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# BOHR'S ABSOLUTE CONVERGENCE PROBLEM FOR $\mathscr{H}_p$-DIRICHLET SERIES IN BANACH SPACES

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The Bohr–Bohnenblust–Hille theorem states that the width of the strip in the complex plane on which an ordinary Dirichlet series $\sum_n a_n n^{-s}$ converges uniformly but not absolutely is less than or equal to $\frac{1}{2}$, and this estimate is optimal. Equivalently, the supremum of the absolute convergence abscissas of all Dirichlet series in the Hardy space $\mathscr{H}_\infty$ equals $\frac{1}{2}$. By a surprising fact of Bayart the same result holds true if $\mathscr{H}_\infty$ is replaced by any Hardy space $\mathscr{H}_p$, $1 \le p < \infty$, of Dirichlet series. For Dirichlet series with coefficients in a Banach space $X$ the maximal width of Bohr's strips depend on the geometry of $X$; Defant, García, Maestre and Pérez-García proved that such maximal width equals $1 - 1/\operatorname{Cot} X$, where $\operatorname{Cot} X$ denotes the maximal cotype of $X$. Equivalently, the supremum over the absolute convergence abscissas of all Dirichlet series in the vector-valued Hardy space $\mathscr{H}_\infty(X)$ equals $1 - 1/\operatorname{Cot} X$. In this article we show that this result remains true if $\mathscr{H}_\infty(X)$ is replaced by the larger class $\mathscr{H}_p(X)$, $1 \le p < \infty$.

## 1. Main result and its motivation

Given a Banach space $X$, an ordinary Dirichlet series in $X$ is a series of the form $D = \sum_n a_n n^{-s}$, where the coefficients $a_n$ are vectors in $X$ and $s$ is a complex variable. Maximal domains where such Dirichlet series converge conditionally, uniformly or absolutely are half planes $[\operatorname{Re} > \sigma]$, where $\sigma = \sigma_c$, $\sigma_u$ or $\sigma_a$ are called the abscissa of conditional, uniform or absolute convergence, respectively. More precisely, $\sigma_\alpha(D)$ is the infimum of all $r \in \mathbb{R}$ such that on $[\operatorname{Re} > r]$ we have convergence of $D$ of the requested type $\alpha = c$, $u$ or $a$. Clearly, we have $\sigma_c(D) \le \sigma_u(D) \le \sigma_a(D)$, and it can be easily shown that $\sup \sigma_a(D) - \sigma_c(D) = 1$, where the supremum is taken over all Dirichlet series $D$ with coefficients in $X$. To determine the maximal width of the strip on which a Dirichlet series in $X$ converges uniformly but not absolutely is more complicated. The main result of [Defant et al. 2008] states, with the notation given below, that

$$S(X) := \sup \sigma_a(D) - \sigma_u(D) = 1 - \frac{1}{\operatorname{Cot} X}. \tag{1}$$

Recall that a Banach space $X$ is of cotype $q$, $2 \le q < \infty$, whenever there is a constant $C \ge 0$ such that for each choice of finitely many vectors $x_1, \dots, x_N \in X$ we have

$$\left( \sum_{k=1}^N \|x_k\|_X^q \right)^{1/q} \le C \left( \int_{\mathbb{T}^N} \left\| \sum_{k=1}^N x_k z_k \right\|_X^2 dz \right)^{1/2}, \tag{2}$$

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where $\mathbb{T} := \{z \in \mathbb{C} \mid |z| = 1\}$ and $\mathbb{T}^N$ is endowed with the $N$-th product of the normalized Lebesgue measure on $\mathbb{T}$; we denote the best of such constants $C$ by $C_q(X)$. As usual we write

$$\operatorname{Cot} X := \inf\{2 \leq q < \infty \mid X \text{ is of cotype } q\},$$

and, although this infimum in general is not attained, we call it the optimal cotype of $X$. If there is no $2 \leq q < \infty$ for which $X$ has cotype $q$, then $X$ is said to have no finite cotype, and we put $\operatorname{Cot} X = \infty$. To see an example,

$$\operatorname{Cot} \ell_q = \begin{cases} q & \text{for } 2 \leq q \leq \infty, \\ 2 & \text{for } 1 \leq q \leq 2\,. \end{cases}$$

