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BOHR’S ABSOLUTE CONVERGENCE PROBLEM FOR $\mathcal{H}_p$-DIRICHLET SERIES IN BANACH SPACES

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The Bohr–Bohnenblust–Hille theorem states that the width of the strip in the complex plane on which an ordinary Dirichlet series $\sum a_n n^{-s}$ converges uniformly but not absolutely is less than or equal to $\frac{1}{2}$, and this estimate is optimal. Equivalently, the supremum of the absolute convergence abscissas of all Dirichlet series in the Hardy space $\mathcal{H}_\infty$ equals $\frac{1}{2}$. By a surprising fact of Bayart the same result holds true if $\mathcal{H}_\infty$ is replaced by any Hardy space $\mathcal{H}_p$, $1 \leq p < \infty$, of Dirichlet series. For Dirichlet series with coefficients in a Banach space $X$ the maximal width of Bohr’s strips depend on the geometry of $X$; Defant, García, Maestre and Pérez-García proved that such maximal width equals $1 - \frac{1}{\text{Cot} X}$, where Cot $X$ denotes the maximal cotype of $X$. Equivalently, the supremum over the absolute convergence abscissas of all Dirichlet series in the vector-valued Hardy space $\mathcal{H}_\infty(X)$ equals $1 - \frac{1}{\text{Cot} X}$. In this article we show that this result remains true if $\mathcal{H}_\infty(X)$ is replaced by the larger class $\mathcal{H}_p(X)$, $1 \leq p < \infty$.

1. Main result and its motivation

Given a Banach space $X$, an ordinary Dirichlet series in $X$ is a series of the form $D = \sum a_n n^{-s}$, where the coefficients $a_n$ are vectors in $X$ and $s$ is a complex variable. Maximal domains where such Dirichlet series converge conditionally, uniformly or absolutely are half planes $\{\text{Re} > r\}$, where $\sigma = \sigma_c$, $\sigma_u$ or $\sigma_a$ are called the abscissa of conditional, uniform or absolute convergence, respectively. More precisely, $\sigma_\alpha(D)$ is the infimum of all $r \in \mathbb{R}$ such that on $\{\text{Re} > r\}$ we have convergence of $D$ of the requested type $\alpha = c, u$ or $a$. Clearly, we have $\sigma_c(D) \leq \sigma_u(D) \leq \sigma_a(D)$, and it can be easily shown that $\sup \sigma_u(D) - \sigma_c(D) = 1$, where the supremum is taken over all Dirichlet series $D$ with coefficients in $X$. To determine the maximal width of the strip on which a Dirichlet series in $X$ converges uniformly but not absolutely is more complicated. The main result of [Defant et al. 2008] states, with the notation given below, that

$$ S(X) := \sup \sigma_a(D) - \sigma_u(D) = 1 - \frac{1}{\text{Cot} X}. $$

Recall that a Banach space $X$ is of cotype $q$, $2 \leq q < \infty$, whenever there is a constant $C \geq 0$ such that for each choice of finitely many vectors $x_1, \ldots, x_N \in X$ we have

$$ \left( \sum_{k=1}^N \|x_k\|_X^q \right)^{1/q} \leq C \left( \int_T \left\| \sum_{k=1}^N x_k z_k \right\|_X^2 \, dz \right)^{1/2}, $$

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where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and $\mathbb{T}^N$ is endowed with the $N$-th product of the normalized Lebesgue measure on $\mathbb{T}$; we denote the best of such constants $C$ by $C_q(X)$. As usual we write

$$\text{Cot } X := \inf\{2 \leq q < \infty \mid X \text{ is of cotype } q\},$$

and, although this infimum in general is not attained, we call it the optimal cotype of $X$. If there is no $2 \leq q < \infty$ for which $X$ has cotype $q$, then $X$ is said to have no finite cotype, and we put $\text{Cot } X = \infty$.

To see an example,

$$\text{Cot } \ell_q = \begin{cases} q & \text{for } 2 \leq q \leq \infty, \\ 2 & \text{for } 1 \leq q \leq 2. \end{cases}$$

The scalar case $X = \mathbb{C}$ in (1) was first studied over a hundred years ago: Bohr [1913a] proved that $S(\mathbb{C}) \leq \frac{1}{2}$, and Bohnenblust and Hille [1931] that $S(\mathbb{C}) \geq \frac{1}{2}$. Clearly, the equality

$$S(\mathbb{C}) = \frac{1}{2},$$

nowadays called the Bohr–Bohnenblust–Hille theorem, fits with (1). Let us give a second formulation of (1). Define the vector space $\mathcal{H}_\infty(X)$ of all Dirichlet series $D = \sum a_n n^{-s}$ in $X$ such that

- $\sigma_c(D) \leq 0$,
- the function $D(s) = \sum a_n (1/n^s)$ on $\text{Re } s > 0$ is bounded.

Then $\mathcal{H}_\infty(X)$ together with the norm

$$\|D\|_{\mathcal{H}_\infty(X)} = \sup_{\text{Re } s > 0} \left\| \sum_{n=1}^\infty a_n \frac{1}{n^s} \right\|_X$$

forms a Banach space. For any Dirichlet series $D$ in $X$ we have

$$\sigma_a(D) = \inf \left\{ \sigma \in \mathbb{R} \left| \sum_{n=1}^\infty a_n \frac{1}{n^\sigma} \in \mathcal{H}_\infty(X) \right. \right\}.$$

In the scalar case $X = \mathbb{C}$, this is (what we call) Bohr’s fundamental theorem [1913b], and for Dirichlet series in arbitrary Banach spaces the proof follows similarly. Together with (4) a simply translation argument gives the following reformulation of (1):

$$S(X) = \sup_{D \in \mathcal{H}_\infty(X)} \sigma_a(D) = 1 - \frac{1}{\text{Cot } X}. \quad (5)$$

