ANALYSIS & PDE

Volume 7 No. 2 2014

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ANALYSIS AND PDE
Vol. 7, No. 2, 2014
dx.doi.org/10.2140/apde.2014.7.461

NONDISPERSIVE DECAY FOR THE CUBIC WAVE EQUATION

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We consider the hyperboloidal initial value problem for the cubic focusing wave equation

$$(-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 = 0, \quad x \in \mathbb{R}^3.$$

Without symmetry assumptions, we prove the existence of a codimension-4 Lipschitz manifold of initial data that lead to global solutions in forward time which do not scatter to free waves. More precisely, for any $\delta \in (0,1)$, we construct solutions with the asymptotic behavior

$$\|v - v_0\|_{L^4(t,2t)L^4(B_{(1-\delta)t})} \lesssim t^{-\frac{1}{2}+}$$

as $t \to \infty$, where $v_0(t,x) = \sqrt{2}/t$ and $B_{(1-\delta)t} := \{x \in \mathbb{R}^3 : |x| < (1-\delta)t\}$.

1. Introduction

We consider the cubic focusing wave equation

$$(-\partial_t^2 + \Delta_x)v(t,x) + v(t,x)^3 = 0 \tag{1-1}$$

in three spatial dimensions. Equation (1-1) admits the conserved energy

$$E(v(t,\cdot), v_t(t,\cdot)) = \tfrac{1}{2}\|(v(t,\cdot), v_t(t,\cdot))\|^2_{\dot{H}^1\times L^2(\mathbb{R}^3)} - \tfrac{1}{4}\|v(t,\cdot)\|^4_{L^4(\mathbb{R}^3)},$$

and it is well-known that solutions with small $\dot{H}^1 \times L^2(\mathbb{R}^3)$-norm exist globally and scatter to zero [Strauss 1981; Mochizuki and Motai 1985; 1987; Pecher 1988], whereas solutions with negative energy blow up in finite time [Glassey 1973; Levine 1974]. There exists an explicit blowup solution $\tilde{v}_T(t,x) = \sqrt{2}/(T-t)$, which describes a stable blowup regime [Donninger and Schörkhuber 2012b] and the blowup speed (but not the profile) of any blowup solution [Merle and Zaag 2005]; see also [Bizoń et al. 2004] for numerical work. By the time translation and reflection symmetries of (1-1) we obtain from $\tilde{v}_T$ the explicit solution $v_0(t,x) = \sqrt{2}/t$, which is now global for $t \geq 1$ and decays in a nondispersive manner. However, in the context of the standard Cauchy problem, where one prescribes data at $t = t_0$ for some t_0 and considers the evolution for $t \geq t_0$, the role of v_0 for the study of global solutions is unclear because v_0 has infinite energy. In the present paper we argue that this is not a defect of the solution v_0 but rather a problem of the usual viewpoint concerning the Cauchy problem. Consequently, we study a different type of initial

The authors would like to thank the Erwin Schrödinger Institute for Mathematical Physics (ESI) in Vienna for hospitality during the workshop "Dynamics of general relativity: black holes and asymptotics" where this work was initiated. Zenginoğlu is supported by the NSF grant PHY-106881 and by a Sherman Fairchild Foundation grant to Caltech.
MSC2010: primary 35L05, 58J45, 35L71; secondary 35Q75, 83C30.
Keywords: nonlinear wave equations, soliton resolution conjecture, hyperboloidal initial value problem, Kelvin coordinates.

value problem for (1-1) where we prescribe data on a spacelike hyperboloid. In this formulation there exists a different "energy" which is finite for v_0.

Hyperboloidal initial value formulations have many advantages over the standard Cauchy problem and are well-known in numerical and mathematical relativity [Eardley and Smarr 1979; Friedrich 1983; Frauendiener 2004; Zenginoğlu 2008]. However, in the mathematical literature on wave equations in flat spacetime, hyperboloidal initial value formulations are less common (with notable exceptions such as [Christodoulou 1986]). We provide a thorough discussion of hyperboloidal methods in Section 2, where we argue that the hyperboloidal initial value problem is natural for hyperbolic equations in view of the underlying Minkowski geometry.

To state our main result, we consider a foliation of the future of the forward null cone emanating from the origin by spacelike hyperboloids

$$\Sigma_T := \left\{(t,x) \in \mathbb{R}\times\mathbb{R}^3 : t = -\frac{1}{2T} + \sqrt{\frac{1}{4T^2} + |x|^2}\right\},$$

where $T \in (-\infty, 0)$. Each Σ_T is parametrized by

$$\Phi_T : B_{|T|} \subset \mathbb{R}^3 \to \mathbb{R}^4, \quad \Phi_T(X) = \left(-\frac{T}{T^2-|X|^2}, \frac{X}{T^2-|X|^2}\right),$$

where $B_R := \{X \in \mathbb{R}^3 : |X| < R\}$ for $R > 0$. The ball $B_{|T|}$ shrinks in time as $T \to 0-$, but its image under Φ_T is an unbounded spacelike hypersurface in Minkowski space. The transformation $(T, X) \mapsto \Phi_T(X)$ has also been used by Christodoulou [1986] to study semilinear wave equations and is known as the Kelvin inversion [Tao 2008]. Note that in four-dimensional notation it can be written as $X^\mu \mapsto -X^\mu/(X^\nu X_\nu)$ (up to a sign in the zero component). To illustrate the resulting initial value problem, we plot the spacelike hyperboloids Σ_T for various values of $T \in (-\infty, 0)$ in a spacetime diagram (left panel) and in a Penrose diagram (right panel) in Figure 1 along with a null surface emanating from the origin. In our formulation of the initial value problem we prescribe data on the hypersurface Σ_{-1} and consider the future development. We refer the reader to Section 2 for a discussion on hyperboloidal foliations and their relation to wave equations.

We define a differential operator ∇_n by

$$\frac{(\nabla_n v)\circ\Phi_T(X)}{T^2-|X|^2} = \partial_T \frac{(v\circ\Phi_T)(X)}{T^2-|X|^2},$$

which one should think of as the normal derivative to the surface Σ_T (although this is not quite correct due to the additional factor $1/(T^2-|X|^2)$). Explicitly, we have

$$\nabla_n v(t,x) = (t^2+|x|^2)\,\partial_t v(t,x) + 2tx^j\,\partial_j v(t,x) + 2tv(t,x).$$

On each leaf Σ_T we define the norms

$$\|v\|^2_{L^2(\Sigma_T)} := \int_{B_{|T|}} \left|\frac{v\circ\Phi_T(X)}{T^2-|X|^2}\right|^2 dX, \quad \|v\|^2_{\dot H^1(\Sigma_T)} := \int_{B_{|T|}} \left|\nabla_X \frac{v\circ\Phi_T(X)}{T^2-|X|^2}\right|^2 dX, \tag{1-2}$$

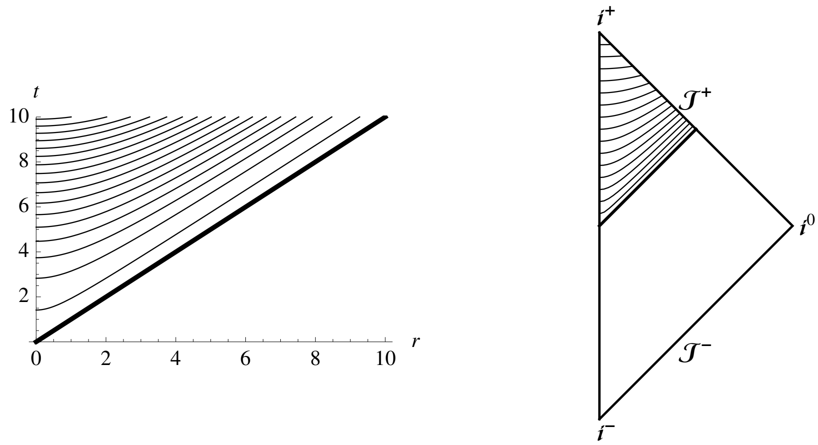

Figure 1. The spacelike hyperboloids Σ_T in a spacetime diagram (left panel) and a Penrose diagram (right panel) together with the null surface emanating from the origin (thick line with 45 degrees to the horizontal). Compare Figure 2.

and we write $\|\cdot\|^2_{H^1(\Sigma_T)} = \|\cdot\|^2_{\dot H^1(\Sigma_T)} + |T|^{-2}\|\cdot\|^2_{L^2(\Sigma_T)}$. We emphasize that

$$v_0 \circ \Phi_T(X) = \sqrt{2}\,\frac{T^2 - |X|^2}{(-T)},$$

and thus, $\|v_0\|_{H^1(\Sigma_T)} + \|\nabla_n v_0\|_{L^2(\Sigma_T)} \simeq |T|^{-\frac{1}{2}}$. Finally, for any subset $A \subset \mathbb{R}^4$ we denote its future domain of dependence by $D^+(A)$. With this notation at hand, we state our main result.

Theorem 1.1. *There exists a codimension-*4 *Lipschitz manifold* $\mathcal{M}$ *of functions in* $H^1(\Sigma_{-1}) \times L^2(\Sigma_{-1})$ *with* $(0, 0) \in \mathcal{M}$ *such that the following holds. For data* $(f, g) \in \mathcal{M}$ *the hyperboloidal initial value problem*

$$\begin{cases} (-\partial_t^2 + \Delta_x)v(t, x) + v(t, x)^3 = 0, \\ v\big|_{\Sigma_{-1}} = v_0\big|_{\Sigma_{-1}} + f, \\ \nabla_n v\big|_{\Sigma_{-1}} = \nabla_n v_0\big|_{\Sigma_{-1}} + g \end{cases}$$

has a unique solution v *defined on* $D^+(\Sigma_{-1})$ *such that*

$$|T|^{\frac{1}{2}}\big(\|v - v_0\|_{H^1(\Sigma_T)} + \|\nabla_n v - \nabla_n v_0\|_{L^2(\Sigma_T)}\big) \lesssim |T|^{\frac{1}{2}-}$$

for all $T \in [-1, 0)$. *As a consequence, for any* $\delta \in (0, 1)$, *we have*

$$\|v - v_0\|_{L^4(t,2t)L^4(B_{(1-\delta)t})} \lesssim t^{-\frac{1}{2}+}$$

as $t \to \infty$, *i.e.*, v *converges to* v_0 *in a localized Strichartz sense.*

Some remarks are in order.

- As usual, by a "solution" we mean a function which solves the equation in an appropriate weak sense, not necessarily in the sense of classical derivatives.

- The manifold $\mathcal{M}$ can be represented as a graph of a Lipschitz function. More precisely, let $\mathcal{H} := H^1(\Sigma_{-1}) \times L^2(\Sigma_{-1})$ and denote by $\mathcal{B}_R(0)$ the open ball of radius $R > 0$ around 0 in $\mathcal{H}$. We prove that there exists a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with $\dim \mathcal{H}_2 = 4$ and a function $F : \mathcal{H}_1 \cap \mathcal{B}_\delta(0) \to \mathcal{H}_2$ such that $\mathcal{M} = \{\vec{u} + F(\vec{u}) : \vec{u} \in \mathcal{H}_1 \cap \mathcal{B}_\delta(0)\}$ provided $\delta > 0$ is chosen sufficiently small. Furthermore, F satisfies

$$\|F(\vec{u}) - F(\vec{v})\|_{\mathcal{H}} \lesssim \delta^{\frac{1}{2}} \|\vec{u} - \vec{v}\|_{\mathcal{H}}$$

  for all $\vec{u}, \vec{v} \in \mathcal{H}_1 \cap \mathcal{B}_\delta(0)$ and $F(\vec{0}) = \vec{0}$.

- The reason for the codimension-4 instability of the attractor v_0 is the invariance of (1-1) under time translations and Lorentz transforms (combined with the Kelvin inversion). The Lorentz boosts do not destroy the nondispersive character of the solution v_0 whereas the time translation does — see the beginning of Section 4 below for a more detailed discussion. In this sense, one may say that there exists a codimension-*one* manifold of data that lead to nondispersive solutions. However, if one fixes v_0, as we have done in our formulation, there are 4 unstable directions.

There was tremendous recent progress in the understanding of universal properties of global solutions to nonlinear wave equations, in particular in the energy critical case; see, for example, [Duyckaerts et al. 2012; 2013; Cote et al. 2012; Kenig et al. 2013]. A guiding principle for all these studies is the soliton resolution conjecture, that is, the idea that global solutions to nonlinear dispersive equations decouple into solitons plus radiation as time tends to infinity. It is known that, in such a strict sense, soliton resolution does not hold in most cases. One possible obstacle is the existence of global solutions which do not scatter. Recently, the first author and Krieger constructed nonscattering solutions for the energy critical focusing wave equation [Donninger and Krieger 2013]; see also [Ortoleva and Perelman 2013] for similar results in the context of the nonlinear Schrödinger equation. These solutions are obtained by considering a rescaled ground state soliton, the existence of which is typical for critical dispersive equations. The cubic wave equation under consideration is energy subcritical and does not admit solitons. Consequently, our result is of a completely different nature. Instead of considering moving solitons, we obtain the nonscattering solutions by perturbing the *self-similar* solution $v_0(t, x) = \sqrt{2}/t$. This can only be done in the framework of a hyperboloidal initial value formulation because the standard energy for the self-similar solution v_0 is infinite.

Another novel feature of our result is a precise description of the data which lead to solutions that converge to v_0: They lie on a Lipschitz manifold of codimension 4. In this respect we believe that our result is also interesting from the perspective of infinite-dimensional dynamical systems theory for wave equations, which is currently a very active field; see, for example, [Krieger et al. 2013a; 2013b; 2012].

Finally, we mention that the present work is motivated by numerical investigations undertaken by Bizoń and the second author [Bizoń and Zenginoğlu 2009]. In particular, the conformal symmetry for the cubic wave equation has been used in [Bizoń and Zenginoğlu 2009] to translate the (linear) stability analysis for blowup to asymptotic results for decay. We exploit this idea in a similar way: If v solves

(1-1) then u, defined by

$$u(T,X)=\frac{1}{T^2-|X|^2}v\left(-\frac{T}{T^2-|X|^2},\frac{X}{T^2-|X|^2}\right)=\frac{v\circ\Phi_T(X)}{T^2-|X|^2},$$

solves $(-\partial_T^2+\Delta_X)u(T,X)+u(T,X)^3=0$. The point is that the coordinate transformation $(t,x)\mapsto(T,X)$ with

$$T=-\frac{t}{t^2-|x|^2},\quad X=\frac{x}{t^2-|x|^2}$$

maps the forward light cone $\{(t,x):|x|<t,\, t>0\}$ to the backward light cone $\{(T,X):|X|<-T,\, T<0\}$ and $t\to\infty$ translates into $T\to 0-$ (see Figure 1). Moreover,

$$\frac{1}{T^2-|X|^2}v_0\left(-\frac{T}{T^2-|X|^2},\frac{X}{T^2-|X|^2}\right)=\frac{\sqrt{2}}{(-T)}=:u_0(T,X)$$

and thus, we are led to the study of the stability of the self-similar blowup solution u_0 in the backward light cone of the origin. In the context of radial symmetry, this problem was recently addressed by Donninger and Schörkhuber [2012b]; see also [Donninger 2011; 2012; Donninger and Schörkhuber 2012a] for similar results in the context of wave maps, Yang–Mills equations, and supercritical wave equations. However, in the present paper we do not assume any symmetry of the data and hence, we develop a stability theory similar to [Donninger and Schörkhuber 2012b] but beyond the radial context. Furthermore, the instabilities of u_0 have a different interpretation in the current setting and lead to the codimension-4 condition in Theorem 1.1 whereas the blowup studied in [Donninger and Schörkhuber 2012b] is stable. The conformal symmetry, although convenient, does not seem crucial for our argument. It appears that one can employ similar techniques to study nondispersive solutions for semilinear wave equations $(-\partial_t^2+\Delta_x)v(t,x)+v(t,x)|v(t,x)|^{p-1}=0$ with more general $p>3$.

Notation. The arguments for functions defined on Minkowski space are numbered by 0, 1, 2, 3 and we write ∂_μ, $\mu\in\{0,1,2,3\}$, for the respective derivatives. Our sign convention for the Minkowski metric η is $(-,+,+,+)$. We use the notation ∂_y for the derivative with respect to the variable y. We employ Einstein's summation convention throughout with Latin indices running from 1 to 3 and Greek indices running from 0 to 3, unless otherwise stated. We denote by $\mathbb{R}_0^+$ the set of positive real numbers including 0.

The letter C (possibly with indices to indicate dependencies) denotes a generic positive constant which may have a different value at each occurrence. The symbol $a\lesssim b$ means $a\le Cb$ and we abbreviate $a\lesssim b\lesssim a$ by $a\simeq b$. We write $f(x)\sim g(x)$ for $x\to a$ if $\lim_{x\to a}f(x)/g(x)=1$.

