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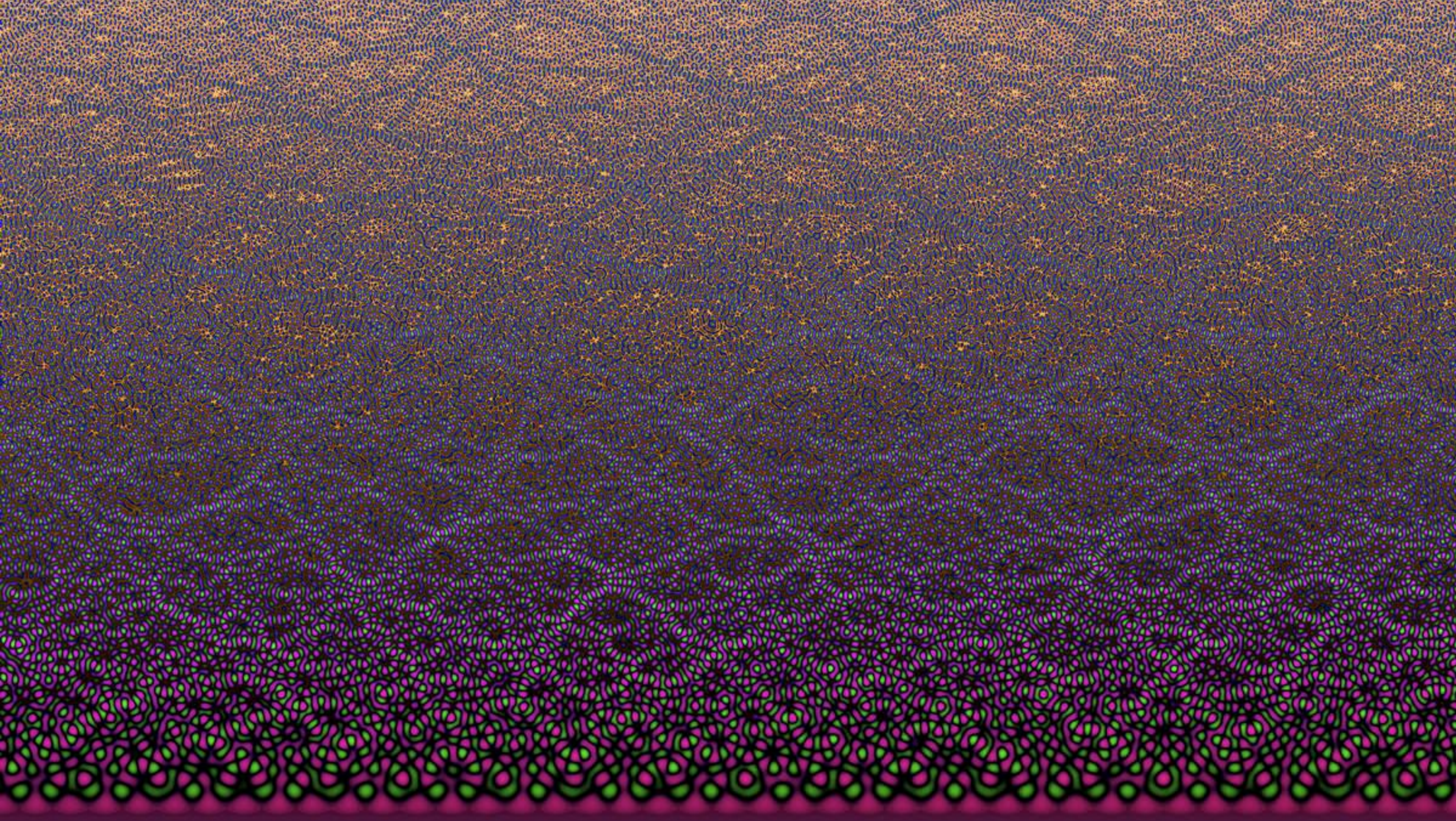

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ANGULAR ENERGY QUANTIZATION FOR LINEAR ELLIPTIC SYSTEMS WITH ANTISYMMETRIC POTENTIALS AND APPLICATIONS

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We establish a quantization result for the angular part of the energy of solutions to elliptic linear systems of Schrödinger type with antisymmetric potentials in two dimensions. This quantization is a consequence of uniform Lorentz–Wente type estimates in degenerating annuli. Moreover this result is optimal in the sense that we exhibit a sequence of functions satisfying our hypothesis whose radial part of the energy is not quantized. We derive from this angular quantization the full energy quantization for general critical points to functionals which are conformally invariant or also for pseudoholomorphic curves on degenerating Riemann surfaces.

Introduction

Conformal invariance is a fundamental property for many problems in physics and geometry. In the last decades it has become an important feature of many questions of nonlinear analysis too. Elliptic conformally invariant Lagrangians for instance share similar analysis behaviors: their Euler–Lagrange equations are critical with respect to the function space naturally given by the Lagrangian and, as a consequence, solutions to these Euler Lagrange equations are subject to *concentration compactness* phenomena. Questions such as the regularity of solutions or energy losses for sequences of solutions cannot be solved by robust general arguments in PDE but require instead a careful study of the interplay between the highest order part of the PDE and its nonlinearity.

For example, in dimension 2, let (Σ, h) be a closed Riemann surface, it has been proved [Rivière 2007, Theorem I.2] that every critical point of a conformally invariant functional, $u : \Sigma \to \mathbb{R}^n$, solves a system

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of the form[1]

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } \Sigma, \tag{1}$$

where $\Omega \in \mathrm{so}(n) \otimes T\Sigma$ and Δ is the negative Laplace–Beltrami operator $\left(1/\sqrt{|h|}\right)\partial_i\left(\sqrt{|h|}\,h^{ij}\partial_j\right)$. The fundamental fact here that has been observed in [Rivière 2007] and exploited in this work to obtain the Hölder continuity of $W^{1,2}$-solutions to (1) is the *antisymmetry* of Ω.

The analysis developed in [Rivière 2007] allowed one to extend to general two-dimensional conformally invariant Lagrangians the use of *integrability by compensation theory* as it was introduced originally by H. Wente in the framework of constant mean curvature immersions in $\mathbb{R}^3$ to solve the *CMC system*

$$\Delta u = 2u_x \wedge u_y \quad \text{on } \Sigma. \tag{2}$$

Solutions to this *CMC system* are in fact critical points to the conformally invariant Lagrangian

$$E(u) = \frac{1}{2}\int_\Sigma |du|_h^2 \, d(\mathrm{vol}_h) + \int_\Sigma u^*\omega,$$

where ω is a 2-form in $\mathbb{R}^3$ satisfying $d\omega = 4\,dx_1 \wedge dx_2 \wedge dx_3$. The natural space to consider (2) is clearly the Sobolev space $W^{1,2}$. The CMC system (2) is critical for $W^{1,2}$ in the following sense: the right-hand side of (2) is *a priori* only in L^1. Classical Calderon Zygmund theory tells us that derivatives of functions in $\Delta^{-1}L^1$ are in the weak L^2 space locally which is "almost" the information we started from. Hence in a sense both the quadratic nonlinearity for the gradient in the right-hand side of the system and the operator in the left-hand side are at the same level from regularity point of view and it requires a more careful analysis in order to decide which one is leading the general dynamic of this system.

H. Wente discovered the special role played by the jacobian in the right-hand side of (2) — see [Hélein 1996] and references therein — and was able to prove that if u satisfies (2) then

$$\|\nabla u\|_2 \le C\|\nabla u\|_2^2, \tag{3}$$

where C is independent on Σ and equals[2] $\sqrt{3/16\pi}$. This inequality implies that if $\sqrt{3/16\pi}\,\|\nabla u\|_2 < 1$ then the solution is constant. This is what we call the *bootstrap test* and it is the key observation for proving Morrey estimates and deduce the Hölder regularity of general solutions to (2) which bootstraps easily in order to establish that solutions to (2) are in fact C^∞.

Another analysis issue for this equation is to understand the behavior of sequences u_k of solutions to the CMC system (2). Inequality (3) tells us again that if the energy does not concentrate at a point then the system will behave locally like a linear system of the form $\Delta u = 0$: the nonlinearity $2u_x \wedge u_y$ in the right-hand side is dominated by the linear highest order term Δu in the left-hand side. As a

[1] In coordinates this system reads

$$-\Delta u_i = \sum_{j=1}^{n} \Omega_i^j \cdot \nabla u_j \quad \text{on } \Sigma \text{ for all } i = 1, \dots, n,$$

where the $\cdot$ operation is the scalar product between the gradient vector fields ∇u_j and the different entries of the vector-valued antisymmetric matrix Ω.

[2] This later fact was discovered later on by Y. Ge [1998]; see also [Hélein 1996].

consequence we deduce that sequences of solutions to (2) with uniformly bounded energy strongly converge in C^p-norm for any $p \in \mathbb{N}$, modulo extraction of a subsequence and possibly away from finitely many points[3] $\{a^1_\infty, \ldots, a^l_\infty\}$ in Σ, where the $W^{1,2}$-norm concentrates, towards a smooth limit that solves also (2):

$$u_k \longrightarrow u_\infty \text{ strongly in } C^p_{\mathrm{loc}}\big(\Sigma \setminus \{a^1_\infty, \ldots, a^l_\infty\}\big) \text{ for all } p \in \mathbb{N}.$$

The question remains to understand how the convergence at the concentration points a^i_∞ fails to be strong, in other words we want to understand how and how much energy has been dissipated at the points a^i_∞. A careful analysis shows that the energy is lost by concentrating solution on $\mathbb{R}^2$ of the CMC system (2), the so-called *bubbles*, that converge to the a^i_∞: there exists points in Σ $a^i_k \to a^i_\infty$ and a family of sequences of radii λ^i_k converging to zero such that

$$u_k(\lambda^i_k x + a^i_k) \longrightarrow \omega^i(x) \text{ strongly in } C^p_{\mathrm{loc}}\big(\mathbb{R}^2 \setminus \{\text{finitely many points}\}\big) \text{ for all } p \in \mathbb{N},$$

where ω^i denote the bubbles, solutions on $\mathbb{R}^2$ of the CMC system (2). Because of the nature of the convergence it is clear that the Dirichlet energy lost in the convergences amount at least to the sum of the Dirichlet energies of the *bubbles* ω^i:

$$\liminf_{k\to+\infty} \int_\Sigma |du_k|^2_h \, d(\mathrm{vol}_h) \geq \int_\Sigma |du_\infty|^2_h \, d(\mathrm{vol}_h) + \sum_{i=1}^{l} \int_{\mathbb{R}^2} |\nabla \omega^i|^2 \, dx_1 \, dx_2. \tag{4}$$

The question remains to understand if the inequality in (4) is strict or is in fact an equality. This question for general conformally invariant problems is known as the *energy quantization question:* is the loss of energy only concentrated in the forming *bubbles* or is there any additional dissipation in the intermediate regions between the *bubbles* and shrinking at the limiting concentration points a^i_∞ in the so-called *neck region*. Since the work of Sacks and Uhlenbeck [1981] where it has been maybe first considered, in the particular framework of minimizing harmonic maps from a Riemann surface into a manifold, this question has generated a special interest, intensive researches and several detailed results have been obtained in the last decades on the subject. We refer to [Rivière 2002] and reference therein for a survey on the *energy quantization results*. Positive results establishing *energy quantization* (that is, the inequality in (4) is in fact an equality) often make use of some special *geometric* objects such as isoperimetric inequality or the Hopf differential, see for instance [Jost 1991] or [Parker 1996]. Rivière, in collaboration with F. H. Lin, introduced [2001; 2002] a more *functional analysis* type technique based on the use of the interpolation Lorentz spaces in order to prove *energy quantization results* in the special cases where the nonlinearity of the conformally invariant PDE can be written as a linear combination of jacobians of $W^{1,2}$-functions. Using this technique we can for instance prove that equality holds in (4): *energy quantization* holds for the CMC system, the whole loss of energy exclusively arises in the bubbles. The main step in the proof consists in using an improvement of Wente inequality (3) which has been obtained by L. Tartar and R. Coifman, P. L. Lions, Y. Meyer and S. Semmes [1993]. This improved *Lorentz–Wente type inequality*

[3] In our notation we can have some a^i_∞ that coincide with another.

reads

$$\|\nabla u\|_{L^{2,1}} \leq C \, \|\nabla u\|_2^2, \tag{5}$$

where this time C depends *a priori* on (Σ, h) and where $L^{2,1}$ denotes the Lorentz space "slightly" smaller than L^2 given by the space of measurable function f on Σ satisfying

$$\int_0^\infty \left|\left\{x \in \Omega \,\middle|\, |f(x)| \geq \lambda\right\}\right|^{1/2} d\lambda < +\infty.$$

The goal of the present paper is to extend energy quantization results to sequences of critical points to general conformally invariant Lagrangians using functional analysis arguments in the style of [Lin and Rivière 2002].

The constant in the inequality (5) depends *a priori* on the domain, at least on its conformal class since the equation is conformally invariant. But our *neck regions* connecting the *bubbles* are conformally equivalent to degenerating annuli. The first task of the present work is to prove different lemma which give some uniform estimates on the $L^{2,1}$-norm of the gradient for solution to Wente-type equations on degenerating annuli. This is the subject of Section 2.

In the following sections, we use these uniform estimates established in Section 2 for proving various quantization phenomena. In particular we get the quantization of the angular part of the gradient for solution of general elliptic second-order systems with antisymmetric potentials. What we mean here by the angular part is the component of the gradient in the orthogonal of the radial direction with respect to the nearest point of concentration. Precisely the first main result in the present work is the following:

Theorem 1. *Let $\Omega_k \in L^2\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$ and let $u_k \in W^{2,1}(B_1, \mathbb{R}^n)$ be a sequence of solutions of*

$$-\Delta u_k = \Omega_k \cdot \nabla u_k,$$

with bounded energy, that is,

$$\int_{B_1} \left(|\nabla u_k|^2 + |\Omega_k|^2\right) dz \leq M. \tag{6}$$

Then there exists $\Omega_\infty \in L^2\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$ and $u_\infty \in W^{2,1}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u_\infty = \Omega_\infty \cdot \nabla u_\infty$ on B_1, $l \in \mathbb{N}^*$ and

(1) *$\omega^1, \ldots, \omega^l$ a family of solutions to system of the form*

$$-\Delta \omega^i = \Omega^i_\infty \cdot \nabla \omega^i \quad \text{on } \mathbb{R}^2,$$

where $\Omega^i_\infty \in L^2\big(\mathbb{R}^2, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$,

(2) *$a^1_k, \ldots, a^l_k$ a family of converging sequences of points of B_1,*

(3) *$\lambda^1_k, \ldots, \lambda^l_k$ a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$\Omega_k \rightharpoonup \Omega_\infty \text{ in } L^2_{\mathrm{loc}}\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big),$$

$$u_k \to u_\infty \text{ on } W^{1,p}_{\mathrm{loc}}\big(B_1 \setminus \{a^1_\infty, \ldots, a^l_\infty\}\big) \text{ for all } p \geq 1,$$

and

$$\Big\|\Big\langle\nabla\Big(u_k - u_\infty - \sum_{i=1}^{l}\omega_k^i\Big),\, X_k\Big\rangle\Big\|_{L^2_{\mathrm{loc}}(B_1)} \to 0,$$

where $\omega_k^i = \omega^i(a_k^i + \lambda_k^i\,\cdot\,)$ *and* $X_k = \nabla^\perp d_k$ *with* $d_k = \min\limits_{1\le i\le l}(\lambda_k^i + d(a_k^i,\cdot))$.

Moreover, if we have $\|\Omega_k\|_\infty = O(1)$ *or even just* $\Omega_k = \Lambda(u_k, \nabla u_k)$ *where* $\Lambda(\,\cdot\,, p) = O(|p|)$ *the convergence to the limit solution* u_∞ *is in fact in* $C^{1,\eta}_{\mathrm{loc}}$ *for all* $\eta \in [0, 1[$.

This theorem is optimal in the sense that we have also exhibited a sequence of functions satisfying the hypothesis of the theorem whose radial part of the energy is not quantized. Moreover, the loss of energy in the neck region is very rigid. We explain these two facts after the proof of Theorem 1.

