ANALYSIS & PDE

Volume 7 No. 1 2014

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ANALYSIS AND PDE
Vol. 7, No. 1, 2014
dx.doi.org/10.2140/apde.2014.7.117

A PARTIAL DATA RESULT FOR THE MAGNETIC SCHRÖDINGER INVERSE PROBLEM

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This article shows that restricting the domain of the Dirichlet–Neumann map to functions supported on a certain part of the boundary, and measuring the output on, roughly speaking, the rest of the boundary, uniquely determines a magnetic Schrödinger operator. If the domain is strongly convex, either the subset on which the Dirichlet–Neumann map is measured or the subset on which the input functions have support may be made arbitrarily small. The key element of the proof is the modification of a Carleman estimate for the magnetic Schrödinger operator using operators similar to pseudodifferential operators.

1. Introduction

Let $n \geq 2$, and let Ω be a simply connected bounded domain in $\mathbb{R}^{n+1}$ with smooth boundary. Suppose W is a C^2 vector field on $\mathbb{R}^{n+1}$ and q is an L^∞ function on $\mathbb{R}^{n+1}$. Then define the magnetic Schrödinger operator $\mathscr{L}_{W,q}$ with magnetic potential W and electric potential q by

$$\mathscr{L}_{W,q} = (D+W)^2 + q,$$

where $D = -i\nabla$. I will assume that q and W are such that zero is not an eigenvalue of $\mathscr{L}_{W,q}$ on Ω. Then the Dirichlet problem

$$\mathscr{L}_{W,q} u = 0, \quad u|_{\partial\Omega} = g$$

has a unique solution $u \in H^1(\Omega)$ for each $g \in H^{1/2}(\partial\Omega)$. Therefore for $g \in H^{1/2}(\partial\Omega)$, we can define the Dirichlet–Neumann map $\Lambda_{W,q}$ by

$$\Lambda_{W,q} g = (\partial_\nu + iW\cdot\nu)u|_{\partial\Omega},$$

where ν is the outward unit normal and u is the unique solution to the Dirichlet problem with boundary value g. This gives a well-defined map from $H^{1/2}(\partial\Omega)$ to $H^{-1/2}(\partial\Omega)$.

The basic inverse problem associated to the magnetic Schrödinger operator $\mathscr{L}_{W,q}$ is to recover the electric potential q and the magnetic field dW from knowledge of $\Lambda_{W,q}$. (Here dW makes sense by identifying W with the 1-form $W_1 dx_1 + \cdots + W_{n+1} dx_{n+1}$.) We cannot hope to recover W itself, since the Dirichlet–Neumann map is invariant under the gauge transformation $W \mapsto W + \nabla\Psi$ whenever $\Psi \in C^1(\overline{\Omega})$ and $\Psi|_{\partial\Omega} = 0$. However, identifying dW identifies W up to this gauge transformation.

MSC2010: primary 35R30; secondary 35S99.
Keywords: inverse problems, partial data, Dirichlet–Neumann map, Carleman estimate, magnetic Schrödinger operator, semiclassical analysis, pseudodifferential operators.

This can be thought of as a generalization of the Calderón problem [1980], which can be written in this form with $W \equiv 0$ in the case of smooth enough conductivity (see [Sylvester and Uhlmann 1987]).

Sylvester and Uhlmann [1987] showed that in the Calderón problem, the Dirichlet–Neumann map determines q. For the magnetic Schrödinger problem, Sun showed that the Dirichlet–Neumann map determines dW and q when W is small enough, in a certain sense. Nakamura, Sun, and Uhlmann [Nakamura et al. 1995] removed the smallness assumption and showed that the Dirichlet–Neumann map determines dW and q for W in C^2 and q in L^∞. Tolmasky [1998] and Salo [2004] improved the regularity conditions on W to $C^{2/3+\varepsilon}$ and Dini continuous, respectively. Salo [2006] also gave a proof for $W \in C^{1+\varepsilon}$ involving a reconstruction method.

Given that $\Lambda_{W,q}$ determines dW and q, a further question might be whether *partial* knowledge of $\Lambda_{W,q}$ determines dW and q. In particular, one might ask whether restricting the domain of the Dirichlet–Neumann map to functions supported on a particular subset of the boundary still gives enough information to determine dW and q. Alternatively, one might ask whether measuring the output of the Dirichlet–Neumann map on a particular subset of the boundary still gives enough information to determine dW and q.

Kenig, Sjöstrand, and Uhlmann [Kenig et al. 2007] proved a result for the Calderón problem addressing both of these questions. Roughly speaking, they proved that restricting the domain of the Dirichlet–Neumann map to functions supported on particular subsets of the boundary and measuring the output on the rest of the boundary determines q. Together with Dos Santos Ferreira, they proved a similar result for the magnetic Schrödinger problem in [Dos Santos Ferreira et al. 2007], but without being able to restrict the domain of $\Lambda_{W,q}$. The main results of this paper are to impose that restriction, and thus show that a result analogous to the one in [Kenig et al. 2007] also holds for the magnetic Schrödinger problem.

In order to describe these results more fully, we need to describe the subsets of the boundary involved. Assume that x_0 is not in the closure of the convex hull of Ω, and define the front and back of $\partial\Omega$ (with respect to x_0) by

$$\partial\Omega_- = \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) \le 0\}, \quad \partial\Omega_+ = \{x \in \partial\Omega \mid (x - x_0) \cdot \nu(x) \ge 0\},$$

where $\nu(x)$ is the outward unit normal at x.

The main results of this paper are the following two theorems.

Theorem 1.1. *Let W_1 and W_2 be C^2 vector fields on $\overline{\Omega}$, and let q_1 and q_2 be L^∞ functions on Ω. Let $\Gamma_- \subset \partial\Omega$ be a neighborhood of $\partial\Omega_-$, and let $\Gamma_+ \subset \partial\Omega$ be a neighborhood of $\partial\Omega_+$. Suppose*

$$\Lambda_{W_1,q_1} g|_{\Gamma_-} = \Lambda_{W_2,q_2} g|_{\Gamma_-}$$

for all $g \in H^{1/2}(\partial\Omega)$ with support contained in Γ_+.

Then $q_1 = q_2$ and $dW_1 = dW_2$.

Theorem 1.2. *Let W_1 and W_2 be C^2 vector fields on $\overline{\Omega}$, and let q_1 and q_2 be L^∞ functions on Ω. Let $\Gamma_+ \subset \partial\Omega$ be a neighborhood of $\partial\Omega_+$, and let $\Gamma_- \subset \partial\Omega$ be a neighborhood of $\partial\Omega_-$. Suppose*

$$\Lambda_{W_1,q_1} g|_{\Gamma_+} = \Lambda_{W_2,q_2} g|_{\Gamma_+}$$

for all $g \in H^{1/2}(\partial\Omega)$ *with support contained in* Γ_-.

Then $q_1 = q_2$ *and* $dW_1 = dW_2$.

The second theorem is essentially the first theorem after the conformal transformation on Ω given by inversion in x_0. Imposing the condition $W_1 \equiv W_2 \equiv 0$ in these theorems would give the results from [Kenig et al. 2007], and removing the restriction on the support of g would give the results from [Dos Santos Ferreira et al. 2007].

Roughly speaking, the first theorem says that if the Dirichlet–Neumann map is known on a neighborhood of the front for functions supported on a neighborhood of the back, then potentials can be determined. The second theorem says something similar, but with the roles of the front and back reversed.

If the domain Ω is nice enough, then the front can be made arbitrarily small. For example, if Ω is strongly convex (convex, and the intersection of the boundary with any tangent hyperplane to the boundary consists only of one point), then the front can be contained in an arbitrarily small open subset of the boundary, for the right choice of x_0. This gives us the following corollary.

Corollary 1.3. *Suppose* Ω *is a smooth bounded strongly convex domain in* $\mathbb{R}^{n+1}$. *Let* W_1 *and* W_2 *be* C^2 *vector fields on* $\overline{\Omega}$, *and let* q_1 *and* q_2 *be* L^∞ *functions on* Ω. *Then for any nonempty open subset* Γ_1 *of the boundary, there exists a neighborhood* Γ_2 *of* $\Gamma_1^c := \partial\Omega \setminus \Gamma_1$ *such that if*

$$\Lambda_{W_1,q_1} g|_{\Gamma_1} = \Lambda_{W_2,q_2} g|_{\Gamma_1}$$

for all $g \in H^{1/2}(\partial\Omega)$ *with support contained in* Γ_2, *then* $q_1 = q_2$ *and* $dW_1 = dW_2$.

Alternatively, for any nonempty open subset Γ_2 *of the boundary, there exists a neighborhood* Γ_1 *of* Γ_2^c *in* Ω *such that if*

$$\Lambda_{W_1,q_1} g|_{\Gamma_1} = \Lambda_{W_2,q_2} g|_{\Gamma_1}$$

for all $g \in H^{1/2}(\partial\Omega)$ *with support contained in* Γ_2, *then* $q_1 = q_2$ *and* $dW_1 = dW_2$.

The first part of the corollary says that in particular, the Dirichlet–Neumann map can be measured on an arbitrarily small subset of the boundary. The second part of the corollary says that alternatively, the input functions may be restricted to an arbitrarily small subset of the boundary.

Theorem 1.2 can either be proved from Theorem 1.1 by the change of variables mentioned above, or proved in the same manner as Theorem 1.1, making the changes indicated at the end of Section 8. Therefore most of this paper will be devoted to proving Theorem 1.1. From here on, unless otherwise noted, I will assume Γ_+, Γ_- and Ω are as in Theorem 1.1.

The key to the proof of Theorem 1.1 is the construction of complex geometrical optics (CGO) solutions to the system

$$\mathscr{L}_{W,q} u = 0 \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0, \tag{1-1}$$

where $\Gamma_+^c := \partial\Omega \setminus \Gamma_+$. This in turn requires a Carleman estimate for $\mathscr{L}_{W,q}$, which can be described as follows.

Let φ be a limiting Carleman weight on Ω; that is, a real-valued smooth function that has nonvanishing gradient on Ω and satisfies

$$\langle \varphi'' \nabla\varphi, \nabla\varphi \rangle + \langle \varphi'' \xi, \xi \rangle = 0$$

whenever $|\xi| = |\nabla\varphi|$ and $\nabla\varphi \cdot \xi = 0$. Define

$$\mathcal{L}_{\varphi,W,q} = h^2 e^{\varphi/h} \mathcal{L}_{W,q} e^{-\varphi/h}.$$

Here h is a semiclassical parameter; from here on, all Sobolev spaces and Fourier transforms in this paper are semiclassical, unless otherwise specified, with h being the semiclassical parameter. Thus $\|u\|_{H^1}$ means the norm defined by

$$\|u\|_{H^1}^2 = \|u\|_{L^2}^2 + \|h\nabla u\|_{L^2}^2,$$

and $\|u\|_{H^{-1}}$ means the dual norm to this, and so forth.

Then we have the following Carleman estimate.

Theorem 1.4. *Suppose Ω' is a smooth domain with $\Omega \subset \Omega'$ and $\partial\Omega' \cap \partial\Omega = \Gamma_+^c$, where Γ_+ is as described in Theorem 1.1. Then if $w \in C_0^\infty(\Omega)$,*

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,W,q} w\|_{H^{-1}(\Omega')}.$$

The proof of this theorem is the main new ingredient in this paper. It differs from the Carleman estimate in [Dos Santos Ferreira et al. 2007] in that this one can be used in a Hahn–Banach argument to give solutions that vanish on E. The rest of the proof of Theorem 1.1 follows the proofs in [Kenig et al. 2007; Dos Santos Ferreira et al. 2007] fairly closely. Thus, the next seven sections will be devoted to the proof of Theorem 1.4. In Section 9, I will use this estimate to construct CGO solutions to (1-1). Once these are constructed, the proof of Theorem 1.1 follows by an argument more or less identical to that in [Dos Santos Ferreira et al. 2007]. This argument is outlined in Section 10 for completeness.

2. Outline of the proof of Theorem 1.4

In order to outline the proof of Theorem 1.4, I will give a rough sketch of the proof for a special case. Choose Cartesian coordinates (x, y) on $\mathbb{R}^{n+1}$ such that $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, and suppose that Ω lies in the set $\mathbb{R}^{n+1}_+ = \{y > 0\}$, with a subset of $\partial\Omega$ lying on the hyperplane $\{y = 0\}$. Label the subset $\partial\Omega \cap \{y = 0\}$ by Γ_+^c. Then I want to show that

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,W,q} w\|_{H^{-1}(\mathbb{R}^{n+1}_+)}$$

for $w \in C_0^\infty(\Omega)$ and $\varphi(x, y) = y$. The starting point is the following estimate. Define

$$\mathcal{L}_\varphi = h^2 e^{\varphi/h} \triangle e^{-\varphi/h}$$

and

$$\mathcal{L}_{\varphi,\varepsilon} = e^{\varphi^2/2\varepsilon} \mathcal{L}_\varphi e^{-\varphi^2/2\varepsilon}.$$

Proposition 2.1 [Dos Santos Ferreira et al. 2007, Equation (2.12)]. *If φ is a limiting Carleman weight, and $w \in C_0^\infty(\Omega)$, then*

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{H^1(\Omega)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w\|_{L^2(\Omega)}.$$

A note on inequalities here: inequalities of the form $F(w,h) \lesssim G(w,h)$ mean that there exists $h_0 > 0$ independent of w such that for $h \le h_0$, the inequality $F(w,h) \le CG(w,h)$ holds for some positive constant C independent of w and h. In the case of this inequality, the constant implied in the $\lesssim$ sign is independent of ε as well.

Now set $\varphi(x,y) = y$ and define a new domain Ω_2 such that $\Omega \subset \Omega_2 \subset \mathbb{R}^{n+1}_+$, with $\Gamma^c_+ \subset \partial\Omega_2$. Proposition 2.1 still holds on Ω_2. Now the objective is to find an operator J with the following properties.

(1) J has a right inverse, denoted by J^{-1}, and J^{-1} preserves smoothness.

(2) J and J^{-1} preserve support with respect to y in the positive direction: if the support of u is in the set $\{y \ge y_0\}$, so are the supports of Ju and $J^{-1}u$.

(3) The commutators of J with differential operators behave as though J were a semiclassical pseudo-differential operator of order 1.

(4) J is bounded from $H^1(\mathbb{R}^{n+1}_+)$ to $L^2(\mathbb{R}^{n+1}_+)$.

(5) $\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u\|_{L^2(\mathbb{R}^{n+1}_+)}$.

If such an operator existed, the argument could go like this: Suppose $w \in C^\infty_0(\Omega)$, and let $\chi \in C^\infty(\mathbb{R}^{n+1}_+)$ be a cutoff function that is identically one on Ω but supported within Ω_2. Then $\chi J^{-1}w \in C^\infty_0(\Omega_2)$, so it can be plugged into Proposition 2.1 to give

$$\frac{h}{\sqrt{\varepsilon}}\|\chi J^{-1}w\|_{H^1(\Omega_2)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon}\chi J^{-1}w\|_{L^2(\Omega_2)}.$$

Here we are using property (1) to get J^{-1} and property (2) to ensure that $\chi J^{-1}w$ has the right support. Now we can use property (4) on the left and (5) on the right to get

$$\frac{h}{\sqrt{\varepsilon}}\|J\chi J^{-1}w\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \|J\mathcal{L}_{\varphi,\varepsilon}\chi J^{-1}w\|_{H^{-1}(\mathbb{R}^{n+1}_+)}.$$

The commutator properties tell us that this is

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon}w\|_{H^{-1}(\mathbb{R}^{n+1}_+)},$$

with error terms small enough to hide in the left side, for ε small enough. Then $\mathcal{L}_{\varphi,\varepsilon}w = \mathcal{L}_{\varphi,\varepsilon,W,q}w$ up to a similarly permissible error, where

$$\mathcal{L}_{\varphi,\varepsilon,W,q} = e^{\varphi^2/2\varepsilon}\mathcal{L}_{\varphi,W,q}e^{-\varphi^2/2\varepsilon},$$

and noting that $e^{\varphi^2/2\varepsilon}$ is smooth and bounded on Ω finishes the proof.

