following. For each m in the range $1 \leq m \leq N$,

$$\Big\| \sum_{n=1}^{N} a_n \psi_n \Big\|_{\Gamma_*} \ll \Big(\sum_{n=1}^{N} a_n^2 \Big)^{1/2}$$

for all vectors $\mathbf{a} \in \mathbb{R}^N$ such that $\mathbf{support}(\mathbf{a}) \leq m$. (*We use* $\mathbf{support}(\mathbf{a})$ *to denote the number of nonzero coordinates of* $\mathbf{a}$.)

The proof will build on arguments from [Bourgain 1989], although the estimates we obtain are substantially stronger. We start by establishing a weaker result. For a fixed m in the range $1 \leq m \leq N$, we let $\mathbb{S}_m \subset \mathbb{R}^N$ denote the subset of vectors $\mathbf{b}$ such that $\|\mathbf{b}\|_{L^2} \leq 1$ and support($\mathbf{b}$) $\leq m$. We then define

$$B(m, \mathbb{O}) := \sup_{\mathbf{a} \in \mathbb{S}_m} \Big\| \sum_{n=1}^{N} a_n \psi_n \Big\|_{\Gamma_*}.$$

Note that both the set $\mathbb{S}_m$ and the function $\Gamma_* := \Gamma_{\sqrt{(2/5)\log((N/m)\log(N/m+1))}}$ depend on m. Our first step will be to establish the following.

Proposition 8. *For any* $1 \le m \le N$ *we have that*

$$\mathbb{E}_{\mathbb{O}(N)} B(m, O) \ll 1,$$

where the implied constant is independent of m *and* N.

This does not quite give Proposition 7, since there the claim is made with large probability and we require the estimates to hold for all m simultaneously. The stronger claim, however, will be deduced later from the weaker statement using the concentration of measure phenomenon on the orthogonal group.

We will need the following result, which is Lemma 5.5 from [Bourgain 1989], where it is attributed to [Benyamini and Gordon 1981]. The result is a concatenation of Lemmas 1.10 and 1.12 in [Benyamini and Gordon 1981]. These are due to [Chevet 1978] and [Marcus and Pisier 1981], respectively.

Lemma 9. *Let* X *and* Y *be Banach spaces, and consider the operator*

$$T_O := \sum_{i,j=1}^{N} o_{ij} (x_i^* \otimes y_j)$$

for $O := (o_{ij})_{1 \le i,j \le N} \in \mathbb{O}(N)$, *and where* $\{x_i^*\}_{i=1}^N$ (*respectively* $\{y_j\}_{j=1}^N$) *are sequences in* X^* (*respectively* Y). *Then*

$$\int_{\mathbb{O}(N)} \|T_O\| \le \frac{C\alpha(\{x_i^*\}_{i=1}^N)}{\sqrt{N}} \int \Big\| \sum_{j=1}^N g_j(\omega) y_j \Big\| \, d\omega + \frac{C\alpha(\{y_j\}_{j=1}^N)}{\sqrt{N}} \int \Big\| \sum_{i=1}^N g_i(\omega) x_i^* \Big\| \, d\omega, \tag{4}$$

where

$$\alpha(\{x_i^*\}) := \sup\bigg\{ \Big(\sum |\langle x_i^*, x\rangle|^2 \Big)^{1/2} : x \in X, \ \|x\| \le 1 \bigg\},$$
$$\alpha(\{y_j\}) := \sup\bigg\{ \Big(\sum |\langle y_j, y^*\rangle|^2 \Big)^{1/2} : y^* \in Y^*, \ \|y^*\| \le 1 \bigg\},$$

and $\{g_i\}_{i=1}^N$ *is a system of independent Gaussians with mean zero and variance one. Note that the norms in* (4) *refer, respectively, to the Banach spaces* $B(X, Y)$, Y, *and* X^*.

Let $\ell^2[N]$ denote the set of real sequences $\mathbf{a} := \{a_n\}_{n=1}^N$. We will denote by X the Banach space obtained by considering this set with the norm $\|\cdot\|_{[m]}$ defined as follows. For a vector $\mathbf{a}$, we define $\|\mathbf{a}\|_{[m]}$ to be the infimum of positive $c \in \mathbb{R}$ such that scaling the convex hull of $\mathbb{S}_m$ by c results in a set containing $\mathbf{a}$. We take Y to be the space of real-valued functions on $\mathbb{T}$ equipped with the Orlicz norm associated to Γ_*.

Let x_i^* ($1 \le i \le N$) denote the canonical unit vectors in $\mathbb{R}^N$ (which is naturally identified with the dual space X^*). We have, from Lemma 9, that

$$\mathbb{E} B(m, \mathbb{O}) \ll \frac{\alpha(\{x_i^*\}_{i=1}^N)}{\sqrt{N}} \mathbb{E} \Big\| \sum g_i \phi_i \Big\|_{\Gamma_*} + \frac{\alpha(\{\phi_i\}_{i=1}^N)}{\sqrt{N}} \mathbb{E} \Big\| \sum g_i x_i^* \Big\|_{X^*}.$$

In order to establish Proposition 8, we need to show the above is $\ll 1$. This follows from the estimates

$$\alpha(\{x_i^*\}_{i=1}^N) \ll 1, \qquad \alpha(\{\phi_i\}_{i=1}^N) \ll \left(\frac{N}{m}\log\left(\frac{N}{m}+1\right)\right)^{1/5},$$

$$\mathbb{E}\left\|\sum g_i\phi_i\right\|_{\Gamma_*} \le \sqrt{N}, \quad \mathbb{E}\left\|\sum g_i x_i^*\right\|_{X^*} \le \sqrt{m}\sqrt{\log\left(\frac{N}{m}+1\right)}.$$

The first estimate above follows from the observation that the convex hull of $\mathbb{S}_m$ is contained in the ℓ^2 unit ball in $\mathbb{R}^N$. We will prove the others in the following lemmas.

Lemma 10. *We have that* $\mathbb{E}\|\sum g_i\phi_i\|_{\Gamma_*} \ll \sqrt{N}$.

Proof.

Letting C be a positive constant, by Fubini's theorem we have that

$$\mathbb{E}\int_{\mathbb{T}} e^{(\sum g_i\phi_i(x))^2/(CN)}\,d\mu = \int_{\mathbb{T}} \mathbb{E}e^{(\sum g_i\phi_i(x))^2/CN}\,d\mu.$$

Now, for each fixed x, we recall that $\sum_i |\phi_i(x)|^2 \le N$, so $(1/\sqrt{CN})\sum g_i\phi_i(x)$ is a Gaussian random variable with mean 0 and variance at most $1/C$. Thus, $\int_{\mathbb{T}} \mathbb{E}e^{(\sum g_i\phi_i(x))^2/(CN)}\,d\mu \ll 1$ for an appropriate choice of C.

Since $e^{f^2/\lambda} \le 1+e^{f^2}/\lambda$ for $\lambda \ge 1$, we have that $\inf_{\lambda\in\mathbb{R}^+}\{\int_{\mathbb{T}} e^{|f/\lambda|^2}\,d\mu \le 2\} \ll 1+\int_{\mathbb{T}} e^{|f|^2}\,d\mu$. Applying this to $f=(1/\sqrt{CN})\sum g_i\phi_i$, we have

$$\left\|\frac{1}{\sqrt{CN}}\sum g_i\phi_i\right\|_{\Gamma_*} \le \int_{\mathbb{T}} e^{(\sum g_i\phi_i(x))^2/(CN)}\,d\mu.$$

Taking expectations on both sides, we have $\mathbb{E}\|\sum g_i\phi_i\|_{\Gamma_*} \ll \sqrt{N}$, as required. □

Lemma 11. *We have that* $\alpha(\{\phi_i\}_{i=1}^n) \ll ((N/m)\log(N/m+1))^{1/5}$.

Proof. From Lemma 4 it follows that $\|f\|_{\Gamma_*} \le ((N/m)\log(N/m+1))^{1/5}\|f\|_{L^2}$. Now

$$\|g\|_{\Gamma_*^*} = \sup_{f\in\Gamma_*}\frac{\langle f,g\rangle}{\|f\|_{\Gamma_*}} \ge \frac{\langle g,g\rangle}{\|g\|_{\Gamma_*}} \gg \frac{\|g\|_2^2}{((N/m)\log(N/m+1))^{1/5}\|g\|_{L^2}} \gg \left(\frac{N}{m}\log\left(\frac{N}{m}+1\right)\right)^{-1/5}\|g\|_{L^2}.$$

Here we have used that each element of the dual space Γ_*^* can be represented as integration against a measurable function. This follows from standard properties of Orlicz spaces. In particular, see Theorem 14.2 of [Krasnosel'skiĭ and Rutickiĭ 1961], since the modulus Γ_* satisfies the Δ_2 condition.

It now follows that if $\|g\|_{\Gamma_*^*} \le 1$, then $\|g\|_{L^2} \ll ((N/m)\log(N/m+1))^{1/5}$. Thus by Bessel's inequality we have

$$\alpha(\{\phi_j\}) := \sup\left\{\left(\sum|\langle\phi_i,g\rangle|^2\right)^{1/2} : g\in\Gamma_*^*,\ \|g\|_{\Gamma_*^*}\le 1\right\} \ll \left(\frac{N}{m}\log\left(\frac{N}{m}+1\right)\right)^{1/5},$$

which completes the proof. □

Lemma 12. $$\mathbb{E}\|\sum g_i x_i^*\|_{X^*} \le \sqrt{m}\sqrt{\log(N/m+1)}.$$

Proof. It follows from the definition of X^* that

$$\mathbb{E}\left\|\sum g_i x_i^*\right\|_{X^*} = \mathbb{E}\sup_{\mathbf{a}\in\mathbb{S}_m}\left|\sum g_i a_i\right|.$$

(Note that taking the supremum over the convex hull of $\mathbb{S}_m$ would yield the same result.)

The latter quantity is well studied in the theory of Gaussian processes. Recall that Dudley's bound [1967] gives

$$\ll \int_0^\infty \sqrt{\log(\mathcal{N}(\mathbb{S}_m,\epsilon))}\,d\epsilon,$$

where $\mathcal{N}(\mathbb{S}_m,\epsilon)$ denotes the number of ℓ^2 balls of radius ϵ needed to cover $\mathbb{S}_m$. Now, clearly $\mathbb{S}_m$ is a subset of the n-dimensional ℓ^2 unit ball. Thus $\log(\mathcal{N}(\mathbb{S}_m,\epsilon)) = 0$ for $\epsilon \ge 1$, and the above quantity is equal to

$$\int_0^1 \sqrt{\log(\mathcal{N}(\mathbb{S}_m,\epsilon))}\,d\epsilon.$$

Lemma 12 now follows from the following.

Lemma 13. *For* $0 < \epsilon \le 1$, *we have that*

$$\mathcal{N}(\mathbb{S}_m,\epsilon) \ll \binom{N}{m}\left(\frac{3}{\epsilon}\right)^m,$$

and thus

$$\log\mathcal{N}(\mathbb{S}_m,\epsilon) \ll m\log\left(\frac{N}{m}+1\right) + m\log\left(\frac{3}{\epsilon}\right).$$

Proof. We prove the first inequality (the second follows by taking logarithms). We let K denote the unit ℓ^2 ball in $\mathbb{R}^m$. Then $\mathcal{N}(K,\epsilon K) \le (3/\epsilon)^m$, where $\mathcal{N}(K,\epsilon K)$ denotes the number of translates of ϵK needed to cover K. To see this, consider a maximal set of disjoint balls of radius $\epsilon/2$ with centers in K. Let T denote the set of their centers. By maximality, taking balls of radius ϵ around each point in T yields a cover of K, and hence the cardinality of T is an upper bound on $\mathcal{N}(K,\epsilon K)$. Now, the union of all the disjoint balls of radius $\epsilon/2$ with centers in T is a set with volume equal to $|T|\operatorname{vol}((\epsilon/2)K)$, where $|T|$ denotes the cardinality of T and $\operatorname{vol}((\epsilon/2)K)$ denotes the volume of the ball of radius $\epsilon/2$. Since this set is contained in $(1+\epsilon/2)K$, we have

$$\mathcal{N}(K,\epsilon K) \le \frac{\operatorname{vol}((1+\epsilon/2)K)}{\operatorname{vol}((\epsilon/2)K)} = \frac{(1+\epsilon/2)^m}{(\epsilon/2)^m} = \left(1+\frac{2}{\epsilon}\right)^m \le \left(\frac{3}{\epsilon}\right)^m$$

whenever $0 < \epsilon \le 1$.

Fix m coordinates and consider the associated m-dimensional ℓ^2 ball. We have shown that this can be covered by $(3/\epsilon)^m$ balls of radius ϵ. Summing over all $\binom{N}{m}$ such balls completes the proof. ☐

This completes the proof of Lemma 12 and hence the proof of Proposition 8. ☐

3.1. *Concentration of measure on $\mathbb{O}(n)$.* In the prior section, we proved that for any $1 \le m \le N$ we have $\mathbb{E}_{\mathbb{O}(N)} B(m, O) \ll 1$. It follows from Markov's inequality that, for some large universal C, we have $\nu(\mathscr{A}(m)) > \frac{1}{2}$, where

$$\mathscr{A}(m) := \{O \in \mathbb{O}(N) : B(m, O) \le C\}$$

and $\nu(\mathscr{A}(m))$ denotes the measure of the set $\mathscr{A}(m)$ in $\mathbb{O}(N)$.

Consider the Hilbert–Schmidt norm on the set of $N \times N$ matrices, $\|A\|_{\mathrm{HS}} := (\sum_{1 \le i,j \le N} |A_{i,j}|^2)^{1/2}$. We recall the concentration of measure inequality on the Orthogonal group; see [Milman and Schechtman 1986].

Lemma 14. *Let ν denote the Haar measure on the orthogonal group $O(N)$ and $A \subset O(N)$ such that $\nu(A) > \frac{1}{2}$. Then*

$$\mathbb{P}[A \in O(N) : \inf_{B \in A_c} \|A - B\|_{HS} > \epsilon] \ll e^{-c\epsilon^2 N}$$

for some absolute positive constant c.

For any $N \times N$ matrix $M = \{m_{i,j}\}$, using the bounds from Lemma 4, we have

$$\begin{aligned}\Big\| \sum_{1 \le i,n \le N} m_{i,n} a_i \phi_n \Big\|_{\Gamma_*} &\ll \left(\frac{N}{m} \log\left(\frac{N}{m}\right)\right)^{1/5} \left(\sum_n \left(\sum_i m_{i,n} a_i\right)^2\right)^{1/2} \\ &\ll \left(\frac{N}{m} \log\left(\frac{N}{m}\right)\right)^{1/5} \|M\|_{HS} \|\mathbf{a}\|_{\ell^2} \qquad (5)\end{aligned}$$

for all $\mathbf{a} \in \mathbb{R}^N$. The final inequality follows from Cauchy–Schwartz.

Now consider $\mathscr{A}(m, \epsilon) \subset \mathbb{O}(N)$, defined to be the set of all orthogonal matrices that differ from an element of $\mathscr{A}(m)$ by a matrix with Hilbert–Schmidt norm at most ϵ. Using (5), we have that for $O \in \mathscr{A}\big(m, (m/(N \log(N/m)))^{1/5}\big)$, $B(m, O) \le C'$, where C' is a new absolute constant. On the other hand, denoting the complement of $\mathscr{A}\big(m, (m/(N \log(N/m)))^{1/5}\big)$ by $\mathscr{A}^c\big(m, (m/(N \log(N/m)))^{1/5}\big)$, by Lemma 14, we have

$$\mathbb{P}\left[O \in \mathscr{A}^c\left(m, \left(\frac{m}{N \log(N/m)}\right)^{1/5}\right)\right] \ll e^{-cN^{2/5}}$$

for some positive constant c.

Now to conclude the proof of Proposition 7, it suffices to find a sufficiently high probability set of elements $O \in \mathbb{O}(N)$ such that for every $1 \le m \le N$ we have $O \in \mathscr{A}\big(m, (m/(N \log(N/m)))^{1/5}\big)$. However, for sufficiently large N, we see from the union bound that

$$\nu\left(\bigcup_{1 \le m \le N} \mathscr{A}^c\left(m, \left(\frac{m}{N \log(N/m)}\right)^{1/5}\right)\right) \le N e^{-cN^{2/5}} \ll e^{-c_2 N^{2/5}}.$$

This completes the proof of Proposition 7.

4. Maximal function decomposition

Proposition 15. *For N fixed, let $\{\phi_n(x)\}_{n=1}^N$ be an ONS such that $\sum_{n=1}^N |\phi_n(x)|^2 \leq N$. There exists $Q \subset \mathbb{O}(N)$ with $\mathbb{P}[Q] \geq 1 - C(e^{-cN^{2/5}})$ such that for $O \in Q$ the associated system $\Psi(O) = \{\psi_n\}_{n=1}^N$ satisfies the following property. For any $f = \sum a_n \psi_n$, letting m denote* support($\{a_n\}$), *we have that the maximal function defined by*

$$\mathcal{M}f := \sup_{I \subseteq [N]} \left| \sum_{n \in I} a_n \psi_n \right|$$

can be decomposed as $\mathcal{M}f := \widetilde{G} + \widetilde{E}$, where $\|\widetilde{G}\|_{\mathcal{G}} \ll \|f\|_{L^2}$ and $\|\widetilde{E}\|_{L^2} \ll (m/N)^c \|f\|_{L^2}$ for some universal constant $c > 0$.

To prove this, we fix $Q \subset \mathbb{O}(N)$ from Proposition 7. We now decompose $[N]$ into a family of subintervals according to a concept of mass defined with respect to the a_i values. We define the *mass* of a subinterval $I \subseteq [N]$ as $M(I) := \sum_{n \in I} |a_n|^2$. By normalization, we may assume that $M([N]) = 1$. We define $I_{0,1} := [N]$ and we iteratively define $I_{k,s}$, for $1 \leq s \leq 2^k$, as follows. Assuming we have already defined $I_{k-1,s}$ for all $1 \leq s \leq 2^{k-1}$, we will define $I_{k,2s-1}$ and $I_{k,2s}$, which are subintervals of $I_{k-1,s}$. $I_{k,2s-1}$ begins at the left endpoint of $I_{k-1,s}$ and extends to the right as far as possible while covering strictly less than half the mass of $I_{k-1,s}$, while $I_{k,2s}$ ends at the right endpoint of $I_{k-1,s}$ and extends to the left as far as possible while covering at most half the mass of $I_{k-1,s}$. More formally, we define $I_{k,2s-1}$ as the maximal subinterval of $I_{k-1,s}$ which contains the left endpoint of $I_{k-1,s}$ and satisfies $M(I_{k,2s-1}) < \frac{1}{2} M(I_{k-1,s})$. We also define $I_{k,2s}$ as the maximal subinterval of $I_{k-1,s}$ which contains the right endpoint of $I_{k-1,s}$ and satisfies $M(I_{k,2s}) \leq \frac{1}{2} M(I_{k-1,s})$. We note that these subintervals are disjoint. We may express $I_{k-1,s}$ as $I_{k,2s-1} \cup I_{k,2s} \cup i_{k,s}$, where $i_{k,s} \in I_{k-1,s}$. In other words, $i_{k,s}$ denotes the single element which lies between $I_{k,2s-1}$ and $I_{k,2s}$ (note that such a point always exists because we have required that $I_{k,2s-1}$ contains strictly less than half of the mass of the interval). Here it is acceptable, and in many instances necessary, for some choices of the intervals in this decomposition to be empty. By construction we have that

$$M(I_{k,s}) \leq 2^{-k}. \tag{6}$$

We call an interval $J \subseteq [N]$ admissible if it is an element of the decomposition given above. We denote the collection of admissible intervals by $\mathcal{A}$. We additionally refer to the subset $\{I_{k,s} : 1 \leq s \leq 2^k\}$ of $\mathcal{A}$ as the admissible intervals on level k and the subset $\{i_{k,s} : 1 \leq s \leq 2^k\}$ as the admissible points on level k. We note that every point in $[N]$ is an admissible point on some level. (Eventually, we have subdivided all intervals down to being single elements.)

Now we write $\mathcal{I}_k := \{I_{k,s} : 1 \leq s \leq 2^k\}$. We decompose this as $\mathcal{I}_k^a := \{I \in \mathcal{I}_k : |I| \leq 2^{-k/2} N\}$ and its complement, $\mathcal{I}_k^b := \{I \in \mathcal{I}_k : |I| > 2^{-k/2} N\}$. Here, $|I|$ denotes the number of nonzero a_i values contained in an interval I.

For $J \subseteq [N]$, we define

$$S_J(x) = \sum_{n \in J} a_n \psi_n(x).$$

We also define

$$\widetilde{S}_J(x) := \max_{I \subseteq J} \left| \sum_{n \in I} a_n \psi_n(x) \right|.$$

From Lemma 5 and Proposition 7, we deduce that $S_J = G_J + E_J$, where $\|G_J\|_{\mathcal{G}} \ll \|S_J\|_{L^2}$ and $\|E_J\|_{L^2} \ll (|J|/N)^{c'} \|S_J\|_{L^2}$ for some positive constant c'. Our purpose now is to show a similar decomposition for $\widetilde{S}_J(x)$. Clearly, it suffices to show such a decomposition for a pointwise majorant. Denote the decomposition of $S_{I_{k,s}}$ by $S_{I_{k,s}} := G_{k,s} + E_{k,s}$, and the decomposition of $S_{i_{k,s}}$ by $S_{i_{k,s}} := G_{i_{k,s}} + E_{i_{k,s}}$. Setting $r = 3$, for an interval J we have the following bound, where the sums below are restricted to values of k, s such that $I_{k,s}, i_{k,s} \subseteq J$:

$$\begin{aligned}
&\widetilde{S}_J(x) \\
&\ll \sum_k \Big(\sum_s |G_{k,s} + E_{k,s}|^r \Big)^{1/r} + \sum_k \Big(\sum_s |G_{i_{k,s}} + E_{i_{k,s}}|^r \Big)^{1/r} \\
&\ll \bigg(\sum_k \Big(\sum_s |G_{k,s}|^r \Big)^{1/r} + \sum_k \Big(\sum_s |G_{i_{k,s}}|^r \Big)^{1/r} \bigg) + \bigg(\sum_k \Big(\sum_s |E_{k,s}|^r \Big)^{1/r} + \sum_k \Big(\sum_s |E_{i_{k,s}}|^r \Big)^{1/r} \bigg) \\
&=: \widetilde{G}_J + \widetilde{E}_J.
\end{aligned} \tag{7}$$

This follows from the observation that, for each point x, the maximizing subinterval $I \subseteq J$ can be decomposed as a union of admissible intervals and points with at most two intervals and points on each level. The contribution on each level can then be bounded by a constant times the contribution from the "worst" interval/point, which is in turn bounded by the quantity inside the sum over k above for each level k.

For an admissible interval J, we let k^* denote the level of J. We note that the sums over k in (7) range only over $k \geq k^*$ (and the sums over s are also appropriately restricted). Next we show that $\|\widetilde{G}_J\|_{\mathcal{G}(c)} \ll \|S_J\|_{L^2}$ for some absolute constant c and $\|\widetilde{E}_J\|_{L^2} \ll (|J|/N)^{c'} \|S_J\|_{L^2}$.

Now let us estimate $\|\widetilde{E}_J\|_{L^2}$. We first estimate the contribution from the admissible points $i_{k,s} \in J$. We observe that

$$\bigg\| \sum_k \Big(\sum_s |E_{i_{k,s}}|^r \Big)^{1/r} \bigg\|_{L^2} \leq \sum_k \bigg\| \Big(\sum_s |E_{i_{k,s}}|^r \Big)^{1/r} \bigg\|_{L^2}.$$

Since $r > 2$, this is at most

$$\sum_k \Big(\sum_s \|E_{i_{k,s}}\|_{L^2}^2 \Big)^{1/2} \ll \Big(\frac{1}{N} \Big)^{c'} \sum_k \Big(\sum_s \|S_{i_{k,s}}\|_{L^2}^2 \Big)^{1/2},$$

where the latter inequality follows from the definition of $E_{i_{k,s}}$.

Now since these sums only range over values of k, s such that $i_{k,s} \in J$, we may split the sum over k into two portions as

$$\sum_k \Big(\sum_s \|S_{i_{k,s}}\|_{L^2}^2 \Big)^{1/2} = \sum_{k=k^*}^{k^*+10\log(N)} \Big(\sum_s \|S_{i_{k,s}}\|_{L^2}^2 \Big)^{1/2} + \sum_{k > k^*+10\log(N)} \Big(\sum_s \|S_{i_{k,s}}\|_{L^2}^2 \Big)^{1/2}. \tag{8}$$

To bound the first quantity in (8), it suffices to observe that the inner quantity for each k is at most $\|S_J\|_{L^2}$, and hence its contribution is $\ll \log(N)\|S_J\|_{L^2} \ll N^{\epsilon}\|S_J\|_{L^2}$, for a constant $\epsilon < c'$. (Thus we will adjust the value of c' for our final estimate by subtracting ϵ.)

To bound the second quantity in (8), we note that, for any $i_{k,s} \in J$ with $k > k^* + 10\log(N)$, we have $\|S_{i_{k,s}}\|_{L^2}^2 \le N^{-10}\|S_J\|_{L^2}^2$. There are at most N points $i_{k,s}$ in the sum, and thus

$$\sum_{k>k^*+10\log(N)} \Bigl(\sum_s \|S_{i_{k,s}}\|_{L^2}^2\Bigr)^{1/2} \ll N^{-4}\|S_J\|_{L^2}.$$

To estimate the contribution from the admissible intervals, we proceed as follows. For each $k \ge k^*$, we define $I_k^a(J)$ to be the set of admissible intervals I on level k contained in J such that $|I| < 2^{-(k-k^*)/2}|J|$ and we let $I_k^b(J)$ denote the set of remaining admissible intervals on level k contained in J. Note that $I_k^a(J)$ and $I_k^b(J)$ are disjoint, and their union is the set of all admissible intervals on level k contained in J. It thus suffices to estimate

$$\widetilde{E}_J^a + \widetilde{E}_J^b := \sum_{k\ge k^*}\Bigl(\sum_{I_{k,s}\in I_k^a(J)} |E_{k,s}|^r\Bigr)^{1/r} + \sum_k \Bigl(\sum_{I_{k,s}\in I_k^b(J)} |E_{k,s}|^r\Bigr)^{1/r}.$$

Now $|I_k^b(J)| \le 2^{(k-k^*)/2}$, and we also have

$$\|E_{k,s}\|_{L^2} \ll \Bigl(\frac{|J|}{N}\Bigr)^{c'}\|S_{k,s}\|_{L^2} \ll \Bigl(\frac{|J|}{N}\Bigr)^{c'} 2^{-(k-k^*)/2}\|S_J\|_{L^2}.$$

Since $r > 2$, we have

$$\Bigl\|\sum_{k\ge k^*}\Bigl(\sum_{s\in I_k^b(J)} |E_{k,s}|^r\Bigr)^{1/r}\Bigr\|_{L^2} \le \sum_{k\ge k^*}\Bigl(\sum_{s\in I_k^b(J)} \|E_{k,s}\|_{L^2}^2\Bigr)^{1/2} \ll \Bigl(\frac{|J|}{N}\Bigr)^{c'}\|S_J\|_{L^2}\sum_{j\ge 0} 2^{-j/4} \ll \Bigl(\frac{|J|}{N}\Bigr)^{c'}\|S_J\|_{L^2}.$$

Next, we recall that $I \in I_k^a(J)$ implies $|I| \le 2^{-(k-k^*)/2}|J|$. We have $\|S_{I_{k,s}}\|_{L^2} \ll 2^{-(k-k^*)/2}\|S_J\|_{L^2}$. Thus $\|E_{k,s}\|_{L^2} \ll (|J|/N)^{c'}2^{-c'(k-k^*)/2}\|S_{I_{k,s}}\|_2 \ll (|J|/N)^{c'}2^{-(c'+1)(k-k^*)/2}\|S_J\|_2$.

We then have

$$\begin{aligned}\Bigl\|\sum_{k\ge k^*}\Bigl(\sum_{I_{k,s}\in I_k^a(J)} |E_{k,s}|^r\Bigr)^{1/r}\Bigr\|_{L^2} &\le \sum_{k\ge k^*}\Bigl(\sum_{I_{k,s}\in I_k^a(J)} \|E_{k,s}\|_{L^2}^2\Bigr)^{1/2} \\ &\ll \Bigl(\frac{|J|}{N}\Bigr)^{c'}\|S_J\|_{L^2}\sum_{k\ge k^*} 2^{k-k^*}2^{-(c'+1)(k-k^*)} \ll \Bigl(\frac{|J|}{N}\Bigr)^{c'}\|S_J\|_{L^2}.\end{aligned}$$

Here we have used the fact that there are at most 2^{k-k^*} values of s such that $I_{k,s} \subseteq J$ for each $k \ge k^*$. We can apply this for $J = [N]$ in particular, recalling that $|J|$ denotes the number of nonzero a_i values contained in J, which in this case is m. This completes the proof that $\|\widetilde{E}\|_{L^2} \ll (m/N)^{c'}\|f\|_{L^2}$ for some positive constant c'.

To show that $\|\widetilde{G}\|_{\mathscr{G}(c)} \ll \|f\|_{L^2}$ for some universal constant $c > 0$, we will use the following lemma. These implications and arguments are well known, however, we include a proof for completeness.

Lemma 16. *Let A denote a fixed, positive constant. For positive constants c, C, we define the following sets of measurable functions*:

$$\begin{aligned} S_1(c) &:= \{f : \mathbb{T} \to \mathbb{R} \text{ s.t. } \|f\|_{L^p} \le c\sqrt{p}A \text{ for all } p \ge 2\}, \\ S_2(c, C) &:= \{f : \mathbb{T} \to \mathbb{R} \text{ s.t. } \mu(|f| \ge \lambda) \le Ce^{-c\lambda^2/A^2} \text{ for all } \lambda \ge 0\}, \\ S_3(c) &:= \{f : \mathbb{T} \to \mathbb{R} \text{ s.t. } \|f\|_{\mathscr{G}(c)} \le A\}, \end{aligned}$$

where $\mu(|f| \ge \lambda)$ denotes the measure of the subset of $x \in \mathbb{T}$ such that $|f(x)| \ge \lambda$. Then, for any $c > 0$, there exist positive constants c', C', c'' (depending only on c) such that $S_1(c) \subseteq S_2(c', C')$ and $S_1(c) \subseteq S_3(c'')$. Similarly, for any $c, C > 0$, there exist positive constants c', c'' (depending only on c, C) such that $S_2(c, C) \subseteq S_1(c')$ and $S_2(c, C) \subseteq S_3(c'')$. Finally, for any $c > 0$, there exist positive constants c', C', c'' (depending only on c) such that $S_3(c) \subseteq S_2(c', C')$ and $S_3(c) \subseteq S_1(c'')$.

Proof. Fixing c, C, we will determine c' such that $S_2(c, C) \subseteq S_3(c')$ (for every A). We consider an $f \in S_2(c, C)$. We consider $c' := d_1 d_2$ as a product of two variables d_1, d_2 whose values will be set later. We assume $d_1 \le 1$. We have

$$\int_{\mathbb{T}} e^{c'|f|^2/A^2}\, d\mu = \int_{\mathbb{T}} e^{d_1 d_2 |f|^2/A^2}\, d\mu \le 1 + d_1 \int_{\mathbb{T}} e^{d_2|f|^2/A^2}\, d\mu, \tag{9}$$

using the inequality $e^{x/a} \le (1/a)e^x + 1$ for all $a \ge 1$ and nonnegative x (this can be seen by considering the Taylor expansion of e^x).

Now, we observe that

$$\int_{\mathbb{T}} e^{d_2|f|^2/A^2}\, d\mu \le \sum_{k\ge 0} \int_{\mathbb{T}} e^{d_2|f|^2/A^2} \cdot \mathbb{I}_{A^2k \le |f|^2 < A^2(k+1)}\, d\mu \le \sum_{k \ge 0} \mu(|f|^2 \ge A^2 k) e^{d_2(k+1)},$$

where $\mathbb{I}_{A^2k \le |f|^2 < A^2(k+1)}$ denotes the characteristic function of the set on which $|f|^2$ takes values between A^2k and $A^2(k+1)$. Since $f \in S_2(c, C)$, we have $\mu(|f|^2 \ge A^2k) \le Ce^{-ck}$ for all $k \ge 0$. Thus, we conclude

$$\int_{\mathbb{T}} e^{d_2|f|^2/A^2}\, d\mu \le \sum_{k\ge 0} Ce^{-ck+d_2(k+1)} = Ce^{d_2} \sum_{k \ge 0} e^{-(c-d_2)k} = \frac{Ce^c}{e^{c-d_2} - 1}$$

whenever $d_2 < c$. Setting $d_2 = c/2$, we obtain the above quantity is $\le Ce^c/(e^{c/2} - 1)$. Letting $d_1 = \min\{1, (e^{c/2} - 1)/(Ce^c)\}$, we have

$$d_1 \int_{\mathbb{T}} e^{d_2|f|^2/A^2} \le 1,$$

and hence $\int_{\mathbb{T}} e^{c'|f|^2/A^2}\, d\mu - 1 \le 1$ for $c' = d_1 d_2$, showing that $f \in S_3(c')$. Note that $c' = d_1 d_2$ depends only on c and C.

Conversely, we observe that, for every $c > 0$, $S_3(c) \subseteq S_2(c, 2)$. To see this, consider $f \in S_3(c)$. Then we have

$$\int_{\mathbb{T}} e^{c|f|^2/A^2}\, d\mu - 1 \le 1 \Rightarrow \int_{\mathbb{T}} e^{c|f|^2/A^2}\, d\mu \le 2.$$

Thus, for any $\lambda > 0$,

$$\mu(|f| \geq \lambda) e^{c\lambda^2/A^2} \leq \int_{\mathbb{T}} e^{c|f|^2/A^2}\, d\mu \leq 2.$$

It follows that $f \in S_2(c, 2)$.

For any $c > 0$, we will now show there exist c', C such that $S_1(c) \subseteq S_2(c', C)$ (for every A). We consider an $f \in S_1(c)$. This means that $\|f\|_p^p \leq c^p p^{p/2} A^p$ for all $p \geq 2$. Thus, for every $\lambda > 0$, $\mu(|f| \geq \lambda)\lambda^p \leq (cA)^p p^{p/2}$, which implies

$$\mu(|f| \geq \lambda) \leq \frac{(cA)^p p^{p/2}}{\lambda^p}. \tag{10}$$

For a fixed λ, we may minimize this quantity over the choices of $p \geq 2$. In the case that $\lambda^2/(ec^2A^2) \geq 2$, we may set p equal to this value, and the quantity in (10) then becomes

$$\left(\frac{cA}{\lambda}\right)^{\lambda^2/(ec^2A^2)} \left(\frac{\lambda^2}{ec^2A^2}\right)^{\lambda^2/(2ec^2A^2)} = e^{-\lambda^2/(2ec^2A^2)}.$$

Hence, by setting $c' = 1/(2ec^2)$, we achieve $\mu(|f| \geq \lambda) \leq e^{-c'\lambda^2/A^2}$ in these cases.

Now, when $\lambda^2/(ec^2A^2) < 2$, we note that $e^{-c'\lambda^2/A^2} \geq e^{-c'(2ec^2)} = e^{-1}$. Thus, setting $C = e$, we have $\mu(|f| \geq \lambda) \leq 1 \leq Ce^{-c'\lambda^2/A^2}$ in these cases. Hence, in all cases, we have that

$$\mu(|f| \geq \lambda) \leq Ce^{-c'\lambda^2/A^2},$$

so $f \in S_2(c', C)$.

Conversely, for any $c, C > 0$, we will show that there exists c' such that $S_2(c, C) \subseteq S_1(c')$ for every A. We consider an $f \in S_2(c, C)$. Then, for every $\lambda \geq 0$, we have $\mu(|f| \geq \lambda) \leq Ce^{-c\lambda^2/A^2}$. We fix $p \geq 2$. We observe that

$$\|f\|_{L^p}^p = p \int_0^\infty \lambda^{p-1} \mu(|f| > \lambda)\, d\lambda \ll p \int_0^\infty \lambda^{p-1} e^{-c\lambda^2/A^2}\, d\lambda.$$

Substituting $\lambda = t^{1/p}$, we see this equals

$$\int_0^\infty e^{-ct^{2/p}/A^2} dt. \tag{11}$$

We note the identity $(p/2)\Gamma(p/2) = \int_0^\infty e^{-s^{2/p}}\, ds$ where Γ denotes the function $\Gamma(z) := \int_0^\infty y^{z-1}e^{-y}\, dy$. Setting $s = (c/A^2)^{p/2} t$, we see that the quantity in (11) is equal to

$$c^{-p/2} A^p \int_0^\infty e^{-s^{2/p}}\, ds = c^{-p/2} A^p \left(\frac{p}{2}\right) \Gamma\left(\frac{p}{2}\right).$$

By Sterling's formula, $\Gamma(p/2) \ll p^{-1/2}(p/(2e))^{p/2}$. Hence,

$$\|f\|_{L^p} \ll A\sqrt{p}(p^{1/(2p)}) \ll A\sqrt{p},$$

as required. $\square$

Appealing to Lemma 16, we see that we may bound the quantity $\|\widetilde{G}_J\|_{\mathcal{G}(c)}$ by considering the p norm. We recall that

$$\widetilde{G}_J = \sum_k \Big(\sum_s |G_{k,s}|^r\Big)^{1/r} + \sum_k \Big(\sum_s |G_{i_{k,s}}|^r\Big)^{1/r},$$

where the sums are restricted to values of k, s such that $I_{k,s}, i_{k,s} \subseteq J$. We let k^* again denote the level of J, so we are only summing over values $k \geq k^*$.

By the triangle inequality, we have

$$\Big\|\sum_k \Big(\sum_s |G_{k,s}|^r\Big)^{1/r} + \sum_k \Big(\sum_s |G_{i_{k,s}}|^r\Big)^{1/r}\Big\|_{L^p} \leq \sum_k \Big\|\Big(\sum_s |G_{k,s}|^r\Big)^{1/r}\Big\|_{L^p} + \sum_k \Big\|\Big(\sum_s |G_{i_{k,s}}|^r\Big)^{1/r}\Big\|_{L^p},$$

which, by another application of the triangle inequality, is equal to

$$\begin{aligned}\sum_k \Big\|\sum_s |G_{k,s}|^r\Big\|_{L^{p/r}}^{1/r} + \sum_k \Big\|\sum_s |G_{i_{k,s}}|^r\Big\|_{L^{p/r}}^{1/r} &\leq \sum_k \Big(\sum_s \|\,|G_{k,s}|^r\,\|_{L^{p/r}}\Big)^{1/r} + \sum_k \Big(\sum_s \|\,|G_{i_{k,s}}|^r\,\|_{L^{p/r}}\Big)^{1/r} \\ &= \sum_k \Big(\sum_s \|G_{k,s}\|_{L^p}^r\Big)^{1/r} + \sum_k \Big(\sum_s \|G_{i_{k,s}}\|_{L^p}^r\Big)^{1/r}.\end{aligned}$$

Now, using that $\|G_{k,s}\|_{L^p} \ll \sqrt{p}\|S_{I_{k,s}}\|_{L^2}$ and $\|G_{i_{k,s}}\|_{L^p} \ll \sqrt{p}\|S_{i_{k,s}}\|_{L^2}$, by Lemma 16 and $\|S_{I_{k,s}}\|_{L^2} \ll \|S_J\|_{L^2} 2^{-(k-k^*)/2}$ and $\|S_{i_{k,s}}\|_{L^2} \ll \|S_J\|_{L^2} 2^{-(k-k^*)/2}$, we have

$$\|\widetilde{G}_J\|_{L^p} \leq \sum_{k \geq k^*} \Big(\sum_s \|G_{k,s}\|_{L^p}^r\Big)^{1/r} + \sum_{k \geq k^*} \Big(\sum_s \|G_{i_{k,s}}\|_{L^p}^r\Big)^{1/r} \ll \sqrt{p}\|S_J\|_{L^2} \sum_{k \geq k^*} \Big(\sum_s 2^{-r(k-k^*)/2}\Big)^{1/r}.$$

Since the sum of s ranges over at most 2^{k-k^*} values (recall we only include values of s such that $I_{k,s} \subseteq J$) and $r > 2$, this is

$$\ll \sqrt{p}\|S_J\|_{L^2} \sum_{k \geq k^*} 2^{(k-k^*)(r^{-1}-2^{-1})} \ll \sqrt{p}\|S_J\|_{L^2}.$$

It thus follows from Lemma 16 that

$$\|\widetilde{G}_J\|_{\mathcal{G}(c)} \ll \|S_J\|_{L^2}$$

for some positive constant c. Lastly, we have that $\|\widetilde{G}_J\|_{\mathcal{G}} \ll \|\widetilde{G}_J\|_{\mathcal{G}(c)}$ from the definition of the Orlicz norm.

5. Proof of the main result

We are now ready to prove the following.

Theorem 17. *Let $\Phi := \{\phi_n(x)\}_{n=1}^N$ be an ONS such that $\sum_{n=1}^N |\phi_n(x)|^2 \leq N$. Then there exists $Q \subset \mathbb{O}(N)$ with $\mathbb{P}[Q] \geq 1 - Ce^{-cN^{2/5}}$ such that, for $O \in Q$, the alternate ONS $\Phi(O)$ satisfies*

$$\|\mathcal{V}^2 f\|_{L^2} \ll \sqrt{\log\log(N)}\|f\|_{L^2}.$$

Proof. Here we use the mass decomposition (into dyadic subintervals $I_{k,s}$) stated previously. We use the following easily verified fact; see Lemma 29 of [Lewko and Lewko 2012a].

Lemma 18. *For every $J \subseteq [N]$ ($J \neq \varnothing$), there exist $\tilde{J}_\ell, \tilde{J}_r \in \mathcal{A}$ and $i_J \in [N]$ such that $\tilde{J} := \tilde{J}_\ell \cup i_J \cup \tilde{J}_r$ is an interval* (*that is*, J_ℓ, i_J, J_ℓ *are adjacent*), $J \subseteq \tilde{J}$, *and* $M(\tilde{J}) \leq 2M(J)$.

Without loss of generality, we set $\|f\|_{L^2} = 1$, and we have the pointwise inequality

$$|\mathcal{V}^2 f(x)|^2 \ll \sum_{k,s} |\widetilde{S}_{I_{k,s}} \mathbb{1}_{B(I_{k,s})}|^2 + \sum_{k,s} |S_{i_{k,s}}|^2 + \log\log(N),$$

where $B(I_{k,s}) \subseteq \mathbb{T}$ is the set such that $|\widetilde{S}_{I_{k,s}}(x)|^2 \geq C \log\log(N) M(I_{k,s})$, for a fixed constant C whose value will be chosen to be sufficiently large. Appealing to Proposition 15, for each $I_{k,s}$ we can decompose $\widetilde{S}_{I_{k,s}} = \widetilde{G}_{I_{k,s}} + \widetilde{E}_{I_{k,s}}$. We then define $B_G(I_{k,s}) \subseteq \mathbb{T}$ by $|\widetilde{G}_{I_{k,s}}(x)|^2 \geq (C/10) \log\log(N) M(I_{k,s})$ and $B_E(I_{k,s}) \subseteq \mathbb{T}$ by $|\widetilde{E}_{I_{k,s}}(x)|^2 \geq (C/10) \log\log(N) M(I_{k,s})$.

Clearly $\int \sum_{k,s} |S_{i_{k,s}}|^2 \leq 1$ is acceptable, so it suffices to show that

$$\int_{\mathbb{T}} \sum_{k,s} |\widetilde{S}_{I_{k,s}} \mathbb{1}_{B(I_{k,s})}|^2 \, d\mu \ll 1.$$

Now, appealing to the decomposition above, we have

$$\int_{\mathbb{T}} \sum_{k,s} |\widetilde{S}_{I_{k,s}} \mathbb{1}_{B(I_{k,s})}|^2 \, d\mu \ll \int_{\mathbb{T}} \sum_{k,s} |\widetilde{G}_{I_{k,s}} \mathbb{1}_{B_G(I_{k,s})}|^2 \, d\mu + \int_{\mathbb{T}} \sum_{k,s} |\widetilde{E}_{I_{k,s}} \mathbb{1}_{B_E(I_{k,s})}|^2 \, d\mu.$$

First we estimate

$$\int_{\mathbb{T}} \sum_{k,s} |\widetilde{E}_{I_{k,s}} \mathbb{1}_{B_E(I_{k,s})}|^2 \, d\mu \ll \int_{\mathbb{T}} \sum_{k,s} |\widetilde{E}_{I_{k,s}}|^2 \, d\mu.$$

Employing notation used above, we let $I_k^a := \{I_{k,s} \text{ s.t. } |I_{k,s}| \leq 2^{-k/2} N\}$ and $I_k^b := \{I_{k,s} \text{ s.t. } |I_{k,s}| > 2^{-k/2} N\}$. Thus $I \in I_k^a$ implies $|I| \leq 2^{-k/2} N$ and $|I_k^b| \leq 2^{k/2}$. We then have

$$\int_{\mathbb{T}} \sum_{k,s} |\widetilde{E}_{I_{k,s}}|^2 \, d\mu = \int_{\mathbb{T}} \sum_{I_{k,s} \in I_k^a} |\widetilde{E}_{I_{k,s}}|^2 \, d\mu + \int_{\mathbb{T}} \sum_{I_{k,s} \in I_k^b} |\widetilde{E}_{I_{k,s}}|^2 \, d\mu.$$

Using that $I \in I_k^a$ implies $|I| \leq 2^{-k/2} N$, we have $\int |\widetilde{E}_{I_{k,s}}|^2 \ll 2^{-c'k/2} \|S_{I_{k,s}}\|_{L^2}^2 \ll 2^{-k-c'k/2}$. Thus

$$\int_{\mathbb{T}} \sum_{I_{k,s} \in I_k^a} |\widetilde{E}_{I_{k,s}}|^2 \, d\mu \ll \sum_k 2^{-c'k/2} \ll 1.$$

Next, using that $|I_k^b| \leq 2^{k/2}$ and $\int |\widetilde{E}_{I_{k,s}}|^2 \ll 2^{-k}$, we have

$$\int_{\mathbb{T}} \sum_{I_{k,s} \in I_k^b} |\widetilde{E}_{I_{k,s}}|^2 \ll \sum_k 2^{-k/2} \ll 1.$$

Finally, we estimate

$$\int_{\mathbb{T}} \sum_{k,s} |\widetilde{G}_{I_{k,s}} \mathbb{1}_{B_G(I_{k,s})}|^2 \, d\mu.$$

We can choose C sufficiently large so that $|B_G(I_{k,s})| \ll 1/\log^{10}(N)$ for all k, s (here, $|B_G(I_{k,s})|$ denotes the μ-measure). To see this, recall that $\|\widetilde{G}_{I_{k,s}}\|_{\mathscr{G}(c)} \ll \sqrt{M(I_{k,s})}$. By Lemma 16, there exists a constant $c' > 0$ such that

$$\mu(|\widetilde{G}_{I_{k,s}}| \geq \lambda) \ll e^{-c'\lambda^2/M(I_{k,s})}$$

for all $\lambda \geq 0$. Setting $\lambda^2 = (C/10)\log\log(N)M(I_{k,s})$, we obtain

$$|B_G(I_{k,s})| \ll \log(N)^{-c'C/10}.$$

We can then choose C sufficiently large with respect to c' to make this estimate $\ll 1/\log^{10}(N)$.

Now we split the sum at $k = 100\log(N)$ so

$$\int_{\mathbb{T}} \sum_{k,s} |\widetilde{G}_{I_{k,s}} \mathbb{1}_{B_G(I_{k,s})}|^2 \, d\mu = \int_{\mathbb{T}} \sum_{\substack{k,s \\ k \geq 100\log(N)}} |\widetilde{G}_{I_{k,s}} \mathbb{1}_{B_G(I_{k,s})}|^2 \, d\mu + \int_{\mathbb{T}} \sum_{\substack{k,s \\ k < 100\log(N)}} |\widetilde{G}_{I_{k,s}} \mathbb{1}_{B_G(I_{k,s})}|^2 \, d\mu.$$

By the Cauchy–Schwarz inequality,

$$\int_{\mathbb{T}} \sum_{\substack{k,s \\ k < 100\log(N)}} |\widetilde{G}_{I_{k,s}} \mathbb{1}_{B_G(I_{k,s})}|^2 \, d\mu \ll \sum_{k,s} \|\widetilde{G}_{I_{k,s}}\|_4^2 \, \|1_{B_G(I_{k,s})}\|_4^2.$$

Now, by Lemma 16, we have $\|\widetilde{G}_{I_{k,s}}\|_4^2 \ll \|S_{I_{k,s}}\|_{L^2}^2 \ll 2^{-k}$, and, by the previous estimate, $\|\mathbb{1}_{B_G(I_{k,s})}\|_4^2 \ll 1/\log^5(N)$. Thus we have shown that the quantity above is

$$\ll \frac{1}{\log^5(N)} \sum_{\substack{k,s \\ k < 100\log(N)}} \|\widetilde{G}_{I_{k,s}}\|_4^2 \ll \frac{1}{\log^4(N)} \ll 1.$$

Lastly, let $T \subset [N]$ denote the set of indices appearing in some $I_{k,s}$ for $k \geq 100\log(N)$. Note that any index will appear in at most N such intervals, and that $M(I_{k,s}) \leq N^{-100}$ if $k \geq 100\log(N)$. Thus $|a_n|^2 \ll N^{-100}$ for $n \in T$. Thus we have

$$\int_{\mathbb{T}} \sum_{\substack{k,s \\ k \geq 100\log(N)}} |\widetilde{G}_{I_{k,s}} \mathbb{1}_{B_G(I_{k,s})}|^2 \, d\mu \ll N^2 \int_{\mathbb{T}} \sum_{n \in T} |a_n \phi_n(x)|^2 \, d\mu \ll N^{-98} \int_{\mathbb{T}} \sum_{n \in T} |\phi_n(x)|^2 \, d\mu \ll 1.$$

This completes the proof. □

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A PARTIAL DATA RESULT FOR THE MAGNETIC SCHRÖDINGER INVERSE PROBLEM

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This article shows that restricting the domain of the Dirichlet–Neumann map to functions supported on a certain part of the boundary, and measuring the output on, roughly speaking, the rest of the boundary, uniquely determines a magnetic Schrödinger operator. If the domain is strongly convex, either the subset on which the Dirichlet–Neumann map is measured or the subset on which the input functions have support may be made arbitrarily small. The key element of the proof is the modification of a Carleman estimate for the magnetic Schrödinger operator using operators similar to pseudodifferential operators.

1. Introduction

Let $n \geq 2$, and let Ω be a simply connected bounded domain in $\mathbb{R}^{n+1}$ with smooth boundary. Suppose W is a C^2 vector field on $\mathbb{R}^{n+1}$ and q is an L^∞ function on $\mathbb{R}^{n+1}$. Then define the magnetic Schrödinger operator $\mathscr{L}_{W,q}$ with magnetic potential W and electric potential q by

$$\mathscr{L}_{W,q} = (D+W)^2 + q,$$

where $D = -i\nabla$. I will assume that q and W are such that zero is not an eigenvalue of $\mathscr{L}_{W,q}$ on Ω. Then the Dirichlet problem

$$\mathscr{L}_{W,q} u = 0, \quad u|_{\partial\Omega} = g$$

has a unique solution $u \in H^1(\Omega)$ for each $g \in H^{1/2}(\partial\Omega)$. Therefore for $g \in H^{1/2}(\partial\Omega)$, we can define the Dirichlet–Neumann map $\Lambda_{W,q}$ by

$$\Lambda_{W,q} g = (\partial_\nu + iW\cdot\nu)u|_{\partial\Omega},$$

where ν is the outward unit normal and u is the unique solution to the Dirichlet problem with boundary value g. This gives a well-defined map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

The basic inverse problem associated to the magnetic Schrödinger operator $\mathscr{L}_{W,q}$ is to recover the electric potential q and the magnetic field dW from knowledge of $\Lambda_{W,q}$. (Here dW makes sense by identifying W with the 1-form $W_1dx_1 + \cdots + W_{n+1}dx_{n+1}$.) We cannot hope to recover W itself, since the Dirichlet–Neumann map is invariant under the gauge transformation $W \mapsto W + \nabla\Psi$ whenever $\Psi \in C^1(\overline{\Omega})$ and $\Psi|_{\partial\Omega} = 0$. However, identifying dW identifies W up to this gauge transformation.

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This can be thought of as a generalization of the Calderón problem [1980], which can be written in this form with $W \equiv 0$ in the case of smooth enough conductivity (see [Sylvester and Uhlmann 1987]).

Sylvester and Uhlmann [1987] showed that in the Calderón problem, the Dirichlet–Neumann map determines q. For the magnetic Schrödinger problem, Sun showed that the Dirichlet–Neumann map determines dW and q when W is small enough, in a certain sense. Nakamura, Sun, and Uhlmann [Nakamura et al. 1995] removed the smallness assumption and showed that the Dirichlet–Neumann map determines dW and q for W in C^2 and q in L^∞. Tolmasky [1998] and Salo [2004] improved the regularity conditions on W to $C^{2/3+\varepsilon}$ and Dini continuous, respectively. Salo [2006] also gave a proof for $W \in C^{1+\varepsilon}$ involving a reconstruction method.

Given that $\Lambda_{W,q}$ determines dW and q, a further question might be whether *partial* knowledge of $\Lambda_{W,q}$ determines dW and q. In particular, one might ask whether restricting the domain of the Dirichlet–Neumann map to functions supported on a particular subset of the boundary still gives enough information to determine dW and q. Alternatively, one might ask whether measuring the output of the Dirichlet–Neumann map on a particular subset of the boundary still gives enough information to determine dW and q.

Kenig, Sjöstrand, and Uhlmann [Kenig et al. 2007] proved a result for the Calderón problem addressing both of these questions. Roughly speaking, they proved that restricting the domain of the Dirichlet–Neumann map to functions supported on particular subsets of the boundary and measuring the output on the rest of the boundary determines q. Together with Dos Santos Ferreira, they proved a similar result for the magnetic Schrödinger problem in [Dos Santos Ferreira et al. 2007], but without being able to restrict the domain of $\Lambda_{W,q}$. The main results of this paper are to impose that restriction, and thus show that a result analogous to the one in [Kenig et al. 2007] also holds for the magnetic Schrödinger problem.

In order to describe these results more fully, we need to describe the subsets of the boundary involved. Assume that x_0 is not in the closure of the convex hull of Ω, and define the front and back of $\partial\Omega$ (with respect to x_0) by

$$\partial\Omega_- = \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) \leq 0\}, \quad \partial\Omega_+ = \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) \geq 0\},$$

where $\nu(x)$ is the outward unit normal at x.

The main results of this paper are the following two theorems.

Theorem 1.1. *Let W_1 and W_2 be C^2 vector fields on $\overline{\Omega}$, and let q_1 and q_2 be L^∞ functions on Ω. Let $\Gamma_- \subset \partial\Omega$ be a neighborhood of $\partial\Omega_-$, and let $\Gamma_+ \subset \partial\Omega$ be a neighborhood of $\partial\Omega_+$. Suppose*

$$\Lambda_{W_1,q_1} g|_{\Gamma_-} = \Lambda_{W_2,q_2} g|_{\Gamma_-}$$

for all $g \in H^{1/2}(\partial\Omega)$ with support contained in Γ_+.

Then $q_1 = q_2$ and $dW_1 = dW_2$.

Theorem 1.2. *Let W_1 and W_2 be C^2 vector fields on $\overline{\Omega}$, and let q_1 and q_2 be L^∞ functions on Ω. Let $\Gamma_+ \subset \partial\Omega$ be a neighborhood of $\partial\Omega_+$, and let $\Gamma_- \subset \partial\Omega$ be a neighborhood of $\partial\Omega_-$. Suppose*

$$\Lambda_{W_1,q_1} g|_{\Gamma_+} = \Lambda_{W_2,q_2} g|_{\Gamma_+}$$

for all $g \in H^{1/2}(\partial\Omega)$ *with support contained in* Γ_-.

Then $q_1 = q_2$ *and* $dW_1 = dW_2$.

The second theorem is essentially the first theorem after the conformal transformation on Ω given by inversion in x_0. Imposing the condition $W_1 \equiv W_2 \equiv 0$ in these theorems would give the results from [Kenig et al. 2007], and removing the restriction on the support of g would give the results from [Dos Santos Ferreira et al. 2007].

Roughly speaking, the first theorem says that if the Dirichlet–Neumann map is known on a neighborhood of the front for functions supported on a neighborhood of the back, then potentials can be determined. The second theorem says something similar, but with the roles of the front and back reversed.

If the domain Ω is nice enough, then the front can be made arbitrarily small. For example, if Ω is strongly convex (convex, and the intersection of the boundary with any tangent hyperplane to the boundary consists only of one point), then the front can be contained in an arbitrarily small open subset of the boundary, for the right choice of x_0. This gives us the following corollary.

Corollary 1.3. *Suppose* Ω *is a smooth bounded strongly convex domain in* $\mathbb{R}^{n+1}$. *Let* W_1 *and* W_2 *be* C^2 *vector fields on* $\overline{\Omega}$, *and let* q_1 *and* q_2 *be* L^∞ *functions on* Ω. *Then for any nonempty open subset* Γ_1 *of the boundary*, *there exists a neighborhood* Γ_2 *of* $\Gamma_1^c := \partial\Omega \setminus \Gamma_1$ *such that if*

$$\Lambda_{W_1,q_1} g|_{\Gamma_1} = \Lambda_{W_2,q_2} g|_{\Gamma_1}$$

for all $g \in H^{1/2}(\partial\Omega)$ *with support contained in* Γ_2, *then* $q_1 = q_2$ *and* $dW_1 = dW_2$.

Alternatively, *for any nonempty open subset* Γ_2 *of the boundary*, *there exists a neighborhood* Γ_1 *of* Γ_2^c *in* Ω *such that if*

$$\Lambda_{W_1,q_1} g|_{\Gamma_1} = \Lambda_{W_2,q_2} g|_{\Gamma_1}$$

for all $g \in H^{1/2}(\partial\Omega)$ *with support contained in* Γ_2, *then* $q_1 = q_2$ *and* $dW_1 = dW_2$.

The first part of the corollary says that in particular, the Dirichlet–Neumann map can be measured on an arbitrarily small subset of the boundary. The second part of the corollary says that alternatively, the input functions may be restricted to an arbitrarily small subset of the boundary.

Theorem 1.2 can either be proved from Theorem 1.1 by the change of variables mentioned above, or proved in the same manner as Theorem 1.1, making the changes indicated at the end of Section 8. Therefore most of this paper will be devoted to proving Theorem 1.1. From here on, unless otherwise noted, I will assume Γ_+, Γ_- and Ω are as in Theorem 1.1.

The key to the proof of Theorem 1.1 is the construction of complex geometrical optics (CGO) solutions to the system

$$\mathscr{L}_{W,q} u = 0 \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0, \tag{1-1}$$

where $\Gamma_+^c := \partial\Omega \setminus \Gamma_+$. This in turn requires a Carleman estimate for $\mathscr{L}_{W,q}$, which can be described as follows.

Let φ be a limiting Carleman weight on Ω; that is, a real-valued smooth function that has nonvanishing gradient on Ω and satisfies

$$\langle \varphi'' \nabla\varphi, \nabla\varphi \rangle + \langle \varphi'' \xi, \xi \rangle = 0$$

whenever $|\xi| = |\nabla\varphi|$ and $\nabla\varphi \cdot \xi = 0$. Define

$$\mathcal{L}_{\varphi,W,q} = h^2 e^{\varphi/h} \mathcal{L}_{W,q} e^{-\varphi/h}.$$

Here h is a semiclassical parameter; from here on, all Sobolev spaces and Fourier transforms in this paper are semiclassical, unless otherwise specified, with h being the semiclassical parameter. Thus $\|u\|_{H^1}$ means the norm defined by

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|h\nabla u\|_{L^2}^2,$$

and $\|u\|_{H^{-1}}$ means the dual norm to this, and so forth.

Then we have the following Carleman estimate.

Theorem 1.4. *Suppose Ω' is a smooth domain with $\Omega \subset \Omega'$ and $\partial\Omega' \cap \partial\Omega = \Gamma_+^c$, where Γ_+ is as described in Theorem 1.1. Then if $w \in C_0^\infty(\Omega)$,*

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,W,q} w\|_{H^{-1}(\Omega')}.$$

The proof of this theorem is the main new ingredient in this paper. It differs from the Carleman estimate in [Dos Santos Ferreira et al. 2007] in that this one can be used in a Hahn–Banach argument to give solutions that vanish on E. The rest of the proof of Theorem 1.1 follows the proofs in [Kenig et al. 2007; Dos Santos Ferreira et al. 2007] fairly closely. Thus, the next seven sections will be devoted to the proof of Theorem 1.4. In Section 9, I will use this estimate to construct CGO solutions to (1-1). Once these are constructed, the proof of Theorem 1.1 follows by an argument more or less identical to that in [Dos Santos Ferreira et al. 2007]. This argument is outlined in Section 10 for completeness.

2. Outline of the proof of Theorem 1.4

In order to outline the proof of Theorem 1.4, I will give a rough sketch of the proof for a special case. Choose Cartesian coordinates (x, y) on $\mathbb{R}^{n+1}$ such that $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, and suppose that Ω lies in the set $\mathbb{R}_+^{n+1} = \{y > 0\}$, with a subset of $\partial\Omega$ lying on the hyperplane $\{y = 0\}$. Label the subset $\partial\Omega \cap \{y = 0\}$ by Γ_+^c. Then I want to show that

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,W,q} w\|_{H^{-1}(\mathbb{R}_+^{n+1})}$$

for $w \in C_0^\infty(\Omega)$ and $\varphi(x, y) = y$. The starting point is the following estimate. Define

$$\mathcal{L}_\varphi = h^2 e^{\varphi/h} \triangle e^{-\varphi/h}$$

and

$$\mathcal{L}_{\varphi,\varepsilon} = e^{\varphi^2/2\varepsilon} \mathcal{L}_\varphi e^{-\varphi^2/2\varepsilon}.$$

Proposition 2.1 [Dos Santos Ferreira et al. 2007, Equation (2.12)]. *If φ is a limiting Carleman weight, and $w \in C_0^\infty(\Omega)$, then*

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{H^1(\Omega)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w\|_{L^2(\Omega)}.$$

A note on inequalities here: inequalities of the form $F(w,h) \lesssim G(w,h)$ mean that there exists $h_0 > 0$ independent of w such that for $h \le h_0$, the inequality $F(w,h) \le CG(w,h)$ holds for some positive constant C independent of w and h. In the case of this inequality, the constant implied in the $\lesssim$ sign is independent of ε as well.

Now set $\varphi(x,y) = y$ and define a new domain Ω_2 such that $\Omega \subset \Omega_2 \subset \mathbb{R}^{n+1}_+$, with $\Gamma_+^c \subset \partial\Omega_2$. Proposition 2.1 still holds on Ω_2. Now the objective is to find an operator J with the following properties.

(1) J has a right inverse, denoted by J^{-1}, and J^{-1} preserves smoothness.

(2) J and J^{-1} preserve support with respect to y in the positive direction: if the support of u is in the set $\{y \ge y_0\}$, so are the supports of Ju and $J^{-1}u$.

(3) The commutators of J with differential operators behave as though J were a semiclassical pseudo-differential operator of order 1.

(4) J is bounded from $H^1(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^{n+1}_+)$.

(5) $\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u\|_{L^2(\mathbb{R}^{n+1}_+)}$.

If such an operator existed, the argument could go like this: Suppose $w \in C_0^\infty(\Omega)$, and let $\chi \in C^\infty(\mathbb{R}^{n+1}_+)$ be a cutoff function that is identically one on Ω but supported within Ω_2. Then $\chi J^{-1} w \in C_0^\infty(\Omega_2)$, so it can be plugged into Proposition 2.1 to give

$$\frac{h}{\sqrt{\varepsilon}} \|\chi J^{-1} w\|_{H^1(\Omega_2)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} \chi J^{-1} w\|_{L^2(\Omega_2)}.$$

Here we are using property (1) to get J^{-1} and property (2) to ensure that $\chi J^{-1} w$ has the right support. Now we can use property (4) on the left and (5) on the right to get

$$\frac{h}{\sqrt{\varepsilon}} \|J\chi J^{-1} w\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \|J\mathcal{L}_{\varphi,\varepsilon} \chi J^{-1} w\|_{H^{-1}(\mathbb{R}^{n+1}_+)}.$$

The commutator properties tell us that this is

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w\|_{H^{-1}(\mathbb{R}^{n+1}_+)},$$

with error terms small enough to hide in the left side, for ε small enough. Then $\mathcal{L}_{\varphi,\varepsilon} w = \mathcal{L}_{\varphi,\varepsilon,W,q} w$ up to a similarly permissible error, where

$$\mathcal{L}_{\varphi,\varepsilon,W,q} = e^{\varphi^2/2\varepsilon} \mathcal{L}_{\varphi,W,q} e^{-\varphi^2/2\varepsilon},$$

and noting that $e^{\varphi^2/2\varepsilon}$ is smooth and bounded on Ω finishes the proof.

It still remains, of course, to find the magic operator J. Consider the operator J defined by

$$\widehat{Ju}(\xi, y) = (h\partial_y + F(\xi))\hat{u}(\xi, y),$$

where the hat $\hat{}$ signifies the semiclassical Fourier transform in the x variables, and F is a smooth function on $\mathbb{R}^n$ such that $\left|F(\xi)-(1+|\xi|)\right| \le \delta$ for some small δ. This has a right inverse J^{-1} given by

$$\widehat{J^{-1}u}(\xi, y) = \frac{1}{h}\int_0^y \hat{u}(\xi, t)e^{F(\xi)(t-y)/h}\,dt,$$

which satisfies property (1). Now it is relatively straightforward to see that properties (2) and (4) are satisfied, and with a little more work, we can obtain the kind of commutator properties needed for property (3).