The proof of Theorem 1 is established through the iteration of the following result. It says that, if the L^2-norm of the potential Ω is below some threshold on every dyadic sub-annulus of a given annulus, the angular part of the Dirichlet energy of u on a slightly smaller annulus is controlled by the maximal contribution of the Dirichlet energy of u on the dyadic sub-annuli. Precisely we prove the following:

Theorem 2. *There exists* $\delta > 0$ *such that for all* $r, R \in \mathbb{R}_+^*$ *with* $4r < R$, *all* $\Omega \in L^2\big(B_R\setminus B_r, \mathrm{so}(n)\otimes\mathbb{R}^2\big)$ *and all* $u \in W^{1,2}\big(B_R\setminus B_r, \mathbb{R}^n\big)$ *satisfying* $-\Delta u = \Omega\cdot\nabla u$, *we have*

$$\sup_{r<\rho<R/2}\int_{B_{2\rho}\setminus B_\rho} |\Omega|^2\,dz \le \delta.$$

Then there exists $C > 0$, *independent of* u, r *and* R, *such that*

$$\Big\|\frac{1}{\rho}\frac{\partial u}{\partial\theta}\Big\|^2_{L^2(B_{R/2}\setminus B_{2r})} \le C\|\nabla u\|_2\Big[\sup_{r<\rho<R/2}\int_{B_{2\rho}\setminus B_\rho}|\nabla u|^2\,dz\Big]^{1/2}.$$

Thanks to the quantization of the angular part for general elliptic systems with antisymmetric potential, we can derive the *energy quantization* for critical points to an arbitrary continuously conformally invariant elliptic Lagrangian with quadratic growth.

Theorem 3. *Let* N^k *be a* C^2 *submanifold of* $\mathbb{R}^m$ *and* ω *be a* C^1 2*-form on* N^k *such that the* L^∞*-norm of* $d\omega$ *is bounded on* N^k. *Let* u_k *be a sequence of critical points in* $W^{1,2}(B_1, N^k)$ *for the Lagrangian*

$$F(u) = \int_{B_1}\big[|\nabla u|^2 + \omega(u)(u_x, u_y)\big]\,dz \tag{7}$$

with uniformly bounded energy, that is,

$$\|\nabla u_k\|_2 \le M.$$

Then there exists $\Lambda \in C^0\big(TN\otimes\mathbb{R}^2, \mathrm{so}(n)\otimes\mathbb{R}^2\big)$ *and* $u_\infty \in W^{1,2}(B_1, \mathbb{R}^n)$ *a solution of* $-\Delta u = \Lambda(u, \nabla u)\cdot\nabla u$ *on* B_1, $l \in \mathbb{N}^*$ *and*

(1) $\omega^1, \dots, \omega^l$ *some nonconstant* Λ*-bubbles, that is, nonconstant solutions of*

$$-\Delta\omega = \Lambda(\omega, \nabla\omega)\cdot\nabla\omega \quad \text{on } \mathbb{R}^2,$$

(2) $a_k^1, \dots, a_k^l$ *a family of converging sequences of points of* B_1,

(3) $\lambda_k^1, \dots, \lambda_k^l$ *a family of sequences of positive reals converging all to zero*,

such that, up to a subsequence,

$$u_k \to u_\infty \text{ on } C^{1,\eta}_{\mathrm{loc}}\big(B_1 \setminus \{a^1_\infty, \dots, a^l_\infty\}\big) \quad \text{for all} \quad \eta \in [0, 1[$$

and

$$\Big\| \nabla \Big(u_k - u_\infty - \sum_{i=1}^{l} \omega_k^i \Big) \Big\|_{L^2_{\mathrm{loc}}(B_1)} \to 0,$$

where $\omega_k^i = \omega^i(a_k^i + \lambda_k^i \,\cdot)$.

Previous works establishing *energy quantizations* for various conformally invariant elliptic Lagrangian usually require more regularity on the Lagrangian (see for instance [Jost 1991; Parker 1996; Struwe 1985; Ding and Tian 1995; Lin and Wang 1998; Zhu 2010]). For instance in [Parker 1996] or [Lin and Wang 1998] the *energy quantization* for harmonic maps in two dimensions is obtained through the application of the maximum principle to an ordinary differential inequality satisfied by the integration over concentric circles of the angular part of the energy. The application of this procedure required an L^∞ bound on the *derivatives of the second fundamental form* [Lin and Wang 1998, Lemma 2.1]. We insist on the fact that, in comparison to the previously existing energy quantization results, Theorem 3 above requires a C^0 bound on the *second fundamental form* only, which is a weakening of the regularity assumption for the target of a magnitude one with respect to derivation. Another application of Theorem 3 is the *energy quantization* for solutions to the prescribed mean curvature system, see Corollary 17, assuming only a C^0 bound on the mean curvature. Again, previous *energy quantization* results were assuming uniform C^1 bounds on H [Bethuel and Rey 1994; Caldiroli and Musina 2006]. Theorem 3 in the prescribed mean curvature system corresponds again for this problem to weakening of the regularity assumption for the target of a magnitude one with respect to derivation in comparison to previous existing result. These weaker assumptions are the minimal ones required in order that the Lagrangian to be continuously differentiable and this is why it coincides with the original one appearing in the formulation of the Heinz–Hildebrandt regularity conjecture in the 1970s.

In a last part, we present some more applications of the uniform Lorentz–Wente estimates we established in Section 2. The first one, for instance, deals with sequences of pseudoholomorphic immersions of sequences of closed Riemann surfaces whose corresponding conformal class degenerate in the moduli space of the underlying two-dimensional manifold. In particular we give a new proof of the Gromov's compactness theorem in all generality, see Theorem 19. We also give some cohomological condition which guarantees the *energy quantization* for sequences of harmonic maps on degenerating surfaces. Finally we give a very brief introduction to the quantization of the Willmore surface established recently in [Bernard and Rivière 2011], where the uniform Lorentz–Wente estimates of Section 2 play a crucial role.

Notation. In the following, if we consider a norm with out specifying its domain, it is implicitly assumed that its domain of definition is the one of the function. We denote $B_R(p)$ the ball of radius R centered at p and we just denote B_R when $p = 0$.

1. Lorentz spaces and standard Wente's inequalities

Lorentz spaces seem to be the good spaces in order to get precise Wente's inequalities. Here we recall some classical facts about these spaces; see [Stein and Weiss 1971] and [Grafakos 2009] for details.

Definition 4. Let D be a domain of $\mathbb{R}^k$, $p \in]1, +\infty[$ and $q \in [1, +\infty]$. The Lorentz space $L^{p,q}(D)$ is the set of measurable functions $f : D \to \mathbb{R}$ such that

$$\|f\|_{p,q} = \left(\int_0^{+\infty} \left(t^{1/p} f^{**}(t)\right)^q \frac{dt}{t} \right)^{1/q} < +\infty \quad \text{if } q < +\infty,$$

or

$$\|f\|_{p,\infty} = \sup \left(t^{1/p} f^{**}(t)\right) \quad \text{if } q = +\infty,$$

where $f^{**}(t) = (1/t)\int_0^t f^*(s)\,ds$ and f^* is the decreasing rearrangement of f.

Each $L^{p,q}$ may be seen as a deformation of L^p. For instance, we have the strict inclusions

$$L^{p,1} \subset L^{p,q'} \subset L^{p,q''} \subset L^{p,\infty}$$

if $1 < q' < q''$. Moreover,

$$L^{p,p} = L^p.$$

Furthermore, if $|D|$ is finite, we have that for all q and q',

$$p > p' \Rightarrow L^{p,q} \subset L^{p',q'}.$$

Finally, for $p \in]1, +\infty[$ and $q \in [1, +\infty]$, we have $L^{p,q} = \left(L^{p/(p-1),q/(q-1)}\right)^*$.

In the case $p, q = 2, 1$ we can give an equivalent definition. First we note that the norm $\| \; \|_{p,q}$ is equivalent to

$$\left(\int_0^{+\infty} \left(t^{1/p} f^{*}(t)\right)^q \frac{dt}{t} \right)^{1/q},$$

which is only a seminorm [Ziemer 1989]. Then, letting $\phi(\lambda) = \left|\left\{t \in [0, |D|] \;\middle|\; f^*(t) \geq \lambda\right\}\right|$, we make the change of variable $t = \phi(\lambda)$ in the definition of the Lorentz-norm, which gives

$$\|f\|_{2,1} \sim 2 \int_{\sup|f|}^{0} \phi^{-1/2}(\lambda)\,\lambda\phi'(\lambda)\,d\lambda.$$

Hence integrating by parts, we get

$$\|f\|_{2,1} \sim 4 \int_0^{+\infty} \left|\left\{x \in \Omega \;\middle|\; |f(x)| \geq \lambda\right\}\right|^{1/2} d\lambda. \tag{8}$$

To finish these preliminaries, we quickly present the standard Wente's inequalities for elliptic system in Jacobian form. Indeed if a and b are in $W^{1,2}$ this is clear that $a_x b_y - a_y b_x$ is in L^1 but in fact thanks to its structure, it is subject to compensated phenomena and $a_x b_y - a_y b_x$ is in $\mathcal{H}^1$ the Hardy space which is a strict subspace of L^1 and has better behavior than L^1 with respect to Calderon–Zygmund theory, since the convolution of a function in $\mathcal{H}^1$ and the Green kernel $\log(|z|)$ is in $W^{2,1}$. This improvement of integrability is summarized in the following theorem.

Lemma 5 [Wente 1969; Tartar 1985; Coifman et al. 1993]. *Let $a, b \in W^{1,2}(B_1)$, and let $\phi \in W_0^{1,1}(B_1)$ be the solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1.$$

Then there exists a constant C independent of ϕ such that

$$\|\phi\|_\infty + \|\nabla\phi\|_{2,1} + \|\nabla^2\phi\|_1 \leq C\|\nabla a\|\|\nabla b\|_2. \tag{9}$$

A consequence of the previous theorem was obtained by Bethuel [1992] using a duality argument.

Lemma 6. *Let a and b be two measurable functions such that $\nabla a \in L^{2,\infty}(B_1)$ and $\nabla b \in L^2(B_1)$, and let $\phi \in W_0^{1,1}(B_1)$ be the solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1.$$

Then there exists a constant C independent of ϕ such that

$$\|\nabla\phi\|_2 \leq C\|\nabla a\|_{2,\infty}\|\nabla b\|_2. \tag{10}$$

2. Wente-type lemmas

In this section we are going to prove some uniform Wente's estimates on annuli whose conformal class is *a priori* not bounded. In fact those estimate were already known for the L^∞-norm and the L^2-norm of the gradient, since it has been proved that the constant is in fact independent of the domain considered, see [Topping 1997] and [Ge 1998]. But this fact is to our knowledge new for the $L^{2,1}$-norm of the gradient.

Lemma 7. *Let $a, b \in W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{2}$, and let $\phi \in W_0^{1,1}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1 \setminus B_\varepsilon.$$

Then $\nabla\phi \in L^{2,1}(B_1 \setminus B_\varepsilon)$, and for each $\lambda > 1$ there exists a positive constant $C(\lambda)$, independent of ε and ϕ, such that

$$\|\nabla\phi\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq C(\lambda)\|\nabla a\|_2\,\|\nabla b\|_2.$$

Proof. First we consider a solution of our equation on the whole disk, that is to say $\varphi \in W_0^{1,1}(B_1)$ which satisfies

$$\Delta\varphi = a_x b_y - a_y b_x \quad \text{on } B_1.$$

Then thanks to the classical Wente's inequality (9), we have

$$\|\varphi\|_\infty + \|\nabla\varphi\|_{2,1} \leq C\|\nabla a\|_2\,\|\nabla b\|_2, \tag{11}$$

where C is a positive constant independent of φ.

Then we set $\psi = \phi - \varphi$, which satisfies

$$\begin{cases} \Delta\psi = 0 & \text{on } B_1 \setminus B_\varepsilon, \\ \psi = 0 & \text{on } \partial B_1, \\ \psi = -\varphi & \text{on } \partial B_\varepsilon. \end{cases}$$

Hence $\widetilde{\psi} = \psi - \left(\int_{\partial B_\varepsilon} \psi \, d\sigma\right) \log(|z|)/(2\pi\varepsilon \log(\varepsilon))$ satisfies the hypothesis of Lemma A.1, then

$$\|\nabla\widetilde{\psi}\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq C(\lambda)\|\nabla\widetilde{\psi}\|_2 \quad \text{for all } \lambda > 1.$$

Hence, computing the L^2-norm of the gradient of the logarithm on $B_1 \setminus B_{\lambda\varepsilon}$, we get that

$$\|\nabla\widetilde{\psi}\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq C(\lambda)\left(\|\nabla\psi\|_2 + \frac{1}{\varepsilon\sqrt{\log(1/\varepsilon)}}\int_{\partial B_\varepsilon} |\psi| \, d\sigma\right). \tag{12}$$

But ψ is the harmonic on $B_1 \setminus B_\varepsilon$ and is equal to $-\varphi$ on the boundary, then

$$\|\nabla\psi\|_2 \leq \|\nabla\varphi\|_2 \quad \text{and} \quad \|\psi\|_\infty \leq \|\varphi\|_\infty. \tag{13}$$

Hence we get that

$$\int_{\partial B_\varepsilon} |\psi| \, d\sigma \leq \varepsilon C(\lambda)\|\nabla a\|_2 \, \|\nabla b\|_2, \tag{14}$$

which gives, using (12) and (13), that

$$\|\nabla\widetilde{\psi}\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq C(\lambda)\|\nabla a\|_2 \, \|\nabla b\|_2. \tag{15}$$

Finally, computing the $L^{2,1}$-norm of the gradient of the logarithm on $B_1 \setminus B_{\lambda\varepsilon}$, we get that

$$\|\nabla \log r\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} = 4\sqrt{\pi}\log(1/\lambda\varepsilon). \tag{16}$$

Hence, thanks to (14), (15) and (16), we get that

$$\|\nabla\psi\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq C(\lambda)\|\nabla a\|_2 \, \|\nabla b\|_2. \tag{17}$$

Then, thanks to (11) and (17), we get the desired estimate. $\square$

Lemma 8. *Let $a, b \in W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, and let $\phi \in W^{1,1}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta\phi = a_x b_y - a_y b_x \quad \text{on } B_1 \setminus B_\varepsilon \tag{18}$$

such that

$$\int_{\partial B_\varepsilon} \phi \, d\sigma = 0 \quad \text{and} \quad \left|\int_{\partial B_1} \phi \, d\sigma\right| \leq K,$$

where K is a constant independent of ε. Then for each $0 < \lambda < 1$ there exists a positive constant $C(\lambda)$, independent of ε, such that

$$\|\nabla\phi\|_{L^{2,1}(B_\lambda\setminus B_{\varepsilon/\lambda})} \leq C(\lambda)(\|\nabla a\|_2 \, \|\nabla b\|_2 + \|\nabla\phi\|_2 + 1).$$

Proof. Let $u \in W^{1,1}(B_1 \setminus B_\varepsilon)$ be the solution of

$$\begin{cases} \Delta u = 0 & \text{on } B_1 \setminus B_\varepsilon, \\ u = \phi & \text{on } \partial B_1 \cup \partial B_\varepsilon. \end{cases}$$

Hence $\|\nabla u\|_2 \leq \|\nabla \phi\|_2$. Moreover, thanks to Lemmas A.2 and 7 we have $\nabla u \in L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})$ and $\nabla(u-\phi) \in L^{2,1}(B_1 \setminus B_{\varepsilon/\lambda})$, with

$$\|\nabla u\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq C(\lambda)\left(\|\nabla \phi\|_2 + 1\right) \quad \text{and} \quad \|\nabla(u-\phi)\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq C(\lambda)\|\nabla a\|_2 \|\nabla b\|_2,$$

which proves Lemma 8. □

Remark. As in Lemma A.2 we cannot control the $L^{2,1}$-norm of $\nabla\phi$ by its L^2-norm, as it is shown by the following example:

$$z \mapsto \frac{\log(|z|/\varepsilon)}{\log(1/\varepsilon)}.$$

Lemma 9. *Let $a, b \in W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, and let $\phi \in W^{1,2}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta \phi = a_x b_y - a_y b_x \quad \textit{on } B_1 \setminus B_\varepsilon.$$

Moreover, assume that

$$\|\phi\|_\infty < +\infty. \tag{19}$$

Then for each $0 < \lambda < 1$ there exists a positive constant $C(\lambda)$, independent of ε and ϕ, such that

$$\|\nabla \phi\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq C(\lambda)\big(\|\nabla a\|_2 \, \|\nabla b\|_2 + \|\phi\|_\infty\big). \tag{20}$$

Proof. We introduce first $\varphi \in W_0^{1,2}(B_1 \setminus B_\varepsilon)$ to be the unique solution to

$$\begin{cases} \Delta \varphi = a_x b_y - a_y b_x & \text{on } B_1 \setminus B_\varepsilon, \\ \varphi = 0 & \text{on } \partial B_1 \cup \partial B_\varepsilon. \end{cases}$$

Then thanks to Lemma 7, we have

$$\|\nabla \varphi\|_{L^{2,1}(B_1 \setminus B_{\varepsilon/\lambda})} \leq C(\lambda)\|\nabla a\|_2 \, \|\nabla b\|_2,$$

where $C(\lambda)$ is a positive constant depending on λ but not on ϕ and ε.