It still remains, of course, to find the magic operator J. Consider the operator J defined by

$$\widehat{Ju}(\xi,y) = (h\partial_y + F(\xi))\hat{u}(\xi,y),$$

where the hat ^ signifies the semiclassical Fourier transform in the x variables, and F is a smooth function on $\mathbb{R}^n$ such that $\left|F(\xi) - (1+|\xi|)\right| \leq \delta$ for some small δ. This has a right inverse J^{-1} given by

$$\widehat{J^{-1}u}(\xi, y) = \frac{1}{h}\int_0^y \hat{u}(\xi, t)e^{F(\xi)(t-y)/h}\,dt,$$

which satisfies property (1). Now it is relatively straightforward to see that properties (2) and (4) are satisfied, and with a little more work, we can obtain the kind of commutator properties needed for property (3).

Unfortunately, property (5) fails to hold in general. Instead we have a new property (5′), that

$$\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u - g_u\|_{L^2(\mathbb{R}^{n+1}_+)}$$

where

$$\hat{g}_u(\xi, y) = \frac{2F(\xi)}{h}\int_0^\infty \hat{u}(\xi, t)e^{-F(\xi)(t+y)/h}\,dt.$$

However, the proof only relies on property (5) applied to functions u of the form $u = \mathscr{L}_{\varphi,\varepsilon}v$, where $v \in C_0^\infty(\Omega_2)$. For these functions,

$$\hat{g}_u(\xi, y) = \frac{2F(\xi)}{h}\int_0^\infty \widehat{\mathscr{L}_{\varphi,\varepsilon}v}(\xi, t)e^{-F(\xi)(t+y)/h}\,dt,$$

where

$$\widehat{\mathscr{L}_{\varphi,\varepsilon}v}(\xi, t) = \left(h^2\partial_t^2 - 2h\partial_t + 1 - |\xi|^2\right)\hat{v}(\xi, t)$$

plus some acceptably small error. The idea is now that by using integration by parts, together with a good choice of F, we can get g_u to be small enough that

$$\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

To do this, we can split up v as $v = v_1 + v_2$, where $\hat{v}_1(\xi, t)$ is supported only for $|\xi| \leq \frac{1}{2}$, and $\hat{v}_2(\xi, t)$ is supported only for $|\xi| > \frac{1}{3}$, say. Then $g_u = \gamma_1 + \gamma_2$, where γ_j is the part that corresponds to v_j. Then for $\hat{\gamma}_1$, integration by parts gives

$$\frac{2F(\xi)}{h}\int_0^\infty \left(F(\xi)^2 - 2F(\xi) + 1 - |\xi|^2\right)\hat{v}_1(\xi, t)e^{-F(\xi)(t+y)/h}\,dt$$

plus an acceptably small error, and then using the fact that F is close to $1 + |\xi|$ gives

$$\|\gamma_1\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|v_1\|_{H^2(\mathbb{R}^{n+1}_+)}.$$

Since v_1 is only supported for small frequencies, the operator $\mathscr{L}_{\varphi,\varepsilon}$ is invertible on the support of v_1, so

$$\|\gamma_1\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|\mathscr{L}_{\varphi,\varepsilon}v_1\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Meanwhile, in the large frequency case, we can factor $\widehat{\mathscr{L}_{\varphi,\varepsilon}}$ as

$$\left(h\partial_t - (1+|\xi|)\right)\left(h\partial_t - (1-|\xi|)\right)$$

up to some acceptably small error, and do integration by parts only with the first factor. (The nonsmoothness of $|\xi|$ will cause trouble in the factoring at small frequencies, which is the reason for splitting up the argument like this.) Then $\hat{\gamma}_2$ becomes

$$\frac{2F(\xi)}{h}\int_0^\infty \big(F(\xi)-(1+|\xi|)\big)\big(h\partial_t-(1-|\xi)\big)\hat{v}_2(\xi,t)e^{-F(\xi)(t+y)/h}\,dt$$

plus some good enough error, and so we get something like

$$\|\gamma_2\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|z\|_{H^1(\mathbb{R}^{n+1}_+)},$$

where $\hat{z}=\big(h\partial_t-(1-|\xi|)\big)\hat{v}_2$. Since $\mathcal{L}_{\varphi,\varepsilon}v \sim \big(h\partial_t-(1+|\xi|)\big)z$, and the operator $h\partial_t-(1+|\xi|)$ is well behaved, we can get

$$\|\gamma_2\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|\mathcal{L}_{\varphi,\varepsilon}v_2\|_{L^2(\mathbb{R}^{n+1}_+)}.$$

Adding these two parts together and using some commutator estimates on the right side gives

$$\|g_u\|_{L^2(\mathbb{R}^{n+1}_+)} \lesssim \delta\|u\|_{L^2(\mathbb{R}^{n+1}_+)},$$

so

$$\|Ju\|_{H^{-1}(\mathbb{R}^{n+1}_+)} \simeq \|u\|_{L^2(\mathbb{R}^{n+1}_+)}$$

for u of this form. This finishes the argument. Changes in $\mathcal{L}_{\varphi,\varepsilon}$ of $O(\delta)$, roughly speaking, do not affect the argument. Therefore the argument still works if Γ_+^c coincides with a graph of the form $y=f(x)$, as long as ∇f is small enough, by using a change of variables that flattens Γ_+^c while making only $O(\delta)$ changes to $\mathcal{L}_{\varphi,\varepsilon}$.

These ideas are the basis of the argument used to prove Theorem 1.4. There are three key changes that make everything much more complicated, however. Firstly, in order to achieve results of the form of Theorems 1.1 and 1.2, we will need to work with the logarithmic weight $\varphi=\log|x-x_0|$, and in spherical coordinates centered at x_0. Then we will work with Γ_+^c's that coincide with graphs of the form $r=f(\theta)$, and work with small subsets on which the spherical coordinates look nearly Euclidean. Secondly, instead of looking at cases where ∇f is small, we will treat cases where ∇f is almost constant. This argument works nearly the same way as the argument outlined above, but requires us to use operators that depend on that constant. In fact, we will need to split the small and large frequency cases much earlier in the argument, and introduce separate operators J_s and J_ℓ for the two cases. Thirdly, we will need to glue together many such estimates at the end of the proof to get Theorem 1.4.

The proof will be presented over the next six sections. In Section 3, I will state the small subset version of the Carleman estimate, and begin the proof by making the change of variables to "flatten" Γ_+^c appropriately. In Section 4, I will split up the problem into separate propositions for the small and large frequency cases, and show that the proofs of these propositions suffice. In Section 5, I will prove analogues of properties (1) through (2) and (5′) for operators of a certain form. Section 6 then contains the small frequency argument, and Section 7 contains the large frequency argument, thus finishing the proof of the small subset version of the Carleman estimate. Finally, in Section 8, I will glue together the small subset estimates in the appropriate way to prove Theorem 1.4.

3. An initial Carleman estimate

For the rest of this paper, we will fix φ to be the logarithmic weight $\varphi(x) = \log|x - x_0|$ unless otherwise stated. Without loss of generality, we will also assume that $x_0 = 0$.

To begin, we should fix coordinates on $\mathbb{R}^{n+1}$. Since 0 is outside the convex hull of Ω, there must exist $r_0 > 0$ such that Ω lies outside the ball of radius r_0 centered at the origin. Moreover, Ω must lie entirely on one side of a hyperplane through the origin. If we choose Cartesian coordinates $x_1, \dots, x_{n+1}$ on $\mathbb{R}^{n+1}$ such that Ω lies entirely in the half-space $\{x_{n+1} > 0\}$, then we can define a map $\sigma : (\mathbb{R}^n \setminus B_{r_0,0}) \cap \{x_{n+1} > 0\} \to [r_0, \infty) \times (0, \pi) \times \cdots \times (0, \pi)$ by

$$\sigma(x_1, \dots, x_{n+1}) = (r, \theta_1, \dots, \theta_n),$$

where

$$\begin{aligned} x_1 &= r\cos\theta_1, \\ x_2 &= r\sin\theta_1\cos\theta_2, \\ &\vdots \\ x_n &= r\sin\theta_1 \dots \sin\theta_{n-1}\cos\theta_n, \\ x_{n+1} &= r\sin\theta_1 \dots \sin\theta_n. \end{aligned}$$

This fixes a set of spherical coordinates on $(\mathbb{R}^n \setminus B_{r_0,0}) \cap \{x_{n+1} > 0\}$. On any compact subset of this space, σ is a diffeomorphism with bounded derivatives; the singularities in σ occur in the other half-space.

Now we can begin by proving the following special version of Theorem 1.4.

Proposition 3.1. *Suppose that $f : S^n \to (r_0, \infty)$ is a C^∞ function such that Ω lies entirely in the region $A_O = \{(r, \theta) \mid r \geq f(\theta)\} \subset \mathbb{R}^{n+1}$, and Γ_+^c is a subset of the graph $r = f(\theta)$. Suppose also that for all $(r, \theta) \in \Omega$,*

$$|\sin\theta_j - 1| \leq \mu \quad \text{for } j = 1, \dots, n-1 \tag{3-1}$$

and

$$|\nabla_{S^n} \log f - K e_n|_{S^n} \leq \mu, \tag{3-2}$$

where e_n is the vector field on S^n given in coordinates by $(0, \dots, 0, 1)$, and ∇_{S^n} and $|\cdot|_{S^n}$ indicate the gradient and metric on the unit sphere. Then if $w \in C_0^\infty(\Omega)$, then

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)}.$$

The inequality (3-1) is designed to force the metric on the unit sphere on the set

$$\{\theta \in S^n \mid (r, \theta) \in \Omega \text{ for some } r\}$$

to be nearly Euclidean, and the inequality (3-2) is designed to ensure that $\nabla_{S^n} \log f$ is nearly constant on Ω.

To prove this, we will need to do some work with a domain Ω_2 that is slightly larger than Ω, but still bounded. Take $\Omega_2 \subset A_O$ to be a smooth bounded domain that contains Ω such that $\Gamma_+^c \subset \partial\Omega_2$. We can pick Ω_2 to lie in $(\mathbb{R}^n \setminus B_{r_0,0}) \cap \{x_{n+1} > 0\}$, with

$$|\sin\theta_j - 1| \le 2\mu \quad \text{for } j = 1, \dots, n-1$$

and

$$|\nabla_{S^n} \log f - K e_n|_{S^n} \le 2\mu$$

for all $(r, \theta) \in \Omega_2$.

Recall that Proposition 2.1, proved in [Dos Santos Ferreira et al. 2007], says that if $w \in C_0^\infty(\Omega_2)$, then

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{H^1(\Omega_2)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{L^2(\Omega_2)}.$$

We can make a change of variables using the map $(r, \theta) \mapsto (r/f(\theta), \theta)$. This is a diffeomorphism from A_O to $\mathbb{R}^{n+1} \setminus B$, where B is the open ball of radius 1 centered at the origin, with the inverse map $(r, \theta) \mapsto (rf(\theta), \theta)$. Let $\tilde{\Omega}$ and $\tilde{\Omega}_2$ be the images of Ω and Ω_2 under this map. This diffeomorphism maps Γ_+^c to a part of the unit sphere S^n, thus "flattening" it out appropriately. This change of variables leaves the θ variables alone, so it is still the case that $\tilde{\Omega}_2$ lies in $(\mathbb{R}^{n+1} \setminus B) \cap \{x_{n+1} > 0\}$, with

$$|\sin\theta_j - 1| \le 2\mu \quad \text{for } j = 1, \dots, n-1$$

and

$$|\nabla_{S^n} \log f - K e_n|_{S^n} \le 2\mu$$

for all $(r, \theta) \in \tilde{\Omega}_2$.

Lemma 3.2. *For $w \in C_0^\infty(\tilde{\Omega}_2)$,*

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{H^1(\tilde{\Omega}_2)} \lesssim \|\tilde{\mathscr{L}}_{\varphi,\varepsilon} w\|_{L^2(\tilde{\Omega}_2)}, \tag{3-3}$$

where

$$\tilde{\mathscr{L}}_{\varphi,\varepsilon} = \big(1 + |\nabla_{S^n} \log f(\theta)|^2_{S^n}\big) h^2 \partial_r^2 - \frac{2}{r}\big(\alpha + (\nabla_{S^n} \log f(\theta)) \cdot_{S^n} h \nabla_{S^n}\big) h \partial_r + \frac{1}{r^2}(\alpha^2 + h^2 \triangle_{S^n})$$

and $\alpha = 1 + (h/\varepsilon)\log(rf(\theta))$. Here ∇_{S^n} is the gradient operator on the unit sphere; $|\cdot|_{S^n}$ and $\cdot_{S^n}$ indicate the use of the Riemannian metric on S^n, and $\triangle_{S^n}$ is the Laplace–Beltrami operator on the unit sphere S^n.

Proof. Let $v \in C_0^\infty(\Omega_2)$, and let

$$\tilde{v}(r, \theta) = v(rf(\theta), \theta).$$

Then $\tilde{v} \in C_0^\infty(\tilde{\Omega}_2)$. By a change of variables,

$$\|\tilde{v}\|_{L^2(\tilde{\Omega}_2)} \simeq \|v\|_{L^2(\Omega_2)}$$

and

$$\|\tilde{v}\|_{H^1(\tilde{\Omega}_2)} \simeq \|v\|_{H^1(\Omega_2)}.$$

The constants implied in the $\simeq$ sign depend on f.

Since $\mathcal{L}_{\varphi,\varepsilon} v \in C_0^\infty(\Omega_2)$, we have that $\widetilde{\mathcal{L}_{\varphi,\varepsilon} v} \in L^2(\tilde\Omega_2)$ and $\|\mathcal{L}_{\varphi,\varepsilon} v\|_{L^2(\Omega_2)} \simeq \|\widetilde{\mathcal{L}_{\varphi,\varepsilon} v}\|_{L^2(\tilde\Omega_2)}$. Therefore, by Proposition 2.1,

$$\frac{h}{\sqrt{\varepsilon}} \|\tilde v\|_{H^1(\tilde\Omega_2)} \lesssim \|\widetilde{\mathcal{L}_{\varphi,\varepsilon} v}\|_{L^2(\tilde\Omega_2)}.$$

Now a calculation shows that

$$\begin{aligned}\mathcal{L}_{\varphi,\varepsilon} = h^2\partial_r^2 - r^{-1}\Big(2 - hn + 2\frac{h}{\varepsilon}\log r\Big)h\partial_r + r^{-2}(1 + h^2\triangle_{S^n})\\ + r^{-2}\Big(h - hn + \frac{h^2}{\varepsilon^2}\big((\log r)^2 - \varepsilon\big) + \frac{h^2}{\varepsilon}\log r + (2 - hn)\frac{h}{\varepsilon}\log r\Big),\end{aligned}$$

and then that

$$\widetilde{\mathcal{L}_{\varphi,\varepsilon} v} = f^{-2}(\theta)\tilde{\mathcal{L}}_{\varphi,\varepsilon}\tilde v - hE\tilde v,$$

where $\tilde{\mathcal{L}}_{\varphi,\varepsilon}$ is as in the statement of the lemma and E is a first-order semiclassical differential operator with coefficients that have bounds independent of h and ε. Therefore

$$\frac{h}{\sqrt{\varepsilon}} \|\tilde v\|_{H^1(\tilde\Omega_2)} \lesssim \big\| f^{-2}(\theta)\tilde{\mathcal{L}}_{\varphi,\varepsilon}\tilde v\big\|_{L^2(\tilde\Omega_2)} + h\|\tilde v\|_{H^1(\tilde\Omega_2)}.$$

For small enough ε, the last term on the right side can be absorbed into the left side. Moreover, $|f^{-2}|$ is bounded above, so

$$\frac{h}{\sqrt{\varepsilon}} \|\tilde v\|_{H^1(\tilde\Omega_2)} \lesssim \|\tilde{\mathcal{L}}_{\varphi,\varepsilon}\tilde v\|_{L^2(\tilde\Omega_2)}$$

for all $v \in C_0^\infty(\Omega_2)$. Now any $w \in C_0^\infty(\tilde\Omega_2)$ can be written as $\tilde v$ for some $v \in C_0^\infty(\Omega_2)$ just by taking $v(r,\theta) = w(r/f(\theta),\theta)$. This finishes the proof. □

We can now make a second change of variables by thinking of the coordinate map σ as a map from $\tilde\Omega_2$ to a subset of $\mathbb{R}^{n+1}_{1+} = \{(r,\theta) \in \mathbb{R}\times\mathbb{R}^n \mid r \ge 1\}$. This gives us that for $w \in C_0^\infty(\sigma(\tilde\Omega_2))$,

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{H^1(\sigma(\tilde\Omega_2))} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w\|_{L^2(\sigma(\tilde\Omega_2))}, \tag{3-4}$$

where

$$\mathcal{L}_{\varphi,\varepsilon,\sigma} = (1 + |\gamma_f|^2)h^2\partial_r^2 - \frac{2}{r}(\alpha + \beta_f \cdot h\nabla_\theta)h\partial_r + \frac{1}{r^2}(\alpha^2 + h^2 L_{S^n}), \tag{3-5}$$

β_f is a vector field on $\mathbb{R}^{n+1}_{1+}$ that equals the coordinate expression of $\nabla_{S^n}\log f(\theta)$ on $\sigma(\tilde\Omega_2)$, γ_f is a function on $\mathbb{R}^{n+1}_{1+}$ that agrees with the coordinate expression of $|\nabla_{S^n}\log f(\theta)|_{S^n}$ on $\sigma(\tilde\Omega_2)$, and L_{S^n} is a second-order differential operator on $\mathbb{R}^{n+1}_{1+}$ that agrees with the coordinate expression of the Laplacian on the sphere on $\sigma(\tilde\Omega_2)$.