For a closed linear operator $\boldsymbol{L}$ on a Banach space we denote its domain by $\mathfrak{D}(\boldsymbol{L})$, its spectrum by $\sigma(\boldsymbol{L})$, and its point spectrum by $\sigma_p(\boldsymbol{L})$. We write $\boldsymbol{R_L}(z):=(z-\boldsymbol{L})^{-1}$ for $z\in\rho(\boldsymbol{L})=\mathbb{C}\backslash\sigma(\boldsymbol{L})$. The space of bounded operators on a Banach space $\mathcal{X}$ is denoted by $\mathcal{B}(\mathcal{X})$.

2. Wave equations and geometry

In this section, we present the motivation for using hyperboloidal coordinates in our analysis and provide some background. We discuss the main arguments and tools in a pedagogical manner to emphasize the relation between spacetime geometry and wave equations for readers not familiar with relativistic terminology.

Geometric preliminaries. A spacetime $(\mathcal{M}, g)$ is a four-dimensional paracompact Hausdorff manifold $\mathcal{M}$ with a time-oriented Lorentzian metric g. The cubic wave equation (1-1) is posed on the Minkowski spacetime $(\mathbb{R}^4, \eta)$. In standard time t and Cartesian coordinates (x, y, z) the Minkowski metric reads

$$\eta = -dt^2 + dx^2 + dy^2 + dz^2, \qquad (t, x, y, z) \in \mathbb{R}^4.$$

Minkowski spacetime is *spherically symmetric*, i.e., the group SO(3) acts nontrivially by isometry on $(\mathbb{R}^4, \eta)$. We introduce the quotient space $\mathfrak{Q} = \mathbb{R}^4 / \mathrm{SO}(3)$ and the area radius $r : \mathfrak{Q} \to \mathbb{R}$ such that the group orbits of points $p \in \mathfrak{Q}$ have area $4\pi r^2(p)$. The area radius can be written as $r = \sqrt{x^2 + y^2 + z^2}$ with respect to Cartesian coordinates. The flat metric can then be written as $\eta = q + r^2\, d\sigma^2$, where q is a rank-2 Lorentzian metric and $d\sigma^2$ is the standard metric on S^2. Choosing the usual angular variables for $d\sigma^2$, we obtain the familiar form of the flat spacetime metric in spherical coordinates

$$\eta = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\phi^2), \quad (t, r, \theta, \phi) \in \mathbb{R} \times \mathbb{R}_+ \times [0, \pi] \times [0, 2\pi).$$

A codimension-one submanifold is called a *hypersurface* and a *foliation* is a one-parameter family of nonintersecting spacelike hypersurfaces. A foliation can also be defined by a time function from $\mathcal{M}$ to the real line $\mathbb{R}$, whose level sets are the hypersurfaces of the foliation.

We can restrict our discussion of the interaction between hyperbolic equations and spacetime geometry to spherical symmetry without loss of generality because the radial direction is sufficient for exploiting the Lorentzian structure. Working in the two-dimensional quotient spacetime $(\mathfrak{Q}, q)$ also allows us to illustrate the geometric definitions in two-dimensional plots.

Compactification and Penrose diagrams. It is useful to introduce Penrose diagrams to depict global features of time foliations in spherically symmetric spacetimes. Penrose presented the construction of the diagrams in his study of the asymptotic behavior of gravitational fields in 1963 [Penrose 2011]. A beautiful exposition of Penrose diagrams has been given in [Dafermos and Rodnianski 2005]. As we are working in Minkowski spacetime only, the main features of Penrose diagrams of interest to us are the compactification and the preservation of the causal structure. See, for example, [Christodoulou 1986; Keel and Tao 1998] for the application of Penrose compactification to study wave equations in flat spacetime.

The image of the Penrose diagram is a two-dimensional Minkowski spacetime with a *bounded* global null coordinate system. Causal concepts extend through the boundary of the map. Consider the rank-2 Minkowski metric q on the quotient manifold $\mathfrak{Q}$

$$q = -dt^2 + dr^2, \quad (t, r) \in \mathbb{R} \times \mathbb{R}_0^+. \tag{2-1}$$

To map this metric to a global, bounded, null coordinate system, define $u = t - r$ and $v = t + r$ for $v \geq u$, and compactify by $U = \arctan u$ and $V = \arctan v$. The quotient metric becomes

$$q = -\frac{1}{\cos^2 V \cos^2 U}\, dV\, dU \quad (-\pi/2 < U \leq V < \pi/2).$$

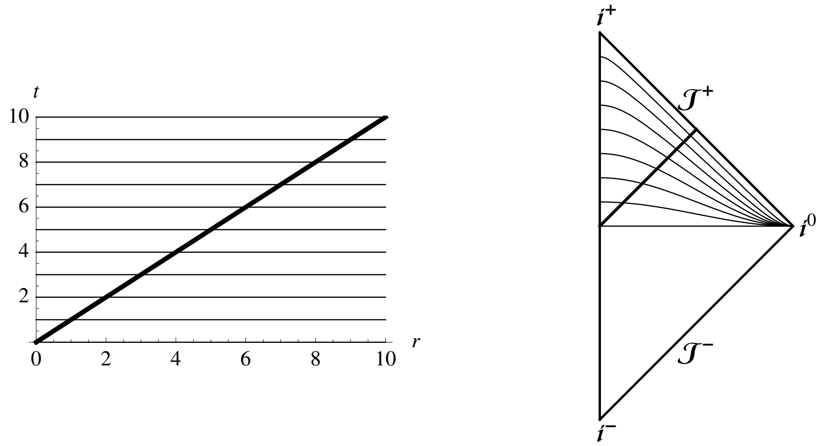

Figure 2. The level sets of the standard time t depicted in a spacetime diagram (left panel) and a Penrose diagram (right panel) together with a characteristic line from the origin. The boundary of the Penrose diagram includes the spatial origin and various notions of infinity. Past and future timelike infinity are depicted by points i^- and i^+. The vertical line connecting i^- and i^+ is the spatial origin $r = 0$. Spatial infinity is denoted by the point i^0. Null curves reach past and future null infinity, denoted by $\mathscr{I}^-$ and $\mathscr{I}^+$, for infinite values of their affine parameter.

Points at infinity with respect to the original coordinates have finite values with respect to the compactifying coordinates. The singular behavior of the metric in compactifying coordinates at the boundary can be compensated by a conformal rescaling with the conformal factor $\Omega = \cos V \cos U$, so that the rescaled metric

$$\bar{q} = \Omega^2 q = -dU\, dV$$

is well defined on the domain $(-\pi/2 \le U \le V \le \pi/2)$ including points that are at infinity with respect to q. We say that q can be conformally extended beyond infinity.

The Penrose diagram is then drawn using time and space coordinates $T = (V+U)/2$ and $R = (V-U)/2$ (see Figure 2). The resulting metric $\bar{q} = -dT^2 + dR^2$ is flat. The combined Penrose map is given by

$$t \mapsto \frac{1}{2}\left(\tan\frac{T+R}{2} + \tan\frac{T-R}{2}\right), \qquad r \mapsto \frac{1}{2}\left(\tan\frac{T+R}{2} - \tan\frac{T-R}{2}\right).$$

The boundary $\partial\bar{\mathfrak{Q}} = \{T = \pm(\pi - R),\ R \in [0, \pi]\}$ corresponds to points at infinity with respect to the original Minkowski metric. Asymptotic behavior of fields on $\mathfrak{Q}$ can be studied using local differential geometry near this boundary where the conformal factor $\Omega = \cos T + \cos R$ vanishes. The part of the boundary without the points at $R = 0, \pi$ is denoted by $\mathscr{I} = \{T = \pm(\pi - R),\ R \in (0, \pi)\}$. This part is referred to as null infinity because null geodesics reach it for an infinite value of their affine parameter. The differential of the conformal factor is nonvanishing at $\mathscr{I}$, $d\Omega|_{\mathscr{I}} \neq 0$, and $\mathscr{I}$ consists of two parts $\mathscr{I}^-$ and $\mathscr{I}^+$ referred to as past and future null infinity.

Hyperboloidal coordinates and wave equations. Equipped with the tools above we now turn to the interplay between wave equations and spacetime geometry. Consider the free wave equation

$$u_{tt} - \Delta u = -\eta^{\mu\nu}\,\partial_\mu\partial_\nu u = 0. \tag{2-2}$$

Radial solutions for the rescaled field $v := ru$ obey the two-dimensional free wave equation

$$v_{tt} - v_{rr} = 0, \tag{2-3}$$

on $(t, r) \in \mathbb{R}^+_0 \times \mathbb{R}^+_0$ with vanishing boundary condition at the origin. Initial data are specified on the $t = 0$ hypersurface. The general solution to this system is such that the data propagate to infinity and leave nothing behind due to the validity of Huygens' principle. Intuitively, this behavior seems to contradict two well-known properties of the free wave equation: *conservation of energy* and *time reversibility*.

The conserved energy for the free wave equation (2-3) reads

$$E(v) = \int_0^\infty \tfrac{1}{2}\big(v_t(t, r)^2 + v_r(t, r)^2\big)\,dr.$$

The conservation of energy is counterintuitive because the waves propagate to infinity leaving nothing behind. One would expect a natural energy norm to decrease rapidly to zero with a nonpositive energy flux at infinity. The conservation of energy, however, implies that at very late times the solution is in some sense similar to the initial state [Tao 2008].

Another counterintuitive property of the free wave equation is its time reversibility, meaning that if $u(t, r)$ solves the equation, so does $u(-t, r)$. Data on a Cauchy hypersurface determine the solution at all future *and past* times in contrast to parabolic (dissipative) equations which are solvable only forward in time due to loss of energy to the future.

Both of these counterintuitive properties depend on our description of the problem. We can choose coordinates in which energy conservation and time reversibility are violated. Of course, it is always possible to find coordinates which break symmetries or hide features of an equation. We argue below that the hyperboloidal coordinates we employ emphasize the intuitive properties of the equation rather than blur them.

The reason behind the conservation of energy integrated along level sets of t can be seen in the Penrose diagram Figure 2. The outgoing characteristic line along which the wave propagates to infinity intersects all leaves of the t-foliation. When the energy expression is integrated globally, the energy of the initial wave will therefore still contribute to the result. The hyperboloidal T-foliation depicted in Figure 1, however, allows for outgoing null rays to leave the leaves of the foliation. Therefore one would expect that the energy flux through infinity is negative when integrated along the leaves of the hyperboloidal foliation.

The wave equation (2-3) has the same form in hyperboloidal coordinates:

$$w_{TT} - w_{RR} = 0,$$

where

$$w(T, R) = v\left(-\frac{T}{T^2 - R^2}, \frac{R}{T^2 - R^2}\right).$$

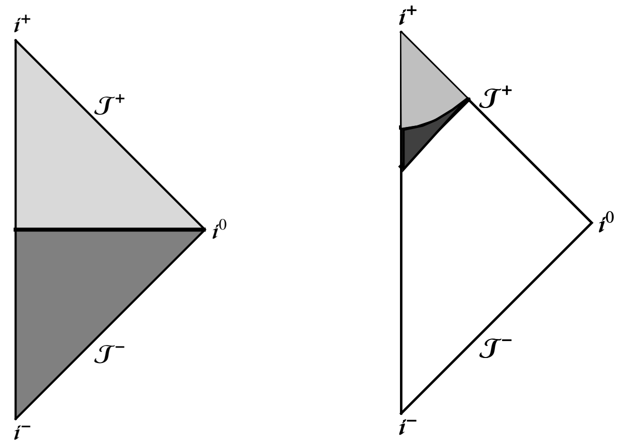

Figure 3. Comparison of the future (light gray) and past (dark gray) domains of dependence for the Cauchy surface $t = 0$ (left) and the hyperboloidal surface $T = -1$ (right).

Energy conservation and time reversibility seem valid for this equation as well, but here we have the shrinking, bounded spatial domain $R \in [0, -T)$ where $T \to 0-$. The energy integrated along the leaves of this domain

$$E(w) = \int_0^{-T} \tfrac{1}{2}\big(w_T(T, R)^2 + w_R(T, R)^2\big)\, dR$$

decays in time. The energy flux reads

$$\frac{\partial E}{\partial T} = -\tfrac{1}{2}\big(w_T(T, -T) - w_R(T, -T)\big)^2 \leq 0.$$

The energy flux through infinity vanishes only if the solution is constant or is propagating along future null infinity. When the solution has an outgoing component through future null infinity, the energy decays in time. This behavior is in accordance with physical intuition.

Consider the time reversibility. The equation in the new coordinates is time-reversible, but the hyperboloidal initial value problem is not. Formally, this is again a consequence of the time dependence of the spatial domain given by $R < -T$. Geometrically, we see in Figure 3 that the union of the past and future domain of dependence of the hyperboloidal surface $T = -1$ covers only a portion of Minkowski spacetime whereas for the Cauchy surface $t = 0$ such a union gives the global spacetime.

In summary, the hyperboloidal foliation given by the Kelvin inversion captures quantitatively the propagation of energy to infinity and leads to a time-irreversible wave propagation problem. Further, the transformation translates asymptotic analysis for $t \to \infty$ to local analysis for $T \to 0-$.

3. Derivation of the equations and preliminaries

First-order formulation and similarity coordinates. We start from $(-\partial_T^2 + \Delta_X)u(T, X) + u(T, X)^3 = 0$ in the hyperboloidal coordinates $T = -t/(t^2 - |x|^2)$, $X = x/(t^2 - |x|^2)$ for the rescaled unknown

$$u(T,X)=\frac{1}{T^2-|X|^2}v\left(-\frac{T}{T^2-|X|^2},\frac{X}{T^2-|X|^2}\right).$$

As discussed in the introduction, the domain we are interested in is $T\in[-1,0)$ and $|X|<|T|$. Our intention is to study the stability of the self-similar solution $u_0(T)=\sqrt{2}/(-T)$. Thus, it is natural to introduce the similarity coordinates

$$\tau=-\log(-T),\quad \xi=\frac{X}{-T} \tag{3-1}$$

with domain $\tau\geq 0$ and $|\xi|<1$. The derivatives transform according to

$$\partial_T=e^{\tau}(\partial_\tau+\xi^j\partial_{\xi^j}),\quad \partial_{X^j}=e^{\tau}\partial_{\xi^j}.$$

This implies

$$\partial_T^2=e^{2\tau}(\partial_\tau^2+\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau+\xi^j\xi^k\partial_{\xi^j}\partial_{\xi^k}+2\xi^j\partial_{\xi^j})$$

and $\partial_{X_j}\partial_{X^j}=e^{2\tau}\partial_{\xi_j}\partial_{\xi^j}$. Consequently, for the function

$$U(\tau,\xi):=u(-e^{-\tau},e^{-\tau}\xi)$$

we obtain from $(-\partial_T^2+\partial_{X_j}\partial_{X^j})u(T,X)+u(T,X)^3=0$ the equation

$$[\partial_\tau^2+\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau-(\delta^{jk}-\xi^j\xi^k)\partial_{\xi^j}\partial_{\xi^k}+2\xi^j\partial_{\xi^j}]U(\tau,\xi)=e^{-2\tau}U(\tau,\xi)^3.$$

To get rid of the time-dependent prefactor on the right-hand side, we rescale and set $U(\tau,\xi)=e^{\tau}\psi(\tau,\xi)$, which yields

$$[\partial_\tau^2+3\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau-(\delta^{jk}-\xi^j\xi^k)\partial_{\xi^j}\partial_{\xi^k}+4\xi^j\partial_{\xi^j}+2]\psi(\tau,\xi)=\psi(\tau,\xi)^3. \tag{3-2}$$

The fundamental self-similar solution is given by

$$\psi_0(\tau,\xi):=e^{-\tau}u_0(-e^{-\tau},e^{-\tau}\xi)=\sqrt{2}.$$

Writing $\psi=\sqrt{2}+\phi$ we find the equation

$$\begin{aligned}[\partial_\tau^2+3\partial_\tau+2\xi^j\partial_{\xi^j}\partial_\tau-(\delta^{jk}-\xi^j\xi^k)\partial_{\xi^j}\partial_{\xi^k}+4\xi^j\partial_{\xi^j}+2]\phi(\tau,\xi)&\\ =6\phi(\tau,\xi)+3\sqrt{2}\phi(\tau,\xi)^2+\phi(\tau,\xi)^3.&\end{aligned} \tag{3-3}$$

In summary, we have applied the coordinate transformation

$$\tau=-\log\frac{t}{t^2-|x|^2},\quad \xi=\frac{x}{t}$$

with inverse

$$t=\frac{e^{\tau}}{1-|\xi|^2},\quad x=\frac{e^{\tau}\xi}{1-|\xi|^2}$$

and $\phi(\tau,\xi)$ solves (3-3) for $\tau>0$ and $|\xi|<1$ if and only if

$$v(t,x)=\frac{\sqrt{2}}{t}+\frac{1}{t}\phi\left(-\log\frac{t}{t^2-|x|^2},\frac{x}{t}\right) \tag{3-4}$$

solves $(-\partial_t^2+\Delta_x)v(t,x)+v(t,x)^3=0$ for $(t,x)\in D^+(\Sigma_{-1})$.