Then we set $\psi = \phi - \varphi$, which is harmonic. Thanks to standard estimates on harmonic functions [Han and Lin 2011], there exists a positive constant $C(\lambda)$ independent of ψ and ε such that

$$\|\psi\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \leq C(\lambda)\|\psi\|_{L^\infty(\partial B_1 \cup \partial B_\varepsilon)} \leq C(\lambda)\|\phi\|_{L^\infty}.$$

This proves the desired inequality, and Lemma 9 is proved. □

Lemma 10. *Let $a, b \in L^2(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, assume that $\nabla a \in L^{2,\infty}(B_1)$ and $\nabla b \in L^2(B_1)$, and let $\phi \in W^{1,(2,\infty)}(B_1 \setminus B_\varepsilon)$ be a solution of*

$$\Delta \phi = a_x b_y - a_y b_x \quad \textit{on } B_1 \setminus B_\varepsilon.$$

For each $\varepsilon \le r \le 1$, *set* $\phi_0(r) = (1/2\pi r)\int_{\partial B_r(0)} \phi \, d\sigma$, *and assume that*

$$\int_{\varepsilon}^{1} \left|\dot{\phi_0}\right|^2 r\, dr < +\infty. \tag{21}$$

Then for each $0 \le \lambda < 1$ *there exists a positive constant* $C(\lambda) > 0$, *independent of* ε *and* ϕ, *such that*

$$\|\nabla \phi\|_{L^2(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\big(\|\nabla a\|_{2,\infty} \|\nabla b\|_2 + \|\nabla \phi_0\|_{L^2(B_1\setminus B_\varepsilon)} + \|\nabla \phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}\big). \tag{22}$$

Proof. First we consider $\varphi \in W_0^{1,2}(B_1)$ to be the solution of

$$\begin{cases} \Delta\varphi = a_x b_y - a_y b_x & \text{on } B_1, \\ \varphi = 0 & \text{on } \partial B_1. \end{cases}$$

Then thanks to the generalized Wente's inequality, see (10), we have

$$\|\nabla\varphi\|_2 \le C\, \|\nabla a\|_{2,\infty} \|\nabla b\|_2. \tag{23}$$

Consider the difference $v := \phi - \varphi - (\phi_0 - \varphi_0)$; it is a harmonic function on $B_1 \setminus B_\varepsilon$ which does not have 0-frequency Fourier modes:

$$v = \sum_{n\in\mathbb{Z}^*} \big(c_n \rho^n + d_n \rho^{-n}\big)\, e^{in\theta},$$

which implies in particular that

$$\int_{\partial B_\rho} \frac{\partial v}{\partial \nu}\, d\sigma = 0 \quad \text{for all } \varepsilon < \rho < 1. \tag{24}$$

Moreover, due to the assumption (21) and due to (23) we have

$$\begin{aligned} \|\nabla v\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)} &\le 2\|\nabla\varphi\|_2 + \|\nabla\phi_0\|_2 + \|\nabla\phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)} \\ &\le C\big(\|\nabla a\|_{2,\infty} \|\nabla b\|_2 + \|\nabla\phi_0\|_2 + \|\nabla\phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}\big). \end{aligned} \tag{25}$$

Here we used the fact that $L^{2,\infty}$-norm is controlled by the L^2-norm on a set of finite measure [Ziemer 1989]. Let $\lambda \in]0, 1[$; then standard elliptic estimates on harmonic functions give that for all $\rho \in (\varepsilon/\lambda, \lambda)$,

$$\begin{aligned} \|\nabla v\|_{L^\infty(\partial B_\rho)} &\le C(\lambda)\rho^{-1}\, \|\nabla v\|_{L^{2,\infty}(B_{\rho/\lambda}\setminus B_{\lambda\rho})} \\ &\le C(\lambda)\rho^{-1}\, \big(\|\nabla a\|_{2,\infty} \|\nabla b\|_2 + \|\nabla\phi_0\|_2 + \|\nabla\phi\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}\big). \end{aligned} \tag{26}$$

Denote $\Omega_\varepsilon := B_\lambda \setminus B_{\varepsilon/\lambda}$. We have that

$$\|\nabla v\|_{L^2(\Omega_\varepsilon)} = \sup_{\|X\|_{L^2(\Omega_\varepsilon)}\le 1} \int_{\Omega_\varepsilon} \nabla v \cdot X\, dz. \tag{27}$$

For such an $X \in L^2(\Omega_\varepsilon)$, we denote by $\widetilde{X}$ its extension by 0 in the complement of Ω_ε in B_1. Let g be the solution of

$$\begin{cases} \Delta g = -\operatorname{div} \widetilde{X}^\perp & \text{in } B_1, \\ g = 0 & \text{on } \partial B_1, \end{cases}$$

where $\widetilde{X}^{\perp} = (-\widetilde{X}_2, \widetilde{X}_1)$. We easily see that

$$\|\nabla g\|_{L^2(B_1)} \leq C\ \|\widetilde{X}\|_{L^2(B_1)} \leq C. \tag{28}$$

The Poincaré lemma gives the existence of $f \in W^{1,2}(B_1)$ such that

$$\widetilde{X} = \nabla f + \nabla^{\perp} g,$$

and we have

$$\|\nabla f\|_{L^2(B_1)} \leq \|\nabla g\|_{L^2(B_1)} + \|\widetilde{X}\|_{L^2(B_1)} \leq C + 1. \tag{29}$$

We have

$$\int_{\Omega_\varepsilon} \nabla v \cdot X\, dz = \int_{\Omega_\varepsilon} \nabla v \cdot \nabla f\, dz + \int_{\Omega_\varepsilon} \nabla v \cdot \nabla^{\perp} g\, dz.$$

We write

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla v \cdot \nabla^{\perp} g\, dz &= \int_{\partial B_\lambda} \partial_\tau v\, g\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\tau v\, g\, d\sigma\\ &= \int_{\partial B_\lambda} \partial_\tau v\, (g - g_\lambda)\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\tau v\, (g - g_{\varepsilon/\lambda})\, d\sigma,\end{aligned} \tag{30}$$

where ∂_τ is the tangential derivative along the circles ∂B_λ and $\partial B_{\varepsilon/\lambda}$, and g_λ and $g_{\varepsilon/\lambda}$ denote the averages of g on ∂B_λ and $\partial B_{\varepsilon/\lambda}$, respectively.

We have for any $\rho \in (0, 1)$

$$\frac{1}{\rho} \int_{\partial B_\rho} |g - g_\rho|\, d\sigma \leq C \|g\|_{H^{1/2}(\partial B_\rho)} \leq C \|\nabla g\|_2 \leq C, \tag{31}$$

where C is independent of ρ. Combining (26), (31) and (30) gives on one hand

$$\left| \int_{\Omega_\varepsilon} \nabla v \cdot \nabla^{\perp} g\, dz \right| \leq C(\lambda) \|\nabla v\|_{L^{2,\infty}(B_1 \setminus B_\varepsilon)}. \tag{32}$$

On the other hand, using the fact that v is harmonic and satisfies (24) we have

$$\begin{aligned}\int_{\Omega_\varepsilon} \nabla v \cdot \nabla f\, dz &= \int_{\partial B_\lambda} \partial_\nu v\, f\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\nu v\, f\, d\sigma\\ &= \int_{\partial B_\lambda} \partial_\nu v\, (f - f_\lambda)\, d\sigma - \int_{\partial B_{\varepsilon/\lambda}} \partial_\nu v\, (f - f_{\varepsilon/\lambda})\, d\sigma.\end{aligned} \tag{33}$$

We have for any $\rho \in (0, 1)$

$$\frac{1}{\rho} \int_{\partial B_\rho} |f - f_\rho|\, d\sigma \leq C \|f\|_{H^{1/2}(\partial B_\rho)} \leq C \|\nabla f\|_2 \leq C. \tag{34}$$

Combining now (26), (33), and (34) we obtain

$$\left| \int_{\Omega_\varepsilon} \nabla v \cdot \nabla f\, dz \right| \leq C(\lambda) \|\nabla v\|_{L^{2,\infty}(B_1 \setminus B_\varepsilon)}. \tag{35}$$

Combining (32), (35), and (27) gives

$$\|\nabla v\|_{L^2(\Omega_\varepsilon)} \leq C(\lambda)\|\nabla v\|_{L^{2,\infty}(B_1\setminus B_\varepsilon)}. \tag{36}$$

This inequality, together with (21) and (23), gives (22), and the lemma is proved. □

3. Angular energy quantization for solutions to elliptic systems with antisymmetric potential

The aim of this section is to prove that the angular part of the energy of a bounded sequence of solutions of an elliptic system with antisymmetric potential is always quantized. But before starting the proof of the quantization, we remind the reader of some facts about elliptic systems with antisymmetric potential which have intensively studied by the second author [Rivière 2007].

Let $\Omega \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$. We consider $u \in W^{1,2}(B_1, \mathbb{R}^n)$ a solution of the equation

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } B_1.$$

One of the fundamental facts about this system is the discovery a conservation law using a Coulomb gauge for Ω when its L^2-norm is small enough which is the aim of the following theorem.

Theorem 11 [Rivière 2007, Theorem I.4]. *There exists $\varepsilon_0 > 0$ such that for all $\Omega \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$ satisfying*

$$\int_{B_1} |\Omega|^2\, dz \leq \varepsilon_0,$$

there exists $A \in W^{1,2} \cap L^\infty(B_1, \mathrm{Gl}_n(\mathbb{R}))$ such that

$$\mathrm{div}(\nabla A - A\Omega) = 0$$

and

$$\int_{B_1} \big(|\nabla A|^2 + |\nabla A^{-1}|^2\big)\, dz + \mathrm{dist}\big(\{A, A^{-1}\}, \mathrm{SO}(n)\big) \leq C \int_{B_1} |\Omega|^2\, dz,$$

where C is a constant independent of Ω.

Then, using this theorem and Poincaré's lemma, we get the existence of $B \in W^{1,2}(B_1, M_n(\mathbb{R}))$ such that

$$\mathrm{div}(A\, \nabla u) = \nabla^\perp B \cdot \nabla u$$

and

$$\int_{B_1} |\nabla B|^2\, dz \leq C \int_{B_1} |\Omega|^2\, dz.$$

Hence the system is rewritten in Jacobian form and we can use standard Wente's estimates. In particular, this permits one to prove three fundamental properties of the solutions of this equation which are the ε-regularity, the energy gap for solutions defined on the whole plane and the passage to the weak limit in the equation. These properties are summarized in the following theorem.

Theorem 12 [Rivière 2007; 2010]. *There exists $\varepsilon_0 > 0$ and $C_q > 0$, depending only on $q \in \mathbb{N}^*$, such that if $\Omega \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$ (respectively, $L^2(\mathbb{R}^2, \mathrm{so}(n) \otimes \mathbb{R}^2)$) satisfies $\|\Omega\|_2^2 \leq \varepsilon_0$, then*:

(1) (ε-regularity) *If $u \in W^{1,2}(B_1, \mathbb{R}^n)$ satisfies*

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } B_1,$$

then we have

$$\|\nabla u\|_{L^q(B_{1/4})} \leq C_q \|\nabla u\|_2 \quad \text{for all } q \in \mathbb{N}^*.$$

(2) (energy gap) *If $u \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^n)$ satisfies*

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } \mathbb{R}^2,$$

then it is constant.

(3) (weak limit property) *Let $\Omega_k \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$ be such that Ω_k weakly converges in L^2 to Ω, and let u_k be a bounded sequence in $W^{1,2}(B_1, \mathbb{R}^n)$ which satisfies*

$$-\Delta u_k = \Omega_k \cdot \nabla u_k \quad \text{on } B_1.$$

Then, there exists a subsequence of u_k which weakly converge in $W^{1,2}(B_1, \mathbb{R}^n)$ to a solution of

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } B_1.$$

For the convenience of the reader we recall the arguments developed in [Rivière 2007] and [Rivière 2010] to prove Theorem 12.

Proof. In order to prove the ε-regularity, let us prove that it suffices to show, for $\alpha > 0$, that we have

$$\sup_{\substack{p \in B_{1/2} \\ 0<\rho<1/2}} \frac{1}{\rho^\alpha} \int_{B_\rho(p)} |\Delta u| \, dz \leq C \|\nabla u\|_{L^2(B_1)}. \tag{37}$$

Indeed, a classical estimate on Riesz potentials gives

$$|\nabla u|(p) \leq C \frac{1}{|x|} * \chi_{B_{1/2}} |\Delta u| + C \|\nabla u\|_{L^2(B_1)} \quad \text{for all } p \in B_{1/4},$$

where $\chi_{B_{1/2}}$ is the characteristic function of the ball $B_{1/2}$. Together with injections proved by Adams [1975] (see also [Grafakos 2009, 6.1.6]), the latter shows that

$$\|\nabla u\|_{L^r(B_{1/4})} \leq C \|\nabla u\|_{L^2(B_1)},$$

for some $r > 1$. Then bootstrapping this estimate (see [Rivière 2010, Lemma IV.1] or [Sharp and Topping 2013, Theorem 1.1]), we get

$$\|\nabla u\|_{L^q(B_{1/4})} \leq C_q \|\nabla u\|_{L^2(B_1)} \quad \text{for all } q \in \mathbb{N}^*,$$

which will prove the ε-regularity.