To avoid the clumsy buildup of modifiers to Ω and Ω_2, I will let U denote $\sigma(\tilde\Omega)$ and U_2 denote $\sigma(\tilde\Omega_2)$.

The hypotheses in Proposition 3.1 imply that on U_2,

$$|\beta_f - (0,\dots,0,K)| \le C_\mu \tag{3-6}$$

and

$$|\gamma_f - K| \le C_\mu \tag{3-7}$$

and if

$$h^2 L_{S^n} = a_1 h^2 \partial_{\theta_1}^2 + \cdots + a_n h^2 \partial_{\theta_n}^2 + b_1 h^2 \partial_{\theta_1} + \cdots + b_n h^2 \partial_{\theta_n},$$

then

$$|a_j - 1| \le C_\mu \tag{3-8}$$

for some constant C_μ that goes to zero if μ goes to zero. C_μ may depend on K, but we are treating K as fixed, so this will be fine. We may as well assume that β_f, γ_f, and the coefficients of L_{S^n} are extended to the rest of $\mathbb{R}^{n+1}_{1+}$ in such a way that these conditions continue to hold. In particular, this means that L_{S^n} is "close" to the ordinary Laplacian on Euclidean space.

4. Small and large frequency cases

To continue the proof of Proposition 3.1, I want to divide w into small and large frequency parts and prove an estimate for each part separately. Recall that $\mathbb{R}^{n+1}_{1+} = \{(r,\theta) \mid \theta \in \mathbb{R}^n, r \ge 1\}$. Let $\mathcal{S}(\mathbb{R}^{n+1}_{1+})$ be the restrictions to $\mathbb{R}^{n+1}_{1+}$ of Schwartz functions on $\mathbb{R}^{n+1}$. Note that functions in $C_0^\infty(U_2)$ are in $\mathcal{S}(\mathbb{R}^{n+1}_{1+})$.

Let c_1 and c_2 be such that

$$\frac{|K|^2}{1+|K|^2} < c_1 < c_2 \le \frac{1}{2} + \frac{|K|^2}{2(1+|K|^2)} < 1,$$

and let δ_1 and δ_2 be such that $\delta_2 > \delta_1 > 0$. Let $\rho \in C_0^\infty(\mathbb{R}^n)$ be a cutoff function such that $\rho(\xi) = 0$ if $|\xi|^2 > c_2$ or $|\xi_n| > \delta_2$, and $\rho(\xi) = 1$ if $|\xi|^2 \le c_1$ or $|\xi_n| \le \delta_1$.

Let the hat ˆ indicate the semiclassical Fourier transform in the θ variables only. (In general, Fourier transforms here will be in the θ variables only unless otherwise indicated.) For $w \in C_0^\infty(U)$, define w_s and w_ℓ by $\hat{w}_s = \rho(\xi)\hat{w}$ and $\hat{w}_\ell = (1-\rho(\xi))\hat{w}$, so $w = w_s + w_\ell$.

Lemma 4.1. *There exist $\mu_0 > 0$ and choices of c_1, c_2, δ_1, and δ_2 such that if* (3-6)–(3-8) *hold for some $\mu \le \mu_0$, then*

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}$$

for all $w \in C_0^\infty(U)$, where w_s is defined as above.

Lemma 4.2. *There exists $\mu_0 > 0$ such that if* (3-6)–(3-8) *hold for some $\mu \le \mu_0$, then*

$$\frac{h}{\sqrt{\varepsilon}} \|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}$$

for all $w \in C_0^\infty(U)$, where w_ℓ is defined as above.

Taken together, these two lemmas imply Proposition 3.1. To see why, first we need a lemma.

Let $m, k \ge 0$ be integers. Suppose $a(x,\xi,y)$ is a smooth function on $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}$ that satisfies the bounds

$$\left|\partial_x^\beta \partial_\xi^\alpha \partial_y^j a(x,\xi,y)\right| \le C_{\alpha,\beta}(1+|\xi|)^{m-\alpha}$$

for all multi-indices α and β, and for $0 \le j \le k$. In other words, each $\partial_y^j a(x, \xi, y)$ is a symbol on $\mathbb{R}^n$ of order m, with bounds uniform in y, for $0 \le j \le k$. Then we can define an operator A on Schwartz functions in $\mathbb{R}^{n+1}$ by applying the pseudodifferential operator on $\mathbb{R}^n$ with symbol $a(x, \xi, y)$ to $f(x, y)$ for each fixed y. More generally, we can also define operators A_j on Schwartz functions in $\mathbb{R}^{n+1}$ by applying the pseudodifferential operator on $\mathbb{R}^n$ with symbol $\partial_y^j a(x, \xi, y)$ to $f(x, y)$ for each fixed y, for $0 \le j \le k$.

Lemma 4.3. *Let A be defined as above. Then A extends to a bounded operator from $H^{k+m}(\mathbb{R}^{n+1})$ to $H^k(\mathbb{R}^{n+1})$.*

Proof. Since $k \in \mathbb{Z}$, $k \ge 0$,

$$\|Af\|^2_{H^k(\mathbb{R}^{n+1})} = \sum_{0 \le |\alpha|+j \le k} \big\|h^{|\alpha|+j} \partial_x^\alpha \partial_y^j Af\big\|^2_{L^2(\mathbb{R}^{n+1})}.$$

Now $\partial_y^j A(f)$ is a sum of terms of the form

$$A_{j_1} \partial_y^{j_2} f,$$

where $j_1 + j_2 = j \le k$. Therefore $\|Af\|^2_{H^k(\mathbb{R}^{n+1})}$ is bounded by a sum of terms of the form

$$\big\|h^{|\alpha|+j_1+j_2} \partial_x^\alpha A_{j_1} \partial_y^{j_2} f\big\|^2_{L^2(\mathbb{R}^{n+1})},$$

where $|\alpha| + j_1 + j_2 \le k$. Then

$$\begin{aligned}\big\|h^{|\alpha|+j_1+j_2} \partial_x^\alpha A_{j_1} \partial_y^{j_2} f\big\|^2_{L^2(\mathbb{R}^{n+1})} &= \int_{\mathbb{R}} \int_{\mathbb{R}^n} \big|h^{|\alpha|+j_1+j_2} \partial_x^\alpha A_{j_1} \partial_y^{j_2} f\big|^2 \, dx \, dy \\ &\le \int_{\mathbb{R}} \big\|h^{j_1+j_2} A_{j_1} \partial_y^{j_2} f\big\|^2_{H^{|\alpha|}(\mathbb{R}^n)} \, dy.\end{aligned}$$

Then by the boundedness of A_{j_1}, this is bounded above by

$$\int_{\mathbb{R}} \big\|h^{j_2} \partial_y^{j_2} f\big\|^2_{H^{|\alpha|+m}(\mathbb{R}^n)} \, dy,$$

which in turn is bounded above by

$$\big\|h^{j_2} \partial_y^{j_2} f\big\|^2_{H^{|\alpha|+m}(\mathbb{R}^{n+1})} \le \|f\|^2_{H^{|\alpha|+m+j_2}(\mathbb{R}^{n+1})} \le \|f\|^2_{H^{k+m}(\mathbb{R}^{n+1})}.$$

Therefore

$$\|Af\|^2_{H^k(\mathbb{R}^{n+1})} \lesssim \|f\|^2_{H^{k+m}(\mathbb{R}^{n+1})}.$$

Then a density argument finishes the proof. □

Proof of Proposition 3.1. Adding the estimates from Lemmas 4.1 and 4.2 gives

$$\frac{h}{\sqrt{\varepsilon}} \big(\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + \|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})}\big) \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}.$$

Since $w_s + w_\ell = w$,

$$\frac{h}{\sqrt{\varepsilon}} \|w\|_{L^2(U)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}.$$

For small enough ε, we can absorb the last term into the left side to give

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Since $(1+|\gamma_f|^2) > 1+K^2-C_\mu$, for μ small enough, we have

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \big\|(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \big\|(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w_\ell\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Now $w_s = Pw$, where P is the semiclassical pseudodifferential operator of order 0 on $\mathbb{R}^n$ with symbol $\rho(\xi)$. The operator P commutes with ∂_r, and its commutators with differential operators in the θ variables are, for each fixed $r \in [1,\infty)$, semiclassical pseudodifferential operators on $\mathbb{R}^n$ that satisfy the conditions of Lemma 4.3. Therefore

$$\begin{aligned}\big\|(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} &= \big\|(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} Pw\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}\\ &\lesssim \big\|P(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|E_0 h\partial_r + E_1 w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})},\end{aligned}$$

where E_1 and E_0, for each fixed $r \in [1,\infty)$, are semiclassical pseudodifferential operators on $\mathbb{R}^n$ of order 1 and 0 and satisfy the conditions of Lemma 4.3. There is no $hE_{-1}h^2\partial_r^2$ in the error term because the coefficient of $h^2\partial_r^2$ in $(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma}$ is just 1. Now E_1^* and E_0^* are also semiclassical pseudodifferential operators on $\mathbb{R}^n$ of order 1 and 0, for each fixed $r \in [1,\infty)$, and satisfy the conditions of Lemma 4.3.

Therefore, by Lemma 4.3, E_1^* is bounded from $H_0^1(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$, so by duality, E_1 is bounded from $L^2(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$.

Also, E_0^* is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^1(\mathbb{R}^{n+1}_{1+})$ and takes functions with trace 0 on the boundary of $\mathbb{R}^{n+1}_{1+}$ to other functions with trace 0 on the boundary of $\mathbb{R}^{n+1}_{1+}$, so by duality, E_0 is bounded from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Therefore

$$\big\|(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|P(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Now by Lemma 4.3, P is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^1(\mathbb{R}^{n+1}_{1+})$. Also, if u has trace zero on the boundary of $\mathbb{R}^{n+1}_{1+}$, then so does Pu, so P is bounded from $H_0^1(\mathbb{R}^{n+1}_{1+})$ to $H_0^1(\mathbb{R}^{n+1}_{1+})$. Since ρ is real-valued, P is also self-adjoint, so by duality, P is bounded from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Therefore

$$\big\|(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|(1+|\gamma_f|^2)^{-1}\mathscr{L}_{\varphi,\varepsilon,\sigma} w\big\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

and thus

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_s\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Similarly,

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w\|_{L^2(U)}.$$

Again the last term can be absorbed into the left side for small enough ε, so

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,\sigma} w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}$$

for each $w \in C_0^\infty(U)$.

Now if the hypotheses of Proposition 3.1 hold, then so do (3-6)–(3-8), and therefore we can obtain this conclusion. Changing variables back gives

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)}$$

for $w \in C_0^\infty(\Omega)$. □

Therefore we need only to establish proofs of Lemmas 4.1 and 4.2. To do this, we will need to introduce the analogues of the operator J described in Section 2.

5. The operators

Suppose $F : \mathbb{R}^n \to \mathbb{C}$ is a smooth function such that $\mathrm{Re}(F(\xi))$, $|F(\xi)| \simeq 1 + |\xi|$ for all $\xi \in \mathbb{R}^n$, and F is a symbol of order one on $\mathbb{R}^n$, so that

$$|\partial_\xi^\alpha F(\xi)| \le C_\alpha (1 + |\xi|)^{1-|\alpha|} \tag{5-1}$$

for all multi-indices α.

Then for $u \in \mathscr{S}(\mathbb{R}^{n+1}_{1+})$, define Ju by

$$\widehat{Ju}(r,\xi) = \left(\frac{F(\xi)}{r} + h\partial_r\right)\hat{u}(r,\xi).$$

This operator has adjoint J^* given by

$$\widehat{J^* u}(r,\xi) = \left(\frac{\overline{F(\xi)}}{r} - h\partial_r\right)\hat{u}(r,\xi).$$

These operators have right inverses defined by

$$\widehat{J^{-1}u}(r,\xi) = h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F(\xi)/h} dt$$

and

$$\widehat{J^{*-1}u}(r,\xi) = h^{-1}\int_r^\infty \hat{u}(t,\xi)\left(\frac{r}{t}\right)^{\overline{F(\xi)}/h} dt.$$

Each of these is well defined as an operator on $\mathscr{S}(\mathbb{R}^{n+1}_{1+})$. We will prove appropriate analogues of the properties (1)–(4) and (5′) from Section 2 for J of this form. Note that J^{-1} is a right inverse, and both J and J^{-1} preserve support in the positive r direction. Therefore it remains to establish analogues of properties (3), (4), and (5′).

To set up the analogue of property (3), define the weighted Sobolev space $H^1_r(\mathbb{R}^{n+1}_{1+})$ by the norm

$$\|u\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})} = \left\|\frac{u}{r}\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + \|h\partial_r u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + \left\|\frac{h}{r}\nabla_\theta u\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Since U_2 lies in the set $1 \le r \le R_0$ for some R_0 depending on U_2, we know H^1 and H^1_r norms are comparable for functions supported on U_2, with constants of comparability depending only on R_0. This holds more generally for any functions supported in $1 \le r \le R_0$.

Now the operators above have the following boundedness properties.

Lemma 5.1. *J, J^*, J^{-1}, and J^{*-1} extend as bounded maps*

$$J,\, J^* : H^1_r(\mathbb{R}^{n+1}_{1+}) \to L^2(\mathbb{R}^{n+1}_{1+})$$

and

$$J^{-1},\, J^{*-1} : L^2(\mathbb{R}^{n+1}_{1+}) \to H^1_r(\mathbb{R}^{n+1}_{1+}).$$

Moreover, the extensions of J^* and J^{*-1} are isomorphisms.

Proof. Consider J first. If $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then

$$\|Ju\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} = h^{-n}\|\widehat{Ju}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} = h^{-n}\left\|\frac{F(\xi)}{r}\hat{u} + h\partial_r \hat{u}\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|_{H^1_r(\mathbb{R}^{n+1}_{1+})}.$$

By a density argument, J extends to a bounded map $J : H^1_r(\mathbb{R}^{n+1}_{1+}) \to L^2(\mathbb{R}^{n+1}_{1+})$. The proof for J^* is similar.