The scalar case $X = \mathbb{C}$ in (1) was first studied over a hundred years ago: Bohr [1913a] proved that $S(\mathbb{C}) \leq \frac{1}{2}$, and Bohnenblust and Hille [1931] that $S(\mathbb{C}) \geq \frac{1}{2}$. Clearly, the equality

$$S(\mathbb{C}) = \tfrac{1}{2}, \tag{3}$$

nowadays called the *Bohr–Bohnenblust–Hille theorem*, fits with (1). Let us give a second formulation of (1). Define the vector space $\mathcal{H}_\infty(X)$ of all Dirichlet series $D = \sum_n a_n n^{-s}$ in $X$ such that

- $\sigma_c(D) \leq 0$,
- the function $D(s) = \sum_n a_n (1/n^s)$ on $\operatorname{Re} s > 0$ is bounded.

Then $\mathcal{H}_\infty(X)$ together with the norm

$$\|D\|_{\mathcal{H}_\infty(X)} = \sup_{\operatorname{Re} s > 0} \Big\| \sum_{n=1}^{\infty} a_n \frac{1}{n^s} \Big\|_X$$

forms a Banach space. For any Dirichlet series $D$ in $X$ we have

$$\sigma_u(D) = \inf\Big\{ \sigma \in \mathbb{R} \;\Big|\; \sum_n \frac{a_n}{n^\sigma} \frac{1}{n^s} \in \mathcal{H}_\infty(X) \Big\}. \tag{4}$$

In the scalar case $X = \mathbb{C}$, this is (what we call) *Bohr's fundamental theorem* [1913b], and for Dirichlet series in arbitrary Banach spaces the proof follows similarly. Together with (4) a simply translation argument gives the following reformulation of (1):

$$S(X) = \sup_{D \in \mathcal{H}_\infty(X)} \sigma_a(D) = 1 - \frac{1}{\operatorname{Cot} X}. \tag{5}$$

Following an ingenious idea of Bohr each Dirichlet series may be identified with a power series in infinitely many variables. More precisely, fix a Banach space $X$ and denote by $\mathfrak{P}(X)$ the vector space of all formal power series $\sum_\alpha c_\alpha z^\alpha$ in $X$ and by $\mathfrak{D}(X)$ the vector space of all Dirichlet series $\sum_n a_n n^{-s}$ in $X$. Let as usual $(p_n)_n$ be the sequence of prime numbers. Since each integer $n$ has a unique prime number decomposition $n = p_1^{\alpha_1} \cdots p_k^{\alpha_k} = p^\alpha$ with $\alpha_j \in \mathbb{N}_0,\ 1 \le j \le k$, the linear mapping

$$\begin{aligned} \mathfrak{B}_X : \mathfrak{P}(X) &\to \mathfrak{D}(X), \\ \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} c_\alpha z^\alpha &\rightsquigarrow \sum_{n=1}^{\infty} a_n n^{-s} \quad \text{if } a_{p^\alpha} = c_\alpha, \end{aligned} \tag{6}$$

is bijective; we call $\mathfrak{B}_X$ the *Bohr transform in $X$*. As discovered by Bayart [2002] this (a priori *very*) formal identification allows us to develop a theory of Hardy spaces of scalar–valued Dirichlet series.

Similarly, we now define Hardy spaces of $X$-valued Dirichlet series. Denote by $dw$ the normalized Lebesgue measure on the infinite-dimensional polytorus $\mathbb{T}^\infty = \prod_{k=1}^{\infty} \mathbb{T}$, that is, the countable product measure of the normalized Lebesgue measure on $\mathbb{T}$. For any multiindex $\alpha = (\alpha_1, \dots, \alpha_n, 0, \dots) \in \mathbb{Z}^{(\mathbb{N})}$ (all finite sequences in $\mathbb{Z}$) the $\alpha$-th Fourier coefficient $\hat{f}(\alpha)$ of $f \in L_1(\mathbb{T}^\infty, X)$ is given by