We have $\partial_T u(T,X)=e^{2\tau}(\partial_\tau+\xi^j\partial_{\xi^j}+1)\psi(\tau,\xi)$ and thus, it is natural to use the variables $\phi_1=\phi$, $\phi_2=\partial_0\phi+\xi^j\partial_j\phi+\phi$ in a first-order formulation. We obtain

$$\begin{aligned}\partial_0\phi_1&=-\xi^j\partial_j\phi_1-\phi_1+\phi_2,\\ \partial_0\phi_2&=\partial_j\partial^j\phi_1-\xi^j\partial_j\phi_2-2\phi_2+6\phi_1+3\sqrt{2}\phi_1^2+\phi_1^3.\end{aligned}\tag{3-5}$$

For later reference we also note that (3-4) implies

$$t^2\partial_t v(t,x)=-\sqrt{2}-\frac{2t^2}{t^2-|x|^2}\left(\frac{x^j}{t}\partial_j\phi_1+\phi_1\right)+\frac{t^2+|x|^2}{t^2-|x|^2}\phi_2,\tag{3-6}$$

where it is understood, of course, that $\phi_1(\tau,\xi)$ and $\phi_2(\tau,\xi)$ are evaluated at $\tau=-\log\frac{t}{t^2-|x|^2}$ and $\xi=x/t$.

Norms. Since our approach is perturbative in nature, the function space in which we study (3-5) should be determined by the *free* version of (3-5), i.e.,

$$\begin{aligned}\partial_0\phi_1&=-\xi^j\partial_j\phi_1-\phi_1+\phi_2,\\ \partial_0\phi_2&=\partial_j\partial^j\phi_1-\xi^j\partial_j\phi_2-2\phi_2.\end{aligned}$$

The natural choice for a norm is derived from the standard energy $\dot{H}^1\times L^2$ of the free wave equation. In the present formulation this translates into

$$\|\phi_1(\tau,\cdot)\|_{\dot{H}^1(B)}+\|\phi_2(\tau,\cdot)\|_{L^2(B)},$$

where $B=\{\xi\in\mathbb{R}^3:|\xi|<1\}$. However, there is a slight technical problem since this is only a seminorm (the point is that we are working on the bounded domain B). In order to go around this difficulty, let us for the moment return to the radial context and consider the free wave equation in $\mathbb{R}^{1+3}$

$$u_{tt}-u_{rr}-\frac{2}{r}u_r=0,$$

in the standard coordinates t and $r=|x|$. Now we make the following observation. The conserved energy is given by

$$E(u)=\frac{1}{2}\int_0^\infty[u_t^2+u_r^2]r^2\,dr.$$

On the other hand, by setting $v=ru$, we obtain

$$v_{tt}-v_{rr}=0$$

with conserved energy $\frac{1}{2}\int_0^\infty[v_t^2+v_r^2]\,dr$, or, in terms of u,

$$E'(u)=\frac{1}{2}\int_0^\infty[r^2u_t^2+(ru_r+u)^2]\,dr.$$

The obvious question now is: how are E and E' related? An integration by parts shows that E and E' are equivalent, up to a boundary term $\lim_{r\to\infty}ru(r)^2$ which may be ignored by assuming some decay at spatial infinity. However, if we consider the *local* energy contained in a ball of radius R, the boundary term can no longer be ignored and one has the identity

$$E'_R(u) := \frac{1}{2}\int_0^R [r^2 u_t^2 + (ru_r + u)^2]\,dr = \frac{1}{2}Ru(R)^2 + \frac{1}{2}\int_0^R [u_t^2 + u_r^2]r^2\,dr.$$

The expression on the right-hand side is the standard energy with the term $\frac{1}{2}Ru(R)^2$ added. This small modification has important consequences because unlike the standard energy, this now defines a norm. Furthermore, $E'_R(u)$ is bounded along the wave flow since it is the local version of a positive definite conserved quantity.

In the nonradial context the above discussion suggests to take

$$\|\phi_1(\tau,\cdot)\|_{\dot H^1(B)} + \|\phi_1(\tau,\cdot)\|_{L^2(\partial B)} + \|\phi_2(\tau,\cdot)\|_{L^2(B)}.$$

This norm is not very handy, but fortunately we have equivalence to $H^1 \times L^2(B)$ as the following result shows.

Lemma 3.1. *We have*[1]

$$\|f\|_{H^1(B)} \simeq \|f\|_{\dot H^1(B)} + \|f\|_{L^2(\partial B)}.$$

Proof. For $x \in \mathbb{R}^3$ we write $r = |x|$ and $\omega = x/|x|$. With this notation we have $f(x) = f(r\omega)$ and

$$\|f\|_{L^2(B)}^2 = \int_0^1 \int_{\partial B} |f(r\omega)|^2\,d\sigma(\omega) r^2\,dr,$$

where $d\sigma$ denotes the surface measure on the sphere. First, we prove $\|f\|_{L^2(B)} \lesssim \|f\|_{\dot H^1(B)} + \|f\|_{L^2(\partial B)}$. By density it suffices to consider $f \in C^\infty(\bar B)$. The fundamental theorem of calculus and Cauchy–Schwarz imply

$$rf(r\omega) = \int_0^r \partial_s[sf(s\omega)]\,ds \le \left(\int_0^1 |\partial_r[rf(r\omega)]|^2\,dr\right)^{1/2}.$$

Expanding the square and integrating by parts yields

$$\begin{aligned}\int_0^1 |\partial_r[rf(r\omega)]|^2\,dr &= \int_0^1 |\partial_r f(r\omega)|^2 r^2\,dr + \int_0^1 r\,\partial_r|f(r\omega)|^2\,dr + \int_0^1 |f(r\omega)|^2\,dr \\ &= |f(\omega)|^2 + \int_0^1 |\partial_r f(r\omega)|^2 r^2\,dr\end{aligned}$$

and thus,

$$r^2|f(r\omega)|^2 \le |f(\omega)|^2 + \int_0^1 |\omega^j \partial_j f(r\omega)|^2 r^2\,dr.$$

Integrating this inequality over the ball B yields the desired estimate. In order to finish the proof, it suffices to show that $\|f\|_{L^2(\partial B)} \lesssim \|f\|_{H^1(B)}$, but this is just the trace theorem (see, e.g., [Evans 1998, p. 258, Theorem 1]). □

[1] As usual, $\|f\|_{L^2(\partial B)}$ has to be understood in the trace sense.

4. Linear perturbation theory

The goal of this section is to develop a functional analytic framework for studying the Cauchy problem for the *linearized* equation

$$\begin{aligned}\partial_0\phi_1 &= -\xi^j\partial_j\phi_1 - \phi_1 + \phi_2,\\ \partial_0\phi_2 &= \partial_j\partial^j\phi_1 - \xi^j\partial_j\phi_2 - 2\phi_2 + 6\phi_1.\end{aligned} \tag{4-1}$$

The main difficulty lies with the fact that the differential operators involved are not self-adjoint. It is thus natural to apply semigroup theory for studying (4-1). Before doing so, however, we commence with a heuristic discussion on instabilities. The equation $(-\partial_T^2 + \Delta_X)u(T, X) + u(T, X)^3 = 0$ is invariant under time translations $T \mapsto T - a$ and the three Lorentz boosts for each direction X^j

$$\begin{cases} T &\mapsto\; T\cosh a - X^j \sinh a,\\ X^j &\mapsto\; -T\sinh a + X^j\cosh a,\\ X^k &\mapsto\; X^k \quad (k \neq j),\end{cases}$$

where $a \in \mathbb{R}$ is a parameter (the rapidity in case of the Lorentz boost). In general, if u_a is a one-parameter family of solutions to a nonlinear equation $F(u_a) = 0$, one obtains (at least formally)

$$0 = \partial_a F(u_a) = DF(u_a)\partial_a u_a$$

and thus, $\partial_a u_a$ is a solution of the linearization of $F(u) = 0$ at $u = u_a$. In our case we linearize around the solution $u_0(T, X) = \sqrt{2}/(-T)$. The time translation symmetry yields the one-parameter family $u_a(T, X) := \sqrt{2}/(a - T)$ and we have $\partial_a u_a(T, X)|_{a=0} = -\sqrt{2}/T^2$. Taking into account the above transformations that led from u to ϕ_1, ϕ_2, we obtain (after a suitable normalization) the functions

$$\phi_1(\tau, \xi) = e^\tau, \quad \phi_2(\tau, \xi) = 2e^\tau, \tag{4-2}$$

and a simple calculation shows that (4-2) indeed solve (4-1). Thus, there exists a growing solution of (4-1). Similarly, for the Lorentz boosts we consider

$$u_{a,j}(T, X) = \frac{\sqrt{2}}{X^j\sinh a - T\cosh a}$$

and thus, $\partial_a u_{a,j}(T, X)|_{a=0} = -(\sqrt{2}/T^2)X^j$. By recalling that $X^j/(-T) = \xi^j$, this yields the functions

$$\phi_1(\tau, \xi) = \xi^j, \quad \phi_2(\tau, \xi) = 2\xi^j, \tag{4-3}$$

and it is straightforward to check that (4-3) indeed solve (4-1). This time the solution (4-3) is not growing in τ but it is not decaying either. It is important to emphasize that in our context, the time translation symmetry leads to a *real* instability. The reason is that $u_a(T, X) = \sqrt{2}/(a - T)$ yields the solution

$$v_a(t, x) = \frac{1}{t^2 - |x|^2}\,\frac{\sqrt{2}}{a + \frac{t}{t^2 - |x|^2}} = \frac{\sqrt{2}}{t + a(t^2 - |x|^2)}$$

of the original problem. This solution is part of a two-parameter family conjectured to describe generic radial solutions of the focusing cubic wave equation [Bizoń and Zenginoğlu 2009]. If $a \neq 0$, $v_a(t,x)$ decays like t^{-2} as $t \to \infty$ for each fixed $x \in \mathbb{R}^3$. This is the generic (dispersive) decay. On the other hand, the Lorentz transforms lead to apparent instabilities since the function $u_{a,j}$ yields the solution $v_{a,j}(t,x) = \sqrt{2}/(t\cosh a + x^j \sinh a)$ of the original problem which still displays the nondispersive decay. Consequently, we expect a codimension-one manifold of initial data that lead to nondispersive decay, as mentioned in the introduction. Since we are working with a fixed u_0, however, there is a four-dimensional unstable subspace of the linearized operator (to be defined below). This observation eventually leads to the codimension-4 statement in our Theorem 1.1. Note that other symmetries of the equation such as scaling, space translations, and space rotations do not play a role in this context as the solution u_0 is invariant under these.

A semigroup formulation for the free evolution. We start the rigorous treatment by considering the free wave equation in similarity coordinates given by the system

$$\begin{aligned} \partial_0\phi_1 &= -\xi^j\partial_j\phi_1 - \phi_1 + \phi_2, \\ \partial_0\phi_2 &= \partial_j\partial^j\phi_1 - \xi^j\partial_j\phi_2 - 2\phi_2. \end{aligned} \tag{4-4}$$

From (4-4) we read off the generator

$$\tilde{\boldsymbol{L}}_0\boldsymbol{u}(\xi) = \begin{pmatrix} -\xi^j\partial_j u_1(\xi) - u_1(\xi) + u_2(\xi) \\ \partial_j\partial^j u_1(\xi) - \xi^j\partial_j u_2(\xi) - 2u_2(\xi) \end{pmatrix},$$

acting on functions in $\mathcal{D}(\tilde{\boldsymbol{L}}_0) := H^2(B)\cap C^2(\bar{B}\backslash\{0\}) \times H^1(B)\cap C^1(\bar{B}\backslash\{0\})$. With this notation we rewrite (4-4) as an ODE

$$\frac{d}{d\tau}\Phi(\tau) = \tilde{\boldsymbol{L}}_0\Phi(\tau).$$

The appropriate framework for studying such a problem is provided by semigroup theory, i.e., our goal is to find a suitable Hilbert space $\mathcal{H}$ such that there exists a map $\boldsymbol{S}_0 : [0,\infty) \to \mathcal{B}(\mathcal{H})$ satisfying

- $\boldsymbol{S}_0(0) = \mathrm{id}_{\mathcal{H}}$,
- $\boldsymbol{S}_0(\tau)\boldsymbol{S}_0(\sigma) = \boldsymbol{S}_0(\tau+\sigma)$ for all $\tau, \sigma \geq 0$,
- $\lim_{\tau\to 0+} \boldsymbol{S}_0(\tau)\boldsymbol{u} = \boldsymbol{u}$ for all $\boldsymbol{u} \in \mathcal{H}$,
- $\lim_{\tau\to 0+}(1/\tau)[\boldsymbol{S}_0(\tau)\boldsymbol{u} - \boldsymbol{u}] = \boldsymbol{L}_0\boldsymbol{u}$ for all $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$, where $\boldsymbol{L}_0$ is the closure of $\tilde{\boldsymbol{L}}_0$.

Given such an $\boldsymbol{S}_0$, the function $\Phi(\tau) = \boldsymbol{S}_0(\tau)\Phi(0)$ solves $d\Phi(\tau)/d\tau = \boldsymbol{L}_0\Phi(\tau)$.

Motivated by the above discussion we define a sesquilinear form on $\tilde{\mathcal{H}} := H^1(B)\cap C^1(B) \times L^2(B)\cap C(B)$ by

$$(\boldsymbol{u} \mid \boldsymbol{v}) := \int_B \partial_j u_1(\xi)\overline{\partial^j v_1(\xi)}\,d\xi + \int_{\partial B} u_1(\omega)\overline{v_1(\omega)}\,d\sigma(\omega) + \int_B u_2(\xi)\overline{v_2(\xi)}\,d\xi.$$

Lemma 3.1 implies that $(\cdot\mid\cdot)$ is an inner product on $\tilde{\mathcal{H}}$, and as usual we denote the induced norm by $\|\cdot\|$. Furthermore, we write $\mathcal{H}$ for the completion of $\tilde{\mathcal{H}}$ with respect to $\|\cdot\|$. We remark that $\mathcal{H}$ is equivalent to $H^1(B) \times L^2(B)$ as a Banach space by Lemma 3.1.

Proposition 4.1. *The operator $\tilde{\boldsymbol{L}}_0 : \mathcal{D}(\tilde{\boldsymbol{L}}_0) \subset \mathcal{H} \to \mathcal{H}$ is closable and its closure, denoted by $\boldsymbol{L}_0$, generates a strongly continuous semigroup $\boldsymbol{S}_0 : [0,\infty) \to \mathcal{B}(\mathcal{H})$ satisfying $\|\boldsymbol{S}_0(\tau)\| \leq e^{-\frac{1}{2}\tau}$ for all $\tau \geq 0$. In particular, we have $\sigma(\boldsymbol{L}_0) \subset \{z \in \mathbb{C} : \operatorname{Re} z \leq -\frac{1}{2}\}$.*

The proof of Proposition 4.1 requires the following technical lemma.

Lemma 4.2. *Let $f \in L^2(B)$ and $\varepsilon > 0$ be arbitrary. Then there exists a function $u \in H^2(B) \cap C^2(\bar{B}\backslash\{0\})$ such that $g \in L^2(B) \cap C(\bar{B}\backslash\{0\})$, defined by*

$$g(\xi) := -(\delta^{jk} - \xi^j \xi^k)\partial_j \partial_k u(\xi) + 5\xi^j \partial_j u(\xi) + \tfrac{15}{4} u(\xi), \tag{4-5}$$

satisfies $\|f - g\|_{L^2(B)} < \varepsilon$.