Following an ingenious idea of Bohr each Dirichlet series may be identified with a power series in infinitely many variables. More precisely, fix a Banach space $X$ and denote by $\mathfrak{P}(X)$ the vector space of all formal power series $\sum_{\alpha} c_\alpha e^\alpha$ in $X$ and by $\mathcal{D}(X)$ the vector space of all Dirichlet series $\sum a_n n^{-s}$ in $X$. Let as usual $(p_n)_n$ be the sequence of prime numbers. Since each integer $n$ has a unique prime
number decomposition \( n = p_1^{a_1} \cdots p_k^{a_k} = p^a \) with \( \alpha_j \in \mathbb{N}_0 \), \( 1 \leq j \leq k \), the linear mapping
\[
\mathcal{B}_X : \mathcal{H}(X) \to \mathcal{D}(X),
\sum_{\alpha \in \mathbb{N}_0^{(\infty)}} c_\alpha z^\alpha \mapsto \sum_{n=1}^\infty a_n n^{-s} \quad \text{if } a_{p^a} = c_\alpha,
\]
is bijective; we call \( \mathcal{B}_X \) the Bohr transform in \( X \). As discovered by Bayart [2002] this (a priori very) formal identification allows us to develop a theory of Hardy spaces of scalar–valued Dirichlet series.

Similarly, we now define Hardy spaces of \( X \)-valued Dirichlet series. Denote by \( dw \) the normalized Lebesgue measure on the infinite-dimensional polytorus \( T^\infty = \prod_{k=1}^{\infty} \mathbb{T} \), that is, the countable product measure of the normalized Lebesgue measure on \( \mathbb{T} \). For any multindex \( \alpha = (\alpha_1, \ldots, \alpha_n, 0, \ldots) \in \mathbb{Z}^{(n)} \) (all finite sequences in \( \mathbb{Z} \)) the \( \alpha \)-th Fourier coefficient \( \hat{f}(\alpha) \) of \( f \in L_1(T^\infty, X) \) is given by
\[
\hat{f}(\alpha) = \int_{T^\infty} f(w) w^{-\alpha} \, dw,
\]
where we as usual write \( w^\alpha \) for the monomial \( w_1^{\alpha_1} \cdots w_n^{\alpha_n} \). Then, given \( 1 \leq p < \infty \), the \( X \)-valued Hardy space on \( T^\infty \) is the subspace of \( L_p(T^\infty, X) \) defined as
\[
H_p(T^\infty, X) = \{ f \in L_p(T^\infty, X) \mid \hat{f}(\alpha) = 0 \text{ for all } \alpha \in \mathbb{Z}^{(n)} \setminus \mathbb{N}_0^{(n)} \}.
\]
Assigning to each \( f \in H_p(T^\infty, X) \) its unique formal power series \( \sum_{\alpha} \hat{f}(\alpha) z^\alpha \) we may consider \( H_p(T^\infty, X) \) as a subspace of \( \mathcal{H}(X) \). We denote the image of this subspace under the Bohr transform \( \mathcal{B}_X \) by
\[
\mathcal{H}_p(X).
\]
This vector space of all (so-called) \( \mathcal{H}_p(X) \)-Dirichlet series \( D \) together with the norm
\[
\| D \|_{\mathcal{H}_p(X)} = \| \mathcal{B}_X^{-1}(D) \|_{H_p(T^\infty, X)}
\]
forms a Banach space; in other words, through Bohr’s transform \( \mathcal{B}_X \) from (6) we by definition identify
\[
\mathcal{H}_p(X) = H_p(T^\infty, X), \quad 1 \leq p < \infty.
\]
For \( p = \infty \) we this way of course could also define a Banach space \( \mathcal{H}_\infty(X) \), and it turns out that at least in the scalar case \( X = \mathbb{C} \) this definition then coincides with the one given above; but we remark that these two \( \mathcal{H}_\infty(X) \)’s are different for arbitrary \( X \). It is important to note that by the Birkhoff–Khinchin ergodic theorem the following internal description of the \( \mathcal{H}_p(X) \)-norm for finite Dirichlet polynomials \( D = \sum_{k=1}^{n} a_k k^{-s} \) holds:
\[
\| D \|_{\mathcal{H}_p(X)} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} \left\| \sum_{k=1}^{n} a_k \frac{1}{k^t} \right\|^p_X \, dt \right)^{1/p}
\]
(see, for example, Bayart [2002] for the scalar case, and the vector-valued case follows exactly the same way).
Motivated by (4) we define for \( D \in \mathfrak{D}(X) \) and \( 1 \leq p < \infty \)

\[
\sigma_{\mathcal{H}_p(X)}(D) := \inf \left\{ \sigma \in \mathbb{R} \left| \sum_n \frac{a_n}{n^\sigma n^s} \in \mathcal{H}_p(X) \right. \right\},
\]

the so-called \( \mathcal{H}_p(X) \)-abscissa of \( D \). In [Aleman et al. \( \geq 2014 \)], Aleman, Olsen, and Saksman prove that the sequence (of Dirichlet series) \( 1/n^s, n \in \mathbb{N} \) is a Schauder basis in \( \mathcal{H}_p(\mathbb{C}) \) for \( 1 < p < \infty \). Hence, for \( 1 < p < \infty \) and any Dirichlet series \( D \in \mathfrak{D}(\mathbb{C}) \) we have

\[
\sigma_{\mathcal{H}_p(\mathbb{C})}(D) = \inf \left\{ \sigma \in \mathbb{R} \left| \left( \sum_{n=1}^N \frac{a_n}{n^\sigma n^s} \right)_N \text{ is Cauchy in } \mathcal{H}_p(\mathbb{C}) \right. \right\},
\]

which (in the scalar case) is the perfect analog of Bohr’s fundamental theorem (i.e., the case \( p = \infty \) from (4), where uniform convergence is precisely being Cauchy in \( \mathcal{H}_p(\mathbb{C}) \)). In [Defant 2013] it is shown that (9) also holds true for \( p = 1 \) (although in this case the \( 1/n^s \) are definitely no Schauder basis in \( \mathcal{H}_1(\mathbb{C}) \)), and even more: The arguments given in [Defant 2013] (inspired by Bohr’s original ideas [1913b]) prove that (9) even holds for any \( 1 \leq p \leq \infty \) and any \( X \)-valued Dirichlet series \( D \in \mathcal{H}_p(X) \). In view of (1) and (5), it therefore seems natural to study

\[
S_p(X) := \sup_{D \in \mathfrak{D}(X)} \sigma_a(D) - \sigma_{\mathcal{H}_p(X)}(D) = \sup_{D \in \mathcal{H}_p(X)} \sigma_a(D)
\]