Now consider J^{-1}. If $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then

$$\begin{aligned}\int_1^\infty \left|\frac{1}{r}\widehat{J^{-1}u}\right|^2 dr &= \int_1^\infty \left|h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F(\xi)/h} dt\right|^2 r^{-2}\,dr \\ &\le \int_1^\infty \left|h^{-1}\int_0^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F(\xi)/h} dt\right|^2 r^{-2}\,dr.\end{aligned}$$

By a change of variables, we get

$$\int_1^\infty \left|\frac{1}{r}\widehat{J^{-1}u}\right|^2 dr = \int_1^\infty \left|h^{-1}\int_0^1 \hat{u}(rt,\xi)t^{F(\xi)/h}\,dt\right|^2 dr.$$

Then using Minkowski's inequality and changing variables again, we get

$$\begin{aligned}\int_1^\infty \left|\frac{1}{r}\widehat{J^{-1}u}\right|^2 dr &\le h^{-2}\left(\int_0^1\left(\int_1^\infty |\hat{u}(r,\xi)|^2\,dr\right)^{1/2} t^{\mathrm{Re}(F(\xi)/h)}t^{-1/2}dt\right)^2 \\ &= h^{-2}\int_1^\infty |\hat{u}(r,\xi)|^2\,dr\left(\frac{h}{\mathrm{Re}\,F(\xi)+h/2}\right)^2 \simeq \int_1^\infty \left|\frac{\hat{u}(r,\xi)}{1+|\xi|}\right|^2 dr.\end{aligned}$$

Therefore

$$\left\|\frac{1}{r}\widehat{J^{-1}u}\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \int_{\mathbb{R}^n}\int_1^\infty \left|\frac{\hat{u}(r,\xi)}{1+|\xi|}\right|^2 dr\,d\xi \lesssim \|\hat{u}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Similarly,

$$\int_1^\infty \left|\frac{\xi}{r}\widehat{J^{-1}u}\right|^2 dr \lesssim \int_1^\infty |\hat{u}(r,\xi)|^2\, dr,$$

so

$$\left\|\frac{h}{r}\nabla_\theta J^{-1}u\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Finally,

$$h\partial_r \widehat{J^{-1}u} = -\left(\frac{F(\xi)}{r}\right)\widehat{J^{-1}u} + \hat{u},$$

so

$$\int_1^\infty \left|h\partial_r \widehat{J^{-1}u}\right|^2 dr \lesssim \int_1^\infty |\hat{u}(r,\xi)|^2\, dr$$

and

$$\|h\partial_r J^{-1}u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})},$$

by the same logic.

Putting all of this together gives

$$\|J^{-1}u\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})} \lesssim \|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})},$$

for $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. Then a density argument shows that J^{-1} extends to a bounded map

$$J^{-1} : L^2(\mathbb{R}^{n+1}_{1+}) \to H^1_r(\mathbb{R}^{n+1}_{1+}).$$

Again, the proof for J^{*-1} is similar.

It remains to show that the extensions of J^* and J^{*-1} are isomorphisms. If $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then

$$J^* J^{*-1} u = u$$

and (using integration by parts)

$$J^{*-1} J^* u = u.$$

Then the result follows from a density argument.

Note that $J^{-1}Ju \neq u$ in general, because integration by parts will pick up a boundary term at $r = 1$. Therefore the extensions of J and J^{-1} are not isomorphisms. □

Let $H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$ denote the subspace of $H^1_r(\mathbb{R}^{n+1}_{1+})$ consisting of functions with trace zero on the hyperplane $r = 1$, and let $H^{-1}_r(\mathbb{R}^{n+1}_{1+})$ denote the dual space to $H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$.

Now we need to prove some commutator properties for J.

Lemma 5.2. *Suppose that $w \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, $\chi \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$ and that Q is a second-order semiclassical differential operator with smooth bounded coefficients on $\mathbb{R}^{n+1}_{1+}$. Then*

$$\|J\chi J^{-1}w\|_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|\chi w\|_{L^2(\mathbb{R}^{n+1}_{1+})} - h\|rw\|_{L^2(\mathbb{R}^{n+1}_{1+})}$$

and

$$\|(JQ - QJ)w\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim h\|rw\|_{H^1(\mathbb{R}^{n+1}_{1+})}.$$

The constants in the $\gtrsim$ and $\lesssim$ signs will depend on the derivatives of F.

Proof. Consider the first statement. If T is the operator on $\mathbb{R}^n$ with symbol $F(\xi)$, interpreted as acting on functions on $\mathbb{R}^{n+1}_{1+}$ by action on the θ variables only, then

$$\|J\chi J^{-1}w\|_{L^2(\mathbb{R}^{n+1}_{1+})} = \left\|\left(h\partial_r + \frac{T}{r}\right)\chi J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \geq \left\|\chi\left(h\partial_r + \frac{T}{r}\right)J^{-1}w\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} - \|hE_0 J^{-1}w\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

where for each fixed r, E_0 is an order-zero pseudodifferential operator on $\mathbb{R}^n$ with bounds that are uniform in r. Therefore, by Lemma 4.3, E_0 is bounded from L^2 to L^2, so

$$\|J\chi J^{-1}w\|_{L^2(\mathbb{R}^{n+1}_{1+})} \geq \|\chi JJ^{-1}w\|_{L^2(\mathbb{R}^{n+1}_{1+})} - h\|J^{-1}w\|_{L^2(\mathbb{R}^{n+1}_{1+})} \geq \|\chi w\|_{L^2(\mathbb{R}^{n+1}_{1+})} - h\|rw\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

The proof of the second statement is similar, but somewhat more involved. First, note that multiplication by $1/r$ is a bounded operator from $H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$ to $H^1_0(\mathbb{R}^{n+1}_{1+})$. Therefore, by duality, it is a bounded operator from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H_r^{-1}(\mathbb{R}^{n+1}_{1+})$, and so

$$\|(J_sQ - QJ_s)w\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \|r(J_sQ - QJ_s)w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Note that $J_s = h\partial_r + r^{-1}T$, where T is a semiclassical pseudodifferential operator on $\mathbb{R}^n$ of order 1. Meanwhile, Q can be written as a combination of ∂_r derivatives and differential operators on $\mathbb{R}^n$:

$$Q = Ah^2\partial_r^2 + Bh\partial_r + C,$$

where A, B, and C are (perhaps r-dependent) differential operators of orders 0, 1, and 2 respectively on $\mathbb{R}^n$ for each fixed r, with bounds uniform in r.

If $w \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, then $Qw \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. Then

$$\|r(J_sQ - QJ_s)w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} = \|r[h\partial_r + r^{-1}T, Ah^2\partial_r^2 + Bh\partial_r + C]w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.$$

Expanding this, and noting that T commutes with ∂_r, we get

$$\begin{aligned}\|r(J_sQ - QJ_s)w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \leq \|r[h\partial_r, Q]w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|[T, A]h^2\partial_r^2 w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}\\ + \|hr^{-1}[T, A]h\partial_r w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|2h^2r^{-2}[T, A]w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}\\ + \|[T, B]h\partial_r w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|r^{-1}[T, B]w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|[T, C]w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.\end{aligned}$$

By the product rule, $r[h\partial_r, Q] = hrE_2' = hE_2'r + h^2E_1'$, where E_2' and E_1' are second- and first-order semiclassical differential operators. Meanwhile, $[T, A] = hE_0$, $[T, B] = hE_1$, and $[T, C] = hE_2$, where

E_0, E_1, and E_2 are semiclassical pseudodifferential operators on $\mathbb{R}^n$ of orders 0, 1, and 2. Therefore

$$\begin{aligned}\|r(J_sQ-QJ_s)w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \le \|hE_2'rw\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|h^2E_1'w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|hE_0h^2\partial_r^2 w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \\ + \|h^2r^{-1}E_0h\partial_r w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|2h^3r^{-2}E_0w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} \\ + \|hE_1h\partial_r w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|hr^{-1}E_1w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + \|hE_2w\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})}.\end{aligned}$$

E_2' is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$, and E_1' is bounded from $L^2(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. In addition, by Lemma 4.3, E_1^* is bounded from $H_0^1(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$, so by duality, E_1 is bounded from $L^2(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Meanwhile, E_2 is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Finally, E_0^* is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $H^1(\mathbb{R}^{n+1}_{1+})$ and maps functions with trace 0 on the boundary of $\mathbb{R}^{n+1}_{1+}$ to other functions with trace 0 on that boundary, so it is bounded from $H_0^1(\mathbb{R}^{n+1}_{1+})$ to $H_0^1(\mathbb{R}^{n+1}_{1+})$. Therefore, by duality, E_0 is bounded from $H^{-1}(\mathbb{R}^{n+1}_{1+})$ to $H^{-1}(\mathbb{R}^{n+1}_{1+})$. Moreover, $1/r \le 1$ on $\mathbb{R}^{n+1}_{1+}$. Applying all of these facts together to the last inequality then finishes the proof. □

To finish this section, we need to prove a property analogous to (5′) from Section 2.

Lemma 5.3. *Suppose $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. If g is defined by*

$$\hat{g}(r,\xi) = \frac{2\operatorname{Re} F(\xi) - h}{h}\int_1^\infty \hat{u}(t,\xi) r^{-F(\xi)/h} t^{-\overline{F(\xi)}/h}\,dt,$$

then

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \simeq \|u-g\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Proof. Suppose $u \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$. Define g as above. A calculation shows that $g \in L^2(\mathbb{R}^{n+1}_{1+})$, and

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \le \|u\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Note that

$$\widehat{Jg} = \left(\frac{F(\xi)}{r} + h\partial_r\right)\hat{g} = 0.$$

Therefore

$$\begin{aligned}\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} &= \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, w\neq 0} \frac{|(Ju,w)|}{\|w\|_{H_r^1(\mathbb{R}^{n+1}_{1+})}} \\ &= \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, w\neq 0} \frac{|(J(u-g),w)|}{\|w\|_{H_r^1(\mathbb{R}^{n+1}_{1+})}} \\ &= \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, w\neq 0} \frac{|(u-g,J^*w)|}{\|w\|_{H_r^1(\mathbb{R}^{n+1}_{1+})}}.\end{aligned}$$

Since $J^* : H_r^1(\mathbb{R}^{n+1}_{1+}) \to L^2(\mathbb{R}^{n+1}_{1+})$ is an isomorphism,

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \simeq \sup_{w\in H^1_{0,r}(\mathbb{R}^{n+1}_{1+}),\, J^*w\neq 0} \frac{|(u-g,J^*w)|}{\|J^*w\|_{L^2(\mathbb{R}^{n+1}_{1+})}}. \tag{5-2}$$

Now $J^* w \in L^2(\mathbb{R}^{n+1}_{1+})$, so

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \lesssim \|u-g\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

On the other hand, $u-g = J^* J^{*-1}(u-g)$. Also $J^{*-1}(u-g) \in H_r^1(\mathbb{R}^{n+1}_{1+})$, and by definition of g, $J^{*-1}(u-g)(x,0) = 0$. Therefore $J^{*-1}(u-g) \in H^1_{r,0}(\mathbb{R}^{n+1}_{1+})$. Then if $u-g=0$, the lemma is true by (5-2). Otherwise, we can pick $w = J^{*-1}(u-g)$ in (5-2) to show that

$$\|Ju\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} \gtrsim \|u-g\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

This finishes the proof. □

6. The small frequency case

To prove Lemma 4.1, we need to define an operator of the form given in Section 5.

Consider the function $\Phi : \mathbb{R}^n \to \mathbb{C}$ given by

$$\overline{\Phi(\xi)} = \frac{1}{1+|K|^2}\Big(1 + iK\xi_n + \sqrt{2iK\xi_n - (K\xi_n)^2 + (1+|K|^2)|\xi|^2 - |K|^2}\Big),$$

where the square root is taken to mean the branch of the square root function with nonnegative imaginary part. We would like to use this function in place of F in Section 5 to define J and the related operators of that section. Unfortunately, Φ is not smooth. However, we can try to construct a function F_s that approximates Φ on the support of $\hat{w}_s$ and has the properties of F from Section 5. To do this, first notice that if c_2 and δ_2 are chosen small enough, then this is nearly continuous on the support of $\hat{w}_s$, or equivalently, on the support of ρ. To be more precise, Φ is smooth except where

$$\tau_K(\xi) = 2iK\xi_n - (K\xi_n)^2 + (1+|K|^2)|\xi|^2 - |K|^2$$

lies on the nonnegative real axis, where this branch of the square root has its branch cut. This occurs when $\xi_n = 0$ and $|\xi|^2 \geq |K|^2/(1+|K|^2)$, and gives a jump discontinuity of size $2\sqrt{(1+|K|^2)|\xi|^2 - |K|^2}$. However, $|\xi|^2 \leq c_2$ on the support of ρ, so for c_2 close to $|K|^2/(1+|K|^2)$, the maximum possible size of the jump discontinuity is small.

Therefore, for any $\delta > 0$ we can define $F_s(\xi)$ on the support of ρ such that

$$|F_s(\xi) - \Phi(\xi)| \leq \delta$$

on the support of ρ, by choosing c_2 small enough. The derivatives of F_s inside the support of ρ may depend on c_1, c_2, δ_1, and δ_2. Since the choice of these in turn depends on δ, the derivatives of F_s are bounded by a quantity that depends on δ.

Now consider the necessary bounds on F_s. On the support of ρ, the imaginary part of τ_K must lie in the interval $[-2K\delta_2, 2K\delta_2]$. The real part of τ_K is given by

$$-(K\xi_n)^2 - |K|^2 + (1+|K|^2)|\xi|^2.$$

We have that $|\xi|^2 \leq c_2$ on the support of ρ. We can choose c_2 so close to $\dfrac{K^2}{1+K^2}$ that

$$(1+K^2)r_2 - K^2 \leq \delta_2.$$

Then the real part of τ_K is bounded above by δ_2 on the support of ρ. Therefore, on the support of ρ, $(\mathrm{Re}(\tau_K), \mathrm{Im}(\tau_K)) \in (-\infty, \delta_2] \times [-2K\delta_2, 2K\delta_2]$, and so by taking δ_2 small enough, we can ensure that the real part of $\sqrt{\tau_K}$ has absolute value less than $\frac{1}{3}$ on the support of ρ.

Therefore, if δ is small enough, $\mathrm{Re}(F_s), |F_s| > 1/(2+2K^2)$ on the support of ρ. We can now define F_s smoothly outside the support of ρ so that $\mathrm{Re}(F_s), |F_s| \geq 1/(2+2K^2)$ for all ξ, and $F_s = (1+|\xi|^2)^{1/2}$ for $|\xi| > 2$, say. Then F_s is smooth, $\mathrm{Re}(F(\xi)), |F(\xi)| \simeq 1+|\xi|$ for all $\xi \in \mathbb{R}^n$, and the conditions (5-1) are satisfied automatically for $|\xi| > 2$, and hence for all ξ.

Therefore F_s satisfies all the conditions given in Section 5, and the operators defined by

$$\widehat{J_s u}(r,\xi) = \left(\frac{F_s(\xi)}{r} + h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_s^* u}(r,\xi) = \left(\frac{\overline{F_s(\xi)}}{r} - h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_s^{-1} u}(r,\xi) = h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F_s(\xi)/h} dt,$$

and

$$\widehat{J_s^{*-1} u}(r,\xi) = h^{-1}\int_r^\infty \hat{u}(t,\xi)\left(\frac{r}{t}\right)^{\overline{F_s(\xi)}/h} dt$$

satisfy all the properties from that section.

Now we are ready to begin the proof of the small frequency case. Suppose $\chi_2(r,\theta) \in C^\infty(\mathbb{R}^{n+1}_{1+})$ is a cutoff function that is 1 on U and has support inside U_2.