Proof. Since $C^\infty(\bar{B}) \subset L^2(B)$ is dense, we can find a $\tilde{g} \in C^\infty(\bar{B})$ such that $\|f - \tilde{g}\|_{L^2(B)} < \varepsilon/2$. We consider the equation

$$-(\delta^{jk} - \xi^j \xi^k)\partial_j \partial_k u(\xi) + 5\xi^j \partial_j u(\xi) + \tfrac{15}{4} u(\xi) = \tilde{g}(\xi). \tag{4-6}$$

In order to solve (4-6) we define $\rho(\xi) = |\xi|$, $\omega(\xi) = \xi/|\xi|$ and note that

$$\partial_j \rho(\xi) = \omega_j(\xi), \quad \partial_j \omega^k(\xi) = \frac{\delta_j{}^k - \omega_j(\xi)\omega^k(\xi)}{\rho(\xi)}.$$

Thus, interpreting ρ and ω as new coordinates, we obtain

$$\begin{aligned}
\xi^j \partial_j u(\xi) &= \rho \partial_\rho u(\rho\omega), \\
\xi^j \xi^k \partial_j \partial_k u(\xi) &= \xi^j \partial_{\xi^j}[\xi^k \partial_{\xi^k} u(\xi)] - \xi^j \partial_j u(\xi) = \rho^2 \partial_\rho^2 u(\rho\omega)
\end{aligned}$$

as well as

$$\partial^j \partial_j u(\rho\omega) = \left[\partial_\rho^2 + \frac{d-1}{\rho}\partial_\rho + \frac{\delta^{jk} - \omega^j \omega^k}{\rho^2}\partial_{\omega^j}\partial_{\omega^k} - \frac{d-1}{\rho^2}\omega^j \partial_{\omega^j}\right] u(\rho\omega),$$

where $d = 3$ is the spatial dimension. Consequently, (4-6) can be written as

$$\left[-(1-\rho^2)\partial_\rho^2 - \frac{2}{\rho}\partial_\rho + 5\rho\partial_\rho + \frac{15}{4} - \frac{1}{\rho^2}\Delta_{S^2}\right] u(\rho\omega) = \tilde{g}(\rho\omega), \tag{4-7}$$

where $-\Delta_{S^2}$ is the Laplace–Beltrami operator on S^2. The operator $-\Delta_{S^2}$ is self-adjoint on $L^2(S^2)$ and we have $\sigma(-\Delta_{S^2}) = \sigma_p(-\Delta_{S^2}) = \{\ell(\ell+1) : \ell \in \mathbb{N}_0\}$. The eigenspace to the eigenvalue $\ell(\ell+1)$ is $(2\ell+1)$-dimensional and spanned by the spherical harmonics $\{Y_{\ell,m} : m \in \mathbb{Z}, -\ell \leq m \leq \ell\}$ which are obtained by restricting harmonic homogeneous polynomials in $\mathbb{R}^3$ to the two-sphere S^2; see, for example, [Atkinson and Han 2012] for an up-to-date account of this classical subject. We may expand $\tilde{g}$ according to

$$\tilde{g}(\rho\omega) = \sum_{\ell,m}^{\infty} g_{\ell,m}(\rho) Y_{\ell,m}(\omega),$$

where $\sum_{\ell,m}^{\infty}$ is shorthand for $\sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}$ and for any fixed $\rho \in [0, 1]$, the sum converges in $L^2(S^2)$; see [Atkinson and Han 2012, p. 66, Theorem 2.34]. The expansion coefficient $g_{\ell,m}(\rho)$ is given by

$$g_{\ell,m}(\rho) = (\tilde{g}(\rho\,\cdot\,) \mid Y_{\ell,m})_{L^2(S^2)} := \int_{S^2} \tilde{g}(\rho\omega)\overline{Y_{\ell,m}(\omega)}\, d\sigma(\omega)$$

and by dominated convergence it follows that $g_{\ell,m} \in C^\infty[0, 1]$. Furthermore, by using the identity $Y_{\ell,m} = [\ell(\ell+1)]^{-1}(-\Delta_{S^2})Y_{\ell,m}$ and the self-adjointness of $-\Delta_{S^2}$ on $L^2(S^2)$, we obtain

$$\begin{aligned} g_{\ell,m}(\rho) &= \frac{1}{\ell(\ell+1)}\big(\tilde{g}(\rho\,\cdot\,)\,\big|\,(-\Delta_{S^2})Y_{\ell,m}\big)_{L^2(S^2)} \\ &= \frac{1}{\ell(\ell+1)}\big((-\Delta_{S^2})\tilde{g}(\rho\,\cdot\,)\,\big|\,Y_{\ell,m}\big)_{L^2(S^2)}. \end{aligned}$$

Consequently, by iterating this argument we see that the smoothness of $\tilde{g}$ implies the pointwise decay $\|g_{\ell,m}\|_{L^\infty(0,1)} \le C_M \ell^{-M}$ for any $M \in \mathbb{N}$ and all $\ell \in \mathbb{N}$. Now we set

$$g_N(\xi) := \sum_{\ell,m}^{N} g_{\ell,m}(|\xi|)Y_{\ell,m}\Big(\frac{\xi}{|\xi|}\Big)$$

and note that $\|g_N(\rho\,\cdot\,) - \tilde{g}(\rho\,\cdot\,)\|_{L^2(S^2)} \to 0$ as $N \to \infty$. Furthermore, by $\|g_{\ell,m}\|_{L^\infty(0,1)} \lesssim \ell^{-2}$ for all $\ell \in \mathbb{N}$ we infer

$$\sup_{\rho\in(0,1)} \|g_N(\rho\,\cdot\,) - \tilde{g}(\rho\,\cdot\,)\|_{L^2(S^2)} \lesssim 1$$

for all $N \in \mathbb{N}$ and dominated convergence yields

$$\|g_N - \tilde{g}\|^2_{L^2(B)} = \int_0^1 \|g_N(\rho\,\cdot\,) - \tilde{g}(\rho\,\cdot\,)\|^2_{L^2(S^2)}\rho^2\, d\rho \to 0$$

as $N \to \infty$. Thus, we may choose N so large that $\|g_N - \tilde{g}\|_{L^2(B)} < \varepsilon/2$.

By making the ansatz $u(\rho\omega) = \sum_{\ell,m}^{N} u_{\ell,m}(\rho)Y_{\ell,m}(\omega)$ we derive from (4-7) the (decoupled) system

$$\Big[-(1-\rho^2)\partial_\rho^2 - \frac{2}{\rho}\partial_\rho + 5\rho\partial_\rho + \frac{15}{4} + \frac{\ell(\ell+1)}{\rho^2}\Big]u_{\ell,m}(\rho) = g_{\ell,m}(\rho) \tag{4-8}$$

for $\ell \in \mathbb{N}_0$, $\ell \le N$, and $-\ell \le m \le \ell$. Equation (4-8) has regular singular points at $\rho = 0$ and $\rho = 1$ with Frobenius indices $\{\ell, -\ell-1\}$ and $\{0, -\frac{1}{2}\}$, respectively. In fact, solutions to (4-8) can be given in terms of hypergeometric functions. In order to see this, define a new variable $v_{\ell,m}$ by $u_{\ell,m}(\rho) = \rho^\ell v_{\ell,m}(\rho^2)$. Then, (4-8) with $g_{\ell,m} = 0$ is equivalent to

$$z(1-z)v''_{\ell,m}(z) + [c - (a+b+1)z]v'_{\ell,m}(z) - abv_{\ell,m}(z) = 0 \tag{4-9}$$

with $a = \frac{1}{2}\big(\frac{3}{2}+\ell\big)$, $b = a + \frac{1}{2}$, $c = \frac{3}{2} + \ell$, and $z = \rho^2$. We immediately obtain the two solutions

$$\phi_{0,\ell}(z) = {}_2F_1\Big(\frac{3+2\ell}{4}, \frac{5+2\ell}{4}, \frac{3+2\ell}{2}; z\Big), \quad \phi_{1,\ell}(z) = {}_2F_1\Big(\frac{3+2\ell}{4}, \frac{5+2\ell}{4}, \frac{3}{2}; 1-z\Big),$$

where ${}_2F_1$ is the standard hypergeometric function; see [Olver et al. 2010; Kristensson 2010]. For later reference we also state a third solution, $\tilde{\phi}_{1,\ell}$, given by

$$\tilde{\phi}_{1,\ell}(z) = (1-z)^{-\frac{1}{2}} {}_2F_1\left(\frac{3+2\ell}{4}, \frac{1+2\ell}{4}, \frac{1}{2}; 1-z\right). \tag{4-10}$$

Note that $\phi_{0,\ell}$ is analytic around $z = 0$ whereas $\phi_{1,\ell}$ is analytic around $z = 1$. As a matter of fact, $\phi_{1,\ell}$ can be represented in terms of elementary functions and we have

$$\phi_{1,\ell}(z) = \frac{1}{(2\ell+1)\sqrt{1-z}}\left[(1-\sqrt{1-z})^{-\ell-\frac{1}{2}} - (1+\sqrt{1-z})^{-\ell-\frac{1}{2}}\right]; \tag{4-11}$$

see [Olver et al. 2010]. This immediately shows that $|\phi_{1,\ell}(z)| \to \infty$ as $z \to 0+$ which implies that $\phi_{0,\ell}$ and $\phi_{1,\ell}$ are linearly independent. Transforming back, we obtain the two solutions $\psi_{j,\ell}(\rho) = \rho^\ell \phi_{j,\ell}(\rho^2)$, $j = 0, 1$, of (4-8) with $g_{\ell,m} = 0$. By differentiating the Wronskian $W(\psi_{0,\ell}, \psi_{1,\ell}) = \psi_{0,\ell}\psi_{1,\ell}' - \psi_{0,\ell}'\psi_{1,\ell}$ and inserting the equation, we infer

$$W(\psi_{0,\ell}, \psi_{1,\ell})'(\rho) = \left(\frac{3\rho}{1-\rho^2} - \frac{2}{\rho}\right) W(\psi_{0,\ell}, \psi_{1,\ell})(\rho),$$

which implies

$$W(\psi_{0,\ell}, \psi_{1,\ell})(\rho) = \frac{c_\ell}{\rho^2(1-\rho^2)^{\frac{3}{2}}} \tag{4-12}$$

for some constant c_ℓ. In order to determine the precise value of c_ℓ, we first note that

$$\psi_{j,\ell}'(\rho) = 2\rho^{\ell+1}\phi_{j,\ell}'(\rho^2) + \ell\rho^{\ell-1}\phi_{j,\ell}(\rho^2).$$

For the following we recall the differentiation formula [Olver et al. 2010]

$$\frac{d}{dz} {}_2F_1(a, b, c; z) = \frac{ab}{c} {}_2F_1(a+1, b+1, c+1; z), \tag{4-13}$$

which is a direct consequence of the series representation of the hypergeometric function. Furthermore, by the formula [Olver et al. 2010]

$$\lim_{z\to 1-} \left[(1-z)^{a+b-c} {}_2F_1(a, b, c; z)\right] = \frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}, \tag{4-14}$$

valid for $\operatorname{Re}(a+b-c) > 0$, we obtain

$$\lim_{z\to 1-} \left[(1-z)^{\frac{1}{2}}\phi_{0,\ell}(z)\right] = \frac{\Gamma\left(\frac{3+2\ell}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{3+2\ell}{4}\right)\Gamma\left(\frac{5+2\ell}{4}\right)} = 2^{\ell+\frac{1}{2}} \tag{4-15}$$

as well as

$$\lim_{z\to 1-} \left[(1-z)^{\frac{3}{2}}\phi_{0,\ell}'(z)\right] = \frac{\Gamma\left(\frac{3+2\ell}{2}\right)\Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{3+2\ell}{4}\right)\Gamma\left(\frac{5+2\ell}{4}\right)} = 2^{\ell-\frac{1}{2}},$$

where we used the identity $\Gamma(x)\Gamma(x+\frac{1}{2})=\pi^{\frac{1}{2}}2^{1-2x}\Gamma(2x)$. This yields

$$\begin{aligned}
c_\ell &= \rho^2(1-\rho^2)^{\frac{3}{2}}W(\psi_{0,\ell},\psi_{1,\ell})(\rho)\\
&= \rho^2(1-\rho^2)^{\frac{3}{2}}\rho^\ell\phi_{0,\ell}(\rho^2)[2\rho^{\ell+1}\phi_{1,\ell}'(\rho^2)+\ell\rho^{\ell-1}\phi_{1,\ell}(\rho^2)]\\
&\qquad\qquad -\rho^2(1-\rho^2)^{\frac{3}{2}}\rho^\ell\phi_{1,\ell}(\rho^2)[2\rho^{\ell+1}\phi_{0,\ell}'(\rho^2)+\ell\rho^{\ell-1}\phi_{0,\ell}(\rho^2)]\\
&= -2\lim_{\rho\to1-}(1-\rho^2)^{\frac{3}{2}}\phi_{0,\ell}'(\rho^2)\\
&= -2^{\ell+\frac{1}{2}}.
\end{aligned}$$

By the variation of constants formula, a solution to (4-8) is given by

$$u_{\ell,m}(\rho)=-\psi_{0,\ell}(\rho)\int_\rho^1\frac{\psi_{1,\ell}(s)}{W(\psi_{0,\ell},\psi_{1,\ell})(s)}\frac{g_{\ell,m}(s)}{1-s^2}\,ds-\psi_{1,\ell}(\rho)\int_0^\rho\frac{\psi_{0,\ell}(s)}{W(\psi_{0,\ell},\psi_{1,\ell})(s)}\frac{g_{\ell,m}(s)}{1-s^2}\,ds. \quad (4\text{-}16)$$

We claim that $u_{\ell,m}\in C^2(0,1]$. By formally differentiating (4-16) we find

$$u_{\ell,m}''(\rho)=-\frac{g_{\ell,m}(\rho)}{1-\rho^2}-\psi_{0,\ell}''(\rho)I_{1,\ell}(\rho)-\psi_{1,\ell}''(\rho)I_{0,\ell}(\rho),$$

where $I_{j,\ell}$, $j=0,1$, denote the respective integrals in (4-16). This implies $u_{\ell,m}\in C^2(0,1)$ but $u_{\ell,m}''(\rho)$ has an apparent singularity at $\rho=1$. We have the asymptotics $\psi_{0,\ell}''(\rho)I_{1,\ell}(\rho)\simeq(1-\rho)^{-1}$ and $\psi_{1,\ell}''(\rho)I_{0,\ell}(\rho)\simeq1$ as $\rho\to1-$. Thus, a necessary condition for $\lim_{\rho\to1-}u_{\ell,m}''(\rho)$ to exist is

$$a_{\ell,m}:=\lim_{\rho\to1-}[(1-\rho^2)\psi_{0,\ell}''(\rho)I_{1,\ell}(\rho)]=-g_{\ell,m}(1).$$

This limit can be computed by l'Hôpital's rule, i.e., we write

$$\begin{aligned}
a_{\ell,m}&=\lim_{\rho\to1-}\frac{I_{1,\ell}(\rho)}{[(1-\rho^2)\psi_{0,\ell}''(\rho)]^{-1}}\\
&=\lim_{\rho\to1-}\frac{I_{1,\ell}'(\rho)}{-[(1-\rho^2)\psi_{0,\ell}''(\rho)]^{-2}[(1-\rho^2)\psi_{0,\ell}^{(3)}(\rho)-2\rho\psi_{0,\ell}''(\rho)]}.
\end{aligned}$$

We have

$$\lim_{\rho\to1-}[(1-\rho^2)^{-\frac{1}{2}}I_{1,\ell}'(\rho)]=-\frac{1}{c_\ell}\lim_{\rho\to1-}[\rho^2\psi_{1,\ell}(\rho)g_{\ell,m}(\rho)]=-\frac{g_{\ell,m}(1)}{c_\ell},$$

and thus it suffices to show that

$$-\frac{1}{c_\ell}=\lim_{\rho\to1-}\frac{(1-\rho^2)\psi_{0,\ell}^{(3)}(\rho)-2\rho\psi_{0,\ell}''(\rho)}{(1-\rho^2)^{\frac{1}{2}}[(1-\rho^2)\psi_{0,\ell}''(\rho)]^2}=\lim_{\rho\to1-}\frac{(1-\rho^2)^{\frac{7}{2}}\psi_{0,\ell}^{(3)}(\rho)-2\rho(1-\rho^2)^{\frac{5}{2}}\psi_{0,\ell}''(\rho)}{[(1-\rho^2)^{\frac{5}{2}}\psi_{0,\ell}''(\rho)]^2}. \quad (4\text{-}17)$$

Note that

$$\begin{aligned}
\psi_{0,\ell}''(\rho)&=4\rho^{\ell+2}\phi_{0,\ell}''(\rho^2)+\text{lower order derivatives},\\
\psi_{0,\ell}^{(3)}(\rho)&=8\rho^{\ell+3}\phi_{0,\ell}^{(3)}(\rho^2)+\text{lower order derivatives}.
\end{aligned}$$

Consequently, from the definition of $\phi_{0,\ell}$ and equations (4-13) and (4-14), we infer

$$\lim_{\rho\to 1-}[(1-\rho^2)^{\frac{5}{2}}\psi''_{0,\ell}(\rho)] = 4\lim_{z\to 1-}[(1-z)^{\frac{5}{2}}\phi''_{0,\ell}(z)] = -3c_\ell,$$
$$\lim_{\rho\to 1-}[(1-\rho^2)^{\frac{7}{2}}\psi^{(3)}_{0,\ell}(\rho)] = 8\lim_{z\to 1-}[(1-z)^{\frac{7}{2}}\phi^{(3)}_{0,\ell}(z)] = -15c_\ell,$$

which proves (4-17). We have $g_{\ell,m}\in C^\infty[0,1]$ and thus, in order to prove the claim $u_{\ell,m}\in C^2(0,1]$, it suffices to show that $\rho\mapsto(1-\rho^2)\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)$ belongs to $C^1(0,1]$. We write the integrand in $I_{1,\ell}$ as

$$\frac{\psi_{1,\ell}(s)}{W(\psi_{0,\ell},\psi_{1,\ell})(s)}\frac{g_{\ell,m}(s)}{1-s^2} = (1-s)^{\frac{1}{2}}O(1),$$

where in the following, $O(1)$ stands for a suitable function in $C^\infty(0,1]$. Consequently, we infer $I_{1,\ell}(\rho) = (1-\rho)^{\frac{3}{2}}O(1)$. We have $\psi_{0,\ell} = a_\ell\psi_{1,\ell}+\tilde{a}_\ell\tilde{\psi}_{1,\ell}$ where $\tilde{\psi}_{1,\ell}(\rho) := \rho^\ell\tilde{\phi}_{1,\ell}(\rho^2)$ — see (4-10) — and $a_\ell,\tilde{a}_\ell\in\mathbb{C}$ are suitable constants. This yields

$$\psi''_{0,\ell}(\rho) = (1-\rho)^{-\frac{5}{2}}O(1)+O(1)$$

and thus, $(1-\rho^2)\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho) = O(1)+(1-\rho)^{\frac{5}{2}}O(1)$. Consequently, $\rho\mapsto(1-\rho^2)\psi''_{0,\ell}(\rho)I_{1,\ell}(\rho)$ belongs to $C^1(0,1]$ and by l'Hôpital's rule we infer $u_{\ell,m}\in C^2(0,1]$ as claimed.