Unfortunately, property (5) fails to hold in general. Instead we have a new property (5′), that

$$\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u - g_u\|_{L^2(\mathbb{R}^{n+1}_+)}$$

where

$$\hat{g}_u(\xi, y) = \frac{2F(\xi)}{h}\int_0^\infty \hat{u}(\xi, t)e^{-F(\xi)(t+y)/h}\,dt.$$

However, the proof only relies on property (5) applied to functions u of the form $u = \mathscr{L}_{\varphi,\varepsilon}v$, where $v \in C_0^\infty(\Omega_2)$. For these functions,

$$\hat{g}_u(\xi, y) = \frac{2F(\xi)}{h}\int_0^\infty \widehat{\mathscr{L}_{\varphi,\varepsilon}v}(\xi, t)e^{-F(\xi)(t+y)/h}\,dt,$$

where

$$\widehat{\mathscr{L}_{\varphi,\varepsilon}v}(\xi, t) = \left(h^2\partial_t^2 - 2h\partial_t + 1 - |\xi|^2\right)\hat{v}(\xi, t)$$

plus some acceptably small error. The idea is now that by using integration by parts, together with a good choice of F, we can get g_u to be small enough that

$$\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

To do this, we can split up v as $v = v_1 + v_2$, where $\hat{v}_1(\xi, t)$ is supported only for $|\xi| \le \frac{1}{2}$, and $\hat{v}_2(\xi, t)$ is supported only for $|\xi| > \frac{1}{3}$, say. Then $g_u = \gamma_1 + \gamma_2$, where γ_j is the part that corresponds to v_j. Then for $\hat{\gamma}_1$, integration by parts gives

$$\frac{2F(\xi)}{h}\int_0^\infty \left(F(\xi)^2 - 2F(\xi) + 1 - |\xi|^2\right)\hat{v}_1(\xi, t)e^{-F(\xi)(t+y)/h}\,dt$$

plus an acceptably small error, and then using the fact that F is close to $1+|\xi|$ gives

$$\|\gamma_1\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|v_1\|_{H^2(\mathbb{R}^{n+1}_+)}.$$

Since v_1 is only supported for small frequencies, the operator $\mathscr{L}_{\varphi,\varepsilon}$ is invertible on the support of v_1, so

$$\|\gamma_1\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|\mathscr{L}_{\varphi,\varepsilon}v_1\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Meanwhile, in the large frequency case, we can factor $\widehat{\mathscr{L}_{\varphi,\varepsilon}}$ as

$$\left(h\partial_t - (1+|\xi|)\right)\left(h\partial_t - (1-|\xi|)\right)$$

up to some acceptably small error, and do integration by parts only with the first factor. (The nonsmoothness of $|\xi|$ will cause trouble in the factoring at small frequencies, which is the reason for splitting up the argument like this.) Then $\hat{\gamma}_2$ becomes

$$\frac{2F(\xi)}{h}\int_0^\infty \big(F(\xi)-(1+|\xi|)\big)\big(h\partial_t-(1-|\xi|)\big)\hat{v}_2(\xi,t)e^{-F(\xi)(t+y)/h}\,dt$$

plus some good enough error, and so we get something like

$$\|\gamma_2\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|z\|_{H^1(\mathbb{R}^{n+1}_+)},$$

where $\hat{z}=\big(h\partial_t-(1-|\xi|)\big)\hat{v}_2$. Since $\mathscr{L}_{\varphi,\varepsilon}v\sim\big(h\partial_t-(1+|\xi|)\big)z$, and the operator $h\partial_t-(1+|\xi|)$ is well behaved, we can get

$$\|\gamma_2\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|\mathscr{L}_{\varphi,\varepsilon}v_2\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Adding these two parts together and using some commutator estimates on the right side gives

$$\|g_u\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|u\|_{L^2(\mathbb{R}^{n+1}_+)},$$

so

$$\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u\|_{L^2(\mathbb{R}^{n+1}_+)}$$

for u of this form. This finishes the argument. Changes in $\mathscr{L}_{\varphi,\varepsilon}$ of $O(\delta)$, roughly speaking, do not affect the argument. Therefore the argument still works if Γ_+^c coincides with a graph of the form $y=f(x)$, as long as ∇f is small enough, by using a change of variables that flattens Γ_+^c while making only $O(\delta)$ changes to $\mathscr{L}_{\varphi,\varepsilon}$.

These ideas are the basis of the argument used to prove Theorem 1.4. There are three key changes that make everything much more complicated, however. Firstly, in order to achieve results of the form of Theorems 1.1 and 1.2, we will need to work with the logarithmic weight $\varphi=\log|x-x_0|$, and in spherical coordinates centered at x_0. Then we will work with Γ_+^c's that coincide with graphs of the form $r=f(\theta)$, and work with small subsets on which the spherical coordinates look nearly Euclidean. Secondly, instead of looking at cases where ∇f is small, we will treat cases where ∇f is almost constant. This argument works nearly the same way as the argument outlined above, but requires us to use operators that depend on that constant. In fact, we will need to split the small and large frequency cases much earlier in the argument, and introduce separate operators J_s and J_ℓ for the two cases. Thirdly, we will need to glue together many such estimates at the end of the proof to get Theorem 1.4.

The proof will be presented over the next six sections. In Section 3, I will state the small subset version of the Carleman estimate, and begin the proof by making the change of variables to "flatten" Γ_+^c appropriately. In Section 4, I will split up the problem into separate propositions for the small and large frequency cases, and show that the proofs of these propositions suffice. In Section 5, I will prove analogues of properties (1) through (2) and (5′) for operators of a certain form. Section 6 then contains the small frequency argument, and Section 7 contains the large frequency argument, thus finishing the proof of the small subset version of the Carleman estimate. Finally, in Section 8, I will glue together the small subset estimates in the appropriate way to prove Theorem 1.4.

3. An initial Carleman estimate

For the rest of this paper, we will fix φ to be the logarithmic weight $\varphi(x) = \log|x - x_0|$ unless otherwise stated. Without loss of generality, we will also assume that $x_0 = 0$.

To begin, we should fix coordinates on $\mathbb{R}^{n+1}$. Since 0 is outside the convex hull of Ω, there must exist $r_0 > 0$ such that Ω lies outside the ball of radius r_0 centered at the origin. Moreover, Ω must lie entirely on one side of a hyperplane through the origin. If we choose Cartesian coordinates $x_1, \dots, x_{n+1}$ on $\mathbb{R}^{n+1}$ such that Ω lies entirely in the half-space $\{x_{n+1} > 0\}$, then we can define a map $\sigma : (\mathbb{R}^n \setminus B_{r_0,0}) \cap \{x_{n+1} > 0\} \to [r_0, \infty) \times (0, \pi) \times \cdots \times (0, \pi)$ by

$$\sigma(x_1, \dots, x_{n+1}) = (r, \theta_1, \dots, \theta_n),$$

where

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ &\vdots \\ x_n &= r\sin\theta_1 \dots \sin\theta_{n-1}\cos\theta_n, \\ x_{n+1} &= r\sin\theta_1 \dots \sin\theta_n. \end{aligned}$$

This fixes a set of spherical coordinates on $(\mathbb{R}^n \setminus B_{r_0,0}) \cap \{x_{n+1} > 0\}$. On any compact subset of this space, σ is a diffeomorphism with bounded derivatives; the singularities in σ occur in the other half-space.

Now we can begin by proving the following special version of Theorem 1.4.

Proposition 3.1. *Suppose that $f : S^n \to (r_0, \infty)$ is a C^∞ function such that Ω lies entirely in the region $A_O = \{(r, \theta) \mid r \geq f(\theta)\} \subset \mathbb{R}^{n+1}$, and Γ_+^c is a subset of the graph $r = f(\theta)$. Suppose also that for all $(r, \theta) \in \Omega$,*

$$|\sin\theta_j - 1| \leq \mu \quad \text{for } j = 1, \dots, n-1 \tag{3-1}$$

and

$$|\nabla_{S^n} \log f - Ke_n|_{S^n} \leq \mu, \tag{3-2}$$

where e_n is the vector field on S^n given in coordinates by $(0, \dots, 0, 1)$, and ∇_{S^n} and $|\cdot|_{S^n}$ indicate the gradient and metric on the unit sphere. Then if $w \in C_0^\infty(\Omega)$, then

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)}.$$

The inequality (3-1) is designed to force the metric on the unit sphere on the set

$$\{\theta \in S^n \mid (r, \theta) \in \Omega \text{ for some } r\}$$

to be nearly Euclidean, and the inequality (3-2) is designed to ensure that $\nabla_{S^n} \log f$ is nearly constant on Ω.

To prove this, we will need to do some work with a domain Ω_2 that is slightly larger than Ω, but still bounded. Take $\Omega_2 \subset A_O$ to be a smooth bounded domain that contains Ω such that $\Gamma_+^c \subset \partial\Omega_2$. We can pick Ω_2 to lie in $(\mathbb{R}^n \setminus B_{r_0,0}) \cap \{x_{n+1} > 0\}$, with

$$|\sin\theta_j - 1| \leq 2\mu \quad \text{for } j = 1, \dots, n-1$$

and

$$|\nabla_{S^n} \log f - K e_n|_{S^n} \leq 2\mu$$

for all $(r, \theta) \in \Omega_2$.

Recall that Proposition 2.1, proved in [Dos Santos Ferreira et al. 2007], says that if $w \in C_0^\infty(\Omega_2)$, then

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{H^1(\Omega_2)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{L^2(\Omega_2)}.$$

We can make a change of variables using the map $(r, \theta) \mapsto (r/f(\theta), \theta)$. This is a diffeomorphism from A_O to $\mathbb{R}^{n+1} \setminus B$, where B is the open ball of radius 1 centered at the origin, with the inverse map $(r, \theta) \mapsto (rf(\theta), \theta)$. Let $\tilde{\Omega}$ and $\tilde{\Omega}_2$ be the images of Ω and Ω_2 under this map. This diffeomorphism maps Γ_+^c to a part of the unit sphere S^n, thus "flattening" it out appropriately. This change of variables leaves the θ variables alone, so it is still the case that $\tilde{\Omega}_2$ lies in $(\mathbb{R}^{n+1} \setminus B) \cap \{x_{n+1} > 0\}$, with

$$|\sin\theta_j - 1| \leq 2\mu \quad \text{for } j = 1, \dots, n-1$$

and

$$|\nabla_{S^n} \log f - K e_n|_{S^n} \leq 2\mu$$

for all $(r, \theta) \in \tilde{\Omega}_2$.

Lemma 3.2. *For $w \in C_0^\infty(\tilde{\Omega}_2)$,*

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{H^1(\tilde{\Omega}_2)} \lesssim \|\tilde{\mathscr{L}}_{\varphi,\varepsilon} w\|_{L^2(\tilde{\Omega}_2)}, \tag{3-3}$$

where

$$\tilde{\mathscr{L}}_{\varphi,\varepsilon} = \big(1 + |\nabla_{S^n} \log f(\theta)|_{S^n}^2\big) h^2 \partial_r^2 - \frac{2}{r}\big(\alpha + (\nabla_{S^n} \log f(\theta)) \cdot_{S^n} h \nabla_{S^n}\big) h \partial_r + \frac{1}{r^2}(\alpha^2 + h^2 \triangle_{S^n})$$

and $\alpha = 1 + (h/\varepsilon)\log(rf(\theta))$. Here ∇_{S^n} is the gradient operator on the unit sphere; $|\cdot|_{S^n}$ and $\cdot_{S^n}$ indicate the use of the Riemannian metric on S^n, and $\triangle_{S^n}$ is the Laplace–Beltrami operator on the unit sphere S^n.

Proof. Let $v \in C_0^\infty(\Omega_2)$, and let

$$\tilde{v}(r, \theta) = v(rf(\theta), \theta).$$

Then $\tilde{v} \in C_0^\infty(\tilde{\Omega}_2)$. By a change of variables,

$$\|\tilde{v}\|_{L^2(\tilde{\Omega}_2)} \simeq \|v\|_{L^2(\Omega_2)}$$

and

$$\|\tilde{v}\|_{H^1(\tilde{\Omega}_2)} \simeq \|v\|_{H^1(\Omega_2)}.$$

The constants implied in the $\simeq$ sign depend on f.

Since $\mathscr{L}_{\varphi,\varepsilon} v \in C_0^\infty(\Omega_2)$, we have that $\widetilde{\mathscr{L}_{\varphi,\varepsilon} v} \in L^2(\tilde{\Omega}_2)$ and $\|\mathscr{L}_{\varphi,\varepsilon} v\|_{L^2(\Omega_2)} \simeq \|\widetilde{\mathscr{L}_{\varphi,\varepsilon} v}\|_{L^2(\tilde{\Omega}_2)}$. Therefore, by Proposition 2.1,

$$\frac{h}{\sqrt{\varepsilon}}\|\tilde{v}\|_{H^1(\tilde{\Omega}_2)} \lesssim \|\widetilde{\mathscr{L}_{\varphi,\varepsilon} v}\|_{L^2(\tilde{\Omega}_2)}.$$

Now a calculation shows that

$$\mathscr{L}_{\varphi,\varepsilon} = h^2\partial_r^2 - r^{-1}\left(2 - hn + 2\frac{h}{\varepsilon}\log r\right)h\partial_r + r^{-2}(1 + h^2\triangle_{S^n}) \\ + r^{-2}\left(h - hn + \frac{h^2}{\varepsilon^2}\big((\log r)^2 - \varepsilon\big) + \frac{h^2}{\varepsilon}\log r + (2-hn)\frac{h}{\varepsilon}\log r\right),$$

and then that

$$\widetilde{\mathscr{L}_{\varphi,\varepsilon} v} = f^{-2}(\theta)\tilde{\mathscr{L}}_{\varphi,\varepsilon}\tilde{v} - hE\tilde{v},$$

where $\tilde{\mathscr{L}}_{\varphi,\varepsilon}$ is as in the statement of the lemma and E is a first-order semiclassical differential operator with coefficients that have bounds independent of h and ε. Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|\tilde{v}\|_{H^1(\tilde{\Omega}_2)} \lesssim \left\|f^{-2}(\theta)\tilde{\mathscr{L}}_{\varphi,\varepsilon}\tilde{v}\right\|_{L^2(\tilde{\Omega}_2)} + h\|\tilde{v}\|_{H^1(\tilde{\Omega}_2)}.$$

For small enough ε, the last term on the right side can be absorbed into the left side. Moreover, $|f^{-2}|$ is bounded above, so

$$\frac{h}{\sqrt{\varepsilon}}\|\tilde{v}\|_{H^1(\tilde{\Omega}_2)} \lesssim \|\tilde{\mathscr{L}}_{\varphi,\varepsilon}\tilde{v}\|_{L^2(\tilde{\Omega}_2)}$$

for all $v \in C_0^\infty(\Omega_2)$. Now any $w \in C_0^\infty(\tilde{\Omega}_2)$ can be written as $\tilde{v}$ for some $v \in C_0^\infty(\Omega_2)$ just by taking $v(r,\theta) = w(r/f(\theta),\theta)$. This finishes the proof. □

We can now make a second change of variables by thinking of the coordinate map σ as a map from $\tilde{\Omega}_2$ to a subset of $\mathbb{R}^{n+1}_{1+} = \{(r,\theta) \in \mathbb{R}\times\mathbb{R}^n \mid r \geq 1\}$. This gives us that for $w \in C_0^\infty(\sigma(\tilde{\Omega}_2))$,

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{H^1(\sigma(\tilde{\Omega}_2))} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w\|_{L^2(\sigma(\tilde{\Omega}_2))}, \tag{3-4}$$

where

$$\mathscr{L}_{\varphi,\varepsilon,\sigma} = (1 + |\gamma_f|^2)h^2\partial_r^2 - \frac{2}{r}(\alpha + \beta_f \cdot h\nabla_\theta)h\partial_r + \frac{1}{r^2}(\alpha^2 + h^2 L_{S^n}), \tag{3-5}$$

β_f is a vector field on $\mathbb{R}^{n+1}_{1+}$ that equals the coordinate expression of $\nabla_{S^n}\log f(\theta)$ on $\sigma(\tilde{\Omega}_2)$, γ_f is a function on $\mathbb{R}^{n+1}_{1+}$ that agrees with the coordinate expression of $|\nabla_{S^n}\log f(\theta)|_{S^n}$ on $\sigma(\tilde{\Omega}_2)$, and L_{S^n} is a second-order differential operator on $\mathbb{R}^{n+1}_{1+}$ that agrees with the coordinate expression of the Laplacian on the sphere on $\sigma(\tilde{\Omega}_2)$.

To avoid the clumsy buildup of modifiers to Ω and Ω_2, I will let U denote $\sigma(\tilde{\Omega})$ and U_2 denote $\sigma(\tilde{\Omega}_2)$.

The hypotheses in Proposition 3.1 imply that on U_2,

$$|\beta_f - (0,\dots,0,K)| \leq C_\mu \tag{3-6}$$

and

$$|\gamma_f - K| \le C_\mu \tag{3-7}$$

and if

$$h^2 L_{S^n} = a_1 h^2 \partial^2_{\theta_1} + \cdots + a_n h^2 \partial^2_{\theta_n} + b_1 h^2 \partial_{\theta_1} + \cdots + b_n h^2 \partial_{\theta_n},$$

then

$$|a_j - 1| \le C_\mu \tag{3-8}$$

for some constant C_μ that goes to zero if μ goes to zero. C_μ may depend on K, but we are treating K as fixed, so this will be fine. We may as well assume that β_f, γ_f, and the coefficients of L_{S^n} are extended to the rest of $\mathbb{R}^{n+1}_{1+}$ in such a way that these conditions continue to hold. In particular, this means that L_{S^n} is "close" to the ordinary Laplacian on Euclidean space.

4. Small and large frequency cases

To continue the proof of Proposition 3.1, I want to divide w into small and large frequency parts and prove an estimate for each part separately. Recall that $\mathbb{R}^{n+1}_{1+} = \{(r,\theta) \mid \theta \in \mathbb{R}^n, r \ge 1\}$. Let $\mathcal{S}(\mathbb{R}^{n+1}_{1+})$ be the restrictions to $\mathbb{R}^{n+1}_{1+}$ of Schwartz functions on $\mathbb{R}^{n+1}$. Note that functions in $C_0^\infty(U_2)$ are in $\mathcal{S}(\mathbb{R}^{n+1}_{1+})$.

Let c_1 and c_2 be such that

$$\frac{|K|^2}{1+|K|^2} < c_1 < c_2 \le \frac{1}{2} + \frac{|K|^2}{2(1+|K|^2)} < 1,$$

and let δ_1 and δ_2 be such that $\delta_2 > \delta_1 > 0$. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function such that $\rho(\xi) = 0$ if $|\xi|^2 > c_2$ or $|\xi_n| > \delta_2$, and $\rho(\xi) = 1$ if $|\xi|^2 \le c_1$ or $|\xi_n| \le \delta_1$.

Let the hat ˆ indicate the semiclassical Fourier transform in the θ variables only. (In general, Fourier transforms here will be in the θ variables only unless otherwise indicated.) For $w \in C_0^\infty(U)$, define w_s and w_ℓ by $\hat{w}_s = \rho(\xi)\hat{w}$ and $\hat{w}_\ell = (1-\rho(\xi))\hat{w}$, so $w = w_s + w_\ell$.

Lemma 4.1. *There exist $\mu_0 > 0$ and choices of c_1, c_2, δ_1, and δ_2 such that if* (3-6)–(3-8) *hold for some $\mu \le \mu_0$, then*

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}$$

for all $w \in C_0^\infty(U)$, where w_s is defined as above.

Lemma 4.2. *There exists $\mu_0 > 0$ such that if* (3-6)–(3-8) *hold for some $\mu \le \mu_0$, then*

$$\frac{h}{\sqrt{\varepsilon}} \|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}$$

for all $w \in C_0^\infty(U)$, where w_ℓ is defined as above.

Taken together, these two lemmas imply Proposition 3.1. To see why, first we need a lemma.

Let $m, k \ge 0$ be integers. Suppose $a(x,\xi,y)$ is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ that satisfies the bounds

$$\left|\partial_x^\beta \partial_\xi^\alpha \partial_y^j a(x,\xi,y)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\alpha}$$

for all multi-indices α and β, and for $0 \le j \le k$. In other words, each $\partial_y^j a(x, \xi, y)$ is a symbol on $\mathbb{R}^n$ of order m, with bounds uniform in y, for $0 \le j \le k$. Then we can define an operator A on Schwartz functions in $\mathbb{R}^{n+1}$ by applying the pseudodifferential operator on $\mathbb{R}^n$ with symbol $a(x, \xi, y)$ to $f(x, y)$ for each fixed y. More generally, we can also define operators A_j on Schwartz functions in $\mathbb{R}^{n+1}$ by applying the pseudodifferential operator on $\mathbb{R}^n$ with symbol $\partial_y^j a(x, \xi, y)$ to $f(x, y)$ for each fixed y, for $0 \le j \le k$.

Lemma 4.3. *Let A be defined as above. Then A extends to a bounded operator from $H^{k+m}(\mathbb{R}^{n+1})$ to $H^k(\mathbb{R}^{n+1})$.*

Proof. Since $k \in \mathbb{Z}$, $k \ge 0$,

$$\|Af\|^2_{H^k(\mathbb{R}^{n+1})} = \sum_{0 \le |\alpha|+j \le k} \big\| h^{|\alpha|+j} \partial_x^\alpha \partial_y^j Af \big\|^2_{L^2(\mathbb{R}^{n+1})}.$$

Now $\partial_y^j A(f)$ is a sum of terms of the form

$$A_{j_1} \partial_y^{j_2} f,$$

where $j_1 + j_2 = j \le k$. Therefore $\|Af\|^2_{H^k(\mathbb{R}^{n+1})}$ is bounded by a sum of terms of the form

$$\big\| h^{|\alpha|+j_1+j_2} \partial_x^\alpha A_{j_1} \partial_y^{j_2} f \big\|^2_{L^2(\mathbb{R}^{n+1})},$$

where $|\alpha| + j_1 + j_2 \le k$. Then

$$\begin{aligned} \big\| h^{|\alpha|+j_1+j_2} \partial_x^\alpha A_{j_1} \partial_y^{j_2} f \big\|^2_{L^2(\mathbb{R}^{n+1})} &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \big| h^{|\alpha|+j_1+j_2} \partial_x^\alpha A_{j_1} \partial_y^{j_2} f \big|^2 \, dx \, dy \\ &\le \int_{\mathbb{R}} \big\| h^{j_1+j_2} A_{j_1} \partial_y^{j_2} f \big\|^2_{H^{|\alpha|}(\mathbb{R}^n)} \, dy. \end{aligned}$$

Then by the boundedness of A_{j_1}, this is bounded above by

$$\int_{\mathbb{R}} \big\| h^{j_2} \partial_y^{j_2} f \big\|^2_{H^{|\alpha|+m}(\mathbb{R}^n)} \, dy,$$

which in turn is bounded above by

$$\big\| h^{j_2} \partial_y^{j_2} f \big\|^2_{H^{|\alpha|+m}(\mathbb{R}^{n+1})} \le \|f\|^2_{H^{|\alpha|+m+j_2}(\mathbb{R}^{n+1})} \le \|f\|^2_{H^{k+m}(\mathbb{R}^{n+1})}.$$

Therefore

$$\|Af\|^2_{H^k(\mathbb{R}^{n+1})} \lesssim \|f\|^2_{H^{k+m}(\mathbb{R}^{n+1})}.$$

Then a density argument finishes the proof. □

Proof of Proposition 3.1. Adding the estimates from Lemmas 4.1 and 4.2 gives

$$\frac{h}{\sqrt{\varepsilon}} \big(\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + \|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \big) \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}.$$

Since $w_s + w_\ell = w$,

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{L^2(U)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}.$$

For small enough ε, we can absorb the last term into the left side to give

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Since $(1+|\gamma_f|^2) > 1 + K^2 - C_\mu$, for μ small enough, we have

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \big\|(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \big\|(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w_\ell\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Now $w_s = Pw$, where P is the semiclassical pseudodifferential operator of order 0 on $\mathbb{R}^n$ with symbol $\rho(\xi)$. The operator P commutes with ∂_r, and its commutators with differential operators in the θ variables are, for each fixed $r \in [1,\infty)$, semiclassical pseudodifferential operators on $\mathbb{R}^n$ that satisfy the conditions of Lemma 4.3. Therefore

$$\begin{aligned}\big\|(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} &= \big\|(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} Pw\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \\ &\lesssim \big\|P(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|E_0 h\partial_r + E_1 w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})},\end{aligned}$$

where E_1 and E_0, for each fixed $r \in [1,\infty)$, are semiclassical pseudodifferential operators on $\mathbb{R}^n$ of order 1 and 0 and satisfy the conditions of Lemma 4.3. There is no $hE_{-1}h^2\partial_r^2$ in the error term because the coefficient of $h^2\partial_r^2$ in $(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma}$ is just 1. Now E_1^* and E_0^* are also semiclassical pseudodifferential operators on $\mathbb{R}^n$ of order 1 and 0, for each fixed $r \in [1,\infty)$, and satisfy the conditions of Lemma 4.3.

Therefore, by Lemma 4.3, E_1^* is bounded from $H_0^1(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$, so by duality, E_1 is bounded from $L^2(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$.

Also, E_0^* is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^1(\mathbb{R}^{n+1}_{1+})$ and takes functions with trace 0 on the boundary of $\mathbb{R}^{n+1}_{1+}$ to other functions with trace 0 on the boundary of $\mathbb{R}^{n+1}_{1+}$, so by duality, E_0 is bounded from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Therefore

$$\big\|(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|P(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Now by Lemma 4.3, P is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^1(\mathbb{R}^{n+1}_{1+})$. Also, if u has trace zero on the boundary of $\mathbb{R}^{n+1}_{1+}$, then so does Pu, so P is bounded from $H_0^1(\mathbb{R}^{n+1}_{1+})$ to $H_0^1(\mathbb{R}^{n+1}_{1+})$. Since ρ is real-valued, P is also self-adjoint, so by duality, P is bounded from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Therefore

$$\big\|(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|(1+|\gamma_f|^2)^{-1}\mathcal{L}_{\varphi,\varepsilon,\sigma} w\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

and thus

$$\|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Similarly,

$$\|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}.$$

Again the last term can be absorbed into the left side for small enough ε, so

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}$$

for each $w \in C_0^\infty(U)$.

Now if the hypotheses of Proposition 3.1 hold, then so do (3-6)–(3-8), and therefore we can obtain this conclusion. Changing variables back gives

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)}$$

for $w \in C_0^\infty(\Omega)$. □

Therefore we need only to establish proofs of Lemmas 4.1 and 4.2. To do this, we will need to introduce the analogues of the operator J described in Section 2.

5. The operators

Suppose $F : \mathbb{R}^n \to \mathbb{C}$ is a smooth function such that $\mathrm{Re}(F(\xi))$, $|F(\xi)| \simeq 1 + |\xi|$ for all $\xi \in \mathbb{R}^n$, and F is a symbol of order one on $\mathbb{R}^n$, so that

$$|\partial_\xi^\alpha F(\xi)| \le C_\alpha (1+|\xi|)^{1-|\alpha|} \tag{5-1}$$

for all multi-indices α.

Then for $u \in \mathscr{S}(\mathbb{R}^{n+1}_{1+})$, define Ju by

$$\widehat{Ju}(r,\xi) = \left(\frac{F(\xi)}{r} + h\partial_r\right)\hat{u}(r,\xi).$$

This operator has adjoint J^* given by

$$\widehat{J^*u}(r,\xi) = \left(\frac{\overline{F(\xi)}}{r} - h\partial_r\right)\hat{u}(r,\xi).$$

These operators have right inverses defined by

$$\widehat{J^{-1}u}(r,\xi) = h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F(\xi)/h} dt$$

and

$$\widehat{J^{*-1}u}(r,\xi) = h^{-1}\int_r^\infty \hat{u}(t,\xi)\left(\frac{r}{t}\right)^{\overline{F(\xi)}/h} dt.$$

Each of these is well defined as an operator on $\mathscr{S}(\mathbb{R}^{n+1}_{1+})$. We will prove appropriate analogues of the properties (1)–(4) and (5′) from Section 2 for J of this form. Note that J^{-1} is a right inverse, and both J and J^{-1} preserve support in the positive r direction. Therefore it remains to establish analogues of properties (3), (4), and (5′).

To set up the analogue of property (3), define the weighted Sobolev space $H^1_r(\mathbb{R}^{n+1}_{1+})$ by the norm

$$\|u\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})} = \left\|\frac{u}{r}\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + \|h\partial_r u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + \left\|\frac{h}{r}\nabla_\theta u\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Since U_2 lies in the set $1 \le r \le R_0$ for some R_0 depending on U_2, we know H^1 and H^1_r norms are comparable for functions supported on U_2, with constants of comparability depending only on R_0. This holds more generally for any functions supported in $1 \le r \le R_0$.

Now the operators above have the following boundedness properties.

Lemma 5.1. *J, J^*, J^{-1}, and J^{*-1} extend as bounded maps*

$$J,\ J^* : H^1_r(\mathbb{R}^{n+1}_{1+}) \to L^2(\mathbb{R}^{n+1}_{1+})$$

and

$$J^{-1},\ J^{*-1} : L^2(\mathbb{R}^{n+1}_{1+}) \to H^1_r(\mathbb{R}^{n+1}_{1+}).$$

Moreover, the extensions of J^* and J^{*-1} are isomorphisms.

Proof. Consider J first. If $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then

$$\|Ju\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} = h^{-n}\|\widehat{Ju}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} = h^{-n}\left\|\frac{F(\xi)}{r}\hat{u} + h\partial_r\hat{u}\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|_{H^1_r(\mathbb{R}^{n+1}_{1+})}.$$

By a density argument, J extends to a bounded map $J : H^1_r(\mathbb{R}^{n+1}_{1+}) \to L^2(\mathbb{R}^{n+1}_{1+})$. The proof for J^* is similar.

Now consider J^{-1}. If $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then

$$\begin{aligned}\int_1^\infty \left|\frac{1}{r}\widehat{J^{-1}u}\right|^2 dr &= \int_1^\infty \left|h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F(\xi)/h} dt\right|^2 r^{-2}\,dr \\ &\le \int_1^\infty \left|h^{-1}\int_0^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F(\xi)/h} dt\right|^2 r^{-2}\,dr.\end{aligned}$$

By a change of variables, we get

$$\int_1^\infty \left|\frac{1}{r}\widehat{J^{-1}u}\right|^2 dr = \int_1^\infty \left|h^{-1}\int_0^1 \hat{u}(rt,\xi)t^{F(\xi)/h}\,dt\right|^2 dr.$$

Then using Minkowski's inequality and changing variables again, we get

$$\begin{aligned}\int_1^\infty \left|\frac{1}{r}\widehat{J^{-1}u}\right|^2 dr &\le h^{-2}\left(\int_0^1\left(\int_1^\infty |\hat{u}(r,\xi)|^2\,dr\right)^{1/2} t^{\mathrm{Re}(F(\xi)/h)}t^{-1/2}dt\right)^2 \\ &= h^{-2}\int_1^\infty |\hat{u}(r,\xi)|^2\,dr\left(\frac{h}{\mathrm{Re}\,F(\xi)+h/2}\right)^2 \simeq \int_1^\infty \left|\frac{\hat{u}(r,\xi)}{1+|\xi|}\right|^2 dr.\end{aligned}$$

Therefore

$$\left\|\frac{1}{r}\widehat{J^{-1}u}\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \int_{\mathbb{R}^n}\int_1^\infty \left|\frac{\hat{u}(r,\xi)}{1+|\xi|}\right|^2 dr\,d\xi \lesssim \|\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Similarly,

$$\int_1^\infty \left| \frac{\xi}{r} \widehat{J^{-1}u} \right|^2 dr \lesssim \int_1^\infty |\hat{u}(r,\xi)|^2 \, dr,$$

so

$$\left\| \frac{h}{r} \nabla_\theta J^{-1} u \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Finally,

$$h\partial_r \widehat{J^{-1}u} = -\left(\frac{F(\xi)}{r}\right) \widehat{J^{-1}u} + \hat{u},$$

so

$$\int_1^\infty \left| h\partial_r \widehat{J^{-1}u} \right|^2 dr \lesssim \int_1^\infty |\hat{u}(r,\xi)|^2 \, dr$$

and

$$\left\| h\partial_r J^{-1} u \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})},$$

by the same logic.

Putting all of this together gives

$$\left\| J^{-1} u \right\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})},$$

for $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. Then a density argument shows that J^{-1} extends to a bounded map

$$J^{-1} : L^2(\mathbb{R}^{n+1}_{1+}) \to H^1_r(\mathbb{R}^{n+1}_{1+}).$$

Again, the proof for J^{*-1} is similar.

It remains to show that the extensions of J^* and J^{*-1} are isomorphisms. If $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then

$$J^* J^{*-1} u = u$$

and (using integration by parts)

$$J^{*-1} J^* u = u.$$

Then the result follows from a density argument.

Note that $J^{-1} J u \neq u$ in general, because integration by parts will pick up a boundary term at $r = 1$. Therefore the extensions of J and J^{-1} are not isomorphisms. □

Let $H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$ denote the subspace of $H^1_r(\mathbb{R}^{n+1}_{1+})$ consisting of functions with trace zero on the hyperplane $r = 1$, and let $H^{-1}_r(\mathbb{R}^{n+1}_{1+})$ denote the dual space to $H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$.

Now we need to prove some commutator properties for J.

Lemma 5.2. *Suppose that $w \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, $\chi \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$ and that Q is a second-order semiclassical differential operator with smooth bounded coefficients on $\mathbb{R}^{n+1}_{1+}$. Then*

$$\left\| J\chi J^{-1} w \right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|\chi w\|_{L^2(\mathbb{R}^{n+1}_{1+})} - h\|rw\|_{L^2(\mathbb{R}^{n+1}_{1+})}$$

and

$$\left\|(JQ-QJ)w\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim h\|rw\|_{H^1(\mathbb{R}^{n+1}_{1+})}.$$

The constants in the $\gtrsim$ and $\lesssim$ signs will depend on the derivatives of F.

Proof. Consider the first statement. If T is the operator on $\mathbb{R}^n$ with symbol $F(\xi)$, interpreted as acting on functions on $\mathbb{R}^{n+1}_{1+}$ by action on the θ variables only, then

$$\left\|J\chi J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} = \left\|\left(h\partial_r+\frac{T}{r}\right)\chi J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \geq \left\|\chi\left(h\partial_r+\frac{T}{r}\right)J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} - \left\|hE_0J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

where for each fixed r, E_0 is an order-zero pseudodifferential operator on $\mathbb{R}^n$ with bounds that are uniform in r. Therefore, by Lemma 4.3, E_0 is bounded from L^2 to L^2, so

$$\left\|J\chi J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \geq \left\|\chi JJ^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} - h\left\|J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \geq \|\chi w\|_{L^2(\mathbb{R}^{n+1}_{1+})} - h\|rw\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

The proof of the second statement is similar, but somewhat more involved. First, note that multiplication by $1/r$ is a bounded operator from $H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$ to $H^1_0(\mathbb{R}^{n+1}_{1+})$. Therefore, by duality, it is a bounded operator from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}_r(\mathbb{R}^{n+1}_{1+})$, and so

$$\left\|(J_sQ-QJ_s)w\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|r(J_sQ-QJ_s)w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Note that $J_s = h\partial_r + r^{-1}T$, where T is a semiclassical pseudodifferential operator on $\mathbb{R}^n$ of order 1. Meanwhile, Q can be written as a combination of ∂_r derivatives and differential operators on $\mathbb{R}^n$:

$$Q = Ah^2\partial_r^2 + Bh\partial_r + C,$$

where A, B, and C are (perhaps r-dependent) differential operators of orders 0, 1, and 2 respectively on $\mathbb{R}^n$ for each fixed r, with bounds uniform in r.

If $w \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then $Qw \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. Then

$$\left\|r(J_sQ-QJ_s)w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} = \left\|r\left[h\partial_r + r^{-1}T,\, Ah^2\partial_r^2 + Bh\partial_r + C\right]w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Expanding this, and noting that T commutes with ∂_r, we get

$$\begin{aligned}\left\|r(J_sQ-QJ_s)w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \leq{}& \left\|r[h\partial_r, Q]w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \left\|[T,A]h^2\partial_r^2 w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \\ &+ \left\|hr^{-1}[T,A]h\partial_r w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \left\|2h^2r^{-2}[T,A]w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \\ &+ \left\|[T,B]h\partial_r w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \left\|r^{-1}[T,B]w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \left\|[T,C]w\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.\end{aligned}$$

By the product rule, $r[h\partial_r, Q] = hrE_2' = hE_2'r + h^2E_1'$, where E_2' and E_1' are second- and first-order semiclassical differential operators. Meanwhile, $[T,A] = hE_0$, $[T,B] = hE_1$, and $[T,C] = hE_2$, where

E_0, E_1, and E_2 are semiclassical pseudodifferential operators on $\mathbb{R}^n$ of orders 0, 1, and 2. Therefore

$$\begin{aligned}\|r(J_sQ-QJ_s)w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \le \|hE_2'rw\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|h^2E_1'w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|hE_0h^2\partial_r^2 w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}\\ + \|h^2r^{-1}E_0h\partial_r w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|2h^3r^{-2}E_0w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}\\ + \|hE_1h\partial_r w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|hr^{-1}E_1w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|hE_2w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.\end{aligned}$$

E_2' is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$, and E_1' is bounded from $L^2(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. In addition, by Lemma 4.3, E_1^* is bounded from $H^1_0(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$, so by duality, E_1 is bounded from $L^2(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Meanwhile, E_2 is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Finally, E_0^* is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^1(\mathbb{R}^{n+1}_{1+})$ and maps functions with trace 0 on the boundary of $\mathbb{R}^{n+1}_{1+}$ to other functions with trace 0 on that boundary, so it is bounded from $H^1_0(\mathbb{R}^{n+1}_{1+})$ to $H^1_0(\mathbb{R}^{n+1}_{1+})$. Therefore, by duality, E_0 is bounded from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Moreover, $1/r \le 1$ on $\mathbb{R}^{n+1}_{1+}$. Applying all of these facts together to the last inequality then finishes the proof. □

To finish this section, we need to prove a property analogous to (5′) from Section 2.

Lemma 5.3. *Suppose $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. If g is defined by*

$$\hat{g}(r,\xi) = \frac{2\operatorname{Re}F(\xi)-h}{h}\int_1^\infty \hat{u}(t,\xi)r^{-F(\xi)/h}t^{-\overline{F(\xi)}/h}\,dt,$$

then

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \simeq \|u-g\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Proof. Suppose $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. Define g as above. A calculation shows that $g \in L^2(\mathbb{R}^{n+1}_{1+})$, and

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \le \|u\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Note that

$$\widehat{Jg} = \left(\frac{F(\xi)}{r} + h\partial_r\right)\hat{g} = 0.$$

Therefore

$$\begin{aligned}\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} &= \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, w\neq 0} \frac{|(Ju,w)|}{\|w\|_{H_r^1(\mathbb{R}^{n+1}_{1+})}}\\ &= \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, w\neq 0} \frac{|(J(u-g),w)|}{\|w\|_{H_r^1(\mathbb{R}^{n+1}_{1+})}}\\ &= \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, w\neq 0} \frac{|(u-g,J^*w)|}{\|w\|_{H_r^1(\mathbb{R}^{n+1}_{1+})}}.\end{aligned}$$

Since $J^* : H_r^1(\mathbb{R}^{n+1}_{1+}) \to L^2(\mathbb{R}^{n+1}_{1+})$ is an isomorphism,

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \simeq \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, J^*w\neq 0} \frac{|(u-g,J^*w)|}{\|J^*w\|_{L^2(\mathbb{R}^{n+1}_{1+})}}. \tag{5-2}$$

Now $J^*w \in L^2(\mathbb{R}^{n+1}_{1+})$, so

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \|u-g\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

On the other hand, $u-g = J^*J^{*-1}(u-g)$. Also $J^{*-1}(u-g) \in H_r^1(\mathbb{R}^{n+1}_{1+})$, and by definition of g, $J^{*-1}(u-g)(x,0)=0$. Therefore $J^{*-1}(u-g) \in H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$. Then if $u-g=0$, the lemma is true by (5-2). Otherwise, we can pick $w = J^{*-1}(u-g)$ in (5-2) to show that

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \gtrsim \|u-g\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

This finishes the proof. $\square$

6. The small frequency case

To prove Lemma 4.1, we need to define an operator of the form given in Section 5.

Consider the function $\Phi : \mathbb{R}^n \to \mathbb{C}$ given by

$$\overline{\Phi(\xi)} = \frac{1}{1+|K|^2}\Big(1 + iK\xi_n + \sqrt{2iK\xi_n - (K\xi_n)^2 + (1+|K|^2)|\xi|^2 - |K|^2}\Big),$$

where the square root is taken to mean the branch of the square root function with nonnegative imaginary part. We would like to use this function in place of F in Section 5 to define J and the related operators of that section. Unfortunately, Φ is not smooth. However, we can try to construct a function F_s that approximates Φ on the support of $\hat{w}_s$ and has the properties of F from Section 5. To do this, first notice that if c_2 and δ_2 are chosen small enough, then this is nearly continuous on the support of $\hat{w}_s$, or equivalently, on the support of ρ. To be more precise, Φ is smooth except where

$$\tau_K(\xi) = 2iK\xi_n - (K\xi_n)^2 + (1+|K|^2)|\xi|^2 - |K|^2$$

lies on the nonnegative real axis, where this branch of the square root has its branch cut. This occurs when $\xi_n = 0$ and $|\xi|^2 \geq |K|^2/(1+|K|^2)$, and gives a jump discontinuity of size $2\sqrt{(1+|K|^2)|\xi|^2 - |K|^2}$. However, $|\xi|^2 \leq c_2$ on the support of ρ, so for c_2 close to $|K|^2/(1+|K|^2)$, the maximum possible size of the jump discontinuity is small.

Therefore, for any $\delta > 0$ we can define $F_s(\xi)$ on the support of ρ such that

$$|F_s(\xi) - \Phi(\xi)| \leq \delta$$

on the support of ρ, by choosing c_2 small enough. The derivatives of F_s inside the support of ρ may depend on c_1, c_2, δ_1, and δ_2. Since the choice of these in turn depends on δ, the derivatives of F_s are bounded by a quantity that depends on δ.

Now consider the necessary bounds on F_s. On the support of ρ, the imaginary part of τ_K must lie in the interval $[-2K\delta_2, 2K\delta_2]$. The real part of τ_K is given by

$$-(K\xi_n)^2 - |K|^2 + (1+|K|^2)|\xi|^2.$$

We have that $|\xi|^2 \leq c_2$ on the support of ρ. We can choose c_2 so close to $\dfrac{K^2}{1+K^2}$ that

$$(1+K^2)r_2 - K^2 \leq \delta_2.$$

Then the real part of τ_K is bounded above by δ_2 on the support of ρ. Therefore, on the support of ρ, $(\mathrm{Re}(\tau_K), \mathrm{Im}(\tau_K)) \in (-\infty, \delta_2] \times [-2K\delta_2, 2K\delta_2]$, and so by taking δ_2 small enough, we can ensure that the real part of $\sqrt{\tau_K}$ has absolute value less than $\frac{1}{3}$ on the support of ρ.

Therefore, if δ is small enough, $\mathrm{Re}(F_s), |F_s| > 1/(2+2K^2)$ on the support of ρ. We can now define F_s smoothly outside the support of ρ so that $\mathrm{Re}(F_s), |F_s| \geq 1/(2+2K^2)$ for all ξ, and $F_s = (1+|\xi|^2)^{1/2}$ for $|\xi| > 2$, say. Then F_s is smooth, $\mathrm{Re}(F(\xi)), |F(\xi)| \simeq 1+|\xi|$ for all $\xi \in \mathbb{R}^n$, and the conditions (5-1) are satisfied automatically for $|\xi| > 2$, and hence for all ξ.

Therefore F_s satisfies all the conditions given in Section 5, and the operators defined by

$$\widehat{J_s u}(r,\xi) = \left(\frac{F_s(\xi)}{r} + h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_s^* u}(r,\xi) = \left(\frac{\overline{F_s(\xi)}}{r} - h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_s^{-1} u}(r,\xi) = h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F_s(\xi)/h} dt,$$

and

$$\widehat{J_s^{*-1} u}(r,\xi) = h^{-1}\int_r^\infty \hat{u}(t,\xi)\left(\frac{r}{t}\right)^{\overline{F_s(\xi)}/h} dt$$

satisfy all the properties from that section.

Now we are ready to begin the proof of the small frequency case. Suppose $\chi_2(r,\theta) \in C^\infty(\mathbb{R}^{n+1}_{1+})$ is a cutoff function that is 1 on U and has support inside U_2.

If $w \in C_0^\infty(U)$, then $w_s \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, supported away from $r = 1$. Therefore $J_s^{-1} w_s \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$ is supported away from $r = 1$. Then $\chi_2 J_s^{-1} w_s$ is in $C_0^\infty(U_2)$. Therefore, by (3-4),

$$\frac{h}{\sqrt{\varepsilon}}\left\|\chi_2 J_s^{-1} w_s\right\|_{H^1(U_2)} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(U_2)}.$$

Since $\chi_2 J_s^{-1} w_s \in C_0^\infty(U_2)$, the H^1 and H^1_r norms are comparable, so

$$\frac{h}{\sqrt{\varepsilon}}\left\|\chi_2 J_s^{-1} w_s\right\|_{H^1_r(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Using the boundedness properties from Lemma 5.1,

$$\frac{h}{\sqrt{\varepsilon}}\left\|J_s\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

so applying the first part of Lemma 5.2,

$$\frac{h}{\sqrt{\varepsilon}}\|\chi_2 w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + C_\delta\frac{h^2}{\varepsilon}\|r w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

The C_δ factor written in front of the last term is to indicate that the constant in the $\lesssim$ sign depends on the derivatives of F_s, and hence on δ. This is fine, because δ is chosen independently of h and ε, but this will help track the δ dependence. Now $\chi_2 w_s = \chi_2 P w$. Since w is only supported on the region where χ_2 is identically one,

$$\chi_2 w_s = Pw + O(h^\infty)Ew = w_s + O(h^\infty)Ew,$$

where E is a pseudodifferential operator of order 0 (actually a smoothing operator) on $\mathbb{R}^n$. Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|\chi_2 w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} - O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

and so

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + C_\delta \frac{h^2}{\varepsilon}\|r w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

For small enough h, the second last term can be absorbed into the left side (r is bounded on the support of w_s) to give

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(U_2))} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

By the product rule, $\mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2 - \mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2$ is a first-order semiclassical differential operator, and thus it is bounded from $H^1(U_2)$ to $L^2(U_2)$. Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\chi_2\mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s\right\|_{L^2(U_2)} + h\left\|J_s^{-1} w_s\right\|_{H^1(U_2)} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

On U_2, the H^1 and H^1_r norms are comparable, so

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + h\left\|J_s^{-1} w_s\right\|_{H^1_r(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Using the boundedness properties again,

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + h\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

The second last term can be absorbed into the left side to give

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}. \tag{6-1}$$

I want to combine this last inequality with Lemma 5.3 to get

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|J_s\mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s\right\|_{H^{-1}_r(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

To do this, I need to show that if $u = \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s$, then the function g defined in Lemma 5.3 satisfies a bound like

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \le \tfrac{1}{2}\|u\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

by using an integration by parts argument like the one described in Section 2.

Let $v = J_s^{-1} w_s$. Then

$$\hat{g} = \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \widehat{\mathscr{L}_{\varphi,\varepsilon,\sigma} v}(t,\xi) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt.$$

Writing out $\mathscr{L}_{\varphi,\varepsilon,\sigma}$ as in (3-5), we can consider the integral for each term of $\mathscr{L}_{\varphi,\varepsilon,\sigma}$ separately. For this equation the hat notation for the Fourier transform will become a little impractical, so let $\mathscr{F}(v) = \hat{v}$. Then

$$\begin{aligned}\hat{g} = \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \mathscr{F}\big((1+|\gamma_f|^2) h^2 \partial_t^2 v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ - \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{2}{t} \mathscr{F}\big((\alpha + \beta_f \cdot h\nabla_\theta) h\partial_t v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ + \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{1}{t^2} \mathscr{F}\big((\alpha^2 + h^2 L_{S^n}) v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt.\end{aligned}$$

We can use the assumptions on β_f, γ_f, and L_{S^n} in equations (3-6), (3-7), and (3-8), together with the fact that $|1-\alpha| \lesssim h\varepsilon^{-1}$, to write this as

$$\begin{aligned}\hat{g} = \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \mathscr{F}\big((1+|K|^2) h^2 \partial_t^2 v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ - \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{2}{t} \mathscr{F}\big((1 + K \cdot h\nabla_\theta) h\partial_t v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ + \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{1}{t^2} \mathscr{F}\big((1 + h^2 \triangle_\theta) v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ + C_\mu \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \mathscr{F}(E_2 v) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt, \quad \text{(6-2)}\end{aligned}$$

where E_2 is a second-order semiclassical differential operator with bounds uniform in μ. Now we can integrate by parts to remove the $h\partial_t$'s.

In the first term, this gives us

$$\begin{aligned}\frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty (1+|K|^2) h^2 \partial_t^2 \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ = \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{\overline{F_s}}{t} (1+K^2) h\partial_t \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ = \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \left(\frac{\overline{F_s}}{t}\right)^2 (1+K^2) \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ + \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty h \frac{\overline{F_s}}{t^2} (1+K^2) \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt.\end{aligned}$$

There are no boundary terms from the integration by parts because w is supported away from $r = 1$, and hence w_s and v are as well. The last term can be absorbed into the last term of (6-2). In the second term, we get

$$\frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \frac{2}{t}(1+iK\xi_n)h\partial_t\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt = \frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \frac{2\overline{F_s}}{t^2}(1+iK\xi_n)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt + \frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \frac{2h}{t^2}(1+iK\xi_n)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt.$$

Again the last term can be absorbed into the last term of (6-2). Therefore, returning to (6-2), we have

$$\begin{aligned}\hat{g} = \frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \left(\frac{\overline{F_s}}{t}\right)^2(1+K^2)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt &- \frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \frac{2\overline{F_s}}{t^2}(1+iK\xi_n)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt \\ &+ \frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \frac{1}{t^2}(1-|\xi|^2)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt \\ &+ C_\mu\frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \widehat{E_2 v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt.\end{aligned}$$

Now $F_s(\xi)$ is designed so that $\overline{F_s(\xi)}$ is very nearly a solution to $(1+K^2)X^2-2(1+iK\xi_n)X+1-|\xi|^2=0$ when $\hat{w}_s\neq 0$, and hence when $\hat{v}\neq 0$. More precisely,

$$\left|(1+K^2)\overline{F_s(\xi)}^2-2(1+iK\xi_n)\overline{F_s(\xi)}+1-|\xi|^2\right| \lesssim \delta\big(|F_s(\xi)|+|\xi_n|\big)\lesssim \delta|F_s(\xi)|.$$

That means that we can write $\hat{g}$ as

$$\hat{g} = \delta\frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty R(\xi)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt + C_\mu\frac{2\operatorname{Re} F_s - h}{h}\int_1^\infty \widehat{E_2 v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt,$$

where $|R(\xi)|\lesssim|F_s(\xi)|\lesssim 1+|\xi|$. Now it follows, as in the proof of Lemma 5.3, that

$$\|\hat{g}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}\lesssim \delta^2\|R(\xi)\hat{v}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}+C_\mu^2\|\widehat{E_2 v}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}\lesssim(\delta+C_\mu)\|v\|_{H^2(\mathbb{R}^{n+1}_{1+})}. \tag{6-3}$$

This gives an estimate for g in terms of v. However, we want the estimate to be in terms of u. We have $u=\mathscr{L}_{\varphi,\varepsilon,\sigma}v$, so

$$\|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}=\|\mathscr{L}_{\varphi,\varepsilon,\sigma}v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}$$

and

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma}v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}\gtrsim\left\|\left((1+K^2)h^2\partial_r^2-\frac{2}{r}(1+Kh\partial_{\theta_n})h\partial_r+\frac{1}{r^2}(1+h^2\triangle_\theta)\right)v\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}-C_\mu^2\|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}.$$

Rewriting in terms of $\hat{v}$, we get

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim h^{-n} \left\| \left((1+K^2)h^2\partial_r^2 - \frac{2}{r}(1+iK\xi_n)h\partial_r + \frac{1}{r^2}(1-|\xi|^2) \right) \hat{v}(r,\xi) \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - C_\mu^2 \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}.$$

Now $\hat{v}(r,\xi) = \mathscr{F}(J_s^{-1}Pw)(r,\xi)$ is only nonzero for ξ such that

$$|\xi|^2 \le \frac{1}{2} + \frac{1}{2}\frac{|K|^2}{1+|K|^2} < 1.$$

The operator

$$(1+K^2)h^2\partial_r^2 - \frac{2}{r}(1+iK\xi_n)h\partial_r + \frac{1}{r^2}(1-|\xi|^2)$$

coincides, for $r > 1$, with a differential operator in r of the form

$$(1+K^2)h^2\partial_r^2 - 2\omega(1+iK\xi_n)h\partial_r + \omega^2(1-|\xi|^2),$$

where ω is a smooth function that coincides with $1/r$ for $r > 1$. This is second-order elliptic for each $|\xi|$ such that $\hat{v}(r,\xi)$ is nonzero, and its symbol (in r) is bounded below; therefore

$$h^{-n} \left\| \left((1+K^2)h^2\partial_r^2 - \frac{2}{r}(1+iK\xi_n)h\partial_r + \frac{1}{r^2}(1-|\xi|^2) \right) \hat{v}(r,\xi) \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \simeq \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})} - C_\mu^2 \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})},$$

and so

$$\|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} = \|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})} - C_\mu^2 \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}$$

for μ small enough.

Substituting this into (6-3) gives

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim (\delta + C_\mu)\|u\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Taking μ and δ small enough means

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \le \tfrac{1}{2}\|u\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Combining this with (6-1) now gives

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| J_s \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s \right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Then using the second part of Lemma 5.2, we get

$$\begin{aligned}\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} &\lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} J_s J_s^{-1} w_s\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + C_\delta h\left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}\\ &\lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + C_\delta h\left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.\end{aligned}$$

Again the C_δ factor is written to track the δ dependence, but again this is fine. $\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s$ is supported in the r direction only for those r that can come from $\tilde{\Omega}_2$, since w_s is. Therefore the H_r^{-1} and H^{-1} norms are comparable, and so

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + hC_\delta\left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}. \tag{6-4}$$

Meanwhile,

$$\widehat{J_s^{-1} w_s}(r,\xi) = \frac{1}{h}\int_1^r \hat{w}_s(t,\xi)\left(\frac{t}{r}\right)^{F_s(\xi)/h} dt,$$

and $\hat{w}_s(t,\xi)$ is supported only for $1 \le t \le C$ for some C depending on $\sigma(\tilde{\Omega}_2)$. Therefore, for $r > 4C$,

$$\left|\widehat{J_s^{-1} w_s}(r,\xi)\right| \le \left|\frac{1}{h}\int_1^C \hat{w}_s(t,\xi)\left(\frac{t}{2C}\right)^{F_s/h} dt\right|\left|\frac{1}{2}\right|^{\mathrm{Re}(F_s/h)}\left|\frac{4C}{r}\right|^{\mathrm{Re}(F_s/h)},$$

so

$$\left|\widehat{J_s^{-1} w_s}(r,\xi)\right|^2 \lesssim \int_1^C |\hat{w}(t,\xi)|^2\, dt\left|\frac{1}{2}\right|^{\mathrm{Re}(2F_s/h)}.$$

Therefore

$$\left\|r J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|r J_s^{-1} w_s\right\|_{L^2(1<r<4C)} + O(h^\infty)\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Similar calculations for derivatives of $J_s^{-1} w$ give

$$\left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|r J_s^{-1} w_s\right\|_{H^1(1<r<4C)} + O(h^\infty)\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

so

$$\left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|J_s^{-1} w_s\right\|_{H_r^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Returning to (6-4), we get

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + hC_\delta\left\|J_s^{-1} w_s\right\|_{H_r^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Applying the boundedness result for J^{-1} gives

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + hC_\delta\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

For small enough ε, the second last term can be absorbed into the left side to give

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

This finishes the proof of Lemma 4.1.

7. The large frequency case

Now we turn to the large frequency case. We will need to define a new operator J_ℓ.

Consider again the function $\Phi : \mathbb{R}^n \to \mathbb{C}$ given by

$$\overline{\Phi(\xi)} = \frac{1}{1+K^2}\Big(1 + iK\xi_n + \sqrt{2iK\xi_n - (K\xi_n)^2 + (1+K^2)|\xi|^2 - |K|^2}\Big),$$

but this time take the branch of the square root that has nonnegative real part. Now Φ is smooth except where

$$\tau_K(\xi) = 2iK\xi_n - (K\xi_n)^2 + (1+|K|^2)|\xi|^2 - |K|^2$$

lies on the nonpositive real axis. This happens when $\xi_n = 0$ and

$$|\xi|^2 \le \frac{|K|^2}{1+|K|^2}.$$

Therefore, on the support of $1-\rho(\xi)$, Φ is smooth. Since the real part of the square root is nonnegative, both $|\Phi|$ and the real part of Φ are bounded below by $1/(1+K^2)$. Therefore we can pick a smooth function F_ℓ such that $F_\ell(\xi) = \Phi(\xi)$ on the support of $1-\rho(\xi)$ and

$$\operatorname{Re} F_\ell(\xi),\ |F_\ell(\xi)| \ge \frac{1}{1+K^2}.$$

In fact, if $\dfrac{K^2}{1+K^2} < c_0 < c_1$ and $0 < \delta_0 < \delta_1$, we can still pick F_ℓ to be equal to Φ on the set

$$\{\xi \in \mathbb{R}^n \mid |\xi|^2 \ge c_0 \text{ or } \xi_n \ge \delta_0\},$$

with F_ℓ smooth and $\operatorname{Re} F_\ell(\xi), |F_\ell(\xi)| \ge (1+K^2)^{-1}$. Now for large $|\xi|$,

$$\operatorname{Re} \Phi(\xi),\ |\Phi_\ell(\xi)| \ge \frac{1}{1+K^2}(1+|\xi|),$$

so F_ℓ then satisfies these inequalities for all ξ. Finally, for large $|\xi|$, Φ is smooth and satisfies the inequalities (5-1), so it follows that F_ℓ satisfies those inequalities for all ξ. Thus F_ℓ satisfies all of the conditions at the beginning of Section 5, and therefore the operators defined by

$$\widehat{J_\ell u}(r,\xi) = \left(\frac{F_\ell(\xi)}{r} + h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_\ell^* u}(r,\xi) = \left(\frac{\overline{F_\ell(\xi)}}{r} - h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_\ell^{-1} u}(r,\xi) = h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F_\ell(\xi)/h} dt,$$

and

$$\widehat{J_\ell^{*-1} u}(r,\xi) = h^{-1}\int_r^\infty \hat{u}(t,\xi)\left(\frac{r}{t}\right)^{\overline{F_\ell(\xi)}/h} dt$$

satisfy all of the properties from that section.

Consider the Carleman estimate (3-4). By a similar argument as in the small frequency case, we get

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathscr{L}_{\varphi,\varepsilon,\sigma} J_\ell^{-1} w_\ell\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}. \tag{7-1}$$

Again, I want to combine this last inequality with Lemma 5.3 to get

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|J_\ell\mathscr{L}_{\varphi,\varepsilon,\sigma} J_\ell^{-1} w_\ell\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

To do this, I need to show that if u is of the form $u = \mathscr{L}_{\varphi,\varepsilon,\sigma} J_\ell^{-1} w_\ell$, then the function g defined in Lemma 5.3 satisfies a bound like

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \le \tfrac{1}{2}\|u\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h)\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

by an appropriate integration by parts argument. The approach used in the small frequency case does not work here, because $\mathscr{L}_{\varphi,\varepsilon,\sigma}$ is not at all elliptic on the support of $\hat{w}_\ell$. However, now $\mathscr{L}_{\varphi,\varepsilon,\sigma}$ can be factored into a composition of two operators, one of which has the desired properties.

Let $\zeta(\xi)$ be a smooth cutoff function that is identically one on the set where $|\xi|^2 \ge c_1$ or $|\xi_n| \ge \delta_1$, and vanishes if $|\xi|^2 \le c_0$ or $|\xi_n| \le \delta_0$. Let

$$G_s = (1-\zeta(\xi))F_\ell(\xi),$$

and consider the symbols

$$G_\pm = \zeta(\xi)\frac{\alpha + i\beta_f\cdot\xi \pm \sqrt{(\alpha+i\beta_f\cdot\xi)^2 - (1+(\gamma_f)^2)(\alpha^2 - L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2} + G_s(\xi),$$

where $L_{S^n}(\theta,\xi)$ represents the symbol of the differential operator L_{S^n}. The square root represents the branch of the square root with nonnegative real part. The argument of the square root lies on the nonpositive real axis only when $\beta_f\cdot\xi = 0$ and

$$L_{S^n}(\theta,\xi) \le \frac{\alpha^2|\gamma_f|^2}{1+|\gamma_f|^2}.$$

Now

$$L_{S^n}(\theta,\xi) = a_1(\theta)\xi_1^2 + \cdots + a_n(\theta)h^2\xi_n + hb_1(\theta)\xi_1 + \cdots + hb_n(\theta)h\xi_n,$$

where by the hypotheses in Proposition 3.1,

$$|a_j - 1| \le C_\mu$$

for C_μ that goes to zero as μ goes to zero. Therefore

$$L_{S^n}(\theta,\xi) \ge (1-C_\mu)|\xi|^2 - hC|\xi| \ge (1-C_\mu-h)|\xi|^2,$$

where C bounds the $b_i(\theta)$. On the support of ζ,

$$|\xi|^2 \ge c_0 > \frac{K^2}{1+K^2},$$

so for small enough μ and h,

$$L_{S^n}(\theta,\xi) > \frac{K^2}{1+K^2}.$$

Then $|\alpha - 1| \lesssim h\varepsilon^{-1}$, and by (3-7),

$$|\gamma_f - K| \le C_\mu,$$

so for small enough μ and h, it follows that

$$L_{S^n}(\theta,\xi) > \frac{\alpha^2|\gamma_f|^2}{1+|\gamma_f|^2}$$

on the support of ζ. Therefore the square root is actually smooth on the support of ζ, and hence $G_\pm$ are smooth and really are symbols of order 1 on $\mathbb{R}^n$.

Now if T_a is the operator associated to the symbol a,

$$\begin{aligned}\Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)&\big(1+|\gamma_f|^2\big)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)\\ &= \big(1+|\gamma_f|^2\big)h^2\partial_r^2 - \frac{2}{r}\big(\alpha+\beta_f\cdot h\nabla_\theta\big)h\partial_r T_\zeta + \frac{1}{r^2}\big(\alpha^2+h^2L_{S^n}\big)T_{\zeta^2}\\ &\quad - \frac{2}{r}\big(1+|\gamma_f|^2\big)T_{G_s} + \frac{1}{r^2}\big(1+|\gamma_f|^2\big)\big(T_{G_+}T_{G_s}+T_{G_-}T_{G_s}+T_{G_s}T_{G_s}\big) + hE_1,\end{aligned}$$

where E_1 is an operator built of first-order semiclassical pseudodifferential operators in $\mathbb{R}^n$ and ∂_r derivatives that is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$.

Now let $v = J_\ell^{-1}w_\ell$. Then

$$\begin{aligned}\Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)&\big(1+|\gamma_f|^2\big)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)v\\ &= \big(1+|\gamma_f|^2\big)h^2\partial_r^2 v - \frac{2}{r}\big(\alpha+\beta_f\cdot h\nabla_\theta\big)h\partial_r T_\zeta v + \frac{1}{r^2}\big(\alpha^2+h^2L_{S^n}\big)T_{\zeta^2}v\\ &\quad - \frac{2}{r}\big(1+|\gamma_f|^2\big)T_{G_s}v + \frac{1}{r^2}\big(1+|\gamma_f|^2\big)\big(T_{G_+}+T_{G_-}+T_{G_s}\big)T_{G_s}v + hE_1v.\end{aligned}$$

Note that $\hat{w}_\ell(r,\xi)$ is only supported for ξ on the support of $1-\rho$, and therefore $v = J_\ell^{-1}w_\ell$ is supported only for ξ on the support of $1-\rho$. Therefore

$$T_\zeta v = v,$$

since $\zeta \equiv 1$ on the support of $1-\rho$. Similarly, $T_{\zeta^2}v = v$. In addition,

$$T_{G_s}v = 0,$$

since G_s is 0 on the support of $1-\rho$. Therefore

$$\begin{aligned}\Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)&\big(1+|\gamma_f|^2\big)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)v\\ &= \big(1+|\gamma_f|^2\big)h^2\partial_r^2 v - \frac{2}{r}\big(\alpha+\beta_f\cdot h\nabla_\theta\big)h\partial_r v + \frac{1}{r^2}\big(\alpha^2+h^2L_{S^n}\big)v + hE_1 v\\ &= \mathscr{L}_{\varphi,\varepsilon,\sigma}v + hE_1 v,\end{aligned}$$

where E_1 is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$.

Therefore

$$\mathscr{L}_{\varphi,\varepsilon,\sigma}v = \Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)z + hE_1 v$$

for some function z, given by

$$z = (1+|\gamma_f|^2)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)v.$$

Then

$$\begin{aligned}\hat{g}(r,\xi) &= \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty \widehat{\mathscr{L}_{\varphi,\varepsilon,\sigma}v}(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt\\ &= \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty \mathscr{F}\Big(\Big(h\partial_t - \frac{1}{t}T_{G_+}\Big)z\Big)(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt\\ &\qquad + \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty h\widehat{E_1 v}(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt.\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}\hat{g}(r,\xi) &= \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty \frac{1}{t}\mathscr{F}\big((T_{\overline{F_\ell}} - T_{G_+})z\big)(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt\\ &\qquad + \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty h\widehat{E_1 v}(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt.\end{aligned}$$

There are no boundary terms because z is supported away from $r=1$. Therefore, using the bounds on g,

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \le \Big\|\frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z\Big\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + h^2\|E_1 v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

We need an estimate for $\Big\|\frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z\Big\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}$. Examine the symbol $\overline{F_\ell}-G_+$.

$$\overline{F_\ell}-G_+ = \zeta\bigg(\overline{F_\ell(\xi)} - \frac{\alpha+i\beta_f\cdot\xi+\sqrt{(\alpha+i\beta_f\cdot\xi)^2-(1+|\gamma_f|^2)(\alpha^2+L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2}\bigg).$$

On the support of ζ,

$$\overline{F_\ell(\xi)} = \frac{1}{1+K^2}\Big(1+iK\xi_n+\sqrt{2iK\xi_n-(K\xi_n)^2+(1+K^2)|\xi|^2-|K|^2}\Big).$$

Therefore

$$\overline{F_\ell} - G_+ = \zeta\left(\frac{1+iK\xi_n}{1+K^2} - \frac{\alpha+i\beta_f\cdot\xi}{1+|\gamma_f|^2}\right) \\ + \zeta\left(\frac{\sqrt{2iK\xi_n-(K\xi_n)^2-(1+K^2)|\xi|^2-|K|^2}}{1+K^2} - \frac{\sqrt{(\alpha+i\beta_f\cdot\xi)^2-(1+(\gamma_f)^2)(\alpha^2+L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2}\right).$$

Consider the first term.

$$\frac{1+iK\xi_n}{1+K^2} - \frac{\alpha+i\beta_f\cdot\xi}{1+|\gamma_f|^2} = \frac{(|\gamma_f|^2-K^2)(1+iK\xi_n)}{(1+K^2)(1+|\gamma_f|^2)} + \frac{(1+K^2)\big((1-\alpha)+i(\beta_f-Ke_n)\cdot\xi\big)}{(1+K^2)(1+|\gamma_f|^2)}.$$

The first-order operators with symbols

$$\frac{(|\gamma_f|^2-K^2)(1+iK\xi_n)}{(1+K^2)(1+|\gamma_f|^2)}$$

and

$$\frac{(1+K^2)\big((1-\alpha)+i(\beta_f-Ke_n)\cdot\xi\big)}{(1+K^2)(1+|\gamma_f|^2)}$$

have bounds $\lesssim C_\mu$, because they involve multiplication by a function of θ that is bounded by $C_K C_\mu$.