If $w \in C_0^\infty(U)$, then $w_s \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$, supported away from $r = 1$. Therefore $J_s^{-1} w_s \in \mathcal{S}(\mathbb{R}^{n+1}_{1+})$ is supported away from $r = 1$. Then $\chi_2 J_s^{-1} w_s$ is in $C_0^\infty(U_2)$. Therefore, by (3-4),

$$\frac{h}{\sqrt{\varepsilon}}\left\|\chi_2 J_s^{-1} w_s\right\|_{H^1(U_2)} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(U_2)}.$$

Since $\chi_2 J_s^{-1} w_s \in C_0^\infty(U_2)$, the H^1 and H^1_r norms are comparable, so

$$\frac{h}{\sqrt{\varepsilon}}\left\|\chi_2 J_s^{-1} w_s\right\|_{H^1_r(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Using the boundedness properties from Lemma 5.1,

$$\frac{h}{\sqrt{\varepsilon}}\left\|J_s\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

so applying the first part of Lemma 5.2,

$$\frac{h}{\sqrt{\varepsilon}}\|\chi_2 w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + C_\delta\frac{h^2}{\varepsilon}\|r w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

The C_δ factor written in front of the last term is to indicate that the constant in the $\lesssim$ sign depends on the derivatives of F_s, and hence on δ. This is fine, because δ is chosen independently of h and ε, but this will help track the δ dependence. Now $\chi_2 w_s = \chi_2 P w$. Since w is only supported on the region where χ_2 is identically one,

$$\chi_2 w_s = P w + O(h^\infty) E w = w_s + O(h^\infty) E w,$$

where E is a pseudodifferential operator of order 0 (actually a smoothing operator) on $\mathbb{R}^n$. Therefore

$$\frac{h}{\sqrt{\varepsilon}} \|\chi_2 w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} - O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

and so

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \mathscr{L}_{\varphi,\varepsilon,\sigma} \chi_2 J_s^{-1} w_s \right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + C_\delta \frac{h^2}{\varepsilon} \|r w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

For small enough h, the second last term can be absorbed into the left side (r is bounded on the support of w_s) to give

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \mathscr{L}_{\varphi,\varepsilon,\sigma} \chi_2 J_s^{-1} w_s \right\|_{L^2(U_2))} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

By the product rule, $\mathscr{L}_{\varphi,\varepsilon,\sigma} \chi_2 - \mathscr{L}_{\varphi,\varepsilon,\sigma} \chi_2$ is a first-order semiclassical differential operator, and thus it is bounded from $H^1(U_2)$ to $L^2(U_2)$. Therefore

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \chi_2 \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s \right\|_{L^2(U_2)} + h \left\| J_s^{-1} w_s \right\|_{H^1(U_2)} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

On U_2, the H^1 and H^1_r norms are comparable, so

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s \right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + h \left\| J_s^{-1} w_s \right\|_{H^1_r(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Using the boundedness properties again,

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s \right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + h \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

The second last term can be absorbed into the left side to give

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s \right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}. \tag{6-1}$$

I want to combine this last inequality with Lemma 5.3 to get

$$\frac{h}{\sqrt{\varepsilon}} \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| J_s \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s \right\|_{H^{-1}_r(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

To do this, I need to show that if $u = \mathscr{L}_{\varphi,\varepsilon,\sigma} J_s^{-1} w_s$, then the function g defined in Lemma 5.3 satisfies a bound like

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \leq \tfrac{1}{2} \|u\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

by using an integration by parts argument like the one described in Section 2.

Let $v = J_s^{-1} w_s$. Then

$$\hat{g} = \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \widehat{\mathscr{L}_{\varphi,\varepsilon,\sigma} v}(t,\xi) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt.$$

Writing out $\mathscr{L}_{\varphi,\varepsilon,\sigma}$ as in (3-5), we can consider the integral for each term of $\mathscr{L}_{\varphi,\varepsilon,\sigma}$ separately. For this equation the hat notation for the Fourier transform will become a little impractical, so let $\mathscr{F}(v) = \hat{v}$. Then

$$\begin{aligned}\hat{g} = \frac{2\operatorname{Re} F_s - h}{h} &\int_1^\infty \mathscr{F}\big((1+|\gamma_f|^2)h^2\partial_t^2 v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &- \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{2}{t}\mathscr{F}\big((\alpha + \beta_f \cdot h\nabla_\theta) h\partial_t v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &+ \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{1}{t^2}\mathscr{F}\big((\alpha^2 + h^2 L_{S^n}) v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt.\end{aligned}$$

We can use the assumptions on β_f, γ_f, and L_{S^n} in equations (3-6), (3-7), and (3-8), together with the fact that $|1-\alpha| \lesssim h\varepsilon^{-1}$, to write this as

$$\begin{aligned}\hat{g} = \frac{2\operatorname{Re} F_s - h}{h} &\int_1^\infty \mathscr{F}\big((1+|K|^2)h^2\partial_t^2 v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &- \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{2}{t}\mathscr{F}\big((1 + K \cdot h\nabla_\theta) h\partial_t v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &+ \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{1}{t^2}\mathscr{F}\big((1 + h^2 \triangle_\theta) v\big) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &+ C_\mu \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \mathscr{F}(E_2 v) r^{-F_s/h} t^{-\overline{F_s}/h}\, dt, \qquad (6\text{-}2)\end{aligned}$$

where E_2 is a second-order semiclassical differential operator with bounds uniform in μ. Now we can integrate by parts to remove the $h\partial_t$'s.

In the first term, this gives us

$$\begin{aligned}\frac{2\operatorname{Re} F_s - h}{h} &\int_1^\infty (1+|K|^2)h^2\partial_t^2 \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &= \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \frac{\overline{F_s}}{t}(1+K^2) h\partial_t \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &= \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty \left(\frac{\overline{F_s}}{t}\right)^2 (1+K^2) \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt \\ &\quad + \frac{2\operatorname{Re} F_s - h}{h} \int_1^\infty h\frac{\overline{F_s}}{t^2}(1+K^2) \hat{v} r^{-F_s/h} t^{-\overline{F_s}/h}\, dt.\end{aligned}$$

There are no boundary terms from the integration by parts because w is supported away from $r = 1$, and hence w_s and v are as well. The last term can be absorbed into the last term of (6-2). In the second term, we get

$$\begin{aligned}\frac{2\operatorname{Re}F_s-h}{h}&\int_1^\infty \frac{2}{t}(1+iK\xi_n)h\partial_t\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt\\ &=\frac{2\operatorname{Re}F_s-h}{h}\int_1^\infty \frac{2\overline{F_s}}{t^2}(1+iK\xi_n)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt\\ &\qquad+\frac{2\operatorname{Re}F_s-h}{h}\int_1^\infty \frac{2h}{t^2}(1+iK\xi_n)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt.\end{aligned}$$

Again the last term can be absorbed into the last term of (6-2). Therefore, returning to (6-2), we have

$$\begin{aligned}\hat{g}=\frac{2\operatorname{Re}F_s-h}{h}&\int_1^\infty \left(\frac{\overline{F_s}}{t}\right)^2(1+K^2)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt\\ &-\frac{2\operatorname{Re}F_s-h}{h}\int_1^\infty \frac{2\overline{F_s}}{t^2}(1+iK\xi_n)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt\\ &+\frac{2\operatorname{Re}F_s-h}{h}\int_1^\infty \frac{1}{t^2}(1-|\xi|^2)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt\\ &\qquad+C_\mu\frac{2\operatorname{Re}F_s-h}{h}\int_1^\infty \widehat{E_2v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt.\end{aligned}$$

Now $F_s(\xi)$ is designed so that $\overline{F_s(\xi)}$ is very nearly a solution to $(1+K^2)X^2-2(1+iK\xi_n)X+1-|\xi|^2=0$ when $\hat{w}_s\neq 0$, and hence when $\hat{v}\neq 0$. More precisely,

$$\left|(1+K^2)\overline{F_s(\xi)}^2-2(1+iK\xi_n)\overline{F_s(\xi)}+1-|\xi|^2\right|\lesssim\delta\big(|F_s(\xi)|+|\xi_n|\big)\lesssim\delta|F_s(\xi)|.$$

That means that we can write $\hat{g}$ as

$$\hat{g}=\delta\frac{2\operatorname{Re}F_s-h}{h}\int_1^\infty R(\xi)\hat{v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt+C_\mu\frac{2\operatorname{Re}F_s-h}{h}\int_1^\infty \widehat{E_2v}r^{-F_s/h}t^{-\overline{F_s}/h}\,dt,$$

where $|R(\xi)|\lesssim|F_s(\xi)|\lesssim 1+|\xi|$. Now it follows, as in the proof of Lemma 5.3, that

$$\|\hat{g}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}\lesssim\delta^2\|R(\xi)\hat{v}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}+C_\mu^2\|\widehat{E_2v}\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}\lesssim(\delta+C_\mu)\|v\|_{H^2(\mathbb{R}^{n+1}_{1+})}. \tag{6-3}$$

This gives an estimate for g in terms of v. However, we want the estimate to be in terms of u. We have $u=\mathscr{L}_{\varphi,\varepsilon,\sigma}v$, so

$$\|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}=\|\mathscr{L}_{\varphi,\varepsilon,\sigma}v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}$$

and

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma}v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}\gtrsim\left\|\left((1+K^2)h^2\partial_r^2-\frac{2}{r}(1+Kh\partial_{\theta_n})h\partial_r+\frac{1}{r^2}(1+h^2\triangle_\theta)\right)v\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}-C_\mu^2\|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}.$$

Rewriting in terms of $\hat{v}$, we get

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim h^{-n}\left\|\left((1+K^2)h^2\partial_r^2 - \frac{2}{r}(1+iK\xi_n)h\partial_r + \frac{1}{r^2}(1-|\xi|^2)\right)\hat{v}(r,\xi)\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - C_\mu^2\|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}.$$

Now $\hat{v}(r,\xi) = \mathscr{F}(J_s^{-1}Pw)(r,\xi)$ is only nonzero for ξ such that

$$|\xi|^2 \le \frac{1}{2} + \frac{1}{2}\frac{|K|^2}{1+|K|^2} < 1.$$

The operator

$$(1+K^2)h^2\partial_r^2 - \frac{2}{r}(1+iK\xi_n)h\partial_r + \frac{1}{r^2}(1-|\xi|^2)$$

coincides, for $r > 1$, with a differential operator in r of the form

$$(1+K^2)h^2\partial_r^2 - 2\omega(1+iK\xi_n)h\partial_r + \omega^2(1-|\xi|^2),$$

where ω is a smooth function that coincides with $1/r$ for $r > 1$. This is second-order elliptic for each $|\xi|$ such that $\hat{v}(r,\xi)$ is nonzero, and its symbol (in r) is bounded below; therefore

$$h^{-n}\left\|\left((1+K^2)h^2\partial_r^2 - \frac{2}{r}(1+iK\xi_n)h\partial_r + \frac{1}{r^2}(1-|\xi|^2)\right)\hat{v}(r,\xi)\right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \simeq \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})} - C_\mu^2\|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})},$$

and so

$$\|u\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} = \|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})} - C_\mu^2\|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})} \gtrsim \|v\|^2_{H^2(\mathbb{R}^{n+1}_{1+})}$$

for μ small enough.

Substituting this into (6-3) gives

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim (\delta + C_\mu)\|u\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Taking μ and δ small enough means

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \le \tfrac{1}{2}\|u\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Combining this with (6-1) now gives

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|J_s\mathscr{L}_{\varphi,\varepsilon,\sigma}J_s^{-1}w_s\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Then using the second part of Lemma 5.2, we get

$$\begin{aligned}\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} &\lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma} J_s J_s^{-1} w_s\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + C_\delta h \left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})} \\ &\lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + C_\delta h \left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.\end{aligned}$$

Again the C_δ factor is written to track the δ dependence, but again this is fine. $\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s$ is supported in the r direction only for those r that can come from $\tilde{\Omega}_2$, since w_s is. Therefore the H_r^{-1} and H^{-1} norms are comparable, and so

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + hC_\delta \left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}. \tag{6-4}$$

Meanwhile,

$$\widehat{J_s^{-1} w_s}(r,\xi) = \frac{1}{h}\int_1^r \hat{w}_s(t,\xi)\left(\frac{t}{r}\right)^{F_s(\xi)/h} dt,$$

and $\hat{w}_s(t,\xi)$ is supported only for $1 \le t \le C$ for some C depending on $\sigma(\tilde{\Omega}_2)$. Therefore, for $r > 4C$,

$$\left|\widehat{J_s^{-1} w_s}(r,\xi)\right| \le \left|\frac{1}{h}\int_1^C \hat{w}_s(t,\xi)\left(\frac{t}{2C}\right)^{F_s/h} dt\right| \left|\frac{1}{2}\right|^{\mathrm{Re}(F_s/h)} \left|\frac{4C}{r}\right|^{\mathrm{Re}(F_s/h)},$$

so

$$\left|\widehat{J_s^{-1} w_s}(r,\xi)\right|^2 \lesssim \int_1^C |\hat{w}(t,\xi)|^2\, dt \left|\frac{1}{2}\right|^{\mathrm{Re}(2F_s/h)}.$$

Therefore

$$\left\|r J_s^{-1} w_s\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|r J_s^{-1} w_s\right\|_{L^2(1<r<4C)} + O(h^\infty)\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Similar calculations for derivatives of $J_s^{-1} w$ give

$$\left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|r J_s^{-1} w_s\right\|_{H^1(1<r<4C)} + O(h^\infty)\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

so

$$\left\|r J_s^{-1} w_s\right\|_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|J_s^{-1} w_s\right\|_{H_r^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Returning to (6-4), we get

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + hC_\delta \left\|J_s^{-1} w_s\right\|_{H_r^1(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Applying the boundedness result for J^{-1} gives

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + hC_\delta \|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

For small enough ε, the second last term can be absorbed into the left side to give

$$\frac{h}{\sqrt{\varepsilon}}\|w_s\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_s\right\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

This finishes the proof of Lemma 4.1.

7. The large frequency case

Now we turn to the large frequency case. We will need to define a new operator J_ℓ.

Consider again the function $\Phi : \mathbb{R}^n \to \mathbb{C}$ given by

$$\overline{\Phi(\xi)} = \frac{1}{1+K^2}\Big(1 + iK\xi_n + \sqrt{2iK\xi_n - (K\xi_n)^2 + (1+K^2)|\xi|^2 - |K|^2}\Big),$$

but this time take the branch of the square root that has nonnegative real part. Now Φ is smooth except where

$$\tau_K(\xi) = 2iK\xi_n - (K\xi_n)^2 + (1+|K|^2)|\xi|^2 - |K|^2$$

lies on the nonpositive real axis. This happens when $\xi_n = 0$ and

$$|\xi|^2 \leq \frac{|K|^2}{1+|K|^2}.$$

Therefore, on the support of $1-\rho(\xi)$, Φ is smooth. Since the real part of the square root is nonnegative, both $|\Phi|$ and the real part of Φ are bounded below by $1/(1+K^2)$. Therefore we can pick a smooth function F_ℓ such that $F_\ell(\xi) = \Phi(\xi)$ on the support of $1-\rho(\xi)$ and

$$\operatorname{Re} F_\ell(\xi),\ |F_\ell(\xi)| \geq \frac{1}{1+K^2}.$$

In fact, if $\dfrac{K^2}{1+K^2} < c_0 < c_1$ and $0 < \delta_0 < \delta_1$, we can still pick F_ℓ to be equal to Φ on the set

$$\{\xi \in \mathbb{R}^n \mid |\xi|^2 \geq c_0 \text{ or } \xi_n \geq \delta_0\},$$

with F_ℓ smooth and $\operatorname{Re} F_\ell(\xi), |F_\ell(\xi)| \geq (1+K^2)^{-1}$. Now for large $|\xi|$,

$$\operatorname{Re} \Phi(\xi),\ |\Phi_\ell(\xi)| \geq \frac{1}{1+K^2}(1+|\xi|),$$

so F_ℓ then satisfies these inequalities for all ξ. Finally, for large $|\xi|$, Φ is smooth and satisfies the inequalities (5-1), so it follows that F_ℓ satisfies those inequalities for all ξ. Thus F_ℓ satisfies all of the conditions at the beginning of Section 5, and therefore the operators defined by

$$\widehat{J_\ell u}(r,\xi) = \left(\frac{F_\ell(\xi)}{r} + h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_\ell^* u}(r,\xi) = \left(\frac{\overline{F_\ell(\xi)}}{r} - h\partial_r\right)\hat{u}(r,\xi),$$

$$\widehat{J_\ell^{-1} u}(r,\xi) = h^{-1}\int_1^r \hat{u}(t,\xi)\left(\frac{t}{r}\right)^{F_\ell(\xi)/h} dt,$$

and

$$\widehat{J_\ell^{*-1} u}(r,\xi) = h^{-1}\int_r^\infty \hat{u}(t,\xi)\left(\frac{r}{t}\right)^{\overline{F_\ell(\xi)}/h} dt$$

satisfy all of the properties from that section.