(for the second equality use again a simple translation argument). The scalar case is completely understood since, by a result of Bayart [2002],

\[
S_p(\mathbb{C}) = \frac{1}{2} \quad \text{for every } 1 \leq p < \infty,
\]

which according to Helson [2005] is surprising since \( \mathcal{H}_\infty(\mathbb{C}) \) is much smaller than \( \mathcal{H}_p(\mathbb{C}) \).

The following theorem unifies and generalizes (1), (3) as well as (10), and it is our main result.

**Theorem 1.1.** For every \( 1 \leq p \leq \infty \) and every Banach space \( X \) we have

\[
S_p(X) = 1 - \frac{1}{\Cot X}.
\]

The proof will be given in Section 3. But before we start let us give an interesting reformulation in terms of the monomial convergence of \( X \)-valued \( H_p \)-functions on \( \mathbb{T}^\infty \). Fix a Banach space \( X \) and \( 1 \leq p \leq \infty \), and define the set of monomial convergence of \( H_p(\mathbb{T}^\infty, X) \):

\[
\text{mon } H_p(\mathbb{T}^\infty, X) = \left\{ z \in B_{c_0} \left| \sum_\alpha \| \hat{f}(\alpha)z^\alpha \|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X) \right. \right\}.
\]

Philosophically, this is the largest set \( M \) on which for each \( f \in H_p(\mathbb{T}^\infty, X) \) the definition \( g(z) = \sum_\alpha \hat{f}(\alpha)z^\alpha \), \( z \in M \) leads to an extension of \( f \) from the distinguished boundary \( \mathbb{T}^\infty \) to its “interior” \( B_{c_0} \) (the open unit ball of the Banach space \( c_0 \) of all null sequences). For a detailed study of sets of monomial convergence in the scalar case \( X = \mathbb{C} \) see [Defant et al. 2009], and in the vector-valued case [Defant and Sevilla-Peris 2011].
We later need the following two basic properties of monomial domains (in the scalar case see [Defant et al. 2008, p. 550; 2014, Lemma 4.3], and in the vector-valued case the proofs follow similar lines).

Remark 1.2. (1) Let \( z \in \text{mon} \ H_p(\mathbb{T}^\infty, X) \). Then \( u = (z_{\sigma(n)})_n \in \text{mon} \ H_p(\mathbb{T}^\infty, X) \) for every permutation \( \sigma \) of \( \mathbb{N} \).

(2) Let \( z \in \text{mon} \ H_p(\mathbb{T}^\infty, X) \) and \( x = (x_n)_n \in \ell^\infty \) be such that \( |x_n| \leq |z_n| \) for all but finitely many \( n \)'s. Then \( x \in \text{mon} \ H_p(\mathbb{T}^\infty, X) \).

Given \( 1 \leq p \leq \infty \) and a Banach space \( X \), the following number measures the size of \( \text{mon} \ H_p(\mathbb{T}^\infty, X) \) within the scale of \( \ell_r \)-spaces:

\[
M_p(X) = \sup\{1 \leq r \leq \infty \mid \ell_r \cap B_{c_0} \subset \text{mon} \ H_p(\mathbb{T}^\infty, X)\}.
\]

The following result is a reformulation of Theorem 1.1 in terms of vector-valued \( H_p \)-functions on \( \mathbb{T}^\infty \) through Bohr’s transform \( \mathfrak{B}_X \). The proof is modeled along ideas from Bohr’s seminal article [1913a, Satz IX].

Corollary 1.3. For each Banach space \( X \) and \( 1 \leq p \leq \infty \) we have

\[
M_p(X) = \frac{\text{Cot} \ X}{\text{Cot} X - 1}.
\]

Proof. We are going to prove that \( S_p(X) = 1/M_p(X) \), and as a consequence the conclusion follows from Theorem 1.1. We begin by showing that \( S_p(X) \leq 1/M_p(X) \). We fix \( q < M_p(X) \) and \( r > 1/q \); then we have that \( (1/p'_n)_n \in \ell_q \cap B_{c_0} \) and, by the very definition of \( M_p(X) \), \( \sum_\alpha \| \hat{f}(\alpha)(1/p')^\alpha \|_X < \infty \) converges absolutely for every \( f \in H_p(\mathbb{T}^\infty, X) \). We choose now an arbitrary Dirichlet series

\[
D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathcal{H}_p(X) \quad \text{with} \quad f \in H_p(\mathbb{T}^\infty, X).
\]

Then

\[
\sum_n \| a_n \|_X \frac{1}{n^r} = \sum_\alpha \| a_{p^\alpha} \|_X \left( \frac{1}{p^\alpha} \right)^r = \sum_\alpha \| \hat{f}(\alpha) \|_X \left( \frac{1}{p^r} \right)^\alpha < \infty.
\]