Similarly, consider the first-order operator with symbol

$$\zeta\left(\frac{\sqrt{2iK\xi_n-(K\xi_n)^2-(1+K^2)|\xi|^2-|K|^2}}{1+K^2} - \frac{\sqrt{(\alpha+i\beta_f\cdot\xi)^2-(1+(\gamma_f)^2)(\alpha^2+L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2}\right).$$

To fit everything horizontally on the page, write

$$\tau_K := 2iK\xi_n-(K\xi_n)^2-(1+K^2)|\xi|^2-|K|^2$$

and

$$\tau_f := (\alpha+i\beta_f\cdot\xi)^2-(1+(\gamma_f)^2)(\alpha^2+L_{S^n}(\theta,\xi)).$$

Then

$$\frac{\sqrt{\tau_K}}{1+K^2} - \frac{\sqrt{\tau_f}}{1+|\gamma_f|^2} = (1+K^2)\frac{\tau_K-\tau_f}{(1+|\gamma_f|^2)\big((1+|\gamma_f|^2)\sqrt{\tau_K}+(1+K^2)\sqrt{\tau_f}\big)} \\ + \frac{\big((1+|\gamma_f|^2)^2-(1+K^2)^2\big)\tau_K}{(1+K^2)(1+|\gamma_f|^2)\big((1+|\gamma_f|^2)\sqrt{\tau_K}+(1+K^2)\sqrt{\tau_f}\big)}.$$

Expanding,

$$\tau_K-\tau_f = 2i(Ke_n-\alpha\beta_f)\cdot\xi + \big((\beta_f\cdot\xi)^2-(Ke_n\cdot\xi)^2\big) + (|\gamma_f|^2-K^2)L(\theta,i\xi) \\ + (|\gamma_f|^2-|K|^2) + (1+K^2)(|\xi|^2-L(\theta,\xi)).$$

Therefore the second term has operator bounds $\lesssim C_\mu$, because each term involves multiplication by a function of θ that is bounded by $C_K C_\mu$.

Therefore

$$\left\| \frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \le \delta^2 \|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}$$

for μ small enough. Then

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + h^2\|E_1 v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2 \|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}.$$

Since

$$\mathscr{L}_{\varphi,\varepsilon,\sigma} v = \left(h\partial_r - \frac{1}{r}T_{G_+}\right)z + hE_1 v,$$

we have

$$\begin{aligned}
\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} &\ge \left\| \left(h\partial_r - \frac{1}{r}T_{G_+}\right)z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - h^2\|E_1 v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \\
&\ge \|J_\ell^* z\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - \left\| \frac{1}{r}T_{\overline{F_\ell}-G_+} z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \\
&\gtrsim \|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} - \delta^2\|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} - h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \\
&\gtrsim \|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} - h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}
\end{aligned}$$

for δ small enough. Therefore

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\big\|J_\ell^{-1}(1-P)w\big\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}.$$

Using similar reasoning as for the small frequency case,

$$h^2\big\|J_s^{-1}(1-P)w\big\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim h^2\big\|J_\ell^{-1}(1-P)w\big\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\big\|J_\ell^{-1}(1-P)w\big\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\|w_\ell\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Then for δ small enough,

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \tfrac{1}{2}\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|_{L^2(\mathbb{R}^{n+1}_{1+})} + h\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Now using (7-1) and Lemma 5.3,

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|J_\ell \mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_\ell^{-1} w_\ell\big\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Absorbing the second last term into the left side gives

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|J_\ell \mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_\ell^{-1} w_\ell\big\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

We can finish the argument as in the small frequency case to get

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

This finishes the proof of Lemma 4.2, and thus of Proposition 3.1.

8. Proof of Theorem 1.4

We will begin by gluing together estimates of the form in Proposition 3.1 to prove the following intermediate proposition.

Proposition 8.1. *Suppose that $f : S^n \to (0,\infty)$ is a C^∞ function such that Ω lies entirely in the region $A_O = \{(r,\theta) \mid r \geq f(\theta)\} \subset \mathbb{R}^{n+1}$, and Γ_+^c is a subset of the graph $r = f(\theta)$. If $w \in C_0^\infty(\Omega)$, then*

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,W,q} w\|_{H^{-1}(A_O)}.$$

Proof. Now let Ω be as in Proposition 8.1. We can take an open cover $U_1,\dots,U_m$ of Ω such that on each $\Omega \cap U_j$, there exists K_j such that under some choice of coordinates, $|\nabla_{S^n}\log f - K_j e_n| \leq \mu_{K_j}$ and $|\sin(\theta_k) - 1| \leq \mu_{K_j}$, where μ_{K_j} is the value of μ from Proposition 3.1 that works for $K = K_j$. (Since $|\nabla_{S^n}\log f|$ must be bounded above, μ_{K_j} must be bounded below, and therefore this is possible with only finitely many U_j.)

Let $\zeta_1,\dots,\zeta_m$ be a smooth partition of unity subordinate to the cover $U_1,\dots,U_m$. Now for $w \in C_0^\infty(\Omega)$,

$$w = \zeta_1 w + \dots + \zeta_m w =: w_1 + \dots + w_m,$$

where each $w_j \in C_0^\infty(\Omega \cap U_j)$. Applying Proposition 3.1 to the domain $\Omega \cap U_j$,

$$\frac{h}{\sqrt{\varepsilon}}\|w_j\|_{L^2(\Omega\cap U_j)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)}$$

for each $j = 1,\dots,m$. Then

$$\sum_j \frac{h}{\sqrt{\varepsilon}}\|w_j\|_{L^2(\Omega)} \lesssim \sum_j \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)},$$

so

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \sum_j \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)}.$$

Now by the product rule,

$$\begin{aligned}\|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)} = \|\mathcal{L}_{\varphi,\varepsilon}\zeta_j w\|_{H^{-1}(A_O)} &\leq \|\zeta_j \mathcal{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)} + Ch\|w\|_{L^2(A_O)} \\ &\leq \|\mathcal{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)} + Ch\|w\|_{L^2(A_O)}.\end{aligned}$$

Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)} \tag{8-1}$$

for ε small enough, for every $w \in C_0^\infty(\Omega)$.

To treat the case where W and q are nonzero, note that

$$\mathscr{L}_{\varphi,\varepsilon,W,q} = \mathscr{L}_{\varphi,\varepsilon} + h\big(W\cdot hD + hD\cdot W\big) + 2ihW\cdot\nabla\left(\log r + h\frac{\log^2 r}{2\varepsilon}\right) + h^2(q+W^2).$$

Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(A_O)} + hC\|w\|_{L^2(A_O)},$$

and the last term can be absorbed into the left side to give

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(A_O)}.$$

This completes the proof. □

Finally, I can prove Theorem 1.4 by gluing together estimates of the form in Proposition 8.1. If Γ_+ is a neighborhood of $\partial\Omega_+$, then let Ω' be a smooth domain containing Ω, with $\partial\Omega\cap\partial\Omega' = \Gamma_+^c$.

Then let $U_1,\dots,U_m$ be an open cover of Ω such that each $\partial U_j\cap\Gamma_+^c$ coincides with a graph of the form $r = f_j(\theta)$. For each U_j, Proposition 3.1 gives us

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U_j)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(A_j)}$$

for $w\in C_0^\infty(U_j)$.

Each A_j is defined by the graph of a function $r = f_j(\theta)$, and since $\partial\Omega'$ is smooth and coincides with $\partial\Omega$ on Γ_+^c, and $\partial_\nu\varphi<0$ on Γ_+^c, $\partial\Omega'$ must be locally a graph in a neighborhood of Γ_+^c. Therefore we can assume that A_j coincides with Ω' in a neighborhood of each U_j, in the sense that their characteristic functions are equal in that neighborhood. Then there is a smooth cutoff function χ_j defined on $A_j\cap\Omega'$ that is identically one on U_j but vanishes outside on the complements of A_j and Ω'. Multiplication by this function provides a bounded map from $H_0^1(A_j)$ to $H_0^1(\Omega')$ and vice versa, and therefore $\|w\|_{H^{-1}(\Omega')}\simeq\|w\|_{H^{-1}(A_j)}$ for $w\in C_0^\infty(U_j)$. Therefore we have

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U_j)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(\Omega')}$$

for $w\in C_0^\infty(U_j)$.

Gluing together these estimates in the manner used above gives

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(\Omega')}$$

for $w\in C_0^\infty(\Omega)$.

Finally, note that if $w\in C_0^\infty(\Omega)$, then $e^{(\log r)^2/\varepsilon}w\in C_0^\infty(\Omega)$, so

$$\frac{h}{\sqrt{\varepsilon}}\left\|e^{(\log r)^2/\varepsilon}w\right\|_{L^2(\Omega)} \lesssim \left\|e^{(\log r)^2/\varepsilon}\mathscr{L}_{\varphi,W,q}w\right\|_{H^{-1}(\Omega')}.$$

On Ω, there exists some C_Ω such that $1\le e^{(\log r)^2/\varepsilon}\le e^{C_\Omega/\varepsilon}$, so

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,W,q}w\|_{H^{-1}(\Omega')},$$

as desired. This establishes Theorem 1.4.

Remark. If we want to prove Theorem 1.2 instead of Theorem 1.1, then we could begin by supposing that $f : S^n \to (0, \infty)$ is a C^∞ function such that Ω lies entirely in the region $A_I = \{(r, \theta) \mid r \le f(\theta)\} \subset \mathbb{R}^{n+1}$, and Γ_-^c is a subset of the graph $r = f(\theta)$. Then by the change of variables $(r, \theta) \mapsto (1/r, \theta)$, Ω maps to a region $\hat{\Omega}$ of the form described in Proposition 8.1. Therefore, by (8-1),

$$h\|w\|_{L^2(\hat{\Omega})} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{H^{-1}(\hat{A}_O)}$$

for $w \in C_0^\infty \hat{\Omega}$, where $\varphi = \log r$. Changing variables back gives the Carleman estimate

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{-\log r,\varepsilon} w\|_{H^{-1}(A_I)}$$

for $w \in C_0^\infty \Omega$. Therefore, by the same kind of argument as above, we get

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,W,q} w\|_{H^{-1}(\Omega')},$$

where $\varphi = -\log r$ and Ω' is a domain containing Ω, with $\Gamma_-^c \subset \partial\Omega' \cap \partial\Omega$ whenever Γ_- is of the form described in Theorem 1.2. Using this Carleman estimate in the place of Theorem 1.4 in the remainder of the argument proves Theorem 1.2 instead of Theorem 1.1.

9. Complex geometric optics solutions

Theorem 1.4 can be used to construct solutions to equations of the system (1-1). The key is the following proposition.

Proposition 9.1. *For every $v \in L^2(\Omega)$, there exists $u \in H^1(\Omega)$ such that*

$$\mathscr{L}^*_{\varphi,W,q} u = v \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0$$

and

$$\|u\|_{H^1(\Omega)} \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)}.$$

Proof. The proof is based on a Hahn–Banach argument. Suppose $v \in L^2(\Omega)$. Then for all $w \in C_0^\infty(\Omega)$,

$$\left|(w|v)_\Omega\right| \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)} h\|w\|_{L^2(\Omega)}.$$

Therefore, by Theorem 1.4,

$$\left|(w|v)_\Omega\right| \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)} \|\mathscr{L}_{\varphi,W,q} w\|_{H^{-1}(\Omega')}. \tag{9-1}$$

Now consider the subspace

$$\{\mathscr{L}_{\varphi,W,q} w \mid w \in C_0^\infty(\Omega)\} \subset H^{-1}(\Omega').$$

By the estimate from Theorem 1.4, the map $\mathscr{L}_{\varphi,W,q} w \mapsto (w|v)_\Omega$ is well defined on this subspace. It is a linear functional, and by (9-1), it is bounded by $(C/h)\|v\|_{L^2(\Omega)}$.

Therefore, by Hahn–Banach, there exists an extension of this functional to the whole space $H^{-1}(\Omega')$ with the same bound. This can be represented by an element of the dual space $H_0^1(\Omega')$, so there exists $u \in H_0^1(\Omega')$ such that

$$\|u\|_{H^1(\Omega')} \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)}$$

and

$$(w|v)_\Omega = (\mathscr{L}_{\varphi,W,q}w|u)_{\Omega'} = (\mathscr{L}_{\varphi,W,q}w|u)_\Omega$$

for all $w \in C_0^\infty(\Omega)$. Note that $u \in H_0^1(\Omega')$ implies that $u|_{\Gamma_+^c} = 0$. Then

$$(w|v)_\Omega = (w|\mathscr{L}^*_{\varphi,W,q}u)_\Omega$$

since $w \in C_0^\infty(\Omega)$, and thus

$$(w|v - \mathscr{L}^*_{\varphi,W,q}u)_\Omega = 0$$

for all $v \in C_0^\infty(\Omega)$. Therefore $v = \mathscr{L}^*_{\varphi,W,q}u$ on Ω, and

$$\|u\|_{H^1(\mathbb{R}^{n+1})} \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)},$$

as desired. □

Now I can construct the complex geometrical optics solutions.

Proposition 9.2. *There exists a solution of the problem*

$$\mathscr{L}_{W,q}u = 0 \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0$$

of the form $u = e^{(1/h)(\varphi+i\psi)}(a+r) - e^{\ell/h}b$, *where* $\varphi(x,y) = \log r$, ψ *is a solution to the eikonal equation* $\nabla\varphi\cdot\nabla\psi = 0$, $|\nabla\varphi| = |\nabla\psi|$, *a and b are* C^2 *functions on* Ω, *and*

$$\operatorname{Re}\ell(x,y) = \varphi(x,y) - k(x,y),$$

where $k(x) \simeq \operatorname{dist}(x,\Gamma_+^c)$ *in a neighborhood of* Γ_+^c *and b has its support in that neighborhood. Finally,* $r \in H^1(\Omega)$, *with* $r|_{\Gamma_+^c} = 0$, $\|r\|_{H^1(\Omega)} = O(h)$, *and* $\|r\|_{L^2(\partial\Omega)} = O(h^{1/2})$.

The proof is a combination of the proofs of the equivalent theorems in [Dos Santos Ferreira et al. 2007; Kenig et al. 2007].

Proof. Let $\varphi(r,\theta) = \log r$, and take $\psi(r,\theta) = d_{S^n}(\theta,\omega)$ for some fixed point $\omega \in S^n$. If $\omega \neq \theta$ for all $(r,\theta) \in \Omega$, then ψ solves the eikonal equation $\nabla\varphi\cdot\nabla\psi = 0$, $|\nabla\varphi| = |\nabla\psi|$. Then

$$h^2\mathscr{L}_{W,q}e^{(1/h)(\varphi+i\psi)} = e^{(1/h)(\varphi+i\psi)}\big(h(D+W)\cdot(\nabla\psi - i\nabla\varphi) + h(\nabla\psi - i\nabla\varphi)\cdot(D+W) + h^2\mathscr{L}_{W,q}\big).$$

Therefore, if a is a C^2 solution to

$$(\nabla\psi - i\nabla\varphi)\cdot Da + (\nabla\psi - i\nabla\varphi)\cdot Wa + \frac{1}{2i}(\triangle\psi - i\triangle\varphi)a = 0,$$

then

$$h^2\mathscr{L}_{W,q}e^{(1/h)(\varphi+i\psi)}a = e^{(1/h)(\varphi+i\psi)}h^2\mathscr{L}_{W,q}a = O(h^2)e^{(1/h)(\varphi+i\psi)}.$$

We can look for an exponential solution $a = e^{\Phi}$, in which case the relevant equation becomes

$$(\nabla\varphi + i\nabla\psi)\cdot\nabla\Phi + i(\nabla\varphi + i\nabla\psi)\cdot W + \tfrac{1}{2}\triangle(\varphi + i\psi) = 0.$$

Now suppose $x \in \mathbb{R}^{n+1}$, and write $x = (x_\omega, x')$, where x_ω is the component of x in the ω direction, and x' are the remaining components. Then by considering $z = x_\omega + i|x'|$ as a complex variable, we get $\varphi = \operatorname{Re}\log z$ and $\psi = \operatorname{Im}\log z$. Now our equation is an inhomogeneous Cauchy–Riemann equation in the z variable, and can be solved by the Cauchy formula. Then a is C^2, since W is. The solution is only unique up to addition of terms g_a with

$$(\nabla\varphi + i\nabla\psi)\cdot\nabla g_a = 0. \tag{9-2}$$

Now I want to construct a (complex-valued) function ℓ to be an approximate solution to the equation

$$\nabla\ell\cdot\nabla\ell = 0, \quad \ell|_{\Gamma_+^c} = \varphi + i\psi.$$

In order to avoid duplicating the solution $\varphi + i\psi$, we can ask for

$$\partial_\nu\ell|_{\Gamma_+^c} = -\partial_\nu(\varphi + i\psi)|_{\Gamma_+^c}.$$

To construct an approximate solution, pick coordinates (t, s) near Γ_+^c such that t are the coordinates along Γ_+^c and s is perpendicular to Γ_+^c. Suppose ℓ takes the form of a power series

$$\ell(t, s) = \sum_{j=0}^{\infty} a_j(t)s^j.$$

Then

$$\nabla\ell = (\nabla_t\ell, \partial_s\ell) = \biggl(\sum_{j=0}^{\infty}\nabla_t a_j(t)s^j, \sum_{j=0}^{\infty} a_j(t)js^{j-1}\biggr).$$

Expanding the equation $\nabla\ell\cdot\nabla\ell = 0$ and considering each power of s separately gives a sequence of equations

$$\sum_{j+k=m}\nabla_t a_j\nabla_t a_k + \sum_{j+k=m+2} jka_ja_k = 0 \tag{9-3}$$

for each $m = 0, 1, 2, \ldots$. The boundary conditions determine a_0 and a_1, so we can solve this recursively. If $m \geq 1$ and all a_j are known for $j \leq m$, the only part of (9-3) that contains an unknown looks like $2(m+1)a_1a_{m+1}$. Note that

$$a_1 = -\partial_\nu(\varphi + i\psi).$$

Since Γ_+^c coincides with a graph $r = f(\theta)$ for some smooth function f, and $\varphi = \log r$, there exists some $\varepsilon_0 > 0$ such that $|a_1| > \varepsilon_0$ on Γ_+^c, so we can divide by a_1 to solve for a_{m+1}.

This gives a formal power series that may or may not converge outside $s = 0$. However, we can construct a C^∞ function ℓ in Ω whose Taylor series in s coincides with this formal power series at $s = 0$, such that

$$\nabla\ell\cdot\nabla\ell = O\bigl(\operatorname{dist}(x, \Gamma_+^c)^\infty\bigr).$$

Moreover,

$$\partial_\nu \operatorname{Re} \ell|_{\Gamma_+^c} = -\partial_\nu \varphi|_{\Gamma_+^c} < -\varepsilon_0$$

and

$$\operatorname{Re} \ell|_{\Gamma_+^c} = \varphi|_{\Gamma_+^c},$$

so in a neighborhood of Γ_+^c,

$$\operatorname{Re} \ell(x, y) = \varphi(x, y) - k(x, y), \tag{9-4}$$

where $k(x) \simeq \operatorname{dist}(x, \Gamma_+^c)$ in a neighborhood of Γ_+^c.

By a similar method, we can construct an approximate solution b for the problem

$$\nabla\ell \cdot Db + \nabla\ell \cdot Wb = 0, \quad b|_{\Gamma_+^c} = a|_{\Gamma_+^c},$$

so

$$\nabla\ell \cdot Db + \nabla\ell \cdot Wb = O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big), \quad b|_{\Gamma_+^c} = a|_{\Gamma_+^c}.$$

Multiplying b by a smooth cutoff function does not change these properties, so we may as well assume that b is only supported close to Γ_+^c for (9-4) to hold. Then

$$-h^2 \mathscr{L}_{W,q}(e^{\ell/h} b) = e^{\ell/h}\big(O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big) + O(h^2)\big),$$

so

$$\big|h^2 \mathscr{L}_{W,q}(e^{\ell/h} b)\big| = e^{\varphi/h} e^{-k/h}\big(O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big) + O(h^2)\big).$$

If $\operatorname{dist}(x, \Gamma_+^c) \le h^{1/2}$, for h small, this is $e^{\varphi/h} O(h^2)$, because of the $O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big)$ term. On the other hand, if $\operatorname{dist}(x, \Gamma_+^c) \ge h^{1/2}$, this is still $e^{\varphi/h} O(h^2)$, because of $e^{-k/h}$.

Now $e^{(1/h)(\varphi+i\psi)} a - e^{\ell/h} b = 0$ on Γ_+^c, and

$$e^{-\varphi/h} h^2 \mathscr{L}_{W,q}\big(e^{(1/h)(\varphi+i\psi)} a + e^{\ell/h} b\big) = v,$$

where $\|v\|_{L^2(\Omega)} = O(h^2)$. By Proposition 9.1, the problem

$$\mathscr{L}^*_{\varphi,W,q} r_1 = e^{-\varphi/h} h^2 \mathscr{L}_{W,q} e^{\varphi/h} r_1 = -v \text{ on } \Omega, \quad r_1|_{\Gamma_+^c} = 0$$

has an H^1 solution r_1 with

$$\|r_1\|_{H^1(\Omega)} \lesssim \frac{1}{h} \|v\|_{L^2(\Omega)} = O(h).$$

Set $r = e^{-i\psi/h} r_1$ and $u = e^{(1/h)(\varphi+i\psi)}(a + r) - e^{\ell/h} b$. Then

$$\|r\|_{H^1(\Omega)} = O(h),$$

so $\|r\|_{L^2(\partial\Omega)} = O(h^{1/2})$ by the trace theorem, and

$$\mathscr{L}_{W,q} u = 0 \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0.$$

This finishes the proof. □

If the boundary condition is not needed, then the result is as follows:

Proposition 9.3. *There exists a solution of the problem*

$$\mathcal{L}_{W,q} u = 0 \text{ on } \Omega$$

of the form $u = e^{(1/h)(\varphi + i\psi)}(a + r)$, *where* $\varphi(x, y)$ *is any limiting Carleman weight*, ψ *is any solution to the eikonal equation*, a *is a* C^2 *function on* Ω, *and* $r \in H^1(\Omega)$, *with* $\|r\|_{H^1(\Omega)} = O(h)$ *and* $\|r\|_{L^2(\partial\Omega)} = O(h^{1/2})$.

This is essentially Lemma 3.4 from [Dos Santos Ferreira et al. 2007]. We can always replace a by γa, where γ is a solution to

$$(\nabla\varphi + i\nabla\psi) \cdot \nabla\gamma = 0 \text{ on } \Omega.$$

10. Proof of Theorem 1.1

For convenience, $\|\cdot\|$ will denote the L^2 norm in this section, unless otherwise indicated. The tilde as used in this section has nothing to do with the notation from Section 3.

Using Proposition 9.2, we can construct $\tilde{u}_2 = e^{(1/h)(\varphi + i\psi)}(a_2 + r_2) - e^{\ell/h} b =: u_2 + u_r$ to be a solution to

$$\mathcal{L}_{W_2,q_2} \tilde{u}_2 = 0 \text{ on } \Omega, \quad \tilde{u}_2|_{\Gamma_+^c} = 0.$$

Then $-\varphi$ is also a Carleman weight, and if φ and ψ satisfy the eikonal equation, then so do $-\varphi$ and ψ. Therefore, using Proposition 9.3, we can construct $u_1 = e^{(1/h)(-\varphi + i\psi)}(a_1 + r_1)$ to be a solution to

$$\mathcal{L}_{W_1,\overline{q_1}} u_1 = 0.$$

Let w be the unique solution to

$$\mathcal{L}_{W_1,q_1} w = 0, \quad w|_{\partial\Omega} = \tilde{u}_2|_{\partial\Omega}.$$

(Here we are using the assumption that $\mathcal{L}_{W_1,q_1}$ does not have a zero eigenvalue.) In particular, $w|_{\Gamma_+^c} = \tilde{u}_2|_{\Gamma_+^c} = 0$, so by the hypothesis on the Dirichlet–Neumann map,

$$\partial_\nu (w - \tilde{u}_2)|_{\Gamma_-} = 0.$$

Now

$$\begin{aligned} \mathcal{L}_{W_1,q_1}(w - \tilde{u}_2) &= -\mathcal{L}_{W_1,q_1} \tilde{u}_2 \\ &= (\mathcal{L}_{W_2,q_2} - \mathcal{L}_{W_1,q_1}) \tilde{u}_2 \\ &= (W_2 - W_1) \cdot D\tilde{u}_2 + D \cdot (W_2 - W_1)\tilde{u}_2 + \big(W_2^2 - W_1^2 + q_2 - q_1\big)\tilde{u}_2. \end{aligned} \tag{10-1}$$

On the other hand, Green's formula from [Dos Santos Ferreira et al. 2007] gives us

$$\int_\Omega \mathcal{L}_{W_1,q_1}(w - \tilde{u}_2)\overline{u_1}\, dV = \int_{\partial\Omega} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS = \int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS. \tag{10-2}$$

Combining (10-1) with (10-2) gives

$$\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS = \int_\Omega (W_2 - W_1) \cdot \big(D\tilde{u}_2\overline{u_1} + \tilde{u}_2\overline{Du_1}\big)\, dV + \int_\Omega \big(W_2^2 - W_1^2 + q_2 - q_1\big)\tilde{u}_2\overline{u_1}\, dV.$$

Expanding $\tilde{u}_2$ as $\tilde{u}_2 = u_2 + u_r$ on the right side gives

$$\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS = \int_\Omega (W_2 - W_1)\cdot\big(Du_2\overline{u_1} + u_2\overline{Du_1}\big)\, dV + \int_\Omega \big(W_2^2 - W_1^2 + q_2 - q_1\big)u_2\overline{u_1}\, dV$$
$$+ \int_\Omega (W_2 - W_1)\cdot\big(Du_r\overline{u_1} + u_r\overline{Du_1}\big)\, dV + \int_\Omega \big(W_2^2 - W_1^2 + q_2 - q_1\big)u_r\overline{u_1}\, dV. \quad (10\text{-}3)$$

To show that $dW_1 = dW_2$, we can apply the reasoning from [Dos Santos Ferreira et al. 2007] verbatim if we can establish that

$$\lim_{h\to 0} h \int_\Omega (W_2 - W_1)\cdot\big(Du_2\overline{u_1} + u_2\overline{Du_1}\big)\, dV = 0. \quad (10\text{-}4)$$

Similarly, to show that $q_1 = q_2$, we can apply the reasoning from [Dos Santos Ferreira et al. 2007] verbatim if we can establish that

$$\lim_{h\to 0} \int_\Omega (q_2 - q_1)u_2\overline{u_1}\, dV = 0. \quad (10\text{-}5)$$

To establish (10-4), label the terms as follows: $T_1 = T_2 + T_3 + T_4 + T_5$. Consider the terms on the right side first. T_2 is bounded above by

$$\big\|(W_2 - W_1)e^{-\varphi/h}Du_2\big\|_\Omega \|a_1 + r_1\|_\Omega + \big\|(W_2 - W_1)e^{\varphi/h}\overline{Du_1}\big\|_\Omega \|a_2 + r_2\|_\Omega.$$

Since $W_2 - W_1$ is bounded on Ω, $\|a_1\|_\Omega$ and $\|a_2\|_\Omega$ are $O(1)$, and $\|r_1\|_\Omega$ and $\|r_2\|_\Omega$ are $O(h)$,

$$|T_2| \lesssim \big\|e^{-\varphi/h}Du_2\big\|_\Omega + \big\|e^{\varphi/h}Du_1\big\|_\Omega.$$

T_3 is bounded above by

$$|T_3| \le \big\|\big(W_2^2 - W_1^2 + q_2 - q_1\big)(a_2 + r_2)\big\|_\Omega \|a_1 + r_1\|_\Omega = O(1).$$

Similarly,

$$|T_4| \lesssim \big\|e^{-\varphi/h}Du_r\big\|_\Omega + \big\|e^{\varphi/h}\overline{Du_1}\big\|_\Omega \big\|e^{-2\beta y/h}\big\|_\Omega \lesssim \big\|e^{-\varphi/h}Du_r\big\|_\Omega + h\big\|e^{\varphi/h}\overline{Du_1}\big\|_\Omega$$

and

$$|T_5| \le \big\|\big(W_2^2 - W_1^2 + q_2 - q_1\big)e^{-2\beta y/h}b\big\|_\Omega \|a_1 + r_1\|_\Omega = O(h).$$

Now examine the term T_1:

$$\left|\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS\right| \le \big\|\partial_\nu(\tilde{u}_2 - w)e^{-\varphi/h}\big\|_{\Gamma_-^c} \|a_1 + r_1\|_{\Gamma_-^c}.$$

The factor $\|a_1 + r_1\|_{\Gamma_-^c}$ is $O(1)$. Furthermore, $\partial_\nu\varphi \ge \varepsilon_1$ on Γ_-^c, so

$$\left|\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS\right| \lesssim \frac{1}{\sqrt{\varepsilon_1}}\big\|\sqrt{\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\Gamma_-^c} \lesssim \frac{1}{\sqrt{\varepsilon_1}}\big\|\sqrt{\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\Gamma_+}.$$

By the Carleman estimate given in Equation (2.13) of [Dos Santos Ferreira et al. 2007],

$$\big\|\sqrt{\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\Gamma_+} \lesssim \sqrt{h}\big\|e^{-\varphi/h}\mathscr{L}_{W_1,q_1}(\tilde{u}_2 - w)\big\|_\Omega + \big\|\sqrt{-\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\partial\Omega_-}.$$

Therefore
$$\frac{C}{\sqrt{\varepsilon_1}}\Big(\sqrt{h}\big\|e^{-\varphi/h}\mathscr{L}_{W_1,q_1}(\tilde{u}_2-w)\big\|_\Omega+\big\|\sqrt{-\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2-w)\big\|_{\partial\Omega_-}\Big).$$

The last term on the right side is zero, because $\partial_\nu(\tilde{u}_2-w)=0$ on Γ_- and $\partial\Omega_-\subset\Gamma_-$. Therefore the upper bound becomes
$$\frac{C}{\sqrt{\varepsilon_1}}\sqrt{h}\big\|e^{-\varphi/h}\mathscr{L}_{W_1,q_1}(\tilde{u}_2-w)\big\|_\Omega.$$

Expanding $\mathscr{L}_{W_1,q_1}(\tilde{u}_2-w)$ and writing $\tilde{u}_2=u_2+u_r$, we obtain that T_1 is bounded above by
$$\begin{aligned}\frac{C\sqrt{h}}{\sqrt{\varepsilon_1}}&\Big(\big\|e^{-\varphi/h}Du_2\big\|_\Omega+\big\|e^{-\varphi/h}u_2\big\|_\Omega+\big\|e^{-\varphi/h}Du_r\big\|_\Omega+\big\|e^{-\varphi/h}u_r\big\|_\Omega\Big)\\&\le\frac{C\sqrt{h}}{\sqrt{\varepsilon_1}}\Big(\big\|e^{-\varphi/h}Du_2\big\|_\Omega+\|a_2+r_2\|_\Omega+\big\|e^{-\varphi/h}Du_r\big\|_\Omega+\big\|e^{-2\beta y/h}b\big\|_\Omega\Big)\\&\le\frac{C\sqrt{h}}{\sqrt{\varepsilon_1}}\Big(\big\|e^{-\varphi/h}Du_2\big\|_\Omega+O(1)+\big\|e^{-\varphi/h}Du_r\big\|_\Omega+O(h)\Big),\end{aligned}$$

where the constant C mutates as necessary to preserve the bound. Therefore, in order to bound the terms T_1, T_2, and T_4, we need to calculate $\big\|e^{\varphi/h}Du_1\big\|_\Omega$, $\big\|e^{-\varphi/h}Du_2\big\|_\Omega$, and $\big\|e^{-\varphi/h}Du_r\big\|_\Omega$. We have
$$\begin{aligned}\big\|e^{\varphi/h}Du_1\big\|_\Omega&=\Big\|e^{\varphi/h}\frac{1}{h}D(-\varphi+i\psi)e^{(1/h)(-\varphi+i\psi)}(a_1+r_1)+e^{i\psi/h}D(a_1+r_1)\Big\|_\Omega\\&\lesssim\frac{1}{h}\big\|D(-\varphi+i\psi)(a_1+r_1)\big\|_\Omega+\big\|D(a_1+r_1)\big\|_\Omega=O(h^{-1}),\end{aligned}$$

since $\|r_1\|_{H^1(\Omega)}$ is $O(h)$. Similarly,
$$\big\|e^{-\varphi/h}Du_2\big\|_\Omega=O(h^{-1}).$$

Finally,
$$\big\|e^{-\varphi/h}Du_r\big\|_\Omega=\Big\|e^{-\varphi/h}\frac{1}{h}D\ell e^{\ell/h}b+e^{-\varphi/h}e^{\ell/h}Db\Big\|_\Omega\lesssim\frac{1}{h}\big\|e^{-k/h}bD\ell\big\|_\Omega+\big\|e^{-k/h}Db\big\|_\Omega=O(1).$$

Putting all of this together gives $T_1=O(h^{-1/2})$, $T_2=O(h^{-1})$, $T_3=O(1)$, $T_4=O(1)$, and $T_5=O(h)$. Therefore, multiplying (10-3) through by h and taking the limit as h goes to zero gives
$$\lim_{h\to 0}h\int_\Omega(W_2-W_1)\cdot\big(Du_2\overline{u_1}+u_2\overline{Du_1}\big)\,dV=0,$$

which establishes (10-4), and thus by the reasoning in [Dos Santos Ferreira et al. 2007], that $dW_1=dW_2$ in Ω and $W_1=W_2$ up to a gauge transformation that leaves the Dirichlet–Neumann maps invariant.

It remains only to prove (10-5). Going back to (10-3), we now have
$$\int_{\Gamma_-^c}\partial_\nu(\tilde{u}_2-w)\overline{u_1}\,dS=\int_\Omega(q_2-q_1)u_2\overline{u_1}\,dx+\int_\Omega(q_2-q_1)u_r\overline{u_1}\,dV.\tag{10-6}$$

The first and second terms on the right side are $O(1)$ and $O(h)$ as before. The left side is now bounded by

$$\frac{\sqrt{h}}{\sqrt{\varepsilon_1}}\Big(\big\|e^{-\varphi/h}(q_1-q_2)u_2\big\|_\Omega+\big\|e^{-\varphi/h}(q_1-q_2)u_r\big\|_\Omega\Big)=\sqrt{h}\big(O(1)+O(h)\big)=O(h^{1/2}),$$

so taking the limit of (10-6) as h goes to zero gives

$$\lim_{h\to 0}\int_\Omega (q_2-q_1)u_2\overline{u_1}\,dV=0.$$

This establishes (10-5), and thus that $q_1=q_2$ on Ω. This finishes the proof.

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SHARP POLYNOMIAL DECAY RATES FOR THE DAMPED WAVE EQUATION ON THE TORUS

NALINI ANANTHARAMAN AND MATTHIEU LÉAUTAUD
APPENDIX BY STÉPHANE NONNENMACHER

We address the decay rates of the energy for the damped wave equation when the damping coefficient b does not satisfy the geometric control condition (GCC). First, we give a link with the controllability of the associated Schrödinger equation. We prove in an abstract setting that the observability of the Schrödinger equation implies that the solutions of the damped wave equation decay at least like $1/\sqrt{t}$ (which is a stronger rate than the general logarithmic one predicted by the Lebeau theorem).

Second, we focus on the 2-dimensional torus. We prove that the best decay one can expect is $1/t$, as soon as the damping region does not satisfy GCC. Conversely, for smooth damping coefficients b vanishing flatly enough, we show that the semigroup decays at least like $1/t^{1-\varepsilon}$, for all $\varepsilon > 0$. The proof relies on a second microlocalization around trapped directions, and resolvent estimates.

In the case where the damping coefficient is a characteristic function of a strip (hence discontinuous), Stéphane Nonnenmacher computes in an appendix part of the spectrum of the associated damped wave operator, proving that the semigroup cannot decay faster than $1/t^{2/3}$. In particular, our study emphasizes that the decay rate highly depends on the way b vanishes.

CONTENTS

MSC2010: primary 35A21, 35B35, 35L05, 35P20, 35S05; secondary 35B37, 93C20.
Keywords: damped wave equation, polynomial decay, observability, Schrödinger group, torus, two-microlocal semiclassical measures, spectrum of the damped wave operator.

Part I. The damped wave equation

1. Decay of energy: a survey of existing results

Let (M, g) be a smooth compact connected Riemannian d-dimensional manifold, with or without boundary ∂M. We denote by Δ the (nonpositive) Laplace–Beltrami operator on M for the metric g. Given a bounded nonnegative function, $b \in L^\infty(M)$, $b(x) \geq 0$ on M, we want to understand the asymptotic behavior as $t \to +\infty$ of the solution u of the problem

$$\begin{cases} \partial_t^2 u - \Delta u + b(x)\,\partial_t u = 0 & \text{in } \mathbb{R}^+ \times M, \\ u = 0 & \text{on } \mathbb{R}^+ \times \partial M \ (\text{if } \partial M \neq \varnothing), \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) & \text{in } M. \end{cases} \tag{1-1}$$

The energy of a solution is defined by

$$E(u,t) = \tfrac{1}{2}(\|\nabla u(t)\|^2_{L^2(M)} + \|\partial_t u(t)\|^2_{L^2(M)}). \tag{1-2}$$

Multiplying (1-1) by $\partial_t u$ and integrating on M yields the dissipation identity

$$\frac{d}{dt}E(u,t) = -\int_M b|\partial_t u|^2\,dx,$$

which, as b is nonnegative, implies a decay of the energy. As soon as $b \geq C > 0$ on a nonempty open subset of M, the decay is strict and $E(u,t) \to 0$ as $t \to +\infty$. The question is then to know at what rate the energy goes to zero.

The first interesting issue concerns uniform stabilization: under which condition does there exist a function $F(t)$, $F(t) \to 0$, such that

$$E(u,t) \leq F(t)E(u,0)\ ? \tag{1-3}$$

The answer was given by Rauch and Taylor [1974] in the case $\partial M = \varnothing$ and by Bardos, Lebeau and Rauch [Bardos et al. 1992] in the general case (see also [Burq and Gérard 1997] for the necessity of this condition): assuming that $b \in \mathscr{C}^0(\overline{M})$, uniform stabilization occurs if and only if the set $\{b > 0\}$ satisfies the geometric control condition (GCC). Recall that a set $\omega \subset M$ is said to satisfy GCC if there exists $L_0 > 0$ such that every geodesic γ (resp. generalized geodesic in the case $\partial M \neq \varnothing$) of M with length larger than L_0 satisfies $\gamma \cap \omega \neq \varnothing$. When (1-3) is satisfied, one can take $F(t) = Ce^{-\kappa t}$ (for some constants

$C, \kappa > 0$) in (1-3), and the energy decays exponentially. Finally, Lebeau [1996] gives the explicit value of the best exponential decay rate κ in terms of the spectral abscissa of the generator of the semigroup and the mean value of the function b along the rays of geometrical optics.

In the case where $\{b > 0\}$ does not satisfy GCC, i.e., in the presence of "trapped rays" that do not meet $\{b > 0\}$, what can be said about the decay rate of the energy? As soon as $b \geq C > 0$ on a nonempty open subset of M, Lebeau [1996] shows that the energy of smoother initial data (satisfying the boundary condition if $\partial M \neq \varnothing$) goes at least logarithmically to zero:

$$E(u,t) \leq C(f(t))^2\big(\|u_0\|^2_{H^2(M)} + \|u_1\|^2_{H^1(M)}\big) \quad \text{for all } t > 0, \tag{1-4}$$

with $f(t) = 1/\log(2+t)$ (see also [Burq 1998]). Note that here, $(f(t))^2$ characterizes the decay of the energy, whereas $f(t)$ is that of the associated semigroup. Moreover, the author constructed a series of explicit examples of geometries for which this rate is optimal, including for instance the case where $M = \mathbb{S}^2$ is the two-dimensional sphere and $\{b > 0\} \cap N_\varepsilon = \varnothing$, where N_ε is a neighborhood of an equator of $\mathbb{S}^2$. This result is generalized in [Lebeau and Robbiano 1997] for a wave equation damped on a (small) part of the boundary. In this paper, the authors also make the following comment about the result they obtain:

> Notons toutefois qu'une étude plus approfondie de la localisation spectrale et des taux de décroissance de l'énergie pour des données régulières doit faire intervenir la dynamique globale du flot géodésique généralisé sur M. Les théorèmes 1 et 2 [de cet article] ne fournissent donc que les bornes a priori qu'on peut obtenir sans aucune hypothèse sur la dynamique, en n'utilisant que les inégalités de Carleman qui traduisent «l'effet tunnel».

In all examples where the optimal decay rate is logarithmic, the trapped ray is a stable trajectory from the point of view of the dynamics of the geodesic flow. This means basically that an important amount of the energy can stay concentrated, for a long time, in a neighborhood of the trapped ray, i.e., away from the damping region.

If the trapped trajectories are less stable, or unstable, one can expect to obtain an intermediate decay rate between exponential and logarithmic. We shall say that the energy decays at rate $f(t)$ if (1-4) is satisfied (more generally, see Definition 2.2 below in the abstract setting). This problem has already been addressed and, in some particular geometries, several different behaviors have been exhibited. Two main directions have been investigated.

On the one hand, Liu and Rao [2005] considered the case where M is a square and the set $\{b > 0\}$ contains a vertical strip. In this situation, the trapped trajectories consist of a family of parallel vertical geodesics; these are unstable, in the sense that nearby geodesics diverge at a linear rate. They proved that the energy decays at rate $(\log(t)/t)^{1/2}$ (i.e., that (1-4) is satisfied with $f(t) = (\log(t)/t)^{1/2}$). This was extended by Burq and Hitrik [2007] (see also [Nishiyama 2009]) to the case of partially rectangular two-dimensional domains, if the set $\{b > 0\}$ contains a neighborhood of the nonrectangular part. Phung [2007] proved a decay at rate $t^{-\delta}$ for some (unspecified) $\delta > 0$ in a three-dimensional domain having two parallel faces. In all these situations, the only obstruction to GCC is due to a "cylinder of periodic orbits".

The geometry is flat and the instabilities of the geodesic flow around the trapped rays are relatively weak (geodesics diverge at a linear rate).

In [Burq and Hitrik 2007], the authors argue that the optimal decay in their geometry should be of the form $1/t^{1-\varepsilon}$, for all $\varepsilon > 0$. They provide conditions on the damping coefficient $b(x)$ under which one can obtain such decay rates, and wonder whether this is true in general. Our main theorem (Theorem 2.6) extends these results to more general damping functions b on the two-dimensional torus.

On the other hand, Christianson [2007] proved that the energy decays at rate $e^{-C\sqrt{t}}$ for some $C > 0$, in the case where the trapped set is a hyperbolic closed geodesic. Schenck [2011] proved an energy decay at rate e^{-Ct} on manifolds with negative sectional curvature, if the trapped set is "small enough" in terms of topological pressure (for instance, a small neighborhood of a closed geodesic), and if the damping is "large enough" (that is, starting from a damping function b, βb will work for any $\beta > 0$ sufficiently large). In these two papers, the geodesic flow near the trapped set enjoys strong instability properties: the flow on the trapped set is uniformly hyperbolic, and in particular all trajectories are exponentially unstable.

These cases confirm the idea that the decay rate of the energy strongly depends on the stability of trapped trajectories.

One may now want to compare these geometric situations to situations where the Schrödinger group is observable (or, equivalently, controllable), i.e., for which there exist $C > 0$ and $T > 0$ such that, for all $u_0 \in L^2(M)$, we have

$$\|u_0\|^2_{L^2(M)} \leq C \int_0^T \|\sqrt{b}\, e^{-it\Delta} u_0\|^2_{L^2(M)}\, dt. \tag{1-5}$$

The conditions under which this property holds are also known to be related to stability of the geodesic flow. In particular, [Bardos et al. 1992], [Liu and Rao 2005], [Burq and Hitrik 2007; Nishiyama 2009] and [Christianson 2007; Schenck 2011] can be seen as counterparts for damped wave equations of the articles [Lebeau 1992], [Haraux 1989a; Jaffard 1990], [Burq and Zworski 2004] and [Anantharaman and Rivière 2012], respectively, in the context of observation of the Schrödinger group.

Our main results are twofold. First, we clarify (in an abstract setting) the link between the observability (or the controllability) of the Schrödinger equation and polynomial decay for the damped wave equation. This follows the spirit of [Haraux 1989b; Miller 2005], exploring the links between the different equations and their control properties, such as observability, controllability, and stabilization. More precisely, we prove that the controllability of the Schrödinger equation implies a polynomial decay at rate $1/\sqrt{t}$ for the damped wave equation (Theorem 2.3).

Second, we study precisely the damped wave equation on the flat torus $\mathbb{T}^2$ in case GCC fails. We give the following a priori lower bound on the decay rate, revisiting the argument of [Burq and Hitrik 2007]: (1-1) is not stable at a better rate than $1/t$, provided that GCC is not satisfied. In this situation, the Schrödinger group is known to be controllable (see [Jaffard 1990; Komornik 1992] and the more recent [Anantharaman and Macià 2010; Burq and Zworski 2012]). Thus, one cannot hope to have a decay better than polynomial in our previous result, i.e., under the mere assumption that the Schrödinger flow is observable.

The remainder of the paper is devoted to studying the gap between the a priori lower and upper bounds given respectively by $1/t$ and $1/\sqrt{t}$ on flat tori. For some *smooth* nonvanishing damping coefficient $b(x)$, we prove that the energy decays at rate $1/t^{1-\varepsilon}$ for all $\varepsilon > 0$. This result holds without making any dynamical assumption on the damping coefficient, but only on the order of vanishing of b. It generalizes a result of [Burq and Hitrik 2007], which holds in the case where b is invariant in one direction. Our analysis is, again, inspired by the recent microlocal approach proposed in [Anantharaman and Macià 2010] and [Burq and Zworski 2012] for the observability of the Schrödinger group. More precisely, we follow here several ideas and tools introduced in [Macià 2010] and [Anantharaman and Macià 2010].

In the situation where b is a characteristic function of a vertical strip of the torus (hence discontinuous), Stéphane Nonnenmacher proves in Appendix B that the decay rate cannot be better than $1/t^{2/3}$. This is done by explicitly computing the high frequency eigenvalues of the damped wave operator which are closest to the imaginary axis; see, for instance, the figures in [Asch and Lebeau 2003; Anantharaman and Léautaud 2012]. That the decay rate $1/t$ is not achieved in this situation was observed in the numerical computations from this last paper.

In contrast to the control problem for the Schrödinger equation on the torus, this result shows that the stabilization of the wave equation is not only sensitive to the global properties of the geodesic flow, but also to the rate at which the damping function vanishes.

2. Main results of the paper

Our first result can be stated in a general abstract setting that we now introduce. We come back to the case of the torus afterwards.

2A. ***The damped wave equation in an abstract setting.*** Let H and Y be two Hilbert spaces (resp. the state space and the observation/control space) with norms $\|\cdot\|_H$ and $\|\cdot\|_Y$, and associated inner products $(\cdot\,,\cdot)_H$ and $(\cdot\,,\cdot)_Y$.

We denote by $A\colon D(A) \subset H \to H$ a *nonnegative* selfadjoint operator with compact resolvent, and by $B \in \mathscr{L}(Y; H)$ a control operator. We recall that $B^* \in \mathscr{L}(H; Y)$ is defined by $(B^*h, y)_Y = (h, By)_H$ for all $h \in H$ and $y \in Y$.

Definition 2.1. We say that the system

$$\partial_t u + iAu = 0, \quad y = B^*u, \tag{2-1}$$

is observable in time T if there exists a constant $K_T > 0$ such that, for all solution of (2-1), we have

$$\|u(0)\|_H^2 \le K_T \int_0^T \|y(t)\|_Y^2 \, dt.$$

We recall that the observability of (2-1) in time T is equivalent to the exact controllability in time T of the adjoint problem

$$\partial_t u + iAu = Bf, \quad u(0) = u_0, \tag{2-2}$$

(see, for instance, [Lebeau 1992] or [Ramdani et al. 2005]). More precisely, given $T > 0$, the exact controllability in time T is the ability of finding for any $u_0, u_1 \in H$ a control function $f \in L^2(0, T; Y)$ so that the solution of (2-2) satisfies $u(T) = u_1$.

We equip $\mathscr{H} = D(A^{\frac{1}{2}}) \times H$ with the graph norm

$$\|(u_0, u_1)\|_{\mathscr{H}}^2 = \|(A + \mathrm{Id})^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2,$$

and define the seminorm

$$|(u_0, u_1)|_{\mathscr{H}}^2 = \|A^{\frac{1}{2}} u_0\|_H^2 + \|u_1\|_H^2.$$

Of course, if A is coercive on H, $|\cdot|_{\mathscr{H}}$ is a norm on $\mathscr{H}$ equivalent to $\|\cdot\|_{\mathscr{H}}$.

We also introduce in this abstract setting the damped wave equation on the space $\mathscr{H}$

$$\begin{cases} \partial_t^2 u + Au + BB^* \partial_t u = 0, \\ (u, \partial_t u)|_{t=0} = (u_0, u_1) \in \mathscr{H}, \end{cases} \tag{2-3}$$

which can be recast on $\mathscr{H}$ as a first order system

$$\begin{cases} \partial_t U = \mathscr{A} U, \\ U|_{t=0} = {}^t(u_0, u_1), \end{cases} \qquad U = \begin{pmatrix} u \\ \partial_t u \end{pmatrix}, \quad \mathscr{A} = \begin{pmatrix} 0 & \mathrm{Id} \\ -A & -BB^* \end{pmatrix}, \quad D(\mathscr{A}) = D(A) \times D(A^{\frac{1}{2}}). \tag{2-4}$$

The compact injections $D(A) \hookrightarrow D(A^{\frac{1}{2}}) \hookrightarrow H$ imply that $D(\mathscr{A}) \hookrightarrow \mathscr{H}$ compactly, and that the operator $\mathscr{A}$ has a compact resolvent.

We define the energy of solutions of (2-3) by

$$E(u, t) = \tfrac{1}{2}\big(\|A^{\frac{1}{2}} u\|_H^2 + \|\partial_t u\|_H^2\big) = \tfrac{1}{2}|(u, \partial_t u)|_{\mathscr{H}^2}^2.$$

Definition 2.2. Let f be a function such that $f(t) \to 0$ when $t \to +\infty$. We say that system (2-3) is stable at rate $f(t)$ if there exists a constant $C > 0$ such that for all $(u_0, u_1) \in D(\mathscr{A})$, we have

$$E(u, t)^{\frac{1}{2}} \le C f(t) |\mathscr{A}(u_0, u_1)|_{\mathscr{H}} \quad \text{for all } t > 0.$$

If it is the case, for all $k > 0$, there exists a constant $C_k > 0$ such that for all $(u_0, u_1) \in D(\mathscr{A}^k)$, we have (see, for instance, [Batty and Duyckaerts 2008, page 767])

$$E(u, t)^{\frac{1}{2}} \le C_k (f(t))^k \|\mathscr{A}^k(u_0, u_1)\|_{\mathscr{H}} \quad \text{for all } t > 0.$$

Theorem 2.3. *Suppose that there exists $T > 0$ such that system* (2-1) *is observable in time T. Then system* (2-3) *is stable at rate $1/\sqrt{t}$.*

Note that the gain of the $\log(t)^{\frac{1}{2}}$ with respect to [Liu and Rao 2005; Burq and Hitrik 2007] is not essential in our work. It is due to the optimal characterization of polynomially decaying semigroups obtained in [Borichev and Tomilov 2010].

This theorem may be compared with the works (both presented in a similar abstract setting) [Haraux 1989b], proving that the controllability of wave-type equations in some time is equivalent to uniform stabilization of (2-3), and [Miller 2005], showing that the controllability of wave-type equations in some time implies the controllability of Schrödinger-type equations in any time.

The link between this abstract setting and that of problem (1-1) is as follows: $H = Y = L^2(M)$; $A = -\Delta$ with $D(A) = H^2(M)$ if $\partial M = \varnothing$ and $H^2(M) \cap H_0^1(M)$ otherwise; B is the multiplication in $L^2(M)$ by the bounded function $\sqrt{b}$.

As a first application of Theorem 2.3 we obtain a different proof of the polynomial decay results for wave equations of [Liu and Rao 2005] and [Burq and Hitrik 2007] as consequences of the associated control results for the Schrödinger equation of [Haraux 1989a] and [Burq and Zworski 2004], respectively.

Moreover, Theorem 2.3 also provides several new stability results for system (1-1) in particular geometric situations; namely, in all following situations, the Schrödinger group is proved to be observable, and Theorem 2.3 gives the polynomial stability at rate $1/\sqrt{t}$ for (1-1):

- For any nonvanishing $b(x) \geq 0$ in the 2-dimensional square (resp. torus), as a consequence of [Jaffard 1990] (resp. [Macià 2010; Burq and Zworski 2012]); for any nonvanishing $b(x) \geq 0$ in the d-dimensional rectangle (resp. d-dimensional torus) as a consequence of [Komornik 1992] (resp. [Anantharaman and Macià 2010]).
- If M is the Bunimovich stadium and $b(x) > 0$ on the neighborhood of one half-disc and on one point of the opposite side, as a consequence of [Burq and Zworski 2004].
- If M is a d-dimensional manifold of constant negative curvature and the set of trapped trajectories (as a subset of S^*M, see [Anantharaman and Rivière 2012, Theorem 2.5] for a precise definition) has Hausdorff dimension lower than d, as a consequence of [Anantharaman and Rivière 2012].

Moreover, Lebeau [1996, Théorème 1(ii)] gives several 2-dimensional examples for which the decay rate $1/\log(2+t)$ is optimal. For all these geometrical situations, Theorem 2.3 implies that the Schrödinger group is not observable.

The proof of Theorem 2.3 relies on the following characterization of polynomial decay for system (2-3). For $z \in \mathbb{C}$, we define on H the operator

$$P(z) = A + z^2 \operatorname{Id} + zBB^*, \quad \text{with domain } D(P(z)) = D(A). \tag{2-5}$$

Proposition 2.4. *Suppose that*

$$\text{for any eigenvector } \varphi \text{ of } A, \text{ we have } B^*\varphi \neq 0. \tag{2-6}$$

Then, for all $\alpha > 0$, the following five assertions are equivalent:

The system (2-3) *is stable at rate* $1/t^{\alpha}$. (2-7)

There exist $C > 0$ and $s_0 \geq 0$ such that $\|(is \operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|s|^{\frac{1}{\alpha}}$ *for all $s \in \mathbb{R}$, $|s| \geq s_0$.* (2-8)

There exist $C > 0$ and $s_0 \geq 0$ such that for all $z \in \mathbb{C}$ satisfying $|z| \geq s_0$ and $|\mathrm{Re}(z)| \leq \dfrac{1}{C|\mathrm{Im}(z)|^{\frac{1}{\alpha}}}$,

we have $\|(z \operatorname{Id} - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\mathrm{Im}(z)|^{\frac{1}{\alpha}}$. (2-9)

There exist $C > 0$ and $s_0 \geq 0$ such that $\|P(is)^{-1}\|_{\mathcal{L}(H)} \leq C|s|^{\frac{1}{\alpha}-1}$ *for all $s \in \mathbb{R}$, $|s| \geq s_0$.* (2-10)

There exist $C > 0$ and $s_0 \geq 0$ such that for all $s \in \mathbb{R}$, $|s| \geq s_0$ and $u \in D(A)$,

$$\text{we have } \|u\|_H^2 \leq C\big(|s|^{\frac{2}{\alpha}-2}\|P(is)u\|_H^2 + |s|^{\frac{1}{\alpha}}\|B^*u\|_Y^2\big). \quad (2\text{-}11)$$

Theorem 2.3 and Proposition 2.4 are proved in Part II, as consequences of the characterization of polynomial decay for general semigroups in terms of resolvent estimates given in [Borichev and Tomilov 2010], providing the equivalence between (2-7) and (2-8). See also [Batty and Duyckaerts 2008] for general decay rates in Banach spaces. Note that the proof of a decay rate is reduced to the proof of a resolvent estimate on the imaginary axes. By the way, this estimate implies the existence of a "spectral gap" between the spectrum of $\mathcal{A}$ and the imaginary axis, given by (2-9).

Note finally that the estimates (2-8), (2-10) and (2-11) can be equivalently restricted to $s > 0$, since $P(-is)\overline{u} = \overline{P(is)u}$ for $s \in \mathbb{R}$.

2B. ***Decay rates for the damped wave equation on the torus.*** The main results of this article deal with the decay rate for problem (1-1) on the torus $\mathbb{T}^2 := (\mathbb{R}/2\pi\mathbb{Z})^2$. In this setting, as well as in the abstract setting, we shall write $P(z) = -\Delta + z^2 + zb(x)$.

First, we give an a priori lower bound for the decay rate of the damped wave equation, on $\mathbb{T}^2$, when GCC is "strongly violated", that is, assuming that supp(b) does not satisfy GCC (instead of $\{b > 0\}$). This theorem is proved by constructing explicit *quasimodes* for the operator $P(is)$.

Theorem 2.5. *Suppose that there exists $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, such that*

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0, \tau \in \mathbb{R}\} = \varnothing.$$

Then there exist two constants $C > 0$ and $\kappa_0 > 0$ such that for all $n \in \mathbb{N}$,

$$\|P(in\kappa_0)^{-1}\|_{\mathcal{L}(L^2(\mathbb{T}^2))} \geq C. \quad (2\text{-}12)$$

As a consequence of Proposition 2.4, polynomial stabilization at rate $1/t^{1+\varepsilon}$ for $\varepsilon > 0$ is not possible if there is a strongly trapped ray (i.e., that does not intersect supp(b)). More precisely, in such geometry, Theorem 2.5 combined with Lemma 4.6 and [Batty and Duyckaerts 2008, Proposition 1.3] shows that $m_1(t) \geq C/(1+t)$, for some $C > 0$ (with the notation of [Batty and Duyckaerts 2008], where $m_1(t)$ denotes the best decay rate).

The main goal of this paper is to explore the gap between the a priori upper bound $1/\sqrt{t}$ for the decay rate, given by Theorem 2.3, and the a priori lower bound $1/t$ of Theorem 2.5. Our results are twofold (somehow in two opposite directions) and concern either the case of smooth damping functions b, or the case $b = \mathbb{1}_U$, with $U \subset \mathbb{T}^2$.

2B1. *The case of smooth damping coefficients.* Our main result deals with the case of smooth damping coefficients. Without any geometric assumption, but with an additional hypothesis on the order of vanishing of the damping function b, we prove a weak converse of Theorem 2.5.

Theorem 2.6. *Let $M = \mathbb{T}^2$ with the standard flat metric. There exists $\varepsilon_0 > 0$ and $k_0 \in \mathbb{N}$ satisfying the following property: Suppose that $b \in W^{k_0,\infty}(\mathbb{T}^2)$ is a nonnegative, nonvanishing function on $\mathbb{T}^2$ and that* *there exist* $\varepsilon \in (0, \varepsilon_0)$ *and* $C_\varepsilon > 0$ *such that*

$$|\nabla b(x)| \leq C_\varepsilon b^{1-\varepsilon}(x) \quad \text{for } x \in \mathbb{T}^2. \tag{2-13}$$

Then there exist $C > 0$ *and* $s_0 \geq 0$ *such that for all* $s \in \mathbb{R}$, $|s| \geq s_0$, *we have*

$$\|P(is)^{-1}\|_{\mathscr{L}(L^2(\mathbb{T}^2))} \leq C|s|^\delta, \quad \text{with } \delta = 4\varepsilon. \tag{2-14}$$

As a consequence of Proposition 2.4, in this situation, the damped wave equation (1-1) *is stable at rate* $1/t^{1/(1+\delta)}$.

Remark 2.7. Following carefully the steps of the proof, one sees that $\varepsilon_0 = \frac{1}{29}$ works, but the proof is not optimized with respect to this parameter, and it is likely that it could be much improved.

The regularity assumption $b \in W^{k_0,\infty}(\mathbb{T}^2)$ is required since we make use of symbolic calculus in the proof of Lemma 7.1 (and only at this point of the paper). We only use the two following properties: (i) that the commutator of b with some Fourier multipliers is given by the usual principal term plus a lower order perturbation; (ii) the sharp Gårding inequality for a symbol depending on ∇b. It seems that (in 2 space dimensions) $k_0 = 8$ suffices in these two different applications of symbolic calculus (see [Sjöstrand 1995, Proposition 5.1] for a Gårding inequality with this regularity or [Lerner 2010, pp. 117–118] for a related discussion).

One of the main difficulties in understanding the decay rates is that there exists no general monotonicity property of the type "$b_1(x) \leq b_2(x)$ for all $x \Rightarrow$ the decay rate associated to the damping b_2 is larger (or smaller) than the decay rate associated to the damping b_1." This makes a significant difference with observability or controllability problems of the type (1-5).

Assumption (2-13) is only a local assumption in a neighborhood of $\partial\{b > 0\}$ (even if it is stated here globally on $\mathbb{T}^2$). Far from this set, i.e., on each compact set $\{b \geq b_0\}$ for $b_0 > 0$, the constant C_ε can be chosen uniformly, depending only on b_0, and not on ε. Hence, ε somehow quantifies the vanishing rate of the damping function b.

An interesting situation is when the smooth function b vanishes like e^{-1/x^α} in smooth local coordinates, for some $\alpha > 0$. In this case, assumption (2-13) is satisfied for any $\varepsilon > 0$, and the associated damped wave equation (1-1) is stable at rate $1/t^{1-\delta}$ for any $\delta > 0$. This shows that the lower bound given by Theorem 2.5 is sharp, in the sense that one cannot improve upon the exponent of t. This phenomenon had already been remarked by Burq and Hitrik [2007] in the case where b is invariant in one direction.

An example of a smooth function not satisfying assumption (2-13) is a function vanishing like $\sin(1/x)^2 e^{-1/x}$. We do not have any idea of the decay rate achieved in this case (except for the a priori upper and lower bounds $1/\sqrt{t}$ and $1/t$).

Theorem 2.6 generalizes the result of [Burq and Hitrik 2007], which only holds if b is assumed to be invariant in one direction. Moreover, our condition (2-13) is weaker than the assumption (3.2) of Burq and Hitrik. Actually their proof only uses the condition $|b'| \leq C_\varepsilon b^{1-\varepsilon}$ and $|b''| \leq C_\varepsilon b^{1-2\varepsilon}$ for some $\varepsilon < \frac{1}{4}$ (which is similar to ours), to obtain the same decay at rate $t^{-1/(1+4\varepsilon)}$.

The proof of Theorem 2.6 occupies Part III and is sketched in its introduction. It is based on ideas and tools developed in [Macià 2010; Anantharaman and Macià 2010] and especially the notion of *two-microlocal* semiclassical measures. One of the key technical points appears in Section 12: we have to construct, for each trapped direction, a cutoff function invariant in that direction and adapted to the damping coefficient b. We do not know how to adapt this technical construction to tori of higher dimension $d > 2$; hence we do not know whether Theorem 2.6 holds in higher dimension (although we have no reason to suspect it should not hold). Only in the particular case where b is invariant in $d-1$ directions can our methods (or those of [Burq and Hitrik 2007]) be applied to prove the analogue of Theorem 2.6.

Note that if GCC is satisfied, one has (on a general compact manifold M) for some $C > 1$ and all $|s| \geq s_0$ the estimate

$$\|P(is)^{-1}\|_{\mathscr{L}(L^2(M))} \leq C|s|^{-1} \tag{2-15}$$

instead of (2-14). Estimate (2-15) is in turn equivalent to uniform stabilization (see [Huang 1985] together with Lemma 4.6).

Remark 2.8. As a consequence of Theorem 2.6 on the torus, we can deduce that the decay rate $t^{-1/(1+\delta)}$ also holds for (1-1) if $M = (0,\pi)^2$ is the square, one considers Dirichlet or Neumann boundary conditions, and the damping function b is smooth, vanishes near ∂M and satisfies assumption (2-13). First, we extend the function b as an even (with respect to both variables) smooth function on the larger square $(-\pi,\pi)^2$, and using the injection $\imath : (-\pi,\pi)^2 \to \mathbb{T}^2$, as a smooth function on $\mathbb{T}^2$, still satisfying (2-13). Moreover, $D(\Delta_D)$ (resp. $D(\Delta_N)$) on $(0,\pi)^2$ can be identified as the closed subspace of odd (resp. even) functions of $D(\Delta_D)$ (resp. $D(\Delta_N)$) on $(-\pi,\pi)^2$. Using again the injection $\imath$, it can also be identified with a closed subspace of $H^2(\mathbb{T}^2)$. The estimate

$$\|u\|_{L^2(\mathbb{T}^2)} \leq C|s|^{\delta}\|P(is)u\|_{L^2(\mathbb{T}^2)} \quad \text{for all } u \in H^2(\mathbb{T}^2)$$

is thus also true on the square $(0,\pi)^2$ for Dirichlet or Neumann boundary conditions. In particular, this strongly improves the results of [Liu and Rao 2005].

The lower bound of Theorem 2.5 can be similarly extended to the case of a square with Dirichlet or Neumann boundary conditions, implying that the rate $1/t$ is optimal if GCC is strongly violated.

2B2. *The case of discontinuous damping functions.* Appendix B (by Stéphane Nonnenmacher) deals with the case where b is the characteristic function of a vertical strip, i.e., $b = \tilde{B}\mathbb{1}_{U\times\mathbb{T}}$, for some $\tilde{B} > 0$ and $U \subset \mathbb{T}$, U a nonempty open interval with $\bar{U} \neq \mathbb{T}$. Due to the invariance of b in one direction, the spectrum of the damped wave operator $\mathscr{A}$ splits into countably many “branches” of eigenvalues. This structure of the spectrum is illustrated in the numerics of [Asch and Lebeau 2003; Anantharaman and Léautaud 2012].

The branch closest to the imaginary axis is explicitly computed; it contains a sequence of eigenvalues $(z_i)_{i\in\mathbb{N}}$ such that $\operatorname{Im} z_i \to \infty$ and $|\operatorname{Re} z_i| \leq C_0/(\operatorname{Im} z_i)^{\frac{3}{2}}$. This result is in agreement with the numerical tests given in [Anantharaman and Léautaud 2012].

As a consequence, for any $\varepsilon > 0$ and $C > 0$, the strip $\{|\mathrm{Re}\, z| \le C|\mathrm{Im}(z)|^{-\frac{3}{2}+\varepsilon}\}$ contains infinitely many poles of the resolvent $(z\,\mathrm{Id} - \mathscr{A})^{-1}$, so item (2-9) in Proposition 2.4 implies the following obstruction to the stability of this damped system:

Corollary 2.9. *For any* $\varepsilon > 0$, *the damped wave equation* (1-1) *on* $\mathbb{T}^2$ *with the damping function* (B-1) *cannot be stable at the rate* $1/t^{\frac{2}{3}+\varepsilon}$.

The same result holds on the square with Dirichlet or Neumann boundary conditions.

More precisely, in this situation, Lemma 4.6 and [Batty and Duyckaerts 2008, Proposition 1.3] yield that $m_1(t) \ge C/(1+t)^{\frac{2}{3}}$, for some $C > 0$ (with the notation of that reference, where $m_1(t)$ denotes the best decay rate).

This corollary shows in particular that the regularity conditions in Theorem 2.6 cannot be completely disposed of if one wants a stability at the rate $1/t^{1-\varepsilon}$ for small ε.

2C. ***Some related open questions.*** The various results obtained in this article lead to several open questions.

(1) In the case where b is the characteristic function of a vertical strip, our analysis shows that the best decay rate lies somewhere between $1/t^{\frac{1}{2}}$ and $1/t^{\frac{2}{3}}$, but the "true" decay rate is not yet clear.

(2) It would also be interesting to investigate the spectrum and the decay rates for damping functions b invariant in one direction, but having a less singular behavior than a characteristic function. In particular, is it possible to give a precise link between the vanishing rate of b and the decay rate?

(3) In the general setting of Section 2A (as well as in the case of the damped wave equation on the torus), is the a priori upper bound $1/t^{\frac{1}{2}}$ for the decay rate optimal?