In order to prove (37), we assume that ε_0 is small enough to apply Theorem 11. Hence there exists $A \in W^{1,2} \cap L^\infty(B_1, \mathrm{Gl}_n(\mathbb{R}))$ and $B \in W^{1,2} \cap L^\infty(B_1, M_n(\mathbb{R}))$ such that

$$\int_{B_1} \big(|\nabla A|^2 + |\nabla B|^2\big)\, dz + \mathrm{dist}\big(\{A, A^{-1}\}, \mathrm{SO}(n)\big) \le C \int_{B_1} |\Omega|^2\, dz,$$

$\mathrm{div}(A\,\nabla u) = \nabla^\perp B \cdot \nabla u$, and $\mathrm{curl}(A\,\nabla u) = \nabla^\perp A \cdot \nabla u$. Let $p \in B_{1/2}$ and $0 < \rho < \frac{1}{2}$; we proceed by introducing on $B_\rho(p)$ the linear Hodge decomposition in L^2 of $A \nabla u$. Namely, there exist two functions C and D, unique up to additive constants, elements of $W_0^{1,2}(B_\rho(p))$ and $W^{1,2}(B_\rho(p))$ respectively, and such that

$$A\,\nabla u = \nabla C + \nabla^\perp D, \tag{38}$$

with

$$\Delta C = \mathrm{div}(A\,\nabla u) = \nabla^\perp B \cdot \nabla u \quad \text{and} \quad \Delta D = -\nabla A \cdot \nabla^\perp u.$$

Wente's Lemma 5 guarantees that C lies in $W^{1,2}$, and moreover

$$\int_{B_\rho(p)} |\nabla C|^2\, dz \le C\Big(\int_{B_\rho(p)} |\nabla B|^2\, dz\Big)\Big(\int_{B_\rho(p)} |\nabla u|^2\, dz\Big). \tag{39}$$

Then, we introduce the decomposition $D = \phi + v$, with ϕ satisfying

$$\begin{cases} \Delta\phi = -\nabla A \cdot \nabla^\perp u & \text{in } B_\rho(p), \\ \phi = 0 & \text{on } \partial B_\rho(p), \end{cases} \tag{40}$$

and with v being harmonic. Once again, Wente's Lemma 5 gives us the estimate

$$\int_{B_\rho(p)} |\nabla \phi|^2\, dz \le C\Big(\int_{B_\rho(p)} |\nabla A|^2\, dz\Big)\Big(\int_{B_\rho(p)} |\nabla u|^2\, dz\Big).$$

Since $\rho \mapsto (1/\rho^2) \int_{B_\rho(p)} |\nabla v|^2\, dz$ is increasing for any harmonic function [Rivière 2010, Lemma II.1], we get, for any $0 \le \delta \le 1$, that

$$\int_{B_{\delta\rho}(p)} |\nabla v|^2\, dz \le \delta^2 \int_{B_\rho(p)} |\nabla v|^2\, dz.$$

Finally, we have

$$\int_{B_{\delta\rho}(p)} |\nabla D|^2\, dz \le 2\delta^2 \int_{B_\rho(p)} |\nabla D|^2\, dz \; + \; 2 \int_{B_\rho(p)} |\nabla \phi|^2\, dz. \tag{41}$$

Bringing together (38), (39), and (41) produces

$$\int_{B_{\delta\rho}(p)} |A\,\nabla u|^2\, dz \le 2\delta^2 \int_{B_\rho(p)} |A\,\nabla u|^2\, dz \; + \; C\varepsilon_0 \int_{B_\rho(p)} |\nabla u|^2\, dz. \tag{42}$$

Using the hypotheses that A and A^{-1} are bounded in L^∞, it follows from (42) that for all $0 < \delta < 1$,

$$\int_{B_{\delta\rho}(p)} |\nabla u|^2\, dz \le 2\|A^{-1}\|_\infty \|A\|_\infty \delta^2 \int_{B_\rho(p)} |\nabla u|^2\, dz + C\, \|A^{-1}\|_\infty\, \varepsilon_0 \int_{B_\rho(p)} |\nabla u|^2\, dz. \tag{43}$$

Next, we choose ε_0 and δ strictly positive, independent of ρ and p, and such that

$$2\|A^{-1}\|_\infty \|A\|_\infty \delta^2 + C\|A^{-1}\|_\infty \varepsilon_0 = \frac{1}{2}.$$

For this particular choice of δ, we have thus obtained the inequality

$$\int_{B_{\delta\rho}(p)} |\nabla u|^2 \, dz \leq \frac{1}{2} \int_{B_\rho(p)} |\nabla u|^2 \, dz.$$

Classical results then yield the existence of some constant $\alpha > 0$ for which

$$\sup_{\substack{p \in B_{1/2}(0) \\ 0<\rho<1/2}} \frac{1}{\rho^\alpha} \int_{B_\rho(p)} |\nabla u|^2 \, dz < +\infty,$$

which proves the ε-regularity as already remarked above.

Then, the energy gap follows easily remarking that, thanks to the conformal invariance, for all $R > 0$ and some $q > 2$, we have

$$\|\nabla u\|_{L^q(B_{R/4})} \leq \frac{C_q}{R^{(q-2)/q}} \|\nabla u\|_{L^2(B_R)}.$$

Finally, the weak limit property is a just a special case of [Rivière 2007, Theorem I.5] which is one of the many consequences of Theorem 11. □

We will be in position to prove Theorem 2 which is the main result of this section once we will have established the following lemma.

Lemma 13. *There exists $\delta > 0$ such that for all $r, R \in \mathbb{R}_+^*$ satisfying $2r < R$, all $\Omega \in L^2(B_R \setminus B_r$, $\mathrm{so}(n) \otimes \mathbb{R}^2)$, and all $u \in W^{1,2}(B_R \setminus B_r, \mathbb{R}^n)$ satisfying*

$$-\Delta u = \Omega \cdot \nabla u \quad \textit{and} \quad \sup_{r<\rho<R/2} \int_{B_{2\rho} \setminus B_\rho} |\Omega|^2 \, dz \leq \delta,$$

there exists $C > 0$, independent of u, r and R, such that

$$\|\nabla u\|_{L^{2,\infty}(B_R \setminus B_r)} \leq C \left[\sup_{r<\rho<R/2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u|^2 \, dz \right]^{1/2}. \tag{44}$$

Proof. Let

$$\varepsilon := \sup_{r<\rho<R/2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u|^2 \, dz.$$

We assume δ to be smaller than ε_0 in the ε-regularity result Theorem 12 in such a way that for any $2r < \rho < R/4$ one has

$$\left[\frac{1}{\rho^2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u|^4 \, dz \right]^{1/4} \leq C \frac{\sqrt{\varepsilon}}{\rho}. \tag{45}$$

Let $\lambda > 0$. Let $f(x) := |\nabla u|$ in $B_{R/2} \setminus B_{2r}$ and $f = 0$ otherwise; then

$$\int_{B_{2\rho} \setminus B_\rho} f^4 \, dz \leq C \frac{\varepsilon^2}{\rho^2} \quad \text{for all } \rho > 0. \tag{46}$$

For any $\rho > 0$ denote

$$U(\lambda, \rho) := \left\{ z \in B_{2\rho} \setminus B_\rho \;\middle|\; f(z) > \lambda \right\}.$$

Let $j \in \mathbb{Z}$ such that $2^j/\rho \le \lambda < 2^{j+1}/\rho$. For any j, using (46), one has that

$$\lambda^4 \, |U(\lambda, \rho)| \le C \frac{\varepsilon^2}{\rho^2}.$$

Let $k \in \mathbb{Z}$. By summing over $j \ge k$ one obtains

$$\lambda^2 \left| \left\{ z \in \mathbb{R}^2 \setminus B_{2^k \lambda^{-1}} \;\middle|\; f(x) > \lambda \right\} \right| \le C \sum_{j=k}^{\infty} 2^{-2j} \varepsilon^2 \le C 2^{-2k} \varepsilon^2.$$

So we deduce that for any $k \in \mathbb{Z}$

$$\lambda^2 \left| \left\{ z \in \mathbb{R}^2 \;\middle|\; f(z) > \lambda \right\} \right| \le C 2^{-2k} \varepsilon^2 + \pi 2^{2k}. \tag{47}$$

Taking $2^{2k} \simeq \varepsilon$ we obtain

$$\|\nabla u\|_{L^{2,\infty}(B_{R/2} \setminus B_{2r})} \le C \left[\sup_{r < \rho < R/2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u|^2 \, dx \right]^{1/2}. \tag{48}$$

Using now the triangle inequality for the $L^{2,\infty}$-norm and the fact that the $L^{2,\infty}$-norm of ∇u is controlled by the L^2-norm of ∇u over $B_R \setminus B_{R/2}$ and $B_{2r} \setminus B_r$, (48) implies (44) and Lemma 13 is proved. □

3.1. *Proofs of Theorems 1 and 2.*

Proof of Theorem 2. Let $\varepsilon_0 > 0$ be as in Theorem 11.

Step 1: We reduce the problem to an $L^{2,1}$ estimate. Indeed, we use the duality $L^{2,1}$–$L^{2,\infty}$ to infer that

$$\int_{B_{R/2} \setminus B_{2r}} \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2 dx \le \left\| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{R/2} \setminus B_{2r})} \left\| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right\|_{L^{2,\infty}(B_{R/2} \setminus B_{2r})}.$$

Combining this inequality with (44) we obtain

$$\int_{B_{R/2} \setminus B_{2r}} \left| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right|^2 dx \le C \left\| \frac{1}{\rho} \frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{R/2} \setminus B_{2r})} \left[\sup_{r < \rho < R/2} \int_{B_{2\rho} \setminus B_\rho} |\nabla u|^2 \, dx \right]^{1/2}. \tag{49}$$

Hence, thanks to duality, it suffices to control the $L^{2,1}$-norm of $(1/\rho)(\partial u/\partial \theta)$ by the L^2-norm of ∇u in the annulus in order to prove the theorem.

Step 2: We prove the theorem assuming that $\int_{B_R \setminus B_r} |\Omega|^2 \, dz < \varepsilon_0$. We start by extending Ω, setting

$$\widetilde{\Omega} = \begin{cases} \Omega & \text{on } B_R \setminus B_r, \\ 0 & \text{on } B_r. \end{cases}$$

Hence, thanks to Theorem 11, there exists $\widetilde{A} \in W^{1,2}(B_R, \mathrm{Gl}_n(\mathbb{R})) \cap L^\infty(B_R, \mathrm{Gl}_n(\mathbb{R}))$ such that

$$\operatorname{div}(\nabla \widetilde{A} - \widetilde{A}\, \widetilde{\Omega}) = 0$$

and

$$\int_{B_R} \big(|\nabla \widetilde{A}|^2 + |\nabla \widetilde{A}^{-1}|^2\big)\, dz + \operatorname{dist}\big(\{\widetilde{A}, \widetilde{A}^{-1}\}, \mathrm{SO}(n)\big) \le C \int_{B_R} |\widetilde{\Omega}|^2\, dz. \tag{50}$$

Then, thanks to Poincaré's lemma, there exists $\widetilde{B} \in W^{1,2}(B_R(0), M_n(\mathbb{R}))$ such that

$$\nabla \widetilde{A} - \widetilde{A}\, \widetilde{\Omega} = \nabla^{\perp} \widetilde{B}, \tag{51}$$

and, thanks to (50) and (51), we get

$$\|\nabla \widetilde{B}\|_{L^2(B_R)} \le C \|\Omega\|_{L^2(B_R \setminus B_r)},$$

where C is a constant independent of Ω. Hence, u satisfies

$$\operatorname{div}(\widetilde{A}\, \nabla u) = \nabla^{\perp} \widetilde{B} \cdot \nabla u \quad \text{on } B_R \setminus B_r.$$

We extend u to B_R by $\widetilde{u}$ using Whitney's extension theorem (see [Adams and Fournier 2003] or [Stein 1970] for instance); then we get $\widetilde{u} \in W^{1,2}(B_R)$ such that

$$\int_{B_R} |\nabla \widetilde{u}|^2\, dz \le C \int_{B_R \setminus B_r} |\nabla u|^2\, dz. \tag{52}$$

We consider the Hodge decomposition of $\widetilde{A} \nabla \widetilde{u}$ on B_R, that is, $C \in W_0^{1,2}(B_R)$ and $D \in W^{1,2}(B_R)$ such that

$$\widetilde{A}\, \nabla \widetilde{u} = \nabla C + \nabla^{\perp} D. \tag{53}$$

Moreover, thanks to (52), we get

$$\int_{B_R} |\nabla C|^2\, dz + \int_{B_R} |\nabla D|^2\, dz = \int_{B_R} |\widetilde{A}\, \nabla \widetilde{u}|^2\, dz \le C \int_{B_R \setminus B_r} |\nabla u|^2\, dz.$$

Here we use the fact that C vanishes on the boundary to get that

$$\int_{B_R} \nabla C \cdot \nabla^{\perp} D\, dz = 0.$$

Then, on $B_R \setminus B_r$, C satisfies

$$\Delta C = \nabla^{\perp} \widetilde{B} \cdot \nabla u.$$

As usual, we write $C = v + \phi$, where $\phi \in W_0^{1,2}(B_R \setminus B_r)$ and $v \in W^{1,2}(B_R \setminus B_r)$ satisfy

$$\Delta \phi = \nabla^{\perp} \widetilde{B} \cdot \nabla u \quad \text{and} \quad \Delta v = 0.$$

On the one hand, thanks to Lemma 7 we get, for $0 < \lambda < 1$, that

$$\|\nabla \phi\|_{L^{2,1}(B_R \setminus B_{r/\lambda})} \le C(\lambda) \|\nabla \widetilde{B}\|_2\, \|\nabla u\|_2.$$

On the other hand, we decompose v as a Fourier series:

$$v = c_0 + d_0 \log(\rho) + \sum_{n \in \mathbb{Z}^*} (c_n \rho^n + d_n \rho^{-n}) e^{in\theta}.$$

Since $(1/\rho)(\partial v/\partial\theta)$ has no logarithm part, we conclude as in Lemma A.2 that for any $0<\lambda<1$ we have

$$\Big\|\frac{1}{\rho}\frac{\partial v}{\partial\theta}\Big\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\leq C(\lambda)\|\nabla v\|_2.$$

The Dirichlet principle implies that

$$\|\nabla v\|_2\leq\|\nabla C\|_2,$$

then we get

$$\Big\|\frac{1}{\rho}\frac{\partial C}{\partial\theta}\Big\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\leq C(\lambda)\|\nabla u\|_{L^2(B_R\setminus B_r)}. \tag{54}$$

Now we estimate D, which satisfies the equation

$$\Delta D=\nabla\widetilde{A}\cdot\nabla^{\perp}\widetilde{u}\quad\text{on } B_R.$$

Then, we also decompose D as $D=v+\phi$, where $\phi\in W_0^{1,2}(B_R)$ and $v\in W^{1,2}(B_R)$ satisfy

$$\Delta\phi=\nabla\widetilde{A}\cdot\nabla^{\perp}\widetilde{u}\quad\text{and}\quad\Delta v=0.$$

On the one hand, thanks to Lemma 5, we have

$$\|\nabla\phi\|_2\leq\|\nabla\phi\|_{L^{2,1}(B_R)}\leq C\|\nabla\widetilde{A}\|_2\,\|\nabla\widetilde{u}\|_2\leq C\|\nabla u\|_{L^2(B_R\setminus B_r)}.$$

On the other hand, since v is harmonic, for any $0<\lambda<1$ we have

$$\|\nabla v\|_{L^{2,1}(B_{\lambda R})}\leq C(\lambda)\|\nabla v\|_{L^2(B_R)}\leq C(\lambda)\|\nabla D\|_{L^2(B_R)}\leq C(\lambda)\|\nabla u\|_2.$$

Finally,

$$\|\nabla D\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\leq C(\lambda)\|\nabla u\|_2. \tag{55}$$

Combining (53), (54) and (55), we get

$$\Big\|\widetilde{A}\,\frac{1}{r}\frac{\partial\widetilde{u}}{\partial\theta}\Big\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\leq C(\lambda)\|\nabla u\|_2.$$

Finally, using (50), we get that

$$\Big\|\frac{1}{\rho}\frac{\partial\widetilde{u}}{\partial\theta}\Big\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})}\leq C(\lambda)\|\nabla u\|_2, \tag{56}$$

which proves, as remarked at the end of Step 1, the theorem under the extra assumption.

Step 3: We prove the general case. We construct two sequences of radii r_i and R_i such that

$$r=r_0<r_1=R_0<\cdots<r_{i+1}=R_i<\cdots<R_N=R,$$

with

$$\int_{B_{R_i}\setminus B_{r_i}}|\Omega|^2\,dz\leq\varepsilon_0\quad\text{and}\quad N\leq\frac{1}{\varepsilon_0}\int_{B_R\setminus B_r}|\Omega|^2\,dz.$$

First, applying (56) of Step 2, we get that

$$\left\| \frac{1}{\rho}\frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{\lambda R_i}\setminus B_{r_i/\lambda})} \leq C(\lambda)\|\nabla u\|_{L^2(B_{R_i}\setminus B_{r_i})}. \tag{57}$$

We choose δ such that $\delta < \varepsilon_0/4$; hence for all i we have

$$\int_{B_{4r_i}\setminus B_{r_i/4}} |\Omega|^2\, dz < 4\delta < \varepsilon_0.$$

Let $S_i = \min(R, 4r_i)$ and $s_i = \max(r, r_i/4)$, then we apply again (56) of Step 2 on $B_{S_i}\setminus B_{s_i}$, which gives

$$\left\| \frac{1}{\rho}\frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{\lambda S_i}\setminus B_{s_i/\lambda})} \leq C(\lambda)\|\nabla u\|_{L^2(B_{S_i}\setminus B_{s_i})}. \tag{58}$$

Finally, summing (57) and (58), for $i = 0$ to N, we get

$$\left\| \frac{1}{\rho}\frac{\partial u}{\partial \theta} \right\|_{L^{2,1}(B_{\lambda R}\setminus B_{r/\lambda})} \leq C(\lambda)\|\nabla u\|_2,$$

which achieves the proof of Theorem 2. □

We shall now make use of the Theorem 2 in order to prove the quantization of the angular part of the energy for solutions to antisymmetric elliptic systems.