Clearly, this implies that \( S_p(X) \leq r \). Since this holds for each \( r > 1/q \), we get that \( S_p(X) \leq 1/q \), and since this now holds for each \( q < M_p(X) \), we have \( S_p(X) \leq 1/M_p(X) \). Conversely, let us take some \( q > M_p(X) \); then there is \( z \in \ell_q \cap B_{c_0} \) and \( f \in H_p(\mathbb{T}^\infty, X) \) such that \( \sum_\alpha \hat{f}(\alpha) z^\alpha \) does not converge absolutely. By Remark 1.2 we may assume that \( z \) is decreasing, and hence \( (z_n n^{1/q})_n \) is bounded. We choose now \( r > q \) and define \( w_n = 1/p_n^{1/r} \). By the prime number theorem we know that there is a universal constant \( C > 0 \) such that

\[
0 < \frac{z_n}{w_n} = \frac{n^{1/q}}{p_n^{1/r}} = \frac{n^{1/q}}{p_n^{1/r}} \frac{p_n^{1/r}}{n^{1/q}} = z_n n^{1/q} \left( \frac{p_n}{n} \right)^{1/r} \leq C z_n n^{1/q} \left( \log n \right)^{1/r}.
\]

The last term tends to 0 as \( n \to \infty \); hence \( z_n \leq w_n \) but for a finite number of \( n \)'s. By Remark 1.2 this implies that \( \sum_\alpha \hat{f}(\alpha) w^\alpha \) does not converge absolutely. But then \( D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathcal{H}_p(X) \)
satisfies
\[
\sum_n \|a_n\|_X \frac{1}{n^{1/r}} = \sum_\alpha \|a_{\alpha}p\|_X \left(\frac{1}{p^{1/r}} \right)^\alpha = \sum_\alpha \|\hat{f}(\alpha)\|_X u^{\alpha} = \infty.
\]
This gives that \(\sigma_a(D) \geq \frac{1}{r}\) for every \(r > q\), hence \(\sigma_a(D) \geq \frac{1}{q}\). Since this holds for every \(q > M_p(X)\), we finally have \(S_p(X) \geq \frac{1}{M_p(X)}\).

We shall use standard notation and notions from Banach space theory, as presented, for example, in [Lindenstrauss and Tzafriri 1977; 1979]. For everything needed on polynomials in Banach spaces see, for example, [Dineen 1999; Floret 1997].

2. Relevant inequalities

The main aim here is to prove a sort of polynomial extension of the notion of cotype. Recall the definition of \(C_q(X)\) from (2). Moreover, from Kahane’s inequality we know that there is a (best) constant \(K \geq 1\) such that, for each Banach space \(X\) and each choice of finitely many vectors \(x_1, \ldots, x_N \in X\),

\[
\left(\int_{T^N} \left\|\sum_{k=1}^N x_k z_k\right\|_X^2 dz\right)^{1/2} \leq K \int_{T^N} \left\|\sum_{k=1}^N x_k z_k\right\|_X^2 dz.
\]

As usual we write \(|\alpha| = \alpha_1 + \cdots + \alpha_N\) and \(\alpha! = \alpha_1! \cdots \alpha_N!\) for every multiindex \(\alpha \in \mathbb{N}_0^N\).

**Proposition 2.1.** Let \(X\) be a Banach space of cotype \(q, 2 \leq q < \infty\), and \(P : \mathbb{C}^N \rightarrow X, \; P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha| = m} c_\alpha z^\alpha\) be an \(m\)-homogeneous polynomial. Let

\[
T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \rightarrow X, \; T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1, \ldots, i_m} z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}
\]

be the unique \(m\)-linear symmetrization of \(P\). Then

\[
\left(\sum_{i_1, \ldots, i_m} \|a_{i_1, \ldots, i_m}\|_X^q \right)^{1/q} \leq (C_q(X)K)^m \frac{m^m}{m!} \int_{T^N} \|P(z)\|_X dz.
\]

Before we give the proof let us note that [Bombal et al. 2004, Theorem 3.2] is an \(m\)-linear result that, combined with polarization, gives (with the previous notation)

\[
\left(\sum_{i_1, \ldots, i_m} \|a_{i_1, \ldots, i_m}\|_X^q \right)^{1/q} \leq C_q(X)^m m! \frac{m^m}{m!} \sup_{z \in D^N} \|P(z)\|.
\]

Our result allows us to replace (up to the constant \(K\)) the \(\|\|_\infty\) norm with the smaller norm \(\|\|_1\). We prepare the proof of Proposition 2.1 with three lemmas. The first one is a complex version of [Defant et al. 2010, Lemma 2.2] with essentially the same proof; we include it for the sake of completeness.
Lemma 2.2. Let \( X \) be a Banach space of cotype \( q \), \( 2 \leq q < \infty \). Then, for every \( m \)-linear form
\[
T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m=1}^N a_{i_1, \ldots, i_m} z^{(1)}_{i_1} \cdots z^{(m)}_{i_m},
\]
we have
\[
\left( \sum_{i_1, \ldots, i_m=1}^N \|a_{i_1, \ldots, i_m}\|_X \right)^{1/q} \leq (C_q(X) K)^m \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X \, dz^{(1)} \cdots dz^{(m)}.
\]

Proof. We prove this result by induction on the degree \( m \). For \( m = 1 \) the result is an immediate consequence of the definition of cotype \( q \) and Kahane’s inequality. Assume that the result holds for \( m - 1 \). By the continuous Minkowski inequality we then conclude that for every choice of finitely many vectors \( a_{i_1, \ldots, i_m} \in X \) with \( 1 \leq i_j \leq N, 1 \leq j \leq m \) we have
\[
\sum_{i_1, \ldots, i_m} \|a_{i_1, \ldots, i_m}\|_X^q = \sum_{i_1, \ldots, i_{m-1}} \sum_{i_m} \|a_{i_1, \ldots, i_m}\|_X^q
\leq C_q(X)^q K^q \left( \sum_{i_1, \ldots, i_{m-1}} \left( \int_{\mathbb{T}^{N-1}} \left\| \sum_{i_m} a_{i_1, \ldots, i_m} z^{(m)}_{i_m} \right\|_X \, dz^{(m)} \right)^{q/q} \right)^{1/q}
\leq C_q(X)^q K^q \left( \int_{\mathbb{T}^N} \left( \sum_{i_1, \ldots, i_{m-1}} \left\| \sum_{i_m} a_{i_1, \ldots, i_m} z^{(m)}_{i_m} \right\|_X \right)^q \, dz^{(m)} \right)^{1/q}
\leq C_q(X)^q K^q \left( \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \left\| \sum_{i_1, \ldots, i_{m-1}} a_{i_1, \ldots, i_{m-1}} z^{(1)}_{i_1} \cdots z^{(m-1)}_{i_{m-1}} \right\|_X \, dz^{(1)} \cdots dz^{(m-1)} \, dz^{(m)} \right)^q,
\]
which is the conclusion.