Next, we turn to the endpoint $\rho=0$. The integrand of $I_{1,\ell}$ is bounded by $C_\ell\rho^{-\ell+1}$ and thus, we obtain

$$\begin{aligned}
&|I_{1,\ell}(\rho)|\lesssim 1 && \text{for } \ell\in\{0,1\},\\
&|I_{1,2}(\rho)|\lesssim|\log\rho|,\\
&|I_{1,\ell}(\rho)|\lesssim\rho^{-\ell+2} && \text{for } \ell\in\mathbb{N},\ 3\le\ell\le N,
\end{aligned}$$

for all $\rho\in(0,1]$. The integrand of $I_{0,\ell}$ is bounded by $C_\ell\rho^{\ell+2}$ and this implies $|I_{0,\ell}(\rho)|\lesssim\rho^{\ell+3}$ for all $\rho\in[0,1]$ and $\ell\in\mathbb{N}_0$, $\ell\le N$. Thus, we obtain for all $\rho\in(0,1]$ and $k\in\{0,1,2\}$ the estimates

$$\begin{aligned}
&|u^{(k)}_{0,m}(\rho)|\lesssim 1,\\
&|u^{(k)}_{1,m}(\rho)|\lesssim\rho^{\max\{1-k,0\}},\\
&|u^{(k)}_{2,m}(\rho)|\lesssim\rho^{2-k}|\log\rho|+\rho^{2-k},\\
&|u^{(k)}_{\ell,m}(\rho)|\lesssim\rho^{2-k}\quad\text{for } \ell\in\mathbb{N},\ 3\le\ell\le N.
\end{aligned}\tag{4-18}$$

Now we define the function $u:\bar{B}\backslash\{0\}\to\mathbb{C}$ by

$$u(\xi) := \sum_{\ell,m}^{N} u_{\ell,m}(|\xi|)Y_{\ell,m}\left(\frac{\xi}{|\xi|}\right).\tag{4-19}$$

From the bounds (4-18) we obtain[2] $|\partial_j\partial_k u(\xi)|\lesssim|\xi|^{-1}$ which implies $u\in H^2(B)\cap C^2(\bar{B}\backslash\{0\})$ and by construction, u satisfies

$$-(\delta^{jk}-\xi^j\xi^k)\partial_j\partial_k u(\xi)+5\xi^j\partial_j u(\xi)+\tfrac{15}{4}u(\xi) = g_N(\xi),$$

[2] Note that $Y_{0,0}(\omega) = 1/\sqrt{4\pi}$.

where $\|f - g_N\|_{L^2(B)} \le \|f - \tilde g\|_{L^2(B)} + \|\tilde g - g_N\|_{L^2(B)} < \varepsilon$. □

Proof of Proposition 4.1. First note that $\tilde{\boldsymbol{L}}_0$ is densely defined. Furthermore, we claim that

$$\operatorname{Re}(\tilde{\boldsymbol{L}}_0\boldsymbol{u} \mid \boldsymbol{u}) \le -\tfrac{1}{2}\|\boldsymbol{u}\|^2 \tag{4-20}$$

for all $\boldsymbol{u} \in \mathfrak{D}(\tilde{\boldsymbol{L}}_0)$. We write $[\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_A$ for the A-th component of $\tilde{\boldsymbol{L}}_0\boldsymbol{u}$, where $A \in \{1, 2\}$. Then we have

$$\partial_k[\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_1(\xi) = -\xi^j\partial_j\partial_k u_1(\xi) - 2\partial_k u_1(\xi) + \partial_k u_2(\xi).$$

By noting that

$$\begin{aligned}\operatorname{Re}[\partial_j\partial_k u_1\overline{\partial^k u_1}] &= \tfrac{1}{2}\partial_j[\partial_k u_1\overline{\partial^k u_1}],\\ \xi^j\partial_j f(\xi) &= \partial_{\xi^j}[\xi^j f(\xi)] - 3f(\xi),\end{aligned}$$

we infer

$$\operatorname{Re}[\xi^j\partial_j\partial_k u_1(\xi)\overline{\partial^k u_1(\xi)}] = \tfrac{1}{2}\partial_{\xi^j}[\xi^j\partial_k u_1(\xi)\overline{\partial^k u_1(\xi)}] - \tfrac{3}{2}\partial_k u_1(\xi)\overline{\partial^k u_1(\xi)},$$

and the divergence theorem implies

$$\begin{aligned}&\operatorname{Re}\int_B \partial_k[\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_1(\xi)\,\overline{\partial^k u_1(\xi)}\,d\xi\\ &\qquad = -\frac{1}{2}\int_{\partial B}\partial_k u_1(\omega)\,\overline{\partial^k u_1(\omega)}\,d\sigma(\omega) - \frac{1}{2}\int_B \partial_k u_1(\xi)\,\overline{\partial^k u_1(\xi)}\,d\xi + \operatorname{Re}\int_B \partial_k u_2(\xi)\,\overline{\partial^k u_1(\xi)}\,d\xi.\end{aligned}$$

Furthermore, we have

$$\int_B \partial_j\partial^j u_1(\xi)\overline{u_2(\xi)}\,d\xi = \int_{\partial B}\omega^j\partial_j u_1(\omega)\overline{u_2(\omega)}\,d\sigma(\omega) - \int_B \partial_j u_1(\xi)\overline{\partial^j u_2(\xi)}\,d\xi$$

and

$$\operatorname{Re}\int_B \xi^j\partial_j u_2(\xi)\overline{u_2(\xi)}\,d\xi = \frac{1}{2}\int_{\partial B}|u_2(\omega)|^2\,d\sigma(\omega) - \frac{3}{2}\int_B |u_2(\xi)|^2\,d\xi,$$

which yields

$$\begin{aligned}&\operatorname{Re}\int_B [\tilde{\boldsymbol{L}}_0\boldsymbol{u}]_2(\xi)\overline{u_2(\xi)}\,d\xi\\ &\qquad = \operatorname{Re}\int_{\partial B}\omega^j\partial_j u_1(\omega)\overline{u_2(\omega)}\,d\sigma(\omega) - \tfrac{1}{2}\|u_2\|^2_{L^2(\partial B)} - \operatorname{Re}\int_B \partial_j u_1(\xi)\overline{\partial^j u_2(\xi)}\,d\xi - \tfrac{1}{2}\|u_2\|^2_{L^2(B)}.\end{aligned}$$

In summary, we infer

$$\operatorname{Re}(\tilde{\boldsymbol{L}}_0\boldsymbol{u} \mid \boldsymbol{u}) = -\tfrac{1}{2}\|u_1\|^2_{\dot H^1(B)} - \tfrac{1}{2}\|u_2\|^2_{L^2(B)} + \int_{\partial B} A(\omega)\,d\sigma(\omega)$$

with

$$\begin{aligned}A(\omega) &= -\tfrac{1}{2}|u_1(\omega)|^2 - \tfrac{1}{2}|u_1(\omega)|^2 - \tfrac{1}{2}|\nabla u_1(\omega)|^2 - \tfrac{1}{2}|u_2(\omega)|^2\\ &\qquad - \operatorname{Re}[\omega^j\partial_j u_1(\omega)\overline{u_1(\omega)}] + \operatorname{Re}[\omega^j\partial_j u_1(\omega)\overline{u_2(\omega)}] + \operatorname{Re}[u_2(\omega)\overline{u_1(\omega)}]\\ &\le -\tfrac{1}{2}|u_1(\omega)|^2,\end{aligned}$$

where we have used the inequality

$$\mathrm{Re}\,(\bar{a}b)+\mathrm{Re}\,(\bar{a}c)-\mathrm{Re}\,(\bar{b}c)\le \tfrac{1}{2}(|a|^2+|b|^2+|c|^2),\quad a,b,c\in\mathbb{C}$$

which follows from $0\le |a-b-c|^2$. This proves (4-20).

The estimate (4-20) implies

$$\begin{aligned}\big\|[\lambda-(\tilde{\boldsymbol{L}}_0+\tfrac{1}{2})]\boldsymbol{u}\big\|^2&=\lambda^2\|\boldsymbol{u}\|^2-2\lambda\mathrm{Re}\,\big((\tilde{\boldsymbol{L}}_0+\tfrac{1}{2})\boldsymbol{u}\,\big|\,\boldsymbol{u}\big)+\big\|(\tilde{\boldsymbol{L}}_0+\tfrac{1}{2})\boldsymbol{u}\big\|^2\\&\ge\lambda^2\|\boldsymbol{u}\|^2\end{aligned}$$

for all $\lambda>0$ and $\boldsymbol{u}\in\mathfrak{D}(\tilde{\boldsymbol{L}}_0)$. Thus, in view of the Lumer–Phillips theorem [Engel and Nagel 2000, p. 83, Theorem 3.15] it suffices to prove density of the range of $\lambda-\tilde{\boldsymbol{L}}_0$ for some $\lambda>-\frac{1}{2}$. Let $\boldsymbol{f}\in\mathcal{H}$ and $\varepsilon>0$ be arbitrary. We consider the equation $(\lambda-\tilde{\boldsymbol{L}}_0)\boldsymbol{u}=\boldsymbol{f}$. From the first component we infer $u_2=\xi^j\partial_j u_1+(\lambda+1)u_1-f_1$ and inserting this in the second component we arrive at the degenerate elliptic problem

$$-(\delta^{jk}-\xi^j\xi^k)\partial_j\partial_k u(\xi)+2(\lambda+2)\xi^j\partial_j u(\xi)+(\lambda+1)(\lambda+2)u(\xi)=f(\xi)\tag{4-21}$$

for $u=u_1$ and $f(\xi):=\xi^j\partial_j f_1(\xi)+(\lambda+2)f_1(\xi)+f_2(\xi)$. Note that by assumption we have $f\in L^2(B)$. Setting $\lambda=\frac{1}{2}$ we infer from Lemma 4.2 the existence of functions $u\in H^2(B)\cap C^2(\bar{B}\backslash\{0\})$ and $g\in L^2(B)$ such that

$$-(\delta^{jk}-\xi^j\xi^k)\partial_j\partial_k u(\xi)+5\xi^j\partial_j u(\xi)+\tfrac{15}{4}u(\xi)=g(\xi)$$

and $\|f-g\|_{L^2(B)}<\varepsilon$. We set $u_1:=u$, $u_2(\xi):=\xi^j\partial_j u(\xi)+\frac{3}{2}u-f_1$, $g_1:=f_1$, and $g_2(\xi):=g(\xi)-\xi^j\partial_j f_1(\xi)-\frac{5}{2}f_1(\xi)$. Then we have $\boldsymbol{u}\in\mathfrak{D}(\tilde{\boldsymbol{L}}_0)$, $\boldsymbol{g}\in\mathcal{H}$,

$$\|\boldsymbol{f}-\boldsymbol{g}\|=\|f_2-g_2\|_{L^2(B)}=\|f-g\|_{L^2(B)}<\varepsilon$$

and by construction, $\big(\frac{1}{2}-\tilde{\boldsymbol{L}}_0\big)\boldsymbol{u}=\boldsymbol{g}$. Since $\boldsymbol{f}\in\mathcal{H}$ and $\varepsilon>0$ were arbitrary, this shows that $\mathrm{rg}\,\big(\frac{1}{2}-\tilde{\boldsymbol{L}}_0\big)$ is dense in $\mathcal{H}$, which finishes the proof. □

Well-posedness for the linearized problem. Next, we include the potential term and consider the system

$$\begin{aligned}\partial_0\phi_1&=-\xi^j\partial_j\phi_1-\phi_1+\phi_2,\\\partial_0\phi_2&=\partial_j\partial^j\phi_1-\xi^j\partial_j\phi_2-2\phi_2+6\phi_1.\end{aligned}\tag{4-22}$$

We define an operator $\boldsymbol{L}'$, acting on $\mathcal{H}$, by

$$\boldsymbol{L}'\boldsymbol{u}(\xi):=\begin{pmatrix}0\\6u_1\end{pmatrix}.$$

Then we may rewrite (4-22) as an ODE

$$\frac{d}{d\tau}\Phi(\tau)=(\boldsymbol{L}_0+\boldsymbol{L}')\Phi(\tau)$$

for a function $\Phi:[0,\infty)\to\mathcal{H}$.

Lemma 4.3. *The operator $\boldsymbol{L} := \boldsymbol{L}_0 + \boldsymbol{L}' : \mathcal{D}(\boldsymbol{L}_0) \subset \mathcal{H} \to \mathcal{H}$ generates a strongly continuous one-parameter semigroup $\boldsymbol{S} : [0, \infty) \to \mathcal{B}(\mathcal{H})$ satisfying $\|\boldsymbol{S}(\tau)\| \leq e^{(-\frac{1}{2}+\|\boldsymbol{L}'\|)\tau}$. Furthermore, for the spectrum of the generator we have $\sigma(\boldsymbol{L}) \backslash \sigma(\boldsymbol{L}_0) = \sigma_p(\boldsymbol{L})$.*

Proof. The first assertion is an immediate consequence of the bounded perturbation theorem of semigroup theory; see [Engel and Nagel 2000, p. 158, Theorem 1.3]. In order to prove the claim about the spectrum, we note that the operator $\boldsymbol{L}' : \mathcal{H} \to \mathcal{H}$ is compact by the compactness of the embedding $H^1(B) \hookrightarrow L^2(B)$ (Rellich–Kondrachov) and Lemma 3.1. Assume that $\lambda \in \sigma(\boldsymbol{L})$ and $\lambda \notin \sigma(\boldsymbol{L}_0)$. Then we may write $\lambda - \boldsymbol{L} = [1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)](\lambda - \boldsymbol{L}_0)$. Observe that the operator $\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)$ is compact. Furthermore, $1 \in \sigma(\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda))$ since otherwise we would have $\lambda \in \rho(\boldsymbol{L})$, a contradiction to our assumption. By the spectral theorem for compact operators we infer $1 \in \sigma_p(\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda))$ which shows that there exists a nontrivial $\boldsymbol{f} \in \mathcal{H}$ such that $[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]\boldsymbol{f} = \boldsymbol{0}$. Thus, by setting $\boldsymbol{u} := \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}$, we infer $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$, $\boldsymbol{u} \neq \boldsymbol{0}$, and $(\lambda - \boldsymbol{L})\boldsymbol{u} = \boldsymbol{0}$ which implies $\lambda \in \sigma_p(\boldsymbol{L})$. □

Spectral analysis of the generator. In order to improve the rough growth bound for $\boldsymbol{S}$ given in Lemma 4.3, we need more information on the spectrum of $\boldsymbol{L}$. Thanks to Lemma 4.3 we are only concerned with point spectrum. To begin with, we need the following result concerning $\mathcal{D}(\boldsymbol{L}_0)$.

Lemma 4.4. *Let $\delta \in (0, 1)$ and $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$. Then*

$$\boldsymbol{u}|_{B_{1-\delta}} \in H^2(B_{1-\delta}) \times H^1(B_{1-\delta}),$$

where $B_{1-\delta} := \{\xi \in \mathbb{R}^3 : |\xi| < 1 - \delta\}$.