We will call a *bubble* a solution $u \in W^{2,1}(\mathbb{R}^2, \mathbb{R}^n)$ of the equation

$$-\Delta u = \Omega \cdot \nabla u \quad \text{on } \mathbb{R}^2,$$

where $\Omega \in L^2(\mathbb{R}^2, \mathrm{so}(n)\otimes\mathbb{R}^2)$.

Proof of Theorem 1. First we are going to separate B_1 into three parts: one where u_k converges to a limit solution; some neighborhoods where the energy concentrates and you blow some bubbles; and some neck regions which join the first two parts. This "bubble-tree" decomposition is by now classical (see [Parker 1996] for instance); hence we just sketch briefly how to proceed.

Step 1: Find the point of concentration. Let ε_0 be the one of Theorem 12 and δ the one of Theorem 2. Then, thanks to (6), we easily prove that there exist finitely many points $a^1, \ldots, a^n$, where

$$\liminf_k \int_{B(a_i,r)} |\Omega_k|^2\, dz \geq \varepsilon_0 \quad \text{for all } r > 0. \tag{59}$$

Moreover, using Theorem 12, we prove that there exists $\Omega_\infty \in L^2(B_1, \mathrm{so}(n)\otimes\mathbb{R}^2)$ and $u_\infty \in W^{2,1}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u = \Omega_\infty \cdot \nabla u$ on B_1, such that, up to a subsequence,

$$\Omega_k \rightharpoonup \Omega_\infty \quad \text{in } L^2_{\mathrm{loc}}(B_1, \mathrm{so}(n)\otimes\mathbb{R}^2),$$

and

$$u_k \to u_\infty \quad \text{in } W^{1,p}_{\mathrm{loc}}(B_1\setminus\{a^1,\ldots,a^n\}) \text{ for all } p \geq 1.$$

Of course, if $\|\Omega_k\|_\infty = O(1)$ or $\Omega_k = \Lambda(u_k, \nabla u_k)$ where $\Lambda(\,\cdot\,, p) = O(|p|)$, then u_k is bounded in $W^{2,\infty}$ which gives the convergence in $C^{1,\eta}_{\mathrm{loc}}$ for all $\eta \in [0, 1[$.

Step 2: Blow-up around a^i. We choose $r_i > 0$ such that

$$\int_{B(a^i,r^i)} |\Omega_\infty|^2\,dz \le \frac{\varepsilon_0}{4}.$$

Then, we define a center of mass of $B(a^i, r^i)$ with respect to Ω_k in the following way:

$$a_k^i = \left(\frac{\displaystyle\int_{B(a^i,r^i)} x^\alpha |\Omega_k|^2\,dz}{\displaystyle\int_{B(a^i,r^i)} |\Omega_k|^2\,dz} \right)_{\alpha=1,2}.$$

Let λ_k^i be a positive real such that

$$\int_{B(a_k^i,r^i)\setminus B(a_k^i,\lambda_k^i)} |\Omega_k|^2\,dz = \min\left(\delta, \frac{\varepsilon_0}{2}\right).$$

If $\lambda_k^i \neq o(1)$, then we restart the process replacing r^i by $\liminf \lambda_k^i$ until $\lambda_k^i = o(1)$. Then we set

$$\widetilde{u}_k(z) = u_k(a_k^i + \lambda_k^i z), \quad \widetilde{\Omega}_k(z) = \lambda_k^i\,\Omega_k(a_k^i + \lambda_k^i z), \quad \text{and} \quad N_k^i = B(a_k^i, r^i) \setminus B(a_k^i, \lambda_k^i).$$

Observe that the scaling we chose for defining $\widetilde{\Omega}_k(z)$ guarantees that

$$\int_{B(0,r^i/\lambda_k^i)} \left(|\widetilde{\Omega}_k|^2 + |\nabla \widetilde{u}_k^i|^2\right) dz = \int_{B(a_k^i,r^i)} \left(|\Omega_k|^2 + |\nabla u_k^i|^2\right)\,dx \le C < +\infty;$$

moreover, we have

$$-\Delta \widetilde{u}_k^i = \widetilde{\Omega}_k^i \cdot \nabla \widetilde{u}_k^i.$$

Modulo extraction of a subsequence, we can assume that for each i

$$\nabla \widetilde{u}_k^i \rightharpoonup \nabla \widetilde{u}_\infty^i \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^n) \quad \text{and} \quad \widetilde{\Omega}_k^i \rightharpoonup \widetilde{\Omega}_\infty^i \quad \text{in } L^2_{\text{loc}}(\mathbb{R}^2, \mathrm{so}(n) \otimes \mathbb{R}^2).$$

The *weak limit property* of Theorem 12 implies that $\widetilde{u}_\infty$ and $\widetilde{\Omega}_\infty$ satisfy what we call a *bubble* equation

$$-\Delta \widetilde{u}_\infty^i = \widetilde{\Omega}_\infty^i \cdot \nabla \widetilde{u}_\infty^i.$$

In fact the convergence of u_k^i to u_∞^i is in $W^{1,p}_{\text{loc}}\left(\mathbb{R}^2 \setminus \{a_i^1, \dots, a_i^n\}\right)$ for all $p \ge 1$, where the a_i^j are possible points of concentration of $\widetilde{\Omega}_k^i$ where

$$\liminf_k \int_{B(a_i^j,r)} |\widetilde{\Omega}_k^i|^2\,dz \ge \varepsilon_0 \quad \text{for all } r > 0, \tag{60}$$

which are necessarily finite in number and in B_1.

Step 3: Iteration. Two cases have to be considered separately: either $\widetilde{\Omega}_k$ is subject to some concentration phenomena as (59), and then we find some new points of concentration, in such a case we apply Step 2 to our new concentration points; or $\widetilde{u}_k$ converges in $W^{1,p}_{\text{loc}}(\mathbb{R}^2)$ to a (possibly trivial) bubble.

Of course this process has to stop, since we are assuming a uniform bound on $\|\Omega_k\|_2$ and each step is consuming at least $\min(\delta, \varepsilon_0/2)$ of energy of Ω_k. This process is sketched in Figure 1.

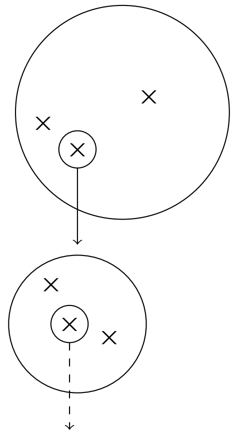

Figure 1. Decomposition of B_1.

Analysis of a neck region. A neck region is a finite union of annuli $N_k^i = B(a_k^i, \mu_k^i) \setminus B(a_k^i, \lambda_k^i)$ such that

$$\lim_{k\to+\infty} \frac{\lambda_k^i}{\mu_k^i} = 0, \qquad X_k = \nabla^\perp d(a_k^i, \cdot),$$

and

$$\int_{N_k^i} |\Omega_k|^2 \, dz \le \min\left(\delta, \frac{\varepsilon_0}{2}\right). \tag{61}$$

In order to prove Theorem 1, we start by proving a weak estimate on the energy of gradient in the region N_k^i. First we remark that, for each $\varepsilon > 0$, there exists $r > 0$ such that for all $\rho > 0$ such that

$$B_{2\rho}(a_k^i) \setminus B_\rho(a_k^i) \subset N_k^i(r),$$

where $N_k^i(r) = B(a_k^i, r\mu_k^i) \setminus B(a_k^i, \lambda_k^i/r)$, we have

$$\int_{B_{2\rho}(a_k^i)\setminus B_\rho(a_k^i)} |\nabla u|^2 \, dz \le \varepsilon. \tag{62}$$

If this were not the case there would exist a sequence $\rho_k^i \to 0$ such that, up to a subsequence, $\hat{u}_k = u_k(a_k^i + \rho_k^i z)$ converges with respect to every $W^{1,p}$-norm to a nontrivial solution of

$$-\Delta \hat{u} = \widehat{\Omega}_\infty \cdot \nabla \hat{u} \quad \text{on } \mathbb{R}^2 \setminus \{0\},$$

where $\widehat{\Omega}_\infty$ is a weak limit, up to a subsequence, of $\widehat{\Omega}_k$. Using the fact that the $W^{1,2}$-norm of $\hat{u}_k$ is bounded, we deduce using Schwartz lemma that it has to be in fact a solution on the whole plane. Using this time the second part of Theorem 12 we deduce that $\hat{\Omega}_\infty$ have energy at least ε_0, which contradicts (61).

Finally, using Theorem 2 on each $N_k^i(r)$, we obtain

$$\lim_{r\to 0} \lim_{k\to+\infty} \left\| \langle \nabla u_k, X_k \rangle \right\|_{L^2(N_k^i(r))} \le C \lim_{r\to 0} \lim_{k\to+\infty} \left(\sup_\rho \int_{B_{2\rho}(a_k^i)\setminus B_\rho(a_k^i)} |\nabla u|^2 \, dz \right) = 0,$$

which achieves the proof of Theorem 1. $\square$

This phenomena of quantization of the angular part of the gradient seems to be quite general for systems with antisymmetric potentials. In a forthcoming paper [Laurain and Rivière 2011] we investigate the quantization for some fourth-order elliptic systems in four dimensions.

3.2. ***Description of the function in the neck regions.*** In this subsection we give a precise description of the behavior of ∇u_k in the neck regions when the radial part is not quantized. In particular we prove that the loss of quantization is due to pure radial part to the form $a(r)/r$ with a uniformly bounded.

Proving Theorem 2, we have proved, see (53) and what follows, that if the L^2-norm of Ω is smaller than a positive constant δ_0 on an annulus $B_R \setminus B_r$, then there exists $A \in W^{1,2} \cap L^\infty(B_1, \mathrm{Gl}_n(\mathbb{R}))$, $h \in L^2(B_1, \mathbb{R}^2 \otimes \mathbb{R}^n)$ and $C \in \mathbb{R}^2 \otimes \mathbb{R}^n$ such that

$$A\nabla u = \frac{C}{r} + h,$$

where C is a constant and $\|h\|_{L^{2,1}(B_{R/2}\setminus B_{2r})}$ is uniformly bounded by the L^2-norm of ∇u, independently of the conformal class of the annulus. Moreover, up to a choice of δ_0 small enough, we can assume that A is very closed to $\mathrm{SO}(n)$. Then using this fact and the fact we can decompose a neck region into a finite number of such regions, we are going to prove that, in the whole neck region,

$$\nabla u = C\frac{a(r)}{r} + h + g, \tag{63}$$

where C is a constant, $a \in L^\infty(B_1, M_n(\mathbb{R}))$ is uniformly bounded by the L^2-norm of ∇u and *radial*, and $\|h\|_{L^{2,1}(B_{R/2}\setminus B_{2r})}$ is uniformly bounded by the L^2-norm of ∇u and $\|g\|_{L^2(B_{R/2}\setminus B_{2r})}$ as the $\|\nabla u\|_{L^{2,\infty}}$ goes to zero.

Indeed, a neck region is an annular region of the form $B_{R_k} \setminus B_{r_k}$. Since the L^2-norm of Ω_k is uniformly bounded we can divide the annulus into a finite number of annuli where the L^2-norm of Ω_k is smaller than $\varepsilon_0/2$. Let $\big(B_{r_k^{i+1}} \setminus B_{r_k^i}\big)_{1\le i\le N}$ be the different annuli, where $r_k^1 = r_k$ and $r_k^{N+1} = R_k$.

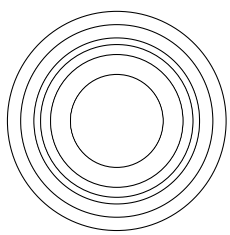

Figure 2. Decomposition of the neck region.

On $B_{r_k^i} \setminus B_{r_k^{i+1}}$ the L^2-norm of Ω_k is smaller than δ_0, so there exist $A_k^i \in W^{1,2} \cap L^\infty\big(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathrm{Gl}_n(\mathbb{R})\big)$, $h_k^i \in L^2\big(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathbb{R}^2 \otimes \mathbb{R}^n\big)$ and $C_k^i \in \mathbb{R}^2 \otimes \mathbb{R}^n$ such that

$$A_k^i \nabla u_k = \frac{C_k^i}{r} + h_k^i \quad \text{on } B_{r_k^{i+2}} \setminus B_{r_k^i},$$

where $\|h_k^i\|_{L^{2,1}}$ is uniformly bounded by the L^2-norm of ∇u_k. Hence we have

$$\nabla u_k = \frac{D_k^i(r)}{r} C_k^i + \tilde{h}_k^i + \tilde{g}_k^i \quad \text{on } B_{r_k^{i+2}} \setminus B_{r_k^i}, \tag{64}$$

where $D_k^i \in L^\infty\big(B_{r_k^{i+2}} \setminus B_{r_k^i}, M_n(\mathbb{R})\big)$ is uniformly bounded by the L^2-norm of ∇u_k and *radial*, $\|\tilde{h}_k^i\|_{L^{2,1}}$ is uniformly bounded, and $\tilde{g}_k^i \in L^2\big(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathbb{R}^2 \otimes \mathbb{R}^n\big)$ with $\|\tilde{g}_k^i\|_{L^2} = o(1)$. Indeed, we have

$$\frac{(A_k^i)^{-1}}{r} = \frac{\overline{(A_k^i)^{-1}}(r)}{r} + \frac{(A_k^i)^{-1} - \overline{(A_k^i)^{-1}}}{r},$$

where $\overline{(A_k^i)^{-1}}$ is the mean value of $(A_k^i)^{-1}$ on each circle. Since $(A_k^i)^{-1}$ is uniformly bounded in $W^{1,2} \cap L^\infty\big(B_{r_k^{i+1}} \setminus B_{r_k^i}, \mathrm{Gl}_n(\mathbb{R})\big)$, we have

$$\begin{aligned}
\left\| \frac{(A_k^i)^{-1} - \overline{(A_k^i)^{-1}}}{r} \right\|^2_{L^2\left(B_{r_k^{i+1}} \setminus B_{r_k^i}\right)} &= \int_{r_k^i}^{r_k^{i+1}} \frac{1}{r} \int_0^{2\pi} \left| (A_k^i)^{-1} - \overline{(A_k^i)^{-1}} \right|^2 d\theta\, dr \\
&\leq \int_{r_k^i}^{r_k^{i+1}} \frac{1}{r} \int_0^{2\pi} \left| \frac{\partial (A_k^i)^{-1}}{\partial \theta} \right|^2 d\theta\, dr \\
&\leq \left\| \nabla (A_k^i)^{-1} \right\|_2^2,
\end{aligned}$$

here we use the Poincaré inequality. Finally, we conclude using the fact that $\|\nabla u_k\|_2$ is bounded, which implies

$$\|C_k^i\| = O\Big(\big(\log\big(r_k^{i+1}/r_k^i\big)\big)^{-1/2}\Big) = o(1),$$

since $\tilde{g}_k^i = \frac{1}{r}\big((A_k^i)^{-1} - \overline{(A_k^i)^{-1}}\big) C_k^i$, this proves (64). Then we glue all the functions to get the whole decomposition.