The following two lemmas are needed to produce a polynomial analog of the preceding result.

Lemma 2.3. Let \( X \) be a Banach space, and \( f : \mathbb{C} \to X \) a holomorphic function. Then for \( R_1, R_2, R \geq 0 \) with \( R_1 + R_2 \leq R \) we have
\[
\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 \leq \int_{\mathbb{T}} \|f(Rz)\|_X \, dz.
\]

Proof. By the rotation invariance of the normalized Lebesgue measure on \( \mathbb{T} \) we get
\[
\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 z_2 + R_2 z_2)\|_X \, d\zeta_1 \, d\zeta_2
\]
\[
= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2(R_1 z_1 + R_2))\|_X \, d\zeta_1 \, d\zeta_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 |R_1 z_1 + R_2|)\|_X \, d\zeta_2 \, d\zeta_1
\]
\[
= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 r(z_1) R)\|_X \, d\zeta_2 \, d\zeta_1 = \int_0^{2\pi} \int_0^{2\pi} \|f(r(\epsilon i) Re^{it})\|_X \, \frac{dt}{2\pi} \, \frac{ds}{2\pi},
\]
where \( r(z) = (1/R)|R_1z + R_2|, \) \( z \in \mathbb{T}. \) We know that for each holomorphic function \( h : \mathbb{C} \to X \) we have
\[
\int_{\mathbb{T}} \|h(z)\|_X \, dz = \sup_{0 \leq r \leq 1} \int_{0}^{2\pi} \|h(re^{it})\|_X \frac{dt}{2\pi}
\]
(see, for example, Blasco and Xu [1991, p. 338]). Define now \( h(z) = f(Rz), \) and note that \( 0 \leq r(z) \leq 1 \) for all \( z \in \mathbb{T}. \) Then
\[
\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1z_1 + R_2z_2)\|_X \, dz_1 \, dz_2 = \int_{0}^{2\pi} \int_{0}^{2\pi} \|h(r(e^{i\theta})e^{i\phi})\|_X \frac{dt}{2\pi} \frac{ds}{2\pi}
\]
\[
\leq \int_{0}^{2\pi} \|h(z)\|_X \, dz \frac{ds}{2\pi} = \int_{\mathbb{T}} \|f(Rz)\|_X \, dz.
\]
This completes the proof. □

A sort of iteration of the preceding result leads to the next:

**Lemma 2.4.** Let \( X \) be a Banach space, and \( f : \mathbb{C}^N \to X \) a holomorphic function. Then, for every \( m, \)
\[
\int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \|f(z^{(1)} + \cdots + z^{(m)})\|_X \, dz^{(1)} \cdots dz^{(m)} \leq \int_{\mathbb{T}^N} \|f(mz)\|_X \, dz.
\]

**Proof.** We fix some \( m, \) and do induction with respect to \( N. \) For \( N = 1 \) we obtain from Lemma 2.3 that
\[
\int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z^{(1)} + \cdots + z^{(m-2)} + z^{(m-1)} + z^{(m)})\|_X \, dz^{(m-1)} \, dz^{(m)} \, dz^{(1)} \cdots dz^{(m-2)}
\]
\[
= \|g_{z^{(1)}, \ldots, z^{(m-2)}, z^{(m-1)}, z^{(m)}}\|_X \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \, dz^{(1)} \cdots dz^{(m-2)}
\]
\[
\leq \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z^{(1)} + \cdots + z^{(m-2)} + 2w)\|_X \, dw \, dz^{(m-2)} \, dz^{(1)} \cdots dz^{(m-3)}
\]
\[
\leq \int_{\mathbb{T}} \cdots \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z^{(1)} + \cdots + z^{(m-3)} + 3w)\|_X \, dw \, dz^{(m-3)} \, dz^{(1)} \cdots dz^{(m-3)}
\]
\[
\leq \cdots \leq \int_{\mathbb{T}} \|f(mz)\|_X \, dz.
\]

We now assume that the conclusion holds for \( N - 1 \) and write each \( z \in \mathbb{T}^N \) as \( z = (u, w), \) with \( u \in \mathbb{T}^{N-1} \) and \( w \in \mathbb{T}. \) Then, using the case \( N = 1 \) in the first inequality and the inductive hypothesis in the second,
we have

\[
\int_T \cdots \int_T \| f(z^{(1)} + \cdots + z^{(m)}) \|_X \, dz^{(1)} \cdots dz^{(m)}
\]

\[
= \int_{T^{N-1}} \cdots \int_{T^{N-1}} \left( \int_T \int_T \| f(u^{(1)}, w_1) + \cdots + (u^{(m)}, w_m) \|_X \, dw_1 \cdots dw_N \right) \, du^{(1)} \cdots du^{(m)}
\]

\[
\leq \int_{T^{N-1}} \cdots \int_{T^{N-1}} \left( \int_T \| f(u^{(1)}, mw) + \cdots + (u^{(m)}, mw) \|_X \, dw \right) \, du^{(1)} \cdots du^{(m)}
\]

\[
= \int_T \int_{T^{N-1}} \| f(u^{(1)}, mw) + \cdots + (u^{(m)}, mw) \|_X \, du \, dw
\]

\[
\leq \int_T \int_{T^{N-1}} \| f(mw, mw) + \cdots + (mu, mw) \|_X \, du \, dw
\]

\[
= \int_T \| f(mz) \|_X \, dz,
\]
as desired. \( \square \)