Proof. Let $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$. By definition of the closure there exists a sequence $(\boldsymbol{u}_n) \subset \mathcal{D}(\tilde{\boldsymbol{L}}_0)$ such that $\boldsymbol{u}_n \to \boldsymbol{u}$ and $\tilde{\boldsymbol{L}}_0\boldsymbol{u}_n \to \boldsymbol{L}_0\boldsymbol{u}$ in $\mathcal{H}$ as $n \to \infty$. We set $\boldsymbol{f}_n := \tilde{\boldsymbol{L}}_0\boldsymbol{u}_n$ and note that $\boldsymbol{f}_n \in H^1(B) \cap C^1(B) \times L^2(B) \cap C(B)$ for all $n \in \mathbb{N}$ by the definition of $\mathcal{D}(\tilde{\boldsymbol{L}}_0)$. We obtain $u_{2n}(\xi) = \xi^j \partial_j u_{1n}(\xi) + u_{1n}(\xi) + f_{1n}(\xi)$ and

$$-(\delta^{jk} - \xi^j \xi^k)\partial_j \partial_k u_{1n}(\xi) + 4\xi^j \partial_j u_{1n}(\xi) + 2u_{1n}(\xi) = f_n(\xi), \tag{4-23}$$

where $f_n(\xi) := -\xi^j \partial_j f_{1n}(\xi) - 2 f_{1n}(\xi) - f_{2n}(\xi)$; compare (4-21). By assumption we have $f_n \to f$ in $L^2(B)$ for some $f \in L^2(B)$. Since

$$(\delta^{jk} - \xi^j \xi^k)\eta_j \eta_k \geq |\eta|^2 - |\xi|^2 |\eta|^2 \geq \frac{\delta}{2}|\eta|^2$$

for all $\xi \in B_{1-\delta/2}$ and all $\eta \in \mathbb{R}^3$, we see that the differential operator in (4-23) is uniformly elliptic on $B_{1-\delta/2}$. Thus, by standard elliptic regularity theory (see [Evans 1998, p. 309, Theorem 1]) we obtain the estimate

$$\|u_{1n}\|_{H^2(B_{1-\delta})} \leq C_\delta (\|u_{1n}\|_{L^2(B_{1-\delta/2})} + \|f_n\|_{L^2(B_{1-\delta/2})}) \tag{4-24}$$

and since $\boldsymbol{u}_n \to \boldsymbol{u}$ in $\mathcal{H}$ implies $u_{1n} \to u_1$ in $L^2(B)$, we infer $u_1|_{B_{1-\delta}} \in H^2(B_{1-\delta})$. Finally, from $u_{2n}(\xi) = \xi^j \partial_j u_{1n}(\xi) + u_{1n}(\xi) + f_{1n}(\xi)$ we conclude $u_2|_{B_{1-\delta}} \in H^1(B_{1-\delta})$. □

The next result allows us to obtain information on the spectrum of $\boldsymbol{L}$ by studying an ODE. For the following we define the space $H^1_{\text{rad}}(a,b)$ by

$$\|f\|^2_{H^1_{\text{rad}}(a,b)} := \int_a^b |f'(\rho)|^2\rho^2\,d\rho + \int_a^b |f(\rho)|^2\rho^2\,d\rho.$$

Lemma 4.5. *Let $\lambda\in\sigma_p(\boldsymbol{L})$. Then there exists an $\ell\in\mathbb{N}_0$ and a nonzero function $u\in C^\infty(0,1)\cap H^1_{\text{rad}}(0,1)$ such that*

$$-(1-\rho^2)u''(\rho)-\frac{2}{\rho}u'(\rho)+2(\lambda+2)\rho u'(\rho)+[(\lambda+1)(\lambda+2)-6]u(\rho)+\frac{\ell(\ell+1)}{\rho^2}u(\rho)=0 \tag{4-25}$$

for all $\rho\in(0,1)$.

Proof. Let $\boldsymbol{u}\in\mathfrak{D}(\boldsymbol{L})=\mathfrak{D}(\boldsymbol{L}_0)$ be an eigenvector associated to the eigenvalue $\lambda\in\sigma_p(\boldsymbol{L})$. The spectral equation $(\lambda-\boldsymbol{L})\boldsymbol{u}=\boldsymbol{0}$ implies $u_2(\xi)=\xi^j\partial_j u_1(\xi)+(\lambda+1)u_1(\xi)$ and

$$-(\delta^{jk}-\xi^j\xi^k)\partial_j\partial_k u_1(\xi)+2(\lambda+2)\xi^j\partial_j u_1(\xi)+[(\lambda+1)(\lambda+2)-6]u_1(\xi)=0 \tag{4-26}$$

(compare (4-21)), but this time the derivatives have to be interpreted in the weak sense since a priori we merely have $u_1\in H^2(B_{1-\delta})\cap H^1(B)$ and $u_2\in H^1(B_{1-\delta})\cap L^2(B)$ by Lemma 4.4. However, by invoking elliptic regularity theory [Evans 1998, p. 316, Theorem 3] we see that in fact $u_1\in C^\infty(B)\cap H^1(B)$. As always, we write $\rho=|\xi|$ and $\omega=\xi/|\xi|$. We expand u_1 in spherical harmonics, i.e.,

$$u_1(\rho\omega)=\sum_{\ell,m}^{\infty}u_{\ell,m}(\rho)Y_{\ell,m}(\omega) \tag{4-27}$$

with $u_{\ell,m}(\rho)=(u_1(\rho\,\cdot)|Y_{\ell,m})_{L^2(S^2)}$ and for each fixed $\rho\in(0,1)$, the sum converges in $L^2(S^2)$. By dominated convergence and $u_1\in C^\infty(B)$ it follows that $u_{\ell,m}\in C^\infty(0,1)$. Similarly, we may expand $\partial_\rho u_1(\rho\omega)$ in spherical harmonics. The corresponding expansion coefficients are given by

$$(\partial_\rho u_1(\rho\,\cdot)\mid Y_{\ell,m})_{L^2(S^2)}=\partial_\rho(u_1(\rho\,\cdot)\mid Y_{\ell,m})_{L^2(S^2)}=\partial_\rho u_{\ell,m}(\rho)$$

where we used dominated convergence and the smoothness of u_1 to pull out the derivative ∂_ρ of the inner product. In other words, we may interchange the operator ∂_ρ with the sum in (4-27). Analogously, we may expand $\Delta_{S^2}u_1(\rho\,\cdot)$ and the corresponding expansion coefficients are

$$(\Delta_{S^2}u_1(\rho\,\cdot)\mid Y_{\ell,m})_{L^2(S^2)}=(u_1(\rho\,\cdot)\mid\Delta_{S^2}Y_{\ell,m})_{L^2(S^2)}=-\ell(\ell+1)u_{\ell,m}(\rho).$$

Thus, the operator Δ_{S^2} commutes with the sum in (4-27). All differential operators that appear in (4-26) are composed of ∂_ρ and Δ_{S^2} and it is therefore a consequence of (4-26) that each $u_{\ell,m}$ satisfies (4-25) for all $\rho\in(0,1)$. Since at least one $u_{\ell,m}$ is nonzero, we obtain the desired function $u\in C^\infty(0,1)$. To complete the proof, it remains to show that $u_{\ell,m}\in H^1_{\text{rad}}(0,1)$. We have

$$|u_{\ell,m}(\rho)|=\left|(u_1(\rho\,\cdot)\mid Y_{\ell,m})_{L^2(S^2)}\right|\le\|u_1(\rho\,\cdot)\|_{L^2(S^2)}$$

and thus,

$$\int_0^1 |u_{\ell,m}(\rho)|^2 \rho^2 \, d\rho \le \int_0^1 \|u_1(\rho\,\cdot\,)\|^2_{L^2(S^2)} \rho^2 \, d\rho = \|u_1\|^2_{L^2(B)}.$$

Similarly, by dominated convergence,

$$|\partial_\rho u_{\ell,m}(\rho)| = \left|(\partial_\rho u_1(\rho\,\cdot\,) \mid Y_{\ell,m})_{L^2(S^2)}\right| \lesssim \|\nabla u_1(\rho\,\cdot\,)\|_{L^2(S^2)}$$

and thus,

$$\int_0^1 |u'_{\ell,m}(\rho)|^2 \rho^2 \, d\rho \lesssim \|\nabla u_1\|^2_{L^2(B)}.$$

Consequently, $u_1 \in H^1(B)$ implies $u_{\ell,m} \in H^1_{\text{rad}}(0, 1)$. □

Proposition 4.6. *For the spectrum of* $\boldsymbol{L}$ *we have*

$$\sigma(\boldsymbol{L}) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -\tfrac{1}{2}\} \cup \{0, 1\}.$$

Furthermore, $\{0, 1\} \subset \sigma_p(\boldsymbol{L})$ *and the* (*geometric*) *eigenspace of the eigenvalue* 1 *is one-dimensional and spanned by*

$$\boldsymbol{u}(\xi; 1) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

whereas the (*geometric*) *eigenspace of the eigenvalue* 0 *is three-dimensional and spanned by*

$$\boldsymbol{u}_j(\xi; 0) = \begin{pmatrix} \xi^j \\ 2\xi^j \end{pmatrix}, \quad j \in \{1, 2, 3\}.$$

Proof. First of all, it is a simple exercise to check that $\boldsymbol{L}\boldsymbol{u}(\xi; 1) = \boldsymbol{u}(\xi; 1)$ and $\boldsymbol{L}\boldsymbol{u}_j(\xi; 0) = \boldsymbol{0}$ for $j = 1, 2, 3$. Since obviously $\boldsymbol{u}(\,\cdot\,; 1), \boldsymbol{u}_j(\,\cdot\,; 0) \in \mathfrak{D}(\tilde{\boldsymbol{L}}_0)$, this implies $\{0, 1\} \subset \sigma_p(\boldsymbol{L})$.

In order to prove the first assertion, let $\lambda \in \sigma(\boldsymbol{L})$ and assume $\operatorname{Re}\lambda > -\frac{1}{2}$. By Proposition 4.1 we have $\lambda \notin \sigma(\boldsymbol{L}_0)$ and thus, Lemma 4.3 implies $\lambda \in \sigma_p(\boldsymbol{L})$. From Lemma 4.5 we infer the existence of a nonzero $u \in C^\infty(0, 1) \cap H^1_{\text{rad}}(0, 1)$ satisfying (4-25) for $\rho \in (0, 1)$. As before, we reduce (4-25) to the hypergeometric differential equation by setting $u(\rho) = \rho^\ell v(\rho^2)$. This yields

$$z(1-z)v''(z) + [c - (a+b+1)z]v'(z) - abv(z) = 0, \tag{4-28}$$

with $a = \frac{1}{2}(-1 + \ell + \lambda)$, $b = \frac{1}{2}(4 + \ell + \lambda)$, $c = \frac{3}{2} + \ell$, and $z = \rho^2$. A fundamental system of (4-28) is given by[3]

$$\begin{aligned} \phi_{1,\ell}(z; \lambda) &= {}_2F_1(a, b, a+b+1-c; 1-z), \\ \tilde{\phi}_{1,\ell}(z; \lambda) &= (1-z)^{c-a-b}{}_2F_1(c-a, c-b, c-a-b+1; 1-z) \end{aligned}$$

and thus, there exist constants $c_\ell(\lambda)$ and $\tilde{c}_\ell(\lambda)$ such that

$$v(z) = c_\ell(\lambda)\phi_{1,\ell}(z; \lambda) + \tilde{c}_\ell(\lambda)\tilde{\phi}_{1,\ell}(z; \lambda).$$

[3] Strictly speaking, this is only true for $c - a - b = -\lambda \ne 0$. In the case $\lambda = 0$ there exists a solution $\tilde{\phi}_{1,\ell}$ which behaves like $\log(1-z)$ as $z \to 1-$.

The function $\phi_{1,\ell}(z;\lambda)$ is analytic around $z=1$ whereas $\tilde{\phi}_{1,\ell}(z;\lambda)\sim(1-z)^{-\lambda}$ as $z\to 1-$ provided $\lambda\neq 0$. In the case $\lambda=0$ we have $\tilde{\phi}_{1,\ell}(z;\lambda)\sim\log(1-z)$ as $z\to 1-$. Since $u\in H^1_{\mathrm{rad}}(0,1)$ implies $v\in H^1\big(\frac{1}{2},1\big)$ and we assume $\operatorname{Re}\lambda>-\frac{1}{2}$, it follows that $\tilde{c}_\ell(\lambda)=0$. Another fundamental system of (4-28) is given by

$$\begin{aligned}\phi_{0,\ell}(z;\lambda)&={}_2F_1(a,b,c;z),\\ \tilde{\phi}_{0,\ell}(z;\lambda)&=z^{1-c}{}_2F_1(a-c+1,b-c+1,2-c;z),\end{aligned}$$

and since $\tilde{\phi}_{0,\ell}(z;\lambda)\sim z^{-\ell-\frac{1}{2}}$ as $z\to 0+$, we see that the function $\rho\mapsto\rho^\ell\tilde{\phi}_{\ell,0}(\rho^2)$ does not belong to $H^1_{\mathrm{rad}}(0,\frac{1}{2})$. As a consequence, we must have $v(z)=d_\ell(\lambda)\phi_{0,\ell}(z;\lambda)$ for some suitable $d_\ell(\lambda)\in\mathbb{C}$. In summary, we conclude that the functions $\phi_{0,\ell}(\,\cdot\,;\lambda)$ and $\phi_{1,\ell}(\,\cdot\,;\lambda)$ are linearly dependent and in view of the connection formula [Olver et al. 2010]

$$\phi_{1,\ell}(z;\lambda)=\frac{\Gamma(1-c)\Gamma(a+b+1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)}\phi_{0,\ell}(z;\lambda)+\frac{\Gamma(c-1)\Gamma(a+b+1-c)}{\Gamma(a)\Gamma(b)}\tilde{\phi}_{0,\ell}(z;\lambda)$$

this is possible only if a or b is a pole of the Γ-function. This yields $-a\in\mathbb{N}_0$ or $-b\in\mathbb{N}_0$ and thus, $\frac{1}{2}(1-\ell-\lambda)\in\mathbb{N}_0$ or $-\frac{1}{2}(4+\ell+\lambda)\in\mathbb{N}_0$. The latter condition is not satisfied for any $\ell\in\mathbb{N}_0$ and the former one is satisfied only if $(\ell,\lambda)=(0,1)$ or $(\ell,\lambda)=(1,0)$, which proves $\sigma(\boldsymbol{L})\subset\{z\in\operatorname{Re}z\le-\frac{1}{2}\}\cup\{0,1\}$. Furthermore, the above argument and the derivation in the proof of Lemma 4.5 also show that the geometric eigenspaces of the eigenvalues 0 and 1 are at most three- and one-dimensional, respectively. □

Remark 4.7. According to the discussion at the beginning of Section 4, the two unstable eigenvalues 1 and 0 emerge from the time translation and Lorentz invariance of the wave equation.

Spectral projections. In order to force convergence to the attractor, we need to "remove" the eigenvalues 0 and 1 from the spectrum of $\boldsymbol{L}$. This is achieved by the spectral projection

$$\boldsymbol{P}:=\frac{1}{2\pi i}\int_\gamma(z-\boldsymbol{L})^{-1}\,dz, \tag{4-29}$$

where the contour γ is given by the curve $\gamma(s)=\frac{1}{2}+\frac{3}{4}e^{2\pi is}$, $s\in[0,1]$. By Proposition 4.6 it follows that $\gamma(s)\in\rho(\boldsymbol{L})$ for all $s\in[0,1]$ and thus, the integral in (4-29) is well-defined as a Riemann integral over a continuous function (with values in a Banach space, though). Furthermore, the contour γ encloses the two unstable eigenvalues 0 and 1. The operator $\boldsymbol{L}$ decomposes into two parts:

$$\begin{aligned}\boldsymbol{L}_u&:\operatorname{rg}\boldsymbol{P}\cap\mathfrak{D}(\boldsymbol{L})\to\operatorname{rg}\boldsymbol{P}, &\quad \boldsymbol{L}_u\boldsymbol{u}&=\boldsymbol{L}\boldsymbol{u},\\ \boldsymbol{L}_s&:\ker\boldsymbol{P}\cap\mathfrak{D}(\boldsymbol{L})\to\ker\boldsymbol{P}, &\quad \boldsymbol{L}_s\boldsymbol{u}&=\boldsymbol{L}\boldsymbol{u},\end{aligned}$$

and for the spectra we have $\sigma(\boldsymbol{L}_u)=\{0,1\}$ as well as $\sigma(\boldsymbol{L}_s)=\sigma(\boldsymbol{L})\backslash\{0,1\}$. We also emphasize the crucial fact that $\boldsymbol{P}$ commutes with the semigroup $\boldsymbol{S}(\tau)$ and thus, the subspaces $\operatorname{rg}\boldsymbol{P}$ and $\ker\boldsymbol{P}$ of $\mathcal{H}$ are invariant under the linearized flow. We refer to [Kato 1995] and [Engel and Nagel 2000] for these standard facts. However, it is important to keep in mind that $\boldsymbol{P}$ is not an orthogonal projection since $\boldsymbol{L}$ is not self-adjoint. Consequently, the following statement on the dimension of $\operatorname{rg}\boldsymbol{P}$ is not trivial.

Lemma 4.8. *The algebraic multiplicities of the eigenvalues* $0,1\in\sigma_p(\boldsymbol{L})$ *equal their geometric multiplicities. In particular*, *we have* $\dim\operatorname{rg}\boldsymbol{P}=4$.