$$\hat{f}(\alpha) = \int_{\mathbb{T}^\infty} f(w) w^{-\alpha}\, dw,$$

where we as usual write $w^\alpha$ for the monomial $w_1^{\alpha_1} \cdots w_n^{\alpha_n}$. Then, given $1 \le p < \infty$, the $X$-valued Hardy space on $\mathbb{T}^\infty$ is the subspace of $L_p(\mathbb{T}^\infty, X)$ defined as

$$H_p(\mathbb{T}^\infty, X) = \{ f \in L_p(\mathbb{T}^\infty, X) \mid \hat{f}(\alpha) = 0 \text{ for all } \alpha \in \mathbb{Z}^{(\mathbb{N})} \setminus \mathbb{N}_0^{(\mathbb{N})} \}. \tag{7}$$

Assigning to each $f \in H_p(\mathbb{T}^\infty, X)$ its unique formal power series $\sum_\alpha \hat{f}(\alpha) z^\alpha$ we may consider $H_p(\mathbb{T}^\infty, X)$ as a subspace of $\mathfrak{P}(X)$. We denote the image of this subspace under the Bohr transform $\mathfrak{B}_X$ by

$$\mathscr{H}_p(X).$$

This vector space of all (so-called) $\mathscr{H}_p(X)$-Dirichlet series $D$ together with the norm

$$\|D\|_{\mathscr{H}_p(X)} = \|\mathfrak{B}_X^{-1}(D)\|_{H_p(\mathbb{T}^\infty, X)}$$

forms a Banach space; in other words, through Bohr's transform $\mathfrak{B}_X$ from (6) we by definition identify

$$\mathscr{H}_p(X) = H_p(\mathbb{T}^\infty, X), \quad 1 \le p < \infty.$$

For $p = \infty$ we this way of course could also define a Banach space $\mathscr{H}_\infty(X)$, and it turns out that at least in the scalar case $X = \mathbb{C}$ this definition then coincides with the one given above; but we remark that these two $\mathscr{H}_\infty(X)$'s are different for arbitrary $X$. It is important to note that by the Birkhoff–Khinchin ergodic theorem the following internal description of the $\mathscr{H}_p(X)$-norm for finite Dirichlet polynomials $D = \sum_{k=1}^{n} a_k k^{-s}$ holds:

$$\|D\|_{\mathscr{H}_p(X)} = \lim_{T \to \infty} \left( \frac{1}{2T} \int_{-T}^{T} \left\| \sum_{k=1}^{n} a_k \frac{1}{k^t} \right\|_X^p dt \right)^{1/p}$$

(see, for example, Bayart [2002] for the scalar case, and the vector-valued case follows exactly the same way).

Motivated by (4) we define for $D \in \mathfrak{D}(X)$ and $1 \leq p < \infty$

$$\sigma_{\mathcal{H}_p(X)}(D) := \inf\left\{\sigma \in \mathbb{R} \;\middle|\; \sum_n \frac{a_n}{n^\sigma}\frac{1}{n^s} \in \mathcal{H}_p(X)\right\}, \tag{8}$$

the so-called $\mathcal{H}_p(X)$-abscissa of $D$. In [Aleman et al. $\geq$ 2014], Aleman, Olsen, and Saksman prove that the sequence (of Dirichlet series) $1/n^s$, $n \in \mathbb{N}$ is a Schauder basis in $\mathcal{H}_p(\mathbb{C})$ for $1 < p < \infty$. Hence, for $1 < p < \infty$ and any Dirichlet series $D \in \mathfrak{D}(\mathbb{C})$ we have

$$\sigma_{\mathcal{H}_p(\mathbb{C})}(D) = \inf\left\{\sigma \in \mathbb{R} \;\middle|\; \left(\sum_{n=1}^{N} \frac{a_n}{n^\sigma}\frac{1}{n^s}\right)_N \text{ is Cauchy in } \mathcal{H}_p(\mathbb{C})\right\}, \tag{9}$$