**Proof of the inequality from Proposition 2.1.** By the polarization formula we know that for every choice of \(z^{(1)}, \ldots, z^{(m)} \in T^N\) we have

\[
T(z^{(1)}, \ldots, z^{(m)}) = \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \varepsilon_i^m P \left( \sum_{i=1}^N \varepsilon_i z^{(i)} \right)
\]

(see, for example, [Dineen 1999] or [Floret 1997]). Hence we deduce from Lemma 2.4

\[
\int_{T^N} \cdots \int_{T^N} \| T(z^{(1)}, \ldots, z^{(m)}) \|_X \, dz^{(1)} \cdots dz^{(m)} \leq \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \int_{T^N} \cdots \int_{T^N} \| P \left( \sum_{i=1}^N \varepsilon_i z^{(i)} \right) \|_X \, dz^{(1)} \cdots dz^{(m)}
\]

\[
= \frac{1}{2^m m!} \sum_{\varepsilon_i = \pm 1} \int_{T^N} \cdots \int_{T^N} \| P \left( \sum_{i=1}^N z^{(i)} \right) \|_X \, dz^{(1)} \cdots dz^{(m)}
\]

\[
= \frac{1}{m!} \int_{T^N} \cdots \int_{T^N} \| P \left( \sum_{i=1}^N z^{(i)} \right) \|_X \, dz^{(1)} \cdots dz^{(m)}
\]

\[
\leq \frac{1}{m!} \int_{T^N} \| P(mz) \|_X \, dz = \frac{m^m}{m!} \int_{T^N} \| P(z) \|_X \, dz.
\]

Then by Lemma 2.2 we obtain

\[
\left( \sum_{i_1, \ldots, i_m} \| a_{i_1, \ldots, i_m} \|_X^q \right)^{1/q} \leq (C_q(X)K)^m \int_{T^\infty} \cdots \int_{T^\infty} \| T(z^{(1)}, \ldots, z^{(m)}) \|_X \, dz^{(1)} \cdots dz^{(m)}
\]

\[
= (C_q(X)K)^m \frac{m^m}{m!} \int_{T^N} \| P(z) \|_X \, dz,
\]

which completes the proof of Proposition 2.1. \( \square \)
A second proposition is needed which allows us to reduce the proof of our main result (Theorem 1.1) to the homogeneous case. It is a vector-valued version of a result of [Cole and Gamelin 1986, Theorem 9.2] with a similar proof (here only given for the sake of completeness).

**Proposition 2.5.** There is a contractive projection

\[ \Phi_m : H_p(\mathbb{T}^N, X) \to H_p(\mathbb{T}^N, X), \quad f \mapsto f_m, \]

such that, for all \( f \in H_p(\mathbb{T}^N, X) \),

\[ \hat{f}(\alpha) = \hat{f}_m(\alpha) \quad \text{for all} \quad \alpha \in \mathbb{N}_0^N \quad \text{with} \quad |\alpha| = m. \]  

(11)

**Proof.** Let \( \mathcal{P}(\mathbb{C}^N, X) \subset H_p(\mathbb{T}^N, X) \) be the subspace of all finite polynomials \( f = \sum_{\alpha \in \Lambda} c_\alpha z^\alpha \); here \( \Lambda \) is a finite set of multiindices in \( \mathbb{N}_0^N \) and the coefficients \( c_\alpha \in X \). Define the linear projection \( \Phi_m^0 \) on \( \mathcal{P}(\mathbb{C}^N, X) \) by

\[ \Phi_m^0(f)(z) = f_m(z) = \sum_{\alpha \in \Lambda, |\alpha| = m} \hat{f}(\alpha)z^\alpha; \]

clearly, we have (11). In order to show that \( \Phi_m^0 \) is a contraction on \( (\mathcal{P}(\mathbb{C}^N, X), \| \cdot \|_p) \) fix some function \( f \in \mathcal{P}(\mathbb{C}^N, X) \) and \( z \in \mathbb{T}^N \), and define

\[ f(z \cdot) : \mathbb{T} \to X, \quad w \mapsto f(z w). \]

Clearly, we have

\[ f(zw) = \sum_k f_k(z) w^k, \]

and hence

\[ f_m(z) = \int_{\mathbb{T}} f(z w) w^{-m} dw. \]

Integration, Hölder’s inequality and the rotation invariance of the normalized Lebesgue measure on \( \mathbb{T}^N \) give

\[
\int_{\mathbb{T}^N} \| f_m(z) \|_X^p \, dz = \int_{\mathbb{T}^N} \left( \int_{\mathbb{T}} \| f(z w) w^{-m} \|_X^p \, dw \right) \, dz \\
\leq \int_{\mathbb{T}^N} \left( \int_{\mathbb{T}} \| f(z w) \|_X \, dw \right)^p \, dz \\
\leq \int_{\mathbb{T}^N} \int_{\mathbb{T}^N} \| f(zw) \|_X^p \, dz \, dw = \int_{\mathbb{T}^N} \| f(z) \|_X^p \, dz,
\]

which proves that \( \Phi_m^0 \) is a contraction on \( (\mathcal{P}(\mathbb{C}^N, X), \| \cdot \|_p) \). By Fejér’s theorem (vector-valued) we know that \( \mathcal{P}(\mathbb{C}^N, X) \) is a dense subspace of \( H_p(\mathbb{T}^N, X) \). Hence \( \Phi_m^0 \) extends to a contractive projection \( \Phi_m \) on \( H_p(\mathbb{T}^N, X) \). This extension \( \Phi_m \) still satisfies (11) since the mapping \( H_p(\mathbb{T}^N, X) \to X, f \mapsto \hat{f}(\alpha) \) is continuous for each multiindex \( \alpha \). \qed
3. Proof of the main result