Proof. We define the two spectral projections $\boldsymbol{P}_0$ and $\boldsymbol{P}_1$ by

$$\boldsymbol{P}_n = \frac{1}{2\pi i}\int_{\gamma_n}(z-\boldsymbol{L})^{-1}\,dz, \quad n \in \{0, 1\}$$

where $\gamma_0(s) = \frac{1}{2}e^{2\pi i s}$ and $\gamma_1(s) = 1+\frac{1}{2}e^{2\pi i s}$ for $s \in [0, 1]$. Note that $\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{P}_1$ and $\boldsymbol{P}_0\boldsymbol{P}_1 = \boldsymbol{P}_1\boldsymbol{P}_0 = \boldsymbol{0}$; see [Kato 1995]. By definition, the algebraic multiplicity of the eigenvalue $n \in \sigma_p(\boldsymbol{L})$ equals $\dim \operatorname{rg} \boldsymbol{P}_n$. First, we exclude the possibility $\dim \operatorname{rg} \boldsymbol{P}_n = \infty$. Suppose this is true. Then n belongs to the essential spectrum of $\boldsymbol{L}$, i.e., $n-\boldsymbol{L}$ fails to be semi-Fredholm [Kato 1995, p. 239, Theorem 5.28]. Since the essential spectrum is invariant under compact perturbations (see [Kato 1995, p. 244, Theorem 5.35]), we infer $n \in \sigma(\boldsymbol{L}_0)$, which contradicts the spectral statement in Proposition 4.1. Consequently, $\dim \operatorname{rg} \boldsymbol{P}_n < \infty$. We conclude that the operators $\boldsymbol{L}_{(n)} := \boldsymbol{L}|_{\operatorname{rg} \boldsymbol{P}_n \cap \mathfrak{D}(\boldsymbol{L})}$ are in fact finite-dimensional and $\sigma(\boldsymbol{L}_{(n)}) = \{n\}$. This implies that $n - \boldsymbol{L}_{(n)}$ is nilpotent and thus, there exist $m_n \in \mathbb{N}$ such that $(n - \boldsymbol{L}_{(n)})^{m_n} = \boldsymbol{0}$. We assume m_n to be minimal with this property. If $m_n = 1$ we are done. Thus, assume $m_n \geq 2$. We first consider $\boldsymbol{L}_{(0)}$. Since $\ker \boldsymbol{L}$ is spanned by $\{\boldsymbol{u}_j(\,\cdot\,;0) : j = 1, 2, 3\}$ by Proposition 4.6, it follows that there exists a $\boldsymbol{u} \in \operatorname{rg} \boldsymbol{P}_0 \cap \mathfrak{D}(\boldsymbol{L})$ and constants $c_1, c_2, c_3 \in \mathbb{C}$, not all of them zero, such that

$$\boldsymbol{L}_{(0)}\boldsymbol{u}(\xi) = \boldsymbol{L}\boldsymbol{u}(\xi) = \sum_{j=1}^{3} c_j \boldsymbol{u}_j(\xi; 0) = \begin{pmatrix} c_j\xi^j \\ 2c_j\xi^j \end{pmatrix}.$$

This implies $u_2(\xi) = \xi^j\partial_j u_1(\xi) + u_1(\xi) + c_j\xi^j$ and thus,

$$-(\delta^{jk} - \xi^j\xi^k)\partial_j\partial_k u_1(\xi) + 4\xi^j\partial_j u_1(\xi) - 4u_1(\xi) = -5c_j\xi^j = |\xi| \sum_{m=-1}^{1} \tilde{c}_m Y_{1,m}\Big(\frac{\xi}{|\xi|}\Big).$$

As before in the proof of Lemma 4.5, we expand u_1 as

$$u_1(\xi) = \sum_{\ell,m}^{\infty} u_{\ell,m}(|\xi|)Y_{\ell,m}\Big(\frac{\xi}{|\xi|}\Big)$$

and find

$$-(1-\rho^2)u''_{1,m}(\rho) - \frac{2}{\rho}u'_{1,m}(\rho) + 4\rho u'_{1,m}(\rho) - 4u_{1,m}(\rho) + \frac{2}{\rho^2}u_{1,m}(\rho) = \tilde{c}_m\rho. \tag{4-30}$$

For at least one $m \in \{-1, 0, 1\}$ we have $\tilde{c}_m \neq 0$ and by normalizing $u_{1,m}$ accordingly, we may assume $\tilde{c}_m = 1$. Of course, (4-30) with $\tilde{c}_m = 0$ is nothing but the spectral equation (4-25) with $\ell = 1$ and $\lambda = 0$. An explicit solution is therefore given by $\psi(\rho) = \rho$ which may of course also be easily checked directly. Another solution is $\tilde{\psi}(\rho) := \tilde{\psi}_{0,1}(\rho; 0) = \rho\tilde{\phi}_{0,1}(\rho^2; 0)$, where $\tilde{\phi}_{1,0}(\,\cdot\,; 0)$ is the hypergeometric function from the proof of Proposition 4.6. We have the asymptotic behavior $\tilde{\psi}(\rho) \sim \rho^{-2}$ as $\rho \to 0+$ and $|\tilde{\psi}(\rho)| \simeq |\log(1-\rho)|$ as $\rho \to 1-$. By the variation of constants formula we infer that $u_{1,m}$ must be of the form

$$u_{1,m}(\rho) = c\psi(\rho) + \tilde{c}\tilde{\psi}(\rho) + \psi(\rho)\int_{\rho_0}^{\rho} \frac{\tilde{\psi}(s)}{W(s)}\frac{s}{1-s^2}\,ds - \tilde{\psi}(\rho)\int_{\rho_1}^{\rho} \frac{\psi(s)}{W(s)}\frac{s}{1-s^2}\,ds \tag{4-31}$$

for suitable constants $c, \tilde{c} \in \mathbb{C}$, $\rho_0, \rho_1 \in [0, 1]$ and

$$W(\rho) = W(\psi, \tilde{\psi})(\rho) = \frac{d}{\rho^2(1-\rho^2)}$$

where $d \in \mathbb{R}\backslash\{0\}$. Recall that $u_1 \in H^1(B)$ implies $u_{1,m} \in H^1_{\mathrm{rad}}(0,1)$ and by considering the behavior of (4-31) as $\rho \to 0+$, we see that necessarily

$$\tilde{c} = \int_{\rho_1}^{0} \frac{\psi(s)}{W(s)} \frac{s}{1-s^2}\, ds,$$

which leaves us with

$$u_{1,m}(\rho) = c\psi(\rho) + \psi(\rho)\int_{\rho_0}^{\rho} \frac{\tilde{\psi}(s)}{W(s)} \frac{s}{1-s^2}\, ds - \tilde{\psi}(\rho)\int_{0}^{\rho} \frac{\psi(s)}{W(s)} \frac{s}{1-s^2}\, ds.$$

Next, we consider the behavior as $\rho \to 1-$. Since

$$\left|\int_{\rho_0}^{\rho} \frac{\tilde{\psi}(s)}{W(s)} \frac{s}{1-s^2}\, ds\right| \lesssim 1$$

for all $\rho \in (0, 1)$ and $\tilde{\psi} \notin H^1_{\mathrm{rad}}\big(\frac{1}{2}, 1\big)$, we must have

$$\lim_{\rho\to 1-} \int_{0}^{\rho} \frac{\psi(s)}{W(s)} \frac{s}{1-s^2}\, ds = 0.$$

This, however, is impossible since

$$\frac{\psi(s)}{W(s)} \frac{s}{1-s^2} = \frac{1}{d} s^4.$$

Thus, we arrive at a contradiction and our initial assumption $m_0 \geq 2$ must be wrong. Consequently, from Proposition 4.6 we infer $\dim \operatorname{rg} \boldsymbol{P}_0 = \dim \ker \boldsymbol{L} = 3$ as claimed. By exactly the same type of argument one proves that $\dim \operatorname{rg} \boldsymbol{P}_1 = 1$. □

Resolvent estimates. Our next goal is to obtain existence of the resolvent $\boldsymbol{R_L}(\lambda) \in \mathcal{B}(\mathcal{H})$ for $\lambda \in H_{-\frac{1}{2}+\epsilon} := \{z \in \mathbb{C} : \operatorname{Re} z \geq -\frac{1}{2} + \epsilon\}$ and $|\lambda|$ large.

Lemma 4.9. *Fix $\epsilon > 0$. Then there exists a constant $C > 0$ such that $\boldsymbol{R_L}(\lambda)$ exists as a bounded operator on $\mathcal{H}$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$ with $|\lambda| > C$.*

Proof. From Proposition 4.1 we know that $\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda) \in \mathcal{B}(\mathcal{H})$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$ with the bound (see [Engel and Nagel 2000, p. 55, Theorem 1.10])

$$\|\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\| \leq \frac{1}{\operatorname{Re}\lambda + \frac{1}{2}}.$$

Furthermore, recall the identity $\boldsymbol{R_L}(\lambda) = \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1}$. By definition of $\boldsymbol{L}'$ we have

$$\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f} = \begin{pmatrix} 0 \\ 6[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1 \end{pmatrix},$$

where we use the notation $[\boldsymbol{g}]_k$ for the k-th component of the vector $\boldsymbol{g}$. Set $\boldsymbol{u} = \boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}$ for a given $\boldsymbol{f} \in \mathcal{H}$. Then we have $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L}_0)$ and $(\lambda - \boldsymbol{L}_0)\boldsymbol{u} = \boldsymbol{f}$, which implies $u_2(\xi) = \xi^j \partial_j u_1(\xi) + (\lambda+1)u_1(\xi) - f_1(\xi)$, or, equivalently,

$$[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1(\xi) = \frac{1}{\lambda+1}\Big[-\xi^j\partial_j[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1(\xi) + [\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2(\xi) + f_1(\xi)\Big].$$

Consequently, we infer

$$\begin{aligned}\|[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1\|_{L^2(B)} &\lesssim \frac{1}{|\lambda+1|}\Big[\|[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_1\|_{H^1(B)} + \|[\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\boldsymbol{f}]_2\|_{L^2(B)} + \|f_1\|_{L^2(B)}\Big]\\ &\lesssim \frac{\|\boldsymbol{f}\|}{|\lambda+1|},\end{aligned}$$

which yields $\|\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)\| \lesssim 1/|\lambda+1|$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$. We conclude that the Neumann series

$$[1 - \boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^{-1} = \sum_{k=0}^{\infty}[\boldsymbol{L}'\boldsymbol{R}_{\boldsymbol{L}_0}(\lambda)]^k$$

converges in norm provided $|\lambda|$ is sufficiently large. This yields the desired result. □

Estimates for the linearized evolution. Finally, we obtain improved growth estimates for the semigroup $\boldsymbol{S}$ from Lemma 4.3 which governs the linearized evolution.

Proposition 4.10. *Fix $\epsilon > 0$. Then the semigroup $\boldsymbol{S}$ from Lemma 4.3 satisfies the estimates*

$$\begin{aligned}\|\boldsymbol{S}(\tau)(1-\boldsymbol{P})\boldsymbol{f}\| &\le Ce^{(-\frac{1}{2}+\epsilon)\tau}\|(1-\boldsymbol{P})\boldsymbol{f}\|,\\ \|\boldsymbol{S}(\tau)\boldsymbol{P}\boldsymbol{f}\| &\le Ce^{\tau}\|\boldsymbol{P}\boldsymbol{f}\|,\end{aligned}$$

for all $\tau \ge 0$ and $\boldsymbol{f} \in \mathcal{H}$.

Proof. The operator $\boldsymbol{L}_s$ is the generator of the subspace semigroup $\boldsymbol{S}_s$ defined by $\boldsymbol{S}_s(\tau) := \boldsymbol{S}(\tau)|_{\ker \boldsymbol{P}}$. We have $\sigma(\boldsymbol{L}_s) \subset \{z \in \mathbb{C} : \operatorname{Re} z \le -\frac{1}{2}\}$ and the resolvent $\boldsymbol{R}_{\boldsymbol{L}_s}(\lambda)$ is the restriction of $\boldsymbol{R}_{\boldsymbol{L}}(\lambda)$ to $\ker \boldsymbol{P}$. Consequently, by Lemma 4.9 we infer $\|\boldsymbol{R}_{\boldsymbol{L}_s}(\lambda)\| \lesssim 1$ for all $\lambda \in H_{-\frac{1}{2}+\epsilon}$ and thus, the Gearhart–Prüss–Greiner theorem (see [Engel and Nagel 2000, p. 302, Theorem 1.11]) yields the semigroup decay $\|\boldsymbol{S}_s(\tau)\| \lesssim e^{(-\frac{1}{2}+\epsilon)\tau}$. The estimate for $\boldsymbol{S}(\tau)\boldsymbol{P}$ follows from the fact that $\operatorname{rg} \boldsymbol{P}$ is spanned by eigenfunctions of $\boldsymbol{L}$ with eigenvalues 0 and 1 (Proposition 4.6 and Lemma 4.8). □

5. Nonlinear perturbation theory

In this section we consider the full problem (3-5),

$$\begin{aligned}\partial_0\phi_1 &= -\xi^j\partial_j\phi_1 - \phi_1 + \phi_2,\\ \partial_0\phi_2 &= \partial_j\partial^j\phi_1 - \xi^j\partial_j\phi_2 - 2\phi_2 + 6\phi_1 + 3\sqrt{2}\phi_1^2 + \phi_1^3,\end{aligned} \tag{5-1}$$

with prescribed initial data at $\tau = 0$. An operator formulation of (5-1) is obtained by defining the nonlinearity

$$\boldsymbol{N}(\boldsymbol{u}) := \begin{pmatrix} 0 \\ 3\sqrt{2}u_1^2 + u_1^3 \end{pmatrix}.$$

It is an immediate consequence of the Sobolev embedding $H^1(B) \hookrightarrow L^p(B)$, $p \in [1, 6]$, that $\boldsymbol{N} : \mathcal{H} \to \mathcal{H}$ and we have the estimate

$$\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \lesssim \|\boldsymbol{u} - \boldsymbol{v}\|(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|) \tag{5-2}$$

for all $\boldsymbol{u}, \boldsymbol{v} \in \mathcal{H}$ with $\|\boldsymbol{u}\|, \|\boldsymbol{v}\| \leq 1$. The Cauchy problem for (5-1) is formally equivalent to

$$\begin{aligned} \frac{d}{d\tau}\Phi(\tau) &= \boldsymbol{L}\Phi(\tau) + \boldsymbol{N}(\Phi(\tau)), \\ \Phi(0) &= \boldsymbol{u}, \end{aligned} \tag{5-3}$$

for a strongly differentiable function $\Phi : [0, \infty) \to \mathcal{H}$ where $\boldsymbol{u}$ are the prescribed data. In fact, we shall consider the weak version of (5-3) which reads

$$\Phi(\tau) = \boldsymbol{S}(\tau)\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau - \sigma)\boldsymbol{N}(\Phi(\sigma))\, d\sigma. \tag{5-4}$$

Since the semigroup $\boldsymbol{S}$ is unstable, one cannot expect to obtain a global solution of (5-4) for general data $\boldsymbol{u} \in \mathcal{H}$. However, on the subspace $\ker \boldsymbol{P}$, the semigroup $\boldsymbol{S}$ is stable (Proposition 4.10). In order to isolate the instability in the nonlinear context, we formally project (5-4) to the unstable subspace $\operatorname{rg} \boldsymbol{P}$ which yields

$$\boldsymbol{P}\Phi(\tau) = \boldsymbol{S}(\tau)\boldsymbol{P}\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau - \sigma)\boldsymbol{P}\boldsymbol{N}(\Phi(\sigma))\, d\sigma.$$

This suggests to subtract the “bad” term

$$\boldsymbol{S}(\tau)\boldsymbol{P}\boldsymbol{u} + \int_0^\infty \boldsymbol{S}(\tau - \sigma)\boldsymbol{P}\boldsymbol{N}(\Phi(\sigma))\, d\sigma$$

from (5-4) in order to force decay. We obtain the equation

$$\Phi(\tau) = \boldsymbol{S}(\tau)(1 - \boldsymbol{P})\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau - \sigma)\boldsymbol{N}(\Phi(\sigma))\, d\sigma - \int_0^\infty \boldsymbol{S}(\tau - \sigma)\boldsymbol{P}\boldsymbol{N}(\Phi(\sigma))\, d\sigma. \tag{5-5}$$

First, we solve (5-5) and then we relate solutions of (5-5) to solutions of (5-4).