Consider the Carleman estimate (3-4). By a similar argument as in the small frequency case, we get

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|\mathcal{L}_{\varphi,\varepsilon,\sigma} J_\ell^{-1} w_\ell\right\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}. \tag{7-1}$$

Again, I want to combine this last inequality with Lemma 5.3 to get

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\|J_\ell\mathcal{L}_{\varphi,\varepsilon,\sigma} J_\ell^{-1} w_\ell\right\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

To do this, I need to show that if u is of the form $u=\mathcal{L}_{\varphi,\varepsilon,\sigma} J_\ell^{-1} w_\ell$, then the function g defined in Lemma 5.3 satisfies a bound like

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \le \tfrac{1}{2}\|u\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h)\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})},$$

by an appropriate integration by parts argument. The approach used in the small frequency case does not work here, because $\mathcal{L}_{\varphi,\varepsilon,\sigma}$ is not at all elliptic on the support of $\hat{w}_\ell$. However, now $\mathcal{L}_{\varphi,\varepsilon,\sigma}$ can be factored into a composition of two operators, one of which has the desired properties.

Let $\zeta(\xi)$ be a smooth cutoff function that is identically one on the set where $|\xi|^2 \ge c_1$ or $|\xi_n| \ge \delta_1$, and vanishes if $|\xi|^2 \le c_0$ or $|\xi_n| \le \delta_0$. Let

$$G_s = (1-\zeta(\xi))F_\ell(\xi),$$

and consider the symbols

$$G_\pm = \zeta(\xi)\frac{\alpha + i\beta_f\cdot\xi \pm \sqrt{(\alpha+i\beta_f\cdot\xi)^2 - (1+(\gamma_f)^2)(\alpha^2 - L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2} + G_s(\xi),$$

where $L_{S^n}(\theta,\xi)$ represents the symbol of the differential operator L_{S^n}. The square root represents the branch of the square root with nonnegative real part. The argument of the square root lies on the nonpositive real axis only when $\beta_f\cdot\xi = 0$ and

$$L_{S^n}(\theta,\xi) \le \frac{\alpha^2|\gamma_f|^2}{1+|\gamma_f|^2}.$$

Now

$$L_{S^n}(\theta,\xi) = a_1(\theta)\xi_1^2 + \cdots + a_n(\theta)h^2\xi_n + hb_1(\theta)\xi_1 + \cdots + hb_n(\theta)h\xi_n,$$

where by the hypotheses in Proposition 3.1,

$$|a_j - 1| \le C_\mu$$

for C_μ that goes to zero as μ goes to zero. Therefore

$$L_{S^n}(\theta,\xi) \ge (1-C_\mu)|\xi|^2 - hC|\xi| \ge (1-C_\mu-h)|\xi|^2,$$

where C bounds the $b_i(\theta)$. On the support of ζ,

$$|\xi|^2 \ge c_0 > \frac{K^2}{1+K^2},$$

so for small enough μ and h,

$$L_{S^n}(\theta,\xi) > \frac{K^2}{1+K^2}.$$

Then $|\alpha-1| \lesssim h\varepsilon^{-1}$, and by (3-7),

$$|\gamma_f - K| \le C_\mu,$$

so for small enough μ and h, it follows that

$$L_{S^n}(\theta,\xi) > \frac{\alpha^2|\gamma_f|^2}{1+|\gamma_f|^2}$$

on the support of ζ. Therefore the square root is actually smooth on the support of ζ, and hence $G_\pm$ are smooth and really are symbols of order 1 on $\mathbb{R}^n$.

Now if T_a is the operator associated to the symbol a,

$$\begin{aligned}\Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)&\big(1+|\gamma_f|^2\big)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)\\ &= \big(1+|\gamma_f|^2\big)h^2\partial_r^2 - \frac{2}{r}\big(\alpha+\beta_f\cdot h\nabla_\theta\big)h\partial_r T_\zeta + \frac{1}{r^2}\big(\alpha^2+h^2L_{S^n}\big)T_{\zeta^2}\\ &\quad - \frac{2}{r}\big(1+|\gamma_f|^2\big)T_{G_s} + \frac{1}{r^2}\big(1+|\gamma_f|^2\big)\big(T_{G_+}T_{G_s}+T_{G_-}T_{G_s}+T_{G_s}T_{G_s}\big) + hE_1,\end{aligned}$$

where E_1 is an operator built of first-order semiclassical pseudodifferential operators in $\mathbb{R}^n$ and ∂_r derivatives that is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$.

Now let $v = J_\ell^{-1}w_\ell$. Then

$$\begin{aligned}\Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)&\big(1+|\gamma_f|^2\big)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)v\\ &= \big(1+|\gamma_f|^2\big)h^2\partial_r^2 v - \frac{2}{r}\big(\alpha+\beta_f\cdot h\nabla_\theta\big)h\partial_r T_\zeta v + \frac{1}{r^2}\big(\alpha^2+h^2L_{S^n}\big)T_{\zeta^2}v\\ &\quad - \frac{2}{r}\big(1+|\gamma_f|^2\big)T_{G_s}v + \frac{1}{r^2}\big(1+|\gamma_f|^2\big)\big(T_{G_+}+T_{G_-}+T_{G_s}\big)T_{G_s}v + hE_1 v.\end{aligned}$$

Note that $\hat{w}_\ell(r,\xi)$ is only supported for ξ on the support of $1-\rho$, and therefore $v = J_\ell^{-1}w_\ell$ is supported only for ξ on the support of $1-\rho$. Therefore

$$T_\zeta v = v,$$

since $\zeta \equiv 1$ on the support of $1-\rho$. Similarly, $T_{\zeta^2}v = v$. In addition,

$$T_{G_s}v = 0,$$

since G_s is 0 on the support of $1-\rho$. Therefore

$$\begin{aligned}\Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)&(1+|\gamma_f|^2)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)v \\ &= (1+|\gamma_f|^2)h^2\partial_r^2 v - \frac{2}{r}(\alpha+\beta_f\cdot h\nabla_\theta)h\partial_r v + \frac{1}{r^2}(\alpha^2+h^2L_{S^n})v + hE_1 v \\ &= \mathscr{L}_{\varphi,\varepsilon,\sigma}v + hE_1 v,\end{aligned}$$

where E_1 is bounded from $H^1(\mathbb{R}^{n+1}_{1+})$ to $L^2(\mathbb{R}^{n+1}_{1+})$.

Therefore

$$\mathscr{L}_{\varphi,\varepsilon,\sigma}v = \Big(h\partial_r - \frac{1}{r}T_{G_+}\Big)z + hE_1 v$$

for some function z, given by

$$z = (1+|\gamma_f|^2)\Big(h\partial_r - \frac{1}{r}T_{G_-}\Big)v.$$

Then

$$\begin{aligned}\hat{g}(r,\xi) &= \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty \widehat{\mathscr{L}_{\varphi,\varepsilon,\sigma}v}(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt \\ &= \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty \mathscr{F}\Big(\Big(h\partial_t - \frac{1}{t}T_{G_+}\Big)z\Big)(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt \\ &\qquad + \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty h\widehat{E_1 v}(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt.\end{aligned}$$

Integrating by parts gives

$$\begin{aligned}\hat{g}(r,\xi) &= \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty \frac{1}{t}\mathscr{F}\big((T_{\overline{F_\ell}} - T_{G_+})z\big)(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt \\ &\qquad + \frac{2\operatorname{Re}F_\ell - h}{h}\int_1^\infty h\widehat{E_1 v}(t,\xi)r^{-F_\ell/h}t^{-\overline{F_\ell}/h}\,dt.\end{aligned}$$

There are no boundary terms because z is supported away from $r=1$. Therefore, using the bounds on g,

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \le \Big\|\frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z\Big\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + h^2\|E_1 v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

We need an estimate for $\Big\|\frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z\Big\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}$. Examine the symbol $\overline{F_\ell} - G_+$.

$$\overline{F_\ell} - G_+ = \zeta\left(\overline{F_\ell(\xi)} - \frac{\alpha + i\beta_f\cdot\xi + \sqrt{(\alpha+i\beta_f\cdot\xi)^2 - (1+|\gamma_f|^2)(\alpha^2+L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2}\right).$$

On the support of ζ,

$$\overline{F_\ell(\xi)} = \frac{1}{1+K^2}\Big(1 + iK\xi_n + \sqrt{2iK\xi_n - (K\xi_n)^2 + (1+K^2)|\xi|^2 - |K|^2}\Big).$$

Therefore

$$\overline{F_\ell} - G_+ = \zeta\left(\frac{1+iK\xi_n}{1+K^2} - \frac{\alpha+i\beta_f\cdot\xi}{1+|\gamma_f|^2}\right) \\ + \zeta\left(\frac{\sqrt{2iK\xi_n-(K\xi_n)^2-(1+K^2)|\xi|^2-|K|^2}}{1+K^2} - \frac{\sqrt{(\alpha+i\beta_f\cdot\xi)^2-(1+(\gamma_f)^2)(\alpha^2+L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2}\right).$$

Consider the first term.

$$\frac{1+iK\xi_n}{1+K^2} - \frac{\alpha+i\beta_f\cdot\xi}{1+|\gamma_f|^2} = \frac{(|\gamma_f|^2-K^2)(1+iK\xi_n)}{(1+K^2)(1+|\gamma_f|^2)} + \frac{(1+K^2)\big((1-\alpha)+i(\beta_f-Ke_n)\cdot\xi\big)}{(1+K^2)(1+|\gamma_f|^2)}.$$

The first-order operators with symbols

$$\frac{(|\gamma_f|^2-K^2)(1+iK\xi_n)}{(1+K^2)(1+|\gamma_f|^2)}$$

and

$$\frac{(1+K^2)\big((1-\alpha)+i(\beta_f-Ke_n)\cdot\xi\big)}{(1+K^2)(1+|\gamma_f|^2)}$$

have bounds $\lesssim C_\mu$, because they involve multiplication by a function of θ that is bounded by $C_K C_\mu$.

Similarly, consider the first-order operator with symbol

$$\zeta\left(\frac{\sqrt{2iK\xi_n-(K\xi_n)^2-(1+K^2)|\xi|^2-|K|^2}}{1+K^2} - \frac{\sqrt{(\alpha+i\beta_f\cdot\xi)^2-(1+(\gamma_f)^2)(\alpha^2+L_{S^n}(\theta,\xi))}}{1+|\gamma_f|^2}\right).$$

To fit everything horizontally on the page, write

$$\tau_K := 2iK\xi_n-(K\xi_n)^2-(1+K^2)|\xi|^2-|K|^2$$

and

$$\tau_f := (\alpha+i\beta_f\cdot\xi)^2-(1+(\gamma_f)^2)(\alpha^2+L_{S^n}(\theta,\xi)).$$

Then

$$\frac{\sqrt{\tau_K}}{1+K^2} - \frac{\sqrt{\tau_f}}{1+|\gamma_f|^2} = (1+K^2)\frac{\tau_K-\tau_f}{(1+|\gamma_f|^2)\big((1+|\gamma_f|^2)\sqrt{\tau_K}+(1+K^2)\sqrt{\tau_f}\big)} \\ + \frac{\big((1+|\gamma_f|^2)^2-(1+K^2)^2\big)\tau_K}{(1+K^2)(1+|\gamma_f|^2)\big((1+|\gamma_f|^2)\sqrt{\tau_K}+(1+K^2)\sqrt{\tau_f}\big)}.$$

Expanding,

$$\tau_K-\tau_f = 2i(Ke_n-\alpha\beta_f)\cdot\xi + \big((\beta_f\cdot\xi)^2-(Ke_n\cdot\xi)^2\big) + (|\gamma_f|^2-K^2)L(\theta,i\xi) \\ + (|\gamma_f|^2-|K|^2) + (1+K^2)(|\xi|^2-L(\theta,\xi)).$$

Therefore the second term has operator bounds $\lesssim C_\mu$, because each term involves multiplication by a function of θ that is bounded by $C_K C_\mu$.

Therefore

$$\left\| \frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \le \delta^2 \|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}$$

for μ small enough. Then

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \left\| \frac{1}{r}\big(T_{\overline{F_\ell}-G_+}\big)z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} + h^2\|E_1 v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}.$$

Since

$$\mathscr{L}_{\varphi,\varepsilon,\sigma} v = \left(h\partial_r - \frac{1}{r}T_{G_+}\right)z + hE_1 v,$$

we have

$$\begin{aligned}
\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} &\ge \left\| \left(h\partial_r - \frac{1}{r}T_{G_+}\right)z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - h^2\|E_1 v\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \\
&\ge \|J_\ell^* z\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - \left\| \frac{1}{r} T_{\overline{F_\ell}-G_+} z \right\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} - h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \\
&\gtrsim \|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} - \delta^2\|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} - h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \\
&\gtrsim \|z\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} - h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}
\end{aligned}$$

for δ small enough. Therefore

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\|v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\big\|J_\ell^{-1}(1-P)w\big\|^2_{H^1(\mathbb{R}^{n+1}_{1+})}.$$

Using similar reasoning as for the small frequency case,

$$h^2\big\|J_s^{-1}(1-P)w\big\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} \lesssim h^2\big\|J_\ell^{-1}(1-P)w\big\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})}.$$

Therefore

$$\|g\|^2_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\big\|J_\ell^{-1}(1-P)w\big\|^2_{H^1_r(\mathbb{R}^{n+1}_{1+})} \lesssim \delta^2\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|^2_{H^1(\mathbb{R}^{n+1}_{1+})} + h^2\|w_\ell\|^2_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Then for δ small enough,

$$\|g\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \tfrac{1}{2}\|\mathscr{L}_{\varphi,\varepsilon,\sigma} v\|_{L^2(\mathbb{R}^{n+1}_{1+})} + h\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Now using (7-1) and Lemma 5.3,

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|J_\ell \mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_\ell^{-1} w_\ell\big\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + h\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

Absorbing the second last term into the left side gives

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \big\|J_\ell \mathscr{L}_{\varphi,\varepsilon,\sigma}\chi_2 J_\ell^{-1} w_\ell\big\|_{H_r^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h^\infty)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

We can finish the argument as in the small frequency case to get

$$\frac{h}{\sqrt{\varepsilon}}\|w_\ell\|_{L^2(\mathbb{R}^{n+1}_{1+})} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,\sigma} w_\ell\|_{H^{-1}(\mathbb{R}^{n+1}_{1+})} + O(h)\|w\|_{L^2(\mathbb{R}^{n+1}_{1+})}.$$

This finishes the proof of Lemma 4.2, and thus of Proposition 3.1.

8. Proof of Theorem 1.4

We will begin by gluing together estimates of the form in Proposition 3.1 to prove the following intermediate proposition.

Proposition 8.1. *Suppose that $f : S^n \to (0,\infty)$ is a C^∞ function such that Ω lies entirely in the region $A_O = \{(r,\theta) \mid r \geq f(\theta)\} \subset \mathbb{R}^{n+1}$, and Γ^c_+ is a subset of the graph $r = f(\theta)$. If $w \in C^\infty_0(\Omega)$, then*

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon,W,q} w\|_{H^{-1}(A_O)}.$$

Proof. Now let Ω be as in Proposition 8.1. We can take an open cover $U_1,\dots,U_m$ of Ω such that on each $\Omega\cap U_j$, there exists K_j such that under some choice of coordinates, $|\nabla_{S^n}\log f - K_j e_n| \leq \mu_{K_j}$ and $|\sin(\theta_k) - 1| \leq \mu_{K_j}$, where μ_{K_j} is the value of μ from Proposition 3.1 that works for $K = K_j$. (Since $|\nabla_{S^n}\log f|$ must be bounded above, μ_{K_j} must be bounded below, and therefore this is possible with only finitely many U_j.)