Hence we have the following theorem:

Theorem 14 (see Theorem 1). *Let $\Omega_k \in L^2\big(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2\big)$ and let $u_k \in W^{2,1}(B_1, \mathbb{R}^n)$ be a sequence of solutions of*

$$-\Delta u_k = \Omega_k \cdot \nabla u_k \tag{65}$$

with bounded energy; that is,

$$\int_{B_1} \big(|\nabla u_k|^2 + |\Omega_k|^2\big)\, dz \leq M.$$

Then there exist $u_\infty \in W^{1,2}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u_\infty = \Lambda(u_\infty, \nabla u_\infty) \cdot \nabla u_\infty$ on B_1, $l \in \mathbb{N}^*$, and

(1) *$\omega^1, \dots, \omega^l$ a family of solutions to system*

$$-\Delta \omega^i = \Omega_i \cdot \nabla \omega^i \quad \text{on } \mathbb{R}^2,$$

where $\Omega_i \in L^2(B_1, \mathrm{so}(n) \otimes \mathbb{R}^2)$,

(2) *$a_k^1, \dots, a_k^l$ a family of converging sequences of points of B_1,*

(3) $\lambda_k^1, \dots, \lambda_k^l$ *a family of sequences of positive reals converging all to zero,*

(4) $C_k^1, \dots, C_k^l$ *a family of sequences of vectors converging all to zero,*

(5) $A_k^1, \dots, A_k^l$ *a family of sequences of uniformly bounded and radial functions from* $\mathbb{R}^2$ *to* $M_n(\mathbb{R})$,

such that, up to a subsequence,

$$u_k \to u_\infty \quad \text{on } C^{1,\eta}_{\mathrm{loc}}\big(B_1 \setminus \{a^1_\infty, \dots, a^l_\infty\}\big) \text{ for all } \eta \in [0, 1[$$

and

$$\left\| \nabla\Big(u_k - u_\infty - \sum_{i=1}^{l} \omega_k^i\Big) + \sum_{i=1}^{l} \frac{A_k^i(d(a_k^i, \cdot))}{d(a_k^i, \cdot)} C_k^i \right\|_{L^2_{\mathrm{loc}}(B_1)} \to 0,$$

where $\omega_k^i = \omega^i(a_k^i + \lambda_k^i \cdot)$.

3.3. ***Counterexample to the quantization of the radial part of the gradient.*** Thanks to the previous subsection, we know that the failure of quantization is given in the neck region by a function of the form $c_k \log(r)$. Hence we look for $u_k : B_1 \to \mathbb{R}^3$ whose third component behaves as $c_k \log(r)$. For this we define the following smooth functions:

$$U_k^3(r) = \begin{cases} 0 & \text{if } 0 \le r \le 1/2, \\ \dfrac{\log(r)}{\log(k)^{1/2}} & \text{if } r \ge 2, \end{cases}$$

such that $|(U_k^3)'(r)| \le \log(k)^{-1/2}$ on $[1/2, 2]$; and

$$\phi(r) = \begin{cases} 2r & \text{if } 0 \le r \le \frac{1}{4}, \\ 1 & \text{if } \frac{1}{2} \le r \le 2, \\ 2/r & \text{if } r \ge 4, \end{cases}$$

such that $|\phi'(r)| \le 4$ on $[1/4, 1/2] \cup [2, 4]$. We set $\psi = r(r\phi')'/\phi - 1$, and we easily see that ψ is a smooth function with compact support in $[1/4, 4]$. Finally we set

$$u_k(r, \theta) = \begin{pmatrix} \cos(\theta)\phi(kr) \\ \sin(\theta)\phi(kr) \\ U_k^3(kr) \end{pmatrix}$$

and

$$\Omega_k^\theta(r, \theta) = \begin{pmatrix} 0 & \psi(kr)/r & \sin(\theta) r \Delta u_k^3 \\ -\psi(kr)/r & 0 & -\cos(\theta) r \Delta u_k^3 \\ -\sin(\theta) r \Delta u_k^3 & \cos(\theta) r \Delta u_k^3 & 0 \end{pmatrix}.$$

We easily verify that $\Delta u_k = \Omega_k \cdot \nabla u_k$ where $\Omega_k = \Omega_k^\theta r d\theta$ and that the L^2-norms of ∇u_k and Ω_k are bounded on B_1. We have a bubble which blows up at radius $1/k$, and

$$\lim_{R \to +\infty} \lim_{k \to +\infty} \int_{B_{1/R} \setminus B_{R/k}} |\Omega_k|^2 \, dz = 0,$$

but

$$\lim_{R\to+\infty}\lim_{k\to+\infty}\int_{B_{1/R}\setminus B_{R/k}}|\nabla u_k|^2\,dz=1,$$

which is a failure of energy quantization and proves the optimality of the conclusion of Theorem 1.

4. Energy quantization for critical points to conformally invariant Lagrangians.

In the present section we are going to use Theorem 1 in order to prove Theorem 3.

In his proof of the Heinz–Hildebrandt's regularity conjecture, the second author prove that the Euler Lagrange equations to general conformally invariant Lagrangians which are coercive and of quadratic growth can be written in the form of an elliptic system with an antisymmetric potential. Precisely we have:

Theorem 15 [Rivière 2007, Theorem I.2]. *Let N^k be a C^2 submanifold of $\mathbb{R}^m$ and ω be a C^1 2-form on N^k such that the L^∞-norm of $d\omega$ is bounded on N^k. Then every critical point in $W^{1,2}(B_1, N^k)$ of the Lagrangian*

$$F(u)=\int_{B_1}\left[|\nabla u|^2+u^*\omega\right]\,dz \tag{66}$$

satisfies

$$-\Delta u=\Omega\cdot\nabla u,$$

with

$$\Omega^i_j=\left[A^i(u)_{j,l}-A^j(u)_{i,l}\right]\nabla u^l+\frac{1}{4}\left[H^i(u)_{j,l}-H^j(u)_{i,l}\right]\nabla^\perp u^l, \tag{67}$$

where A and H are in $C^0(N, M_m(\mathbb{R})\otimes\wedge^1\mathbb{R}^2)$ and satisfy

$$\sum_{j=1}^m A^j_{i,l}\nabla u^j=0$$

and $H^i_{j,l}:=d(\pi^*\omega)(\varepsilon_i,\varepsilon_j,\varepsilon_l)$ where, in a neighborhood of N^k, π is the orthogonal projection onto N^k and $(\varepsilon_i)_{i=1,\dots,m}$ is the canonical basis of $\mathbb{R}^m$.

From (67) we observe that for critical points to a conformally invariant C^1-Lagrangian, there exists

$$\Lambda\in C^0(TN\otimes\mathbb{R}^2,\mathrm{so}(n)\otimes\mathbb{R}^2) \tag{68}$$

such that

$$\Lambda(v)=O(|v|); \tag{69}$$

moreover we remark that $\Lambda(u,\nabla u)\cdot\nabla u$ is always orthogonal to ∇u in the following sense:

$$\left\langle\frac{\partial u}{\partial x_k},\Lambda(u,\nabla u)\cdot\nabla u\right\rangle=0\quad\text{for } k=1,2. \tag{70}$$

For $\Lambda\in C^0(TN\otimes\mathbb{R}^2,\mathrm{so}(n)\otimes\mathbb{R}^2)$, we call a Λ-*bubble* a solution $\omega\in W^{2,1}(\mathbb{R}^2,\mathbb{R}^n)$ of the equation

$$-\Delta\omega=\Lambda(\omega,\nabla\omega)\cdot\nabla\omega\quad\text{on }\mathbb{R}^2.$$

Theorem 16. *Let $u_k \in W^{1,2}(B_1, \mathbb{R}^n)$ be a sequence of critical points of a functional which is conformally invariant, which satisfies*

$$-\Delta u_k = \Lambda(u_k, \nabla u_k) \cdot \nabla u_k, \tag{71}$$

where Λ satisfies (68), (69) *and* (70). *Moreover, assume that u_k has a bounded energy, that is,*

$$\|\nabla u_k\|_2 \leq M.$$

Then there exists $u_\infty \in W^{1,2}(B_1, \mathbb{R}^n)$ a solution of $-\Delta u_\infty = \Lambda(u_\infty, \nabla u_\infty) \cdot \nabla u_\infty$ on B_1, $l \in \mathbb{N}^*$ and

(1) *$\omega^1, \dots, \omega^l$ some nonconstant Λ-bubbles,*
(2) *$a_k^1, \dots, a_k^l$ a family of converging sequences of points of B_1,*
(3) *$\lambda_k^1, \dots, \lambda_k^l$ a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$u_k \to u^\infty \quad \text{on } C^{1,\eta}_{\mathrm{loc}}\big(B_1 \setminus \{a_\infty^1, \dots, a_\infty^l\}\big) \quad \text{for all } \eta \in [0, 1[$$

and

$$\Big\|\nabla\Big(u_k - u_\infty - \sum_{i=1}^{l} \omega_k^i\Big)\Big\|_{L^2_{\mathrm{loc}}(B_1)} \to 0,$$

where $\omega_k^i = \omega(a_k^i + \lambda_k^i z)$.

Since (70) holds for any system issued from a Lagrangian of the form (66), it is clear that Theorem 3 is a consequence of Theorem 16.

Proof. From the previous section, we have the quantization of the angular part of the gradient. To prove Theorem 16 it suffices then to prove the energy quantization for the radial part of the energy. Since u_k satisfies (71) then $u_k \in W^{2,p}(B_{\mu_k^i}(a_k^i))$ for all $p < \infty$ (see [Rivière 2010, Theorem IV.3] or [Sharp and Topping 2013, Lemma 7.1]); hence we can multiply (71) by $\rho(\partial u_k/\partial \rho)$ and integrate. Using (70) we have, for any $r \in [0, \mu_k^i]$,

$$0 = \int_{B_r} \Big\langle \rho \frac{\partial u_k}{\partial \rho}, \Omega \cdot \nabla u_k \Big\rangle \, dz = \int_{B_r} \Big\langle \rho \frac{\partial u_k}{\partial \rho}, \Delta u_k \Big\rangle \, dz.$$

Using Pohozaev identity, we get for all $r \in [0, \mu_k^i]$

$$\int_{\partial B_r} \Big| \frac{\partial u_k}{\partial \rho} \Big|^2 \, d\sigma = \int_{\partial B_r} \Big| \frac{1}{\rho} \frac{\partial u_k}{\partial \theta} \Big|^2 \, d\sigma.$$

Finally, we have

$$\lim_{r \to 0} \lim_{k \to +\infty} \|\nabla u_k\|_{L^2(N_k^i(r))} = 0,$$

which concludes the proof of the theorem. $\square$

In particular we get the quantization for the solution of the problem of prescribed mean curvature. Indeed, an immersion of a Riemann surface Σ into $\mathbb{R}^3$ with prescribed mean curvature $H \in C^0(\mathbb{R}^3, \mathbb{R})$ satisfies the H-system

$$\Delta u = 2H(u)\, u_x \wedge u_y, \tag{72}$$

where $z = x + iy$ are some local conformal coordinates on Σ.

In order to state precisely our theorem, we define the notion of H-bubble as being a map $\omega \in W^{1,2}(\mathbb{R}^2, \mathbb{R}^3)$ satisfying

$$\Delta\omega = 2H(\omega)\, \omega_x \wedge \omega_y \quad \text{on } \mathbb{R}^2.$$

We shall also rescale the Riemann surface around a point. To that aim we will introduce some conformal chart. Precisely there exists $\delta > 0$ such that for any $a \in \Sigma$ and $0 < \lambda < \delta$ there exists a map $\Phi_{a,\lambda}$: $B(a, \delta) \to \mathbb{R}^2$ which is a conformal-diffeomorphism, sends a to 0 and $B(a, \lambda)$ to $B(0, 1)$. We also associate to each point a cut-off function $\chi_a \in C^\infty(\Sigma)$ which satisfies

$$\begin{cases} \chi_a \equiv 1 & \text{on } B(a, \delta/2), \\ \chi_a \equiv 0 & \text{on } \Sigma \setminus B(a, \delta). \end{cases}$$

Corollary 17. *Let Σ be a closed Riemann surface, $H \in C^0(\mathbb{R}^3, \mathbb{R})$ and $u_k \in W^{2,1}(\Sigma, \mathbb{R}^3)$ a sequence of nonconstant solution of* (72) *on Σ then there exists, $u_\infty \in W^{2,1}(\Sigma, \mathbb{R}^3)$ a solution of* (72), *$k \in \mathbb{N}^*$ and*

(1) *$\omega^1, \dots, \omega^l$ a family of H-bubbles,*

(2) *$a_k^1, \dots, a_k^l$ a family of converging sequences of point of Σ,*

(3) *$\lambda_k^1, \dots, \lambda_k^l$ a family of sequences of positive reals converging all to zero,*

such that, up to a subsequence,

$$u_k \to u^\infty \quad \textit{on } C^{1,\eta}_{\mathrm{loc}}\big(\Sigma \setminus \{a_1^\infty, \dots, a_k^\infty\}\big) \quad \textit{for all } \eta \in [0, 1[$$

and moreover

$$\left\| \nabla\Big(u_k - u_\infty - \sum_{i=1}^{l} \chi_{a_k^i}\big(\omega^i \circ \Phi_{a_k^i, \lambda_k^i}\big)\Big) \right\|_2 \to 0.$$

We end up this section by mentioning recent work by Da Lio [2011] in which energy quantization results for fractional harmonic maps (which are also conformally invariant in some dimension) are established using also Lorentz space uniform estimates.

5. Other applications to pseudoholomorphic curves, harmonic maps and Willmore surfaces

In this section we give some more applications of the uniform Lorentz–Wente estimates of Section 2 to problems where the conformal invariance play again a central role. We are interested in Wente's type estimate for first-order system of the form

$$\nabla\phi = \sum_{i=1}^{n} a_i \, \nabla^{\perp} b_i . \tag{73}$$

Taking the divergence of this system gives the classical second-order Wente system

$$\Delta\phi = \sum_{i=1}^{n} \nabla a_i \cdot \nabla^{\perp} b_i . \tag{74}$$

The gain of information provided by a first-order system of the form (73) in comparison to classical second-order system (74) is illustrated by the fact that, in the first-order case, no assumption on the behavior of the solution ϕ at the boundary of the annulus is needed in order to obtain the Lorentz–Wente-type estimates of Section 2. This is proved in Lemma 18. This fact can be applied to geometrically interesting situations that we will describe at the end of the present section.

5.1. ***Lorentz–Wente-type estimates for first-order Wente-type equations.*** The goal of this subsection is to prove the following lemma.

Lemma 18. *Let $n \in \mathbb{N}^*$, let $(a_i)_{1\le i\le n}$ and $(b_i)_{1\le i\le n}$ be two families of maps in $W^{1,2}(B_1)$, let $0 < \varepsilon < \frac{1}{4}$, and assume that $\phi \in W^{1,2}(B_1 \setminus B_\varepsilon)$ satisfies*

$$\nabla\phi = \sum_{i=1}^{n} a_i \, \nabla^{\perp} b_i . \tag{75}$$

Then for each $0 < \lambda < 1$ there exists a positive constant $C(\lambda)$, independent of ϕ, a_i, and b_i, such that

$$\|\nabla\phi\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le C(\lambda)\Big(\sum_{i=1}^{n} \|\nabla a_i\|_2 \, \|\nabla b_i\|_2 + \|\nabla\phi\|_2\Big).$$

Proof. Taking the divergence of (75) gives

$$\Delta\phi = \sum_{i=1}^{n} \nabla a_i \cdot \nabla^{\perp} b_i .$$

Hence, as in the previous lemma, we start by considering a solution of this equation on the whole disk and equal to zero on the boundary. Let $\varphi \in W_0^{1,1}(B_1)$ be the solution of

$$\Delta\varphi = \sum_{i=1}^{n} \nabla a_i \cdot \nabla^{\perp} b_i .$$

Then, thanks to the improved Wente's inequality (9), we have

$$\|\nabla\varphi\|_{L^{2,1}(B_1)} \le C \sum_{i=1}^{n} \|\nabla a_i\|_2 \, \|\nabla b_i\|_2 . \tag{76}$$

We now consider the difference $v = \phi - \varphi$, which is a harmonic function on $B_1 \setminus B_\varepsilon$. Following the proof of Lemma A.2, it suffices to control the logarithmic part of the decomposition in Fourier series. To that aim we set

$$\overline{\phi}(\rho) = \frac{1}{2\pi}\int_0^{2\pi} \phi(\rho,\theta)\, d\theta.$$

We have

$$\frac{d\overline{\phi}}{d\rho} = \frac{1}{2\pi}\int_0^{2\pi} \frac{\partial \phi}{\partial \rho}(\rho,\theta)\, d\theta = \frac{1}{2\pi}\sum_{i=1}^n \int_0^{2\pi} a_i \frac{\partial b_i}{\partial \theta}\frac{d\theta}{\rho} = \frac{1}{2\pi}\sum_{i=1}^n \int_0^{2\pi} (a_i - \overline{a}_i)\frac{\partial b_i}{\partial \theta}\frac{d\theta}{\rho}.$$

Hence

$$\left|\frac{d\overline{\phi}}{d\rho}\right| \le \frac{1}{2\pi}\sum_{i=1}\left(\int_0^{2\pi} \left|a_i - \overline{a}_i\right|^2 d\theta\right)^{1/2}\left(\int_0^{2\pi}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial \theta}\right|^2 d\theta\right)^{1/2}.$$

Which gives, thanks to Poincaré's inequality on the circle,

$$\left|\frac{d\overline{\phi}}{d\rho}\right| \le C\sum_{i=1}\left(\int_0^{2\pi} \left|\frac{\partial a_i}{\partial \theta}\right|^2 d\theta\right)^{1/2}\left(\int_0^{2\pi}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial \theta}\right|^2 d\theta\right)^{1/2},$$

where C is a constant independent of ϕ.