(4) For smooth damping functions vanishing like $e^{-1/x^{\alpha}}$, Theorem 2.6 yields stability at rate $1/t^{1-\delta}$ for all $\delta > 0$. Is the decay rate $1/t$ reached in this situation? Can one find a damping function b such that the decay rate is exactly $1/t$?

(5) The lower bound of Theorem 2.5 is still valid in higher-dimensional tori. Is there an analogue of Theorem 2.6 (i.e., for general "smooth" damping functions) for $\mathbb{T}^d$, with $d \ge 3$?

Part II. Resolvent estimates and stabilization in the abstract setting

3. Proof of Theorem 2.3 assuming Proposition 2.4

To prove Theorem 2.3, we express the observability condition as a resolvent estimate (also known as the Hautus test), as introduced by Burq and Zworski [2004], and further developed by Miller [2005] and Ramdani, Takahashi, Tenenbaum and Tucsnak [Ramdani et al. 2005]. For a survey of this notion, we refer to the book [Tucsnak and Weiss 2009, Section 6.6].

In particular [Miller 2005, Theorem 5.1] (or [Tucsnak and Weiss 2009, Theorem 6.6.1]) yields that system (2-1) is observable in some time $T > 0$ if and only if there exists a constant $C > 0$ such that we have

$$\|u\|_H^2 \le C\big(\|(A - \lambda\,\mathrm{Id})u\|_H^2 + \|B^*u\|_Y^2\big) \quad \text{for all } \lambda \in \mathbb{R} \text{ and } u \in D(A).$$

As a first consequence, assumption (2-6) is satisfied and Proposition 2.4 applies in this context. Moreover, recalling that $P(z)$ is defined in (2-5), we have, for all $s \in \mathbb{R}$ and $u \in D(A)$,

$$\begin{aligned}\|u\|_H^2 &\leq C\big(\|(A - s^2 \operatorname{Id} + isBB^* - isBB^*)u\|_H^2 + \|B^*u\|_Y^2\big)\\ &\leq C\big(\|P(is)u\|_H^2 + s^2\|BB^*u\|_H^2 + \|B^*u\|_Y^2\big)\end{aligned} \tag{3-1}$$

Since $B \in \mathscr{L}(Y; H)$, we obtain for $s \geq 1$ and for some $C > 0$,

$$\|u\|_H^2 \leq C\big(\|P(is)u\|_H^2 + s^2\|B^*u\|_Y^2\big) \leq C\big(s^2\|P(is)u\|_H^2 + s^2\|B^*u\|_Y^2\big).$$

Proposition 2.4 then yields the polynomial stability at rate $1/\sqrt{t}$ for (2-3). This concludes the proof of Theorem 2.3. □

4. Proof of Proposition 2.4

Our proof relies strongly on the characterization of polynomially stable semigroups given in [Borichev and Tomilov 2010, Theorem 2.4], which can be reformulated as follows.

Theorem 4.1 (Borichev and Tomilov). *Let $(e^{t\dot{\mathscr{A}}})_{t\geq 0}$ be a bounded $\mathscr{C}^0$-semigroup on a Hilbert space $\dot{\mathscr{H}}$, generated by $\dot{\mathscr{A}}$. Suppose that $i\mathbb{R} \cap \operatorname{Sp}(\dot{\mathscr{A}}) = \varnothing$. Then, the following conditions are equivalent*:

$$\|e^{t\dot{\mathscr{A}}}\dot{\mathscr{A}}^{-1}\|_{\mathscr{L}(\dot{\mathscr{H}})} = \mathbb{O}(t^{-\alpha}) \quad \text{as } t \to +\infty, \tag{4-1}$$

$$\|(is \operatorname{Id} - \dot{\mathscr{A}})^{-1}\|_{\mathscr{L}(\dot{\mathscr{H}})} = \mathbb{O}(|s|^{\frac{1}{\alpha}}) \quad \text{as } s \to \infty. \tag{4-2}$$

Let us first describe some spectral properties of the operator $\mathscr{A}$ defined in (2-4).

Lemma 4.2. *The spectrum of $\mathscr{A}$ contains only isolated eigenvalues, and we have*

$$\operatorname{Sp}(\mathscr{A}) \subset \big((-\tfrac{1}{2}\|B^*\|^2_{\mathscr{L}(H;Y)}, 0) + i\mathbb{R}\big) \cup ([-\|B^*\|^2_{\mathscr{L}(H;Y)}, 0] + 0i),$$

with $\ker(\mathscr{A}) = \ker(A) \times \{0\}$.

Moreover, the operator $P(z)$ is an isomorphism from $D(A)$ onto H if and only if $z \notin \operatorname{Sp}(\mathscr{A})$. If this is satisfied, we have

$$(z \operatorname{Id} - \mathscr{A})^{-1} = \begin{pmatrix} P(z)^{-1}(BB^* + z \operatorname{Id}) & P(z)^{-1} \\ P(z)^{-1}(zBB^* + z^2 \operatorname{Id}) - \operatorname{Id} & zP(z)^{-1} \end{pmatrix}. \tag{4-3}$$

The localization properties for the spectrum of $\mathscr{A}$ stated in the first part of this lemma are illustrated, for instance, in [Asch and Lebeau 2003] or [Anantharaman and Léautaud 2012].

This lemma leads us to introduce the spectral projector of $\mathscr{A}$ on $\ker(\mathscr{A})$, given by

$$\Pi_0 = \frac{1}{2i\pi}\int_\gamma (z \operatorname{Id} - \mathscr{A})^{-1}\, dz \in \mathscr{L}(\mathscr{H}),$$

where γ denotes a positively oriented circle centered on 0 with a radius so small that 0 is the single eigenvalue of $\mathscr{A}$ in the interior of γ. We set $\dot{\mathscr{H}} = (\operatorname{Id} - \Pi_0)\mathscr{H}$ and equip this space with the norm

$$\|(u_0, u_1)\|^2_{\dot{\mathscr{H}}} := |(u_0, u_1)|^2_{\mathscr{H}} = \|A^{\frac{1}{2}}u_0\|_H^2 + \|u_1\|_H^2,$$

and associated inner product. This is indeed a norm on $\dot{\mathscr{H}}$ since $\|(u_0,u_1)\|_{\dot{\mathscr{H}}}=0$ is equivalent to $(u_0,u_1)\in\ker(A)\times\{0\}=\Pi_0\mathscr{H}$.

We also set $\dot{\mathscr{A}}=\mathscr{A}|_{\dot{\mathscr{H}}}$ with domain $D(\dot{\mathscr{A}})=D(\mathscr{A})\cap\dot{\mathscr{H}}$. A first remark is that $\mathrm{Sp}(\dot{\mathscr{A}})=\mathrm{Sp}(\mathscr{A})\setminus\{0\}$, so that $\mathrm{Sp}(\dot{\mathscr{A}})\cap i\mathbb{R}=\varnothing$.

The remainder of the proof consists in applying Theorem 4.1 to the operator $\dot{\mathscr{A}}$ in $\dot{\mathscr{H}}$. We first check the assumptions of Theorem 4.1 and describe the solutions of the evolution problem (2-4) (or equivalently (2-3)).

Lemma 4.3. *The operator $\dot{\mathscr{A}}$ generates a contraction $\mathscr{C}^0$-semigroup on $\dot{\mathscr{H}}$, denoted $(e^{t\dot{\mathscr{A}}})_{t\ge0}$. Moreover, for all initial data $U_0\in\mathscr{H}$, problem* (2-4) (*or equivalently* (2-3)) *has a unique solution $U\in\mathscr{C}^0(\mathbb{R}^+;\mathscr{H})$, issued from U_0, that can be decomposed as*

$$U(t)=e^{t\dot{\mathscr{A}}}(\mathrm{Id}-\Pi_0)U_0+\Pi_0U_0\quad\textit{for all } t\ge0. \tag{4-4}$$

As a consequence, we can apply Theorem 4.1 to the semigroup generated by $\dot{\mathscr{A}}$. The proof of Proposition 2.4 will be achieved when the following lemmata are proved.

Lemma 4.4. *Conditions* (2-7) *and* (4-1) *are equivalent.*

Lemma 4.5. *Conditions* (2-10) *and* (2-11) *are equivalent. Conditions* (2-8) *and* (2-9) *are equivalent.*

Lemma 4.6. *There exist $C>1$ and $s_0>0$ such that for $s\in\mathbb{R}$, $|s|\ge s_0$, we have*

$$\|(is\,\mathrm{Id}-\dot{\mathscr{A}})^{-1}\|_{\mathscr{L}(\dot{\mathscr{H}})}-\frac{C}{|s|}\le\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}\le\|(is\,\mathrm{Id}-\dot{\mathscr{A}})^{-1}\|_{\mathscr{L}(\dot{\mathscr{H}})}+\frac{C}{|s|}, \tag{4-5}$$

and

$$C^{-1}|s|\|P(is)^{-1}\|_{\mathscr{L}(H)}\le\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}\le C(1+|s|\|P(is)^{-1}\|_{\mathscr{L}(H)}). \tag{4-6}$$

In particular this implies that (4-2), (2-8) *and* (2-10) *are equivalent.*

The proof of Lemma 4.6 is more or less classical and we follow [Lebeau 1996; Burq and Hitrik 2007].

Proof of Lemma 4.2. Since $\mathscr{A}$ has compact resolvent, its spectrum contains only isolated eigenvalues. Suppose that $z\in\mathrm{Sp}(\mathscr{A})$; then, for some $(u_0,u_1)\in D(\mathscr{A})\setminus\{0\}$, we have

$$u_1=zu_0,\quad -Au_0-BB^*u_1=zu_1,$$

and in particular

$$Au_0+z^2u_0+zBB^*u_0=0, \tag{4-7}$$

with $u_0\in D(A)\setminus\{0\}$.

Suppose that $z\in i\mathbb{R}$; then, this yields $Au_0-\mathrm{Im}(z)^2u_0+i\,\mathrm{Im}(z)BB^*u_0=0$. Following [Lebeau 1996], taking the inner product of this equation with u_0 yields $i\,\mathrm{Im}(z)\|B^*u_0\|_Y^2=0$. Hence, either $\mathrm{Im}(z)=0$ or $B^*u_0=0$. In the first case, $Au_0=0$, i.e., $u_0\in\ker(A)$, and $u_1=0$. This yields $\ker(\mathscr{A})\subset\ker(A)\times\{0\}$ (and the other inclusion is clear). In the second case, u_0 is an eigenvector of A associated to the eigenvalue $\mathrm{Im}(z)^2$ and satisfies $B^*u_0=0$, which is absurd, according to assumption (2-6). Thus, $\mathrm{Sp}(\mathscr{A})\cap i\mathbb{R}\subset\{0\}$.

Now, for a general eigenvalue $z \in \mathbb{C}$, taking the inner product of (4-7) with u_0 yields

$$\begin{aligned}(Au_0, u_0)_H + (\operatorname{Re}(z)^2 - \operatorname{Im}(z)^2)\|u_0\|_H^2 + \operatorname{Re}(z)\|B^*u_0\|_Y^2 = 0,\\ 2\operatorname{Re}(z)\operatorname{Im}(z)\|u_0\|_H^2 + \operatorname{Im}(z)\|B^*u_0\|_Y^2 = 0.\end{aligned} \tag{4-8}$$

If $\operatorname{Im}(z) \neq 0$, then the second equation of (4-8) together with $\operatorname{Sp}(\dot{\mathscr{A}}) \cap i\mathbb{R} \subset \{0\}$ gives

$$0 > \operatorname{Re}(z) = -\frac{1}{2}\frac{\|B^*u_0\|_Y^2}{\|u_0\|_H^2} \geq -\frac{1}{2}\|B^*\|^2_{\mathscr{L}(H;Y)}.$$

If $\operatorname{Im}(z) = 0$, then the first equation of (4-8) together with $(\dot{A}u_0, u_0)_H \geq 0$ gives

$$-\operatorname{Re}(z)\|B^*u_0\|_Y^2 \geq \operatorname{Re}(z)^2\|u_0\|_H^2,$$

which yields

$$0 \geq \operatorname{Re}(z) \geq -\|B^*\|^2_{\mathscr{L}(H;Y)}.$$

Following [Lebeau 1996], we now give the link between $P(z)^{-1}$ and $(z\operatorname{Id} - \mathscr{A})^{-1}$ for $z \notin \operatorname{Sp}(\mathscr{A})$. Taking $F = (f_0, f_1) \in \mathscr{H}$, and $U = (u_0, u_1)$, we have

$$F = (z\operatorname{Id} - \mathscr{A})U \iff \begin{cases} u_1 = zu_0 - f_0,\\ P(z)u_0 = f_1 + (BB^* + z\operatorname{Id})f_0.\end{cases} \tag{4-9}$$

As a consequence, we obtain that $P(z) : D(A) \to H$ is invertible if and only if $(z\operatorname{Id} - \mathscr{A}) : D(\mathscr{A}) \to \mathscr{H}$ is invertible, i.e., if and only if $z \notin \operatorname{Sp}(\mathscr{A})$. Moreover, for such values of z, the condition on the right-hand side of (4-9) is equivalent to

$$u_0 = P(z)^{-1}f_1 + P(z)^{-1}(BB^* + z\operatorname{Id})f_0 \quad \text{and} \quad u_1 = zP(z)^{-1}f_1 + zP(z)^{-1}(BB^* + z\operatorname{Id})f_0 - f_0,$$

which can be rewritten as (4-3). This concludes the proof of Lemma 4.2. □

Proof of Lemma 4.3. Let us check that $\dot{\mathscr{A}}$ is a maximal dissipative operator on $\dot{\mathscr{H}}$ [Pazy 1983]. First, it is dissipative since, for $U = (u_0, u_1) \in D(\dot{\mathscr{A}})$,

$$(\dot{\mathscr{A}}U, U)_{\dot{\mathscr{H}}} = (A^{\frac{1}{2}}u_1, A^{\frac{1}{2}}u_0)_H - (Au_0, u_1)_H - (BB^*u_1, u_1)_H = -\|B^*u_1\|_Y^2 \leq 0.$$

Next, the fact that $\mathscr{A} - \operatorname{Id}$ is onto is a consequence of Lemma 4.2. Hence, for all $F \in \dot{\mathscr{H}} \subset \mathscr{H}$, there exists $U \in D(\mathscr{A})$ such that $(\mathscr{A} - \operatorname{Id})U = F$. Applying $(\operatorname{Id} - \Pi_0)$ to this identity yields $(\dot{\mathscr{A}} - \operatorname{Id})(\operatorname{Id} - \Pi_0)U = F$, so $\dot{\mathscr{A}} - \operatorname{Id} : D(\dot{\mathscr{A}}) \to \dot{\mathscr{H}}$ is onto. According to the Lumer–Phillips theorem (see, for instance, [Pazy 1983, Chapter 1, Theorem 4.3]) $\dot{\mathscr{A}}$ generates a contraction $\mathscr{C}^0$-semigroup on $\dot{\mathscr{H}}$. Then, formula (4-4) directly comes from the linearity of (2-4) (or equivalently (2-3)) together with the decomposition of the initial condition $U_0 = (I - \Pi_0)U_0 + \Pi_0 U_0$. □

Proof of Lemma 4.4. Condition (4-1) is equivalent to the existence of $C > 0$ such that for all $t > 0$, and $\dot{U}_0 \in \dot{\mathscr{H}}$, we have

$$\|e^{t\dot{\mathscr{A}}}\dot{\mathscr{A}}^{-1}\dot{U}_0\|_{\dot{\mathscr{H}}} \leq \frac{C}{t^\alpha}\|\dot{U}_0\|_{\dot{\mathscr{H}}}.$$

This can be rephrased as

$$\|e^{t\dot{\mathscr{A}}}\dot{U}_0\|_{\dot{\mathscr{H}}} \leq \frac{C}{t^\alpha}\|\dot{\mathscr{A}}\dot{U}_0\|_{\dot{\mathscr{H}}}, \tag{4-10}$$

for all $t > 0$, and $\dot{U}_0 \in D(\dot{\mathscr{A}})$. Now, take any $U_0 = (u_0, u_1) \in D(\mathscr{A})$, and associated projection $\dot{U}_0 = (\mathrm{Id} - \Pi_0)U_0 \in D(\dot{\mathscr{A}})$. According to (4-4), we have

$$E(u,t) = \tfrac{1}{2}\big(\|A^{\frac{1}{2}}u(t)\|_H^2 + \|\partial_t u(t)\|_H^2\big) = \tfrac{1}{2}|e^{t\dot{\mathscr{A}}}\dot{U}_0 + \Pi_0 U_0|_{\mathscr{H}}^2 = \tfrac{1}{2}\|e^{t\dot{\mathscr{A}}}\dot{U}_0\|_{\dot{\mathscr{H}}}^2,$$

and

$$|\mathscr{A}U_0|_{\mathscr{H}} = |\dot{\mathscr{A}}\dot{U}_0 + \mathscr{A}\Pi_0 U_0|_{\mathscr{H}} = \|\dot{\mathscr{A}}\dot{U}_0\|_{\dot{\mathscr{H}}}.$$

This shows that (4-10) is equivalent to (2-7), and concludes the proof of Lemma 4.4. ☐

Proof of Lemma 4.5. First, (2-10) clearly implies (2-11). To prove the converse, for $u \in D(\mathscr{A})$, we have

$$(P(is)u, u)_H = \big((A - s^2\,\mathrm{Id})u, u\big)_H + is\|B^* u\|_Y^2.$$

Taking the imaginary part of this identity gives $s\|B^*u\|_Y^2 = \mathrm{Im}(P(is)u,u)_H$, so that, using the Young inequality, we obtain for all $\varepsilon > 0$,

$$|s|^{\frac{1}{\alpha}}\|B^*u\|_Y^2 = |s|^{\frac{1}{\alpha}-1}|\mathrm{Im}(P(is)u,u)_H| \le \frac{|s|^{\frac{2}{\alpha}-2}}{4\varepsilon}\|P(is)u\|_H^2 + \varepsilon\|u\|_H^2.$$

Plugging this into (2-11) and taking ε sufficiently small, we obtain that for some $C > 0$ and $s_0 \ge 0$, for any $s \in \mathbb{R}$, $|s| \ge s_0$, we have

$$\|u\|_H^2 \le C|s|^{\frac{2}{\alpha}-2}\|P(is)u\|_H^2,$$

which yields (2-10). Hence (2-10) and (2-11) are equivalent.

Second, Condition (2-9) clearly implies (2-8) and it only remains to prove the converse. For $z \in \mathbb{C}$, we write $r = \mathrm{Re}(z)$ and $s = \mathrm{Im}(z)$. We have the identity

$$((r+is)\,\mathrm{Id} - \mathscr{A})^{-1} = (is\,\mathrm{Id} - \mathscr{A})^{-1}\big(\mathrm{Id} + r(is\,\mathrm{Id} - \mathscr{A})^{-1}\big)^{-1}. \tag{4-11}$$

Hence, assuming

$$\|r(is\,\mathrm{Id} - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \le \tfrac{1}{2}, \tag{4-12}$$

this gives

$$\big\|\big(\mathrm{Id} + r(is\,\mathrm{Id} - \mathscr{A})^{-1}\big)^{-1}\big\|_{\mathscr{L}(\mathscr{H})} = \Big\|\sum_{k=0}^{\infty}(-1)^k\big(r(is\,\mathrm{Id} - \mathscr{A})^{-1}\big)^k\Big\|_{\mathscr{L}(\mathscr{H})} \le 2.$$

As a consequence of (4-11) and (2-8), we then obtain

$$\|((r+is)\,\mathrm{Id} - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \le 2\|(is\,\mathrm{Id} - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \le 2C|s|^{\frac{1}{\alpha}},$$

for all $s \ge s_0$, under condition (4-12). Finally, (2-8) also yields

$$\|r(is\,\mathrm{Id} - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \le |r|C|s|^{\frac{1}{\alpha}},$$

so that condition (4-12) is realized as soon as $|r| \le 1/(2C|s|^{\frac{1}{\alpha}})$. This proves (2-9) and concludes the proof of Lemma 4.5. ☐

Proof of Lemma 4.6. To prove (4-5), we first remark that the norms $\|\cdot\|_{\dot{\mathscr{H}}}$ and $\|\cdot\|_{\mathscr{H}}$ are equivalent on $\dot{\mathscr{H}}$, so that the norms $\|\cdot\|_{\mathscr{L}(\dot{\mathscr{H}})}$ and $\|\cdot\|_{\mathscr{L}(\mathscr{H})}$ are equivalent on $\mathscr{L}(\dot{\mathscr{H}})$. Next, we have

$$(is\,\mathrm{Id}-\dot{\mathscr{A}})^{-1}(\mathrm{Id}-\Pi_0)=(is\,\mathrm{Id}-\mathscr{A})^{-1}(\mathrm{Id}-\Pi_0)$$

and

$$\begin{aligned}\|(is\,\mathrm{Id}-\dot{\mathscr{A}})^{-1}\|_{\mathscr{L}(\mathscr{H})}=\|(is\,\mathrm{Id}-\dot{\mathscr{A}})^{-1}(\mathrm{Id}-\Pi_0)\|_{\mathscr{L}(\mathscr{H})}&=\|(is\,\mathrm{Id}-\mathscr{A})^{-1}(\mathrm{Id}-\Pi_0)\|_{\mathscr{L}(\mathscr{H})}\\&\le\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}+\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\Pi_0\|_{\mathscr{L}(\mathscr{H})},\end{aligned}$$

together with

$$\begin{aligned}\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}&=\|(is\,\mathrm{Id}-\dot{\mathscr{A}})^{-1}(\mathrm{Id}-\Pi_0)+(is\,\mathrm{Id}-\mathscr{A})^{-1}\Pi_0\|_{\mathscr{L}(\mathscr{H})}\\&\le\|(is\,\mathrm{Id}-\dot{\mathscr{A}})^{-1}\|_{\mathscr{L}(\mathscr{H})}+\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\Pi_0\|_{\mathscr{L}(\mathscr{H})}.\end{aligned}$$

Moreover, for $|s|\ge 1$, we have

$$\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\Pi_0\|_{\mathscr{L}(\mathscr{H})}=\|(is)^{-1}\Pi_0\|_{\mathscr{L}(\mathscr{H})}=\frac{1}{|s|}\|\Pi_0\|_{\mathscr{L}(\mathscr{H})}=\frac{C}{|s|},$$

which concludes the proof of (4-5).

Let us now prove (4-6). For concision, we set $H_1=D(A^{\frac12})$ endowed with the graph norm $\|u\|_{H_1}=\|(A+\mathrm{Id})^{\frac12}u\|_H$ and denote by $H_{-1}=D(A^{\frac12})'$ its dual space. The operator A can be uniquely extended as an operator $\mathscr{L}(H_1;H_{-1})$, still denoted A for simplicity. With this notation, the space H_{-1} can be equipped with the natural norm $\|u\|_{H_{-1}}=\|(A+\mathrm{Id})^{-\frac12}u\|_H$.

As a consequence of formula (4-3), and using the fact that $\mathrm{Sp}(\mathscr{A})\cap i\mathbb{R}\subset\{0\}$, there exist constants $C>1$ and $s_0>0$ such that for all $s\in\mathbb{R}$, $|s|\ge s_0$, we have

$$C^{-1}M(s)\le\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})}\le CM(s),\tag{4-13}$$

with

$$\begin{aligned}M(s)=\big(\|P(is)^{-1}(BB^*+is\,\mathrm{Id})\|_{\mathscr{L}(H_1)}+\|P(is)^{-1}\|_{\mathscr{L}(H;H_1)}\qquad&\\+\|P(is)^{-1}(isBB^*-s^2\,\mathrm{Id})-\mathrm{Id}\,\|_{\mathscr{L}(H_1;H)}+\|sP(is)^{-1}\|_{\mathscr{L}(H)}\big).&\end{aligned}\tag{4-14}$$

On the one hand, this directly yields

$$|s|\|P(is)^{-1}\|_{\mathscr{L}(H)}\le C\|(is\,\mathrm{Id}-\mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})},$$

for $s\in\mathbb{R}$, $|s|\ge s_0$. This proves that (4-2) implies (2-10).

On the other hand, we have to estimate each term of (4-14). First, using $Au=P(is)u+s^2u-isBB^*u$, we have

$$\begin{aligned}\|u\|_{H_1}^2=\|A^{\frac12}u\|_H^2+\|u\|_H^2&=(P(is)u+s^2u-isBB^*u,u)_H+\|u\|_H^2\\&=\mathrm{Re}(P(is)u,u)_H+(s^2+1)\|u\|_H^2\le C\big(\|P(is)u\|_H^2+(s^2+1)\|u\|_H^2\big)\\&\le C\big(1+(s^2+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}^2\big)\|P(is)u\|_H^2,\end{aligned}$$

so that

$$\|P(is)^{-1}\|_{\mathscr{L}(H;H_1)} \leq C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-15}$$

Second, the same computation for $(P(is)^{-1})^* = (A - s^2\,\mathrm{Id} - isBB^*)^{-1}$ (the adjoint of $P(is)^{-1}$ in the space H) in place of $P(is)^{-1}$ leads to

$$(P(is)^{-1})^* \in \mathscr{L}(H; H_1),$$

together with the estimate

$$\|(P(is)^{-1})^*\|_{\mathscr{L}(H;H_1)} \leq C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big).$$

By transposition, we have ${}^t(P(is)^{-1})^* \in \mathscr{L}(H_{-1}; H)$, together with the estimate

$$\|{}^t(P(is)^{-1})^*\|_{\mathscr{L}(H_{-1};H)} \leq \|(P(is)^{-1})^*\|_{\mathscr{L}(H;H_1)} \leq C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-16}$$

Moreover, ${}^t(P(is)^{-1})^*$ is defined, for every $u \in H$, $v \in H_{-1}$, by

$$\big({}^t(P(is)^{-1})^*v, u\big)_H = \langle v, (P(is)^{-1})^*u\rangle_{H_{-1},H_1} = \big((A+\mathrm{Id})^{-\frac{1}{2}}v, (A+\mathrm{Id})^{\frac{1}{2}}(P(is)^{-1})^*u\big)_H.$$

In particular, taking $v \in H$ gives

$$\big({}^t(P(is)^{-1})^*v, u\big)_H = \big(P(is)^{-1}v, u\big)_H,$$

which implies that the restriction of the operator ${}^t(P(is)^{-1})^*$ to H coincides with $P(is)^{-1}$. For simplicity, we will denote $P(is)^{-1}$ for ${}^t(P(is)^{-1})^*$.

Equation (4-16) can thus be rewritten

$$\|P(is)^{-1}\|_{\mathscr{L}(H_{-1};H)} \leq C\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-17}$$

Then, we have $P(is)^{-1}(isBB^* - s^2\,\mathrm{Id}) - \mathrm{Id} = P(is)^{-1}A$, so that

$$\begin{aligned}\|P(is)^{-1}(isBB^* - s^2\,\mathrm{Id}) - \mathrm{Id}\,\|_{\mathscr{L}(H_1;H)} = \|P(is)^{-1}A\|_{\mathscr{L}(H_1;H)} &\leq \|P(is)^{-1}\|_{\mathscr{L}(H_{-1};H)}\|A\|_{\mathscr{L}(H_1;H_{-1})}\\ &\leq \big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big).\end{aligned} \tag{4-18}$$

Third, for $|s| \geq 1$ we write

$$P(is)^{-1}(BB^* + is\,\mathrm{Id}) = \frac{i}{s}(P(is)^{-1}A - \mathrm{Id}), \tag{4-19}$$

and it remains to estimate the term $\|P(is)^{-1}A\|_{\mathscr{L}(H_1)}$ in (4-14). For $f \in H_1$, we set $u = P(is)^{-1}Af$. We have $u \in H_1$, together with

$$(A - s^2\,\mathrm{Id} + isBB^*)u = Af.$$

Taking the real part of the inner product of this identity with u, we find

$$\|A^{\frac{1}{2}}u\|_H^2 - s^2\|u\|_H^2 = \mathrm{Re}(Af, u)_H \leq \|Af\|_{H_{-1}}\|u\|_{H_1} \leq C\|f\|_{H_1}\|u\|_{H_1},$$

since $A \in \mathscr{L}(H_1, H_{-1})$. Hence

$$\|u\|_{H_1}^2 \le C(1+s^2)\|u\|_H^2 + C\|f\|_{H_1}^2.$$

Using (4-17), this gives

$$\begin{aligned}\|u\|_{H_1}^2 &\le C(1+s^2)\|P(is)^{-1}A\|_{\mathscr{L}(H_1;H)}^2\|f\|_{H_1}^2 + C\|f\|_{H_1}^2 \\ &\le C(1+s^2)\|P(is)^{-1}\|_{\mathscr{L}(H_{-1};H)}^2\|f\|_{H_1}^2 + C\|f\|_{H_1}^2 \\ &\le C(1+s^2)\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big)^2\|f\|_{H_1}^2,\end{aligned}$$

and finally $\|P(is)^{-1}A\|_{\mathscr{L}(H_1)} \le C(1+|s|)\big(1+(|s|+1)\|P(is)^{-1}\|_{\mathscr{L}(H)}\big)$. Coming back to (4-19), we have, for $|s| \ge 1$,

$$\|P(is)^{-1}(BB^* + is\,\mathrm{Id})\|_{\mathscr{L}(H_1)} \le C\big(1+|s|\|P(is)^{-1}\|_{\mathscr{L}(H)}\big). \tag{4-20}$$

Finally, combining (4-15), (4-18) and (4-20), together with (4-13)–(4-14), we obtain for $|s| \ge 1$,

$$\|(is\,\mathrm{Id} - \mathscr{A})^{-1}\|_{\mathscr{L}(\mathscr{H})} \le C\big(1+|s|\|P(is)^{-1}\|_{\mathscr{L}(H)}\big).$$

This concludes the proof of Lemma 4.6. □

Part III. Proof of Theorem 2.6: smooth damping coefficients on the torus

To prove Theorem 2.6, we argue by contradiction, assuming that estimate (2-10) does not hold (which provides a sequence of "quasimodes" defined in Section 5). The proof of Theorem 2.6 then relies on the study of semiclassical measures (a semiclassical version of microlocal defect measures) associated to quasimodes. This standard technique originates in the work of Lebeau [1996], but the novelty here is that we introduce a second microlocalization which allows us to study different scales of concentration around periodic orbits.

Sections 5 and 6 are preliminaries: Section 5 deals with the notion of semiclassical measures in a general setting, while Section 6 specializes to the torus case. Lemmata 6.1 and 6.4 reduce everything to understanding the semiclassical measure μ restricted to frequencies of rational slopes.

From Section 7 on, a frequency of rational slope is fixed; it is parametrized by a submodule Λ of $\mathbb{Z}^2$ of rank 1 (rather than by the slope). More precisely, we study the restriction $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$. The main outcome of this section is the technical Proposition 7.3: it says that a quasimode which is small in the support of b must also be small in a whole strip of direction $\Lambda^\perp$.

The core of the proof occupies Sections 8–10. Section 8 introduces tools of second microlocal calculus. The idea is to study in a finer way the rate of concentration of our quasimodes on $\mathbb{T}^2 \times (\Lambda^\perp \setminus \{0\})$. Section 9 is inspired by [Anantharaman and Macià 2010]: the two-microlocal defect measures gain some additional structure, which depends on the rate of concentration. The final argument is in Section 10, showing that the semiclassical measure μ must vanish everywhere, thus obtaining a contradiction since it was by construction a probability measure.

The last two sections are devoted to more technical lemmata.

5. The invariant semiclassical measure μ

5A. ***Quasimodes.*** To prove Theorem 2.6, we shall instead prove estimate (2-10) with $\alpha = 1/(1+\delta)$ (which, according to Proposition 2.4, is equivalent to the statement of Theorem 2.6). Let us first recast (2-10) with $\alpha = 1/(1+\delta)$ in the semiclassical setting: taking $h = s^{-1}$, we are left to prove that there exist $C > 1$ and $h_0 > 0$ such that for all $h \leq h_0$, for all $u \in H^2(\mathbb{T}^2)$, we have

$$\|u\|_{L^2(\mathbb{T}^2)} \leq Ch^{-\delta}\|P(i/h)u\|_{L^2(\mathbb{T}^2)}, \tag{5-1}$$

where $P(z)$ is defined in (2-5).

We prove this inequality by contradiction, using the notion of semiclassical measures. The idea of developing such a strategy for proving energy estimates, together with the associated technology, originates from Lebeau [1996].

We assume that (5-1) is not satisfied, and will obtain a contradiction at the end of Section 10. Hence, for all $n \in \mathbb{N}$, there exists $0 < h_n \leq 1/n$ and $u_n \in H^2(\mathbb{T}^2)$ such that

$$\|u_n\|_{L^2(\mathbb{T}^2)} > \frac{n}{h_n^\delta}\|P(i/h_n)u_n\|_{L^2(\mathbb{T}^2)}.$$

Setting $v_n = u_n/\|u_n\|_{L^2(\mathbb{T}^2)}$ and

$$P_b^{h_n} = -h_n^2\Delta - 1 + ih_nb(x) = h_n^2P(i/h_n),$$

we then have, as $n \to \infty$,

$$h_n \to 0^+, \quad \|v_n\|_{L^2(\mathbb{T}^2)} = 1, \quad h_n^{-2-\delta}\|P_b^{h_n}v_n\|_{L^2(\mathbb{T}^2)} \to 0.$$

Our goal is now to associate to the sequence (u_n, h_n) a semiclassical measure on the cotangent bundle μ on $T^*\mathbb{T}^2 = \mathbb{T}^2 \times (\mathbb{R}^2)^*$ (where $(\mathbb{R}^2)^*$ is the dual space of $\mathbb{R}^2$). To obtain a contradiction, we shall prove both that $\mu(T^*\mathbb{T}^2) = 1$, and that $\mu = 0$ on $T^*\mathbb{T}^2$.

From now on, we drop the subscript n of the sequences above, and write h in place of h_n and v_h in place of v_n. We study sequences (h, v_h) such that $h \to 0^+$ and

$$\begin{cases} \|v_h\|_{L^2(\mathbb{T}^2)} = 1, \\ \|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}) \quad \text{as } h \to 0^+. \end{cases} \tag{5-2}$$

We call such sequences "sequences of $o(h^{2+\delta})$-quasimodes," or simply "quasimodes of order $2+\delta$." In particular, this last equation also yields the key information

$$(bv_h, v_h)_{L^2(\mathbb{T}^2)} = h^{-1}\,\mathrm{Im}(P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta}) \quad \text{as } h \to 0^+.$$

In the following, it will be convenient to identify $(\mathbb{R}^2)^*$ and $\mathbb{R}^2$ through the usual inner product. In particular, the cotangent bundle $T^*\mathbb{T}^2 = \mathbb{T}^2 \times (\mathbb{R}^2)^*$ will be identified with $\mathbb{T}^2 \times \mathbb{R}^2$.

5B. ***Semiclassical measures.*** We denote by $\overline{T^*\mathbb{T}^2}$ the compactification of $T^*\mathbb{T}^2$ obtained by adding a point at infinity to each fiber (i.e., the set $\mathbb{T}^2 \times (\mathbb{R}^2 \cup \{\infty\})$). A neighborhood of $(x, \infty) \in \overline{T^*\mathbb{T}^2}$ is a set

$U \times (\{\infty\} \cup \mathbb{R}^2 \setminus K)$, where U is a neighborhood of x in $\mathbb{T}^2$ and K a compact set in $\mathbb{R}^2$. Endowed with this topology, the set $\overline{T^*\mathbb{T}^2}$ is compact.

We denote by $S^0(T^*\mathbb{T}^2)$, S^0 for short, the space of functions $a(x, \xi)$ that satisfy the following properties:

(1) $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$.

(2) There exists a compact set $K \subset \mathbb{R}^2$ and a constant $k_0 \in \mathbb{C}$ such that $a(x, \xi) = k_0$ for all $\xi \in \mathbb{R}^2 \setminus K$.

Note that we have in particular $\mathscr{C}_c^\infty(T^*\mathbb{T}^2) \subset S^0(T^*\mathbb{T}^2)$.

To a symbol $a \in S^0(T^*\mathbb{T}^2)$, we associate its semiclassical Weyl quantization $\mathrm{Op}_h(a)$ by formula (A-1), which according to the Calderón–Vaillancourt theorem (see Appendix A) defines a uniformly bounded operator on $L^2(\mathbb{T}^2)$.

From the sequence (v_h, h) (see, for instance, [Gérard and Leichtnam 1993]), we can define (using again the Calderón–Vaillancourt theorem) the associated Wigner distribution $V^h \in (S^0)'$ by

$$\langle V^h, a\rangle_{(S^0)', S^0} = (\mathrm{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)} \quad \text{for all } a \in S^0(T^*\mathbb{T}^2). \tag{5-3}$$

Decomposing v_h and a in Fourier series,

$$\hat{v}_h(k) = \frac{1}{2\pi}\int_{\mathbb{T}^2} e^{-ik\cdot x} v_h(x)\,dx, \quad \hat{a}(h, k, \xi) = \frac{1}{2\pi}\int_{\mathbb{T}^2} e^{-ik\cdot x} a(h, x, \xi)\,dx,$$

the expression (5-3) can be more explicitly rewritten as

$$\langle V^h, a\rangle_{(S^0)', S^0} = \frac{1}{2\pi}\sum_{k, j \in \mathbb{Z}^2} \hat{a}\Big(h, j-k, \frac{h}{2}(k+j)\Big)\hat{v}_h(k)\overline{\hat{v}_h(j)}.$$

Proposition 5.1. *The family (V^h) is bounded in $(S^0)'$. Hence, there exists a subsequence of the sequence (h, v_h) and an element $\mu \in (S^0)'$ such that $V^h \rightharpoonup \mu$ weakly in $(S^0)'$, that is,*

$$(\mathrm{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)} \to \langle \mu, a\rangle_{(S^0)', S^0} \quad \textit{for all } a \in S^0(T^*\mathbb{T}^2). \tag{5-4}$$

In addition, $\langle \mu, a\rangle_{(S^0)', S^0}$ is nonnegative if a is; in other words, μ may be identified with a nonnegative Radon measure on $\overline{T^*\mathbb{T}^2}$.

Notation: in what follows, we shall denote by $\mathcal{M}^+(\overline{T^*\mathbb{T}^2})$ the set of nonnegative Radon measures on $\overline{T^*\mathbb{T}^2}$.

Proof. The proof is an adaptation from the original proof of Gérard [1991] (see also [Gérard and Leichtnam 1993] in the semiclassical setting).

The fact that the Wigner distributions V^h are uniformly bounded in $(S^0)'$ follows from the Calderón–Vaillancourt theorem (see Appendix A), and from the boundedness of (v_h) in $L^2(\mathbb{T}^2)$.

The sharp Gårding inequality (see for instance [Sjöstrand 1995, Proposition 5.1] or [Lerner 2010, Section 2.5.2]) gives the existence of $C > 0$ such that, for all $a \geq 0$ and $h > 0$,

$$\big(\mathrm{Op}_h(a)v_h, v_h\big)_{L^2(\mathbb{T}^2)} \geq -Ch\|v_h\|^2_{L^2(\mathbb{T}^2)},$$

so that the distribution μ is nonnegative (and hence is a measure). $\square$

5C. ***Properties of μ for zeroth and first order quasimodes.*** To simplify the notation, we set

$$P_b^h = P_0^h + ihb(x), \quad \text{with } P_0^h = -h^2\Delta - 1 = \mathrm{Op}_h(|\xi|^2 - 1).$$

The geodesic flow on the torus $\phi_\tau : T^*\mathbb{T}^2 \to T^*\mathbb{T}^2$ for $\tau \in \mathbb{R}$ is the flow generated by the Hamiltonian vector field associated to the symbol $\frac{1}{2}(|\xi|^2 - 1)$, i.e., by the vector field $\xi \cdot \partial_x$ on $T^*\mathbb{T}^2$. Explicitly, we have

$$\phi_\tau(x, \xi) = (x + \tau\xi, \xi), \quad \tau \in \mathbb{R},\ (x, \xi) \in T^*\mathbb{T}^2.$$

Note that ϕ_τ preserves the ξ-component, and in particular every energy layer $\{|\xi|^2 = C > 0\} \subset T^*\mathbb{T}^2$.

Now, we describe the first properties of the measure μ implied by (5-2).

We recall that for $\nu \in \mathscr{D}'(T^*\mathbb{T}^2)$, $(\phi_\tau)_*\nu \in \mathscr{D}'(T^*\mathbb{T}^2)$ is defined by $\langle (\phi_\tau)_*\nu, a\rangle = \langle \nu, a \circ \phi_\tau\rangle$ for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$. In particular, $(\phi_\tau)_*\nu$ is a measure if ν is. We shall say that ν is an *invariant measure* if it is invariant by the geodesic flow, i.e., $(\phi_\tau)_*\nu = \nu$ for all $\tau \in \mathbb{R}$.

Proposition 5.2. *Let μ be as in Proposition 5.1 with v_h satisfying* (5-2). *We have*

(1) $\mathrm{supp}(\mu) \subset \{|\xi|^2 = 1\}$ (*hence is compact in $T^*\mathbb{T}^2$*),

(2) $\mu(T^*\mathbb{T}^2) = 1$,

(3) *μ is invariant by the geodesic flow*, *i.e.*, $(\phi_\tau)_*\mu = \mu$,

(4) $\langle \mu, b\rangle_{\mathscr{M}_c(T^*\mathbb{T}^2), \mathscr{C}^0(T^*\mathbb{T}^2)} = 0$, *where $\mathscr{M}_c(T^*\mathbb{T}^2)$ denotes the space of compactly supported measures on $T^*\mathbb{T}^2$.*

In other words, *μ is an invariant probability measure on $T^*\mathbb{T}^2$ vanishing on* $\{b > 0\}$.

These are standard arguments that we reproduce here for the reader's comfort. In particular, we recover all information required to prove the Bardos–Lebeau–Rauch–Taylor uniform stabilization theorem under GCC. The proof of this proposition only uses that μ is a measure associated to a $o(h)$-quasimode, and not the full information in (5-2) (which is the key point to prove Theorem 2.6).

Proof. First, we take $\chi \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ depending only on the ξ variable, such that $\chi \geq 0$, $\chi(\xi) = 0$ for $|\xi| \leq 2$, and $\chi(\xi) = 1$ for $|\xi| \geq 3$. Hence, $\chi(\xi)/(|\xi|^2 - 1) \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ and we have the exact composition formula

$$\mathrm{Op}_h(\chi) = \mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_0^h,$$

since both operators are Fourier multipliers. Moreover, $\mathrm{Op}_h(\chi(\xi)/(|\xi|^2 - 1))$ is a bounded operator on $L^2(\mathbb{T}^2)$. As a consequence, we have

$$\langle V^h, \chi\rangle_{(S^0)', S^0} \to \langle \mu, \chi\rangle_{\mathscr{M}(\overline{T^*\mathbb{T}^2}), \mathscr{C}^0(\overline{T^*\mathbb{T}^2})},$$

together with

$$\begin{aligned}\langle V^h, \chi\rangle_{(S^0)', S^0} &= \left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_0^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} \\ &= \left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) P_b^h v_h, v_h\right)_{L^2(\mathbb{T}^2)} - ih\left(\mathrm{Op}_h\left(\frac{\chi(\xi)}{|\xi|^2 - 1}\right) b v_h, v_h\right)_{L^2(\mathbb{T}^2)}.\end{aligned}$$

Since $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(1)$ and $\|v_h\|_{L^2(\mathbb{T}^2)} = 1$, both terms in this expression vanish in the limit $h \to 0^+$. This implies $\langle \mu, \chi \rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}), \mathscr{C}^0(\overline{T^*\mathbb{T}^2})} = 0$. Since this holds for all χ as above, we have $\operatorname{supp}(\mu) \subset \{|\xi|^2 = 1\}$, which proves item (1).

In particular, this implies $\mu(\overline{T^*\mathbb{T}^2} \setminus T^*\mathbb{T}^2) = 0$. Now, item (2) is a direct consequence of item (1) and $1 = \|v_h\|^2_{L^2(\mathbb{T}^2)} \to \langle \mu, 1 \rangle_{\mathcal{M}(\overline{T^*\mathbb{T}^2}), \mathscr{C}^0(\overline{T^*\mathbb{T}^2})}$. Item (4) is a direct consequence of $(bv_h, v_h)_{L^2(\mathbb{T}^2)} = o(1)$.

Finally, for $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$, we recall that

$$[P_0^h, \operatorname{Op}_h(a)] = \frac{h}{i} \operatorname{Op}_h(\{|\xi|^2 - 1, a\}) = \frac{2h}{i} \operatorname{Op}_h(\xi \cdot \partial_x a)$$

is a consequence of the Weyl quantization (any other quantization would have left an error term of order $\mathbb{O}(h^2)$). Hence, (5-3) yields

$$\langle V^h, \xi \cdot \partial_x a \rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)} \to \langle \mu, \xi \cdot \partial_x a \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)}, \tag{5-5}$$

together with

$$\begin{aligned} \langle V^h, \xi \cdot \partial_x a \rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)} &= \frac{i}{2h} ([P_0^h, \operatorname{Op}_h(a)] v_h, v_h)_{L^2(\mathbb{T}^2)} \\ &= \frac{i}{2h} (\operatorname{Op}_h(a) v_h, P_0^h v_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h} (\operatorname{Op}_h(a) P_0^h v_h, v_h)_{L^2(\mathbb{T}^2)} \\ &= \frac{i}{2h} (\operatorname{Op}_h(a) v_h, P_b^h v_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h} (\operatorname{Op}_h(a) P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} \\ &\qquad - \frac{1}{2} (\operatorname{Op}_h(a) v_h, b v_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2} (\operatorname{Op}_h(a) b v_h, v_h)_{L^2(\mathbb{T}^2)}. \end{aligned} \tag{5-6}$$

In this expression, we have $(1/h)(\operatorname{Op}_h(a) v_h, P_b^h v_h)_{L^2(\mathbb{T}^2)} \to 0$ and $(1/h)(\operatorname{Op}_h(a) P_b^h v_h, v_h)_{L^2(\mathbb{T}^2)} \to 0$ since $\|P_b^h v_h\|_{L^2(\mathbb{T}^2)} = o(h)$. Moreover, the last two terms can be estimated by

$$|(\operatorname{Op}_h(a) b v_h, v_h)_{L^2(\mathbb{T}^2)}| \le \|\sqrt{b} v_h\|_{L^2(\mathbb{T}^2)} \|\sqrt{b} \operatorname{Op}_h(a) v_h\|_{L^2(\mathbb{T}^2)} = o(1), \tag{5-7}$$

since $(bv_h, v_h)_{L^2(\mathbb{T}^2)} = o(1)$. This yields $\langle V^h, \xi \cdot \partial_x a \rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)} \to 0$, so that, using (5-5),

$$\langle \mu, \xi \cdot \partial_x a \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)} = 0 \quad \text{for all } a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2).$$

Replacing a by $a \circ \phi_\tau$ and integrating with respect to the parameter τ gives $(\phi_\tau)_* \mu = \mu$, which concludes the proof of item (3). □

6. Geometry on the torus and decomposition of invariant measures

The results of Section 5 were valid on arbitrary manifolds. We now turn to specific properties of the geodesic flow on the torus (and related facts of Fourier analysis). In Lemma 6.1 we use the partition of the cotangent bundle into resonant and nonresonant vectors to decompose any invariant measure according to the long-time behavior of geodesics.

6A. ***Resonant and nonresonant vectors on the torus.*** In this section, we collect several facts concerning the geometry of $T^*\mathbb{T}^2$ and its resonant subspaces. Most of the setting and the notation comes from [Anantharaman and Macià 2010, Section 2].

We shall say that a submodule $\Lambda \subset \mathbb{Z}^2$ is primitive if $\langle\Lambda\rangle \cap \mathbb{Z}^2 = \Lambda$, where $\langle\Lambda\rangle$ denotes the linear subspace of $\mathbb{R}^2$ spanned by Λ. The family of all primitive submodules will be denoted by $\mathcal{P}$.

Let us denote by $\Omega_j \subset \mathbb{R}^2$, for $j = 0, 1, 2$, the following sets:

$$\Omega_j := \{\xi \in \mathbb{R}^2 \text{ such that } \operatorname{rk}(\Lambda_\xi) = 2 - j\}, \quad \text{with } \Lambda_\xi := \{k \in \mathbb{Z}^2 \text{ such that } \xi \cdot k = 0\} = \xi^\perp \cap \mathbb{Z}^2.$$

The set $\Omega_0 \cup \Omega_1$ is referred to as the set of resonant directions, whereas $\Omega_2 = \mathbb{R}^2 \setminus (\Omega_0 \cup \Omega_1)$ is referred to as the set of nonresonant vectors.

Note that the sets Ω_j form a partition of $\mathbb{R}^2$, and that we have

- $\Omega_0 = \{0\}$ (resonance of order 0);
- $\xi \in \Omega_1$ if and only if the geodesic issued from any $x \in \mathbb{T}^2$ in the direction ξ is periodic (resonances of order 1);
- $\xi \in \Omega_2$ if and only if the geodesic issued from any $x \in \mathbb{T}^2$ in the direction ξ is dense in $\mathbb{T}^2$.

On the Fourier analysis side, we will use the following facts. For $\Lambda \in \mathcal{P}$ let us define

$$\Lambda^\perp := \{\xi \in \mathbb{R}^2 \text{ such that } \xi \cdot k = 0 \text{ for all } k \in \Lambda\}.$$

For a function f on $\mathbb{T}^2$ with Fourier coefficients $(\hat{f}(k))_{k\in\mathbb{Z}^2}$, and $\Lambda \in \mathcal{P}$, we shall say that f has only Fourier modes in Λ if $\hat{f}(k) = 0$ for $k \notin \Lambda$. This means that f is constant in the direction $\Lambda^\perp$, or equivalently, that $\sigma \cdot \partial_x f = 0$ for all $\sigma \in \Lambda^\perp$. This is a trivial property if $\operatorname{rk}\Lambda = 2$, but means that f is constant if $\operatorname{rk}\Lambda = 0$ and that f is constant along the 1-dimensional tori

$$\mathbb{T}_{\Lambda^\perp} := \Lambda^\perp/(2\pi\mathbb{Z}^2 \cap \Lambda^\perp)$$

if $\operatorname{rk}\Lambda = 1$.

We shall use the following notation: $L^p_\Lambda(\mathbb{T}^2)$ will stand for the subspace of $L^p(\mathbb{T}^2)$ consisting of functions having only Fourier modes in Λ. For a function $f \in L^2(\mathbb{T}^2)$ (resp. a symbol $a \in S^0(T^*\mathbb{T}^2)$), we denote by $\langle f\rangle_\Lambda$ its orthogonal projection on $L^2_\Lambda(\mathbb{T}^2)$, i.e., the average of f along $\Lambda^\perp$:

$$\langle f\rangle_\Lambda(x) := \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi}\hat{f}(k) \quad \left(\text{resp. } \langle a\rangle_\Lambda(x,\xi) := \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi}\hat{a}(k,\xi)\right).$$

If $\operatorname{rk}(\Lambda) = 1$ and v is a vector in $\Lambda^\perp \setminus \{0\}$, we also have

$$\langle f\rangle_\Lambda(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(x+tv)\,dt. \tag{6-1}$$

In particular, note that $\langle f\rangle_\Lambda$ (resp. $\langle a\rangle_\Lambda$) is nonnegative if f (resp. a) is, and that $\langle f\rangle_\Lambda \in \mathscr{C}^\infty(\mathbb{T}^2)$ (resp. $\langle a\rangle_\Lambda \in S^0(T^*\mathbb{T}^2)$) if $f \in \mathscr{C}^\infty(\mathbb{T}^2)$ (resp. $a \in S^0(T^*\mathbb{T}^2)$).

Finally, given $f \in L^\infty_\Lambda(\mathbb{T}^2)$, we denote by m_f the bounded operator on $L^2_\Lambda(\mathbb{T}^2)$, consisting in the multiplication by f.

6B. ***Decomposition of invariant measures.*** We denote by $\mathcal{M}^+(T^*\mathbb{T}^2)$ the set of finite, nonnegative measures on $T^*\mathbb{T}^2$. With the definitions above, we have the following decomposition lemmata, proved in [Macià 2010] or [Anantharaman and Macià 2010, Section 2]. These properties are given for general measures $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$. Of course, they apply in particular to the measure μ defined by Proposition 5.1.

Lemma 6.1. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$. Then μ decomposes as a sum of nonnegative measures*

$$\mu = \mu|_{\mathbb{T}^2\times\{0\}} + \mu|_{\mathbb{T}^2\times\Omega_2} + \sum_{\Lambda\in\mathcal{P},\mathrm{rk}(\Lambda)=1} \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}. \tag{6-2}$$

This decomposition simply comes from partitioning $\mathbb{R}^2$ into the disjoint, countable union of $\{0\}$, Ω_2 and the sets $\Lambda^\perp \setminus \{0\}$, which for $\mathrm{rk}(\Lambda) = 1$ are punctured lines of rational slopes. For such Λ, note that $\xi \in \Lambda^\perp \setminus \{0\}$ implies $\Lambda_\xi = \Lambda$.

Given $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$, we define its Fourier coefficients by the complex measures on $\mathbb{R}^2$:

$$\hat{\mu}(k,\cdot) := \int_{\mathbb{T}^2} \frac{e^{-ik\cdot x}}{2\pi}\mu(dx,\cdot), \quad k \in \mathbb{Z}.$$

One has, in the sense of distributions, the Fourier inversion formula

$$\mu(x,\xi) = \sum_{k\in\mathbb{Z}^2} \frac{e^{ik\cdot x}}{2\pi}\hat{\mu}(k,\xi).$$

Lemma 6.2. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$ and $\Lambda \in \mathcal{P}$. Then the distribution*

$$\langle\mu\rangle_\Lambda(x,\xi) := \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi}\hat{\mu}(k,\xi)$$

is in $\mathcal{M}^+(T^*\mathbb{T}^2)$ and satisfies, for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$,

$$\langle\langle\mu\rangle_\Lambda, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle\mu, \langle a\rangle_\Lambda\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)}.$$

Lemma 6.3. *Let $\mu \in \mathcal{M}^+(T^*\mathbb{T}^2)$ be an* invariant *measure. Then, for all $\Lambda \in \mathcal{P}$, the measure $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$ is also a nonnegative invariant measure and*

$$\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})} = \langle\mu\rangle_\Lambda|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}.$$

Let us now come back to the measure μ given by Proposition 5.1, which satisfies all properties listed in Proposition 5.2. In particular, this measure vanishes on the nonempty open subset of $\mathbb{T}^2$ given by $\{b > 0\}$ (see item (4) in Proposition 5.2). As a consequence of Proposition 5.2 and of the three lemmata above, this yields the following lemma.

Lemma 6.4. *We have $\mu = \sum_{\Lambda\in\mathcal{P},\mathrm{rk}(\Lambda)=1} \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$.*

As a consequence of Proposition 5.2, we have indeed that the measure μ is supported in $\{|\xi| = 1\}$, which implies $\mu|_{\mathbb{T}^2\times\{0\}} = 0$. In addition, Lemma 6.3 applied with $\Lambda = \{0\}$ implies that $\mu|_{\mathbb{T}^2\times\Omega_2}$ is constant in x, and thus vanishes everywhere since it vanishes on $\{b > 0\}$.

Remark 6.5. Since the measure μ is supported in $\{|\xi| = 1\}$ (Proposition 5.2(1)), we have

$$\mu|_{\mathbb{T}^2\times\Lambda^\perp} = \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$$

(which simplifies the notation).

As a consequence of these lemmata and the last remark, the study of the measure μ is now reduced to that of all nonnegative invariant measures $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ with $\mathrm{rk}(\Lambda) = 1$.

The aim of the next sections is to prove that the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ vanishes identically, for each periodic direction $\Lambda^\perp$.

6C. ***Adapted coordinates for resonant directions of order* 1.** For each $\Lambda \in \mathcal{P}$, we define

$$\begin{aligned}\Lambda^\perp &:= \{\xi \in \mathbb{R}^2 \text{ such that } \xi\cdot k = 0 \text{ for all } k \in \Lambda\},\\ \mathbb{T}_\Lambda &:= \langle\Lambda\rangle/2\pi\Lambda,\\ \mathbb{T}_{\Lambda^\perp} &:= \Lambda^\perp/(2\pi\mathbb{Z}^2\cap\Lambda^\perp).\end{aligned}$$

Note that if $\mathrm{rk}(\Lambda) = 1$, $\mathbb{T}_\Lambda$ and $\mathbb{T}_{\Lambda^\perp}$ are two submanifolds of $\mathbb{T}^2$ diffeomorphic to one-dimensional tori. Their cotangent bundles admit the global trivializations $T^*\mathbb{T}_\Lambda = \mathbb{T}_\Lambda \times \langle\Lambda\rangle$ and $T^*\mathbb{T}_{\Lambda^\perp} = \mathbb{T}_{\Lambda^\perp} \times \Lambda^\perp$.

To study the measure $\mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}$ for $\Lambda \in \mathcal{P}$, $\mathrm{rk}(\Lambda) = 1$, we need to work in adapted coordinates.

We define the linear isomorphism

$$\chi_\Lambda : \Lambda^\perp \times \langle\Lambda\rangle \to \mathbb{R}^2$$

by $(s, y) \mapsto s + y$, and denote by $\tilde{\chi}_\Lambda : T^*\Lambda^\perp \times T^*\langle\Lambda\rangle \to T^*\mathbb{R}^2$ its extension to the cotangent bundle. This map can be defined as follows: for $(s, \sigma) \in T^*\Lambda^\perp = \Lambda^\perp \times (\Lambda^\perp)^*$ and $(y, \eta) \in T^*\langle\Lambda\rangle = \langle\Lambda\rangle \times \langle\Lambda\rangle^*$, we can extend σ to a covector of $\mathbb{R}^2$ vanishing on $\langle\Lambda\rangle$ and η to a covector of $\mathbb{R}^2$ vanishing on $\Lambda^\perp$. Remember that we identify $(\mathbb{R}^2)^*$ with $\mathbb{R}^2$ through the usual inner product; thus we can also see σ as an element of $\Lambda^\perp$ and η as an element of $\langle\Lambda\rangle$. Then we have

$$\tilde{\chi}_\Lambda(s, \sigma, y, \eta) = (s + y, \sigma + \eta) \in T^*\mathbb{R}^2 = \mathbb{R}^2 \times (\mathbb{R}^2)^*.$$

Conversely, any $\xi \in (\mathbb{R}^2)^*$ can be decomposed into $\xi = \sigma + \eta$, where $\sigma \in \Lambda^\perp$ and $\eta \in \langle\Lambda\rangle$. We denote by P_Λ the orthogonal projection of $\mathbb{R}^2$ onto $\langle\Lambda\rangle$, that is,

$$P_\Lambda\xi = \eta. \tag{6-3}$$

Next, the map χ_Λ goes to the quotient, giving a smooth Riemannian covering of $\mathbb{T}^2$:

$$\pi_\Lambda : \mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda \to \mathbb{T}^2, \quad (s, y) \mapsto s + y.$$

We shall denote by $\tilde{\pi}_\Lambda$ its extension to cotangent bundles:

$$\tilde{\pi}_\Lambda : T^*\mathbb{T}_{\Lambda^\perp} \times T^*\mathbb{T}_\Lambda \to T^*\mathbb{T}^2.$$

As the map π_Λ is not an injection (because the torus $\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda$ contains several copies of $\mathbb{T}^2$), we introduce its degree p_Λ, which is also equal to $\mathrm{Vol}(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)/\mathrm{Vol}(\mathbb{T}^2)$.

Then, the map

$$T_\Lambda u := \frac{1}{\sqrt{p_\Lambda}} u \circ \chi_\Lambda$$

defines a linear isomorphism $L^2_{\mathrm{loc}}(\mathbb{R}^2) \to L^2_{\mathrm{loc}}(\Lambda^\perp \times \langle\Lambda\rangle)$. Note that because of the factor $1/\sqrt{p_\Lambda}$, T_Λ maps $L^2(\mathbb{T}^2)$ isometrically into a subspace of $L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)$. Moreover, T_Λ maps $L^2_\Lambda(\mathbb{T}^2)$ into $L^2(\mathbb{T}_\Lambda) \subset L^2(\mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda)$, since the nonvanishing Fourier modes of $u \in L^2_\Lambda(\mathbb{T}^2)$ correspond only to frequencies $k \in \Lambda$. This reads

$$T_\Lambda u(s, y) = \frac{1}{\sqrt{p_\Lambda}} u(y) \quad \text{for } (s, y) \in \mathbb{T}_{\Lambda^\perp} \times \mathbb{T}_\Lambda. \tag{6-4}$$

Since $\tilde{\chi}_\Lambda$ is linear, we have

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h(a \circ \tilde{\chi}_\Lambda) T_\Lambda, \tag{6-5}$$

for any $a \in \mathscr{C}^\infty(T^*\mathbb{R}^2)$, where on the left Op_h is the Weyl quantization on $\mathbb{R}^2$ (A-1), and on the right Op_h is the Weyl quantization on $\Lambda^\perp \times \langle\Lambda\rangle$. Next, we denote by $\operatorname{Op}_h^{\Lambda^\perp}$ and $\operatorname{Op}_h^\Lambda$ the Weyl quantization operators defined on smooth test functions on $T^*\Lambda^\perp \times T^*\langle\Lambda\rangle$ and acting only on the variables in $T^*\Lambda^\perp$ and $T^*\langle\Lambda\rangle$, respectively, leaving the other frozen. For any $a \in \mathscr{C}_c^\infty(T^*\Lambda^\perp \times T^*\langle\Lambda\rangle)$, we have

$$\operatorname{Op}_h(a) = \operatorname{Op}_h^{\Lambda^\perp} \circ \operatorname{Op}_h^\Lambda(a) = \operatorname{Op}_h^\Lambda \circ \operatorname{Op}_h^{\Lambda^\perp}(a). \tag{6-6}$$

Now, if the symbol $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ has only Fourier modes in Λ, we remark, in view of (6-4), that $a \circ \tilde{\pi}_\Lambda$ does not depend on $s \in \mathbb{T}_{\Lambda^\perp}$. Therefore, we sometimes write $a \circ \tilde{\pi}_\Lambda(\sigma, y, \eta)$ for $a \circ \tilde{\pi}_\Lambda(s, \sigma, y, \eta)$, and (6-5) and (6-6) give

$$T_\Lambda \operatorname{Op}_h(a) = \operatorname{Op}_h^\Lambda \circ \operatorname{Op}_h^{\Lambda^\perp}(a \circ \tilde{\pi}_\Lambda) T_\Lambda = \operatorname{Op}_h^\Lambda\big(a \circ \tilde{\pi}_\Lambda(hD_s, \cdot, \cdot)\big) T_\Lambda. \tag{6-7}$$

Note that for every $\sigma \in \Lambda^\perp$, the operator $\operatorname{Op}_h^\Lambda(a \circ \tilde{\pi}_\Lambda(\sigma, \cdot, \cdot))$ maps $L^2(\mathbb{T}_\Lambda)$ into itself. More precisely, it maps the subspace $T_\Lambda(L^2_\Lambda(\mathbb{T}^2))$ into itself.

7. Change of quasimode and construction of an invariant cutoff function

In this section, we first construct from the quasimode v_h another quasimode w_h that will be easier to handle when studying the measure $\mu|_{\mathbb{T}^2 \times \Lambda^\perp}$. Indeed, w_h is basically a microlocalization of v_h in the direction $\Lambda^\perp$ at a precise concentration rate.

Moreover, we introduce a cutoff function $\chi_h^\Lambda(x) = \chi_h^\Lambda(y, s)$, well adapted to the damping coefficient b and to the invariance of the measure $\mu|_{\mathbb{T}^2 \times \Lambda^\perp}$ in the direction $\Lambda^\perp$ (this cutoff function plays the role of the function $\chi(b/h)$ used in [Burq and Hitrik 2007] in the case where b is itself invariant in the direction $\Lambda^\perp$). Its construction is a key point in the proof of Theorem 2.6.

Let $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ be a nonnegative function such that $\chi = 1$ in a neighborhood of the origin. With P_Λ defined in (6-3), we first set

$$w_h := \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) v_h,$$

which implicitly depends on $\alpha \in (0, 1)$. The following lemma implies that, for δ and α sufficiently small, w_h is also a $o(h^{2+\delta})$-quasimode for P_b^h.

Lemma 7.1. *For any $\alpha > 0$ such that*

$$2\alpha + \delta \leq 1 \quad \text{and} \quad 3\alpha + 2\delta < 1, \tag{7-1}$$

we have

$$\| P_b^h w_h \|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}).$$

As a consequence of this lemma, the semiclassical measures associated to w_h satisfy in particular the conclusions of Proposition 5.2. Moreover, the following proposition implies that the sequence w_h contains all the information in the direction $\Lambda^\perp$.

Proposition 7.2. *Suppose that $\| P_b^h w_h \|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta})$ and $0 < \alpha < (1+\delta)/2$. For any $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$, we have*

$$\langle \mu|_{\mathbb{T}^2 \times \Lambda^\perp}, a \rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}_c^0(T^*\mathbb{T}^2)} = \lim_{h \to 0} (\mathrm{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)}.$$

Note that under condition (7-1), both assumptions of Proposition 7.2 are satisfied since in particular $\alpha < \frac{1}{3}$.

Next, we state the desired properties of the cutoff function χ_h^Λ. The proof of its existence is a crucial point in the proof of Theorem 2.6.

Proposition 7.3. *For $\delta = 4\varepsilon$ and $\varepsilon < \frac{1}{29}$, there exists α satisfying* (7-1), *such that for any constant $c_0 > 0$, there exists a cutoff function $\chi_h^\Lambda \in \mathscr{C}^\infty(\mathbb{T}^2)$ valued in* $[0, 1]$, *such that*

(1) $\chi_h^\Lambda = \chi_h^\Lambda(y)$ *does not depend on the variable s (i.e., χ_h^Λ is $\Lambda^\perp$-invariant)*,

(2) $\|(1 - \chi_h^\Lambda) w_h\|_{L^2(\mathbb{T}^2)} = o(1)$,

(3) $b \leq c_0 h$ *on* $\mathrm{supp}(\chi_h^\Lambda)$,

(4) $\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$,

(5) $\|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$.

Note that the function χ_h^Λ implicitly depends on the constant c_0, which will be taken arbitrarily small in Section 9.

In the particular case where the damping function b is invariant in one direction, this proposition is not needed. In this case, one can take as in [Burq and Hitrik 2007] $\chi_h^\Lambda = \chi(b/(c_0 h))$. In the d-dimensional torus, this cutoff function works as well if b is invariant in $d-1$ directions, and an analogue of Theorem 2.6 can be stated in this setting. Unfortunately, our construction of the function χ_h^Λ (see the proof of Proposition 7.3 in Section 12) strongly relies on the fact that all trapped directions are periodic, and fails in higher dimensions.

We give here a proof of Lemma 7.1. Because of their technicality, we postpone the proofs of Propositions 7.2 and 7.3 to Sections 11 and 12, respectively.

Proof of Lemma 7.1. First, we develop

$$P_b^h w_h = P_b^h \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) v_h = \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) P_b^h v_h + ih\Big[b, \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big)\Big] v_h, \tag{7-2}$$

since P_0^h and $\operatorname{Op}_h(\chi(|P_\Lambda \xi|/h^\alpha))$ are both Fourier multipliers. We know that

$$\Big\| \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) P_b^h v_h \Big\|_{L^2(\mathbb{T}^2)} \leq \| P_b^h v_h \|_{L^2(\mathbb{T}^2)} = o(h^{2+\delta}).$$

It only remains to study the operator

$$\Big[b, \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big)\Big] = ih^{1-\alpha} \operatorname{Op}_h\Big(\partial_y b \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) + \mathcal{O}_{\mathcal{L}(L^2)}(h^{2(1-\alpha)}) \tag{7-3}$$

according to the symbolic calculus.