Solution of the modified equation. We solve (5-5) by a fixed point argument. To this end we define

$$\boldsymbol{K}_{\boldsymbol{u}}(\Phi)(\tau) := \boldsymbol{S}(\tau)(1 - \boldsymbol{P})\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau - \sigma)\boldsymbol{N}(\Phi(\sigma))\, d\sigma - \int_0^\infty \boldsymbol{S}(\tau - \sigma)\boldsymbol{P}\boldsymbol{N}(\Phi(\sigma))\, d\sigma$$

and show that $\boldsymbol{K}_{\boldsymbol{u}}$ defines a contraction mapping on (a closed subset of) the Banach space $\mathcal{X}$, given by

$$\mathcal{X} := \left\{ \Phi \in C([0, \infty), \mathcal{H}) : \sup_{\tau > 0} e^{(\frac{1}{2} - \epsilon)\tau} \|\Phi(\tau)\| < \infty \right\}$$

with norm

$$\|\Phi\|_{\mathcal{X}} := \sup_{\tau>0} e^{(\frac{1}{2}-\epsilon)\tau} \|\Phi(\tau)\|,$$

where $\epsilon \in (0, \frac{1}{2})$ is arbitrary but fixed. We further write

$$\mathcal{X}_\delta := \{\Phi \in \mathcal{X} : \|\Phi\|_{\mathcal{X}} \le \delta\}$$

for the closed ball of radius $\delta > 0$ in $\mathcal{X}$.

Proposition 5.1. *Let $\delta > 0$ be sufficiently small and suppose $\boldsymbol{u} \in \mathcal{H}$ with $\|\boldsymbol{u}\| < \delta^2$. Then $\boldsymbol{K_u}$ maps $\mathcal{X}_\delta$ to $\mathcal{X}_\delta$ and we have the estimate*

$$\|\boldsymbol{K_u}(\Phi) - \boldsymbol{K_u}(\Psi)\|_{\mathcal{X}} \le C\delta\|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$.

Proof. First observe that $\boldsymbol{K_u} : \mathcal{X}_\delta \to C([0,\infty), \mathcal{H})$ since $\|\boldsymbol{N}(\Phi(\tau))\| \lesssim e^{(-1+2\epsilon)\tau}$ for any $\Phi \in \mathcal{X}_\delta$. We have

$$\boldsymbol{P}\boldsymbol{K_u}(\Phi)(\tau) = -\int_\tau^\infty \boldsymbol{S}(\tau-\sigma)\boldsymbol{P}\boldsymbol{N}(\Phi(\sigma))\,d\sigma, \tag{5-6}$$

which yields

$$\begin{aligned}\big\|\boldsymbol{P}[\boldsymbol{K_u}(\Phi)(\tau) - \boldsymbol{K_u}(\Psi)(\tau)]\big\| &\lesssim \int_\tau^\infty e^{\tau-\sigma}\|\Phi(\sigma)-\Psi(\sigma)\|\big(\|\Phi(\sigma)\|+\|\Psi(\sigma)\|\big)\,d\sigma \\ &\lesssim \|\Phi-\Psi\|_{\mathcal{X}}(\|\Phi\|_{\mathcal{X}}+\|\Psi\|_{\mathcal{X}})e^\tau \int_\tau^\infty e^{(-2+2\epsilon)\sigma}\,d\sigma \\ &\lesssim \delta e^{(-1+2\epsilon)\tau}\|\Phi-\Psi\|_{\mathcal{X}}\end{aligned}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$ by Proposition 4.10. On the stable subspace we have

$$(1-\boldsymbol{P})\boldsymbol{K_u}(\Phi)(\tau) = \boldsymbol{S}(\tau)(1-\boldsymbol{P})\boldsymbol{u} + \int_0^\tau \boldsymbol{S}(\tau-\sigma)(1-\boldsymbol{P})\boldsymbol{N}(\Phi(\sigma))\,d\sigma$$

and thus,

$$\begin{aligned}\big\|(1-\boldsymbol{P})[\boldsymbol{K_u}(\Phi)(\tau) - \boldsymbol{K_u}(\Psi)(\tau)]\big\| &\lesssim \int_0^\tau e^{(-\frac{1}{2}+\epsilon)(\tau-\sigma)}\|\Phi(\sigma)-\Psi(\sigma)\|\big(\|\Phi(\sigma)\|+\|\Psi(\sigma)\|\big)\,d\sigma \\ &\lesssim \|\Phi-\Psi\|_{\mathcal{X}}\delta e^{(-\frac{1}{2}+\epsilon)\tau} \int_0^\tau e^{(-\frac{1}{2}+\epsilon)\sigma}\,d\sigma \\ &\lesssim \delta e^{(-\frac{1}{2}+\epsilon)\tau}\|\Phi-\Psi\|_{\mathcal{X}}\end{aligned}$$

again by Proposition 4.10. We conclude that

$$\|\boldsymbol{K_u}(\Phi) - \boldsymbol{K_u}(\Psi)\|_{\mathcal{X}} \lesssim \delta\|\Phi-\Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}$. By a slight modification of the above argument one similarly proves $\|\boldsymbol{K_u}(\Phi)\|_{\mathcal{X}} \le \delta$ for all $\Phi \in \mathcal{X}_\delta$ (here $\|\boldsymbol{u}\| \le \delta^2$ is used). ☐

Now we can conclude the existence of a solution to (5-5) by invoking the contraction mapping principle.

Lemma 5.2. *Let $\delta > 0$ be sufficiently small. Then, for any $\boldsymbol{u} \in \mathcal{H}$ with $\|\boldsymbol{u}\| \le \delta^2$, there exists a unique solution $\Phi_{\boldsymbol{u}} \in \mathcal{X}_\delta$ to* (5-5).

Proof. By Proposition 5.1 we may choose $\delta > 0$ so small that

$$\|\boldsymbol{K}_{\boldsymbol{u}}(\Phi) - \boldsymbol{K}_{\boldsymbol{u}}(\Psi)\|_{\mathcal{X}} \le \tfrac{1}{2}\|\Phi - \Psi\|_{\mathcal{X}}$$

for all $\Phi, \Psi \in \mathcal{X}_\delta$ and thus, the contraction mapping principle implies the existence of a unique $\Phi_{\boldsymbol{u}} \in \mathcal{X}_\delta$ with $\Phi_{\boldsymbol{u}} = \boldsymbol{K}_{\boldsymbol{u}}(\Phi_{\boldsymbol{u}})$. By the definition of $\boldsymbol{K}_{\boldsymbol{u}}$, $\Phi_{\boldsymbol{u}}$ is a solution to (5-5). $\square$

***Solution of* (5-4).** Recall that rg $\boldsymbol{P}$ is spanned by eigenfunctions of $\boldsymbol{L}$ with eigenvalues 0 and 1; see Lemma 4.8. As in the proof of Lemma 4.8 we write $\boldsymbol{P} = \boldsymbol{P}_0 + \boldsymbol{P}_1$, where $\boldsymbol{P}_n$, $n \in \{0, 1\}$, projects to the geometric eigenspace of $\boldsymbol{L}$ associated to the eigenvalue $n \in \sigma_p(\boldsymbol{L})$. Consequently, we infer $\boldsymbol{S}(\tau)\boldsymbol{P}_n = e^{n\tau}\boldsymbol{P}_n$. This shows that the "bad" term we subtracted from (5-4) may be written as

$$\boldsymbol{S}(\tau)\boldsymbol{P}\boldsymbol{u} + \int_0^\infty \boldsymbol{S}(\tau - \sigma)\boldsymbol{P}\boldsymbol{N}(\Phi_{\boldsymbol{u}}(\sigma))\,d\sigma = \boldsymbol{S}(\tau)[\boldsymbol{P}\boldsymbol{u} - \boldsymbol{F}(\boldsymbol{u})],$$

where $\boldsymbol{F}$ is given by

$$\boldsymbol{F}(\boldsymbol{u}) := -\boldsymbol{P}_0 \int_0^\infty \boldsymbol{N}(\Phi_{\boldsymbol{u}}(\sigma))\,d\sigma - \boldsymbol{P}_1 \int_0^\infty e^{-\sigma}\boldsymbol{N}(\Phi_{\boldsymbol{u}}(\sigma))\,d\sigma.$$

According to Lemma 5.2, the function $\boldsymbol{F}$ is well-defined on $\mathcal{B}_{\delta^2} := \{\boldsymbol{u} \in \mathcal{H} : \|\boldsymbol{u}\| < \delta^2\}$ with values in rg $\boldsymbol{P}$ and this shows that we have effectively modified the *initial data* by adding an element of the four-dimensional subspace rg $\boldsymbol{P}$ of $\mathcal{H}$. Note, however, that the modification depends on the solution itself. Consequently, if the initial data for (5-4) are of the form $\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u})$ for $\boldsymbol{u} \in \ker \boldsymbol{P}$, (5-4) and (5-5) are equivalent and Lemma 5.2 yields the desired solution of (5-4). We also remark that $\boldsymbol{F}(\boldsymbol{0}) = \boldsymbol{0}$. The following result implies that the graph

$$\{\boldsymbol{u} + \boldsymbol{F}(\boldsymbol{u}) : \boldsymbol{u} \in \ker \boldsymbol{P}, \|\boldsymbol{u}\| < \delta^2\} \subset \ker \boldsymbol{P} \oplus \operatorname{rg} \boldsymbol{P} = \mathcal{H}$$

defines a Lipschitz manifold of codimension 4.

Lemma 5.3. *Let $\delta > 0$ be sufficiently small. Then the function $\boldsymbol{F} : \mathcal{B}_{\delta^2} \to$ rg $\boldsymbol{P} \subset \mathcal{H}$ satisfies*

$$\|\boldsymbol{F}(\boldsymbol{u}) - \boldsymbol{F}(\boldsymbol{v})\| \le C\delta\|\boldsymbol{u} - \boldsymbol{v}\|.$$

Proof. First, we claim that $\boldsymbol{u} \mapsto \Phi_{\boldsymbol{u}} : \mathcal{B}_{\delta^2} \to \mathcal{X}_\delta \subset \mathcal{X}$ is Lipschitz-continuous. Indeed, since $\Phi_{\boldsymbol{u}} = \boldsymbol{K}_{\boldsymbol{u}}(\Phi_{\boldsymbol{u}})$ we infer

$$\begin{aligned}\|\Phi_{\boldsymbol{u}} - \Phi_{\boldsymbol{v}}\|_{\mathcal{X}} &\le \|\boldsymbol{K}_{\boldsymbol{u}}(\Phi_{\boldsymbol{u}}) - \boldsymbol{K}_{\boldsymbol{u}}(\Phi_{\boldsymbol{v}})\|_{\mathcal{X}} + \|\boldsymbol{K}_{\boldsymbol{u}}(\Phi_{\boldsymbol{v}}) - \boldsymbol{K}_{\boldsymbol{v}}(\Phi_{\boldsymbol{v}})\|_{\mathcal{X}} \\ &\lesssim \delta\|\Phi_{\boldsymbol{u}} - \Phi_{\boldsymbol{v}}\|_{\mathcal{X}} + \|\boldsymbol{u} - \boldsymbol{v}\|\end{aligned}$$

by Proposition 5.1 and the fact that

$$\|\boldsymbol{K}_{\boldsymbol{u}}(\Phi_{\boldsymbol{v}})(\tau) - \boldsymbol{K}_{\boldsymbol{v}}(\Phi_{\boldsymbol{v}})(\tau)\| = \|\boldsymbol{S}(\tau)(1 - \boldsymbol{P})(\boldsymbol{u} - \boldsymbol{v})\| \lesssim e^{(-\frac{1}{2}+\epsilon)\tau}\|\boldsymbol{u} - \boldsymbol{v}\|.$$

The claim now follows from $\|\boldsymbol{N}(\boldsymbol{u}) - \boldsymbol{N}(\boldsymbol{v})\| \lesssim \|\boldsymbol{u} - \boldsymbol{v}\|(\|\boldsymbol{u}\| + \|\boldsymbol{v}\|)$. $\square$

We summarize our results in a theorem.

Theorem 5.4. *Let $\delta > 0$ be sufficiently small. There exists a codimension-4 Lipschitz manifold $\mathcal{M} \subset \mathcal{H}$ with $\mathbf{0} \in \mathcal{M}$ such that for any $\boldsymbol{u} \in \mathcal{M}$,* (5-4) *has a solution $\Phi \in \mathcal{X}_\delta$. Moreover, Φ is unique in $C([0,\infty),\mathcal{H})$. If, in addition, $\boldsymbol{u} \in \mathcal{D}(\boldsymbol{L})$ then $\Phi \in C^1([0,\infty),\mathcal{H})$ and Φ solves* (5-3) *with $\Phi(0) = \boldsymbol{u}$.*

Proof. The last statement follows from standard results of semigroup theory. Uniqueness in $C([0,\infty),\mathcal{H})$ is a simple exercise. □

Proof of Theorem 1.1. Theorem 1.1 is now a consequence of Theorem 5.4: (3-4) implies

$$\frac{v\circ\Phi_T(X)-v_0\circ\Phi_T(X)}{T^2-|X|^2} = \frac{1}{(-T)}\phi\Big(-\log(-T),\,\frac{X}{(-T)}\Big) \tag{5-7}$$

and thus,

$$\begin{aligned} |T|^{-1}\|v-v_0\|_{L^2(\Sigma_T)} &= |T|^{-2}\Big\|\phi_1\Big(-\log(-T),\,\frac{\cdot}{|T|}\Big)\Big\|_{L^2(B_{|T|})} \\ &= |T|^{-\frac{1}{2}}\big\|\phi_1(-\log(-T),\,\cdot\,)\big\|_{L^2(B)} \\ &\lesssim |T|^{-\epsilon}. \end{aligned}$$

Similarly, we obtain

$$\partial_{X^j}\frac{v\circ\Phi_T(X)-v_0\circ\Phi_T(X)}{T^2-|X|^2} = \frac{1}{T^2}\partial_j\phi_1\Big(-\log(-T),\,\frac{X}{(-T)}\Big),$$

which yields

$$\|v-v_0\|_{\dot H^1(\Sigma_T)} = T^{-2}\Big\|\nabla\phi_1\Big(-\log(-T),\,\frac{\cdot}{|T|}\Big)\Big\|_{L^2(B_{|T|})} \lesssim |T|^{-\epsilon}.$$

For the time derivative we infer

$$\begin{aligned} \partial_T\frac{v\circ\Phi_T(X)-v_0\circ\Phi_T(X)}{T^2-|X|^2} &= \frac{1}{T^2}\Big(\partial_0\phi+\frac{X^j}{(-T)}\partial_j\phi+\phi\Big)\Big(-\log(-T),\,\frac{X}{(-T)}\Big) \\ &= \frac{1}{T^2}\phi_2\Big(-\log(-T),\,\frac{X}{(-T)}\Big) \end{aligned}$$

and hence,

$$\|\nabla_n v-\nabla_n v_0\|_{L^2(\Sigma_T)} = T^{-2}\Big\|\phi_2\Big(-\log(-T),\,\frac{\cdot}{|T|}\Big)\Big\|_{L^2(B_{|T|})} \lesssim |T|^{-\epsilon}.$$

Finally, we turn to the Strichartz estimate. First, note that the modulus of the determinant of the Jacobian of $(T,X)\mapsto(t,x)$ is $(T^2-|X|^2)^{-4}$. This is easily seen by considering the transformation

$$X^\mu \mapsto y^\mu = -\frac{X^\mu}{X_\sigma X^\sigma},$$

which has the same Jacobian determinant (up to a sign) since $t=-y^0$ and $x^j=y^j$. We obtain

$$\partial_\nu y^\mu = -\frac{X_\sigma X^\sigma\delta_\nu{}^\mu - 2X_\nu X^\mu}{(X_\sigma X^\sigma)^2}$$

and hence,

$$\partial_\nu y^\mu \partial_\mu y^\lambda = \frac{\delta_\nu{}^\lambda}{(X_\sigma X^\sigma)^2}$$

which yields $|\det(\partial_\nu y^\mu)| = (X_\sigma X^\sigma)^{-4} = (T^2 - |X|^2)^{-4}$. Furthermore, note that $s \in [t, 2t]$ and $x \in B_{(1-\delta)t}$ imply

$$\begin{aligned} S &:= -\frac{s}{s^2 - |x|^2} \geq -\frac{t}{t^2 - |x|^2} \geq -\frac{c_\delta}{t}, \\ S &\leq -\frac{2t}{4t^2 - |x|^2} \leq -\frac{1}{2t}. \end{aligned}$$

Consequently, by (5-7) and Sobolev embedding we infer

$$\begin{aligned} \|v - v_0\|^4_{L^4(t,2t)L^4(B_{(1-\delta)t})} &\leq \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} \int_{B_{(1-\delta)|S|}} \left| \frac{v \circ \Phi_S(X) - v_0 \circ \Phi_S(X)}{S^2 - |X|^2} \right|^4 dX\, dS \\ &\lesssim \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{-4} \left\| \phi\Big(-\log(-S), \frac{\cdot}{|S|}\Big) \right\|^4_{L^4(B_{|S|})} dS \\ &= \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{-1} \left\| \phi(-\log(-S), \cdot) \right\|^4_{L^4(B)} dS \\ &\lesssim \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{-1} \left\| \phi(-\log(-S), \cdot) \right\|^4_{H^1(B)} dS \\ &\lesssim \int_{-\frac{c_\delta}{t}}^{-\frac{1}{2t}} |S|^{1-4\epsilon}\, dS \simeq t^{-2+4\epsilon} \end{aligned}$$

as claimed.

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Received 24 Apr 2013. Accepted 22 Aug 2013.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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