Then integrating over $[1,\varepsilon]$, we get

$$\begin{aligned}\int_\varepsilon^1 \left|\frac{d\overline{\phi}}{d\rho}\right| d\rho &\le C\sum_{i=1}^n \int_\varepsilon^1 \left(\int_0^{2\pi} \left|\frac{\partial a_i}{\partial \theta}\right|^2 d\theta\right)^{1/2}\left(\int_0^{2\pi}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial \theta}\right|^2 d\theta\right)^{1/2} d\rho \\ &\le C\sum_{i=1}^n \left(\int_{\mathrm{D}(0,1)\setminus B_\varepsilon} \left|\frac{1}{\rho}\frac{\partial a_i}{\partial \theta}\right|^2 \rho\, d\rho\, d\theta\right)^{1/2}\left(\int_{\mathrm{D}(0,1)\setminus B_\varepsilon}\left|\frac{1}{\rho}\frac{\partial b_i}{\partial \theta}\right|^2 \rho\, d\rho\, d\theta\right)^{1/2} \\ &\le C\sum_{i=1}^n \|\nabla a_i\|_2\, \|\nabla b_i\|_2.\end{aligned} \tag{77}$$

Moreover, by duality, we obtain

$$\int_\varepsilon^1 \left|\frac{d\overline{\varphi}}{d\rho}\right| d\rho \le \left\|\nabla\varphi\frac{1}{\rho}\right\|_1 \le \|\nabla\varphi\|_{L^{2,1}}\left\|\frac{1}{\rho}\right\|_{L^{2,\infty}} \le C\|\nabla\varphi\|_{L^{2,1}}. \tag{78}$$

The combination of (76), (77) and (78) gives then

$$\int_\varepsilon^1 \left|\frac{d\overline{v}}{d\rho}\right| d\rho \le C\sum_{i=1}^n \|\nabla a_i\|_2\|\nabla b_i\|_2. \tag{79}$$

Following the approaches we used in the proofs of the various lemmas in Section 2, we decompose v as a Fourier series, which gives

$$v(\rho,\theta) = c_0 + d_0\log(\rho) + \sum_{n\in\mathbb{Z}^*}(c_n\rho^n + d_n\rho^{-n})e^{in\theta}.$$

We have

$$\bar{v}(\rho) = c_0 + d_0 \log(\rho).$$

Thanks to (79), we get that

$$|d_0| \ \log\frac{1}{\varepsilon} \le C \sum_{i=1}^{n} \|\nabla a_i\|_2 \ \|\nabla b_i\|_2. \tag{80}$$

We have moreover

$$\begin{aligned}
\|\nabla\bar{v}\|_{L^{2,1}(B_1\setminus B_\varepsilon)} &\simeq |d_0| \int_0^\infty \left|\left\{x \in B_1 \setminus B_\varepsilon \;\middle|\; |x|^{-1} > t\right\}\right|^{1/2} dt \\
&= |d_0| \int_0^\infty \left|(B_1 \setminus B_\varepsilon) \cap B_{1/t}\right|^{1/2} dt \\
&\le \pi\,|d_0| \int_0^{1/\varepsilon} \frac{dt}{\max\{t,\,1\}} = \pi\ |d_0| \left[1 + \log\frac{1}{\varepsilon}\right].
\end{aligned} \tag{81}$$

Thus combining (80) and (81) we have on one hand

$$\|\nabla\bar{v}\|_{L^{2,1}(B_1\setminus B_\varepsilon)} \le C \sum_{i=1}^{n} \|\nabla a_i\|_2 \ \|\nabla b_i\|_2; \tag{82}$$

on the other hand, as in Lemma A.2, we have

$$\Big\| \sum_{n\in\mathbb{Z}^*} (c_n \rho^n + d_n \rho^{-n}) e^{in\theta} \Big\|_{L^{2,1}(B_\lambda\setminus B_{\varepsilon/\lambda})} \le C(\lambda)\|\nabla v\|_2 \le C(\lambda)\|\nabla\phi\|_2. \tag{83}$$

Combining (82), (83) we have for any $\lambda \in (0, 1)$ the existence of a positive constant $C(\lambda) > 0$ such that

$$\|\nabla v\|_{L^{2,1}(B_\lambda\setminus B_{\varepsilon/\lambda})} \le C(\lambda)\Big(\sum_{i=1}^{n} \|\nabla a_i\|_2 \ \|\nabla b_i\|_2 + \|\nabla\phi\|_2\Big). \tag{84}$$

Finally summing (76) and (84) gives the desired inequality and Lemma 18 is proved. □

5.2. ***Quantization of pseudoholomorphic curves on degenerating Riemann surfaces.*** We consider a closed Riemann surface (Σ, h), where Σ is a smooth compact surface without boundary, and is h a metric on Σ. Since we are only interested in the conformal structure of Σ, we can assume, thanks to the uniformization theorem [Hubbard 2006] that h has constant scalar curvature. We consider (N, J) to be a smooth almost-complex manifold and we look at pseudoholomorphic curves between (Σ, h) and (N, J); in other words we consider applications $u \in W^{1,2}(\Sigma, N)$ satisfying

$$\frac{\partial u}{\partial x} = J(u)\frac{\partial u}{\partial y}, \tag{85}$$

where $z = x + iy$ are some local conformal coordinates on Σ. These objects are fundamental in symplectic geometry [McDuff and Salamon 2004]. In the study of the *moduli space* of pseudoholomorphic curves in an almost complex manifold, the compactification question comes naturally. In other words it is of first importance to understand and describe how sequences of pseudoholomorphic curves with possibly degenerating conformal class behave at the limit.

The so-called Gromov's compactness theorem [Gromov 1985] (see also [Parker and Wolfson 1993; Sikorav 1994; Hummel 1997]) provides an answer to this question.

Theorem 19. *Let (N, J) be a compact almost complex manifold, Σ a closed surface and (j_n) a sequence of complex structures on Σ. Assume $u_n : (\Sigma, j_n) \to (N, J)$ is a sequence of pseudoholomorphic curves of bounded area with respect to an arbitrary metric on N. Then u_n converges weakly to some cusp curve*[4] *$\bar{u} : \bar{\bar{\Sigma}} \to (N, J)$ and there exist finitely many bubbles, holomorphic maps $(\omega^i)_{i=1,\dots,l}$ from S^2 into (N, J), such that, modulo extraction of a subsequence,*

$$\lim_{n\to+\infty} E(u_n) = E(\bar{u}) + \sum_{i=1}^{l} E(\omega^i).$$

In fact the bound on the energy is not necessary assuming that the target manifold is symplectic, that is, if there is ω a closed 2-form on N compatible with J. Indeed, in that case (see [McDuff and Salamon 2004, Chapter 2] for instance), all $u : \Sigma \to N(J, \omega)$, regular enough, satisfies

$$A(u) = \int_\Sigma d(\mathrm{vol}_{u^* g}) \geq \int_\Sigma u^*\omega,$$

where $g = \omega(\,\cdot\,, J)$, with equality if and only if u is pseudoholomorphic. Hence, for symplectic manifolds, pseudoholomorphic curves are area-minimizing in their homology class. In particular, they are minimal surfaces, that is, conformal and harmonic, and we can use the general theory of harmonic maps; see [Zhu 2010, Remark 4.2].

We propose below a proof of Theorem 19 that follows the main lines of the most classical one (that is, we shall decompose our curves into thin and thick parts at the limit) but the argument we provide in order to prove that there is no energy in the neck and collar regions is new. We don't make use of the standard isoperimetric machinery but we simply apply the first-order Wente's estimate on annuli given by Lemma 18 which fits in an optimal way the particular structure of the pseudoholomorphic equation (85).

Proof of Theorem 19. The proof consists in splitting the surface in several pieces where the sequence converges either strongly to a nonconstant limiting map or weakly to a constant. Then in a second step, we prove that there is in fact no energy in the pieces where the converge is weak. Note that in contrast to the previous section, in the present case the complex structure of the surface is not fixed and is *a priori* free to degenerate.

Our aim is to show how Lemma 18 can be used in this context and therefore we shall be more brief on the classical parts such as the limiting Deligne–Mumford *thin-thick* decomposition which is described for instance in [Hummel 1997] or in [Zhu 2010]. Observe that due to the structure of the equation the ε-regularity theorem for pseudoholomorphic curves is a consequence of Theorem 12.

For simplicity, we will also assume that we have a surface of genus g greater or equal to 2. Hence let h_n be the hyperbolic metric of volume 1 associated to the complex structure j_n.

[4] We refer to [Hummel 1997, Chapter 5] for precise definitions.

According to the Deligne–Mumford compactification of Riemann surfaces [Hummel 1997, Chapter 4], modulo extraction of a subsequence, (Σ, h_n) converges to a hyperbolic Riemannian (Σ, h) surface by collapsing p $(0 \le p \le 3g-3)$ pairwise disjoint simple closed geodesics (γ_n^i).

Far from the collapsing geodesics, the metric uniformly converges, and we have a classical "bubble-tree" decomposition, that is to say u_n converges to a pseudoholomorphic curves of the (Σ, h) except possibly at finitely many points where, as in the previous section, u_n is forming bubbles (pseudoholomorphic curves from $\mathbb{C}$ to N) which are "connected" to each other by some *neck regions* $N_n^i = B(a_n^i, \mu_n^i) \setminus B(a_n^i, \lambda_n^i)$ where the weak L^2 energy goes to zero,

$$\lim_{r\to 0} \lim_{n\to+\infty} \|\nabla u_n\|_{L^{2,\infty}(N_n^i(r))} = 0,$$

where $N_n^i(r) = B(a_n^i, r\mu_n^i) \setminus B(a_n^i, \lambda_n^i/r)$. This can be established by combining the fact that, on such annular regions, the maximal L^2 energy of ∇u_n on dyadic annuli has to vanish (otherwise we would have another bubble) and the fact that Lemmas 13 and 18 apply to this situation.

Near the collapsing geodesics, our surface becomes asymptotically isometric to a hyperbolic cylinder of the form

$$A_l = \left\{ z = re^{i\phi} \in \mathbb{H} \;\middle|\; 1 \le r \le e^l,\ \arctan(\sinh(l/2)) < \phi < \pi - \arctan(\sinh(l/2)) \right\},$$

where the geodesic corresponds to $\{re^{i\pi/2} \in \mathbb{H} \mid 1 \le r \le e^l\}$, and the lines $\{r = 1\}$ and $\{r = e^l\}$ are identified via $z \mapsto e^l z$. This is the *collar region*. It is sometimes easier to consider the following cylindrical parametrization:

$$P_l = \left\{ (t, \theta) \;\middle|\; \frac{2\pi}{l} \arctan(\sinh(l/2)) < t < \frac{2\pi}{l}\big(\pi - \arctan(\sinh(l/2))\big),\ 0 \le \theta \le 2\pi \right\}.$$

In this parametrization the constant scalar curvature metric reads

$$ds^2 = \left(\frac{l}{2\pi \sin(lt/2\pi)} \right)^2 (dt^2 + d\theta^2),$$

where the geodesic corresponds to $\{t = \pi^2/l\}$, and the lines $\{\theta = 0\}$ and $\{\theta = 2\pi\}$ are identified.

Then, as the length l_n of the degenerating geodesic goes to zero, $P_{l_n} = [0, T_n] \times S^1$ up to translation, which can be decomposed as follows [Zhu 2010, Proposition 3.1]. For each such a thin part, one can extract a subsequence such that the following decomposition holds. There $p \in \mathbb{N}$ and $2p$ sequences (a_n^1), (b_n^1), (a_n^2), (b_n^2),..., (a_n^p), (b_n^p) of positive numbers between 0 and T_n such that

$$\lim_{n\to+\infty} \frac{b_n^i - a_n^i}{T_n} = 0$$

and up to rescaling and identifying $]-\infty, +\infty[\times S^1$ with $\mathbb{C} \setminus \{0\}$, there exists a bubble ω^i (that is, a pseudoholomorphic curve from $\mathbb{C}$ to N) such that

$$u^n\left(\frac{a_n^i + b_n^i}{2} + \frac{t}{b_n^i - a_n^i}, \theta \right) \to \omega^i \qquad \text{on } C^2_{\mathrm{loc}}(\mathbb{C} \setminus \{0\}).$$

Moreover, for any $\varepsilon > 0$, there exists $r > 0$ such that for any $T \in [b_n^i + r^{-1}, a_n^{i+1} - r^{-1}]$,

$$\int_{[T,T+1]\times S^1} |\nabla u_n|^2 \leq \varepsilon. \tag{86}$$

Denoting

$$J_n^i = [a_n^i, b_n^i] \times S^1, \quad I_n^i = [b_n^i, a_n^{i+1}] \times S^1, \quad I_n^0 = [0, a_n^1] \times S^1,$$
$$I_n^p = [b_n^p, T_n] \times S^1, \quad \text{and} \quad I_n^i(r) = [b_n^i + r^{-1}, a_n^{i+1} - r^{-1}],$$

equation (86) combined with Lemma 13 implies that

$$\lim_{r\to 0} \lim_{n\to+\infty} \|\nabla u_n\|_{L^{2,\infty}(I_n^i(r))} = 0. \tag{87}$$

This decomposition is illustrated by Figure 3.

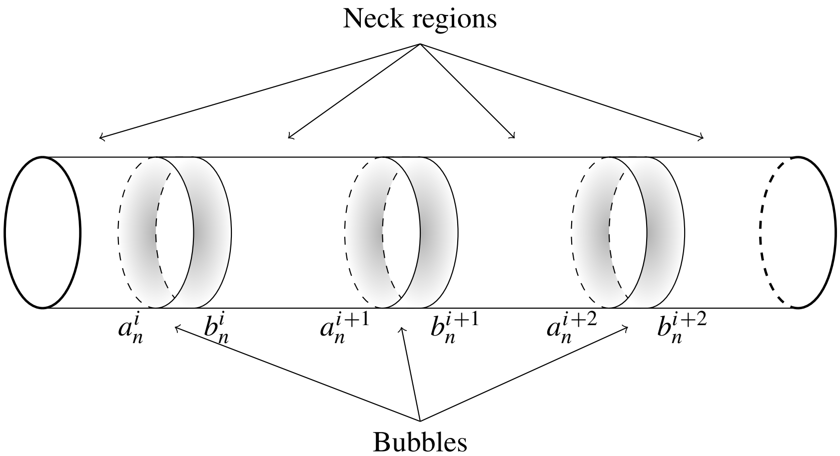

Figure 3. Decomposition into necks and bubbles.