which (in the scalar case) is the perfect analog of Bohr's fundamental theorem (i.e., the case $p = \infty$ from (4), where uniform convergence is precisely being Cauchy in $\mathcal{H}_p(\mathbb{C})$). In [Defant 2013] it is shown that (9) also holds true for $p = 1$ (although in this case the $1/n^s$ are definitely no Schauder basis in $\mathcal{H}_1(\mathbb{C})$), and even more: The arguments given in [Defant 2013] (inspired by Bohr's original ideas [1913b]) prove that (9) even holds for any $1 \leq p \leq \infty$ and any $X$-valued Dirichlet series $D \in \mathcal{H}_p(X)$. In view of (1) and (5), it therefore seems natural to study

$$S_p(X) := \sup_{D \in \mathfrak{D}(X)} \sigma_a(D) - \sigma_{\mathcal{H}_p(X)}(D) = \sup_{D \in \mathcal{H}_p(X)} \sigma_a(D)$$

(for the second equality use again a simple translation argument). The scalar case is completely understood since, by a result of Bayart [2002],

$$S_p(\mathbb{C}) = \tfrac{1}{2} \qquad \text{for every } 1 \leq p < \infty, \tag{10}$$

which according to Helson [2005] is surprising since $\mathcal{H}_\infty(\mathbb{C})$ is much smaller than $\mathcal{H}_p(\mathbb{C})$.

The following theorem unifies and generalizes (1), (3) as well as (10), and it is our main result.

**Theorem 1.1.** *For every $1 \leq p \leq \infty$ and every Banach space $X$ we have*

$$S_p(X) = 1 - \frac{1}{\operatorname{Cot} X}.$$

The proof will be given in Section 3. But before we start let us give an interesting reformulation in terms of the monomial convergence of $X$-valued $H_p$-functions on $\mathbb{T}^\infty$. Fix a Banach space $X$ and $1 \leq p \leq \infty$, and define the set of monomial convergence of $H_p(\mathbb{T}^\infty, X)$:

$$\operatorname{mon} H_p(\mathbb{T}^\infty, X) = \left\{z \in B_{c_0} \;\middle|\; \sum_\alpha \|\hat{f}(\alpha) z^\alpha\|_X < \infty \text{ for all } f \in H_p(\mathbb{T}^\infty, X)\right\}.$$

Philosophically, this is the largest set $M$ on which for each $f \in H_p(\mathbb{T}^\infty, X)$ the definition $g(z) = \sum_\alpha \hat{f}(\alpha) z^\alpha$, $z \in M$ leads to an extension of $f$ from the distinguished boundary $\mathbb{T}^\infty$ to its "interior" $B_{c_0}$ (the open unit ball of the Banach space $c_0$ of all null sequences). For a detailed study of sets of monomial convergence in the scalar case $X = \mathbb{C}$ see [Defant et al. 2009], and in the vector-valued case [Defant and Sevilla-Peris 2011].

We later need the following two basic properties of monomial domains (in the scalar case see [Defant et al. 2008, p. 550; 2014, Lemma 4.3], and in the vector-valued case the proofs follow similar lines).

**Remark 1.2.** (1) Let $z \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$. Then $u = (z_{\sigma(n)})_n \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$ for every permutation $\sigma$ of $\mathbb{N}$.

(2) Let $z \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$ and $x = (x_n)_n \in \mathbb{D}^\infty$ be such that $|x_n| \le |z_n|$ for all but finitely many $n$'s. Then $x \in \operatorname{mon} H_p(\mathbb{T}^\infty, X)$.

Given $1 \le p \le \infty$ and a Banach space $X$, the following number measures the size of $\operatorname{mon} H_p(\mathbb{T}^\infty, X)$ within the scale of $\ell_r$-spaces:

$$M_p(X) = \sup\{1 \le r \le \infty \mid \ell_r \cap B_{c_0} \subset \operatorname{mon} H_p(\mathbb{T}^\infty, X)\}.$$

The following result is a reformulation of Theorem 1.1 in terms of vector-valued $H_p$-functions on $\mathbb{T}^\infty$ through Bohr's transform $\mathfrak{B}_X$. The proof is modeled along ideas from Bohr's seminal article [1913a, Satz IX].