Moreover, using the pointwise inequality[1] $|\nabla b(x)|^2 \leq 2|b|_{W^{2,\infty}} b(x)$ (holding for any nonnegative $W^{2,\infty}$ function b), we have, for some $C > 0$,

$$Cb - \Big|\partial_y b \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big|^2 \geq 0 \quad \text{on } \mathbb{T}^2 \times \mathbb{R}^2.$$

The sharp Gårding inequality applied to this nonnegative symbol then yields

$$\Big(\operatorname{Op}_h\Big(Cb - \Big|\partial_y b \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big|^2\Big) v_h, v_h\Big)_{L^2(\mathbb{T}^2)} \geq -Ch^{1-\alpha},$$

and hence

$$\Big\| \operatorname{Op}_h\Big(\partial_y b \; \chi'\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big) v_h \Big\|^2_{L^2(\mathbb{T}^2)} \leq C(bv_h, v_h)_{L^2(\mathbb{T}^2)} + \mathcal{O}(h^{1-\alpha}).$$

Combining this estimate together with (7-3) gives

$$\Big\| ih\Big[b, \operatorname{Op}_h\Big(\chi\Big(\frac{|P_\Lambda \xi|}{h^\alpha}\Big)\Big)\Big] v_h \Big\|_{L^2(\mathbb{T}^2)} = o(h^{2-\alpha+\frac{1+\delta}{2}}) + \mathcal{O}(h^{\frac{5-3\alpha}{2}}).$$

Coming back to the expression of $P_b^h w_h$ given in (7-2), this concludes the proof of Lemma 7.1. ☐

8. Second microlocalization of μ on a resonant affine subspace by ν^Λ and ρ_Λ

We want to analyze precisely the structure of the restriction $\mu|_{\mathbb{T}^2 \times (\Lambda^\perp \setminus \{0\})}$, using the full information contained in $o(h^{2+\delta})$-quasimodes like v_h and w_h.

From now on, we want to take advantage of the family w_h of $o(h^{2+\delta})$-quasimodes constructed in Section 7, which are microlocalized in the direction $\Lambda^\perp$. Hence, we define the Wigner distribution $W^h \in \mathcal{D}'(T^*\mathbb{T}^2)$ associated to the functions w_h and the scale h, by

$$\langle W^h, a \rangle_{(S^0)', S^0} = (\operatorname{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)} \quad \text{for all } a \in S^0(T^*\mathbb{T}^2).$$

[1]To prove this inequality, we denote by Hf the Hessian of f, take $v \in \mathbb{R}^2$ and write the Taylor formula

$$b(x+tv) = b(x) + tv \cdot \nabla b(x) + \int_0^t (t-s) v \cdot Hf(x+sv) v \, ds.$$

Taking $t > 0$ and using that $b(x+tv) \geq 0$, we obtain $-v \cdot \nabla b(x) \leq \frac{1}{t} b(x) + \frac{t|v|^2}{2} \|Hf\|_{L^\infty}$ for all $(x, v) \in \mathbb{T}^2 \times \mathbb{R}^2$ and $t > 0$. The conclusion follows when optimizing in t.

According to Proposition 7.2, we recover

$$\langle W^h, a\rangle_{(S^0)', S^0} \to \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2), \mathscr{C}^0_c(T^*\mathbb{T}^2)}$$

in the limit $h \to 0$, for any $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$ (and α satisfying (7-1)).

To provide a precise study of $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$, we shall introduce as in [Macià 2010; Anantharaman and Macià 2010] two-microlocal semiclassical measures, describing at a finer level the concentration of the sequence v_h on the resonant subspace

$$\Lambda^\perp = \{\xi \in \mathbb{R}^2 \text{ such that } P_\Lambda \xi = 0\},$$

where P_Λ is defined in (6-3). These objects were introduced in the local Euclidean case in [Nier 1996; Fermanian-Kammerer 2000a; 2000b]. A specific concentration scale may also be chosen in the in the two-microlocal variable, giving rise to the two-scales semiclassical measures studied in [Miller 1996; 1997; Fermanian-Kammerer and Gérard 2002].

We first have to describe the adapted symbol class (inspired by [Fermanian-Kammerer 2000a] and used in [Anantharaman and Macià 2010]). According to Lemma 6.3 (see also Remark 6.5), it suffices to test the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ with functions constant in the direction $\Lambda^\perp$ (or equivalently, having only x-Fourier modes in Λ, in the sense of the following definition).

Definition 8.1. Given $\Lambda \in \mathscr{P}$, we shall say that $a \in S^1_\Lambda$ if $a = a(x, \xi, \eta) \in \mathscr{C}^\infty(T^*\mathbb{T}^2 \times \langle\Lambda\rangle)$ and

(1) there exists a compact set $K_a \subset T^*\mathbb{T}^2$ such that, for all $\eta \in \langle\Lambda\rangle$, the function $(x, \xi) \mapsto a(x, \xi, \eta)$ is compactly supported in K_a;

(2) a is homogeneous of order zero at infinity in the variable $\eta \in \langle\Lambda\rangle$; i.e., if we denote by $\mathbb{S}_\Lambda := \mathbb{S}^1 \cap \langle\Lambda\rangle$ the unit sphere in $\langle\Lambda\rangle$, there exists $R_0 > 0$ (depending on a) and $a_{\text{hom}} \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)$ such that

$$a(x, \xi, \eta) = a_{\text{hom}}\Big(x, \xi, \frac{\eta}{|\eta|}\Big) \quad \text{for } |\eta| \geq R_0 \text{ and } (x, \xi) \in T^*\mathbb{T}^2;$$

for $\eta \neq 0$, we will also use the notation $a(x, \xi, \infty\eta) := a_{\text{hom}}(x, \xi, \eta/|\eta|)$.

(3) a has only x-Fourier modes in Λ, that is,

$$a(x, \xi, \eta) = \sum_{k\in\Lambda} \frac{e^{ik\cdot x}}{2\pi} \hat{a}(k, \xi, \eta).$$

This last assumption is equivalent to saying that $\sigma \cdot \partial_x a = 0$ for any $\sigma \in \Lambda^\perp$. We denote by ${S^1_\Lambda}'$ the topological dual space of S^1_Λ.

Let $\chi \in \mathscr{C}^\infty_c(\mathbb{R})$ be a nonnegative function such that $\chi = 1$ in a neighborhood of the origin. Let $R > 0$. The previous remark allows us to define, for $a \in S^1_\Lambda$ the two following elements of ${S^1_\Lambda}'$:

$$\langle W^{h,\Lambda}_R, a\rangle_{{S^1_\Lambda}', S^1_\Lambda} := \Big\langle W^h, \Big(1 - \chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)\Big) a\Big(x, \xi, \frac{P_\Lambda \xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)}, \tag{8-1}$$

$$\langle W^h_{R,\Lambda}, a\rangle_{{S^1_\Lambda}', S^1_\Lambda} := \Big\langle W^h, \chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big) a\Big(x, \xi, \frac{P_\Lambda \xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}^\infty_c(T^*\mathbb{T}^2)}. \tag{8-2}$$

In particular, for any $R > 0$ and $a \in S^1_\Lambda$, we have

$$\Big\langle W^h, a\Big(x, \xi, \frac{P_\Lambda \xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2), \mathscr{C}_c^\infty(T^*\mathbb{T}^2)} = \langle W_R^{h,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} + \langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda}. \tag{8-3}$$

The next two propositions are the analogues of [Fermanian-Kammerer 2000a] in our context. They state the existence of two-microlocal semiclassical measures, as the limit objects of $W_R^{h,\Lambda}$ and $W^h_{R,\Lambda}$.

Proposition 8.2. *There exists a subsequence (h, w_h) and a nonnegative measure $\nu^\Lambda \in \mathscr{M}^+(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)$ such that, for all $a \in S^1_\Lambda$, we have*

$$\lim_{R\to\infty}\lim_{h\to 0}\langle W_R^{h,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} = \Big\langle \nu^\Lambda, a_{\mathrm{hom}}\Big(x, \xi, \frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathscr{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda), \mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

To define the limit of the distributions $W^h_{R,\Lambda}$, we need first to introduce operator spaces and operator-valued measures, following [Gérard 1991]. Given a Hilbert space H (in the following, we shall use $H = L^2(\mathbb{T}_\Lambda)$), we denote respectively by $\mathscr{K}(H)$, $\mathscr{L}^1(H)$ the spaces of compact and trace class operators on H. We recall that they are both two-sided ideals of the ring $\mathscr{L}(H)$ of bounded operators on H. We refer for instance to [Reed and Simon 1980, Chapter VI.6] for a description of the space $\mathscr{L}^1(H)$ and its basic properties. Given a Polish space T (in the following, we shall use $T = T^*\mathbb{T}_{\Lambda^\perp}$), we denote by $\mathscr{M}^+(T; \mathscr{L}^1(H))$ the space of nonnegative measures on T, taking values in $\mathscr{L}^1(H)$. More precisely, we have $\rho \in \mathscr{M}^+(T; \mathscr{L}^1(H))$ if ρ is a bounded linear form on $\mathscr{C}^0_c(T)$ such that, for every nonnegative function $a \in \mathscr{C}^0_c(T)$, $\langle \rho, a\rangle_{\mathscr{M}(T), \mathscr{C}^0_c(T)} \in \mathscr{L}^1(H)$ is a nonnegative hermitian operator. As a consequence of [Reed and Simon 1980, Theorem VI.26], these measures can be identified in a natural way to nonnegative linear functionals on $\mathscr{C}^0_c(T; \mathscr{K}(H))$.

Proposition 8.3. *There exists a subsequence (h, w_h) and a nonnegative measure*

$$\rho_\Lambda \in \mathscr{M}^+\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathscr{L}^1(L^2(\mathbb{T}_\Lambda))\big),$$

such that, for all $K \in \mathscr{C}^\infty_c\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$, we have

$$\lim_{h\to 0}(K(s, hD_s)T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} = \operatorname{tr}\Big\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} K(s, \sigma)\rho_\Lambda(ds, d\sigma)\Big\}. \tag{8-4}$$

Moreover (for the same subsequence), for all $a \in S^1_\Lambda$, we have

$$\lim_{R\to\infty}\lim_{h\to 0}\langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} = \operatorname{tr}\Big\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathrm{Op}_1^\Lambda\big(a(\tilde\pi_\Lambda(\sigma, y, 0), \eta)\big)\rho_\Lambda(ds, d\sigma)\Big\}. \tag{8-5}$$

In the left-hand side of (8-4), the inner product actually means

$$\begin{aligned}(K(s, hD_s)&T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp} L^2(\mathbb{T}_\Lambda))}\\ &= \int_{s\in\mathbb{T}_{\Lambda^\perp},\, s'\in\Lambda^\perp,\, \sigma\in\Lambda^\perp} e^{\frac{i}{h}(s-s')\cdot\sigma}\Big(K\Big(\frac{s+s'}{2}, \sigma\Big)T_\Lambda w_h(s', y), T_\Lambda w_h(s, y)\Big)_{L^2_y(\mathbb{T}_\Lambda)}\, ds\, ds'\, d\sigma.\end{aligned}$$

In the expression (8-5), remark that for each $\sigma \in \Lambda^\perp$, the operator $\mathrm{Op}_1^\Lambda\big(a(\tilde\pi_\Lambda(\sigma, y, 0), \eta)\big)$ is in $\mathscr{L}(L^2(\mathbb{T}_\Lambda))$. Hence, its product with the operator $\rho_\Lambda(ds, d\sigma)$ defines a trace-class operator.

Before proving Propositions 8.2 and 8.3, we explain how to reconstruct the measure $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ from the two-microlocal measures ν^Λ and ρ_Λ. This reduces the study of the measure μ to that of all two-microlocal measures ν^Λ and ρ_Λ, for $\Lambda \in \mathcal{P}$.

We denote by $\mathcal{M}_c^+(T)$ the set of compactly supported measures on T, and by $\langle\cdot,\cdot\rangle_{\mathcal{M}_c(T),\mathscr{C}^0(T)}$ the associated duality bracket.

Proposition 8.4. *For all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ having only x-Fourier modes in Λ (i.e., for all $a \in S^1_\Lambda$ independent of the third variable $\eta \in \langle\Lambda\rangle$), we have*

$$\langle \mu, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} = \langle \nu^\Lambda, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} + \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds,d\sigma)\right\}, \tag{8-6}$$

and

$$\begin{aligned}\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle&_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}_c^0(T^*\mathbb{T}^2)} \\ &= \langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} + \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds,d\sigma)\right\},\end{aligned} \tag{8-7}$$

where for $\sigma \in \Lambda^\perp$, $m_{a\circ\tilde{\pi}_\Lambda}(\sigma)$ denotes the multiplication in $L^2(\mathbb{T}_\Lambda)$ by the function $y \mapsto a\circ\tilde{\pi}_\Lambda(\sigma,y)$.

Moreover, we have $\nu^\Lambda \in \mathcal{M}_c^+(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ and $\rho_\Lambda \in \mathcal{M}_c^+(T^*\mathbb{T}_{\Lambda^\perp};\mathscr{L}^1(L^2(\mathbb{T}_\Lambda)))$ (i.e., both measures are compactly supported).

Formula (8-7) follows immediately from (8-6) by restriction. By the definition of the measure ρ_Λ, we see that it is already supported on $\mathbb{T}^2\times\Lambda^\perp$ (see expression (8-2)).

The end of this section is devoted to the proofs of the three propositions, inspired by [Fermanian-Kammerer 2000a; Anantharaman and Macià 2010].

Proof of Proposition 8.2. The Calderón–Vaillancourt theorem implies that the operators

$$\mathrm{Op}_h\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right) = \mathrm{Op}_1\left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{R}\right)\right)a\left(x,h\xi,P_\Lambda\xi\right)\right)$$

are uniformly bounded as $h\to 0$ and $R\to+\infty$. It follows that the family $W_R^{h,\Lambda}$ is bounded in ${S^1_\Lambda}'$, and thus there exists a subsequence (h, w_h) and a distribution $\tilde{\mu}^\Lambda$ such that

$$\lim_{R\to\infty}\lim_{h\to 0}\langle W_R^{h,\Lambda}, a\rangle_{{S^1_\Lambda}',S^1_\Lambda} = \langle\tilde{\mu}^\Lambda, a(x,\xi,\eta)\rangle_{{S^1_\Lambda}',S^1_\Lambda}.$$

Because of the support properties of the function χ, we notice that $\langle\tilde{\mu}^\Lambda, a\rangle_{{S^1_\Lambda}',S^1_\Lambda} = 0$ as soon as the support of a is compact in the variable η. Hence, there exists a distribution $\nu^\Lambda \in \mathcal{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ such that

$$\langle\tilde{\mu}^\Lambda, a(x,\xi,\eta)\rangle_{{S^1_\Lambda}',S^1_\Lambda} = \left\langle \nu^\Lambda, a_{\mathrm{hom}}\left(x,\xi,\frac{\eta}{|\eta|}\right)\right\rangle_{\mathcal{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

Next, suppose that $a > 0$ (and that $\sqrt{1-\chi}$ is smooth). Then, using [Anantharaman and Macià 2010, Corollary 35], and setting

$$b^R(x,\xi) = \left(\left(1-\chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right)a\left(x,\xi,\frac{P_\Lambda\xi}{h}\right)\right)^{\frac{1}{2}},$$

there exists $C > 0$ such that for all $h \leq h_0$ and $R \geq 1$, we have

$$\left\| \mathrm{Op}_h\Big(\Big(1-\chi\Big(\frac{|P_\Lambda \xi|}{Rh}\Big)\Big)a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big) - \mathrm{Op}_h(b^R)^2\right\|_{\mathscr{L}(L^2(\mathbb{T}^2))} \leq \frac{C}{R}.$$

As a consequence, we have,

$$\langle W_R^{h,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} \geq \|\mathrm{Op}_h(b^R)w_h\|^2_{L^2(\mathbb{T}^2)} - \frac{C}{R}\|w_h\|^2_{L^2(\mathbb{T}^2)},$$

so that the limit $\langle \nu^\Lambda, a_{\mathrm{hom}}(x,\xi,\frac{\eta}{|\eta|})\rangle_{\mathscr{D}'(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^\infty(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}$ is nonnegative. The distribution ν^Λ is nonnegative, and is hence a measure. This concludes the proof of Proposition 8.2. □

Proof of Proposition 8.3. First, the proof of the existence of a subsequence (h, w_h) and the measure ρ_Λ satisfying (8-4) is the analogue of Proposition 5.1 in the context of operator valued measures, viewing the sequence w_h as a bounded sequence of $L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))$. It follows the lines of this result, after the adaptation of the symbolic calculus to operator-valued symbols (or more precisely, of [Gérard 1991] in the semiclassical setting).

Second, using the definition (8-2) together with (6-7), we have

$$\begin{aligned}\langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} &= \Big(\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} \\ &= \Big(\mathrm{Op}_h^{\Lambda^\perp}\circ \mathrm{Op}_h^\Lambda\Big(\chi\Big(\frac{|\eta|}{Rh}\Big)a\Big(\tilde{\pi}_\Lambda(\sigma,y,\eta),\frac{\eta}{h}\Big)\Big)T_\Lambda w_h, T_\Lambda w_h\Big)_{L^2(\mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda)}.\end{aligned}$$

Hence, setting

$$a^h_{R,\Lambda}(\sigma,y,\eta) = \chi\Big(\frac{|\eta|}{R}\Big)a(\tilde{\pi}_\Lambda(\sigma,y,h\eta),\eta),$$

we obtain

$$\langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} = \big(\mathrm{Op}_h^{\Lambda^\perp}\circ\mathrm{Op}_1^\Lambda(a^h_{R,\Lambda}(\sigma,y,\eta))T_\Lambda w_h, T_\Lambda w_h\big)_{L^2(\mathbb{T}_{\Lambda^\perp}\times\mathbb{T}_\Lambda)}.$$

We also notice that $\mathrm{Op}_1^\Lambda(a^h_{R,\Lambda}) \in \mathscr{K}(L^2(\mathbb{T}_\Lambda))$, for any $\sigma \in \Lambda^\perp$, since $a^h_{R,\Lambda}$ has compact support with respect to η. Moreover, for any $R > 0$ fixed and $a \in S^1_\Lambda$, the Calderón–Vaillancourt theorem yields

$$\mathrm{Op}_1^\Lambda(a^h_{R,\Lambda}) = \mathrm{Op}_1^\Lambda(a^0_{R,\Lambda}) + hB,$$

for some $B \in \mathscr{L}(L^2(\mathbb{T}_\Lambda))$, uniformly bounded with respect to h. Using (8-4), this implies that for any $R > 0$ fixed and $a \in S^1_\Lambda$, we have

$$\lim_{h\to 0}\langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} = \mathrm{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} \mathrm{Op}_1^\Lambda(a^0_{R,\Lambda})\rho_\Lambda(ds,d\sigma)\right\}.$$

Moreover, we have

$$\lim_{R\to+\infty}\mathrm{Op}_1^\Lambda(a^0_{R,\Lambda}) = \mathrm{Op}_1^\Lambda(a^0_{\infty,\Lambda}) = \mathrm{Op}_1^\Lambda\big(a(\tilde{\pi}_\Lambda(\sigma,y,0),\eta)\big),$$

in the strong topology of $\mathscr{C}_c^\infty\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathscr{L}(L^2(\mathbb{T}_\Lambda))\big)$. This proves (8-5) and concludes the proof of Proposition 8.3. □

Proof of Proposition 8.4. Taking $a \in S^1_\Lambda$ independent of the third variable $\eta \in \langle \Lambda \rangle$ gives

$$\langle W^h, a(x,\xi)\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}^\infty_c(T^*\mathbb{T}^2)} \to \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathscr{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)},$$

together with

$$\langle W^{h,\Lambda}_R, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} \to \langle \nu^\Lambda, a\rangle_{\mathscr{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)},$$

(according to Proposition 8.2) and

$$\langle W^h_{R,\Lambda}, a\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} \to \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} \operatorname{Op}^\Lambda_1\left(a(\tilde{\pi}_\Lambda(\sigma,y,0))\right)\rho_\Lambda(ds,d\sigma)\right\} = \operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{a\circ\tilde{\pi}_\Lambda}(\sigma)\rho_\Lambda(ds,d\sigma)\right\}$$

(according to Proposition 8.3). Now, using the last three equations together with (8-3) directly gives (8-7).

As both terms in the right hand-side of (8-7) are nonnegative measures and the left-hand side is a compactly supported nonnegative measure, this implies that ν^Λ and ρ_Λ are both compactly supported. □

9. Propagation laws for the two-microlocal measures ν^Λ and ρ_Λ

In this section, we study the propagation properties of ν^Λ and ρ_Λ defined in Propositions 8.2 and 8.3, respectively. The key point here is the use of the cutoff function introduced in Proposition 7.3.

We will use repeatedly this fact, which follows from item (2) in Proposition 7.3: if A is a bounded operator on $L^2(\mathbb{T}^2)$, we have

$$(Aw_h, w_h)_{L^2(\mathbb{T}^2)} = (A\chi^\Lambda_h w_h, \chi^\Lambda_h w_h)_{L^2(\mathbb{T}^2)} + \|A\|_{\mathscr{L}(L^2)}\, o(1). \tag{9-1}$$

To simplify the notation, we shall write $A_{c_0,h}$ for $\chi^\Lambda_h A \chi^\Lambda_h$.

9A. *Propagation of ν^Λ.* We define for $(x,\xi,\eta) \in T^*\mathbb{T}^2 \times \langle\Lambda\rangle$ and $\tau \in \mathbb{R}$ the flows

$$\phi^0_\tau(x,\xi,\eta) := (x+\tau\xi, \xi, \eta),$$

generated by the vector field $\xi \cdot \partial_x$ and, for $\eta \neq 0$,

$$\phi^1_\tau(x,\xi,\eta) := \left(x + \tau\frac{\eta}{|\eta|}, \xi, \eta\right)$$

generated by the vector field $(\eta/|\eta|)\cdot\partial_x$. With these definitions, we have the following propagation laws for the two-microlocal measure ν^Λ.

Proposition 9.1. *The measure ν^Λ is ϕ^0_τ- and ϕ^1_τ-invariant, that is,*

$$(\phi^0_\tau)_*\nu^\Lambda = \nu^\Lambda \quad \textit{and} \quad (\phi^1_\tau)_*\nu^\Lambda = \nu^\Lambda \quad \textit{for every } \tau \in \mathbb{R}.$$

The key result here is the additional "transverse propagation law" given by the flow ϕ^1_τ. The measure ν^Λ not only propagates along the geodesic flow ϕ^0_τ, but also along directions transverse to $\Lambda^\perp$.

Proof. Fix $a \in S^1_\Lambda$. The computation done in (5-6) is still valid replacing a by

$$\left(1 - \chi\left(\frac{|P_\Lambda\xi|}{Rh}\right)\right) a\left(x, \xi, \frac{P_\Lambda\xi}{h}\right),$$

since it only uses the fact that $\mathrm{Op}_h((1-\chi(|P_\Lambda\xi|/Rh))a(x,\xi,P_\Lambda\xi/h))$ is bounded and that $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)}=o(h)$ and $(bw_h,w_h)_{L^2(\mathbb{T}^2)}=o(1)$. This yields

$$\lim_{h\to 0}\langle W_R^{h,\Lambda},\xi\cdot\partial_x a\rangle_{S_\Lambda^{1\,\prime},S_\Lambda^1}=\lim_{h\to 0}\Big\langle W^h,\xi\cdot\partial_x\Big\{\Big(1-\chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)\Big)a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big\}\Big\rangle_{\mathcal{D}'(T^*\mathbb{T}^2),\mathscr{C}_c^\infty(T^*\mathbb{T}^2)}=0,$$

and hence, in the limit $R\to+\infty$, we obtain

$$\Big\langle \nu^\Lambda,\xi\cdot\partial_x a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0.$$

Replacing a_{hom} by $a_{\mathrm{hom}}\circ\phi_\tau^0$ and integrating with respect to the parameter τ gives $(\phi_\tau^0)_*\nu^\Lambda=\nu^\Lambda$, which concludes the first part of the proof.

Second, to prove the ϕ_τ^1-invariance of ν^Λ we compute

$$\Big\langle \nu^\Lambda,\frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=\lim_{R\to\infty}\lim_{h\to 0}\Big\langle W_R^{h,\Lambda},\frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S_\Lambda^{1\,\prime},S_\Lambda^1}. \tag{9-2}$$

Setting

$$a^R(x,\xi,\eta)=\frac{1}{|\eta|}\Big(1-\chi\Big(\frac{|\eta|}{R}\Big)\Big)a(x,\xi,\eta)$$

and

$$A^R:=\mathrm{Op}_h\Big(a^R\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big) \tag{9-3}$$

we have the relation

$$\Big\langle W_R^{h,\Lambda},\frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S_\Lambda^{1\,\prime},S_\Lambda^1}=-\frac{i}{2}([\Delta_\Lambda,A^R]w_h,w_h)_{L^2(\mathbb{T}^2)},$$

where $\Delta_\Lambda=\partial_y^2$ is the laplacian in the direction Λ.

Lemma 9.2. *For any given $c_0>0$ and $R>0$, we have*

$$([\Delta_\Lambda,A^R]w_h,w_h)_{L^2(\mathbb{T}^2)}=([\Delta_\Lambda,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}+o(1).$$

We postpone the proof of Lemma 9.2 and first indicate how it allows us to prove Proposition 9.1. We now know that

$$\Big\langle \nu^\Lambda,\frac{\eta}{|\eta|}\cdot\partial_x a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}_c^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=\lim_{R\to\infty}\lim_{h\to 0}-\frac{i}{2}([\Delta_\Lambda,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}.$$

Recall that $a\in S^1_\Lambda$ implies that a has only x-Fourier modes in Λ, i.e., $P_\Lambda\xi\cdot\partial_x a=\xi\cdot\partial_x a$. We have also assumed in this section that b has only x-Fourier modes in Λ. As a consequence, we have

$$\begin{aligned}-\frac{i}{2}([\Delta_\Lambda,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}&=-\frac{i}{2}([\Delta,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}\\&=\frac{i}{2h^2}([P_0^h,A^R_{c_0,h}]w_h,w_h)_{L^2(\mathbb{T}^2)}.\end{aligned} \tag{9-4}$$

Developing the last expression of (9-4), we obtain

$$\frac{i}{2h^2}([P_0^h, A_{c_0,h}^R]w_h, w_h)_{L^2(\mathbb{T}^2)} = \frac{i}{2h^2}(A_{c_0,h}^R w_h, P_b^h w_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h^2}(A_{c_0,h}^R P_b^h w_h, w_h)_{L^2(\mathbb{T}^2)} \\ - \frac{1}{2h}(A_{c_0,h}^R w_h, bw_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A_{c_0,h}^R bw_h, w_h)_{L^2(\mathbb{T}^2)}. \quad (9\text{-}5)$$

Since $A_{c_0,h}^R$ is bounded in $\mathscr{L}(L^2(\mathbb{T}^2))$, its adjoint $A_{c_0,h}^R$ is also bounded so that the first two terms in the last expression vanish in the limit $h \to 0$, using $\|P_b^h w_h\|_{L^2(\mathbb{T}^2)} = o(h^2)$. To estimate the last two terms, we use again the boundedness of A^R and $(A^R)^*$ and write

$$|(A_{c_0,h}^R w_h, bw_h)_{L^2(\mathbb{T}^2)}| \le \|A^R\| \|\chi_h^\Lambda bw_h\|_{L^2(\mathbb{T}^2)} \le 2c_0 h \; \|A^R\|,$$

according to item (3) in Proposition 7.3. It follows that

$$\limsup_{h\to 0} \left| \frac{1}{2h}(A_{c_0,h}^R w_h, bw_h)_{L^2(\mathbb{T}^2)} + \frac{1}{2h}(A_{c_0,h}^R bw_h, w_h)_{L^2(\mathbb{T}^2)} \right| \le 2c_0 \sup \|A^R\|.$$

Coming back to the expression (9-2), we obtain

$$\left| \left\langle \nu^\Lambda, \frac{\eta}{|\eta|} \cdot \partial_x a_{\mathrm{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right) \right\rangle_{\mathcal{M}(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda), \mathscr{C}_c^0(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)} \right| \le 2c_0 \sup \|A^R\|$$

and since c_0 was arbitrary,

$$\left\langle \nu^\Lambda, \frac{\eta}{|\eta|} \cdot \partial_x a_{\mathrm{hom}}\left(x, \xi, \frac{\eta}{|\eta|}\right) \right\rangle_{\mathcal{M}(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda), \mathscr{C}_c^0(T^*\mathbb{T}^2 \times \mathbb{S}_\Lambda)} = 0.$$

Replacing a_{hom} by $a_{\mathrm{hom}} \circ \phi_\tau^1$ and integrating with respect to the parameter τ gives $(\phi_\tau^1)_* \nu^\Lambda = \nu^\Lambda$, which concludes the proof of Proposition 9.1. □

Proof of Lemma 9.2. We are going to show that

$$([\Delta_\Lambda, A_{c_0,h}^R]w_h, w_h)_{L^2(\mathbb{T}^2)} = ([\Delta_\Lambda, A^R]_{c_0,h} w_h, w_h)_{L^2(\mathbb{T}^2)} + o(1). \quad (9\text{-}6)$$

Then, using the fact that $[\Delta_\Lambda, A^R]$ is a bounded operator (its symbol is $\left(1 - \chi\left(\frac{|\eta|}{R}\right)\right)\frac{\eta}{|\eta|} \cdot \partial_x a(x, \xi, \eta))$ together with (9-1), this is also $([\Delta_\Lambda, A^R]w_h, w_h)_{L^2(\mathbb{T}^2)} + o(1)$.

To prove (9-6), we develop the difference $[\Delta_\Lambda, A_{c_0,h}^R] - [\Delta_\Lambda, A^R]_{c_0,h}$ as

$$[\Delta_\Lambda, A_{c_0,h}^R] - [\Delta_\Lambda, A^R]_{c_0,h} = [\partial_y^2, \chi_h^\Lambda] A^R \chi_h^\Lambda + \chi_h^\Lambda A^R [\partial_y^2, \chi_h^\Lambda]. \quad (9\text{-}7)$$

Then, writing

$$[\partial_y^2, \chi_h^\Lambda] = \partial_y^2 \chi_h^\Lambda + 2\partial_y \chi_h^\Lambda \partial_y,$$

we have

$$([\partial_y^2, \chi_h^\Lambda] A^R \chi_h^\Lambda w_h, w_h)_{L^2(\mathbb{T}^2)} = (A^R \chi_h^\Lambda w_h, \partial_y^2 \chi_h^\Lambda w_h)_{L^2(\mathbb{T}^2)} + (\partial_y \circ A^R \chi_h^\Lambda w_h, 2\partial_y \chi_h^\Lambda w_h)_{L^2(\mathbb{T}^2)}.$$

Recalling that the operator $\partial_y \circ A^R$ is bounded, and using items (4) and (5) in Proposition 7.3, we obtain

$$\left|([\partial_y^2, \chi_h^\Lambda] A^R \chi_h^\Lambda w_h, w_h)_{L^2(\mathbb{T}^2)}\right| \le C\|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} + C\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1).$$

The last term in (9-7) is handled similarly. This finally implies (9-6), concluding the proof. □

9B. ***Propagation of ρ_Λ.*** We denote by $(\omega_\Lambda^j, e_\Lambda^j)_{j\in\mathbb{N}}$ the eigenvalues and associated eigenfunctions of the operator $-\Delta_\Lambda = -\partial_y^2$ forming a Hilbert basis of $L^2(\mathbb{T}_\Lambda)$. We shall use the projector onto low frequencies of $-\Delta_\Lambda$, that is, for any $\omega \in \mathbb{R}_+$, the operator

$$\Pi_\Lambda^\omega := \sum_{\omega_\Lambda^j \le \omega} (\,\cdot\,, e_\Lambda^j)_{L^2(\mathbb{T}_\Lambda)} e_\Lambda^j,$$

which has finite rank.

We have the following propagation laws for the two-microlocal measure ρ_Λ.

Proposition 9.3. (1) *For any $K \in \mathscr{C}_c^\infty\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$ independent of s (i.e., $K(s,\sigma) = K(\sigma)$) and any $\omega > 0$, we have*

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega]\, \rho_\Lambda(ds, d\sigma)\right\} = 0.$$

(2) *Defining*

$$M_\Lambda := \int_{\mathbb{T}_{\Lambda^\perp}\times\Lambda^\perp} \rho_\Lambda(ds, d\sigma) \in \mathscr{L}^1(L^2(\mathbb{T}_\Lambda)),$$

we have

$$[\Delta_\Lambda, M_\Lambda] = 0.$$

Remark that for any $\sigma \in \Lambda^\perp$, the operator

$$[\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega] = \Pi_\Lambda^\omega [\Delta_\Lambda, K(\sigma)]\, \Pi_\Lambda^\omega$$

has finite rank, so the right-hand side of item (1) is well defined. Note that the definition of M_Λ is meaningful since ρ_Λ has a compact support according to Proposition 8.4.

The commutation relations of items (1) and (2) in this proposition correspond to propagation laws at the operator level. They are formulated here in a "derivated form", which, for item (2) for instance, is equivalent to

$$e^{i\tau\Delta_\Lambda} M_\Lambda e^{-i\tau\Delta_\Lambda} = M_\Lambda \quad \text{for all } \tau \in \mathbb{R},$$

in the "integrated form".

Proof of Proposition 9.3. For $K \in \mathscr{C}_c^\infty\big(\Lambda^\perp; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$ (in other words $K \in \mathscr{C}_c^\infty\big(T^*\mathbb{T}_{\Lambda^\perp}; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$ independent of $s \in \mathbb{T}_{\Lambda^\perp}$), we denote

$$K^\omega(\sigma) := \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega$$

and we note that K^ω is also in $\mathscr{C}_c^\infty\big(\Lambda^\perp; \mathscr{K}(L^2(\mathbb{T}_\Lambda))\big)$. Hence, we have

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, \Pi_\Lambda^\omega K(\sigma)\Pi_\Lambda^\omega]\, \rho_\Lambda(ds, d\sigma)\right\} = -\lim_{h\to 0}([-\Delta_\Lambda, K^\omega(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}.$$

To show that this limit vanishes, we proceed as in (9-4), (9-5) and in the subsequent calculation, replacing the operator A^R by $K^\omega(hD_s)$.

With the notation $\Delta_\Lambda = \partial_y^2$ and $\Delta_{\Lambda^\perp} = \partial_s^2$, we first note that

$$([-\Delta_\Lambda, K^\omega(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} = ([-\Delta, K^\omega(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))},$$

since $\Delta = \Delta_\Lambda + \Delta_{\Lambda^\perp}$ and since $[\Delta_{\Lambda^\perp}, K^\omega(hD_s)] = 0$. As a matter of fact, $K^\omega(hD_s) = \mathrm{Op}_h^\Lambda(K^\omega(\sigma))$ and $\Delta_{\Lambda^\perp} = -h^{-2}\,\mathrm{Op}_h^\Lambda(|\sigma|^2)$ are both Fourier multipliers.

The following lemma is proved the same way as Lemma 9.2.

Lemma 9.4. *For any given $c_0 > 0$, we have*

$$([\Delta_\Lambda, K^\omega(\sigma)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}^2)} = ([\Delta_\Lambda, K^\omega_{c_0,h}(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}^2)} + o(1).$$

Here $K^\omega_{c_0,h}(hD_s)$ means $\chi_h^\Lambda K^\omega(hD_s)\chi_h^\Lambda$.

Writing

$$-h^2\Delta = T_\Lambda P_b^h T_\Lambda^* - ihb\circ\pi_\Lambda,$$

we have

$$\begin{aligned}
&([-\Delta, K^\omega_{c_0,h}(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}\\
&\quad = \frac{1}{h^2}(K^\omega_{c_0,h}(hD_s) T_\Lambda w_h, T_\Lambda P_b^h w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} - \frac{1}{h^2}(K^\omega_{c_0,h}(hD_s) T_\Lambda P_b^h w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}\\
&\qquad + \frac{i}{h}(K^\omega_{c_0,h}(hD_s) T_\Lambda w_h, T_\Lambda(bw_h))_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))} + \frac{i}{h}(K^\omega_{c_0,h}(hD_s) T_\Lambda(bw_h), T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}.
\end{aligned}$$

It follows, as in (9-5), that

$$\limsup_{h\to 0}\left|([-\Delta, K^\omega_{c_0,h}(hD_s)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}_{\Lambda^\perp}; L^2(\mathbb{T}_\Lambda))}\right| \le 2c_0\|K\|$$

and since c_0 was arbitrary, we can conclude that

$$\lim_{h\to 0}([\Delta_\Lambda, K^\omega(\sigma)]\, T_\Lambda w_h, T_\Lambda w_h)_{L^2(\mathbb{T}^2)} = 0,$$

which concludes the proof of item (1).

Item (1) gives, for all $K \in \mathcal{K}(L^2(\mathbb{T}_\Lambda))$ constant (which is possible since $\rho_\Lambda(ds, d\sigma)$ has compact support),

$$0 = \mathrm{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} [\Delta_\Lambda, K^\omega]\,\rho_\Lambda(ds, d\sigma)\right\} = \mathrm{tr}\left\{[\Delta_\Lambda, K^\omega]\int_{T^*\mathbb{T}_{\Lambda^\perp}} \rho_\Lambda(ds, d\sigma)\right\} = \mathrm{tr}\{[\Delta_\Lambda, K^\omega]\, M_\Lambda\}.$$

Using that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for all $A \in \mathcal{L}^1$ and $B \in \mathcal{L}$ together with the linearity of the trace (see [Reed and Simon 1980, Theorem VI.25]), we now obtain, for all $K \in \mathcal{K}(L^2(\mathbb{T}_\Lambda))$ and all $\omega > 0$,

$$0 = \mathrm{tr}\{[\Delta_\Lambda, \Pi^\omega_\Lambda K \Pi^\omega_\Lambda]\, M_\Lambda\} = \mathrm{tr}\{K\Pi^\omega_\Lambda[\Delta_\Lambda, M_\Lambda]\,\Pi^\omega_\Lambda\}.$$

Consequently, we have $\Pi^\omega_\Lambda[\Delta_\Lambda, M_\Lambda]\,\Pi^\omega_\Lambda = 0$ for all $\omega > 0$ (see [Reed and Simon 1980, Theorem VI.26]). Letting ω go to $+\infty$, this yields $[\Delta_\Lambda, M_\Lambda] = 0$ and concludes the proof of item (2). □

10. The measures ν^Λ and ρ_Λ vanish identically. End of the proof of Theorem 2.6

In this section, we prove that both measures ν^Λ and ρ_Λ vanish when paired with the function $\langle b\rangle_\Lambda$. Then, we deduce that these two measures vanish identically. In turn, this implies that $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$, and finally that $\mu=0$, which will conclude the proof of Theorem 2.6.

Proposition 10.1. *We have*

$$\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0 \quad \textit{and} \quad \operatorname{tr}\{m_{\langle b\rangle_\Lambda}M_\Lambda\}=0.$$

As a consequence, we prove that ρ_Λ and $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}$ vanish.

Proposition 10.2. *We have $\rho_\Lambda=0$ and $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}=0$. Hence $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$.*

This allows us to conclude the proof of Theorem 2.6. Indeed, as a consequence of the decomposition formula of Proposition 8.4, we obtain $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$ for all $\Lambda\in\mathcal{P}$ such that $\operatorname{rk}(\Lambda)=1$. Using the decomposition of the measure μ given in Lemma 6.1 together with Lemma 6.4, this yields $\mu=0$ on $\mathbb{T}^2$. This is in contradiction with $\mu(T^*\mathbb{T}^2)=1$ (Proposition 5.2), and this contradiction proves Theorem 2.6.

Proof of Proposition 10.1. First, (5-2) implies that $(bv_h, v_h)_{L^2(\mathbb{T}^2)}\to 0$, and hence

$$\langle \mu, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0.$$

Then the decomposition given in Lemma 6.1 into a sum of nonnegative measures yields that, for all $\Lambda\in\mathcal{P}$,

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0, \tag{10-1}$$

since b is also nonnegative. Lemmata 6.2, 6.3 and 6.4 (see also Remark 6.5), then give

$$\begin{aligned}\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)} &= \langle \mu|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}\\ &= \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}=0,\end{aligned} \tag{10-2}$$

where the function $\langle b\rangle_\Lambda$ is also nonnegative. The decomposition formula of Proposition 8.4 into the two-microlocal semiclassical measures then yields

$$\begin{aligned}&\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2),\mathscr{C}^0(T^*\mathbb{T}^2)}\\ &\qquad= \langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}+\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}.\end{aligned}$$

Since the measure $\nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}$ is nonnegative, we get $\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}\geq 0$. Similarly, $\rho_\Lambda\in\mathcal{M}_c^+(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{L}^1(L^2(\mathbb{T}_\Lambda)))$ and the operator $m_{\langle b\rangle_\Lambda}\in\mathcal{L}(L^2(\mathbb{T}_\Lambda))$ is selfadjoint and nonnegative, which gives $\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}\geq 0$. Using (10-1) and (10-2), this yields

$$\langle \nu^\Lambda|_{\mathbb{T}^2\times\Lambda^\perp\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}=0$$

and

$$\operatorname{tr}\left\{\int_{T^*\mathbb{T}_{\Lambda^\perp}} m_{\langle b\rangle_\Lambda}\rho_\Lambda(ds,d\sigma)\right\}=0.$$

In this expression, the operator $m_{\langle b\rangle_\Lambda}$ does not depend on (s,σ), so

$$0=\operatorname{tr}\left\{m_{\langle b\rangle_\Lambda}\int_{T^*\mathbb{T}_{\Lambda^\perp}}\rho_\Lambda(ds,d\sigma)\right\}=\operatorname{tr}\{m_{\langle b\rangle_\Lambda}M_\Lambda\},$$

which concludes the proof of Proposition 10.1. □

Proof of Proposition 10.2. Let us first prove that $\rho_\Lambda=0$. We recall that the operator M_Λ is a selfadjoint nonnegative trace-class operator. Moreover, Proposition 9.3 implies that the operators M_Λ and Δ_Λ commute. As a consequence, there exists a Hilbert basis $(\tilde e^j_\Lambda)_{j\in\mathbb{N}}$ of $L^2(\mathbb{T}_\Lambda)$ in which M_Λ and Δ_Λ are simultaneously diagonal, i.e., such that

$$-\Delta_\Lambda\tilde e^j_\Lambda=\omega^j_\Lambda\tilde e^j_\Lambda\quad\text{and}\quad M_\Lambda\tilde e^j_\Lambda=\gamma^j_\Lambda\tilde e^j_\Lambda,$$

where $(\gamma^j_\Lambda)_{j\in\mathbb{N}}$ are the associated eigenvalues of M_Λ. In particular, we have $\gamma^j_\Lambda\geq 0$ for all $j\in\mathbb{N}$ (and $\gamma^j_\Lambda\in\ell^1$). Note that the basis $(\tilde e^j_\Lambda)_{j\in\mathbb{N}}$ is not necessarily the same as the basis $(e^j_\Lambda)_{j\in\mathbb{N}}$ introduced in Section 9B.

Using Proposition 10.1, together with the definition of the trace (see, for instance, [Reed and Simon 1980, Theorem VI.18]) we have

$$0=\operatorname{tr}\{m_{\langle b\rangle_\Lambda}M_\Lambda\}=\sum_{j\in\mathbb{N}}(m_{\langle b\rangle_\Lambda}M_\Lambda\tilde e^j_\Lambda,\tilde e^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)}=\sum_{j\in\mathbb{N}}\gamma^j_\Lambda(\langle b\rangle_\Lambda\tilde e^j_\Lambda,\tilde e^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)}.$$

Since all terms in this sum are nonnegative (because both γ^j_Λ and $\langle b\rangle_\Lambda$ are), we deduce that for all $j\in\mathbb{N}$,

$$\gamma^j_\Lambda(\langle b\rangle_\Lambda\tilde e^j_\Lambda,\tilde e^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)}=0.$$

Suppose that $\gamma^j_\Lambda\neq 0$ for some $j\in\mathbb{N}$. Then, $(\langle b\rangle_\Lambda\tilde e^j_\Lambda,\tilde e^j_\Lambda)_{L^2(\mathbb{T}_\Lambda)}=0$ where $\langle b\rangle_\Lambda$ is nonnegative and not identically zero on $\mathbb{T}_\Lambda$. This yields $\tilde e^j_\Lambda=0$ on the nonempty open set $\{\langle b\rangle_\Lambda>0\}$. Using a unique continuation property for eigenfunctions of the Laplace operator on $\mathbb{T}_\Lambda$, we finally obtain that the eigenfunction $\tilde e^j_\Lambda$ vanishes identically on $\mathbb{T}_\Lambda$. This is absurd, and thus we must have $\gamma^j_\Lambda=0$ for all $j\in\mathbb{N}$, so that $M_\Lambda=0$. Since $\rho_\Lambda\in\mathcal{M}^+(T^*\mathbb{T}_{\Lambda^\perp};\mathcal{L}^1(L^2(\mathbb{T}_\Lambda)))$, this directly gives $\rho_\Lambda=0$.

Next, we prove that $\nu^\Lambda=0$. This is a consequence of the additional propagation law of ν^Λ with respect to the flow ϕ^1_τ (see Section 9A). Indeed the torus $\mathbb{T}_\Lambda$ has dimension one, $(\phi^1_\tau)_*\nu^\Lambda=\nu^\Lambda$ (according to Proposition 9.1) and, using Proposition 10.1, ν^Λ vanishes on the (nonempty) set $\{\langle b\rangle_\Lambda>0\}\times\mathbb{R}^2\times\mathbb{S}_\Lambda$ (with $\{\langle b\rangle_\Lambda>0\}$ clearly satisfying GCC on $\mathbb{T}_\Lambda$). Hence, $\nu^\Lambda=0$.

To conclude the proof of Proposition 10.2, it only remains to use the decomposition formula (8-7) which directly yields $\mu|_{\mathbb{T}^2\times\Lambda^\perp}=0$. □

11. Proof of Proposition 7.2

In this section, we prove Proposition 7.2. For this, we consider two-microlocal semiclassical measures at the scale h^α. The setting is close to that of [Fermanian Kammerer 2005].

We shall see that the concentration rate of the sequence v_h towards the direction $\Lambda^\perp$ is of the form h^α for all $\alpha\leq(1+\delta)/2$.

First, Lemma 6.3 yields $\mu|_{\mathbb{T}^2\times\Lambda^\perp} = \langle\mu\rangle_\Lambda|_{\mathbb{T}^2\times\Lambda^\perp}$ (see also Remark 6.5); that is,

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, \langle a\rangle_\Lambda\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)},$$

and it suffices to characterize the action of $\mu|_{\mathbb{T}^2\times\Lambda^\perp}$ on $\Lambda^\perp$-invariant symbols. Recall that, for all $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$,

$$\langle \mu, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}(\mathrm{Op}_h(a)v_h, v_h)_{L^2(\mathbb{T}^2)}.$$

As in (8-1) and (8-2), let us define

$$\langle V_R^{h,\Lambda}, a\rangle_{S^{1\prime}_\Lambda,S^1_\Lambda} := \Big\langle V^h, \Big(1-\chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)\Big)a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}^\infty_c(T^*\mathbb{T}^2)}, \tag{11-1}$$

$$\langle V_{R,\Lambda}^{h}, a\rangle_{S^{1\prime}_\Lambda,S^1_\Lambda} := \Big\langle V^h, \chi\Big(\frac{|P_\Lambda\xi|}{Rh}\Big)a\Big(x,\xi,\frac{P_\Lambda\xi}{h}\Big)\Big\rangle_{\mathscr{D}'(T^*\mathbb{T}^2),\mathscr{C}^\infty_c(T^*\mathbb{T}^2)}, \tag{11-2}$$

for $a \in S^1_\Lambda$.

We take $R = R(h) = h^{-(1-\alpha)}$ for some $\alpha \in (0,1)$, so that $Rh = h^\alpha$. The proof of Proposition 8.2 applies verbatim and shows the existence of a subsequence (h, v_h) and a nonnegative measure $\nu_\alpha^\Lambda \in \mathcal{M}^+(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)$ such that, for all $a \in S^1_\Lambda$, we have

$$\lim_{h\to 0}\langle V_{R(h)}^{h,\Lambda}, a\rangle_{S^{1\prime}_\Lambda,S^1_\Lambda} = \Big\langle \nu_\alpha^\Lambda, a_{\mathrm{hom}}\Big(x,\xi,\frac{\eta}{|\eta|}\Big)\Big\rangle_{\mathcal{M}(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)}.$$

Proposition 11.1. *Let $R(h) = h^{-(1-\alpha)}$ with $\alpha \le (1+\delta)/2$. Then*

$$\nu_\alpha^\Lambda|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda} = 0.$$

The proof of Proposition 11.1 relies on the following propagation result.

Lemma 11.2. *For $\alpha \le (1+\delta)/2$ the measure ν_α^Λ is ϕ^0_τ- and ϕ^1_τ-invariant*:

$$(\phi^0_\tau)_*\nu_\alpha^\Lambda = \nu_\alpha^\Lambda \quad \textit{and} \quad (\phi^1_\tau)_*\nu_\alpha^\Lambda = \nu_\alpha^\Lambda \quad \textit{for every } \tau \in \mathbb{R}.$$

The proof is very similar to that of Proposition 9.1 but does not use assumption (2-13).

Proof. The proof of ϕ^0_τ-invariance is strictly identical to what has been done for Proposition 9.1 and thus we focus on the ϕ^1_τ-invariance. Equation (9-5) still holds with $R(h) = h^{-(1-\alpha)}$, now reading

$$\Big\langle V_{R(h)}^{h,\Lambda}, \frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S^{1\prime}_\Lambda,S^1_\Lambda} = \frac{i}{2h^2}(A^{R(h)}v_h, P^h_b v_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h^2}(A^{R(h)}P^h_b v_h, v_h)_{L^2(\mathbb{T}^2)} \\ - \frac{1}{2h}(A^{R(h)}v_h, bv_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A^{R(h)}bv_h, v_h)_{L^2(\mathbb{T}^2)},$$

where A^R was defined in (9-3). Using $\|P^h_b v_h\|_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$ together with the boundedness of $A^{R(h)}$, it follows that

$$\lim_{h\to 0}\Big\langle V_{R(h)}^{h,\Lambda}, \frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S^{1\prime}_\Lambda,S^1_\Lambda} = \lim_{h\to 0}\Big(-\frac{1}{2h}(A^{R(h)}v_h, bv_h)_{L^2(\mathbb{T}^2)} - \frac{1}{2h}(A^{R(h)}bv_h, v_h)_{L^2(\mathbb{T}^2)}\Big). \tag{11-3}$$

Recall from (5-2) that $\|\sqrt{b}v_h\|_{L^2(\mathbb{T}^2)} = o(h^{(1+\delta)/2})$. In addition, with $R(h) = h^{-(1-\alpha)}$ we have

$$A^{R(h)} = \mathrm{Op}_1(\tilde{a}_h), \quad \tilde{a}_h(x,\xi) = \frac{1}{|P_\Lambda \xi|}\big(1 - \chi(h^{(1-\alpha)}|P_\Lambda\xi|)\big)a(x, h\xi, P_\Lambda\xi),$$

where $a \in S^1_\Lambda$ is homogeneous of order zero in the third variable and P_Λ is defined in (6-3). Since $h^{1-\alpha}|P_\Lambda\xi| \geq 1$ on $\mathrm{supp}(1-\chi)$, the symbol $\tilde{a}_h$ satisfies

$$|\partial_x^{\beta'}\partial_\xi^{\beta}\tilde{a}_h| \leq C_{\beta,\beta'}h^{1-\alpha}h^{|\beta|(1-\alpha)}.$$

Hence, the Calderón–Vaillancourt theorem (see for instance Theorem A.1) yields $\|A^{R(h)}\|_{\mathscr{L}(L^2)} \leq Ch^{1-\alpha}$, which implies

$$\left|\frac{1}{2h}(A^{R(h)}v_h, bv_h)_{L^2(\mathbb{T}^2)}\right| \leq Ch^{-1}\|A^{R(h)}\|_{\mathscr{L}(L^2)}\|v_h\|_{L^2(\mathbb{T}^2)}\|\sqrt{b}v_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{1+\delta}{2}-\alpha}).$$

Coming back to (11-3), this finally gives

$$\lim_{h\to 0}\Big\langle V^{h,\Lambda}_{R(h)}, \frac{\eta}{|\eta|}\cdot\partial_x a\Big\rangle_{S^{1\,\prime}_\Lambda, S^1_\Lambda} = 0,$$

as soon as $\alpha \leq (1+\delta)/2$. □

Proof of Proposition 11.1. We have $\langle \nu^\Lambda_\alpha|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda}, \langle b\rangle_\Lambda\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0$, since ν^Λ_α is (ϕ^0_τ)-invariant and $\langle \nu^\Lambda_\alpha, b\rangle_{\mathcal{M}_c(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda),\mathscr{C}^0(T^*\mathbb{T}^2\times\mathbb{S}_\Lambda)} = 0$. Then, the ϕ^1_τ-invariance of ν^Λ_α implies that $\nu^\Lambda_\alpha|_{\mathbb{T}^2\times(\Lambda^\perp\setminus\{0\})\times\mathbb{S}_\Lambda}$ vanishes. □

Proof of Proposition 7.2. Proposition 11.1 implies that

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\Big(\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)a(x,\xi)\Big)v_h, v_h\Big)_{L^2(\mathbb{T}^2)}$$

for all $\alpha \leq (1+\delta)/2$ and $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$. The same holds if we replace χ by χ^2:

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\Big(\mathrm{Op}_h\Big(\chi^2\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)a(x,\xi)\Big)v_h, v_h\Big)_{L^2(\mathbb{T}^2)}.$$

Since

$$\mathrm{Op}_h\Big(\chi^2\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)a(x,\xi)\Big) = \mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big)\,\mathrm{Op}_h(a)\,\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big) + \mathbb{O}(h^{1-\alpha}), \tag{11-4}$$

we obtain

$$\langle \mu|_{\mathbb{T}^2\times\Lambda^\perp}, a\rangle_{\mathcal{M}(T^*\mathbb{T}^2),\mathscr{C}^0_c(T^*\mathbb{T}^2)} = \lim_{h\to 0}\Big(\mathrm{Op}_h(a)\,\mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big)v_h, \mathrm{Op}_h\Big(\chi\Big(\frac{|P_\Lambda\xi|}{h^\alpha}\Big)\Big)v_h\Big)_{L^2(\mathbb{T}^2)},$$

for all $\alpha \leq (1+\delta)/2$ and $a \in \mathscr{C}^\infty_c(T^*\mathbb{T}^2)$. □

12. Proof of Proposition 7.3: existence of the cutoff function

Given a constant $c_0 > 0$, we define the following subsets of $\mathbb{T}^2$:

$$\mathscr{E}_h = \langle \{b > c_0 h\} \rangle_\Lambda, \quad \mathscr{F}_h = \left\langle \bigcup_{x \in \{b > c_0 h\}} B(x, (c_0 h)^{2\varepsilon}) \right\rangle_\Lambda = \bigcup_{x \in \mathscr{E}_h} B(x, (c_0 h)^{2\varepsilon}), \quad \mathscr{G}_h = \mathscr{F}_h \setminus \mathscr{E}_h,$$

where for $U \subset \mathbb{T}^2$, we denote $\langle U \rangle_\Lambda := \bigcup_{\tau \in \mathbb{R}} \{U + \tau\sigma\}$ for some $\sigma \in \Lambda^\perp \setminus \{0\}$. Remark that $\mathscr{E}_h \subset \mathscr{F}_h$ and that $\mathbb{T}^2 = \mathscr{E}_h \cup \mathscr{G}_h \cup (\mathbb{T}^2 \setminus \mathscr{F}_h)$. Note also that the sets $\mathscr{E}_h, \mathscr{F}_h$ are nonempty for h small enough, and that $\mathscr{G}_h$ is nonempty (for h small enough) as soon as b vanishes somewhere on $\mathbb{T}^2$ (this condition is assumed here, since otherwise GCC is satisfied).

In this section, we construct the cutoff function χ_h^Λ needed to prove the propagation results of Section 9. In particular, this function will be $\Lambda^\perp$-invariant and will satisfy $\chi_h^\Lambda = 0$ on $\mathscr{E}_h$ and $\chi_h^\Lambda = 1$ on $\mathbb{T}^2 \setminus \mathscr{F}_h$.

The proof of Proposition 7.3 relies on three key lemmata. The first key lemma is a precised version of Proposition 5.2 concerning the localization in $T^*\mathbb{T}^2$ of the semiclassical measure μ. It is an intermediate step towards the propagation result stated in Lemma 12.2.

Lemma 12.1. *For any $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$ such that $\chi = 1$ in a neighborhood of the origin, for all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$, and any $\gamma \le (3+\delta)/2$, we have*

$$(\mathrm{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)} = \left(\mathrm{Op}_h(a)\, \mathrm{Op}_h\left(\chi\left(\frac{|\xi|^2 - 1}{h^\gamma}\right)\right) w_h, w_h\right)_{L^2(\mathbb{T}^2)} + o(h^{\frac{3+\delta}{2} - \gamma}) \| \mathrm{Op}_h(a) \|_{\mathscr{L}(L^2)}. \quad (12\text{-}1)$$

For all $a \in \mathscr{C}_c^\infty(T^*\mathbb{T}^2)$ and all $\tau \in \mathbb{R}$,

$$(\mathrm{Op}_h(a \circ \phi_\tau) w_h, w_h)_{L^2(\mathbb{T}^2)} = (\mathrm{Op}_h(a) w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}}) \| \mathrm{Op}_h(a \circ \phi_t) \|_{L^\infty(0,\tau; \mathscr{L}(L^2(\mathbb{T}^2)))}.$$

In this statement, we used the notation

$$\| \mathrm{Op}_h(a \circ \phi_t) \|_{L^\infty(0,\tau;\mathscr{L}(L^2(\mathbb{T}^2)))} := \sup_{t \in (0,\tau)} \| \mathrm{Op}_h(a \circ \phi_t) \|_{\mathscr{L}(L^2(\mathbb{T}^2))}.$$

In turn, this lemma implies the following transport property.

Lemma 12.2. *Suppose that the coefficients α, ε satisfy*

$$0 < 3\varepsilon \le \alpha \quad \textit{and} \quad \alpha + \varepsilon \le 1. \quad (12\text{-}2)$$

Then, for any time $\tau \in \mathbb{R}$ uniformly bounded with respect to h and any h-family of functions $\psi = \psi_h \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ satisfying

$$\| \partial_x^k \psi \|_{L^\infty(\mathbb{T}^2)} \le C_k h^{-\varepsilon |k|} \quad \textit{for all } k \in \mathbb{N}^2, \quad (12\text{-}3)$$

we have

$$\begin{aligned}(\psi(s, y) w_h, w_h)_{L^2(\mathbb{T}^2)} \\ = (\psi(s+\tau, y) w_h, w_h)_{L^2(\mathbb{T}^2)} + (\psi(s-\tau, y) w_h, w_h)_{L^2(\mathbb{T}^2)} + \mathbb{O}(h^{\alpha - 3\varepsilon}) + \mathbb{O}(h^{1-\alpha-\varepsilon}) + o(h^{\frac{1+\delta}{2}}), \quad (12\text{-}4)\end{aligned}$$

where the coordinates (s, y) are the ones introduced in Section 6C.

In view of Proposition 7.3, this lemma will allow us to propagate the smallness of the sequence w_h above the set $\{b > c_0 h\}$ to all $\mathscr{E}_h$.

The third key lemma states a property of the damping function b, as a consequence of (2-13).

Lemma 12.3. *For all $\varepsilon \in (0, 1]$, $x \in \mathbb{T}^2$ and all $z \in B(x, \frac{1}{2}b(x)^\varepsilon)$, we have $\frac{1}{2}b(x) \le b(z) \le e^{\frac{1}{2}} b(x)$.*

Assumption (2-13) is used here. We denoted by $B(x, \frac{1}{2}b(x)^\varepsilon)$ the Euclidean ball in $\mathbb{T}^2$ centered at x of radius $\frac{1}{2}b(x)^\varepsilon$. Note that only the left inequality is used in this paper.

With these three lemmata, we are now able to prove Proposition 7.3.

Proof of Proposition 7.3. In the coordinates (s, y) of Section 6C, we can write

$$\mathscr{E}_h = \mathbb{T}_{\Lambda^\perp} \times E_h, \quad \mathscr{F}_h = \mathbb{T}_{\Lambda^\perp} \times F_h, \quad \text{with } E_h \subset F_h \subset \mathbb{T}_\Lambda.$$

Here, F_h is a union of intervals and has uniformly bounded total length. We can hence cover F_h with $C_1 h^{-\varepsilon}$ subsets of length of order $(c_0 h)^\varepsilon/4$, overlapping on intervals of length of order $(c_0 h)^\varepsilon/10$. Associated to this covering, we denote by $(\psi_j)_{j\in\{1,\dots,J\}}$, $J = J(h)$, a smooth partition of unity on E_h, also satisfying

- $\psi_j \in \mathscr{C}_c^\infty(F_h)$;
- $\sum_{j=1}^J \psi_j(y) = 1$ for $y \in E_h$;
- $\|\partial_y^m \psi_j\|_{L^\infty(\mathbb{T}_\Lambda)} \le C_m h^{-\varepsilon m}$ for all $m \in \mathbb{N}$;
- $J = J(h) \le C h^{-\varepsilon}$.

Similarly, we cover $\mathbb{T}_{\Lambda^\perp}$ with $C_2 h^{-\varepsilon}$ subsets of length of order $(c_0 h)^\varepsilon/4$, overlapping on intervals of length of order $(c_0 h)^\varepsilon/10$, and define $(\psi_k)_{k\in\{1,\dots,K\}}$ an associated partition of unity on $\mathbb{T}_{\Lambda^\perp}$ satisfying

- $\psi_k \in \mathscr{C}_c^\infty(\mathbb{T}_{\Lambda^\perp})$;
- $\sum_{k=1}^K \psi_k(s) = 1$ for $s \in \mathbb{T}_{\Lambda^\perp}$;
- $\|\partial_s^m \psi_k\|_{L^\infty(\mathbb{T}_{\Lambda^\perp})} \le C_m h^{-\varepsilon m}$, for all $m \in \mathbb{N}$;
- $K = K(h) \le C h^{-\varepsilon}$;
- for any $k, k_0 \in \{1, \dots, K\}^2$, there exists τ_k satisfying $|\tau_k| \le \text{Length}(\mathbb{T}_{\Lambda^\perp}) \le C$ and $\psi_k(s+\tau_k) = \psi_{k_0}(s)$.

We set

$$\psi_{kj}(s, y) := \psi_k(s)\psi_j(y) \quad \text{and} \quad \chi_h^\Lambda(s, y) = 1 - \sum_{j=1}^J \sum_{k=1}^K \psi_{kj}(s, y) \in \mathscr{C}^\infty(\mathbb{T}^2),$$

which satisfies $\partial_s \chi_h^\Lambda(s, y) = 0$, i.e., χ_h^Λ is $\Lambda^\perp$-invariant, together with

- $\chi_h^\Lambda = 0$ on $\mathscr{E}_h$ and hence $b \le c_0 h$ on $\text{supp}(\chi_h^\Lambda)$;
- $\chi_h^\Lambda = 1$ on $\mathbb{T}^2 \setminus \mathscr{F}_h$;
- $\chi_h^\Lambda \in [0, 1]$ on $\mathscr{G}_h$, with $|\partial_y \chi_h^\Lambda| \le C h^{-\varepsilon}$ and $|\partial_y^2 \chi_h^\Lambda| \le C h^{-2\varepsilon}$.

To conclude the proof of Proposition 7.3, it remains to check item (2) ($\|(1-\chi_h^\Lambda)w_h\|_{L^2(\mathbb{T}^2)} = o(1)$), item (4) ($\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$) and item (5) ($\|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(1)$).

Now, let us fix $j_0 \in \{1, \dots, J\}$. Because of the definition of the set $\mathscr{E}_h$, there exists $k_0 \in \{1, \dots, K\}$ and $x_0 \in \{b > c_0 h\}$ such that $\operatorname{supp}(\psi_{k_0 j_0}) \subset B(x_0, (c_0h)^\varepsilon/2)$. According to Lemma 12.3, we have

$$B\Big(x_0, \frac{(c_0h)^\varepsilon}{2}\Big) \subset B\Big(x_0, \frac{b(x_0)^\varepsilon}{2}\Big) \subset \Big\{b > \frac{b(x_0)}{2}\Big\} \subset \Big\{b > \frac{c_0 h}{2}\Big\},$$

so that $\operatorname{supp}(\psi_{k_0 j_0}) \subset \{b > c_0 h/2\}$. This yields

$$\frac{c_0 h}{2}(\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} \le (b\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta}),$$

and hence $(\psi_{k_0 j_0} w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^\delta)$. Moreover, for any $k \in \{1, \dots, K\}$, there exists τ_k satisfying $|\tau_k| \le C_2$ with

$$\psi_{k j_0}(s+\tau_k, y) = \psi_{k_0 j_0}(s, y).$$

Hence, using (12-4), we obtain

$$\begin{aligned} o(h^\delta) &= (\psi_{k_0 j_0}(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)} = (\psi_{k j_0}(s+\tau_k, y) w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &= (\psi_{k j_0}(s+2\tau_k, y) w_h, w_h)_{L^2(\mathbb{T}^2)} + (\psi_{k j_0}(s, y) w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &\qquad + \mathbb{O}(h^{\alpha-3\varepsilon}) + \mathbb{O}(h^{1-\alpha-\varepsilon}) + o(h^{\frac{1+\delta}{2}}). \end{aligned} \tag{12-5}$$

Since both terms on the right-hand side are nonnegative, this implies $(\psi_{k j_0}(s,y) w_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^\delta)$ as long as

$$\alpha - 3\varepsilon > \delta, \quad 1 - \alpha - \varepsilon > \delta, \quad \text{and} \quad \frac{1+\delta}{2} \ge \delta$$

(which implies (12-2)). From now on we will take $\delta = 4\varepsilon$ (the reason for this choice will become apparent in the following lines). The existence of α satisfying this condition together with (7-1) is equivalent to having $\varepsilon < \frac{1}{29}$.