As in the previous section, in order to prove that there is no energy at the limit in the *neck regions* of the thin parts, we combine the vanishing of the $L^{2,\infty}$-norm given by (87) with a uniform estimate on the $L^{2,1}$-norm of $|\nabla u^n|$ on each $I_n^i(r)$, which is a direct consequence of Lemma 18 applied to the pseudoholomorphic equation

$$\nabla u_n = J(u_n)\nabla^\perp u_n.$$

This concludes the proof of Theorem 19. □

Remark 20. Here again, in addition to the fact that our argument is not specific to J-holomorphic curves, our proof, in comparison with previous ones such as the one given in [Zhu 2010], has the advantage to require less regularity on the target manifold N. In fact, following the approach of [Parker 1996] or [Lin and Wang 1998], in order to establish the angular energy quantization, M. Zhu goes through a lower estimate of the second derivative

$$\frac{d^2}{d\theta^2}\int_{S^1\times\{t\}} |u_\theta|^2\, d\theta.$$

Such an estimate requires for the metric of N to be at least C^2. In the alternative proof we are providing, in order to apply Lemma 18, we only require the almost complex structure and the compatible metric to be C^1 which corresponds to a weakening of the assumption of magnitude 1 in the derivative.

5.3. ***Quantification for harmonic maps on a degenerating surface, a cohomological condition.*** The aim of this section is to shed a new light on the quantization for harmonic maps on a degenerating surfaces, which has been fully described by M. Zhu in [2010].

The main result in the present subsection is the following result, which connects energy quantization for harmonic maps into spheres with a cohomological condition.

Theorem 21. *Let (Σ, h_n) be a sequence of closed Riemann surfaces equipped with their constant scalar curvature metric with volume* 1. *Let u_n be a sequence of harmonic maps from (Σ, h_n) into the unit sphere S^{m-1} of the euclidean space $\mathbb{R}^m$. Assume that*

$$\limsup_{n\to+\infty} E(u_n) < +\infty,$$

and assume that the closed forms

$$\star(u_n^i\, du_n^j - u_n^j\, du_n^i)$$

are exact for all $i, j = 1, \dots, m$. Then the energy quantization holds: *modulo extraction of a subsequence*, *on each component of the limiting thick part*, *u_n converges strongly*, *away from the punctures*, *to some limiting harmonic map u and there exists finitely many bubbles*, *holomorphic maps $(\omega^i)_{i=1,\dots,l}$ from S^2 into S^{m-1} —forming possibly both on the thick and the thin parts— such that*, *modulo extraction of a subsequence*

$$\lim_{n\to+\infty} E(u_n) = E(u) + \sum_{i=1}^{l} E(\omega^i). \tag{88}$$

Proof. In fact, assuming that our sequence of harmonic maps u_n get valued into S^{m-1} the equation simply written

$$\Delta u_n^i = \left(u_n^i \nabla (u_n)_j - (u_n)_j \nabla u_n^i\right) \nabla u_n^j.$$

But div $\left(u_n^i \nabla (u_n)_j - (u_n)_j \nabla u_n^i\right) = 0 = d(*u_n \wedge du_n)$. Hence *assuming that the closed* $\wedge^2 \mathbb{R}^m$*-valued* 1*-form* $\star(u_n \wedge du_n)$ *is exact* , there exists $b_n \in W^{1,2}$ such that

$$\star(u_n \wedge du_n) = db_n \quad \text{and} \quad \|b_n\|_{W^{1,2}} = O(\|u_n\|_{W^{1,2}}).$$

Then we have

$$\operatorname{div}(\nabla u_n - \nabla^\perp b_n\, u_n) = 0.$$

If we are on a neck region such as $B_1 \setminus D(0, \varepsilon_n)$, it can be integrated as

$$\nabla u_n = \nabla^\perp b_n\, u_n + \nabla^\perp c_n + d_n \nabla \log(\rho), \tag{89}$$

where $c_n \in W^{1,2}(B_1)$ and $d_n \in \mathbb{R}$. Then we try to control the gradient of the logarithmic part, remarking that

$$\frac{d}{d\rho}\int_0^{2\pi} u_n\, d\theta = \int_0^{2\pi} \frac{1}{\rho}\frac{\partial b_n}{\partial\theta}\, u_n\, d\theta + 2\pi\,\frac{d_n}{\rho} = \int_0^{2\pi} \frac{1}{\rho}\frac{\partial b_n}{\partial\theta}\big(u_n - \bar{u}_n^{\rho}\big)\, d\theta + 2\pi\,\frac{d_n}{\rho},$$

where $\bar{u}_n^{\rho}$ is the mean value of u_n over ∂B_ρ. Integrating the previous identity from ε_n to an arbitrary ρ gives

$$2\pi\big(\bar{u}_n^{\rho} - \bar{u}_n^{\varepsilon_n}\big) = \int_{\varepsilon_n}^{\rho}\int_0^{2\pi} \frac{1}{t}\frac{\partial b_n}{\partial\theta}\big(u_n - \bar{u}_n^{t}\big)\, d\theta\, dt + 2\pi d_n\, \log(\rho/\varepsilon_n)\,. \tag{90}$$

And, thanks to Poincaré's inequality, we get

$$\left|\int_{\varepsilon_n}^{\rho}\int_0^{\pi} \frac{1}{t}\frac{\partial b_n}{\partial\theta}\big(u_n - \bar{u}_n^{t}\big)\, d\theta\, dt\right| \le C\|\nabla b_n\|_2\|\nabla u_n\|_2. \tag{91}$$

Then, combining (90) and (91), we finally obtain that

$$d_n = O\left(\frac{1}{\log(1/\varepsilon_n)}\right).$$

Which implies, as in the proof of Lemma 18, that the $L^{2,1}$-norm of $d_n\nabla\log(\rho)$ in $B_1 \setminus D(0,\varepsilon_n)$ is uniformly bounded. By Lemma 18 and thanks to (89), we see that he $L^{2,1}$-norm of $\nabla(u_n - d_n\log(\rho))$ is also uniformly bounded and these two uniform bounds imply the uniform $L^{2,1}$ bound of ∇u_n in neck regions. Combining the uniform $L^{2,1}$ bound of ∇u_n in neck regions together with the Lemma 13 gives the desired energy quantization (88) and Theorem 21 is proved. □

More generally we can raise the following question: *Considering a sequence of harmonic maps from a degenerating surface to a general target manifolds, is there is a simple cohomological condition similar to the one in Theorem 21 ensuring the quantization of the energy in collar region?*

5.4. ***Energy Quantization for Willmore Surfaces.*** Finally we would like to recall a last application of Lemma 18 that has been used in a recent work by Y. Bernard and T. Rivière in [2011] for proving energy quantization for sequences of Willmore surfaces with uniformly bounded energy and nondegenerating conformal classes. The problem can be described as follows: for a sufficiently smooth immersion $u : \Sigma \to \mathbb{R}^m$, where Σ is a closed two-dimensional Riemannian surface, we can define its mean curvature vector $\vec{H}$ and we consider the functional

$$W(u) = \int_\Sigma |\vec{H}|^2\, u^*(dy),$$

where $u^*(dy)$ denotes the metric induced on Σ by the immersion u. This functional is called, the Willmore functional and is known to be conformally invariant [Rivière 2010]. Critical points to the functional W are called *Willmore immersions* or *Willmore surfaces*. Hence as for harmonic maps or pseudoholomorphic curves the question of the quantization of sequences of Willmore surfaces arise naturally. The second author has developed appropriate tools to study weak critical points to W in [Rivière 2008] and [Rivière] and proved the ε-regularity for these weak critical points. Using in particular Lemma 18 the following energy quantization has been established:

Theorem 22 [Bernard and Rivière 2011]. *Let u_n be a sequence of Willmore immersions of a closed surface Σ. Assume that*

$$\limsup_{n\to+\infty} W(u_n) < +\infty,$$

and that the conformal class of $u_n^*(\xi_{\mathbb{R}^m})$ remains within a compact subdomain of the moduli space of Σ. Then, modulo extraction of a subsequence, the following energy identity holds:

$$\lim_{n\to+\infty} W(u_n) = W(u_\infty) + \sum_{l=1}^{L} W(\omega_l) + \sum_{k=1}^{K} (W(\Omega_k) - 4\pi\theta_k),$$

where u_∞ is a possibly branched smooth Willmore immersion of Σ. The maps ω_l and Ω_k are smooth, possibly branched, Willmore immersions of S^2 and θ_k is the integer density of the current $(\Omega_k)_*(S^2)$ at some point $p_k \in \Omega_k(S^2)$, namely

$$\theta_k = \lim_{\rho\to 0} \frac{\mathcal{H}^2\big(B_\rho(p_k) \cap \Omega_k(S^2)\big)}{\pi\rho^2}.$$

Appendix A. Lorentz estimates on harmonic functions.

Here we prove two lemmas on harmonic functions which insure that we can control the $L^{2,1}$-norm by the L^2-norm on a smaller domain up to some appropriate boundary condition.

Lemma A.1. *Let $0 < \varepsilon < \frac{1}{2}$ and let $f : B_1 \setminus B_\varepsilon \to \mathbb{R}$ be a harmonic function which satisfies*

$$f = 0 \text{ on } \partial B_1 \quad \text{and} \quad \int_{\partial B_\varepsilon} f\, d\sigma = 0. \tag{92}$$

Then for each $\lambda > 1$ there exists positive a constant $C(\lambda)$, independent of ε and f, such that

$$\|\nabla f\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le C(\lambda)\|\nabla f\|_2.$$

Proof. We start by decomposing f as a Fourier series, which gives

$$f(\rho,\theta) = c_0 + d_0\log(\rho) + \sum_{n\in\mathbb{Z}^*} (c_n\rho^n + d_n\rho^{-n})e^{in\theta}.$$

Hence, using (92), we easily prove that $c_0 = d_0 = c_n + d_n = 0$; then we get

$$f(\rho,\theta) = \sum_{n\in\mathbb{Z}^*} c_n(\rho^n - \rho^{-n})e^{in\theta}.$$

Then we estimate the gradient as follows:

$$|\nabla f(\rho,\theta)| \le 2\sum_{n\in\mathbb{Z}^*} |nc_n|(\rho^{n-1} + \rho^{-n-1}).$$

Then, we estimate the $L^{2,1}$-norm of the $f_m(z) = |z|^m$ on $B_1 \setminus B_{\lambda\varepsilon}$, for $m \in \mathbb{Z}\setminus\{-1\}$ and $\lambda \in]1,2]$, which gives

$$\|f_m\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \le \sqrt{\pi}\int_0^{(\lambda\varepsilon)^m} t^{1/m}\, dt \le 2\sqrt{\pi}(\lambda\varepsilon)^{m+1} \quad \text{for } m < -1,$$

and $\|f_m\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq \sqrt{\pi}$ for $m \geq 0$. Here we use the characterization of the $L^{2,1}$ norm given in (8). Hence we get

$$\|\nabla f\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq 4\sqrt{\pi}\Bigg(\sum_{n>0}|n\, c_n|\,\big((\lambda\varepsilon)^{-n}+1\big)+\sum_{n<0}|n\, c_n|\,\big((\lambda\varepsilon)^{n}+1\big)\Bigg).$$

Hence, thanks to the Cauchy–Schwarz and the fact that $\lambda > 1$, we get

$$\|\nabla f\|_{L^{2,1}(B_1\setminus B_{\lambda\varepsilon})} \leq 8\sqrt{\pi}\bigg(\sum_{n\neq 0}|n|\lambda^{-2|n|}\bigg)\bigg(\sum_{n\neq 0}|n|\,|c_n|^2\varepsilon^{-2|n|}\bigg)^{1/2}.$$

Finally we compute the L^2-norm of ∇f:

$$\|\nabla f\|_2 = \bigg(2\pi\int_\varepsilon^1\sum_{n\neq 0}|n\, c_n|^2\big(\rho^{2n-2}+\rho^{-2n-2}\big)\rho\, d\rho\bigg)^{1/2} \geq \sqrt{\frac{\pi}{2}}\bigg(\sum_{n\neq 0}|n|\,|c_n|^2\varepsilon^{-2|n|}\bigg)^{1/2},$$

which achieves the proof of Lemma A.1. □

Lemma A.2. *Let $0<\varepsilon<\frac{1}{4}$ and let $f : B_1\setminus B_\varepsilon \to \mathbb{R}$ be a harmonic function which satisfies*

$$\int_{\partial B_\varepsilon} f\, d\sigma = 0 \quad \textit{and} \quad \left|\int_{\partial B_1} f\, d\sigma\right| \leq K, \tag{93}$$

where K is a constant independent of ε. Then for each $0<\lambda<1$ there exists positive constant $C(\lambda)$, independent of ε and f, such that

$$\|\nabla f\|_{L^{2,1}(B_\lambda\setminus B_{\varepsilon/\lambda})} \leq C(\lambda)(\|\nabla f\|_2+1).$$

Proof. We start by decomposing f as a Fourier series, which gives

$$f(\rho,\theta) = c_0 + d_0\log(\rho) + \sum_{n\in\mathbb{Z}^*}(c_n\rho^n + d_n\rho^{-n})e^{in\theta}.$$

Hence, using (93), we easily prove that $c_0 + d_0\log(\varepsilon) = 0$ and $|c_0| = O(1)$. Hence

$$d_0 = O\left(-\frac{1}{\log(\varepsilon)}\right). \tag{94}$$

Then we estimate the gradient as follows:

$$|\nabla f(\rho,\theta)| \leq |d_0|\,\frac{1}{\rho} + \sum_{n\in\mathbb{Z}^*}|n\, c_n|\rho^{n-1} + |n\, d_n|\rho^{-n-1}.$$

Then we estimate the $L^{2,1}$-norm of $f_m(z) = |z|^m$ on $B_\lambda\setminus B_{\varepsilon/\lambda}$ for $m\in\mathbb{Z}\setminus\{-1\}$ and $0<\lambda<1$, which gives

$$\|f_m\|_{2,1} \leq \sqrt{\pi}\int_0^{(\varepsilon/\lambda)^m} t^{1/m}\, dt \leq 2\sqrt{\pi}(\varepsilon/\lambda)^{m+1} \quad \text{for } m<-1,$$

$\|f_m\|_{2,1} \le \sqrt{\pi}\lambda^m$ for $m \ge 0$, and $\|f_{-1}\|_{2,1} = O(-\log(\varepsilon))$. Here we use the following characterization (8). Thanks to (94) and the above, we get

$$\|\nabla f\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le 2\sqrt{\pi}\Bigg(\sum_{n>0} \big(|n\, c_n|\lambda^n + |n\, d_n|(\varepsilon/\lambda)^{-n}\big) + \sum_{n<0} \big(|n\, c_n|(\varepsilon/\lambda)^n + |n\, d_n|\lambda^{-n}\big)\Bigg) + O(1).$$

Hence, thanks to Cauchy–Schwarz and the fact that $0 < \lambda < 1$, we get

$$\|\nabla f\|_{L^{2,1}(B_\lambda \setminus B_{\varepsilon/\lambda})} \le 4\sqrt{\pi}\bigg(\sum_{n\neq 0} |n|\lambda^{2|n|}\bigg)\bigg(\sum_{n<0} |n|\,\frac{|c_n|^2 + |d_{-n}|^2}{\varepsilon^{2|n|}} + \sum_{n>0} |n|\,\frac{|c_n|^2 + |d_{-n}|^2}{2^n}\bigg)^{1/2} + O(1).$$

Finally we compute the L^2-norm of ∇f:

$$\begin{aligned}\|\nabla f\|_2 &= |d_0|\bigg(\int_\varepsilon^1 \frac{1}{\rho}\,d\rho\bigg)^{1/2} + \bigg(2\pi \int_\varepsilon^1 \sum_{n\neq 0}\big(|n\, c_n|^2\rho^{2n-2} + |n\, d_n|^2\rho^{-2n-2}\big)\,\rho\, d\rho\bigg)^{1/2} \\ &\ge \sqrt{\frac{\pi}{2}}\bigg(\sum_{n<0} |n|\,\frac{|c_n|^2 + |d_{-n}|^2}{\varepsilon^{2|n|}} + \sum_{n>0} |n|\,\frac{|c_n|^2 + |d_{-n}|^2}{2^n}\bigg)^{1/2},\end{aligned}$$

which achieves the proof of Lemma A.2. □

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