Let $\zeta_1,\dots,\zeta_m$ be a smooth partition of unity subordinate to the cover $U_1,\dots,U_m$. Now for $w \in C^\infty_0(\Omega)$,

$$w = \zeta_1 w + \cdots + \zeta_m w =: w_1 + \cdots + w_m,$$

where each $w_j \in C^\infty_0(\Omega\cap U_j)$. Applying Proposition 3.1 to the domain $\Omega\cap U_j$,

$$\frac{h}{\sqrt{\varepsilon}}\|w_j\|_{L^2(\Omega\cap U_j)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)}$$

for each $j = 1,\dots,m$. Then

$$\sum_j \frac{h}{\sqrt{\varepsilon}}\|w_j\|_{L^2(\Omega)} \lesssim \sum_j \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)},$$

so

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \sum_j \|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)}.$$

Now by the product rule,

$$\begin{aligned}\|\mathcal{L}_{\varphi,\varepsilon} w_j\|_{H^{-1}(A_O)} = \|\mathcal{L}_{\varphi,\varepsilon}\zeta_j w\|_{H^{-1}(A_O)} &\leq \|\zeta_j\mathcal{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)} + Ch\|w\|_{L^2(A_O)}\\ &\leq \|\mathcal{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)} + Ch\|w\|_{L^2(A_O)}.\end{aligned}$$

Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathcal{L}_{\varphi,\varepsilon} w\|_{H^{-1}(A_O)} \tag{8-1}$$

for ε small enough, for every $w \in C^\infty_0(\Omega)$.

To treat the case where W and q are nonzero, note that

$$\mathscr{L}_{\varphi,\varepsilon,W,q} = \mathscr{L}_{\varphi,\varepsilon} + h\big(W\cdot hD + hD\cdot W\big) + 2ihW\cdot\nabla\left(\log r + h\frac{\log^2 r}{2\varepsilon}\right) + h^2(q+W^2).$$

Therefore

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(A_O)} + hC\|w\|_{L^2(A_O)},$$

and the last term can be absorbed into the left side to give

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(A_O)}.$$

This completes the proof. □

Finally, I can prove Theorem 1.4 by gluing together estimates of the form in Proposition 8.1. If Γ_+ is a neighborhood of $\partial\Omega_+$, then let Ω' be a smooth domain containing Ω, with $\partial\Omega\cap\partial\Omega' = \Gamma_+^c$.

Then let $U_1,\dots,U_m$ be an open cover of Ω such that each $\partial U_j\cap\Gamma_+^c$ coincides with a graph of the form $r = f_j(\theta)$. For each U_j, Proposition 3.1 gives us

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U_j)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(A_j)}$$

for $w\in C_0^\infty(U_j)$.

Each A_j is defined by the graph of a function $r = f_j(\theta)$, and since $\partial\Omega'$ is smooth and coincides with $\partial\Omega$ on Γ_+^c, and $\partial_\nu\varphi<0$ on Γ_+^c, $\partial\Omega'$ must be locally a graph in a neighborhood of Γ_+^c. Therefore we can assume that A_j coincides with Ω' in a neighborhood of each U_j, in the sense that their characteristic functions are equal in that neighborhood. Then there is a smooth cutoff function χ_j defined on $A_j\cap\Omega'$ that is identically one on U_j but vanishes outside on the complements of A_j and Ω'. Multiplication by this function provides a bounded map from $H_0^1(A_j)$ to $H_0^1(\Omega')$ and vice versa, and therefore $\|w\|_{H^{-1}(\Omega')}\simeq\|w\|_{H^{-1}(A_j)}$ for $w\in C_0^\infty(U_j)$. Therefore we have

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(U_j)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(\Omega')}$$

for $w\in C_0^\infty(U_j)$.

Gluing together these estimates in the manner used above gives

$$\frac{h}{\sqrt{\varepsilon}}\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,\varepsilon,W,q}w\|_{H^{-1}(\Omega')}$$

for $w\in C_0^\infty(\Omega)$.

Finally, note that if $w\in C_0^\infty(\Omega)$, then $e^{(\log r)^2/\varepsilon}w\in C_0^\infty(\Omega)$, so

$$\frac{h}{\sqrt{\varepsilon}}\left\|e^{(\log r)^2/\varepsilon}w\right\|_{L^2(\Omega)} \lesssim \left\|e^{(\log r)^2/\varepsilon}\mathscr{L}_{\varphi,W,q}w\right\|_{H^{-1}(\Omega')}.$$

On Ω, there exists some C_Ω such that $1\le e^{(\log r)^2/\varepsilon}\le e^{C_\Omega/\varepsilon}$, so

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,W,q}w\|_{H^{-1}(\Omega')},$$

as desired. This establishes Theorem 1.4.

Remark. If we want to prove Theorem 1.2 instead of Theorem 1.1, then we could begin by supposing that $f : S^n \to (0, \infty)$ is a C^∞ function such that Ω lies entirely in the region $A_I = \{(r, \theta) \mid r \le f(\theta)\} \subset \mathbb{R}^{n+1}$, and Γ_-^c is a subset of the graph $r = f(\theta)$. Then by the change of variables $(r, \theta) \mapsto (1/r, \theta)$, Ω maps to a region $\hat{\Omega}$ of the form described in Proposition 8.1. Therefore, by (8-1),

$$h\|w\|_{L^2(\hat{\Omega})} \lesssim \|\mathscr{L}_{\varphi,\varepsilon} w\|_{H^{-1}(\hat{A}_O)}$$

for $w \in C_0^\infty \hat{\Omega}$, where $\varphi = \log r$. Changing variables back gives the Carleman estimate

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{-\log r,\varepsilon} w\|_{H^{-1}(A_I)}$$

for $w \in C_0^\infty \Omega$. Therefore, by the same kind of argument as above, we get

$$h\|w\|_{L^2(\Omega)} \lesssim \|\mathscr{L}_{\varphi,W,q} w\|_{H^{-1}(\Omega')},$$

where $\varphi = -\log r$ and Ω' is a domain containing Ω, with $\Gamma_-^c \subset \partial\Omega' \cap \partial\Omega$ whenever Γ_- is of the form described in Theorem 1.2. Using this Carleman estimate in the place of Theorem 1.4 in the remainder of the argument proves Theorem 1.2 instead of Theorem 1.1.

9. Complex geometric optics solutions

Theorem 1.4 can be used to construct solutions to equations of the system (1-1). The key is the following proposition.

Proposition 9.1. *For every $v \in L^2(\Omega)$, there exists $u \in H^1(\Omega)$ such that*

$$\mathscr{L}^*_{\varphi,W,q} u = v \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0$$

and

$$\|u\|_{H^1(\Omega)} \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)}.$$

Proof. The proof is based on a Hahn–Banach argument. Suppose $v \in L^2(\Omega)$. Then for all $w \in C_0^\infty(\Omega)$,

$$\left|(w|v)_\Omega\right| \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)} h\|w\|_{L^2(\Omega)}.$$

Therefore, by Theorem 1.4,

$$\left|(w|v)_\Omega\right| \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)} \|\mathscr{L}_{\varphi,W,q} w\|_{H^{-1}(\Omega')}. \tag{9-1}$$

Now consider the subspace

$$\{\mathscr{L}_{\varphi,W,q} w \mid w \in C_0^\infty(\Omega)\} \subset H^{-1}(\Omega').$$

By the estimate from Theorem 1.4, the map $\mathscr{L}_{\varphi,W,q} w \mapsto (w|v)_\Omega$ is well defined on this subspace. It is a linear functional, and by (9-1), it is bounded by $(C/h)\|v\|_{L^2(\Omega)}$.

Therefore, by Hahn–Banach, there exists an extension of this functional to the whole space $H^{-1}(\Omega')$ with the same bound. This can be represented by an element of the dual space $H_0^1(\Omega')$, so there exists $u \in H_0^1(\Omega')$ such that

$$\|u\|_{H^1(\Omega')} \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)}$$

and

$$(w|v)_\Omega = (\mathscr{L}_{\varphi,W,q} w|u)_{\Omega'} = (\mathscr{L}_{\varphi,W,q} w|u)_\Omega$$

for all $w \in C_0^\infty(\Omega)$. Note that $u \in H_0^1(\Omega')$ implies that $u|_{\Gamma_+^c} = 0$. Then

$$(w|v)_\Omega = (w|\mathscr{L}^*_{\varphi,W,q} u)_\Omega$$

since $w \in C_0^\infty(\Omega)$, and thus

$$(w|v - \mathscr{L}^*_{\varphi,W,q} u)_\Omega = 0$$

for all $v \in C_0^\infty(\Omega)$. Therefore $v = \mathscr{L}^*_{\varphi,W,q} u$ on Ω, and

$$\|u\|_{H^1(\mathbb{R}^{n+1})} \lesssim \frac{1}{h}\|v\|_{L^2(\Omega)},$$

as desired. □

Now I can construct the complex geometrical optics solutions.

Proposition 9.2. *There exists a solution of the problem*

$$\mathscr{L}_{W,q} u = 0 \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0$$

of the form $u = e^{(1/h)(\varphi+i\psi)}(a+r) - e^{\ell/h} b$, *where* $\varphi(x,y) = \log r$, ψ *is a solution to the eikonal equation* $\nabla\varphi\cdot\nabla\psi = 0$, $|\nabla\varphi| = |\nabla\psi|$, a *and* b *are* C^2 *functions on* Ω, *and*

$$\operatorname{Re}\ell(x,y) = \varphi(x,y) - k(x,y),$$

where $k(x) \simeq \operatorname{dist}(x,\Gamma_+^c)$ *in a neighborhood of* Γ_+^c *and* b *has its support in that neighborhood. Finally,* $r \in H^1(\Omega)$, *with* $r|_{\Gamma_+^c} = 0$, $\|r\|_{H^1(\Omega)} = O(h)$, *and* $\|r\|_{L^2(\partial\Omega)} = O(h^{1/2})$.

The proof is a combination of the proofs of the equivalent theorems in [Dos Santos Ferreira et al. 2007; Kenig et al. 2007].

Proof. Let $\varphi(r,\theta) = \log r$, and take $\psi(r,\theta) = d_{S^n}(\theta,\omega)$ for some fixed point $\omega \in S^n$. If $\omega \neq \theta$ for all $(r,\theta) \in \Omega$, then ψ solves the eikonal equation $\nabla\varphi\cdot\nabla\psi = 0$, $|\nabla\varphi| = |\nabla\psi|$. Then

$$h^2\mathscr{L}_{W,q} e^{(1/h)(\varphi+i\psi)} = e^{(1/h)(\varphi+i\psi)}\big(h(D+W)\cdot(\nabla\psi - i\nabla\varphi) + h(\nabla\psi - i\nabla\varphi)\cdot(D+W) + h^2\mathscr{L}_{W,q}\big).$$

Therefore, if a is a C^2 solution to

$$(\nabla\psi - i\nabla\varphi)\cdot Da + (\nabla\psi - i\nabla\varphi)\cdot Wa + \frac{1}{2i}(\triangle\psi - i\triangle\varphi)a = 0,$$

then

$$h^2\mathscr{L}_{W,q} e^{(1/h)(\varphi+i\psi)} a = e^{(1/h)(\varphi+i\psi)} h^2\mathscr{L}_{W,q} a = O(h^2) e^{(1/h)(\varphi+i\psi)}.$$

We can look for an exponential solution $a = e^{\Phi}$, in which case the relevant equation becomes

$$(\nabla\varphi + i\nabla\psi)\cdot\nabla\Phi + i(\nabla\varphi + i\nabla\psi)\cdot W + \tfrac{1}{2}\triangle(\varphi + i\psi) = 0.$$

Now suppose $x \in \mathbb{R}^{n+1}$, and write $x = (x_\omega, x')$, where x_ω is the component of x in the ω direction, and x' are the remaining components. Then by considering $z = x_\omega + i|x'|$ as a complex variable, we get $\varphi = \operatorname{Re}\log z$ and $\psi = \operatorname{Im}\log z$. Now our equation is an inhomogeneous Cauchy–Riemann equation in the z variable, and can be solved by the Cauchy formula. Then a is C^2, since W is. The solution is only unique up to addition of terms g_a with

$$(\nabla\varphi + i\nabla\psi)\cdot\nabla g_a = 0. \tag{9-2}$$

Now I want to construct a (complex-valued) function ℓ to be an approximate solution to the equation

$$\nabla\ell\cdot\nabla\ell = 0, \quad \ell|_{\Gamma_+^c} = \varphi + i\psi.$$

In order to avoid duplicating the solution $\varphi + i\psi$, we can ask for

$$\partial_\nu\ell|_{\Gamma_+^c} = -\partial_\nu(\varphi + i\psi)|_{\Gamma_+^c}.$$

To construct an approximate solution, pick coordinates (t, s) near Γ_+^c such that t are the coordinates along Γ_+^c and s is perpendicular to Γ_+^c. Suppose ℓ takes the form of a power series

$$\ell(t, s) = \sum_{j=0}^{\infty} a_j(t)s^j.$$

Then

$$\nabla\ell = (\nabla_t\ell, \partial_s\ell) = \biggl(\sum_{j=0}^{\infty}\nabla_t a_j(t)s^j, \sum_{j=0}^{\infty} a_j(t)js^{j-1}\biggr).$$

Expanding the equation $\nabla\ell\cdot\nabla\ell = 0$ and considering each power of s separately gives a sequence of equations

$$\sum_{j+k=m}\nabla_t a_j\nabla_t a_k + \sum_{j+k=m+2} jka_ja_k = 0 \tag{9-3}$$

for each $m = 0, 1, 2, \ldots$. The boundary conditions determine a_0 and a_1, so we can solve this recursively. If $m \geq 1$ and all a_j are known for $j \leq m$, the only part of (9-3) that contains an unknown looks like $2(m+1)a_1a_{m+1}$. Note that

$$a_1 = -\partial_\nu(\varphi + i\psi).$$

Since Γ_+^c coincides with a graph $r = f(\theta)$ for some smooth function f, and $\varphi = \log r$, there exists some $\varepsilon_0 > 0$ such that $|a_1| > \varepsilon_0$ on Γ_+^c, so we can divide by a_1 to solve for a_{m+1}.

This gives a formal power series that may or may not converge outside $s = 0$. However, we can construct a C^∞ function ℓ in Ω whose Taylor series in s coincides with this formal power series at $s = 0$, such that

$$\nabla\ell\cdot\nabla\ell = O\bigl(\operatorname{dist}(x, \Gamma_+^c)^\infty\bigr).$$

Moreover,

$$\partial_\nu \operatorname{Re} \ell|_{\Gamma_+^c} = -\partial_\nu \varphi|_{\Gamma_+^c} < -\varepsilon_0$$

and

$$\operatorname{Re} \ell|_{\Gamma_+^c} = \varphi|_{\Gamma_+^c},$$

so in a neighborhood of Γ_+^c,

$$\operatorname{Re} \ell(x, y) = \varphi(x, y) - k(x, y), \tag{9-4}$$

where $k(x) \simeq \operatorname{dist}(x, \Gamma_+^c)$ in a neighborhood of Γ_+^c.

By a similar method, we can construct an approximate solution b for the problem

$$\nabla \ell \cdot Db + \nabla \ell \cdot Wb = 0, \quad b|_{\Gamma_+^c} = a|_{\Gamma_+^c},$$

so

$$\nabla \ell \cdot Db + \nabla \ell \cdot Wb = O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big), \quad b|_{\Gamma_+^c} = a|_{\Gamma_+^c}.$$

Multiplying b by a smooth cutoff function does not change these properties, so we may as well assume that b is only supported close to Γ_+^c for (9-4) to hold. Then

$$-h^2 \mathscr{L}_{W,q}(e^{\ell/h} b) = e^{\ell/h}\big(O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big) + O(h^2)\big),$$

so

$$\big|h^2 \mathscr{L}_{W,q}(e^{\ell/h} b)\big| = e^{\varphi/h} e^{-k/h}\big(O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big) + O(h^2)\big).$$

If $\operatorname{dist}(x, \Gamma_+^c) \le h^{1/2}$, for h small, this is $e^{\varphi/h} O(h^2)$, because of the $O\big(\operatorname{dist}(x, \Gamma_+^c)^\infty\big)$ term. On the other hand, if $\operatorname{dist}(x, \Gamma_+^c) \ge h^{1/2}$, this is still $e^{\varphi/h} O(h^2)$, because of $e^{-k/h}$.