To conclude the proof of Proposition 7.3, we first compute

$$((1-\chi_h^\Lambda) w_h, w_h)_{L^2(\mathbb{T}^2)} = \sum_{j=1}^{J} \sum_{k=1}^{K} (\psi_{kj} w_h, w_h)_{L^2(\mathbb{T}^2)} = C h^{-2\varepsilon} o(h^\delta) = o(1),$$

since $\delta \ge 2\varepsilon$. This proves item (2). Next, we have by construction $\operatorname{supp}(\partial_y^2 \chi_h^\Lambda) \subset \operatorname{supp}(\partial_y \chi_h^\Lambda) \subset \mathscr{G}_h$, with $\|\partial_y \chi_h^\Lambda\|_{L^\infty(\mathbb{T}^2)} = \mathbb{O}(h^{-\varepsilon})$, $\|\partial_y^2 \chi_h^\Lambda\|_{L^\infty(\mathbb{T}^2)} = \mathbb{O}(h^{-2\varepsilon})$. Hence, covering $\operatorname{supp}(\partial_y \chi_h^\Lambda)$) by balls of radius $(c_0 h)^\varepsilon$ and using a propagation argument similar to (12-5) shows that we have $\|w_h\|_{L^2(\operatorname{supp}(\partial_y \chi_h^\Lambda))} = o(h^{\frac{\delta}{2}})$. We thus obtain

$$\|\partial_y \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{\delta}{2}-\varepsilon}) = o(1), \quad \|\partial_y^2 \chi_h^\Lambda w_h\|_{L^2(\mathbb{T}^2)} = o(h^{\frac{\delta}{2}-2\varepsilon}) = o(1),$$

(since $\delta \ge 4\varepsilon$), which concludes the proof of items (4) and (5), and that of Proposition 7.3. $\square$

To conclude this section, it remains to prove Lemmata 12.2, 12.1 and 12.3. In the following proofs, we shall systematically write η in place of $P_\Lambda \xi$ and σ in place of $(1-P_\Lambda)\xi$ to lighten the notation. Hence, $\xi \in \mathbb{R}^2$ is decomposed as $\xi = \eta + \sigma$, with $\eta \in \langle \Lambda \rangle$ and $\sigma \in \Lambda^\perp$, in accordance to Section 6C.

Proof of Lemma 12.2 from Lemma 12.1. First, given a function $\psi \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ satisfying (12-3), we have

$$\begin{aligned}(\psi w_h, w_h)_{L^2(\mathbb{T}^2)} &= (\mathrm{Op}_h(\psi\circ\phi_\tau)w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}})\|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))}\\ &= \Big(\mathrm{Op}_h(\psi\circ\phi_\tau)\,\mathrm{Op}_h\Big(\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)\,\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)}\\ &\qquad + \big(o(\tau h^{\frac{1+\delta}{2}}) + o(\tau h^{\frac{3+\delta}{2}-\gamma})\big)\|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))},\end{aligned}$$

when using Lemma 12.1 together with $\mathrm{Op}_h(\chi(\eta/(2h^\alpha)))w_h = w_h$. Next, the pseudodifferential calculus yields

$$\begin{aligned}(\psi w_h, w_h)_{L^2(\mathbb{T}^2)} &= \Big(\mathrm{Op}_h\Big(\psi\circ\phi_\tau\,\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\chi\Big(\frac{\eta}{2h^\alpha}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} + \mathcal{O}(h^{2-\gamma-\varepsilon}) + \mathcal{O}(h^{1-\alpha-\varepsilon})\\ &\qquad + \big(o(\tau h^{\frac{1+\delta}{2}}) + o(\tau h^{\frac{3+\delta}{2}-\gamma})\big)\|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))}. \end{aligned}\tag{12-6}$$

A particular feature of the Weyl quantization in the Euclidean setting is that the Egorov theorem provides an exact formula (see, for instance, [Dimassi and Sjöstrand 1999]): $\mathrm{Op}_h(\psi\circ\phi_t) = e^{-ith\frac{\Delta}{2}}\,\mathrm{Op}_h(\psi)e^{ith\frac{\Delta}{2}}$, so that $\|\mathrm{Op}_h(\psi\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))} \le C_0$ uniformly with respect to h. Now, remark that the cutoff function $\chi(\eta/(2h^\alpha))\chi((|\xi|^2-1)/h^\gamma)$ can be decomposed (for h small enough) as

$$\chi\Big(\frac{\eta}{2h^\alpha}\Big)\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big) = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\big(\tilde\chi_\eta^h(\sigma) + \tilde\chi_\eta^h(-\sigma)\big)$$

for some nonnegative function $\tilde\chi_\eta^h$ such that $(\sigma,\eta)\mapsto \tilde\chi_\eta^h(\sigma) \in \mathscr{C}_c^\infty(\mathbb{R}^2)$, such that $\tilde\chi_\eta^h(\sigma) = \chi((|\xi|^2-1)/h^\gamma)$ for $\eta \in \operatorname{supp}\chi(\,\cdot\,/(2h^\alpha))$ and $\sigma > 0$, and $\tilde\chi_\eta^h(\sigma) = 0$ for $\eta \notin \operatorname{supp}\chi(\,\cdot\,/(2h^\alpha))$ or $\sigma \le 0$.

Choosing $\gamma = \alpha$, we have in particular

$$|\sigma - 1| \le Ch^\alpha \quad \text{on } \operatorname{supp}\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde\chi_\eta^h(\sigma)\Big).$$

Next, we recall that $\psi\circ\phi_\tau(s,y,\sigma,\eta) = \psi(s+\tau\sigma, y+\tau\eta)$, and we focus on the first term (corresponding to $\sigma > 0$) in the right-hand side of the identity

$$\chi\Big(\frac{|\xi|^2-1}{h^\alpha}\Big)\chi\Big(\frac{\eta}{2h^\alpha}\Big)\psi\circ\phi_\tau = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\big(\tilde\chi_\eta^h(\sigma) + \tilde\chi_\eta^h(-\sigma)\big)\psi\circ\phi_\tau. \tag{12-7}$$

We set

$$\zeta_\tau^{(1)}(s,y,\sigma,\eta) = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde\chi_\eta^h(\sigma)\psi(s+\tau\sigma, y+\tau\eta) \quad\text{and}\quad \zeta_\tau^{(2)}(s,y,\sigma,\eta) = \chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde\chi_\eta^h(\sigma)\psi(s+\tau, y),$$

and we want to compare $\mathrm{Op}_h(\zeta_\tau^{(1)})$ and $\mathrm{Op}_h(\zeta_\tau^{(2)})$. For this, let us estimate, for multiindices $\ell, m \in \mathbb{N}^2$,

$$\begin{aligned}&|\partial_{(s,y)}^\ell \partial_{(\sigma,\eta)}^m(\zeta_\tau^{(2)} - \zeta_\tau^{(1)})(s,y,\sigma,\eta)|\\ &\qquad \le C_m \sum_{\nu\le m}\Big|\partial_{(\sigma,\eta)}^{m-\nu}\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde\chi_\eta^h(\sigma)\Big)\partial_{(s,y)}^\ell\partial_{(\sigma,\eta)}^\nu\big(\psi(s+\tau\sigma, y+\tau\eta) - \psi(s+\tau, y)\big)\Big|. \end{aligned}\tag{12-8}$$

On the one hand, we have

$$\left|\partial_{(\sigma,\eta)}^{m-\nu}\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\Big)\right| \le C_{m,\nu}h^{-\alpha|m-\nu|}. \tag{12-9}$$

On the other hand, for $|\nu|>0$ we can also write

$$\begin{aligned}\left|\partial_{(s,y)}^\ell\partial_{(\sigma,\eta)}^\nu\big(\psi(s+\tau\sigma,y+\tau\eta)-\psi(s+\tau,y)\big)\right| &= |\partial_{(s,y)}^\ell\partial_{(\sigma,\eta)}^\nu\psi(s+\tau\sigma,y+\tau\eta)|\\ &\le C_{\ell,\nu}|\tau|^{|\nu|}h^{-\varepsilon(|\ell|+|\nu|)} \le C_{\ell,\nu}h^{-\varepsilon(|\ell|+|\nu|)},\end{aligned}$$

since $|\tau|\le C$.

Finally, for $|\nu|=0$, we apply the mean value theorem to the function $(\sigma,\eta)\mapsto\partial_{(s,y)}^\ell\psi(s+\tau\sigma,y+\tau\eta)$ and write

$$\left|\partial_{(s,y)}^\ell\big(\psi(s+\tau\sigma,y+\tau\eta)-\psi(s+\tau,y)\big)\right| \le (|\eta|+|\sigma-1|)\sup_{T^*\mathbb{T}^2}|\nabla_{(\sigma,\eta)}\partial_{(s,y)}^\ell(\psi(s+\tau\sigma,y+\tau\eta))|.$$

With (12-3), this yields

$$\begin{aligned}\left|\partial_{(s,y)}^\ell\big(\psi(s+\tau\sigma,y+\tau\eta)-\psi(s+\tau,y)\big)\right| &\le (|\eta|+|\sigma-1|)C_\ell h^{-\varepsilon|\ell|}|\tau|h^{-\varepsilon}\\ &\le (|\eta|+|\sigma-1|)C_\ell h^{-\varepsilon(|\ell|+1)},\end{aligned} \tag{12-10}$$

for $|\tau|\le C$.

Using now that $|\eta|\le Ch^\alpha$ and $|\sigma-1|\le Ch^\alpha$ on $\operatorname{supp}(\chi(\eta/(2h^\alpha))\tilde{\chi}_\eta^h(\sigma))$, and combining (12-8), (12-9) and (12-10), we obtain, for all $m\in\mathbb{N}^2$, $\ell\in\mathbb{N}^2$ and $0<h\le h_0$ sufficiently small,

$$\begin{aligned}h^{|m|}|\partial_{(s,y)}^\ell\partial_{(\sigma,\eta)}^m(\zeta_\tau^{(2)}-\zeta_\tau^{(1)})(s,y,\sigma,\eta)| &\le C_{\ell,m}h^{\alpha-\varepsilon(|\ell|+1)}h^{|m|}h^{-\alpha|m|}+C_{\ell,m}\sum_{0<\nu\le m}h^{|m|}h^{-\varepsilon(|\ell|+|\nu|)}h^{-\alpha|m-\nu|}\\ &\le C_{\ell,m}(h^{(1-\alpha)|m|}h^{\alpha-\varepsilon(|\ell|+1)}+|m|h^{|m|(1-\alpha)}h^{-\varepsilon|\ell|}h^{\alpha-\varepsilon})\\ &\le C_{\ell,m}h^{\alpha-\varepsilon(|\ell|+1)}.\end{aligned}$$

Using a precised version of the Calderón–Vaillancourt theorem, as presented in Theorem A.1 below (in which only $|\ell|=2$ derivations are needed with respect to x in dimension two), we obtain

$$\mathrm{Op}_h(\zeta_\tau^{(2)})=\mathrm{Op}_h(\zeta_\tau^{(1)})+\mathcal{O}_{\mathcal{L}(L^2)}(h^{\alpha-3\varepsilon}).$$

Similarly, we have

$$\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(-\sigma)\psi(s+\tau\sigma,y+\tau\eta)\Big)=\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(-\sigma)\psi(s-\tau,y)\Big)+\mathcal{O}_{\mathcal{L}(L^2)}(h^{\alpha-3\varepsilon}).$$

Coming back to (12-6) and using (12-7), we finally obtain, for all $|\tau|\le C$,

$$\begin{aligned}(\psi w_h,w_h)_{L^2(\mathbb{T}^2)} &= \Big(\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(\sigma)\psi(s+\tau,y)\Big)w_h,w_h\Big)_{L^2(\mathbb{T}^2)}\\ &\quad+\Big(\mathrm{Op}_h\Big(\chi\Big(\frac{\eta}{2h^\alpha}\Big)\tilde{\chi}_\eta^h(-\sigma)\psi(s-\tau,y)\Big)w_h,w_h\Big)_{L^2(\mathbb{T}^2)}\\ &\quad+\mathcal{O}(h^{\alpha-3\varepsilon})+\mathcal{O}(h^{1-\alpha-\varepsilon})+o(h^{\frac{1+\delta}{2}})+o(h^{\frac{3+\delta}{2}-\alpha}).\end{aligned}$$

With the pseudodifferential calculus, this yields (12-4), which concludes the proof of Lemma 12.2. □

Proof of Lemma 12.1. Here, we only have to make more precise some arguments in the proof of Proposition 5.2. Recall that according to Lemma 7.1, w_h satisfies $P_b^h w_h = o(h^{2+\delta})$.

First, we take $\chi \in \mathscr{C}_c^\infty(\mathbb{R})$, such that $\chi = 1$ in a neighborhood of the origin. Hence, $(1-\chi(r))/r \in \mathscr{C}^\infty(\mathbb{R})$ and we have the exact composition formula

$$\operatorname{Op}_h\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big) = \operatorname{Op}_h\Big(\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)\frac{h^\gamma}{|\xi|^2-1}\Big)\frac{P_0^h}{h^\gamma},$$

since both operators are Fourier multipliers. Moreover, $\operatorname{Op}_h\big((1-\chi((|\xi|^2-1)/h^\gamma))h^\gamma/(|\xi|^2-1)\big)$ is uniformly bounded as an operator of $\mathscr{L}(L^2(\mathbb{T}^2))$. As a consequence, we have

$$\begin{aligned}\Big(\operatorname{Op}_h(a)\operatorname{Op}_h\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} \\ &= \Big(\operatorname{Op}_h(a)\operatorname{Op}_h\Big(\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)\frac{h^\gamma}{|\xi|^2-1}\Big)\frac{P_0^h}{h^\gamma}w_h, w_h\Big)_{L^2(\mathbb{T}^2)} \\ &= \Big(A\frac{P_b^h}{h^\gamma}w_h, w_h\Big)_{L^2(\mathbb{T}^2)} - \Big(A\frac{ihb}{h^\gamma}w_h, w_h\Big)_{L^2(\mathbb{T}^2)},\end{aligned}$$

where $A = \operatorname{Op}_h(a)\operatorname{Op}_h\big((1-\chi((|\xi|^2-1)/h^\gamma))h^\gamma/(|\xi|^2-1)\big)$ is bounded on $L^2(\mathbb{T}^2)$. Using $P_b^h w_h = o(h^{2+\delta})$ and $(bw_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$, this gives

$$\Big(\operatorname{Op}_h(a)\operatorname{Op}_h\Big(1-\chi\Big(\frac{|\xi|^2-1}{h^\gamma}\Big)\Big)w_h, w_h\Big)_{L^2(\mathbb{T}^2)} = o(h^{\frac{3+\delta}{2}-\gamma})\|\operatorname{Op}_h(a)\|_{\mathscr{L}(L^2)},$$

which in turn implies (12-1).

Next, identity (5-6) yields, for all $a \in \mathscr{C}_c^\infty(\mathbb{T}^2)$,

$$\begin{aligned}(\operatorname{Op}_h(\xi\cdot\partial_x a)w_h, w_h)_{L^2(\mathbb{T}^2)} &= \frac{i}{2h}(\operatorname{Op}_h(a)w_h, P_b^h w_h)_{L^2(\mathbb{T}^2)} - \frac{i}{2h}(\operatorname{Op}_h(a)P_b^h w_h, w_h)_{L^2(\mathbb{T}^2)} \\ &\quad - \tfrac{1}{2}(\operatorname{Op}_h(a)w_h, bw_h)_{L^2(\mathbb{T}^2)} - \tfrac{1}{2}(\operatorname{Op}_h(a)bw_h, w_h)_{L^2(\mathbb{T}^2)} \\ &= o(h^{1+\delta})\|\operatorname{Op}_h(a)\|_{\mathscr{L}(L^2)} + o(h^{\frac{1+\delta}{2}})\|\operatorname{Op}_h(a)\|_{\mathscr{L}(L^2)},\end{aligned}$$

as a consequence of $P_b^h w_h = o(h^{2+\delta})$ and $(bw_h, w_h)_{L^2(\mathbb{T}^2)} = o(h^{1+\delta})$. Applying this identity to $a \circ \phi_t$ in place of a, and integrating on $t \in [0, \tau]$ finally gives

$$(\operatorname{Op}_h(a\circ\phi_\tau)w_h, w_h)_{L^2(\mathbb{T}^2)} = (\operatorname{Op}_h(a)w_h, w_h)_{L^2(\mathbb{T}^2)} + o(\tau h^{\frac{1+\delta}{2}})\|\operatorname{Op}_h(a\circ\phi_t)\|_{L^\infty(0,\tau;\mathscr{L}(L^2))},$$

which concludes the proof of Lemma 12.1. □

Proof of Lemma 12.3. First, we have $\nabla(b^\varepsilon) = 0$ on $\{b = 0\}$ and $\nabla(b^\varepsilon)(x) = \varepsilon b(x)^{\varepsilon-1}\nabla b(x)$ on $\{b > 0\}$. Assumption (2-13) then yields $|\nabla(b^\varepsilon)| \le \varepsilon$ uniformly on $\mathbb{T}^2$. The mean value theorem hence gives, for all $z \in B(x, \frac{1}{2}b(x)^\varepsilon)$,

$$b(x)^\varepsilon \le b(z)^\varepsilon + \varepsilon|x-z| \le b(z)^\varepsilon + \frac{\varepsilon}{2}b(x)^\varepsilon.$$

Hence we obtain $b(z) \geq b(x)(1-\varepsilon/2)^{1/\varepsilon}$. On the interval $(0, 1]$, the function $\varepsilon \mapsto (1/\varepsilon)(1-2^{-\varepsilon})$ is decreasing so that for $\varepsilon \in (0, 1]$, we have $(1/\varepsilon)(1-2^{-\varepsilon}) \geq \frac{1}{2}$. This gives $0 < \varepsilon/2 \leq 1-2^{-\varepsilon}$ so that $b(z) \geq b(x)(2^{-\varepsilon})^{1/\varepsilon}$ for $\varepsilon \in (0, 1]$, which concludes the proof of the left inequality.

The right inequality follows from the same arguments. □

Part IV. An *a priori* lower bound for decay rates on the torus

13. Proof of Theorem 2.5

Under the assumption

$$\overline{\{b > 0\}} \cap \{x_0 + \tau\xi_0,\ \tau \in \mathbb{R}\} = \varnothing, \tag{13-1}$$

for some $(x_0, \xi_0) \in T^*\mathbb{T}^2$, $\xi_0 \neq 0$, we construct in this section a constant $\kappa_0 > 0$ and a sequence $(\varphi_n)_{n\in\mathbb{N}}$ of $\mathbb{O}(1)$-quasimodes in the limit $n \to +\infty$ for the family of operators $P(in\kappa_0)$.

We use the notation introduced in Sections 6A and 8. First, note that, as a consequence of (13-1), ξ_0 is necessarily a rational direction, and the set $\{x_0 + \tau\xi_0,\ \tau \in \mathbb{R}\}$ is a one-dimensional subtorus of $\mathbb{T}^2$, given by

$$\{x_0 + \tau\xi_0,\ \tau \in \mathbb{R}\} = \overline{\{x_0 + \tau\xi_0,\ \tau \in \mathbb{R}\}} = x_0 + \mathbb{T}_{\Lambda_{\xi_0}^{\perp}}, \quad \text{with } \Lambda_{\xi_0} \in \mathcal{P}.$$

Let $\chi \in \mathscr{C}_c^\infty(\mathbb{T}^2)$ such that χ has only x-Fourier modes in Λ_{ξ_0}, $\chi = 0$ on a neighborhood of $\overline{\{b > 0\}}$ and $\chi = 1$ on $x_0 + \mathbb{T}_{\Lambda_{\xi_0}^{\perp}}$.

From assumption (13-1), we have $\mathrm{rk}(\Lambda_{\xi_0}) = 1$, so that one can find $k \in \Lambda_{\xi_0}^{\perp} \cap \mathbb{Z}^2 \setminus \{0\}$. Besides, for all $n \in \mathbb{N}$ we have $nk \in \Lambda_{\xi_0}^{\perp} \cap \mathbb{Z}^2 \setminus \{0\}$.

We then define the sequence of *quasimodes* $(\varphi_n)_{n\in\mathbb{N}}$ by

$$\varphi_n(x) = \chi(x)e^{ink\cdot x}, \quad n \in \mathbb{N},\ x \in \mathbb{T}^2.$$

We have $\varphi_n \in \mathscr{C}^\infty(\mathbb{T}^2)$, together with the decoupling

$$\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) = \chi(y)e^{ink\cdot s}, \quad n \in \mathbb{N},\ (s, y) \in \mathbb{T}_{\Lambda_{\xi_0}^{\perp}} \times \mathbb{T}_{\Lambda_{\xi_0}}.$$

This yields

$$\begin{aligned} -\big(T_{\Lambda_{\xi_0}} \Delta T^*_{\Lambda_{\xi_0}}\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) &= -\big(\Delta_{\Lambda_{\xi_0}} + \Delta_{\Lambda_{\xi_0}^{\perp}}\big)\varphi_n \circ \pi_{\Lambda_{\xi_0}}(s, y) \\ &= -e^{ink\cdot s}\Delta_{\Lambda_{\xi_0}}\chi(y) + n^2|k|^2\chi(y)e^{ink\cdot s}. \end{aligned}$$

Moreover, $b\varphi_n = 0$ since their supports are disjoint. Hence, recalling that

$$P(in|k|) = -\Delta - n^2|k|^2 + in|k|b(x),$$

we have

$$(T_{\Lambda_{\xi_0}} P(in|k|) T^*_{\Lambda_{\xi_0}})\varphi_n \circ \pi_{\Lambda_{\xi_0}} = -e^{ink\cdot s}\Delta_{\Lambda_{\xi_0}}\chi(y),$$

and

$$\|P(in|k|)\varphi_n\|_{L^2(\mathbb{T}^2)} = \|(T_{\Lambda_{\xi_0}} P(in|k|) T^*_{\Lambda_{\xi_0}})\varphi_n \circ \pi_{\Lambda_{\xi_0}}\|_{L^2(\mathbb{T}_{\Lambda^\perp_{\xi_0}} \times \mathbb{T}_{\Lambda_{\xi_0}})} = C_0 \|\Delta_{\Lambda_{\xi_0}} \chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}.$$

Since we also have $\|\varphi_n\|_{L^2(\mathbb{T}^2)} = \|T_{\Lambda_{\xi_0}} \varphi_n\|_{L^2(\mathbb{T}_{\Lambda^\perp_{\xi_0}} \times \mathbb{T}_{\Lambda_{\xi_0}})} = C_0 \|\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}$, we obtain, for all $n \in \mathbb{N}$,

$$\|P^{-1}(in|k|)\|_{\mathscr{L}(L^2(\mathbb{T}^2))} \geq \frac{\|\varphi_n\|_{L^2(\mathbb{T}^2)}}{\|P(in|k|)\varphi_n\|_{L^2(\mathbb{T}^2)}} = \frac{\|\chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}}{\|\Delta_{\Lambda_{\xi_0}} \chi\|_{L^2(\mathbb{T}_{\Lambda_{\xi_0}})}} = C > 0,$$

which concludes the proof of Theorem 2.5. □

Appendix A: Pseudodifferential calculus

In the main part of the article, we use the semiclassical Weyl quantization associating to a function a on $T^*\mathbb{R}^2$ an operator $\mathrm{Op}_h(a)$ defined by

$$(\mathrm{Op}_h(a)u)(x) := \frac{1}{(2\pi h)^2} \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} e^{\frac{i}{h}\xi\cdot(x-y)} a\Big(\frac{x+y}{2}, \xi\Big) u(y)\, dy\, d\xi. \tag{A-1}$$

For smooth functions a with uniformly bounded derivatives, $\mathrm{Op}_h(a)$ defines a continuous operator on $\mathscr{S}(\mathbb{R}^2)$, and also by duality on $\mathscr{S}'(\mathbb{R}^2)$. On a manifold, the quantization Op_h may be defined by working in local coordinates with a partition of unity. On the torus, formula (A-1) still makes sense: taking $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ is equivalent to taking $a \in \mathscr{C}^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$, $(2\pi\mathbb{Z})^2$-periodic with respect to the x-variable. Then the operator defined by (A-1) preserves the space of $(2\pi\mathbb{Z})^2$-periodic distributions on $\mathbb{R}^2$, and hence $\mathscr{D}'(\mathbb{T}^2)$.

We sometimes write, with $D := (1/i)\partial$,

$$a(x, hD) = \mathrm{Op}_h(a).$$

We also note that $\mathrm{Op}_1(a)$ is the classical Weyl quantization, and that we have the relation

$$a(x, hD) = \mathrm{Op}_h(a(x, \xi)) = \mathrm{Op}_1(a(x, h\xi)).$$

Theorem A.1. *There exists a constant $C > 0$ such that for any $a \in \mathscr{C}^\infty(T^*\mathbb{T}^2)$ with uniformly bounded derivatives, we have*

$$\|\mathrm{Op}_1(a)\|_{\mathscr{L}(L^2(\mathbb{T}^2))} \leq C \sum_{\alpha\in\{0,1\}^2, \beta\in\{0,1\}^2} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(T^*\mathbb{T}^2)}.$$

Equivalently, this can be rewritten as

$$\|\mathrm{Op}_h(a)\|_{\mathscr{L}(L^2(\mathbb{T}^2))} \leq C \sum_{\alpha\in\{0,1\}^2, \beta\in\{0,1\}^2} h^{|\beta|} \|\partial_x^\alpha \partial_\xi^\beta a\|_{L^\infty(T^*\mathbb{T}^2)}.$$

This precised version of the Calderón–Vaillancourt theorem for the Weyl quantization is needed in Section 12, and proved in [Boulkhemair 1999, Theorem 1.2]. Here in dimension two, this means that only $|\alpha| = 2$ derivations are needed with respect to the space variable x.

Appendix B: Spectrum of $P(z)$ for a piecewise constant damping (by Stéphane Nonnenmacher)

In this appendix we provide an explicit description of some part of the *spectrum* of the damped wave equation (1-1) on $\mathbb{T}^2$, for a damping function proportional to the characteristic function of a vertical strip. We identify the torus $\mathbb{T}^2$ with the square $\{-1/2 \le x < 1/2,\ 0 \le y < 1\}$. We choose some half-width $\sigma \in (0, 1/2)$, and consider a vertical strip of width 2σ. Due to translation symmetry of $\mathbb{T}^2$, we may center this strip on the axis $\{x = 0\}$. Choosing a damping strength $\tilde{B} > 0$, we then get the damping function

$$b(x, y) = b(x) = \begin{cases} 0 & \text{for } |x| \le \sigma, \\ \tilde{B} & \text{for } \sigma < |x| \le 1/2. \end{cases} \tag{B-1}$$

The reason for centering the strip at $x = 0$ is the parity of the problem with respect to that axis, which greatly simplifies the computations.

We are interested in the spectrum of the operator $\mathscr{A}$ generating the evolution equation (1-1), which amounts (see Lemma 4.2) to solving the eigenvalue problem

$$P(z)u = 0 \quad \text{for } P(z) = -\Delta + zb(x) + z^2,\ z \in \mathbb{C},\ u \in L^2(\mathbb{T}^2),\ u \not\equiv 0.$$

This spectrum consists in a discrete set $\{z_j\}$, which is symmetric with respect to the horizontal axis: indeed, any solution (z, u) admits a "sister" solution $(\bar{z}, \bar{u})$. Furthermore, any solution with $\operatorname{Im} z \neq 0$ satisfies

$$\operatorname{Re} z = -\frac{1}{2}\frac{(u, bu)_{L^2(\mathbb{T}^2)}}{\|u\|^2_{L^2(\mathbb{T}^2)}}, \quad \text{and thus } -\tilde{B}/2 \le \operatorname{Re} z \le 0\,. \tag{B-2}$$

We may thus restrict ourselves to the half-strip $\{-\tilde{B}/2 \le \operatorname{Re} z \le 0, \operatorname{Im} z > 0\}$.

Our aim is to find high-frequency eigenvalues ($\operatorname{Im} z \gg 1$) which are as close as possible to the imaginary axis.

Proposition B.1. *There exists $C_0 > 0$ such that the spectrum* (B-2) *for the damping function* (B-1) *contains an infinite subsequence $\{z_i\}$ such that $\operatorname{Im} z_i \to \infty$ and $|\operatorname{Re} z_i| \le C_0/(\operatorname{Im} z_i)^{3/2}$.*

The proof of the proposition will actually give an explicit value for C_0, as a function of $\tilde{B}, \sigma$.

Proof. To study the high-frequency limit $\operatorname{Im} z \to \infty$ we will change variables and take

$$z = i(1/h + \tilde{\zeta}),$$

where $h \in (0, 1]$ will be a small parameter, while $\tilde{\zeta} \in \mathbb{C}$ is assumed to be uniformly bounded when $h \to 0$. The eigenvalue equation then takes the form

$$(-h^2\Delta + ih(1 + h\tilde{\zeta})b)u = \big(1 + 2h\tilde{\zeta}(1 + h\tilde{\zeta}/2)\big)u\,.$$

Having chosen b independent of y, we may naturally Fourier transform along this direction, that is look for solutions of the form $u(x, y) = e^{2i\pi ny}v(x)$, $n \in \mathbb{Z}$. For each n, we now have to solve the 1-dimensional problem

$$\big(-h^2\partial^2/\partial_x^2 + ih(1 + h\tilde{\zeta})b(x)\big)v = \big(1 - (2\pi hn)^2 + 2h\tilde{\zeta}(1 + h\tilde{\zeta}/2)\big)v.$$

Let us call

$$B \stackrel{\text{def}}{=} \tilde{B}(1+h\tilde{\zeta}), \quad \zeta \stackrel{\text{def}}{=} \tilde{\zeta}(1+h\tilde{\zeta}/2).$$

In terms of these parameters, the above equation reads

$$(-h^2\partial^2/\partial_x^2 + ihB\mathbb{1}_{\{\sigma<|x|\le 1/2\}}(x))v = Ev, \quad \text{with } E = 1-(2\pi hn)^2 + 2h\zeta. \tag{B-3}$$

Since we will assume throughout that $\tilde{\zeta} = \mathbb{O}(1)$, we will have in the semiclassical limit

$$B = \tilde{B} + \mathbb{O}(h), \quad \tilde{\zeta} = \zeta(1 - h\zeta/2 + \mathbb{O}(h^2)). \tag{B-4}$$

At leading order we may forget that the variables B, ζ are not independent from one another, and consider (B-3) as a bona fide linear eigenvalue problem.

Since the function $b(x)$ is even, we may separately search for even (resp. odd) solutions $v(x)$. Let us start with the even solutions. Since $b(x)$ is piecewise constant, any even and periodic solution $v(x)$ takes the following form on $\left[-\frac{1}{2}, \frac{1}{2}\right]$ (up to a global normalization factor):

$$v(x) = \begin{cases} \cos(kx) & \text{for } |x| \le \sigma, \\ \beta \cos\left(k'(\frac{1}{2} - |x|)\right) & \text{for } \sigma < |x| \le \frac{1}{2}, \end{cases} \tag{B-5}$$

$$k = \frac{E^{1/2}}{h}, \quad k' = \frac{(E-ihB)^{1/2}}{h}. \tag{B-6}$$

We notice that k, k' are defined modulo a change of sign, so we may always assume that $\operatorname{Re} k \ge 0$, $\operatorname{Re} k' \ge 0$. The factor β is obtained by imposing the continuity of v and of its derivative v' at the discontinuity point $x = \sigma$ (we use the notation $\sigma' \stackrel{\text{def}}{=} \frac{1}{2} - \sigma$):

$$\cos(k\sigma) = \beta\cos(k'\sigma'), \quad -k\sin(k\sigma) = \beta k'\sin(k'\sigma').$$

The ratio of these two equations provides the quantization condition for the even solutions:

$$\tan(k\sigma) = -\frac{k'}{k}\tan(k'\sigma'). \tag{B-7}$$

Similarly, any odd eigenfunction takes the form (modulo a global normalization factor)

$$v(x) = \begin{cases} \sin(kx) & \text{for } |x| \le \sigma, \\ \beta\,\operatorname{sgn}(x)\sin(k'(\frac{1}{2} - |x|)) & \text{for } \sigma < |x| \le \frac{1}{2}, \end{cases} \tag{B-8}$$

so the associated eigenvalues should satisfy the condition

$$\tan(k\sigma) = -\frac{k}{k'}\tan(k'\sigma'). \tag{B-9}$$

We will now study the solutions of the quantization conditions (B-7) and (B-9), taking into account the relations (B-6) between the wavevectors k, k' and the energy E. To describe the full spectrum (which we plan to present in a separate publication), we would need to consider several régimes, depending on the relative scales of E and h. However, since we are only interested here in proving Proposition B.1, we will focus on the régime leading to the smallest possible values of $|\operatorname{Im}\tilde{\zeta}| = |\operatorname{Re} z|$. What characterizes the corresponding eigenmodes $v(x)$? From (B-2) we see that the mass of $v(x)$ in the damped region,

$2\int_\sigma^{1/2}|v(x)|^2\,dx$, should be small compared to its full mass. Intuitively, if such a mode were carrying a large horizontal "momentum" $\mathrm{Re}(hk)$ in the undamped region, it would then strongly penetrate the damped region, because the boundary at $x=\sigma$ is not reflecting. As a result, the mass in the damped region would be of the same order of magnitude as the one in the undamped one. This hand-waving argument explains why we choose to investigate the eigenmodes for which hk is the smallest possible, namely of order $\mathbb{O}(h)$. This implies that $E=(hk)^2=\mathbb{O}(h^2)$, which means that almost all of the energy is carried by the vertical momentum:

$$hn=(2\pi)^{-1}+\mathbb{O}(h).$$

The study of the full spectrum actually confirms that the smallest values of $\mathrm{Im}\,\tilde{\zeta}$ are obtained in this régime.

Equation (B-6) implies that the wavevector k' in the damped region is then much larger than k:

$$k'=\frac{(-ihB+(hk)^2)^{1/2}}{h}=e^{-i\pi/4}(B/h)^{1/2}+\mathbb{O}(h^{1/2}).$$

$\mathrm{Im}\,k'\sigma'\approx-\sigma'(B/2h)^{1/2}$ is negative and large, so that $\tan(k'\sigma')=-i+\mathbb{O}(e^{2\,\mathrm{Im}(k'\sigma')})$, uniformly with respect to $\mathrm{Re}(k'\sigma')$.

Even eigenmodes. In this situation the even quantization condition (B-7) reads

$$\tan(k\sigma)=i\frac{k'}{k}\big(1+\mathbb{O}(e^{-\sigma'(2B/h)^{1/2}})\big). \tag{B-10}$$

Since the right-hand side is large, $k\sigma$ must be close to a pole of the tangent function. Hence, for each integer m in a bounded interval[2] $0\le m\le M$ we look for a solution of the form

$$k_{m+1/2}=\frac{\pi(m+\frac{1}{2})}{\sigma}+\delta k_{m+1/2},\quad \text{with } |\delta k_{m+1/2}|\ll 1.$$

The quantization condition (B-10) then reads

$$\begin{aligned}\sigma\,\delta k_{m+1/2}+\mathbb{O}((\delta k_{m+1/2})^2)&=i\frac{k_{m+1/2}}{e^{-i\pi/4}(B/h)^{1/2}+\mathbb{O}(h^{1/2})}\big(1+\mathbb{O}(e^{-\sigma'(2B/h)^{1/2}})\big)\\ \Longrightarrow\quad k_{m+1/2}&=\frac{\pi(m+\frac{1}{2})}{\sigma}\Big(1+h^{1/2}\frac{e^{i3\pi/4}}{\sigma B^{1/2}}+\mathbb{O}(h)\Big).\end{aligned}$$

Using (B-3), the corresponding spectral parameter ζ is then given by

$$\begin{aligned}\zeta_{n,m+1/2}&=\frac{(hk_{m+1/2})^2+(2\pi hn)^2-1}{2h}\\ &=\frac{(2\pi hn)^2-1}{2h}+\frac{h}{2}\Big(\frac{\pi(m+\frac{1}{2})}{\sigma}\Big)^2+h^{3/2}\Big(\frac{\pi(m+\frac{1}{2})}{\sigma}\Big)^2\frac{e^{i3\pi/4}}{\sigma B^{1/2}}+\mathbb{O}(h^2).\end{aligned}$$

From the assumptions on the quantum numbers n,m, we check that $\zeta_{n,m+1/2}=\mathbb{O}(1)$. We may now go back to the original variables $\tilde{\zeta}$, $\tilde{B}$, using the relations (B-4). The spectral parameter $\tilde{\zeta}$ has an imaginary

[2] Recall that we only need to study values $\mathrm{Re}\,k\ge 0$.

part

$$\operatorname{Im}\tilde{\zeta}_{n,m+1/2} = \operatorname{Im}\zeta_{n,m+1/2}(1-h\operatorname{Re}\zeta_{n,m+1/2}) + \mathcal{O}(h^2) = h^{3/2}\frac{(\pi(m+\frac{1}{2}))^2}{\sigma^3(2\tilde{B})^{1/2}} + \mathcal{O}(h^2). \tag{B-11}$$

Returning to the spectral variable z, the above expression gives a string of eigenvalues $\{z_{n,m+1/2}\}$ with $\operatorname{Im} z_{n,m+1/2} = h^{-1} + \mathcal{O}(1)$, $\operatorname{Re} z_{n,m+1/2} = -\operatorname{Im}\tilde{\zeta}_{n,m+1/2}$. These even-parity eigenvalues prove Proposition B.1, and one can take for C_0 any value greater than $(\pi/2)^2/(\sigma^3(2\tilde{B})^{1/2})$. □

We remark that the leading order of $k_{m+1/2}$ corresponds to the even spectrum of the operator $-h^2\partial^2/\partial_x^2$ on the undamped interval $[-\sigma,\sigma]$, with Dirichlet boundary conditions. The eigenmode $v_{n,m+1/2}$ associated with $\tilde{\zeta}_{n,m+1/2}$ is indeed essentially supported on that interval, where it resembles the Dirichlet eigenmode $\cos(x\pi(\frac{1}{2}+m)/\sigma)$. At the boundary of that interval, it takes the value

$$v_{n,m+1/2}(\sigma) = (-1)^{m+1}e^{i3\pi/4}h^{1/2}\frac{\pi(m+\frac{1}{2})}{\sigma\tilde{B}^{1/2}} + \mathcal{O}(h),$$

and decays exponentially fast inside the damping region, with a "penetration length" $(\operatorname{Im} k')^{-1} \approx (2h/\tilde{B})^{1/2}$. From (B-2) we see that the intensity $|v_{n,m+1/2}(\sigma)|^2 \sim C\,h$ penetrating on a distance $\sim h^{1/2}$ exactly accounts for the size $\sim h^{3/2} = hh^{1/2}$ of the $\operatorname{Re} z_{n,m+1/2}$.

We notice that the smallest damping occurs for the state $v_{n,1/2}$ resembling the ground state of the Dirichlet Laplacian.

Odd eigenmodes. For completeness we also investigate the odd-parity eigenmodes with $k = \mathcal{O}(1)$. The computations are very similar as in the even-parity case. The odd quantization condition reads in this régime

$$\tan(k\sigma) = i\frac{k}{k'}\big(1 + \mathcal{O}(e^{-(2B/h)^{1/2}})\big).$$

The right-hand side is then very small, showing σk is close to a zero of the tangent, so we may take $k_m = \pi m/\sigma + \delta k_m$ with $|\delta k_m| \ll 1$ and $0 \le m \le M$. We easily see that the case $m = 0$ does not lead to a solution. For the case $m > 0$ we get

$$\delta k_m = e^{3i\pi/4}h^{1/2}\frac{\pi m}{\sigma^2 B^{1/2}} + \mathcal{O}(h),$$

and thus

$$k_m = \frac{\pi m}{\sigma}\Big(1 + h^{1/2}\frac{e^{3i\pi/4}}{\sigma B^{1/2}} + \mathcal{O}(h)\Big), \quad 1 \le m \le M.$$

These values k_m approximately sit on the same "line" $\{s(1 + h^{1/2}e^{3i\pi/4}/(\sigma B^{1/2})), s \in \mathbb{R}\}$ as the values $k_{m+1/2}$ corresponding to the even eigenmodes, both types of eigenvalues appearing successively. The corresponding energy parameter $\tilde{\zeta}_{n,m}$ satisfies

$$\operatorname{Im}\tilde{\zeta}_{n,m} = h^{3/2}\frac{(\pi m)^2}{\sigma^3(2\tilde{B})^{1/2}} + \mathcal{O}(h^2). \tag{B-12}$$

As in the even parity case, the eigenmodes $v_{n,m}$ are close to the odd eigenmodes $\sin(x\pi m/\sigma)$ of the semiclassical Dirichlet Laplacian on $[-\sigma,\sigma]$, and penetrate on a length $\sim h^{1/2}$ inside the damped region.

The case of the square. If the torus is replaced by the square $\left[-\frac{1}{2}, \frac{1}{2}\right] \times [0, 1]$ with Dirichlet boundary conditions, with the same damping function (B-1), the eigenmodes $P(z)$ can as well be factorized into $u(x, y) = \sin(2\pi ny)v(x)$, with $n \in \frac{1}{2}\mathbb{N}\setminus 0$, and $v(x)$ must be an eigenmode of the operator (B-3) vanishing at $x = \pm\frac{1}{2}$. We notice that the odd-parity eigenstates (B-8) satisfy these boundary conditions, so the eigenvalues $z_{n,m}$ (with real parts given by (B-12)) belong to the spectrum of the damped Dirichlet problem.

Similarly, the eigenmodes factorize as $u(x, y) = \cos(2\pi ny)v(x)$, with $n \in \frac{1}{2}\mathbb{N}$, in the case of Neumann boundary conditions. The even-parity states (B-5) satisfy the Neumann boundary conditions at $x = \pm 1/2$, so that the eigenvalues $z_{n,m+1/2}$ described in (B-11) belong to the Neumann spectrum.

As a result, the Dirichlet and Neumann spectra also satisfy Proposition B.1.

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THE J-FLOW ON KÄHLER SURFACES: A BOUNDARY CASE

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We study the J-flow on Kähler surfaces when the Kähler class lies on the boundary of the open cone for which global smooth convergence holds and satisfies a nonnegativity condition. We obtain a C^0 estimate and show that the J-flow converges smoothly to a singular Kähler metric away from a finite number of curves of negative self-intersection on the surface. We discuss an application to the Mabuchi energy functional on Kähler surfaces with ample canonical bundle.

1. Introduction

The J-flow is a parabolic flow on Kähler manifolds with two Kähler classes. It was defined by Donaldson [1999] in the setting of moment maps and by Chen [2000] as the gradient flow of the $\mathcal{J}$-functional appearing in his formula for the Mabuchi energy [1986].

The J-flow is defined as follows. Let X be a compact Kähler manifold with two Kähler metrics ω and χ in different Kähler classes $[\omega]$ and $[\chi]$. Let $\mathcal{P}_\chi$ be the space of smooth χ-plurisubharmonic functions on X:

$$\mathcal{P}_\chi = \{\varphi \mid \chi_\varphi := \chi + dd^c\varphi > 0\}.$$

Then the J-flow is a flow defined in $\mathcal{P}_\chi$ by

$$\frac{\partial}{\partial t}\varphi = c - \frac{n\chi_\varphi^{n-1}\wedge\omega}{\chi_\varphi^n}, \quad \varphi(0) = \varphi_0 \in \mathcal{P}_\chi, \tag{1-1}$$

where c is the topological constant given by

$$c = \frac{n[\chi]^{n-1}\cdot[\omega]}{[\chi]^n}.$$

A stationary point of (1-1) gives a critical Kähler metric $\tilde\chi \in [\chi]$ satisfying

$$c\tilde\chi^n = n\tilde\chi^{n-1}\wedge\omega. \tag{1-2}$$

Donaldson [1999] noted that a smooth critical metric exists only if the cohomological condition $[c\chi - \omega] > 0$ holds. In complex dimension 2, Chen [2000] showed that this necessary condition is sufficient for the existence of a smooth critical metric by observing that in this case, (1-2) is equivalent to the complex Monge–Ampère equation solved by Yau [1978] (see (2-2) below). Chen [2004] also

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established the long time existence for the J-flow (1-1) with any initial data. Weinkove [2004; 2006] showed that the J-flow converges to a critical metric if the cohomological condition $[c\chi - (n-1)\omega] > 0$ holds. In particular, if X is a Kähler surface, a necessary and sufficient condition for convergence of the flow to a smooth critical metric is Donaldson's cohomological condition $[c\chi - \omega] > 0$.

Song and Weinkove [2008] found a necessary and sufficient condition for the convergence of the J-flow in higher dimensions, which we now explain. Define

$$\mathscr{C}_\omega := \big\{[\chi] > 0 \mid \text{there exists } \chi' \in [\chi] \text{ such that } c{\chi'}^{n-1} - (n-1){\chi'}^{n-2} \wedge \omega > 0\big\}. \tag{1-3}$$

Then the J-flow (1-1) converges smoothly to the critical metric solving (1-2) if and only if $[\chi] \in \mathscr{C}_\omega$.

In [Fang et al. 2011; Fang and Lai 2012b], the J-flow was generalized to the general inverse σ_k flow. An analogous necessary and sufficient condition is found to ensure the smooth convergence of the flow.

The behavior of the J-flow in the case when the condition $[\chi] \in \mathscr{C}_\omega$ does *not* hold is still largely open. However, recent progress was made by Fang and Lai [2012a] in the case of a family of Kähler manifolds satisfying the Calabi symmetry condition. It was shown (in the more general case of the inverse σ_k flow) that if the initial metric satisfies the Calabi symmetry, the flow converges to a Kähler current which is the sum of a Kähler metric with a conic singularity and a current of integration along a divisor.

We consider the case when X is a Kähler surface. As discussed above, a necessary and sufficient condition for convergence of the flow to a smooth critical metric is

$$[c\chi - \omega] > 0. \tag{1-4}$$

Donaldson [1999] remarked that if this condition fails, then one might expect the J-flow to blow up over some curves of negative self-intersection. It was observed in [Song and Weinkove 2008, Proposition 4.5] that, applying the results of Buchdahl [1999] and Lamari [1999], there exist a finite number $N \geq 0$, say, of irreducible curves C_i with $C_i^2 < 0$ on X and positive real numbers a_i such that $[c\chi - \omega] - \sum_{i=1}^N a_i[C_i]$ is Kähler. It was shown in [Song and Weinkove 2008] that at least for some sequence of points approaching some C_i, the quantity $|\varphi| + |\Delta_\omega \varphi|$ blows up.

In this paper we describe the behavior of the J-flow for certain classes $[\chi]$ on the boundary of $\mathscr{C}_\omega$. First we introduce some notation: given a closed $(1,1)$-form α, write $[\alpha] \geq 0$ if there exists a smooth closed nonnegative $(1,1)$-form cohomologous to α. We consider any Kähler class $[\chi]$ satisfying

$$[c\chi - \omega] \geq 0. \tag{1-5}$$

All such classes $[\chi]$ lie in the closure of $\mathscr{C}_\omega$. The boundary of $\mathscr{C}_\omega$ consists of Kähler classes $[\chi]$ such that $[c\chi - \omega]$ is *nef*, which means that for every $\varepsilon > 0$ there exists a representative of $[c\chi - \omega]$ which is bounded below by $-\varepsilon\omega$. Further, since

$$[c\chi - \omega]^2 = [\omega]^2 > 0,$$

the class $[c\chi - \omega]$ is nef and big. Nevertheless, to our knowledge, this does not imply that it satisfies (1-5) — see Question 4.1 below. However, at least in many cases the condition (1-5) is equivalent to $[\chi]$ belonging to the closure of $\mathscr{C}_\omega$ in the Kähler cone. This holds for all Hirzebruch surfaces, for example, since explicit nonnegative (1, 1)-forms can be found representing all classes on the boundary of the Kähler cone (see the discussion in [Calabi 1982]).

Our main result is this:

Theorem 1.1. *Let X be a compact Kähler surface with Kähler metrics ω and χ such that*

$$[c\chi - \omega] \geq 0, \quad \textit{where } c = \frac{2[\chi]\cdot[\omega]}{[\chi]^2}.$$

Then there exist a finite number of curves C_i on X of negative self-intersection such that the solution $\varphi(t)$ of the J-flow (1-1) *converges in $C^\infty_{\mathrm{loc}}(X \setminus \bigcup C_i)$ to a continuous function φ_∞, smooth on $X \setminus \bigcup C_i$, satisfying*

$$c\chi^2_{\varphi_\infty} = 2\chi_{\varphi_\infty} \wedge \omega, \quad \textit{for} \quad \chi_{\varphi_\infty} = \chi + dd^c\varphi_\infty \geq 0. \tag{1-6}$$

Moreover, φ_∞ is the unique continuous solution of (1-6) *up to the addition of a constant.*

Our result makes use of some recent works in the study of complex Monge–Ampère equations that appeared after the breakthrough of Kołodziej [1998]. Indeed, the existence of a unique weak solution to the critical equation (1-6) is a direct consequence of a result of Eyssidieux, Guedj, and Zeriahi [Eyssidieux et al. 2009] and Zhang [2006], who generalized Kołodziej's theorem to the degenerate complex Monge–Ampère equation. By comparing with this solution, we obtain our key uniform estimate for $\varphi(t)$ along the J-flow (Proposition 2.2 below). In addition, we use the viscosity methods introduced in [Eyssidieux et al. 2011] to give a second proof of our key estimate. The results of [Eyssidieux et al. 2011] allow us to conclude that the solution of (1-6) is continuous, and that (1-6) can be understood in both the pluripotential and the viscosity senses.

We have an application of our result to the *Mabuchi energy* [1986], a functional which is closely connected to the problem of algebraic stability and existence of constant scalar curvature Kähler (cscK) metrics [Yau 1993; Tian 1997; Donaldson 2002]. Given a Kähler surface (X, χ), the Mabuchi energy is the functional $\mathrm{Mab} : \mathcal{P}_\chi \to \mathbb{R}$ given by

$$\mathrm{Mab}(\varphi) = -\int_0^1 \int_X \frac{\partial\varphi_t}{\partial t}(R_{\chi_{\varphi_t}} - \mu)\chi^n_{\varphi_t}\, dt,$$

where $\{\varphi_t\}_{0\leq t\leq 1}$ is a path in $\mathcal{P}_\chi$ between 0 and φ, $R_{\chi_{\varphi_t}}$ is the scalar curvature of the metric χ_{φ_t}, and μ is the average of the scalar curvature of χ. The value $\mathrm{Mab}(\varphi)$ is independent of the choice of path.

It was conjectured by Tian [1997], assuming X has no nontrivial holomorphic vector fields, that the existence of a cscK metric is equivalent to the *properness* of the Mabuchi energy, meaning that there exists an increasing function $f : [0, \infty) \to \mathbb{R}$ with $\lim_{x\to\infty} f(x) = \infty$ such that

$$\mathrm{Mab}(\varphi) \geq f(E(\varphi)), \quad \text{where} \quad E(\varphi) = \int_X \sqrt{-1}\,\partial\varphi \wedge \bar\partial\varphi \wedge (\chi_0 + \chi_\varphi).$$

This conjecture holds whenever $[\chi] = -c_1(X) > 0$ or if $[\chi] = c_1(X) > 0$ and X has no nontrivial holomorphic vector fields [Tian 1997; 2000; Tian and Zhu 2000]. It also holds on all manifolds with $c_1(X) = 0$, even in the presence of holomorphic vector fields [Tian 2000]. In fact in each case, the function f can be taken to be linear [Tian 2000; Phong et al. 2008]. Chen [2000] showed that on manifolds with

$c_1(X) < 0$, or equivalently, with ample canonical bundle K_X, the Mabuchi energy can be written as a sum of two terms: the first is the $\mathcal{J}$-functional with reference metric ω in $[K_X]$, and the second is a term which is bounded below. In fact, the second term is proper [Tian 2000] (see the discussion in [Song and Weinkove 2008]), and under the cohomological condition $[c\chi - \omega] \geq 0$, the $\mathcal{J}$-functional has a lower bound, as shown in Corollary 3.3 below. Hence we obtain:

Corollary 1.2. *Suppose that X is a compact Kähler surface with ample canonical bundle K_X. Then the Mabuchi energy is proper on the classes $[\chi]$ satisfying*

$$\left(\frac{2[\chi]\cdot[K_X]}{[\chi]^2}\right)[\chi] - [K_X] \geq 0. \tag{1-7}$$

Moreover, the function f in the definition of properness can be taken to be linear.

Thus, since the condition of K_X being ample implies that X has no nontrivial holomorphic vector fields, conjecturally, classes $[\chi]$ in the cone given by (1-7) should admit cscK metrics. The class $[K_X]$ is inside this cone and admits a cscK metric [Aubin 1978; Yau 1978]. The same is true for classes sufficiently close to $[K_X]$ (see [LeBrun and Simanca 1994]). On the other hand, Ross [2006] found Kähler classes on surfaces with K_X ample that do not admit cscK metrics. Corollary 1.2, together with the arguments of [LeBrun and Simanca 1994], suggests that the set of classes that admit cscK metrics is strictly larger than those lying in the cone (1-7).

An outline of the paper is as follows. In Section 2, we prove the key C^0 estimate. We provide two proofs: the first uses smooth maximum principle arguments and the second uses the notion of viscosity solutions from [Eyssidieux et al. 2011]. We complete the proof of the main theorem in Section 3, and in the last section we finish with some questions for further study.

2. The C^0 estimate

For convenience of notation, we assume from now on that $c = 1$. We may do this by considering $(1/c)[\chi]$ instead of $[\chi]$. In addition, we may assume, by modifying the initial data if necessary, that $\chi - \omega \geq 0$.

The key estimate we need is a uniform C^0 estimate for the solution $\varphi(t)$ of the J-flow. We need the following theorem on the degenerate complex Monge–Ampère equation (the C^0 estimate was proved independently in [Zhang 2006] under slightly less general hypotheses).

Theorem 2.1 [Eyssidieux et al. 2009; 2011]. *Let (M, ω) be a compact Kähler manifold of complex dimension n and let α be a semipositive $(1,1)$-form with $\int_M \alpha^n > 0$. For any nonnegative $f \in L^p(M, \omega^n)$, for $p > 1$, with $\int_M f\omega^n = \int_M \alpha^n$, there exists a unique continuous function φ on M with $\alpha + dd^c\varphi \geq 0$ and*

$$(\alpha + dd^c\varphi)^n = f\omega^n, \quad \sup_M \varphi = 0. \tag{2-1}$$

Moreover, $\|\varphi\|_{C^0(M)}$ is uniformly bounded by a constant depending only on p, M, ω, α and $\|f\|_{L^p(M)}$.

Given this, we immediately obtain a solution φ_∞ to (1-6), using the observation of Chen [2000] that the critical equation can be rewritten as a complex Monge–Ampère equation:

$$\chi_\varphi^2 = 2\chi_\varphi \wedge \omega \iff (\chi_\varphi - \omega)^2 = \omega^2. \tag{2-2}$$

Writing $\alpha := \chi - \omega \geq 0$ on the Kähler surface X, we can apply Theorem 2.1 to see that there exists a continuous function φ_∞ solving (1-6). Moverover, φ_∞ is unique up to the addition of a constant.

Next we use the uniform C^0 bound from Theorem 2.1 to obtain:

Proposition 2.2. *We assume that $\chi - \omega \geq 0$ as discussed above. Let $\varphi(t)$ be the solution of J-flow* (1-1) *on the compact Kähler surface X. Then there exists C depending only on the initial data such that for all $t \geq 0$,*

$$\|\varphi(t)\|_{C^0(X)} \leq C. \tag{2-3}$$

Proof. From the introduction, we know

$$[\chi - \omega] - \sum_{i=1}^{N} a_i [C_i] > 0, \tag{2-4}$$

for positive real numbers a_i and irreducible curves C_i of negative self-intersection. Since we are assuming $[\chi - \omega] \geq 0$, we may take the constants a_i to be arbitrarily small. However, we will not need to make use of this last fact.

It follows that there exist Hermitian metrics h_i on the line bundles $[C_i]$ associated to C_i such that

$$\chi - \omega - \sum_{i=1}^{N} a_i R_{h_i} > 0, \tag{2-5}$$

where $R_{h_i} = -dd^c \log h_i$ is the curvature of h_i. Let s_i be a holomorphic section of $[C_i]$ vanishing along C_i to order 1. Recall that we denote $\chi - \omega$ by α.

Next, we apply Theorem 2.1 and write ψ for the solution to the degenerate complex Monge–Ampère equation

$$(\alpha + dd^c \psi)^2 = \omega^2, \quad \alpha + dd^c \psi \geq 0, \tag{2-6}$$

subject to the condition $\sup_X \psi = 0$. We have $\|\psi\|_{C^0(X)} \leq C$.

It follows from a trick of Tsuji [1988], as used in [Eyssidieux et al. 2009], that ψ is smooth away from the curves C_i. Although the proof is the same, the precise statement we need does not seem to be quite contained in [Eyssidieux et al. 2009], so we briefly outline the idea here for the convenience of the reader. For $\delta > 0$, let ψ_δ be Yau's solution of the complex Monge–Ampère equation

$$(\alpha + \delta\omega + dd^c \psi_\delta)^2 = c_\delta \omega^2, \quad \alpha_\delta := \alpha + \delta\omega + dd^c \psi_\delta > 0, \tag{2-7}$$

for a constant c_δ chosen so that the integrals of both sides are equal. From Theorem 2.1, ψ_δ is uniformly bounded in C^0. To obtain a second-order estimate for ψ_δ, uniform in δ, we consider, for a constant $A > 0$,

$$Q_\delta = \log \operatorname{tr}_\omega \alpha_\delta - A\left(\psi_\delta - \sum_i a_i \log |s_i|^2_{h_i}\right), \tag{2-8}$$

which is well-defined on $X \setminus \bigcup C_i$ and tends to $-\infty$ on $\bigcup C_i$. Compute, at a point in $X \setminus \bigcup C_i$,

$$\Delta_{\alpha_\delta} Q_\delta \geq -C \operatorname{tr}_{\alpha_\delta} \omega - 2A + A \operatorname{tr}_{\alpha_\delta}\left(\alpha - \sum_i a_i R_{h_i}\right).$$

Then using (2-5), we may choose a uniform A sufficiently large that

$$A\Bigl(\alpha - \sum_i a_i R_{h_i}\Bigr) \geq (C+1)\omega.$$

The quantity Q_δ achieves a maximum at some point $x \in X \setminus \bigcup C_i$, and at this point we have $\Delta_{\alpha_\delta} Q_\delta \leq 0$. Hence, at x,

$$0 \geq \operatorname{tr}_{\alpha_\delta} \omega - 2A,$$

so $\operatorname{tr}_{\alpha_\delta} \omega$ is uniformly bounded from above. But by (2-7) we have at x

$$\operatorname{tr}_\omega \alpha_\delta = \left(\frac{\alpha_\delta^2}{\omega^2}\right) \operatorname{tr}_{\alpha_\delta} \omega = c_\delta \operatorname{tr}_{\alpha_\delta} \omega \leq C',$$

for some uniform C'. Since ψ_δ is uniformly bounded in C^0, we see that Q_δ is uniformly bounded from above at x, and hence everywhere.

This establishes a uniform upper bound for $\operatorname{tr}_\omega \alpha_\delta$ (and again by (2-7), also for $\operatorname{tr}_{\alpha_\delta} \omega$) on any compact subset of $X \setminus \bigcup C_i$. It follows that on such a fixed compact set, ω and α_δ are uniformly equivalent. Hence we have estimates, uniform in δ, for $dd^c\psi_\delta$ on compact subsets of $X \setminus \bigcup C_i$. The $C^\infty_{\mathrm{loc}}\bigl(X \setminus \bigcup C_i\bigr)$ estimates for ψ_δ then follow from the usual Evans–Krylov local theory for the complex Monge–Ampère equation [Evans 1982; Krylov 1982]. Taking a limit as $\delta \to 0$ shows that ψ is smooth away from the C_i.

Fix $\varepsilon \in (0, 1)$. We will apply the maximum principle to the quantity

$$\theta_\varepsilon = \varphi - (1+\varepsilon)\psi + \varepsilon \sum_{i=1}^N a_i \log |s_i|^2_{h_i} - A\varepsilon t,$$

where A is a constant to be determined. Observe that θ_ε is smooth on $X \setminus \bigcup C_i$ and tends to negative infinity along $\bigcup C_i$, and hence θ_ε achieves a maximum in the interior of $X \setminus \bigcup C_i$ for each time t.

We rewrite (1-1) as

$$\frac{\partial\varphi}{\partial t} = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} = \frac{\chi_\varphi^2 - 2\chi_\varphi \wedge \omega}{\chi_\varphi^2} = \frac{(\chi_\varphi - \omega)^2 - \omega^2}{\chi_\varphi^2} = \frac{\omega^2}{\chi_\varphi^2}\left(\frac{(\chi_\varphi - \omega)^2}{\omega^2} - 1\right) = \frac{\omega^2}{\chi_\varphi^2}\left(\frac{\alpha_\varphi^2}{\alpha_\psi^2} - 1\right). \tag{2-9}$$

Compute on $X \setminus \bigcup C_i$, using (2-9),

$$\begin{aligned}\frac{\partial}{\partial t}\theta_\varepsilon &= \frac{\omega^2}{\chi_\varphi^2}\left(\frac{(\alpha + dd^c\varphi)^2}{(\alpha + dd^c\psi)^2} - 1\right) - A\varepsilon \\ &= \frac{\omega^2}{\chi_\varphi^2}\left(\frac{\bigl((1+\varepsilon)\alpha + (1+\varepsilon)dd^c\psi - \varepsilon(\alpha - \sum a_i R_{h_i}) + dd^c\theta_\varepsilon\bigr)^2}{(\alpha + dd^c\psi)^2} - 1\right) - A\varepsilon.\end{aligned}$$

But $\alpha - \sum a_i R_{h_i} \geq 0$, and at the maximum of θ_ε, we have $dd^c\theta_\varepsilon \leq 0$. Hence at the maximum of θ_ε,

$$\frac{\partial}{\partial t}\theta_\varepsilon \leq \frac{\omega^2}{\chi_\varphi^2}\left((1+\varepsilon)^2\frac{(\alpha + dd^c\psi)^2}{(\alpha + dd^c\psi)^2} - 1\right) - A\varepsilon < 0, \tag{2-10}$$

if we choose

$$A = \sup_{X\times[0,\infty)} \frac{3\omega^2}{\chi_\varphi^2},$$

which is a uniform constant since χ_φ is always uniformly bounded from below away from zero along the J-flow. Indeed, this follows immediately from taking a time derivative of the J-flow equation and applying the maximum principle (see Lemma 4.1 in [Chen 2004]). Then (2-10) implies that θ_ε must achieve its maximum at time zero, and hence θ_ε is uniformly bounded from above by a constant independent of ε. Letting $\varepsilon \to 0$, we obtain the upper bound for φ.

The lower bound of φ is similar: just replace ε by $-\varepsilon$ and consider the minimum instead of the maximum. □

We provide a second proof. The proof is based on the equivalence of two notions of weak solution of (2-2): the pluripotential sense and the viscosity sense.

Second proof of Proposition 2.2. As in the first proof, write ψ for the solution to (2-6) with $\sup_X \psi = 0$. The function ψ is continuous on X and is smooth away from the curves C_i. We now apply Theorem 3.6 of [Eyssidieux et al. 2011], which states that ψ satisfies (2-6) in the viscosity sense as defined in that paper.

We refer to [Eyssidieux et al. 2011] for the precise definition of a viscosity solution to (2-6) and state two consequences of this definition which are sufficient for our purposes:

(i) If x_0 is any point on X and q is any smooth function defined in a neighborhood of x_0 such that

$$\psi - q \text{ has a local maximum at } x_0,$$

then $(\alpha + dd^c q)^2 \geq \omega^2$ at x_0.

(ii) If x_0 is any point on X and q is any smooth function defined in a neighborhood of x_0 such that

$$\psi - q \text{ has a local minimum at } x_0,$$

then $(\alpha + dd^c q)^2 \leq \omega^2$ at x_0.

Indeed, (i) follows from the definition of a viscosity subsolution, and (ii) from the definition of a viscosity supersolution (see Section 2 in [Eyssidieux et al. 2011]).

We first find an upper bound for φ. Let $\varepsilon > 0$ and define $H_\varepsilon = \varphi - \psi - \varepsilon t$. We wish to show that H_ε attains its maximum value at $t = 0$. Note that H_ε satisfies the equation

$$\frac{\partial H_\varepsilon}{\partial t} = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon.$$

Suppose that H_ε attains a maximum at a point (x_0, t_0) on $X \times [0, T]$ for some finite $T > 0$, and assume for a contradiction that $t_0 > 0$. Then $\partial H_\varepsilon/\partial t\,(x_0, t_0) \geq 0$. Define a smooth function q on X by $q(x) = \varphi(x, t_0) - H_\varepsilon(x_0, t_0) - \varepsilon t_0$. The function

$$x \mapsto (\psi - q)(x) = -H_\varepsilon(x, t_0) + H_\varepsilon(x_0, t_0)$$

achieves its minimum at x_0. Then we can apply (ii) to see that $(\alpha + dd^c q)^2 \le \omega^2$ at x_0, or in other words

$$(\chi - \omega + dd^c\varphi)^2 \le \omega^2, \quad \text{at } (x_0, t_0),$$

which is equivalent to

$$\chi_\varphi^2 \le 2\chi_\varphi \wedge \omega \quad \text{at } (x_0, t_0).$$

It follows that

$$\frac{\partial H_\varepsilon}{\partial t}(x_0, t_0) = 1 - \frac{2\chi_\varphi \wedge \omega}{\chi_\varphi^2} - \varepsilon < 0,$$

contradicting the fact that $\partial H_\varepsilon/\partial t\,(x_0, t_0) \ge 0$. Hence H_ε attains its maximum value at $t = 0$ and is uniformly bounded from above independent of ε. Letting $\varepsilon \to 0$ gives the desired upper bound for φ.

Applying a similar argument, using (i) instead of (ii), gives a uniform lower bound for φ. □

We can now apply Theorem 1.3 of [Song and Weinkove 2008] together with the standard local theory for (1-1) to obtain higher-order estimates.

Proposition 2.3. *As above, assume that $\chi - \omega \ge 0$ on the compact Kähler surface X and let $\varphi(t)$ be the solution of the J-flow* (1-1). *For any compact subset $K \subset X \setminus \bigcup C_i$ and any $k \ge 0$, there exists a constant $C_{k,K}$ such that for all t,*

$$\|\varphi(t)\|_{C^k(K)} \le C_{k,K}.$$

Here, the C_i are the irreducible curves of negative self-intersection chosen to satisfy (2-4).

3. Proof of the main theorem

Again we assume in this section that $[\chi]$ is scaled so that $c = 1$. Before proving the main theorem we first discuss the $\mathcal{J}$ and $\mathcal{I}$-functionals. Define $\mathcal{J}_{\omega,\chi}$ and $\mathcal{I}_{\omega,\chi}$ by

$$\mathcal{J}_{\omega,\chi}(\varphi) := \int_0^1 \int_X \dot{\varphi}_t \big(2\chi_{\varphi_t} \wedge \omega - \chi_{\varphi_t}^2\big)\, dt,$$

$$\mathcal{I}_{\omega,\chi}(\varphi) := \int_0^1 \int_X \dot{\varphi}_t \chi_{\varphi_t}^2\, dt,$$

where φ_t is a smooth path in $\mathcal{P}_\chi$ connecting 0 and φ. For simplicity, we will omit the subscripts.

If $\varphi(t)$ is the solution of the J-flow, then

$$\frac{d}{dt}\mathcal{J}(\varphi(t)) = -\int_X \dot{\varphi}(t)^2 \chi_{\varphi(t)}^2, \quad \frac{d}{dt}\mathcal{I}(\varphi(t)) = 0. \tag{3-1}$$

In particular, the J-flow is the gradient flow of $\mathcal{J}$.

One can write explicit formulae for $\mathcal{J}$, $\mathcal{I}$ as follows:

$$\mathcal{J}(\varphi) = \int_X \varphi\big(\chi_\varphi \wedge \omega + \chi \wedge \omega\big) - \frac{1}{3}\int_X \varphi\big(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2\big), \tag{3-2}$$

$$\mathcal{I}(\varphi) = \frac{1}{3}\int_X \varphi\big(\chi_\varphi^2 + \chi_\varphi \wedge \chi + \chi^2\big). \tag{3-3}$$

Thus an immediate corollary of Proposition 2.2 is:

Proposition 3.1. *There exists a uniform constant C such that, for $\varphi(t)$ the solution of the J-flow, we have*

$$\mathcal{J}(\varphi(t)) \geq -C$$

for all $t \geq 0$.