We are now ready to prove Theorem 1.1. Let $1 \leq p < \infty$, and recall from (1) that

$$1 - \frac{1}{\text{Cot } X} = S_\infty(X) \leq S_p(X);$$

see Remark 3.1 for a direct argument. Hence it suffices to concentrate on the upper estimate in Theorem 1.1: Since we obviously have $S_p(X) \leq S_1(X)$, we are going to prove that

$$S_1(X) \leq 1 - \frac{1}{\text{Cot } X}. \quad (12)$$

Suppose first that $X$ has no finite cotype, i.e., $\text{Cot } X = \infty$. For $D = \sum_n a_n n^{-s} \in \mathcal{H}_1(X)$ we take $f \in H_1(\mathbb{T}^\infty, X)$ with $D = \mathbb{B}_X f$. Note that

$$\|\hat{f}(\alpha)\|_X \leq \int_{\mathbb{T}^\infty} \|f(w)w^{-\alpha}\|_X \, dw = \|f\|_{L_1(\mathbb{T}^\infty, X)} < \infty;$$

hence, by the definition of $\mathbb{B}_X$, the coefficients of $D$ are also bounded by $\|f\|_{L_1(\mathbb{T}^\infty, X)}$. As a consequence, for every $\sigma > 1$ we have

$$\sum\limits_{n=1}^{\infty} \|a_n\|_X \frac{1}{n^{\sigma}} \leq \sum\limits_{n=1}^{\infty} \|f\|_{L_1(\mathbb{T}^\infty, X)} \frac{1}{n^{\sigma}} < \infty.$$

This means that $S_1(X) \leq 1$ and as a consequence (12) holds.

Now if $X$ has finite cotype, take $q > \text{Cot } X$ and $\varepsilon > 0$, and put $s = (1 - 1/q)(1 + 2\varepsilon)$. Choose an integer $k_0$ such that $p_{k_0}^{\varepsilon/q'} > eC_q(X)K \left(\sum_{j=1}^{\infty} 1/p_j^{1+\varepsilon}\right)^{1/q'}$ and define

$$\tilde{p} = (p_{k_0}, \ldots, p_{k_0}, p_{k_0+1}, p_{k_0+2}, \ldots).$$

We are going to show that there is a constant $C(q, X, \varepsilon) > 0$ such that for every $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum\limits_{\alpha \in \mathcal{N}_{\tilde{p}/q}(\eta)} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{\varepsilon\alpha}} \leq C(q, X, \varepsilon)\|f\|_{H_1(\mathbb{T}^\infty, X)}. \quad (13)$$

This finishes the argument: By Remark 1.2 the sequence $1/p^s$ is in mon $H_1(\mathbb{T}^\infty, X)$. But in view of Bohr’s transform from (6), this means that for every Dirichlet series $D = \sum_n a_n n^{-s} = \mathbb{B}_X f \in \mathcal{H}_1(X)$ with $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum\limits_{n=1}^{\infty} \|a_n\|_X \frac{1}{n^s} = \sum\limits_{\alpha \in \mathcal{N}_{\tilde{p}/q}(\eta)} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{\varepsilon\alpha}} < \infty.$$

Therefore $\sigma_\varepsilon(D) \leq (1 - 1/q)(1 + 2\varepsilon)$ for each such $D$ which, since $\varepsilon > 0$ was arbitrary, is what we wanted to prove.

It remains to check (13); the idea is to show first that (13) holds for all $X$-valued $H_1$-functions which only depend on $N$ variables: There is a constant $C(q, X, \varepsilon) > 0$ such that for all $N$ and every
We finally give the proof of (14): Take \( f \) we now apply (14) to this \( f \). Then it can be easily shown that \( f \) is \( P \)-homogeneous polynomial \( m \). Indeed, take such a polynomial \( m \) in order to understand that (14) implies (13) (and hence the conclusion), assume that (14) holds and take some \( f \in H_1(\mathbb{T}^\infty, X) \). Given an arbitrary \( N \), define

\[
f_N : \mathbb{T}^N \to X, \quad f_N(w) = \int_{\mathbb{T}^\infty} f(w, \bar{w}) \, d\bar{w}.
\]

Then it can be easily shown that \( f_N \in L_1(\mathbb{T}^N, X) \), \( \|f_N\|_1 \leq \|f\|_1 \), and \( \hat{f}_N(\alpha) = \hat{f}(\alpha) \) for all \( \alpha \in \mathbb{Z}^N \). If we now apply (14) to this \( f_N \), we get

\[
\sum_{\alpha \in \mathbb{N}_0^N : |\alpha| = m} \|\hat{f}(\alpha)\|_X \frac{1}{p^{s\alpha}} \leq C(q, X, \varepsilon) \|f\|_{H_1(\mathbb{T}^N, X)}.
\]

which, after taking the supremum over all possible \( N \) on the left side, leads to (13).

We turn to the proof of (14), and here in a first step will show the following: For every \( N \), every \( m \)-homogeneous polynomial \( P : \mathbb{C}^N \to X \) and every \( u \in \ell_q \) we have

\[
\sum_{\alpha \in \mathbb{N}_0^N : |\alpha| = m} \|\hat{P}(\alpha)u^\alpha\|_X \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X \, dz \left( \sum_{j=1}^{\infty} |u_j|^{q'} \right)^{m/q'}.
\]

Indeed, take such a polynomial \( P(z) = \sum_{\alpha \in \mathbb{N}_0^N : |\alpha| = m} \hat{P}(\alpha)z^\alpha \), \( z \in \mathbb{T}^N \), and look at its unique \( m \)-linear symmetrization

\[
T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m = 1}^N a_{i_1, \ldots, i_m} z^{(1)}_{i_1} \cdots z^{(m)}_{i_m}.
\]

Then we know from Proposition 2.1 that

\[
\left( \sum_{i_1, \ldots, i_m = 1}^N \|a_{i_1, \ldots, i_m}\|^{q'}_X \right)^{1/q'} \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X \, dz.
\]