Now $e^{(1/h)(\varphi + i\psi)} a - e^{\ell/h} b = 0$ on Γ_+^c, and

$$e^{-\varphi/h} h^2 \mathscr{L}_{W,q}\big(e^{(1/h)(\varphi+i\psi)} a + e^{\ell/h} b\big) = v,$$

where $\|v\|_{L^2(\Omega)} = O(h^2)$. By Proposition 9.1, the problem

$$\mathscr{L}^*_{\varphi,W,q} r_1 = e^{-\varphi/h} h^2 \mathscr{L}_{W,q} e^{\varphi/h} r_1 = -v \text{ on } \Omega, \quad r_1|_{\Gamma_+^c} = 0$$

has an H^1 solution r_1 with

$$\|r_1\|_{H^1(\Omega)} \lesssim \frac{1}{h} \|v\|_{L^2(\Omega)} = O(h).$$

Set $r = e^{-i\psi/h} r_1$ and $u = e^{(1/h)(\varphi+i\psi)}(a + r) - e^{\ell/h} b$. Then

$$\|r\|_{H^1(\Omega)} = O(h),$$

so $\|r\|_{L^2(\partial\Omega)} = O(h^{1/2})$ by the trace theorem, and

$$\mathscr{L}_{W,q} u = 0 \text{ on } \Omega, \quad u|_{\Gamma_+^c} = 0.$$

This finishes the proof. □

If the boundary condition is not needed, then the result is as follows:

Proposition 9.3. *There exists a solution of the problem*

$$\mathscr{L}_{W,q} u = 0 \text{ on } \Omega$$

of the form $u = e^{(1/h)(\varphi + i\psi)}(a + r)$, *where* $\varphi(x, y)$ *is any limiting Carleman weight,* ψ *is any solution to the eikonal equation,* a *is a* C^2 *function on* Ω, *and* $r \in H^1(\Omega)$, *with* $\|r\|_{H^1(\Omega)} = O(h)$ *and* $\|r\|_{L^2(\partial\Omega)} = O(h^{1/2})$.

This is essentially Lemma 3.4 from [Dos Santos Ferreira et al. 2007]. We can always replace a by γa, where γ is a solution to

$$(\nabla\varphi + i\nabla\psi) \cdot \nabla\gamma = 0 \text{ on } \Omega.$$

10. Proof of Theorem 1.1

For convenience, $\|\cdot\|$ will denote the L^2 norm in this section, unless otherwise indicated. The tilde as used in this section has nothing to do with the notation from Section 3.

Using Proposition 9.2, we can construct $\tilde{u}_2 = e^{(1/h)(\varphi+i\psi)}(a_2 + r_2) - e^{\ell/h} b =: u_2 + u_r$ to be a solution to

$$\mathscr{L}_{W_2,q_2} \tilde{u}_2 = 0 \text{ on } \Omega, \quad \tilde{u}_2|_{\Gamma_+^c} = 0.$$

Then $-\varphi$ is also a Carleman weight, and if φ and ψ satisfy the eikonal equation, then so do $-\varphi$ and ψ. Therefore, using Proposition 9.3, we can construct $u_1 = e^{(1/h)(-\varphi+i\psi)}(a_1 + r_1)$ to be a solution to

$$\mathscr{L}_{W_1,\overline{q_1}} u_1 = 0.$$

Let w be the unique solution to

$$\mathscr{L}_{W_1,q_1} w = 0, \quad w|_{\partial\Omega} = \tilde{u}_2|_{\partial\Omega}.$$

(Here we are using the assumption that $\mathscr{L}_{W_1,q_1}$ does not have a zero eigenvalue.) In particular, $w|_{\Gamma_+^c} = \tilde{u}_2|_{\Gamma_+^c} = 0$, so by the hypothesis on the Dirichlet–Neumann map,

$$\partial_\nu (w - \tilde{u}_2)|_{\Gamma_-} = 0.$$

Now

$$\begin{aligned}\mathscr{L}_{W_1,q_1}(w - \tilde{u}_2) &= -\mathscr{L}_{W_1,q_1}\tilde{u}_2 \\ &= (\mathscr{L}_{W_2,q_2} - \mathscr{L}_{W_1,q_1})\tilde{u}_2 \\ &= (W_2 - W_1) \cdot D\tilde{u}_2 + D \cdot (W_2 - W_1)\tilde{u}_2 + \big(W_2^2 - W_1^2 + q_2 - q_1\big)\tilde{u}_2. \end{aligned} \tag{10-1}$$

On the other hand, Green's formula from [Dos Santos Ferreira et al. 2007] gives us

$$\int_\Omega \mathscr{L}_{W_1,q_1}(w - \tilde{u}_2)\overline{u_1}\, dV = \int_{\partial\Omega} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS = \int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS. \tag{10-2}$$

Combining (10-1) with (10-2) gives

$$\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS = \int_\Omega (W_2 - W_1) \cdot \big(D\tilde{u}_2\overline{u_1} + \tilde{u}_2\overline{Du_1}\big)\, dV + \int_\Omega \big(W_2^2 - W_1^2 + q_2 - q_1\big)\tilde{u}_2\overline{u_1}\, dV.$$

Expanding $\tilde{u}_2$ as $\tilde{u}_2 = u_2 + u_r$ on the right side gives

$$\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS = \int_\Omega (W_2 - W_1)\cdot\big(Du_2\overline{u_1} + u_2\overline{Du_1}\big)\, dV + \int_\Omega \big(W_2^2 - W_1^2 + q_2 - q_1\big)u_2\overline{u_1}\, dV$$
$$+ \int_\Omega (W_2 - W_1)\cdot\big(Du_r\overline{u_1} + u_r\overline{Du_1}\big)\, dV + \int_\Omega \big(W_2^2 - W_1^2 + q_2 - q_1\big)u_r\overline{u_1}\, dV. \quad (10\text{-}3)$$

To show that $dW_1 = dW_2$, we can apply the reasoning from [Dos Santos Ferreira et al. 2007] verbatim if we can establish that

$$\lim_{h\to 0} h \int_\Omega (W_2 - W_1)\cdot\big(Du_2\overline{u_1} + u_2\overline{Du_1}\big)\, dV = 0. \quad (10\text{-}4)$$

Similarly, to show that $q_1 = q_2$, we can apply the reasoning from [Dos Santos Ferreira et al. 2007] verbatim if we can establish that

$$\lim_{h\to 0} \int_\Omega (q_2 - q_1)u_2\overline{u_1}\, dV = 0. \quad (10\text{-}5)$$

To establish (10-4), label the terms as follows: $T_1 = T_2 + T_3 + T_4 + T_5$. Consider the terms on the right side first. T_2 is bounded above by

$$\big\|(W_2 - W_1)e^{-\varphi/h}Du_2\big\|_\Omega \|a_1 + r_1\|_\Omega + \big\|(W_2 - W_1)e^{\varphi/h}\overline{Du_1}\big\|_\Omega \|a_2 + r_2\|_\Omega.$$

Since $W_2 - W_1$ is bounded on Ω, $\|a_1\|_\Omega$ and $\|a_2\|_\Omega$ are $O(1)$, and $\|r_1\|_\Omega$ and $\|r_2\|_\Omega$ are $O(h)$,

$$|T_2| \lesssim \big\|e^{-\varphi/h}Du_2\big\|_\Omega + \big\|e^{\varphi/h}Du_1\big\|_\Omega.$$

T_3 is bounded above by

$$|T_3| \le \big\|\big(W_2^2 - W_1^2 + q_2 - q_1\big)(a_2 + r_2)\big\|_\Omega \|a_1 + r_1\|_\Omega = O(1).$$

Similarly,

$$|T_4| \lesssim \big\|e^{-\varphi/h}Du_r\big\|_\Omega + \big\|e^{\varphi/h}\overline{Du_1}\big\|_\Omega \big\|e^{-2\beta y/h}\big\|_\Omega \lesssim \big\|e^{-\varphi/h}Du_r\big\|_\Omega + h\big\|e^{\varphi/h}\overline{Du_1}\big\|_\Omega$$

and

$$|T_5| \le \big\|\big(W_2^2 - W_1^2 + q_2 - q_1\big)e^{-2\beta y/h}b\big\|_\Omega \|a_1 + r_1\|_\Omega = O(h).$$

Now examine the term T_1:

$$\left|\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS\right| \le \big\|\partial_\nu(\tilde{u}_2 - w)e^{-\varphi/h}\big\|_{\Gamma_-^c} \|a_1 + r_1\|_{\Gamma_-^c}.$$

The factor $\|a_1 + r_1\|_{\Gamma_-^c}$ is $O(1)$. Furthermore, $\partial_\nu\varphi \ge \varepsilon_1$ on Γ_-^c, so

$$\left|\int_{\Gamma_-^c} \partial_\nu(\tilde{u}_2 - w)\overline{u_1}\, dS\right| \lesssim \frac{1}{\sqrt{\varepsilon_1}}\big\|\sqrt{\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\Gamma_-^c} \lesssim \frac{1}{\sqrt{\varepsilon_1}}\big\|\sqrt{\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\Gamma_+}.$$

By the Carleman estimate given in Equation (2.13) of [Dos Santos Ferreira et al. 2007],

$$\big\|\sqrt{\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\Gamma_+} \lesssim \sqrt{h}\big\|e^{-\varphi/h}\mathcal{L}_{W_1,q_1}(\tilde{u}_2 - w)\big\|_\Omega + \big\|\sqrt{-\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2 - w)\big\|_{\partial\Omega_-}.$$

Therefore

$$\frac{C}{\sqrt{\varepsilon_1}}\Big(\sqrt{h}\big\|e^{-\varphi/h}\mathscr{L}_{W_1,q_1}(\tilde{u}_2-w)\big\|_\Omega+\big\|\sqrt{-\partial_\nu\varphi}e^{-\varphi/h}\partial_\nu(\tilde{u}_2-w)\big\|_{\partial\Omega_-}\Big).$$

The last term on the right side is zero, because $\partial_\nu(\tilde{u}_2-w)=0$ on Γ_- and $\partial\Omega_-\subset\Gamma_-$. Therefore the upper bound becomes

$$\frac{C}{\sqrt{\varepsilon_1}}\sqrt{h}\big\|e^{-\varphi/h}\mathscr{L}_{W_1,q_1}(\tilde{u}_2-w)\big\|_\Omega.$$

Expanding $\mathscr{L}_{W_1,q_1}(\tilde{u}_2-w)$ and writing $\tilde{u}_2=u_2+u_r$, we obtain that T_1 is bounded above by

$$\begin{aligned}\frac{C\sqrt{h}}{\sqrt{\varepsilon_1}}\Big(\big\|e^{-\varphi/h}Du_2\big\|_\Omega+\big\|e^{-\varphi/h}u_2\big\|_\Omega+\big\|e^{-\varphi/h}Du_r\big\|_\Omega+\big\|e^{-\varphi/h}u_r\big\|_\Omega\Big)&\\ \le\frac{C\sqrt{h}}{\sqrt{\varepsilon_1}}\Big(\big\|e^{-\varphi/h}Du_2\big\|_\Omega+\|a_2+r_2\|_\Omega+\big\|e^{-\varphi/h}Du_r\big\|_\Omega+\big\|e^{-2\beta y/h}b\big\|_\Omega\Big)&\\ \le\frac{C\sqrt{h}}{\sqrt{\varepsilon_1}}\Big(\big\|e^{-\varphi/h}Du_2\big\|_\Omega+O(1)+\big\|e^{-\varphi/h}Du_r\big\|_\Omega+O(h)\Big),&\end{aligned}$$

where the constant C mutates as necessary to preserve the bound. Therefore, in order to bound the terms T_1, T_2, and T_4, we need to calculate $\big\|e^{\varphi/h}Du_1\big\|_\Omega$, $\big\|e^{-\varphi/h}Du_2\big\|_\Omega$, and $\big\|e^{-\varphi/h}Du_r\big\|_\Omega$. We have

$$\begin{aligned}\big\|e^{\varphi/h}Du_1\big\|_\Omega&=\bigg\|e^{\varphi/h}\frac{1}{h}D(-\varphi+i\psi)e^{(1/h)(-\varphi+i\psi)}(a_1+r_1)+e^{i\psi/h}D(a_1+r_1)\bigg\|_\Omega\\ &\lesssim\frac{1}{h}\big\|D(-\varphi+i\psi)(a_1+r_1)\big\|_\Omega+\big\|D(a_1+r_1)\big\|_\Omega=O(h^{-1}),\end{aligned}$$

since $\|r_1\|_{H^1(\Omega)}$ is $O(h)$. Similarly,

$$\big\|e^{-\varphi/h}Du_2\big\|_\Omega=O(h^{-1}).$$

Finally,

$$\big\|e^{-\varphi/h}Du_r\big\|_\Omega=\bigg\|e^{-\varphi/h}\frac{1}{h}D\ell e^{\ell/h}b+e^{-\varphi/h}e^{\ell/h}Db\bigg\|_\Omega\lesssim\frac{1}{h}\big\|e^{-k/h}bD\ell\big\|_\Omega+\big\|e^{-k/h}Db\big\|_\Omega=O(1).$$

Putting all of this together gives $T_1=O(h^{-1/2})$, $T_2=O(h^{-1})$, $T_3=O(1)$, $T_4=O(1)$, and $T_5=O(h)$. Therefore, multiplying (10-3) through by h and taking the limit as h goes to zero gives

$$\lim_{h\to0}h\int_\Omega(W_2-W_1)\cdot\big(Du_2\overline{u_1}+u_2\overline{Du_1}\big)\,dV=0,$$

which establishes (10-4), and thus by the reasoning in [Dos Santos Ferreira et al. 2007], that $dW_1=dW_2$ in Ω and $W_1=W_2$ up to a gauge transformation that leaves the Dirichlet–Neumann maps invariant.

It remains only to prove (10-5). Going back to (10-3), we now have

$$\int_{\Gamma_-^c}\partial_\nu(\tilde{u}_2-w)\overline{u_1}\,dS=\int_\Omega(q_2-q_1)u_2\overline{u_1}\,dx+\int_\Omega(q_2-q_1)u_r\overline{u_1}\,dV. \tag{10-6}$$

The first and second terms on the right side are $O(1)$ and $O(h)$ as before. The left side is now bounded by

$$\frac{\sqrt{h}}{\sqrt{\varepsilon_1}}\Big(\big\|e^{-\varphi/h}(q_1-q_2)u_2\big\|_\Omega+\big\|e^{-\varphi/h}(q_1-q_2)u_r\big\|_\Omega\Big)=\sqrt{h}\big(O(1)+O(h)\big)=O(h^{1/2}),$$

so taking the limit of (10-6) as h goes to zero gives

$$\lim_{h\to 0}\int_\Omega (q_2-q_1)u_2\overline{u_1}\,dV=0.$$

This establishes (10-5), and thus that $q_1=q_2$ on Ω. This finishes the proof.

Acknowledgements

This research was partially supported by a Doctoral Postgraduate Scholarship from the Natural Science and Engineering Research Council of Canada. The referee for this paper made several helpful suggestions, which were appreciated by the author. Finally, the author would also like to thank Carlos Kenig for his guidance, support, and patience throughout this work.

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Received 22 May 2012. Revised 20 Jun 2013. Accepted 22 Aug 2013.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

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ANALYSIS & PDE

Volume 7 No. 1 2014

2157-5045(2014)7:1;1-F