In what follows, we will need to make use of a simple continuity-type result for the $\mathcal{I}$ and $\mathcal{J}$ functionals.

Lemma 3.2. *Let $\varphi_j \in \mathcal{P}_\chi$ and let φ be a continuous function on X satisfying $\chi + dd^c\varphi \geq 0$. Let Y be a proper subvariety of X. Suppose that*

(a) *there exists C such that $\|\varphi_j\|_{C^0(X)} \leq C$;*

(b) *$\varphi_j \to \varphi$ in $C^\infty_{\mathrm{loc}}(X \setminus Y)$ as $j \to \infty$.*

Then

$$\mathcal{J}(\varphi_j) \to \mathcal{J}(\varphi) \quad \textit{and} \quad \mathcal{I}(\varphi_j) \to \mathcal{I}(\varphi) \quad \textit{as } j \to \infty.$$

Proof. The proof is a simple exercise in pluripotential theory (we refer the reader to [Kołodziej 2005] for an introduction to this theory). For the convenience of the reader, we sketch the proof here. For φ continuous with $\chi + dd^c\varphi \geq 0$, the quantities χ_φ^2, $\chi \wedge \chi_\varphi$ and $\chi_\varphi \wedge \omega$ define finite measures on X and hence by (3-2) and (3-3), the functionals $\mathcal{I}(\varphi)$ and $\mathcal{J}(\varphi)$ are well-defined.

We may choose a sequence of open tubular neighborhoods Y_k of Y such that $Y_k \downarrow Y$ as $k \to \infty$. Since Y is pluripolar, the capacity $\mathrm{Cap}_\chi(Y)$ of Y with respect to χ (in the sense of [Kołodziej 1998]) is zero. By the properties of this capacity (see [Guedj and Zeriahi 2005], for example) we have

$$\lim_{k\to\infty} \mathrm{Cap}_\chi(Y_k) = \mathrm{Cap}_\chi(Y) = 0.$$

Since the φ_j are uniformly bounded, it follows that $\int_{Y_k} \varphi_j \beta \wedge \gamma \to 0$ as $k \to \infty$, uniformly in j, where β, γ are each one of ω, χ or χ_{φ_j}. The same holds if we replace φ_j by φ. The result then follows from the expressions (3-2) and (3-3) together with condition (b). □

Proof of Theorem 1.1. Since $\mathcal{J}$ is decreasing and bounded from below, there exists a constant C such that

$$\int_0^\infty \int_X \dot{\varphi}(t)^2 \chi^2_{\varphi(t)}\, dt < C. \tag{3-4}$$

We claim that for each fixed point $p \in X \setminus \bigcup C_i$, we have $\dot{\varphi}(p, t) \to 0$ as $t \to \infty$. Suppose not. Then there exists $\varepsilon > 0$ and a sequence of times $t_i \to \infty$ such that $|\dot{\varphi}(t_i)| > \varepsilon$ for all i. But since we have bounds for $\dot{\varphi}$ and all its time and space derivatives in a fixed neighborhood U, say, of p with $U \subset X \setminus \bigcup C_i$, it follows that $|\dot{\varphi}(t)| > \varepsilon/2$ for $t \in [t_i, t_i + \delta]$ for a uniform $\delta > 0$. This contradicts (3-4) and establishes the claim.

Since we have $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$ bounds for $\dot{\varphi}$, the uniqueness of limits implies that $\dot{\varphi}$ converges to zero in $C^\infty_{\mathrm{loc}}\big(X \setminus \bigcup C_i\big)$.

We have uniform C^∞ bounds for $\varphi(t)$ on compact subsets of $X \setminus \bigcup C_i$, and hence we can apply the Arzelà–Ascoli theorem to see that for a sequence of times $t_i \to \infty$, we have $\varphi(t_i) \to \varphi_\infty$ for a smooth (bounded) function φ_∞ on $X \setminus \bigcup C_i$. Since $\dot{\varphi} \to 0$, φ_∞ satisfies the equation $\chi^2_{\varphi_\infty} = 2\chi_{\varphi_\infty} \wedge \omega$ as in the statement of the theorem.

We also have $\mathcal{I}(\varphi_\infty) = \lim_{t\to\infty} \mathcal{I}(\varphi(t)) = \mathcal{I}(\varphi_0)$, using Lemma 3.2 and the fact that $\mathcal{I}$ is constant along the flow. Applying Theorem 2.1, we know that (1-6) has a unique solution up to the addition of a constant. Thus φ_∞ is the unique solution of (1-6) subject to the condition $\mathcal{I}(\varphi_\infty) = \mathcal{I}(\varphi_0)$.

Finally we claim that $\varphi(t)$ converges in $C^\infty_{\mathrm{loc}}(X \setminus \bigcup C_i)$ to φ_∞. Suppose not. Then there exist $\varepsilon > 0$ and a sequence of times $t_i \to \infty$ such that $\|\varphi(t_i) - \varphi_\infty\|_{C^k(K)} > \varepsilon$ for all i, for some integer k and compact $K \subset X \setminus \bigcup C_i$. Since we have uniform C^∞ bounds for $\varphi(t)$ on K, we can pass to a subsequence and assume that $\varphi(t_i)$ converges to a function $\varphi'_\infty \neq \varphi_\infty$. But φ'_∞ will also satisfy the equations $\chi^2_{\varphi'_\infty} = 2\chi_{\varphi'_\infty} \wedge \omega$ and $\mathcal{I}(\varphi'_\infty) = \mathcal{I}(\varphi_0)$, contradicting the uniqueness. □

As a consequence:

Corollary 3.3. *The $\mathcal{J}$-functional is bounded from below on $\mathcal{P}_\chi$.*

Proof. Take any $\varphi_0 \in \mathcal{P}_\chi$. Then running the J-flow from φ_0, which by Theorem 1.1 converges to φ_∞, we obtain (applying Lemma 3.2)

$$\mathcal{J}(\varphi_0) \geq \lim_{t\to\infty} \mathcal{J}(\varphi(t)) = \mathcal{J}(\varphi_\infty),$$

since $\mathcal{J}$ is decreasing along the flow. □

Proof of Corollary 1.2. Combine Corollary 3.3 and Lemma 4.1 of [Song and Weinkove 2008]. □

4. Further questions

Question 4.1. In general, it does not appear to be known whether a nef and big class on a Kähler surface can always be represented by a smooth nonnegative (1, 1)-form (for a counterexample in higher dimensions, see Example 5.4 in [Boucksom et al. 2010]). However, an example of Zariski shows that a nef and big class is not necessarily semiample (see Section 2.3A of [Lazarsfeld 2004]). Also, the nef condition alone is not sufficient for the existence of a nonnegative representative (see Example 1.7 of [Demailly et al. 1994]). What can be proved if we assume only that $[\chi - \omega]$ is nef and big? In this case, by [Boucksom et al. 2010], we know that we can produce a solution ψ of (2-2) with very mild singularities along C_i (less than any log pole). Can it be translated into an estimate for the solution $\varphi(t)$ of the J-flow? Does it imply that the J-functional is bounded from below?

Question 4.2. The results of [Fang and Lai 2012a] indicate a possible picture when $[\chi]$ is outside of $\mathcal{C}_\omega$. But they assume both ω and χ are of Calabi ansatz. Can one prove a general result on Kähler surfaces? In this case, presumably the $\mathcal{J}$-functional is not bounded from below.

Question 4.3. For general n, it would be interesting to investigate the weak solution of the critical equation (1-2) when $[\chi]$ does not lie in $\mathcal{C}_\omega$.

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A PRIORI ESTIMATES FOR COMPLEX HESSIAN EQUATIONS

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We prove some L^∞ a priori estimates as well as existence and stability theorems for the weak solutions of the complex Hessian equations in domains of $\mathbb{C}^n$ and on compact Kähler manifolds. We also show optimal L^p integrability for m-subharmonic functions with compact singularities, thus partially confirming a conjecture of Błocki. Finally we obtain a local regularity result for $W^{2,p}$ solutions of the real and complex Hessian equations under suitable regularity assumptions on the right-hand side. In the real case the method of this proof improves a result of Urbas.

Introduction

Hessian equations. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be the set of eigenvalues of a Hermitian $n \times n$ matrix A. By $S_m(A)$ denote the m-th elementary symmetric function of λ:

$$S_m(A) = \sum_{0<j_1<\cdots<j_m\le n} \lambda_{j_1} \lambda_{j_2} \dots \lambda_{j_m}.$$

If A is the complex Hessian of a real valued C^2 function u defined in $\Omega \subset \mathbb{C}^n$ then we have a pointwise defined function

$$\sigma_m(u_{z_j\bar{z}_k})(z) = S_m\big((u_{z_j\bar{z}_k}(z))\big).$$

In terms of differential forms, with $d = \partial + \bar{\partial}$, $d^c = i(\bar{\partial} - \partial)$ and $\beta = dd^c\|z\|^2$ this function satisfies

$$(dd^c u)^m \wedge \beta^{n-m} = \frac{m!(n-m)!}{n!} \sigma_m(u_{z_j\bar{z}_k})\beta^n.$$

We call a C^2 function $u : \Omega \to \mathbb{C}^n$ *m-subharmonic*, or *m-sh*, if the forms

$$(dd^c u)^k \wedge \beta^{n-k}$$

are positive for $k = 1, \dots, m$ (in particular u is subharmonic). If u is subharmonic but not smooth, one can define m-sh function via inequalities for currents (see definitions in Section 1).

As shown by Błocki [2005] m-sh functions are the right class of admissible solutions to the complex Hessian equation

$$(dd^c u)^m \wedge \beta^{n-m} = f\beta^n \tag{0-1}$$

for a given nonnegative function f. Observe that for $m = 1$ this is the Poisson equation and for $m = n$ the complex Monge–Ampère equation.

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Analogously, one can define *m-subharmonic functions with respect to a Kähler form ω* (abbreviatied *m-ω-sh*) and the corresponding Hessian equation just replacing β with ω in the preceding definitions. This definition can also be extended to subharmonic functions. Then one can consider such functions on Kähler manifolds.

Since on compact Kähler manifolds the sets of m-ω-sh functions are trivial we define in this case ω-m-subharmonic (ω-m-sh) functions requiring that

$$(dd^c u + \omega)^k \wedge \omega^{n-k} \geq 0, \quad k = 1, \dots, m.$$

and consider the Hessian equation on a compact Kähler manifold X, as in [Hou 2009; Hou et al. 2010; Dinew and Kołodziej 2012]:

$$(dd^c u + \omega)^m \wedge \omega^{n-m} = f\omega^n, \quad \int_X f\omega^n = \int_X \omega^n. \tag{0-2}$$

Solving the equation we look for ω-m-sh solutions u. The normalization of f is necessary because of Stokes' theorem and the Kähler condition $d\omega = 0$.

Background. The real Hessian equation has been studied in many papers, for example [Caffarelli et al. 1985; Ivochkina et al. 2004; Krylov 1995; Trudinger 1995; Trudinger and Wang 1999; Labutin 2002; Chou and Wang 2001; Urbas 2001]. In particular the Dirichlet problem is solvable for smooth and strictly positive right-hand side under natural convexity assumptions on the boundary of the considered domain [Caffarelli et al. 1985]. This result is the starting point of study of degenerate Hessian equations [Ivochkina et al. 2004] and regularity of weak solutions [Urbas 2001]. A nonlinear potential theory has also been developed [Trudinger and Wang 1999; Labutin 2002]. We refer to [Wang 2009] for a survey of the real Hessian equation theory. It is interesting that the real and complex theories are very different, and attempts to apply "real" methods directly to the complex Hessian equation often fail. See [Błocki 2003; 2009] for a detailed study of those discrepancies.

The complex Hessian equation (0-1) in domains of $\mathbb{C}^n$ was first considered by S.-Y. Li [2004]. His main result says that if Ω is smoothly bounded and $(m-1)$-pseudoconvex (that means that S_j, $j = 1, \dots, m-1$, applied to the Levi form of $\partial\Omega$ are positive on the complex tangent to $\partial\Omega$) then, for smooth boundary data and for smooth, positive right-hand side there exists a unique smooth solution of the Dirichlet problem for the Hessian equation. The proof is in the spirit of the one in [Caffarelli et al. 1985].

Błocki [2005] considered also weak solutions of the equation, for possibly degenerate right-hand side, introducing some elements of potential theory for m-sh functions based on positivity of currents which are used in the definition. He proved that a m-sh function u is maximal in this class if and only if

$$(dd^c u)^m \wedge \beta^{n-m} = 0.$$

Furthermore he described the maximal domain of definition of the Hessian operator.

As for the equation on compact Kähler manifolds (0-2), Hou [2009] has shown that the solutions, for smooth positive f, exist under the assumption that the metric has nonnegative holomorphic bisectional curvature. Similar results were independently obtained in [Kokarev 2010; Jbilou 2010]. Finally in [Dinew and Kołodziej 2012] the authors removed the curvature assumptions thus obtaining an analogue of the Calabi–Yau theorem for the complex Hessian equations.

New results. The m-subharmonic functions for $m < n$ are much more difficult to handle than the plurisubharmonic ones ($m = n$). They lack a nice geometric description by the mean value property along planes, there is no invariance of the family under holomorphic mappings, and so forth. The cones of m-ω-sh functions are even worse — they are not invariant under translations. Despite that, the pluripotential theory methods developed in [Bedford and Taylor 1982; Kołodziej 1996; 1998; 2003] for the Monge–Ampère equation can be adapted to the Hessian equations. The crucial estimate between volume and capacity in Proposition 2.1 allowed us to prove a sharp integrability statement (conjectured in a stronger form in [Błocki 2005]): m-subharmonic functions, $m < n$, belong to L^q for any $q < mn/(n-m)$, if their level sets are relatively compact in the domain where they are defined. For a plurisubharmonic function u much stronger statement is true: $\exp(-au)$ is locally integrable for some $a > 0$. This accounts for the difference in statements of L^∞ estimates for the Hessian equations and the Monge–Ampère equation. We show a priori L^∞ bounds for the solutions of

$$(dd^c u)^m \wedge \beta^{n-m} = f\omega^n \tag{0-3}$$

(with continuous boundary data) and those of (0-2) with f belonging to L^q, $q > n/m$. We also get strong stability theorems for those solutions. As a consequence one obtains that the families of solutions corresponding to data uniformly bounded in L^q norms are equicontinuous.

The a priori estimates lead to the (continuous) solution of the Dirichlet problem in $(m-1)$-pseudoconvex domains for nonnegative right-hand side in the same L^q spaces as above (Theorem 2.10). The corresponding existence result is also true on compact Kähler manifolds (Theorem 3.3). Those are the extensions of theorems in [Li 2004] and [Hou 2009]. Finally we prove the local regularity statement in Theorem 4.1 which in the case of the Monge–Ampère equation is due to Błocki and Dinew [2011]. It is worth noting that our methods applied to the real Hessian equations yield improvement of the regularity exponent obtained by Urbas [2001].

1. Preliminaries

We briefly recall the notions that we shall need later on. We start with a linear algebra toolkit.

Linear algebra preliminaries. Consider the set $\mathcal{M}_n$ of all Hermitian symmetric $n \times n$ matrices. For a given matrix $M \in \mathcal{M}_n$ let $\lambda(M) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ be its eigenvalues arranged in the decreasing order and let

$$S_k(M) = S_k(\lambda(M)) = \sum_{0<j_1<\cdots<j_m\le n} \lambda_{j_1}\lambda_{j_2}\dots\lambda_{j_m}$$

be the k-th elementary symmetric polynomial applied to the vector $\lambda(M)$.

Then one can define the positive cones Γ_m as

$$\Gamma_m = \{\lambda \in \mathbb{R}^n \mid S_1(\lambda) > 0, \dots, S_m(\lambda) > 0\}. \tag{1-1}$$

The definition of Γ_m is nonlinear if $m > 1$; hence a priori it is unclear whether these sets are indeed convex cones. But the vectors in Γ_m, and hence the set of matrices with corresponding eigenvalues enjoy several convexity properties resembling the properties of positive definite matrices, and in particular the convexity of Γ_m.

Now let V be a fixed positive definite Hermitian matrix and $\lambda_i(V)$ be the eigenvalues of a Hermitian matrix M with respect to V. The we can analogously define the sets $\Gamma_k(V)$.

We list the properties of these cones that will be used later on:

(1) Maclaurin's inequality: If $\lambda \in \Gamma_m$ then $\left(S_j/\binom{n}{j}\right)^{1/j} \geq \left(S_i/\binom{n}{i}\right)^{1/i}$ for $1 \leq j \leq i \leq m$.

(2) Gårding's inequality [1959]: Γ_m is a convex cone for any m and the function $S_m^{1/m}$ is concave when restricted to Γ_m.

(3) [Wang 2009]: Let $S_{k;i}(\lambda) := S_k(\lambda)_{\lambda_i=0} = \dfrac{\partial S_{k+1}}{\partial \lambda_i}(\lambda)$. For any $\lambda, \mu \in \Gamma_m$,

$$\sum_{i=1}^{n} \mu_i S_{m-1;i}(\lambda) \geq m S_m(\mu)^{1/m} S_m(\lambda)^{(m-1)/m}. \tag{1-2}$$

We refer to [Błocki 2005] or [Wang 2009] for further properties of these cones.

Potential theoretic aspects of m-subharmonic functions. Let us fix a relatively compact domain $\Omega \Subset \mathbb{C}^n$. Let also $d = \partial + \bar{\partial}$ and $d^c := i(\bar{\partial} - \partial)$ be the standard exterior differentiation operators. By $\beta := dd^c\|z\|^2$ we denote the Euclidean Kähler form in $\mathbb{C}^n$.

Given a $\mathscr{C}^2(\Omega)$ function u we call it $m-\beta$-subharmonic if for any $z \in \Omega$ the Hessian matrix $(\partial^2 u/\partial z_i \partial \bar{z}_j)(z)$ has eigenvalues forming a vector in the closure of the cone Γ_m. Analogously if ω is any other Kähler form in Ω, u is m-ω-subharmonic if the Hessian matrix has eigenvalues at z forming a vector in $\overline{\Gamma}_m(\omega(z))$ (the latter set will depend on z in general).

Since the $\omega = \beta$ is the most natural case in the flat domains we shall call $m-\beta$-subharmonic functions just m-subharmonic or m-sh for short.

Observe that in the language of differential forms u is m-ω-subharmonic if and only if the following inequalities hold:

$$(dd^c u)^k \wedge \omega^{n-k} \geq 0, \quad k = 1, \dots, m.$$

It was observed by Błocki [2005] that, following the ideas of Bedford and Taylor [1976; 1982], one can relax the smoothness requirement on u and develop a nonlinear version of potential theory for Hessian operators.

The relevant definitions are as follows:

Definition 1.1. Let u be a subharmonic function on a domain $\Omega \Subset \mathbb{C}^n$. Then u is called m-subharmonic (m-sh for short) if for any collection of $\mathscr{C}^2$-smooth m-sh functions $v_1, \dots, v_{m-1}$ the inequality

$$dd^c u \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{m-1} \wedge \beta^{n-m} \geq 0$$

holds in the weak sense of currents. For a general Kähler form ω the notion of m-ω-subharmonic function is defined by formally stronger condition: locally, in a neighborhood of any given point, there exists a decreasing to u sequence of $\mathscr{C}^2$-smooth m-ω-sh functions u_j such that for any set of $\mathscr{C}^2$-smooth m-ω-sh functions $v_1, \dots, v_{m-1}$ the inequality

$$dd^c u_j \wedge dd^c v_1 \wedge \cdots \wedge dd^c v_{m-1} \wedge \omega^{n-m} \geq 0$$

is satisfied. (For $\omega = \beta$ this condition is satisfied due to Proposition 1.3(4).)

The set of all m-ω-sh functions is denoted by $\mathscr{SH}_m(\omega, \Omega)$.

Remark 1.2. It is enough to test m-subharmonicity of u against a collection of m-sh quadratic polynomials (see [Błocki 2005]).

Using the approximating sequence u_j from the definition one can follow the Bedford and Taylor construction [1982] of the wedge products of currents given by locally bounded m-ω-sh functions. They are defined inductively by

$$u_1\, dd^c u_2 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-m} := dd^c(u_1 \wedge \cdots \wedge dd^c u_p \wedge \omega^{n-m}).$$

It can be shown (see [Błocki 2005]) that analogously to the pluripotential setting these currents are continuous under monotone or uniform convergence of their potentials.

Here we list some basic facts about m-subharmonicity (assuming $\mathscr{C}^2$ smoothness).

Proposition 1.3. *Let $\Omega \subset \mathbb{C}^n$ be a domain. Then*:

(1) $\mathscr{SH}_1(\omega, \Omega) \subset \mathscr{SH}_2(\omega, \Omega) \subset \cdots \subset \mathscr{SH}_n(\omega, \Omega)$.

(2) $\mathscr{SH}_m(\omega, \Omega)$ *is a convex cone.*

(3) *If $u \in \mathscr{SH}_m(\omega, \Omega)$ and $\gamma : \mathbb{R} \to \mathbb{R}$ is a $\mathscr{C}^2$-smooth convex, increasing function then $\gamma \circ u \in \mathscr{SH}_m(\omega, \Omega)$.*

(4) *the standard regularizations $u * \rho_\varepsilon$ of a m-sh function is again m-sh.*

Proof. The first claim is trivial. Second claim is proved in [Błocki 2005], with the use of Gårding's inequality [Gårding 1959]. Last two claims are more or less standard and their proofs are analogous to corresponding results for psh (plurisubharmonic) functions. Observe that the last property does fail for a general Kähler form ω. □

The following two theorems, known as comparison principles in pluripotential theory, follow essentially from the same arguments as in the case $m = n$:

Theorem 1.4. *Let u, v be continuous m-ω-sh functions in a domain $\Omega \Subset \mathbb{C}^n$. Suppose that*

$$\liminf_{z \to \partial\Omega} (u - v)(z) \geq 0.$$

Then

$$\int_{\{u<v\}} (dd^c v)^m \wedge \omega^{n-m} \leq \int_{\{u<v\}} (dd^c u)^m \wedge \omega^{n-m}.$$

Theorem 1.5. *Let u, v be continuous m-ω-sh functions in a domain $\Omega \Subset \mathbb{C}^n$. Suppose that*

$$\liminf_{z \to \partial\Omega} (u - v)(z) \geq 0 \quad \text{and} \quad (dd^c v)^m \wedge \omega^{n-m} \geq (dd^c u)^m \wedge \omega^{n-m}.$$

Then $v \leq u$ in Ω.

The last result yields, in particular, uniqueness of bounded weak solutions of the Dirichlet problem. As for the existence we have the following fundamental theorem:

Theorem 1.6 [Li 2004]. *Let Ω be a smoothly bounded relatively compact domain in $\mathbb{C}^n$. Suppose that $\partial\Omega$ is $(m-1)$-pseudoconvex (that means that Levi form at any point $p \in \partial\Omega$ has its eigenvalues in the cone Γ_{m-1}). Let φ be a smooth function on $\partial\Omega$ and f a strictly positive and smooth function in Ω. Then the Dirichlet problem*

$$\begin{cases} u \in \mathcal{SH}_m(\Omega, \beta) \cap \mathcal{C}(\bar{\Omega}), \\ (dd^c u)^m \wedge \beta^{n-m} = f, \\ u|_{\partial\Omega} = \varphi, \end{cases}$$

has a smooth solution u.

The convexity properties of the cones Γ_m yield the following mixed Hessian inequalities:

Proposition 1.7. *Let $u_1, \dots, u_m$ be m-sh $\mathcal{C}^2$ functions in a domain $\Omega \Subset \mathbb{C}^n$. Suppose $(dd^c u_j)^m \wedge \beta^{n-m} = f_j$ for some continuous nonnegative functions f_j. Then*

$$dd^c u_1 \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m} \geq (f_1 \cdots f_m)^{1/m} \beta^n.$$

Proof. Pointwise this reduces to the Gårding inequality; see also (1-2) for the case $u_2 = u_3 = \cdots = u_m$. □

Later on in Theorem 2.12 we shall see that the smoothness assumptions here can be considerably relaxed.

Kähler setting. Given a compact Kähler manifold (X, ω) we can define the cones $\mathcal{SH}_m(X, \omega)$ consisting of those functions u for which, in a local chart Ω where ω has a potential ρ, the function $u + \rho$ belongs to $\mathcal{SH}_m(\Omega, \omega)$. The definition is independent of the choice of the chart and the potential. This essentially allows us to carry over all local results to this setting. We refer to [Kołodziej 2005] for the plurisubharmonic ($m = n$) case.

The Dirichlet problem for smooth nondegenerate data was recently solved:

Proposition 1.8 [Dinew and Kołodziej 2012]. *Let (X, ω) be a compact Kähler manifold and $1 < m < n$ be an integer number. Given a strictly positive smooth function f satisfying the condition $\int_X f\omega^n = \int_X \omega^n$ there is an unique function $u \in \mathcal{SH}_m(X, \omega) \cap \mathcal{C}^\infty(X)$ solving the Dirichlet problem*

$$(dd^c u + \omega)^m \wedge \omega^{n-m} = f\omega^n, \quad \sup_X u = 0.$$

The comparison principle on compact manifolds reads as follows:

Proposition 1.9. *Let (X, ω) be a compact Kähler manifold and u, v continuous functions in $\mathcal{SH}_m(X, \omega)$. Then*

$$\int_{\{u<v\}} (\omega + dd^c v)^m \wedge \omega^{n-m} \leq \int_{\{u<v\}} (\omega + dd^c u)^m \wedge \omega^{n-m}.$$

Proof. One can repeat the proof for psh functions from [Kołodziej 2003] or [Kołodziej 2005]. □

Observe that the cones $\Gamma_k(\omega)$ are not fixed but according to an observation of Hou [2009] these are invariant under the parallel transport defined by the Levi-Civita connection associated to ω.

2. L^∞ estimates and existence of weak solutions in domains

In this section we state the results for $0 < m < n$. Let us denote by $B(a, r)$ the ball in $\mathbb{C}^n$ with center a and radius r. Let also ω be a Kähler form defined in a neighborhood of the closure of a set Ω considered below and $V = \omega^n$ be the volume form associated to ω.

Let $\mathcal{SH}_m(\omega, \overline{\Omega})$ denote the class of m-ω-sh functions which are continuous in $\overline{\Omega}$.

Proposition 2.1. *For $p < n/(n-m)$ and an open set $\Omega \subset B(0,1) = B$ there exists $C(p)$ such that for any $K \Subset \Omega$,*

$$V(K) \le C(p)\,\mathrm{cap}_m^p(K, \Omega),$$

where

$$\mathrm{cap}_m(K, \Omega) = \sup\left\{ \int_K (dd^c u)^m \wedge \omega^{n-m},\ u \in \mathcal{SH}_m(\omega, \overline{\Omega}), 0 \le u \le 1 \right\}.$$

Proof. If $V(K) = 0$ then the inequality trivially holds. Assume from now on that $V(K) > 0$. Fix any $\epsilon \in (0, 1/2)$ and set $f = [V(K)]^{2\epsilon-1}\chi_K$, where χ_K denotes the characteristic function of the set K. Solve the complex Monge–Ampère equation in B to find $v \in PSH_\omega(B) \cap C(\overline{\Omega})$ with $v = 0$ on ∂B and

$$(dd^c v)^n = f\omega^n.$$

By the inequality between mixed Monge–Ampère measures (see [Kołodziej 2005; Dinew 2009])

$$(dd^c v)^m \wedge \omega^{n-m} \ge [V(K)]^{(2\epsilon-1)m/n}\chi_K\omega^n. \tag{2-1}$$

For $q = 1 + \epsilon$

$$\int_B f^q dV = [V(K)]^{(2\epsilon-1)(1+\epsilon)+1} = [V(K)]^{\epsilon+2\epsilon^2} \le V(B).$$

So, by [Kołodziej 1996], there exists $c > 0$, independent of K (though dependent on ϵ), such that $\|v\| \le 1/c$. Take $u = cv$. Then, using (2-1)

$$\mathrm{cap}_m(K, \Omega) \ge \int_K (dd^c u)^m \wedge \omega^{n-m} \ge c^m [V(K)]^{(2\epsilon-1)(m/n)+1}.$$

Therefore

$$V(K) \le C\,\mathrm{cap}_m^{n/(n-m+2m\epsilon)}(K, \Omega),$$

which proves the claim. □

Proposition 2.2. *Let Ω and p be as above and consider $u \in \mathcal{SH}_m(\omega, \overline{\Omega})$ with $u = 0$ on $\partial\Omega$ and*

$$\int_\Omega (dd^c u)^m \wedge \omega^{n-m} \le 1.$$

Then for $U(s) = \{u < -s\}$ we have

$$\mathrm{cap}_m(U(s), \Omega) \le s^{-m} \quad \textit{and} \quad V_{2n}(U(s)) \le C(p)s^{-pm}.$$

In particular $u \in L^q(\Omega)$ for any $q < mn/(n-m)$, and this remains true whenever u is bounded in some neighborhood of the boundary of Ω.

Proof. Fix $\epsilon > 0$, $t > 1$ and $K \subset U(s)$ and find $v \in \mathcal{SH}_m(\omega, \overline{\Omega})$ with $-1 \le v \le 0$ and

$$\int_K (dd^c v)^m \wedge \omega^{n-m} \ge \operatorname{cap}_m(K, \Omega) - \epsilon.$$

Then, using the comparison principle [Bedford and Taylor 1976; Błocki 2005], we obtain

$$\operatorname{cap}_m(K, \Omega) - \epsilon \le \int_K (dd^c v)^m \wedge \omega^{n-m} \le \int_{\{-\frac{t}{s}u < v\}} (dd^c v)^m \wedge \omega^{n-m} \le \left(\frac{t}{s}\right)^m \int_\Omega (dd^c u)^m \wedge \omega^{n-m} \le \left(\frac{t}{s}\right)^m.$$

To finish the proof of the first estimate recall that $\operatorname{cap}_m(U(s), \Omega)$ is the supremum of $\operatorname{cap}_m(K, \Omega)$ over compact $K \subset U(s)$ and let $\epsilon \to 0$ and $t \to 1$. Then the estimate of the volume follows from Proposition 2.1. □

Remark 2.3. The bound for q above is optimal as the function

$$G(z) = -|z|^{2-2n/m}$$

is m-sh and belongs to L^q_{loc} if and only if $q < mn/(n-m)$.

Błocki [2005] conjectured that any m-sh function belongs to $L^q_{\text{loc}}(\Omega)$ for any $q < mn/(n-m)$. He proved this for $q < n/(n-m)$. The above proposition confirms partially the conjecture — under the extra assumption of boundedness near the boundary. Still the question about the local integrability remains open.

We now proceed to proving the L^∞ a priori estimates for the Hessian equation with the right-hand side controlled in terms of the capacity.

Lemma 2.4. *For $p \in (1, n/(n-m))$ and an open set $\Omega \subset B$ consider $u, v \in \mathcal{SH}_m(\omega, \overline{\Omega})$ satisfying*

$$\int_K (dd^c u)^m \wedge \omega^{n-m} \le A \operatorname{cap}_m^p(K, \Omega)$$

for some $A > 0$ and any compact $K \subset \Omega$. If the sets $U(s) = \{u - s < v\}$ are nonempty and relatively compact in Ω for $s \in (s_0, s_0 + t_0)$ then there exists a constant $C(p, A)$ such that

$$t_0 \le C(p, A) \operatorname{cap}_m^{p/n}(U(s_0 + t_0), \Omega).$$

Proof. Using the notation

$$a(s) = \operatorname{cap}_m(U(s), \Omega), \quad b(s) = \int_{U(s)} (dd^c u)^m \wedge \omega^{n-m}$$

we claim that

$$t^m a(s) \le b(s+t), \quad t \in (0, s_0 + t_0 - s). \tag{2-2}$$

Indeed, for fixed compact $K \subset U(s)$, take $w_1 \in \mathcal{SH}_m(\omega, \overline{\Omega})$, $-1 \le w_1 \le 0$, such that

$$\int_K (dd^c w_1)^m \wedge \omega^{n-m} \ge \operatorname{cap}_m(K, \Omega) - \epsilon.$$

Then for $w_2 = (u - s - t)/t$ one readily verifies that $K \subset V \subset U(s+t)$, where $V = \{w_2 < w_1 + v/t\}$.

So, by the comparison principle,

$$\begin{aligned}\operatorname{cap}_m(K,\Omega)-\epsilon &\le \int_K \Big(dd^c\Big(w_1+\frac{1}{t}v\Big)\Big)^m\wedge\omega^{n-m} \le \int_V \Big(dd^c\Big(w_1+\frac{1}{t}v\Big)\Big)^m\wedge\omega^{n-m}\\ &\le \int_V (dd^c w_2)^m\wedge\omega^{n-m} \le t^{-m} b(s+t).\end{aligned}$$

Having (2-2) one proceeds as in the proof of Lemma 4.3 in [Kołodziej 2002] (with $h(x)=x^{m(p-1)}$) to reach the conclusion. □

Coupling this with the volume estimate in Proposition 2.1 we obtain a priori estimates for the solutions of Hessian equations with the right-hand side in some L^q spaces.

Theorem 2.5. *Take $q>n/m$. Then the conjugate q' of q satisfies $q'<n/(n-m)$. Fix $p'\in(q',n/(n-m))$ and $p=p'/q'>1$. Consider $u,v\in\mathcal{SH}_m(\omega,\overline{\Omega})$ such that $u\ge v$ on $\partial\Omega$, $\{u<v\}\neq\varnothing$ and*

$$(dd^c u)^m\wedge\omega^{n-m}=f\omega^n,$$

for some $f\in L^q(\Omega,dV)$. Then

$$\sup(v-u)\le c\big(p',q,\|f\|_{L^q(\Omega)}\big)\,\big\|(v-u)_+\big\|_{L^{q'}(\Omega)}^{p/(n+p(m+1))},\qquad (v-u)_+:=\max(v-u,0).$$

Proof. By the Hölder inequality and Proposition 2.1, for a compact set $K\subset\Omega$ we have

$$\int_K f\omega^n\le\|f\|_q V(K)^{1/q'}\le C(p)\|f\|_{L^q(\Omega)}\operatorname{cap}_m^p(K,\Omega).$$

Therefore, by Lemma 2.4, we get for $t=\frac{1}{2}\sup(v-u)$ and $E(t)=\{u+t<v\}$,

$$t\le c(p',q,\|f\|_{L^q(\Omega)})\operatorname{cap}_m^{p/n}(E(t),\Omega). \tag{2-3}$$

To shorten notation set $a(t)=\operatorname{cap}_m(E(2t),\Omega)$. Take $w\in\mathcal{SH}_m(\omega,\overline{\Omega})$, $-1\le w\le 0$ such that

$$\int_{E(2t)}(dd^c w)^m\wedge\omega^{n-m}\ge\frac{1}{2}a(t).$$

Observe that for $V=\{u<tw+v-t\}$ we have $E(2t)\subset V\subset E(t)$. Applying the comparison principle we thus get

$$\frac{1}{2}a(t)t^m\le\int_{E(2t)}[dd^c(tw+v)]^m\omega^{n-m}\le\int_V(dd^c u)^m\wedge\omega^{n-m}\le\int_{E(t)}f\,dV.$$

Hence from the Hölder inequality one infers

$$a(t)t^{m+1}\le 2\int_\Omega(v-u)_+f\,dV\le\|f\|_{L^q(\Omega)}\|(v-u)_+\|_{q'}.$$

Inserting this estimate into (2-3) we arrive at

$$t\le c_1(p',q,\|f\|_{L^q(\Omega)})\big[\|f\|_q\,\|(v-u)_+\|_{L^{q'}(\Omega)}t^{-m-1}\big]^{p/n},$$

and consequently

$$t\le c_2(p',q,\|f\|_{L^q(\Omega)})\big\|(v-u)_+\big\|_{L^{q'}(\Omega)}^{p/(n+p(m+1))}.$$ □

Corollary 2.6. *The last theorem gives a priori L^∞ estimate for the solutions of the Hessian equation* (0-3) *with the right-hand side in L^q and a fixed boundary condition.*

Indeed, we apply the theorem for the solution u of

$$(dd^c u)^m \wedge \omega^{n-m} = f\omega^n,$$

with given continuous boundary data φ and for v, which is the maximal function in $\mathcal{SH}_m(\omega, \overline{\Omega})$ matching the boundary condition (it exists by [Błocki 2005]). Then u is bounded by a constant depending on Ω, $\|\varphi\| = \|v\|$, and $\|f\|_q$ since $\|(v-u)_+\|_{L^{q'}(\Omega)}$ is bounded (Proposition 2.2).

Corollary 2.7. *The solutions of the Hessian equation with the right-hand sides uniformly bounded in L^q $q > n/m$ and given continuous boundary data form an equicontinuous family.*

Proof. As in [Kołodziej 2005, p. 35], which deals with the Monge–Ampère case. □

Below we state yet another stability theorem which we shall need later. Given the estimates we have already proven its proof follows the arguments from [Kołodziej 1996].

Theorem 2.8. *Let $q > n/m$. Consider $u, v \in \mathcal{SH}_m(\omega, \overline{\Omega})$ such that $\{u < v\} \neq \varnothing$ and*

$$(dd^c u)^m \wedge \omega^{n-m} = f\omega^n, \quad (dd^c v)^m \wedge \omega^{n-m} = g\omega^n$$

for some $f, g \in L^q(\Omega, dV)$. Then

$$\sup_\Omega(v-u) \le \sup_{\partial\Omega}(v-u) + c(q, m, n, \operatorname{diam}(\Omega))\|f-g\|_{L^q(\Omega)}^{1/m}.$$

Remark 2.9. The analogous stability theorem for the real m-Hessian equation ($m < n/2$) can be found in [Wang 2009, Theorem 5.5] (see also [Chou and Wang 2001]). There the optimal exponent q is equal to $n/2m$.

Next we obtain a theorem on the existence of weak, continuous solutions when $\omega = \beta$ and the right-hand side is in L^q, $q > n/m$.

Theorem 2.10. *Let Ω be smoothly bounded* (m-1)*-pseudoconvex domain* (*as in Theorem 1.6*)*. Then for $q > n/m$, $f \in L^q(\Omega, dV)$ and continuous φ on $\partial\Omega$ there exists $u \in \mathcal{SH}_m(\omega, \overline{\Omega})$ satisfying*

$$(dd^c u)^m \wedge \beta^{n-m} = f\beta^n$$

and $u = \varphi$ on $\partial\Omega$.

Proof. For smooth, positive f this is the result of Li [2004] (Theorem 1.6). With our assumptions we approximate f in $L^q(\Omega, dV)$ by smooth positive f_j and approximate uniformly φ by smooth φ_j. The solutions u_j corresponding to f_j, φ_j are equicontinuous and uniformly bounded (Corollaries 2.6, 2.7). Thus we can pick up a subsequence converging uniformly to some $u \in \mathcal{SH}_m(\omega, \overline{\Omega})$. By the convergence theorem u solves the equation. □

Remark 2.11. For $\omega = \beta$, the plurisubharmonic function $u(z) = \log\|z\|$ has a m-Hessian density in L^p for any $p < n/m$ which shows that the exponent n/m is optimal.

Equipped with the existence and stability of weak solutions we can also prove the weak Gårding inequality announced in Section 1:

Theorem 2.12. *Let $u_1, \dots, u_m$ be locally bounded m-sh functions in some domain $\Omega \Subset \mathbb{C}^n$. Suppose $(dd^c u_j)^m \wedge \beta^{n-m} = f_j \beta^n$ for some nonnegative functions $f_j \in L^q(\Omega)$, $q > n/m$. Then*

$$dd^c u_1 \wedge \wedge \cdots \wedge dd^c u_m \wedge \beta^{n-m} \geq (f_1 \cdots f_m)^{1/m} \beta^n.$$

Proof. We can essentially follow the lines of the proof of the analogous result for psh functions from [Kołodziej 2003] (see also [Kołodziej 2005]). First observe that the inequality is purely local hence it suffices to prove it under the additional assumptions that Ω is a ball and all the functions u_i are defined in a slightly bigger ball. Hence one can use convolutions with smoothing kernel to produce a decreasing to u_i sequence of m-sh functions $\{u_{i,j}\}_{j=1}^{\infty}$ (compare Proposition 1.3). Then given any collection of smooth positive functions $f_{i,k} \in L^q(\Omega)$, $q > n/m$, by [Li 2004] we can solve the Dirichlet problems

$$\begin{cases} v_{i,j,k} \in \mathcal{SH}_m(\Omega) \cap \mathcal{C}^{\infty}(\Omega), \\ (dd^c v_{i,j,k})^m \wedge \beta^{n-m} = f_{i,k} \beta^n, \\ v_{i,j,k}|_{\partial\Omega} = u_{i,j}. \end{cases}$$

For those smooth functions we can apply pointwise the Gårding inequality to conclude that

$$dd^c v_{1,j,k} \wedge \cdots \wedge dd^c v_{m,j,k} \wedge \beta^{n-m} \geq (f_{1,k} \cdots f_{m,k})^{1/m} \beta^n$$

for any $j, k \geq 1$. Then given any nonnegative $f_i \in L^q(\Omega)$, $q > n/m$, we can find an approximating sequence of smooth positive $\{f_{i,k}\}_{k=1}^{\infty}$ which converge in L^q to f_i. By the stability theorem the corresponding solutions $v_{i,j,k}$ (recall they the same boundary values $u_{i,j}$) converge uniformly as $k \to \infty$ to the m-sh functions $v_{i,j}$ (solving the limiting weak equation), and hence the inequality follows from the continuity of Hessian currents under uniform convergence of their potentials. Now if we let $j \to \infty$ the boundary values decrease towards u_i and hence so do the functions $v_{i,j}$ by the comparison principle. The convergence is not uniform but monotonicity is still sufficient to guarantee the continuity and hence in the limit we obtain the claimed inequality. □

Remark 2.13. The weak Gårding inequality can be further generalized similarly to the $m = n$ case as in [Dinew 2009].

3. L^{∞} estimates and existence of weak solutions on compact Kähler manifolds

The a priori estimates from the previous section can be carried over to the case of compact Kähler manifolds as it was done in [Kołodziej 2003] or [2005] for the Monge–Ampère equation. Let us consider a compact n-dimensional Kähler manifold X equipped with the fundamental form ω and recall that a continuous function u is ω-m-subharmonic (ω-m-sh) on X if

$$(\omega + dd^c u)^k \wedge \omega^{n-k} \geq 0, \quad k = 1, 2, \dots, m.$$

The set of such functions is denoted by $\mathcal{SH}_m(X,\omega)$. We study the complex m-Hessian equation

$$(\omega+dd^cu)^m\wedge\omega^{n-m}=f\omega^n \tag{3-1}$$

with given nonnegative function $f\in L^1(M)$, which is normalized by the condition

$$\int_X f\,\omega^n=\int_X\omega^n.$$

The solution is required to be ω-m-sh. By Proposition 1.8 the solutions of the equation exist, at least for smooth positive f. Our a priori estimates will also give the existence of weak solutions for $f\geq 0$ in L^q, $q>n/m$.

We define for a compact set $K\subset X$ its capacity

$$\operatorname{cap}_m(K)=\sup\left\{\int_K(\omega+dd^cu)^m\wedge\omega^{n-m}:u\in\mathcal{SH}_m(X,\omega),\,0\leq u\leq 1\right\}.$$

To use the local results we need also a capacity defined as follows. Let us consider two finite coverings by strictly pseudoconvex sets $\{B_s\}$, $\{B_s'\}$, $s=1,2,\dots,N$, of X such that $\bar{B}_s'\subset B_s$ and in each B_s there exists $v_s\in PSH(B_s)$ with $dd^cv_s=\omega$ and $v_s=0$ on ∂B_s. Given a compact set $K\subset X$ define $K_s=K\cap\bar{B}_s'$. Set

$$\operatorname{cap}_m'(K)=\sum_s\operatorname{cap}_m(K_s,B_s),$$

where $\operatorname{cap}_m(K,B)$ denotes the relative capacity from the previous section. As in [Kołodziej 2003] one can show that $\operatorname{cap}_m(K)$ is comparable with $cap_m'(K)$: There exists $C>0$ such that

$$\frac{1}{C}\operatorname{cap}_m(K)\leq\operatorname{cap}_m'(K)\leq C\operatorname{cap}_m(K).$$

Hence, by Proposition 2.1 we have

$$V(K)\leq C(p,X)\operatorname{cap}_m^p(K),$$

for $p<n/(n-m)$ and V the volume measured by ω^n.

With this estimate at our disposal we can obtain the same a priori estimates as in domains in $\mathbb{C}^n$. The proofs are almost identical. In the compact setting one has to make sure that instead of just a sum of m-sh functions one considers a convex combination of ω-m-sh functions (see [Kołodziej 2005]). In particular the following theorems hold.

Theorem 3.1. *Consider $q>n/m$, its conjugate q' and $p'\in(q',n/(n-m))$. Write $p=p'/q'>1$. Consider $u,v\in\mathcal{SH}_m(X,\omega)$ such that $\{u<v\}\neq\varnothing$ and*

$$(\omega+dd^cu)^m\wedge\omega^{n-m}=f\omega^n,$$

for some $f\in L^q(dV)$. Then

$$\sup(v-u)\leq c(p',q,\|f\|_{L^q(X)})\big\|(v-u)_+\big\|_{q'}^{\frac{p}{n+p(m+1)}},\quad\textit{where}\ (v-u)_+:=\max(v-u,0).$$

Corollary 3.2. *The family of solutions of the Hessian equation* (3-1) *with right-hand sides uniformly bounded in L^q, $q > n/m$, is equicontinuous.*

Applying Proposition 1.8 and the statements above one immediately gets this existence theorem:

Theorem 3.3. *Let X be a compact Kähler manifold. For $q > n/m$ and $f \in L^q(dV)$ there exists a unique function $u \in \mathcal{SH}_m(X, \omega)$ satisfying*

$$(\omega + dd^c u)^m \wedge \omega^{n-m} = f\omega^n \quad \textit{and} \quad \max u = 0.$$

4. Local regularity

In this section we prove a counterpart of the main result in [Błocki and Dinew 2011], where the case of the Monge–Ampère equation was studied. We shall treat only the $\omega = \beta$ case and use PDE notation (with σ_m defined in the Introduction).

Theorem 4.1. *Assume that $n \geq 2$ and $p > n(m-1)$. Let $u \in W^{2,p}(\Omega)$, where Ω is a domain in $\mathbb{C}^n$, be a m-subharmonic solution of*

$$\sigma_m(u_{z_j\bar{z}_k}) = \psi > 0. \tag{4-1}$$

Assume that $\psi \in C^{1,1}(\Omega)$. Then for $\Omega' \Subset \Omega$

$$\sup_{\Omega'} \Delta u \leq C,$$

where C is a constant depending only on n, m, p, $\operatorname{dist}(\Omega', \partial\Omega)$, $\inf_\Omega \psi$, $\sup_\Omega \psi$, $\|\psi\|_{C^{1,1}(\Omega)}$ and $\|\Delta u\|_{L^p(\Omega)}$.

Proof. By $C_1, C_2, \ldots$ we will denote possibly different constants depending only on the required quantities. Without loss of generality we may assume that $\Omega = B$ is the unit ball in $\mathbb{C}^n$ and that u is defined in some neighborhood of $\bar{B}$. We will use the notation $u_j = u_{z_j}$, $u_{\bar{j}} = u_{\bar{z}_j}$ with the notable exception of $u_{(\epsilon)}$ which is defined below.

Following [Bedford and Taylor 1976], we define the Laplacian approximating operator

$$T = T_\epsilon(u) = \frac{n+1}{\epsilon^2}(u_{(\epsilon)} - u),$$

where

$$u_{(\epsilon)}(z) = \frac{1}{V(B(z,\epsilon))} \int_{B(z,\epsilon)} u \, dV.$$

Since $T_\epsilon u \to \Delta u$ weakly as $\epsilon \to 0$, it is enough to show a uniform upper bound for T independent of ϵ. Observe that since u is subharmonic we have $T_\epsilon(u) \geq 0$.

Before we continue let us state two lemmas. The first one is classical.

Lemma 4.2. *Let $u \in W^{2,p}(\Omega)$ (Ω is a domain in $\mathbb{C}^n$) be a subharmonic function. Given any $\Omega' \Subset \Omega$ the operator $T_\epsilon(z)$ is well defined on Ω' for any sufficiently small $\epsilon > 0$. Furthermore,*

$$\|T_\epsilon\|_{L^p(\Omega')} \to \|\Delta u\|_{L^p(\Omega')};$$

in particular, $\|T_\epsilon\|_{L^p(\Omega')}$ is uniformly bounded for all $0 < \epsilon < \epsilon_0$.

Lemma 4.3. *The function $T_\epsilon(u)(z)$ for any $\epsilon > 0$ satisfies the subharmonicity condition*

$$\frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} T_{\epsilon,i\bar{j}} \geq -C_1,$$

where $(\partial \sigma_m(u_{j\bar{k}}))/\partial u_{i\bar{j}}$ is the (i, j)-th $(m-1)$-cominor of the matrix $u_{i\bar{j}}(z)$ and C_1 is a constant dependent only on n, m, $\inf_\Omega \psi$, $\sup_\Omega \psi$, and $\|\psi\|_{C^{1,1}(\Omega)}$.

Proof. Observe that $u_{(\epsilon)}$ is a convex combination of m-subharmonic functions, hence it is m-subharmonic. Therefore one has the inequality

$$(dd^c u_{(\epsilon)})^m \wedge \omega^{n-m} \geq 0.$$

In fact following the lines of the same argument in [Bedford and Taylor 1976] (where it was applied to the Monge–Ampère operator) one can prove the stronger inequality

$$(dd^c u_{(\epsilon)})^m \wedge \omega^{n-m} \geq ((\psi^{1/m})_{(\epsilon)})^m. \tag{4-2}$$

Indeed, for smooth u this is just a consequence of the concavity of $\sigma_m^{1/m}$. For nonsmooth solutions one can repeat the Goffman–Serrin formalism, again following Bedford and Taylor.

Thus using the weak Gårding inequality (Theorem 2.12) one has

$$(dd^c u)^{m-1} \wedge dd^c u_{(\epsilon)} \wedge \omega^{n-m} \geq \psi^{(m-1)/m} (\psi^{1/m})_{(\epsilon)} \, dV.$$

Next, identifying (n, n) forms and their densities one gets, up to a multiplicative numerical constant $c_{n,m}$, the following string of inequalities

$$\begin{aligned} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} T_{\epsilon,i\bar{j}} &= c_{n,m} 1/\epsilon^2 dd^c(u_{(\epsilon)} - u) \wedge (dd^c u)^{m-1} \wedge \omega^{n-m} \\ &\geq c_{n,m} 1/\epsilon^2 \psi^{(m-1)/m} ((\psi^{1/m})_{(\epsilon)} - \psi^{1/m}) = c_{n,m} \psi^{(m-1)/m} T_\epsilon(\psi^{1/m}). \end{aligned}$$

But ψ is a strictly positive $\mathscr{C}^{1,1}$ function hence $T_\epsilon(\psi^{1/m}) \geq -C_1(\|\psi\|, \|\psi^{1/m}\|_{C^{1,1}})$. Combining all those inequalities we obtain the claimed estimate. □

From now on we drop the index ϵ in what follows. We will use the same calculations as in [Błocki and Dinew 2011], which in turn relied on [Trudinger 1980]. For some $\alpha, \beta \geq 2$ to be determined later set

$$w := \eta(T)^\alpha, \quad \text{where } \eta(z) := (1 - |z|^2)^\beta.$$

Then

$$w_i = \eta_i (T)^\alpha + \alpha \eta (T)^{\alpha-1} (T)_i$$

and

$$\begin{aligned} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} w_{i\bar{j}} = \alpha \eta (T)^{\alpha-1} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} (T)_{i\bar{j}} + \alpha(\alpha-1) \eta (T)^{\alpha-2} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} (T)_i (T)_{\bar{j}} \\ + 2\alpha (T)^{\alpha-1} \mathrm{Re}\left(\frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} \eta_i (T)_{\bar{j}} \right) + (T)^\alpha \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} \eta_{i\bar{j}}. \end{aligned}$$

By Lemma 4.3 and the Schwarz inequality for $t > 0$,

$$\frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} w_{i\bar{j}} \geq -C_1 \alpha \eta (T)^{\alpha-1} + \alpha(\alpha-1)\eta(T)^{\alpha-2} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} (T)_i (T)_{\bar{j}}$$
$$-t\alpha(T)^{\alpha-1} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} (T)_i (T)_{\bar{j}} - \frac{1}{t}\alpha(T)^{\alpha-1} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} \eta_i \eta_{\bar{j}} + (T)^{\alpha} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} \eta_{i\bar{j}}.$$

Therefore with $t = (\alpha-1)\eta/T$ we get

$$\frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} w_{i\bar{j}} \geq -C_1 \alpha \eta (T)^{\alpha-1} + (T)^{\alpha} \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} \left(\eta_{i\bar{j}} - \frac{\alpha}{\alpha-1} \frac{\eta_i \eta_{\bar{j}}}{\eta} \right).$$

We now have

$$\eta_i = -\beta z_i \eta^{1-1/\beta},$$
$$\eta_{i\bar{j}} = -\beta \delta_{i\bar{j}} \eta^{1-1/\beta} + \beta(\beta-1)\bar{z}_i z_j \eta^{1-2/\beta},$$

and thus

$$|\eta_{i\bar{j}}|, \left| \frac{\eta_i \eta_{\bar{j}}}{\eta} \right| \leq C(\beta)\, \eta^{1-2/\beta}.$$

Coupling the above inequalities we get

$$\frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} w_{i\bar{j}} \geq -C_2 (T)^{\alpha-1} - C_3 w^{1-2/\beta} (T)^{2\alpha/\beta} \sum_{i,j} \left| \frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} \right|.$$

Fix q with $n/m < q < p/m(m-1)$ (by our assumption on p such a choice is possible). By Lemma 4.2 $\|T\|_p$ and $\|\Delta u\|_p$ are under control. By Calderón–Zygmund inequalities we control $\|u_{i\bar{j}}\|_p$ too. Observe that $\partial \sigma_m(u_{j\bar{k}})/\partial u_{i\bar{j}}$ is a sum of products of $m-1$ factors of the type $u_{i\bar{j}}$, and therefore its $p/(m-1)$-norm is also under control. It follows that for

$$\alpha = 1 + \frac{p}{qm}, \quad \beta = 2\left(\frac{qm+p}{p-qm(m-1)} \right),$$

we have

$$\left\| \left(\frac{\partial \sigma_m(u_{j\bar{k}})}{\partial u_{i\bar{j}}} w_{i\bar{j}} \right)_- \right\|_{qm} \leq C_3 (1 + (\sup_B\ w)^{1-2/\beta}),$$

where $f_- := -\min(f, 0)$. By Theorem 2.10 we can find continuous m-subharmonic v vanishing on ∂B and such that

$$\sigma_m(v_{i\bar{j}}) = ((u^{i\bar{j}} w_{i\bar{j}})_-)^m.$$

Then the weak Gårding inequality yields

$$\begin{aligned}\frac{\partial\sigma_m(u_{j\bar k})}{\partial u_{i\bar j}}v_{i\bar j} &= c_{n,m}(dd^c u)^{m-1}\wedge dd^c v\wedge\omega^{n-m}\\ &\geq c_{n,m}\big(\sigma_m(u_{i\bar j})^{(m-1)/m}(\sigma_m(v_{i\bar j}))^{1/m}\big)\geq\frac{1}{C_4}\bigg(\frac{\partial\sigma_m(u_{j\bar k})}{\partial u_{i\bar j}}w_{i\bar j}\bigg)_-\\ &\geq -\frac{1}{C_4}\frac{\partial\sigma_m(u_{j\bar k})}{\partial u_{i\bar j}}w_{i\bar j}.\end{aligned}$$

By maximum principle we obtain that $w\leq -C_4v$, since this inequality holds on ∂B. Applying the stability theorem (Theorem 2.8), with $u=0$, we get

$$\begin{aligned}\sup_B w\leq C_4\|v\|\leq C_5(\|\sigma_m(v_{i\bar j})\|_q^{1/m}) &= C_5\bigg\|\bigg(\frac{\partial\sigma_m(u_{z_j\bar z_k})}{\partial u_{i\bar j}}w_{i\bar j}\bigg)_-\bigg\|_{qn}\\ &\leq C_6(1+(\sup_B\ w)^{1-2/\beta}).\end{aligned}$$

Therefore $w\leq C_7$ and thus

$$T^\alpha\leq C_7/\eta,$$

which is the desired bound. □

Remark 4.4. An analogous reasoning can be applied to the real m-Hessian equation (using Wang stability theorem and existence of weak solutions). It turns out that for $m<n/2$ the corresponding exponent in the $W^{2,p}$ Sobolev space is equal to $n(m-1)/2$. Observe that this improves the $m(n-1)/2$ exponent obtained by different methods by Urbas [2001]. Whether this exponent is optimal is however still unclear and would require construction of suitable Pogorelov type Hessian examples.

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THE AHARONOV–BOHM EFFECT IN SPECTRAL ASYMPTOTICS OF THE MAGNETIC SCHRÖDINGER OPERATOR

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In memory of Hans Duistermaat

We show that in the absence of a magnetic field the spectrum of the magnetic Schrödinger operator in an annulus depends on the cosine of the flux associated with the magnetic potential. This result follows from an analysis of a singularity in the "wave trace" for this Schrödinger operator, and hence shows that even in the absence of a magnetic field the magnetic potential can change the asymptotics of the Schrödinger spectrum; that is, the Aharonov–Bohm effect takes place. We also study the Aharonov–Bohm effect for the magnetic Schrödinger operator on a torus.

1. Introduction

Let Ω be the exterior of a bounded region in $\mathbb{R}^2$ with smooth boundary, and let

$$H_{A,V} = \tfrac{1}{2}(i\partial_{x_1} + A_1(x))^2 + \tfrac{1}{2}(i\partial_{x_2} + A_2(x))^2 - V(x).$$

This is the Schrödinger operator for a particle of mass 1 and charge -1 moving in Ω under the influence of the magnetic potential $A = (A_1, A_2)$ and the electric potential V. We assume that

$$\partial_{x_2} A_1 - \partial_{x_1} A_2 = 0 \quad \text{in } \Omega, \tag{1-1}$$

that is, the magnetic field vanishes in Ω. Given a simple, closed curve γ in Ω encircling the complement of Ω, we define the magnetic flux by

$$\alpha_\gamma = \int_\gamma A(x) \cdot dx.$$

In view of (1-1) α_γ only depends on the orientation of γ.

In a seminal paper, Aharonov and Bohm [1959] showed that if $\alpha_\gamma \neq 0 \bmod 2\pi$, then one can detect the cosine of the magnetic flux in the scattering of particles in this quantum system, that is, the magnetic potential has a physical impact even when the magnetic field is zero in Ω. This is called the Aharonov–Bohm effect. Aharonov and Bohm found this by computing the scattering cross-section explicitly for $\Omega = \mathbb{R}^2 \setminus \{0\}$, when $A(x) = (-x_2/|x|^2, x_1/|x|^2)$ and $V(x) = 0$. They also proposed an experiment to demonstrate this effect. However, the first generally accepted experimental verification of the Aharonov–Bohm (AB) effect was done many years later in [Tonomura et al. 1986]. For further mathematical work

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on the AB effect, see [Nicoleau 2000; Weder 2002; Roux and Yafaev 2002; Eskin 2013; Eskin et al. 2010].

Helffer [1988] showed that $A(x)$ can influence the spectrum of $H_{A,V}$ when the magnetic field is zero in Ω. In the semiclassical setting with $V(x) \to \infty$, as $|x| \to \infty$, and $\Omega = \{|x| > 1\}$ he showed that the lowest Dirichlet eigenvalue depended on the cosine of the magnetic flux. Earlier related results on magnetic Schrödinger operators are due to Lavine and O'Carroll [1977].

In this paper we study the Schrödinger operator in the domain $\Omega_R = \Omega \cap \{|x| < R\}$ with Dirichlet boundary conditions on $|x| = R$ and $\partial\Omega$. We compute the singularity at $t = 3R\sqrt{3}$ of the distribution trace of the fundamental solution of the initial-boundary value problem

$$\begin{aligned} &u_{tt} + H_{A,V}u = 0 && \text{in } \Omega_R \times \mathbb{R}_t, \\ &u(x,0) = f(x) \text{ and } u_t(x,0) = 0 && \text{in } \Omega_R, \\ &u(x,t) = 0 && \text{when } x \in \partial\Omega_R. \end{aligned} \tag{1-2}$$

This distribution trace is known as the "wave trace" for this problem, and it is given by

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j}),$$

where $\{\lambda_j\}_{j=1}^{\infty}$ are the Dirichlet eigenvalues of $H_{A,V}$ in Ω_R. Hence its singularities are determined by the behavior of the λ_j as $j \to \infty$. These singularities are well-known to appear only at the lengths of periodic broken ray paths in Ω_R. The singularity at $t = 3R\sqrt{3}$ comes from equilateral triangles in Ω_R with vertices on $|x| = R$. To compute this singularity we need to know that $3R\sqrt{3}$ is isolated in the set of lengths of broken periodic rays. To ensure that we assume that the complement of Ω, Ω^c is strictly convex and contained in $\{|x| < 1\}$ and $R \geq 8$ (see Remark 1.1), but any assumption that makes the length of the inscribed equilateral triangles isolated in the lengths of periodic reflected ray paths will suffice. The geometry that we have chosen makes the singularity unchanged when one changes the sign of α_γ. Hence we cannot recover more than the cosine of α_γ from it (see Remark 1.2).

A definitive computation of leading singularities in wave traces was given by Duistermaat and Guillemin [1975] for manifolds without boundary. For manifolds with boundary the analogous computation has not been done in that generality. To carry it out in here we have taken this opportunity to present a different method of computation that replaces Fourier integral operators with superpositions of Gaussian beams (see [Combescure et al. 1999] and Chapter 5 of [Combescure and Robert 2012]). In Section 5 we briefly discuss the computation of wave trace singularities using the global theory of Fourier integral operators (see [Hörmander 2003; 2005; 2007; 2009; Duistermaat 1974; Maslov and Fedoriuk 1976; Eskin 2011]). Both approaches lead to the following:

Theorem 1.1. *The distribution*

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j})$$

has an isolated singularity at $t = L = 3R\sqrt{3}$. The leading term in that singularity is the distribution

$$-2^{-5/2}3^{1/4}R^{3/2}\cos\left(\int_\gamma A(x)\cdot dx\right)(t-L)_+^{-3/2}. \tag{1-3}$$

Hence the wave trace determines the cosine of the magnetic flux.

In the final section of this paper we consider $H_{A,V}$ on (flat) 2-torus and obtain essentially the same result: under a nondegeneracy assumption on the torus the singularities in the wave trace at times equal to the lengths of curves in a homology basis determine the cosines of magnetic fluxes around those curves (see Theorem 6.1).

Remark 1.1. The only fact from geometry needed here — we only need it for circles — is: a ray and its reflections inside an ellipse are all tangent to an ellipse confocal with the boundary ellipse. So rays in $|x| \le R$ tangent to a circle $|x| = r > 1$ will never enter $|x| < 1$ after reflection in $|x| = R$, while rays that enter $|x| < 1$ will always reenter $|x| < 1$ after reflection in $|x| = R$. Since the boundary curve C is convex, rays entering $|x| < 1$ will leave $|x| < 1$ after at most one reflection. This gives the following bounds on the length L of periodic ray paths that hit C. For rays that close after entering $|x| < 1$ k times

$$2kR - 2k < L < 2kR + 2k.$$

So periodic rays that enter $|x| < 1$ more than three times have lengths are greater than $8R - 8$, and the equilateral triangles are the (isolated) shortest periodic rays that never enter $|x| < 1$ (assuming $R > 2$). So we need $4R + 4 < 3R\sqrt{3} < 6R - 6$. That happens as soon as $R \ge 8$ (picking the first whole number that works).

Remark 1.2. If $\Omega = \{|x| > 1\}$ and $V \equiv 0$, the mapping $u(x) \to u(-x)$ sends eigenfunctions of $H_{A,0}$ to eigenfunctions of $H_{-A,0}$ bijectively. Thus the wave traces of these operators must be identical. The leading singularity in the wave trace at $t = 3\sqrt{3}R$ does not depend on the boundary of Ω or $V(x)$, hence it will be unchanged when A is replaced by $-A$ in these cases, too. Therefore, one cannot distinguish α_γ and $-\alpha_\gamma$ using the leading singularity. The same ambiguity arises in the results in [Aharonov and Bohm 1959; Helffer 1988].

2. Singularities of the wave trace

Let $E(x, y, t)$ denote the fundamental solution for the initial-boundary value problem (1-2). The wave front set of the distribution kernel of E is contained in the canonical relation for the bicharacteristic flow (see [Melrose and Sjöstrand 1978; 1982]). For this problem the canonical relation is defined as follows: Let $\nu(x)$ denote the outer unit normal to $\partial\Omega_R$ at x. Given (y_0, η_0) with $y_0 \in \Omega_R$ and $|\eta_0| = 1$, define $(x(s, y_0, \eta_0), \xi(s, y_0, \eta_0)) = (y_0 + s\eta_0, \eta_0)$ until, at $s = s_1$, $y_1 = x(s_1, \eta_0, y_0) \in \partial\Omega_R$. Then, if $\eta_0 \cdot \nu(y_1) \neq 0$, continue $(x(s, y_0, \eta_0), \xi(s, y_0, \eta_0))$ for $s > s_1$ as $(y_1 + s\eta_1, \eta_1)$, where $\eta_1 = \eta_0 - 2(\nu(y_1)\cdot\eta_0)\nu(y_1)$. Continue the bicharacteristic this way, reflecting when $x(s, y_0, \eta_0)$ hits $\partial\Omega_R$, as long as $x(s, y_0, \eta_0)$ does not intersect $\partial\Omega_R$ tangentially. At points of tangential intersection one has to distinguish grazing and gliding points. However, since we assume that the boundary of Ω^c is strictly convex, points of tangential intersection with $\partial\Omega$ are grazing points and bicharacteristics continue unaffected by these intersections. When y_0 is in the interior of Ω_R, a bicharacteristic with initial data (y_0, η_0) will never intersect $|x| = R$ tangentially. Hence, the wave front set of the kernel of $E(\cdot, \cdot, t)$ is the union over $y_0 \in \Omega_R$ and $\eta_0 \in \mathbb{S}^1$ of the points

$$\big(x(t, y_0, \eta_0), \xi(t, y_0, \eta_0), y_0, -\eta_0\big),$$

where $\big(x(t, y_0, \eta_0), \xi(t, y_0, \eta_0)\big)$ are the reflected bicharacteristics described above. Strictly speaking, the wave front set is the closure of that set and includes a "boundary wave front set" over $|x| = R$ (see [Melrose and Sjöstrand 1978; 1982] for details).