Hence (15) follows by Hölder’s inequality:

\[
\sum_{\alpha \in \mathbb{N}_0^N : |\alpha| = m} \|\hat{P}(\alpha)u^\alpha\|_X = \sum_{i_1, \ldots, i_m = 1}^N \|a_{i_1, \ldots, i_m}\|_X |u_{i_1} \cdots u_{i_m}| \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X \, dz \left( \sum_{j=1}^{\infty} |u_j|^{q'} \right)^{m/q'}.
\]

We finally give the proof of (14): Take \( f \in H_1(\mathbb{T}^N, X) \), and recall from Proposition 2.5 that for each integer \( m \) there is an \( m \)-homogeneous polynomial \( P_m : \mathbb{C}^N \to X \) such that \( \|P_m\|_{H_1(\mathbb{T}^N, X)} \leq \|f\|_{H_1(\mathbb{T}^N, X)} \).
and \( \hat{P}_m(\alpha) = \hat{f}(\alpha) \) for all \( \alpha \in \mathbb{N}_0^N \) with \( |\alpha| = m \). From (15), the definition of \( s \), and the fact that \( \max\{p_{k_0}, p_j\} \leq \tilde{p}_j \) for all \( j \) we have

\[
\sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^s} = \sum_{m=1}^\infty \sum_{\alpha \in \mathbb{N}_0^N \atop |\alpha| = m} \|\hat{P}_m(\alpha)\|_X \frac{1}{\tilde{p}^s} \leq \sum_{m=1}^\infty (eCq(X)K)^m \|P_m\|_{H_1(\mathbb{T}^N, X)} \left( \sum_{j=1}^{\infty} \frac{1}{\tilde{p}^{q_j}} \right) \frac{m}{q'} \]

\[
= \sum_{m=1}^\infty (eCq(X)K)^m \|f\|_{H_1(\mathbb{T}^N, X)} \left( \sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+2\epsilon}} \right) \frac{m}{q'} \]

\[
= \sum_{m=1}^\infty (eCq(X)K)^m \|f\|_{H_1(\mathbb{T}^N, X)} \left( \sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+\epsilon}} \frac{1}{\tilde{p}_j^s} \right) \frac{m}{q'} \]

\[
\leq \|f\|_{H_1(\mathbb{T}^N, X)} \sum_{m=1}^\infty \left( \frac{eCq(X)K(\sum_{j=1}^{\infty} \frac{1}{p_j^{(1+\epsilon)}})^{1/q'}}{p_{k_0}^{s/q'}} \right)^m < 1.
\]

This completes the proof of Theorem 1.1. □

**Remark 3.1.** We end this note with a direct proof of the fact

\[
1 - \frac{1}{\text{Cot } X} \leq S_p(X), \quad 1 \leq p < \infty,
\]

in which we do not use the inequality

\[
1 - \frac{1}{\text{Cot } X} \leq S_\infty(X)
\]

from [Defant et al. 2008] (here repeated in (1)). The proof of (17) given in that reference shows in a first step that \( 1 - 1/\Pi(X) \leq S_\infty(X) \) where

\[
\Pi(X) = \inf\{r \geq 2 \mid \text{id}_X \text{ is } (r, 1)\text{-summing}\},
\]

and then, in a second step, applies a fundamental theorem of Maurey and Pisier stating that \( \Pi(X) = \text{Cot } X \).

The following argument for (16) is very similar to the original one from [Defant et al. 2008] but does not use the Maurey–Pisier theorem (since we here consider \( \mathcal{H}_p(X) \), \( 1 \leq p < \infty \) instead of \( \mathcal{H}_\infty(X) \)): By the proof of Corollary 1.3, inequality (16) is equivalent to

\[
M_p(X) \leq \frac{\text{Cot } X}{\text{Cot } X - 1}.
\]

Take \( r < M_p(X) \), so that \( \ell_r \cap B_{c_0} \subset \text{mon } H_p(\mathbb{T}^\infty, X) \). Let \( H_1^1(\mathbb{T}^\infty, X) \) be the subspace of \( H_p(\mathbb{T}^\infty, X) \) formed by all 1-homogeneous polynomials (i.e., linear operators). We can define a bilinear operator
\[ \ell_r \times H^1_p (\mathbb{T}^\infty, X) \rightarrow \ell_1 (X) \] by \((z, f) \mapsto (z_j f(e_j))_j\) which, by a closed graph argument, is continuous. Therefore, there is a constant \(M\) such that for all \(z \in \ell_r\) and all \(f \in H^1_p (\mathbb{T}^\infty, X)\) we have
\[
\sum_j |z_j| \| f(e_j) \|_X \leq M \| z \|_{\ell_r} \| f \|_{H^1_p(\mathbb{T}^\infty, X)}.
\]

Taking the supremum over all \(z \in B_{\ell_r}\) we obtain for all \(f \in H^1_p (\mathbb{T}^\infty, X)\)
\[
\left( \sum_j \| f(e_j) \|_X^{r'} \right)^{1/r'} \leq M \| f \|_{H^1_p(\mathbb{T}^\infty, X)}.
\]

Now, take \(x_1, \ldots, x_N \in X\), define \(f \in H^1_p (\mathbb{T}^\infty, X)\) by
\[
f(e_j) = \begin{cases} x_j & \text{if } 1 \leq j \leq N, \\ 0 & \text{if } j > N \end{cases}
\]
and extend it by linearity. By the previous inequality and Proposition 2.5 we have
\[
\left( \sum_{j=1}^N \| x_j \|_X^{r'} \right)^{1/r'} \leq M \left( \int_{\mathbb{T}^N} \left| \sum_{j=1}^N x_j z_j \right|^{r'} \, dz \right)^{1/r'}.
\]
By Kahane’s inequality, \(X\) has cotype \(r'\), which means that \(r' > \text{Cot}_X\) or, equivalently, \(r < \frac{\text{Cot}_X}{\text{Cot}_X - 1}\). Since \(r < M_p(X)\) was arbitrary, we obtain (16).

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