Since $E(x, y, t)$ is a distribution in t depending smoothly on $(x, y) \in \Omega_R \times \Omega_R$, $\int_{\Omega_R} E(x, x, t)\, dx$ is well-defined, and we have the relation

$$T =_{\text{def}} \sum_{j=1}^{\infty} \cos(t\sqrt{\lambda}_j) = \int_{\Omega_R} E(x, x, t)\, dx.$$

The singular support of T is contained in the set of t such that $(y_0, \eta_0, y_0, -\eta_0) \in WF(E(x, y, t))$ for some $y_0 \in \overline{\Omega}_R$, (see [Guillemin and Melrose 1979]). The choice of Ω and R here implies that, for t in a sufficiently small neighborhood of $3R\sqrt{3}$, $(y_0, \eta_0, y_0, -\eta_0) \in WF(E(x, y, t))$ only if the ray $x(s, y_0, \eta_0)$ traces an inscribed equilateral triangle.

To compute the singularities in the wave trace we need a parametrix for the initial-boundary value problem (1-2). Since this parametrix will differ from $E(x, y, t)$ by an integral operator with a smooth kernel, we can use it to compute singularities. Since we are only interested in singularities arising from inscribed equilateral triangles, we only need a parametrix which captures the singularities of $\int_{\Omega_R} E(x, y, t) f(y)\, dy$ when $WF(f) \subset \{y, \eta) : y \in \Omega_R, |y \cdot \eta^{\perp}| = R/2\}$, where $(\eta_1, \eta_2)^{\perp} = (\eta_2, -\eta_1)$. These singularities hit $\partial\Omega_R$ nontangentially, and hence this parametrix construction can be done with reflection at the boundary. This observation applies equally well to constructions with Fourier integral operators and the Gaussian beam superpositions used here.

3. The Gaussian beam construction

Here we will outline the construction of a parametrix for (1-3), for initial data with wave fronts projecting onto the inscribed equilateral triangles. We will continue to let η have length one. The Gaussian beam method allows one to do the following (see [Ralston 1982] for more details):

(i) For any ray, $(x(t), t) = (z + t\eta, t)$, in space-time, one can construct a function $\phi(x, t; z, \eta)$ satisfying:

(a) For any given integer N, $(\phi_t)^2 - |\phi_x|^2$ vanishes to order N on $(x(t), t)$ and $\mathrm{Im}\{\phi_{xx}\}$ is positive definite on $(x(t), t)$.

(b) $\phi(x, 0; z, \eta) = x \cdot \eta + \frac{i}{2}|x - z|^2$ on $|x - z| < \delta$.

(c) $\phi_t(x, 0; z, \eta) = -1$.

Moreover, if Γ is a curve with unit normal ν at $x(t_0)$ and η is not tangent to Γ, then one can construct $\phi^r = \phi$ on Γ, satisfying (a) for the reflected ray $(x(t_0)+(t-t_0)\eta^r, t)$, where $\eta^r = \omega - 2(\nu\cdot\eta)\nu$. Reflection of beams is discussed in [Ralston 1982, Section 2.2].

(ii) Once ϕ has been constructed, for any given integer N, one can solve the transport equations

$$\begin{aligned} &2\phi_t(a_0)_t - 2\phi_x\cdot(a_0)_x + \big(2iA(x)\cdot\phi_x + \phi_{tt} - \Delta\phi\big)a_0 = 0,\\ &2\phi_t(a_j)_t - 2\phi_x\cdot(a_j)_x + \big(2iA(x)\cdot\phi_x + \phi_{tt} - \Delta\phi\big)a_j = -\big(\partial_t^2 - (\partial_x + iA(x))^2\big)a_{j-1}, \quad j>0 \end{aligned} \tag{3-1}$$

to order N on $(x(t), t)$, and impose the initial conditions $a_0(0, x; z, \eta) = 1$ and $a_j(0, x; z, \eta) = 0$ for $j > 0$ on $|x - z| < \delta$.

For the singularity computation we need to know the leading amplitude a_0 on the ray beginning at z in direction η.

We define $a(x, t; z, \eta, r)$ to be the formal sum

$$a(x, t; z, \eta, r) = \sum_{j\ge 0} a_j(x, t; z, \eta) r^{-j}. \tag{3-2}$$

As before one can reflect in a plane curve Γ which is transverse to the ray, and we impose $a^r = -a$ on Γ to satisfy Dirichlet boundary conditions.

Using the preceding constructions we can construct the operator

$$[V(t)f](x) = \tfrac{1}{2}\big([V_+(t)f](x) + [V_-(t)f](x)\big),$$

where

$$[V_\pm(t)f](x) = \sum_{k\ge 0} \frac{1}{(2\pi)^3}\int_{\mathbb{R}_+\times S^1\times\{|z|<R+\delta\}} e^{ir\phi^k(x,\pm t;z,\eta)}\cdot a^k(x, \pm t; z, \eta, r)\hat{f}(r\eta)r^2\,dr\,d\eta\,dz. \tag{3-3}$$

Here, ϕ^0 is the phase function with $\phi^0(x, 0; z, \eta) = x\cdot\eta + i/2|x-z|^2$, and for $k > 0$,

$$e^{ir\phi^k(x,t;z,\eta)}a^k(x, t; z, \eta, r)$$

is the (Dirichlet) reflection of $e^{ir\phi^{k-1}(x,t;z,\eta)}a^{k-1}(x, t; z, \eta, r)$ in the circle $|x| = R$. Since Gaussian beams can be constructed to for any finite ray segment, we can assume that each term in (3-3) is defined on $\{|x| \le 2R\}$ when necessary. Note that in this notation the variables (z, η) in ϕ^k remain the initial data **at t=0** for the ray where $\mathrm{Im}\{\phi^k\} = 0$. Note also that the integration in r in (3-3) is in the sense of distributions.

For the parametrix construction we need $V(0)f = f + Kf$ where K is an operator with a smooth kernel. From (3-3) we have

$$[V(0)f](x) = \frac{1}{(2\pi)^3}\int_{\mathbb{R}_+\times S^1\times\{|z|<2R\}} e^{irx\cdot\eta - r|x-z|^2/2}\hat{f}(r\eta)r^2\,dr\,d\eta\,dz.$$

Since

$$\frac{1}{(2\pi)^3}\int_{\mathbb{R}_+\times S^1\times\mathbb{R}^2_z} e^{irx\cdot\eta - r|x-z|^2/2}\hat{f}(r\eta)r^2\,dr\,d\eta\,dz = f(x)$$

and f is supported in $\{|x| < R\}$, it follows that omitting the contribution from $\{|z| > R+\delta\}$ in (3-3) only adds an operator with a smooth kernel.

To compute singularities of the wave trace we need to make the kernels of the operators $V_\pm(t)$ explicit. The distribution kernels of these operators are sums of terms of the form

$$S(t) = \int_{\mathbb{R}_+ \times S^1 \times \mathbb{R}^2_z} e^{ir\phi(x,t;z,\eta)-ir\eta\cdot y} a(x,t;z,\eta,r) r^2\, dr\, d\eta\, dz. \tag{3-4}$$

As was stated earlier, these operators are smooth in (x, y), and we can compute their traces by integrating these kernels over the diagonal $y = x$. Thus the (distribution) trace of $V(t)$ is a sum of terms of the form

$$\operatorname{Tr}(\phi, a) = \int_{D\times\mathbb{R}_+ \times S^1 \times \mathbb{R}^2_z} e^{ir\phi(x,t;z,\omega)-ir\eta\cdot x} a(x,t;z,\eta,r) r^2\, dr\, d\eta\, dz\, dx. \tag{3-5}$$

We want to compute the singularity in t of this trace at $t = L = 3R\sqrt{3}$, and we only need to consider t in $|t - L| < \delta$, where δ is small enough that $\{t : |t - L| < \delta\}$ contains no other lengths of periodic rays in the disk $|x| < R$.

4. Calculation of the singularity at $t = L = 3\sqrt{3}R$

For $\eta = (\eta_1, \eta_2)$ with $|\eta| = 1$ define $\eta^\perp = (\eta_2, -\eta_1)$, the "right hand" normal. To compute the singularity at $t = L$ we only need the parametrix restricted to $R/2 - \epsilon < |z\cdot\eta^\perp| < R/2+\epsilon$ for any fixed positive ϵ. Since the broken ray $x(t, z, \eta)$ is initially of the form $x = z + t\eta$, $\eta^\perp \cdot z > 0$ corresponds to rays going counterclockwise around $z = 0$, and $\eta^\perp \cdot z < 0$ corresponds to rays going clockwise around $z = 0$.

In the preceding section we concluded that the singularity in the wave trace at $t = L$ could be calculated from a sum of integrals of the form

$$\frac{1}{2}\sum_{\pm}\int_0^\infty r^2 dr \int_{\mathcal{S}^1} d\eta \left(\int a_0(x, \pm t, z, \eta) e^{ir(\phi(x,\pm t,z,\eta)-x\cdot\eta)}\, dx\, dz\right). \tag{4-1}$$

The integral in r is to be taken in distribution sense. Until the end of this section we will consider (4-1) in the case that the phase ϕ is the beam phase resulting from reflecting the bicharacteristic with initial data $(x, \xi) = (z, \eta)$ three times in $|x| = R$. The amplitudes $a_0(x, t, z, \eta)$ are determined by the transport Equation (3-1). The contributions to the singularity from the $+$ and $-$ terms in (4-1) are complex conjugates of each other, and from here one we only consider the "$+$" term.

We assume that a_0 vanishes when $|z\cdot\eta^\perp|$ is not close to $R/2$. Note that we can assume that $\phi(x, t, z, \eta)$ is defined for all (x, z, t) when $|z\cdot\eta^\perp|$ is sufficiently close to $R/2$.

The main step in isolating the singularity is an application of the method of stationary phase to (4-1). For that we introduce the change of coordinates

$$x = u + v\eta + w\eta^\perp, \quad z = v\eta + w\eta^\perp, \quad u \in \mathbb{R}^2,\ v, w \in \mathbb{R}.$$

Our objective is the elimination of the integral in (u, w) by stationary phase. To see when the phase is real and stationary in these variables, note that

(i) the phase is real only when $x = x(t, z, \eta)$;

(ii) the derivative of the phase with respect to u at $x = x(t, z, \eta)$ is

$$\phi_x - \eta = \xi(t, z, \eta) - \eta,$$

which vanishes precisely when three reflections have made ξ return to its initial value. That implies $|z \cdot \eta^\perp| = R/2$. Since the reflected ray will return to z when $t = L$ and it is propagating in the direction η, $x(t, z, \eta) = z + (t - L)\eta$. Hence $u = (t - L)\eta$ and $|w| = R/2$ on the stationary set in u. The derivative of the phase with respect to w at $x = x(t, z, \eta)$ is

$$\eta^\perp \cdot \phi_x + \eta^\perp \cdot \phi_z - \eta \cdot \eta^\perp,$$

which vanishes, since $\phi_z(x(t, z, \eta), t, z, \eta) = \phi_z(x(0, z, \eta), 0, z, \eta) = \partial_z(x \cdot \eta + i|x - z|^2/2)|_{x=z} = 0$. Thus we will need to do the stationary phase computation at $(u, w) = ((t - L)\eta, \pm R/2)$.

Calculation of asymptotics by stationary phase requires the computation of the determinant of the Hessian of the phase, and here this computation is rather long. We have found it useful to consider the phase and the bicharacteristics defined for all $\eta \neq 0$ by homogeneity. That makes the Jacobian matrix

$$F(t) = \begin{pmatrix} \partial x/\partial z(t, z, \eta) & \partial x/\partial \eta(t, z, \eta) \\ \partial \xi/\partial z(t, z, \eta) & \partial \xi/\partial \eta(t, z, \eta) \end{pmatrix} =_{\text{def}} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

symplectic. Using $\phi_x(x(t, z, \eta), t, z, \eta) = \xi(t, z, \eta)$ and $\phi_z(x(t, z, \eta), t, z, \eta) = 0$, and setting $M = \phi_{xx}(x(t, z, \eta), t, z, \eta)$, one computes directly that at $x = x(t, z, \eta)$:

$$H =_{\text{def}} \begin{pmatrix} \phi_{xx} & \phi_{xz} \\ \phi_{zx} & \phi_{zz} \end{pmatrix} = \begin{pmatrix} M & c - Ma \\ c^t - a^t M & a^t Ma - a^t c \end{pmatrix}.$$

Letting O_η be the matrix with columns η and $\eta^\perp$, one sees that the Hessian of the phase in (4-1) with respect to the variables (u, v, w) is $B^t H B$ where

$$B = \begin{pmatrix} I & O_\eta \\ 0 & O_\eta \end{pmatrix}.$$

However, we need the Hessian with respect to (u, w). We will see that $\binom{\eta}{\eta}$ is a null vector for H, and we have

$$B \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \eta \\ \eta \end{pmatrix}.$$

Moreover, letting P_η denote the orthogonal projection of $\mathbb{R}^2$ onto $\langle \eta \rangle$, one computes

$$B^t \begin{pmatrix} 0 & 0 \\ 0 & P_\eta \end{pmatrix} B = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence,

$$\det\begin{pmatrix}\phi_{u_1u_1} & \phi_{u_1u_2} & 0 & \phi_{u_1w}\\ \phi_{u_2u_1} & \phi_{u_2u_2} & 0 & \phi_{u_2w}\\ 0 & 0 & 1 & 0\\ \phi_{wu_1} & \phi_{wu_2} & 0 & \phi_{ww}\end{pmatrix} = \det\begin{pmatrix} M & c-Ma\\ c^t-a^tM & a^tMa-a^tc+P_\eta\end{pmatrix}. \tag{4-2}$$

To proceed with this computation we need to know $F(t)$. The computation begins with the formulas for $x(t,z,\eta)$ and $\xi(t,z,\eta)$ after three reflections:

$$x(t,z,\eta)=w\frac{\xi^\perp}{|\xi|}+\Big(t+\frac{z\cdot\eta}{|\eta|}-6\sqrt{R^2-w^2}\Big)\frac{\xi}{|\xi|}.$$

Setting $\eta=|\eta|(\cos\theta,\sin\theta)$, we get

$$\xi(t,z,\eta)=|\eta|\Big(\cos\Big(\theta+\pi-6\sin^{-1}\frac{w}{R}\Big),\sin\Big(\theta+\pi-6\sin^{-1}\frac{w}{R}\Big)\Big).$$

One checks that $\partial_z w=\eta^\perp/|\eta|$ and $\partial_\eta w=-(z\cdot\eta)(\eta^\perp/|\eta|^3)$, and this implies that the Jacobian $\partial\xi/\partial z$ at $w=\pm R/2$ is $(4\sqrt{3}/R)|\eta|P_{\eta^\perp}$. So $c=(4\sqrt{3}/R)|\eta|P_{\eta^\perp}$. Using $\partial_\eta\theta=-\eta^\perp/|\eta|^2$, one finds that at $w=\pm R/2$

$$\frac{\partial\xi}{\partial\eta}=P_\eta+P_\eta^\perp-\frac{4\sqrt{3}}{R}\frac{z\cdot\eta}{\eta}P_{\eta^\perp}=I-\frac{4\sqrt{3}}{R}\frac{z\cdot\eta}{|\eta|}P_{\eta^\perp}.$$

So $d=I-(4\sqrt{3}/R)vP_{\eta^\perp}$.

The computations of the derivatives of $x(t,z,\eta)$ are longer, but they are simplified by the observation that $|\xi(t,z,\eta)|=|\eta|$. At $w=\pm R/2$ one has

$$\begin{aligned}\frac{\partial x}{\partial z}&=P_{\eta^\perp}\mp 2\sqrt{3}\frac{\eta}{|\eta|}\Big\langle\frac{\eta^\perp}{|\eta|},\cdot\Big\rangle+\frac{\eta}{|\eta|}\Big\langle\frac{\eta}{|\eta|}\pm 2\sqrt{3}\frac{\eta^\perp}{|\eta|},\cdot\Big\rangle+\Big(t-L+\frac{z\cdot\eta}{|\eta|}\Big)\frac{4\sqrt{3}}{R}P_{\eta^\perp}\\ &=I+\Big(t-L+\frac{z\cdot\eta}{|\eta|}\Big)\frac{4\sqrt{3}}{R}P_{\eta^\perp}.\end{aligned}$$

So $a=I+(t-L+v)(4\sqrt{3}/R)P_{\eta^\perp}$.

To compute $\partial x/\partial\eta$ at $w=\pm R/2$ one uses

$$\Big(\frac{\xi}{|\xi|}\Big)_\eta=\frac{1}{|\eta|}\Big(1-\frac{4\sqrt{3}}{R}\frac{z\cdot\eta}{|\eta|}\Big)P_{\eta^\perp}$$

at $w=\pm R/2$, and the less obvious result that

$$\Big(\frac{\xi^\perp}{|\xi|}\Big)_\eta=\Big(-1+\frac{4\sqrt{3}}{R}\frac{z\cdot\eta}{|\eta|}\Big)\frac{\eta}{|\eta|^2}\Big\langle\frac{\eta^\perp}{|\eta|},\cdot\Big\rangle.$$

Combining those with $\partial_\eta v = (z\cdot\eta^\perp)(\eta^\perp/|\eta|^3) = \pm(R/2|\eta|^2)\eta^\perp$, one has

$$\begin{aligned}\frac{\partial x}{\partial\eta} &= \frac{\eta^\perp}{|\eta|}\Big\langle -(z\cdot\eta)\frac{\eta^\perp}{|\eta|^3},\cdot\Big\rangle \pm \frac{R}{2}\Big(-1+\frac{4\sqrt{3}}{R}\frac{z\cdot\eta}{|\eta|}\Big)\frac{\eta}{|\eta|^2}\Big\langle\frac{\eta^\perp}{|\eta|},\cdot\Big\rangle \\ &\qquad + \frac{\eta}{|\eta|}\Big\langle \pm\frac{R}{2|\eta|^2}\eta^\perp \mp 2\sqrt{3}(z\cdot\eta)\frac{\eta^\perp}{|\eta|^3},\cdot\Big\rangle + \frac{1}{|\eta|}\Big(t-L+\frac{z\cdot\eta}{|\eta|}\Big)\Big(1-\frac{4\sqrt{3}}{R}\frac{z\cdot\eta}{|\eta|}\Big)P_{\eta^\perp} \\ &= \frac{\big(t-L+\frac{z\cdot\eta}{|\eta|}\big)}{|\eta|}\Big(1-\frac{4\sqrt{3}}{R}\frac{z\cdot\eta}{|\eta|}\Big)P_{\eta^\perp} - \frac{(z\cdot\eta)}{|\eta|^2}P_{\eta^\perp}.\end{aligned}$$

Thus, when $(5\sqrt{3})/2 - v < t < (7\sqrt{3})/2 - v$,

$$F(t) = \begin{pmatrix} I + (t-L+v)\frac{4\sqrt{3}}{R}P_{\eta^\perp} & \frac{(t-L)}{|\eta|}(1-\frac{4\sqrt{3}}{R}v)P_{\eta^\perp} - \frac{4\sqrt{3}}{R}\frac{v^2}{|\eta|}P_{\eta^\perp} \\ \frac{4\sqrt{3}}{R}|\eta|P_{\eta^\perp} & I - \frac{4\sqrt{3}}{R}vP_{\eta^\perp}\end{pmatrix}. \tag{4-3}$$

From this point onward we will assume that $|\eta| = 1$, that is, $\eta = (\cos\theta, \sin\theta)$. Note that this implies $|\xi(t,z,\eta)| \equiv 1$.

Now we can resume the computation of the Hessian. First we compute the determinant of the Hessian. For this the only facts that we need from the computation of the symplectic matrix $F(t)$ — it is a good check on the computation to verify that it *is* symplectic — are that a, b, c and d commute with P_η with $aP_\eta = dP_\eta = P_\eta$ and $bP_\eta = bP_\eta = 0$. We will also eventually use the exact form of c. Note that since $F(t)$ is symplectic a^tc and d^tb are symmetric and $a^td - c^tb = I$.

Returning to (4-2) we have

$$\begin{pmatrix} M & c - Ma \\ c^t - a^tM & a^tMa - a^tc + P_\eta\end{pmatrix}\begin{pmatrix} I & a \\ 0 & I\end{pmatrix} = \begin{pmatrix} M & c \\ c^t - a^tM & P_\eta\end{pmatrix},$$

$$\begin{pmatrix} I & 0 \\ a^t & I\end{pmatrix}\begin{pmatrix} M & c \\ c^t - a^tM & P_\eta\end{pmatrix} = \begin{pmatrix} M & c \\ c^t & a^tc + P_\eta\end{pmatrix}.$$

Since $M = (c+id)(a+ib)^{-1}$ (see [Combescure et al. 1999]),

$$\begin{pmatrix} M & c \\ c^t & a^tc + P_\eta\end{pmatrix}\begin{pmatrix} a+ib & 0 \\ 0 & I\end{pmatrix} = \begin{pmatrix} c+id & c \\ c^ta + ic^tb & a^tc + P_\eta\end{pmatrix}$$

$$\begin{pmatrix} -a^t & I \\ I & 0\end{pmatrix}\begin{pmatrix} c+id & c \\ c^ta+ic^tb & a^tc+P_\eta\end{pmatrix} = \begin{pmatrix} i(c^tb - a^td) & P_\eta \\ c+id & c\end{pmatrix} = \begin{pmatrix} -iI & P_\eta \\ c+id & c\end{pmatrix}.$$

Finally

$$\begin{pmatrix} -ic+d & I \\ I & 0\end{pmatrix}\begin{pmatrix} -iI & P_\eta \\ c+id & c\end{pmatrix} = \begin{pmatrix} 0 & P_\eta + c \\ -iI & P_\eta\end{pmatrix}.$$

From the preceding, using the exact form of c, one can read off the determinant of the Hessian of the phase (at $u = (t-L)\eta$, $w = \pm R/2$). It is

$$(-1)\Big(\frac{4\sqrt{3}}{R}\Big)\det((a+ib)^{-1}). \tag{4-4}$$

At this point it is convenient to calculate the amplitude a_0. Note that

$$\phi_t (a_0)_t - \phi_x \cdot (a_0)_x = -(d/dt) a_0(x(t,z,\eta),t,z,\eta).$$

Hence (3-1) implies that, after three reflections,

$$a_0(x(t,z,\eta),t,z,\eta) = (-1)^3 e^{i\int_0^t A(x(s))\dot{x}(s)\,ds} e^{(\int_0^t [\phi_{tt}-\Delta\phi](x(s),s)\,ds)/2}. \tag{4-5}$$

Note that $|\phi_x| + \phi_t$ vanishes to second order when $x = x(t,z,\eta)$ and thus $\phi_{tt} + \phi_{tx}\cdot\dot{x} = 0$ and $\phi_x = \xi(t,z,\eta)$ when $x = x(t,z,\eta)$. Differentiating $|\phi_x| + \phi_t = 0$ with respect to x and using $\phi_{tt} = -\phi_{tx}\cdot\dot{x}$, we have $\phi_{tt} - \Delta\phi = \xi\cdot M\xi - \operatorname{trace}(M)$, when $x = x(t,z,\eta)$.

Differentiating $\dot{x} = \xi/|\xi|$ with respect to z and η and restricting to $|\eta| = 1$ one sees that $\dot{a} + i\dot{b} = (I - P_\xi)(c + id)$. Hence, using $M = (c+id)(a+ib)^{-1}$, we see that, when $x(t,z,\eta)$ is not a reflection point,

$$(d/dt)(\log\det(a+ib)) = \operatorname{trace}((\dot{a}+i\dot{b})(a+ib)^{-1}) = \operatorname{trace}((I-P_\xi)M) = \Delta\phi - \phi_{tt}. \tag{4-6}$$

At reflection points $a+ib$ jumps to $(1-2P_\nu)(a+ib)$, where ν is normal to the boundary. Thus $\det(a+ib)$ is multiplied by -1. Note that, since the imaginary part of M is positive definite and the trace of $(I-P_\xi)M$ equals the trace of $(I-P_\xi)M(I-P_\xi)$, (4-6) shows that the argument of $\det(a+ib)$ is strictly increasing away from reflection points. Thus we can make the argument of $(\det(a+ib))^{1/2}$ increasing by defining it to be 1 when $t = 0$, to be multiplied by i at each reflection point, and to be continuous between reflection points. With this definition of $(\det(a+ib))^{1/2}$, we can conclude that after three reflections

$$a_0(x(t,z,\eta),t,z,\eta) = i(\det(a+ib))^{-1/2} e^{i\int_0^t A(x(s))\dot{x}(s)\,ds}. \tag{4-7}$$

We have $\int_0^L A(x(s))\dot{x}(s)\,ds = \alpha_\gamma$, where γ is the equilateral triangle traced by $x(s,z,\eta)$ with $z = v\eta + (R/2)\eta^\perp$ or $z = v\eta - (R/2)\eta^\perp$. Since the magnetic field vanishes in Ω, α_γ is independent of v and η, and its value when $z = v\eta + (R/2)\eta^\perp$ is the negative of its value when $z = v\eta - (R/2)\eta^\perp$.

Now we can evaluate the integral in (u,w) asymptotically by the method of stationary phase. The standard form of the stationary phase lemma [Hörmander 2003, Theorem 7.7.5], gives the following: if $f(y)$ is a smooth function such that $\operatorname{Im}\{f\} \geq 0$, $f_y(y_0) = 0$ and the Hessian $f_{yy}(y_0)$ is nonsingular, then for a smooth with support in a sufficiently small neighborhood of y_0, one has the asymptotic expansion

$$\int_{\mathbb{R}^n} e^{irf(y)} a(y)\,dy = \left(\frac{2\pi}{r}\right)^{n/2} \sum_{j=0}^\infty c_j r^{-j},$$

and the leading coefficient is given by

$$c_0 = e^{irf(y_0)} a(y_0)(\det(-if_{yy}(y_0)))^{-1/2}. \tag{4-8}$$

Here the square root of the determinant in $(\det(-if_{yy}(y_0)))^{-1/2}$ is the analytic continuation to symmetric matrices with nonnegative real part of the positive square root for positive definite matrices (see [Hörmander 2003, Theorem 7.7.5]).

In our case we will use stationary phase to eliminate the integrations in u and w in (4-1) — recall that $z = v\eta + w\eta^\perp$ and $x = u + v\eta + w\eta^\perp$. The stationary point y_0 in (4-5) is either $(u, w) = ((t-L)\eta, R/2)$ or $(u, w) = ((t-L)\eta, -R/2)$. Since

$$\phi(x(t, z, \eta), t, z, \eta) = \phi(x(0, z, \eta), 0, z, \eta) = z \cdot \eta,$$

and we have

$$f(y_0) = \phi(x(t, z, \eta), t, z, \eta) - x(t, z, \eta) \cdot \eta$$

evaluated at $(u, w) = ((t-L)\eta, R/2)$ or $(u, w) = ((t-L)\eta, -R/2)$, it follows that $f(y_0) = -(t-L)$. The domain of integration in (u, v, w, η) is

$$\left\{(u, v, w, \eta) : |\eta| = 1,\ |u + v\eta + w\eta^\perp| \le R \text{ and } \sqrt{w^2 + v^2} < R + \delta\right\}. \tag{4-9}$$

We consider (4-1) as an iterated integral with the integrations in (u, w) done first. After we use the stationary phase lemma in those integrations, the resulting integrand is evaluated at $(u, w) = ((t-L)\eta, \pm R/2)$, and, since we can assume that $|t-L|$ is smaller than δ, the domain of integration in (v, η) becomes

$$D =_{\text{def}} \left[-\frac{\sqrt{3}}{2}R - (t-L),\ \frac{\sqrt{3}}{2}R - (t-L)\right] \times S^1.$$

The stationary phase argument needs to be modified when v is near $\pm\sqrt{3}R/2$. There, since the integration in (u, w) should not cross $|x| = R$, the stationary phase lemma does not apply. However, there is a simple remedy for this. Let $\rho = |u + v\eta + w\eta^\perp|$. On the sphere $\rho = R$ we can introduce coordinates $(\theta_1, \theta_2, \theta_3)$, functions of (u, w) depending on v as a parameter, near the points $(u, v, w) = ((t-L)\eta, \pm\sqrt{3}R/2, \pm R/2)$. Next using smooth cutoffs one can write the trace integral as the sum of an integral over a region when $\rho < R - \delta$, where the stationary phase argument applies as given earlier, and a region where $R - 2\delta < \rho < R$. In the second region, near the points where the phase is stationary, one writes the integral in the variables $(\theta_1, \theta_2, \theta_3, v, \eta)$, and applies stationary phase in $(\theta_1, \theta_2, \theta_3)$. The stationary set will be the image in these coordinates of $(u, w) = ((t-L)\eta, \pm R/2)$ and it will depend on v. Likewise, letting Q denote the Hessian in (u, w) of the phase at the stationary points, the Hessian at the stationary points will now be $J^t Q J$, where J is the Jacobian matrix of (u, w) with respect to $(\theta_1, \theta_2, \theta_3)$. Since the θ variables are tangential, one can use the stationary phase expansion *uniformly* in v. The leading term will be an integral over the stationary set. On that set $(\det Q)^{-1/2}$ will be replaced by $(\det J^t Q J)^{-1/2} = |\det J|^{-1}(\det Q)^{-1/2}$. However, the new factor $|\det J|^{-1}$ is canceled by the Jacobian in the volume form (we have $du\,dw = |\det J|\,d\theta_1 d\theta_2 d\theta_3$). Hence, the stationary phase expansion holds uniformly up to $v = \pm\sqrt{3}R/2$. The result is that (4-4), (4-7) and (4-8) give, uniformly for $(v, \eta) \in D$,

$$\int_{D(v,\eta)} a_0(x, t, z, \eta) e^{ir(\phi(x,t,z,\eta) - x\cdot\eta)}\, du\, dw = \pm\frac{c(R)}{r^{3/2}} K(t) e^{-ir(t-L)} + O\Big(\frac{1}{r^{5/2}}\Big), \tag{4-10}$$

where $D(v, \eta) = \left\{(u, w) : |u + v\eta + w\eta^\perp| \le R\right\}$, and $c(R) = (2\pi)^{3/2}(R/4\sqrt{3})^{1/2} e^{3\pi i/4}$. The choice of sign $\pm$ is determined by (4-7) and (4-8): it is $+1$ when the square roots of $\det(a + ib)$ implicit in (4-7) and (4-8) agree and -1 when they do not. The factor

$$K(t)=\exp\biggl(i\int_0^t A(x^+(s))\cdot\dot{x}^+(s)\,ds\biggr)+\exp\biggl(i\int_0^t A(x^-(s))\cdot\dot{x}^-(s)\,ds\biggr)$$

arises from adding the contributions from stationary points with $w=-R/2$ and $w=R/2$. The path $x^-(s)$ with $w=-R/2$ goes clockwise around the origin, and the path $x^+(s)$ with $w=R/2$ is counterclockwise. Hence $K(L)=2\cos(\int_\gamma A(x)\cdot dx)$.

To compute the singularity we need the distribution calculation

$$\int_0^\infty e^{-i(t-L)r}r^{1/2}\,dr=\frac{e^{-3\pi i/4}\Gamma(3/2)}{(t-L-i0)^{3/2}}=e^{-3\pi i/4}\Gamma(3/2)(t-L)_+^{-3/2}+e^{3\pi i/4}\Gamma(3/2)(t-L)_-^{-3/2},\quad (4\text{-}11)$$

where the homogeneous distributions $(s)_\pm^{-3/2}$ are defined by integration by parts and vanish on functions supported in $\mp s>0$. Note that the contribution to the trace from $V_-(t)$ is the complex conjugate of the contribution from $V_+(t)$. Hence, integrating over (v,η,r), and adding the contributions from $V_-(t)$ and $V_+(t)$ gives the leading singularity in the trace at $t=L$ as

$$\pm 2^{-5/2}R^{3/2}3^{1/4}\cos\biggl(\int_\gamma A(x)\cdot dx\biggr)(t-L)_+^{-3/2}.\qquad (4\text{-}12)$$

The computation up to this point has not determined the choice of sign ($\pm$) in (4-12). That will be done in Remark 4.1, and there is an alternative derivation in Section 5. However, since the choice of sign in (4-12) does not depend on A, (4-12) is sufficient to conclude that the trace determines the cosine of the magnetic flux.

The final step in this argument is showing that (4-12) really is the leading term in the singularity. We have not discussed the contributions of the beams with phases ϕ^j in (3-3) for $j\neq 3$. However, those phases are never stationary near the periodic orbits, and give smooth contributions to the trace by the "nonstationary phase" argument. Note that we can apply that argument up to $|x|=R$ by using the coordinates $(\theta_1,\theta_2,\theta_3)$ as before.

Remark 4.1. The sign "$\pm$" in the leading singularity is actually "$-$". To verify that we need to determine the signs of $(\det(a+ib))^{1/2}$ in both the stationary phase computation and the amplitude computation.

We begin with the stationary phase calculation. The matrix on the right in (4-2) can be rewritten as

$$\tilde{H}=\begin{pmatrix} M & c-Ma\\ c^t-a^tM & a^tMa-a^tc+P_\eta\end{pmatrix}=\begin{pmatrix}(c+id)(a+ib)^{-1} & -i(a+ib)^{-1}\\ -i(a^t+ib^t)^{-1} & i(a+ib)^{-1}a+P_\eta\end{pmatrix}.$$

This is a consequence of $F(t)$ being a symplectic matrix. Then, using (4-3) with $t=L$, one sees that $\tilde{H}$ has the invariant subspaces $V_1=\langle(\eta,\eta),(\eta,-\eta)\rangle$ and $V_2=\langle(\eta^\perp,\eta^\perp),(\eta^\perp,-\eta^\perp)\rangle$. The product of the eigenvalues of $\tilde{H}$ from eigenvectors in V_1 is i (the eigenvalues are $1/2+(1\pm\sqrt{3}/2)i$) and the product of the eigenvalues from eigenvectors in V_2 is $iC(A+iB)^{-1}$ where $A=\eta^\perp\cdot a\eta^\perp$, $B=\eta^\perp\cdot b\eta^\perp$ and $C=\eta^\perp\cdot c\eta^\perp$. Since all the eigenvalues have nonnegative imaginary parts, this makes

$$(\det(-i\tilde{H}))^{-1/2}=\frac{\sqrt{A+iB}}{\sqrt{C}}e^{i\pi/4}=\tfrac{1}{2}R^{1/2}3^{-1/4}e^{i\pi/4}\sqrt{A+iB},$$

in the stationary phase formula, where $\sqrt{A+iB}$ is in the lower half-plane. That $\sqrt{A+iB}$ here is in $\mathrm{Im}\{z\}<0$ is the point of the calculation, note that $A+iB=\det(a+ib)$ at $t=L$.

To calculate $(det(a+ib))^{-1/2}$ in the amplitude we need to consider the entire ray path tracing an equilateral triangle beginning at $z=(z\cdot\eta)\eta\pm(R/2)\eta^{\perp}$ when $t=0$ and returning to that point when $t=L$. Without loss of generality we will assume that $z=(z\cdot\eta)\eta+(R/2)\eta^{\perp}$. Recall that $a(t)+ib(t)=(\partial x/\partial z)(t,z,\eta)+i(\partial x/\partial\eta)(t,z,\eta)$. As we observed in the calculation of the amplitude a_0, $\det(a+ib))$ is multiplied by -1 at each reflection. Geometric optics, following the reflection rule in Remark 1.1, shows that, after the first reflection at $(x,t)=\big((\sqrt{3}R/2)\eta+(R/2)\eta^{\perp},\sqrt{3}R/2-z\cdot\eta\big)$, there is exactly one "focal point" where $\det(\partial x/\partial z)=0$ on each side of the triangle. Moreover, the homogeneity of $x(t,z,\eta)$ in η of degree zero, implies that $(\partial x/\partial\eta)\eta\equiv 0$. That implies that the real part of $\det(a(t)+ib(t))$ changes sign from negative to positive at the points where $\det(\partial x/\partial z)=0$. Since the argument of $\det(a(t)+ib(t))$ is increasing, this makes it possible to track the its change as t goes from 0 to L: the total change when the path reaches the third focal point is $2\pi+2\pi+3\pi/2$. Since the argument of $\big(\det(a(0)+ib(0))\big)^{1/2}$ was chosen to be zero, this means that at the third focal point, its argument will be $3\pi/4$ and $\big(\det(a(L)+ib(L))\big)^{1/2}$ will be in the upper half plane. Thus, the choices of $(\det(a(L)+ib(L))^{1/2}$ in the stationary phase computation and the amplitude computations have opposite signs, and the sign of the leading singularity in (4-10) is "$-$".

Remark 4.2. We used triangular periodic orbits here because it was easy to give conditions that would make their lengths isolated in the set of lengths of periodic orbits (Remark 4.2). However, it is easy to extend the trace formulas for periodic orbits that are regular N-gons. These would give the same results when one can show that their lengths are isolated in the lengths of periodic orbits.

For a regular inscribed N-gon the length of a side is $h_N=2R\sin\pi/N$, and its total length is $L_N=Nh_N$. For the N-gon the entries in the first column of the Jacobian from (4-3) become

$$\frac{\partial x}{\partial z}(t,z,\eta)=I+(t+v-L_N)\frac{4N}{h_N}P_{\eta^{\perp}}\quad\text{and}\quad\frac{\partial\xi}{\partial z}(z,\eta)=\frac{4N}{h_N}P_{\eta^{\perp}}.$$

One can use either the analysis in Remark 4.1 or the Fourier integral approach in Section 5 to show that the only changes this makes in the leading singularity are the following. The factor of $(\sqrt{3}R)(R/4\sqrt{3})^{1/2}$, which arose from integration in v and $(\det(\partial\xi/\partial z))^{-1/2}$ from the stationary phase, is replaced by $(h_N)(h_N/4N)^{1/2}$. The initial ±1 in (4-12) — note that this is -1 by Remark 4.1 — is replaced by $(i)^{N-1}$, since there is one focal point on each side. If one combines that with (4-11) and (4-12), the result is that the leading singularity in the trace is

$$(-1)^{(N-1)/2}C(N,\alpha_\gamma)(t-L_N)_+^{-3/2}\quad\text{for } N \text{ odd}\tag{4-13a}$$

and

$$(-1)^{N/2-1}C(N,\alpha_\gamma)(t-L)_-^{-3/2}\quad\text{for } N \text{ even,}\tag{4-13b}$$

where

$$C(N,\alpha_\gamma)=2^{-5/2}h_N^{3/2}N^{-1/2}\cos(\alpha_\gamma)=\tfrac{1}{2}N^{-1/2}(R\sin(\pi/N))^{3/2}\cos(\alpha_\gamma).$$

5. A Fourier integral operator approach

This problem provides an opportunity for direct comparison of Gaussian beam superpositions and Fourier integral operators. In this section we describe the computation of the singularities in the wave trace using global Fourier integral operators as in [Hörmander 2003; 2005; 2007; 2009; Duistermaat 1974; Maslov and Fedoriuk 1976; Eskin 2011]. This method requires a detailed description of the singularities in the projection of bicharacteristics to x-space, but in a simple situations like ours one can arrive at the formula for the leading singularity quickly. There are analytical arguments needed to justify that computation, and we will sketch them. Both methods make essential use of the computations of $\partial x/\partial z$ and $\partial\xi/\partial z$ in (4-3).

Let $E(t)$ be the fundamental solution for the boundary value problem (1-3). We will construct a parametrix for $E(t)$, microlocalized near the periodic rays, as a global Fourier integral operator. For f supported in Ω_R let

$$[W(t)f](x) = [W_+(t)f](x) + [W_-(t)f](x) = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2}(W_+(x,t,\eta)+W_-(x,t,\eta))\hat{f}(\eta)\,d\eta,$$

where

$$[W_\pm(0)f](x) = \frac{1}{2(2\pi)^2}\int_{\mathbb{R}^2} e^{ix\cdot\eta}\hat{f}(\eta)\,d\eta = \frac{1}{2}f(x).$$

Since the analysis of $W_+(t)$ and $W_-(t)$ is the same, we will work with $W_+(t)$ from here on.

The kernel $W_+(x,t,\eta)$ is given by $\exp(-it|\eta|+ix\cdot\eta)$ plus terms arising from reflection in $|x|=R$, Of course, the phase and amplitude develop singularities, and in a neighborhood of those the form of $W_+(t)$ is more complicated, involving integrals over auxiliary variables. The Schwartz kernel of $W_+(t)$ is given by

$$\int_{\mathbb{R}^2} W_+(x,t,\eta)e^{-iy\cdot\eta}\,d\eta.$$

This is a distribution in t depending smoothly on (x,y). Hence, the distribution trace of $W_+(t)$ is given by

$$\int_{\Omega_R}\left(\int_{\mathbb{R}^2} e^{-ix\cdot\eta}W_+(x,t,\eta)\,d\eta\right)dx. \tag{5-1}$$

Denote the reflected bicharacteristics with initial data $(x(0),\xi(0))=(z,\eta)$ by $(x(t,z,\eta),\xi(t,z,\eta))$ as in Section 2. We will write $\eta=|\eta|\hat{\eta}$ with $\hat{\eta}=(\cos\theta,\sin\theta)$ and $\hat{\eta}^\perp=(\sin\theta,-\cos\theta)$. Note that, since $x(t,z,\eta)$ is homogeneous of degree zero in η, we have $x(t,z,\eta)=x(t,z,\hat{\eta})$. In what follows $\hat{\eta}$ will be treated as a parameter; all estimates will be uniform in $\hat{\eta}\in\mathbb{S}^1$. We will use the coordinates (v,w) in x-space, where $x=v\hat{\eta}+w\hat{\eta}^\perp$, and the coordinates $(\tilde{v},\tilde{w})$ in z-space, where $z=\tilde{v}\hat{\eta}+\tilde{w}\hat{\eta}^\perp$. Since only periodic ray paths contribute to the singularities of the wave trace, we only need to consider $(\tilde{v},\tilde{w})$ with $|\tilde{w}-R/2|<\delta$ or $|\tilde{w}+R/2|<\delta$. Since the analysis is identical in both cases, we will only consider $|\tilde{w}-R/2|<\delta$. We are only interested in t close to L. For convenience of notation we will use $(x(\tilde{v},\tilde{w}),\xi(\tilde{v},\tilde{w})) =_{\text{def}} (x(L,\tilde{v}\hat{\eta}+\tilde{w}\hat{\eta}^\perp,\hat{\eta}),\xi(L,\tilde{v}\hat{\eta}+\tilde{w}\hat{\eta}^\perp,\hat{\eta}))$.

We will use the formulas for bicharacteristics after three reflections that were used to derive (4-3). From those formulas one sees that when $t = L$ the Jacobian $\partial(v, w)/\partial(\tilde{v}, \tilde{w})$ vanishes on the set $\tilde{\Sigma}$ where $\tilde{v} = (35/6)\sqrt{R^2 - \tilde{w}^2} - L$. We define Σ to be the image under the mapping $x = x(\tilde{v}, \tilde{w})$ of the intersection of $\tilde{\Sigma}$ with $|\tilde{w} - R/2| < \delta$. The set Σ is usually called the "caustic set" for the bicharacteristics.

Let $\chi_0(z, \hat{\eta})$, $\chi_\pm(z, \hat{\eta})$ be C^∞ functions in $\tilde{U} = \{|\tilde{w} - (R/2)| < \delta, |\tilde{v}| < \sqrt{R^2 - \tilde{w}^2}\}$ equal to zero near $|\tilde{w} - (R/2)| = \delta$ and such that $\chi_0(z, \hat{\eta}) = 0$ for $|\tilde{v} - \tilde{v}(\tilde{w})| > 2\epsilon$, $\chi_+(z, \hat{\eta}) = 0$ for $\tilde{v} - \tilde{v}(\tilde{w}) < \epsilon$, and $\chi_-(z, \hat{\eta}) = 0$ for $\tilde{v} - \tilde{v}(\tilde{w}) > -\epsilon$, where $\tilde{v} = \tilde{v}(\tilde{w})$ is the equation of $\tilde{\Sigma}$, and ϵ is fixed. We assume also that $\chi_0 + \chi_+ + \chi_- = 1$ for $|\tilde{w} - (R/2)| < \delta/2$. Denote by $\tilde{G}_\pm$ the supports of χ_0, $\chi_\pm$, respectively, and let $G_\pm$ be the images of $\tilde{G}_\pm$ under the mapping $x = x(\tilde{v}, \tilde{w})$. Denote by $V_0(x, t, \eta)e^{-iz\eta}$, $V_\pm(x, t, \eta)e^{-iz\eta}$ the distribution kernels corresponding to the initial conditions

$$\frac{1}{2(2\pi)^2}\chi_0(z, \hat{\eta})e^{i(x-z)\cdot\eta} \quad \text{and} \quad \frac{1}{2(2\pi)^2}\chi_\pm(z, \hat{\eta})e^{i(x-z)\cdot\eta},$$

respectively. Note that the difference $W_+(x, t, \eta) - \big(V_0(x, t, \eta) + V_+(x, t, \eta) + V_-(x, t, \eta)\big)$ does not contribute to the singularity near $t = L$.

It follows from [Maslov and Fedoriuk 1976] and [Eskin 2011, Section 66], that $V_\pm(x, t, \eta)$ has the following form on $G_\pm : V_\pm(x, t, \eta) = V_\pm^0(x, t, \eta)(1 + R^\pm(x, t, \eta))$, where

$$V_\pm^0(x, t, \eta) = \frac{(-1)^3}{8\pi^2}\chi_\pm(z^\pm(x, t, \hat{\eta}), \hat{\eta})\Big|\det\frac{\partial x^\pm}{\partial z}\Big|^{-1/2}\cdot\exp\Big(i\Big[\frac{\pi}{4}\sigma^\pm + \alpha(t) + \phi^\pm(x, t, \eta)\Big]\Big), \tag{5-2}$$

and $R^\pm \approx \sum_{k\geq 1} r_k^\pm(x, t, \hat{\eta})|\eta|^{-k}$ is an asymptotic series in $|\eta|$. Here $\phi^\pm(x, t, \eta) = z^\pm(x, t, \hat{\eta})\cdot\hat{\eta}$, where $z = z^\pm(x, t, \eta)$ is the inverse function to $x = x(t, z, \eta)$ in $\tilde{G}_\pm$, and $\partial x^\pm/\partial z = (\partial x/\partial z)(t, z^\pm(x, t, \hat{\eta}), \hat{\eta})$. The piecewise constant function $\sigma^\pm$ in (5-2) is the sum of the "phase shifts" at the focal points on the ray paths used to define $\phi^\pm$. The sum of these phase shifts along the curve $x(t, z, \eta)$, $0 \leq t \leq L$ is called "Maslov index" of this curve (see [Maslov and Fedoriuk 1976, Section 1.7] or [Eskin 2011, Section 66]). The computation of the phase shifts at the focal points here can be done as in [Eskin 2011, Section 66.46–66.48], and the result is that the contribution to σ is -2 for each focal point that $x(t, \tilde{v}\eta + R/2\eta^\perp, \eta)$ has passed through up to time t. This makes $\sigma^+ = \sigma^- - 2$. The function $\alpha(t) = \int_0^t A(x(s, z, \eta))\cdot\dot{x}(s, z, \eta)\,ds$, and the factor $(-1)^3$ comes from the three reflections of a ray on $0 \leq t \leq L$. Note that $V_\pm$ decay rapidly in $|\eta|$ outside $G_\pm$, respectively.

We denote the leading term of $\int_{\Omega_R}(V_1 + V_2)e^{-ix\cdot\eta}dx$ by $I(t, \eta) = I_+ + I_-$, where

$$I_\pm(t, \eta) = e^{-i|\eta|(t-L)}\int_{G_\pm} V_\pm^0(x, L, \eta)e^{-ix\cdot\eta}dx.$$

The phase in $I_\pm(t, \eta)$ is $\Phi_\pm(x, L, \eta) = \phi_\pm(x, L, \eta) - x\cdot\eta$. The phase functions $\phi^\pm(x, t, \eta)$ satisfy $\phi_t^\pm + |\phi_x|^2 = 0$, and we have

$$\phi_x^\pm(x(t, z, \eta), t, \eta) = \xi(t, z, \eta), \ \ \phi_\eta^\pm(x(t, z, \eta), t, \eta) = z. \tag{5-3}$$

Since $|\phi_x^\pm| = |\eta|$ we have $\phi_t^\pm = -|\eta|$. Therefore $\phi^\pm(x, t, \eta) = \phi^\pm(x, L, \eta) - |\eta|(t - L)$. The critical points of $\Phi^\pm(x, L, \eta)$ are solutions of $\phi_x^\pm(x, L, \eta) - \eta = 0$, $\phi_\eta^\pm(x, L, \eta) - x = 0$. It follows from (5-3) that $\xi(L, z, \eta) = \eta$ and $z = x(L, z, \eta)$. In the geometry here this means that the periodic orbit is an equilateral triangle inscribed in $|x| \le R$, and $L = 3R\sqrt{3}$. Since any point of this triangle is a critical point, we need to use the stationary phase expansion in the transversal variable w.

Note that $z^{\pm}(x, L, \eta) = x = v\hat{\eta} + (R/2)\hat{\eta}^{\perp}$, $x \in G_{\pm}$. Hence

$$\Phi^{\pm}(v\hat{\eta} + (R/2)\hat{\eta}^{\perp}, L, \eta) = \phi^{\pm}(v\hat{\eta} + (R/2)\hat{\eta}^{\perp}, L, \eta) - x \cdot \eta = 0.$$

Also

$$\Phi^{\pm}_{w}(v\hat{\eta} + (R/2)\hat{\eta}^{\perp}, L, \eta) = \phi^{\pm}_{x}(v\hat{\eta} + (R/2)\hat{\eta}^{\perp}, L, \eta) \cdot \hat{\eta}^{\perp} = 0,$$

since $\phi^{\pm}_{x} - \eta = 0$ and $\eta \cdot \hat{\eta}^{\perp} = 0$. Compute now

$$\Phi^{\pm}_{ww}(v\hat{\eta} + (R/2)\hat{\eta}^{\perp}, L, \eta) = \hat{\eta}^{\perp} \cdot \phi^{\pm}_{xx}(v\hat{\eta} + (R/2)\hat{\eta}^{\perp}, L, \eta)\hat{\eta}^{\perp}.$$

Differentiating $\phi^{\pm}_{x}(x, L, \eta) = \xi(L, z^{\pm}(x, L, \eta), \eta)$ in x we get

$$\phi^{\pm}_{xx} = \frac{\partial \xi}{\partial z}\Big(\frac{\partial x}{\partial z}\Big)^{-1} \quad \text{at } x = v\hat{\eta} + (R/2)\hat{\eta}^{\perp},\ x \in G_{\pm}.$$

It follows from (4-3) that

$$\Phi^{\pm}_{ww}\Big(v\hat{\eta} + \frac{R}{2}\hat{\eta}^{\perp}, L, \eta\Big) = \frac{4\sqrt{3}}{R}\Big(1 + v\frac{4\sqrt{3}}{R}\Big)^{-1}.$$

Note that $\Phi^{\pm}_{ww} > 0$ when $v > -R/4\sqrt{3}$ and $\Phi^{\pm}_{ww} < 0$ when $v < -R/4\sqrt{3}$.

At this point we have the data needed in the stationary phase formula, but we need to consider the behavior of the amplitude that comes from (5-2). Since $\det(\partial x/\partial z)(L, z, \eta) = 1 + v(4\sqrt{3}/R)$, the factor $|\det(\partial x/\partial z)|^{-1/2}$ in the amplitude is canceled by part of the factor $|\Phi^{\pm}_{ww}|^{-1/2}$ in the stationary phase formula. Hence the stationary phase expansion in w has the leading terms

$$\frac{(-1)^3}{8\pi^2}\Big(\frac{2\pi}{|\eta|}\Big)^{1/2}\Big(\frac{R}{4\sqrt{3}}\Big)^{1/2}\chi_{-}\Big(v, \frac{R}{2}, \hat{\eta}\Big)\exp\Big(i\Big[(L-t)|\eta| + \frac{\pi}{4}\sigma^{-} + \alpha(L) - \pi/4\Big]\Big), \quad \text{for } v < -R/(4\sqrt{3}),$$
$$\frac{(-1)^3}{8\pi^2}\Big(\frac{2\pi}{|\eta|}\Big)^{1/2}\Big(\frac{R}{4\sqrt{3}}\Big)^{1/2}\chi_{+}\Big(v, \frac{R}{2}, \hat{\eta}\Big)\exp\Big(i[(L-t)|\eta| + \frac{\pi}{4}\sigma^{+} + \alpha(L) + \pi/4]\Big), \quad \text{for } v > -R/(4\sqrt{3}),$$

where σ^{-} and σ^{+} are the values of σ before and after crossing the focal point at $v = -R/(4\sqrt{3})$. Since $\sigma^{-} = -4$ and $\sigma_{+} = -6$, the two formulas above can be combined to give the leading term in the integrand in (5-2) after integration in w

$$\frac{2(\chi_{+} + \chi_{-})}{8\pi^2}\cos(\alpha(L))\Big(\frac{2\pi}{|\eta|}\Big)^{1/2}\Big(\frac{R}{4\sqrt{3}}\Big)^{1/2}\exp\big(i[(L-t)|\eta| - \pi/4]\big). \tag{5-4}$$

Here we have included the contributions from both $w = R/2$ and $w = -R/2$ which have $\alpha(L)$ with opposite signs.

Now we will find the contribution of $\int_{\Omega_R} V_0(x, t, \eta)e^{-ix\cdot\eta}\,dx$. The caustic set Σ is a fold-type singularity (see [Duistermaat 1974] and [Eskin 2011, Example 66.1]). Therefore $V_0(x, t, \eta)$ is given by an integral representation (see [Eskin 2011, Section 66.53] and also [Ludwig 1966]):

$$V_0(x,t,\eta)=\frac{|\eta|^{1/2}e^{i(L-t)|\eta|}}{(2\pi)^{1/2}}\int_{-\infty}^{\infty}a(v,\xi_2,|\eta|)e^{i|\eta|(S(v,\xi_2,L)+w\xi_2)}d\xi_2. \tag{5-5}$$

Computing the stationary points in (5-5) for $x\in G_-\cap\{d(x,\Sigma)<\epsilon\}$ we see that the stationary points are given by $S_{\xi_2}(v,p^-(v,w),L)+w=0$ and the phase is $S(v,p^-,L)+wp^-=\phi^-(x,t,\eta)$, where $\phi^-(x,t,\eta)$ is the same as in (5-2). The amplitude $a(v,\xi_2,|\eta|)$ in (5-5) is an asymptotic series $\sum_{k\ge 0}a_k(v,\xi_2)|\eta|^{-k}$, where

$$a_0(v,\xi_2)=\frac{(-1)^3}{8\pi^2}\chi_0(z(v,\xi_1),\hat{\eta})e^{i[\alpha(L)+(\pi/4)\sigma_--(\pi/4)]}\left|\det\frac{\partial(v,\xi_2)}{\partial z}\right|^{-1/2}. \tag{5-6}$$

Note that the factor $e^{-i(\pi/4)}$ arises because

$$S_{\xi_2^2}(v,p^-(v,w),L)>0;$$

see [Eskin 2011, Section 66.44].

To evaluate the contribution of $\int_{\Omega_R}V_0e^{-x\cdot\eta}\,dv\,dw$ we apply the stationary phase method to the double integral in ξ_2 and w. The phase function is $S(v,\xi_2,t)+w\xi_2-v$. The equations for the stationary points are

$$S_{\xi_2}(v,\xi_2,t)+w=0,\quad \xi_2=0.$$

Note that $t=L$. We will show that $w=-S_{\xi_2}(v,0,L)=R/2$: Let $\xi_2-\alpha(v)=0$ be the equation of the caustic set, that is,

$$S_{\xi_2^2}(v,\alpha(v),L)=0.$$

In our situation

$$S_{\xi_2^3}(v,\alpha(v),L)\ne 0.$$

Expand $S_{\xi_2}(v,\xi_2,L)$ by the Taylor's formula with a remainder at $\xi_2=\alpha(v)$. When $\xi_2=0$, that gives

$$S_{\xi_2}(v,0,L)=S_{\xi_2}(v,\alpha(v),L)+c(v)(0-\alpha(v))^2.$$

Therefore

$$S_{\xi_2}(v,\alpha(v),L)=S_{\xi_2}(v,0,L)-c(v)\alpha^2(v).$$

The equation of the caustic set in (v,w) coordinates is

$$w=-S_{\xi_2}(v,\alpha(v),L)=-S_{\xi_2}(v,0,L)+c(v)\alpha^2(v).$$

On the other hand, using the mapping $x(\tilde{v},\tilde{w})$, one sees that near $(v,w)=(v_0,R/2)$ with $v_0=-R/(4\sqrt{3})$, the caustic set Σ is given by

$$w=(R/2)-c_1(v)(v-v_0)^2.$$

Comparing these two expressions for the caustic set we get

$$-S_{\xi_2}(v,0,L)=R/2\quad\text{and}\quad\alpha(v)=c_2(v)(v-v_0)^2.$$

Note that the determinant of the Hessian at the critical point $(0, R/2)$ is -1. Therefore the standard stationary phase lemma in (ξ_2, w) gives the asymptotic expansion $\sum_{i\geq 0} r_k^0(v)|\xi|^{-(1/2)-k}$, where

$$r_0^0 = \frac{(-1)^3}{8\pi^2}\left(\frac{2\pi}{|\eta|}\right)^{1/2} \chi_0\left(v\hat{\eta} + \frac{R}{2}\hat{\eta}^\perp, \eta\right) e^{i(\alpha(L)+(\pi/4)\sigma_- - \pi/4)} \left(\frac{4\sqrt{3}}{R}\right)^{-1/2}. \tag{5-7}$$

In (5-7) we substituted the value of the Jacobian in (5-6). By (4-3) that is equal to $4\sqrt{3}/R$ at $\xi_2 = 0$, $w = R/2$.

Combining the contributions of (5-7) for $w = R/2$ and $w = -R/2$ with the contribution of (5-4) and then integrating in (v, θ) we get the leading terms of the contribution of $W_+(t)$ to the trace:

$$\left(\frac{1}{(2\pi)^2}(R\sqrt{3})(2\pi)\right)(2\pi)^{1/2}\left(\frac{R}{4\sqrt{3}}\right)^{1/2} \int_0^\infty \cos(\alpha(L)) e^{i[(L-t)|\eta| - \pi/4]} |\eta|^{1/2} d|\eta|. \tag{5-8}$$

This is consistent with (4-10), and therefore the final form of the singularity is again the one given in (1-3).

Note that contributions from neighborhoods on reflection points can be treated by introduction of the natural angular coordinate place of w as in the final part of Section 4.

6. The Aharonov–Bohm effect on a torus

The Aharonov–Bohm effect only arises when the underlying domain is not simply connected. In the previous sections the domain was an annulus. Here we consider the Schrödinger operator on a torus. Let $L = \{m_1 e_1 + m_2 e_2 : m \in \mathbb{Z}^2\}$, where $\{e_1, e_2\}$ is a basis for $\mathbb{R}^2$. We assume that the lattice L has the property: For $d, d' \in L$, if $|d'| = |d|$, then $d' = \pm d$. This is a generic condition that implies that the group of isometries of L consists of lattice translations and the inversion $d \to -d$. Associated to L one has the dual lattice $L^* = \{\delta \in \mathbb{R}^2 : \delta \cdot d \in \mathbb{Z} \text{ for all } d \in L\}$.

We consider the Schrödinger operator,

$$H_{A,V} = \tfrac{1}{2}(i\partial_{x_1} + A_1(x))^2 + \tfrac{1}{2}(i\partial_{x_2} + A_2(x))^2 - V(x),$$

acting on functions on $\mathbb{T}^2 = \mathbb{R}^2/L$. The functions $A = (A_1, A_2)$ and V are assumed to be smooth on $\mathbb{T}^2$ and hence they have smooth extensions to $\mathbb{R}^2$ satisfying $A(x+d) = A(x)$ and $V(x+d) = V(x)$ for all $d \in L$. As before we assume that the magnetic field vanishes

$$\partial_{x_2} A_1 - \partial_{x_1} A_2 = 0 \quad \text{on } \mathbb{T}^2. \tag{6-1}$$

Thus for any closed curve γ on $\mathbb{T}^2$ the flux

$$\alpha_\gamma = \int_\gamma A(x) \cdot dx,$$

is determined by the homology class of γ. We let γ_1 and γ_2 be a basis for the homology group, for instance

$$\gamma_j = \{te_j, t \in [0, 1)\}, \quad j = 1, 2, \tag{6-2}$$

and denote the corresponding fluxes by α_1 and α_2.

Let $g(x) \in C^\infty(\mathbb{T}^2)$ be such that $|g(x)| = 1$. The conjugation of $H_{A,V}$ by the unitary operator of multiplication by $g(x)$ transforms $H_{A,V}$ to $H_{\tilde{A},V}$, where $\tilde{A} = A + ig^{-1}\nabla g$. The condition $|g(x)| = 1$ on $\mathbb{T}^2$ implies that $g(x) = \exp(2\pi i\delta\cdot x + \varphi(x))$, where $\delta \in L^*$ and $\varphi(x)$ is periodic. Hence

$$\alpha_1(\tilde{A}) = \alpha_1(A) - 2\pi\delta\cdot e_1, \alpha_2(\tilde{A}) = \alpha_2(A) - 2\pi\delta\cdot e_2.$$

Therefore if A and $\tilde{A}$ are gauge equivalent we have

$$\alpha_j(\tilde{A}) = \alpha_j(A) \ \text{ modulo } 2\pi, \ \ j = 1, 2. \tag{6-3}$$

Expanding $A(x)$ in a Fourier series we have

$$A(x) = A_0 + \sum_{\delta\in L^*\setminus\{0\}} A_\delta e^{2\pi i\delta\cdot x},$$

where $A_0 = |\mathbb{T}^2|^{-1}\int_{\mathbb{T}^2} A(x)\,dx$, $|\mathbb{T}^2|$ denotes the area of $\{se_1 + te_2;\ 0 \le s, t \le 1\}$. Since $\partial_{x_2}A_1 = \partial_{x_1}A_1$ we have $A(x) = A_0 + \nabla\varphi(x)$, where

$$\varphi(x) = \sum_{\delta\in L^*\setminus\{O\}} \frac{\delta\cdot A_\delta}{2\pi i\delta\cdot\delta} e^{2\pi i\delta\cdot x}.$$

Therefore when (6-1) holds $A(x)$ is gauge equivalent to the constant potential A_0. Two constant magnetic potentials A_0 and $\tilde{A}_0$ are not gauge equivalent if (6-3) does not hold. When $\tilde{A}_0$ is not gauge equivalent to either A_0 or $-A_0$ the potentials A_0 and $\tilde{A}_0$ have a different physical impact, in particular, the spectra of $H_{A_0,V}$ and $H_{\tilde{A}_0,V}$ are not the same.

The last assertion is a consequence of the following theorem.

Theorem 6.1. *Suppose* (6-1) *holds. The spectrum of $H_{A,V}$ as a self-adjoint operator on $L^2(\mathbb{T}^2)$ determines* $\cos\alpha_1$ *and* $\cos\alpha_2$, *where* $\alpha_j = \int_{\gamma_j} A(x)\cdot dx$, $j = 1, 2$.

Theorem 6.1 complements the results of [Guillemin 1990; Eskin and Ralston 2009; Eskin 1989]. In particular it shows that, if A and $\tilde{A}$ give rise to zero magnetic fields on $\mathbb{T}^2$ but different values for $\cos\alpha_1$ and $\cos\alpha_2$, the Schrödinger operators, $H_{A,V}$ and $H_{\tilde{A},V}$ will have different spectra. This proves the Aharonov–Bohm effect on the torus.

Proof of Theorem 6.1. As in the preceding sections we start with the wave trace formula

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\lambda_j}) = \int_{\mathbb{T}^2} E_{\mathbb{T}^2}(x, x, t)\,dx,$$

where $\{\lambda_j\}_{j=1}^{\infty}$ is the spectrum of $H_{A,V}$ on $\mathbb{T}^2$ and $E_{\mathbb{T}^2}(x, y, t)$ is the solution to $E_{tt} + H_{A,V}E = 0$ on $\mathbb{T}^2 \times \mathbb{R}$ satisfying $E(x, y, 0) = \delta(x - y)$ and $E_t(x, y, 0) = 0$. Note that

$$E_{\mathbb{T}^2}(x, y, t) = \sum_{d\in L} E_{\mathbb{R}^2}(x + d, y, t),$$

where $E_{\mathbb{R}^2}$ is the solution to $E_{tt}+H_{A,V}E=0$ on $\mathbb{R}^2\times\mathbb{R}$ satisfying $E(x,y,0)=\delta(x-y)$ and $E_t(x,y,0)=0$ once $H_{A,V}$ is extended to $\mathbb{R}^2$ by making its coefficients periodic: $A(x+d)=A(x)$ and $V(x+d)=V(x)$ for all $d\in L$. Hence

$$\int_{\mathbb{T}^2} E_{\mathbb{T}^2}(x,x,t)\,dx=\sum_{d\in L}\int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x+d,x,t)\,dx.$$

Since $E_{\mathbb{R}^2}$ is smooth off the cone $|x-y|^2=t^2$, and our assumption on L implies that only two lattice vectors can have $|d|^2=t^2$ for a fixed value of t, the singularity in the wave trace at $t=|d|$, must come from (compare [Eskin et al. 1984a; 1984b; Eskin and Ralston 2007])

$$\int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x+d,x,t)\,dx+\int_{\mathbb{T}^2} E_{\mathbb{R}^2}(x-d,x,t)\,dx.$$

To compute the leading singularities in this trace we will use the Hadamard–Hörmander parametrix [Hörmander 2003; 2005; 2007; 2009]. We have

$$E_{\mathbb{R}^2}(x,y,t)=\partial_t\big(E_+(x,y,t)-E_+(x,y,-t)\big),$$

where E_+ is the forward fundamental solution.

The Hadamard–Hörmander parametrix construction for E_+ writes E_+ as an asymptotic sum of terms with increasing regularity. The first term is $a_0(x,y)e_0(|x-y|,t)$, where

$$e_0=\frac{1}{2\sqrt{\pi}}(t^2-|x-y|^2)_+^{-1/2}\quad\text{when } t>0 \text{ and } e_0=0 \text{ when } t<0,$$

and

$$a_0(x,y)=\exp\biggl(i\int_0^1(x-y)\cdot A(y+s(x-y))\,ds\biggr).$$

Therefore, by [Eskin and Ralston 2009], the singularity of the trace at $t=|d|$ determines $I(d)+I(-d)$ where

$$I(d)=\int_{\mathbb{T}^2}\exp\biggl(i\int_0^1 d\cdot A(x+sd)\,ds\biggr)dx.$$

Since $A(x)=A_0+\nabla\varphi(x)$, where $\varphi(x)$ is periodic, we have

$$\int_0^1 d\cdot A(x+sd)\,ds=d\cdot A_0\quad\text{since}\quad\int_0^1 d\cdot\nabla\varphi(x+sd)\,ds=0.$$

Therefore $I(d)=e^{id\cdot A_0}|\mathbb{T}^2|$ and hence the singularity of the wave trace at $t=|d|$ determines $\cos(A_0\cdot d)$ for all $d\in L$. In particular, when $d=e_j$ and $\gamma_j=\{te_j,t\in[0,1)\}$, $j=1,2$, we get

$$\alpha_j=\int_{\gamma_j}A(x)\cdot dx=e_j\cdot A_0.$$

Thus the singularities of the wave trace when $t=|e_j|$ determine $\cos\alpha_j$ for $j=1,2$. When $V(x)=V(-x)$, then $H_{A_0,V}$ and $H_{-A_0,V}$ are isospectral and one can only recover $\cos\alpha_j$, $j=1,2$, from the spectrum. When V is not even, the question of whether one could recover $\exp(i\alpha_j)$, $j=1,2$, is open. □

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