**Corollary 1.3.** *For each Banach space $X$ and $1 \le p \le \infty$ we have*

$$M_p(X) = \frac{\operatorname{Cot} X}{\operatorname{Cot} X - 1}.$$

*Proof.* We are going to prove that $S_p(X) = 1/M_p(X)$, and as a consequence the conclusion follows from Theorem 1.1. We begin by showing that $S_p(X) \le 1/M_p(X)$. We fix $q < M_p(X)$ and $r > 1/q$; then we have that $(1/p_n^r)_n \in \ell_q \cap B_{c_0}$ and, by the very definition of $M_p(X)$, $\sum_\alpha \|\hat{f}(\alpha)(1/p^r)^\alpha\|_X < \infty$ converges absolutely for every $f \in H_p(\mathbb{T}^\infty, X)$. We choose now an arbitrary Dirichlet series

$$D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathcal{H}_p(X) \quad \text{with } f \in H_p(\mathbb{T}^\infty, X).$$

Then

$$\sum_n \|a_n\|_X \frac{1}{n^r} = \sum_\alpha \|a_{p^\alpha}\|_X \Big(\frac{1}{p^\alpha}\Big)^r = \sum_\alpha \|\hat{f}(\alpha)\|_X \Big(\frac{1}{p^r}\Big)^\alpha < \infty.$$

Clearly, this implies that $S_p(X) \le r$. Since this holds for each $r > 1/q$, we get that $S_p(X) \le 1/q$, and since this now holds for each $q < M_p(X)$, we have $S_p(X) \le 1/M_p(X)$. Conversely, let us take some $q > M_p(X)$; then there is $z \in \ell_q \cap B_{c_0}$ and $f \in H_p(\mathbb{T}^\infty, X)$ such that $\sum_\alpha \hat{f}(\alpha) z^\alpha$ does not converge absolutely. By Remark 1.2 we may assume that $z$ is decreasing, and hence $(z_n n^{1/q})_n$ is bounded. We choose now $r > q$ and define $w_n = 1/p_n^{1/r}$. By the prime number theorem we know that there is a universal constant $C > 0$ such that

$$0 < \frac{z_n}{w_n} = z_n p_n^{1/r} = z_n n^{1/q} \frac{p_n^{1/r}}{n^{1/q}} = z_n n^{1/q} \Big(\frac{p_n}{n}\Big)^{1/r} \frac{1}{n^{1/q-1/r}} \le C z_n n^{1/q} \frac{(\log n)^{1/r}}{n^{1/q-1/r}}.$$

The last term tends to 0 as $n \to \infty$; hence $z_n \le w_n$ but for a finite number of $n$'s. By Remark 1.2 this implies that $\sum_\alpha \hat{f}(\alpha) w^\alpha$ does not converge absolutely. But then $D = \mathfrak{B}_X f = \sum_n a_n n^{-s} \in \mathcal{H}_p(X)$

satisfies

$$\sum_n \|a_n\|_X \frac{1}{n^{1/r}} = \sum_\alpha \|a_{p^\alpha}\|_X \Big(\frac{1}{p^{1/r}}\Big)^\alpha = \sum_\alpha \|\hat{f}(\alpha)\|_X w^\alpha = \infty.$$

This gives that $\sigma_a(D) \geq 1/r$ for every $r > q$, hence $\sigma_a(D) \geq 1/q$. Since this holds for every $q > M_p(X)$, we finally have $S_p(X) \geq 1/M_p(X)$. □

We shall use standard notation and notions from Banach space theory, as presented, for example, in [Lindenstrauss and Tzafriri 1977; 1979]. For everything needed on polynomials in Banach spaces see, for example, [Dineen 1999; Floret 1997].

## 2. Relevant inequalities

The main aim here is to prove a sort of polynomial extension of the notion of cotype. Recall the definition of $C_q(X)$ from (2). Moreover, from Kahane's inequality we know that there is a (best) constant $K \geq 1$ such that, for each Banach space $X$ and each choice of finitely many vectors $x_1, \dots, x_N \in X$,

$$\bigg(\int_{\mathbb{T}^N} \Big\| \sum_{k=1}^N x_k z_k \Big\|_X^2 \, dz\bigg)^{1/2} \leq K \int_{\mathbb{T}^N} \Big\| \sum_{k=1}^N x_k z_k \Big\|_X \, dz.$$

As usual we write $|\alpha| = \alpha_1 + \cdots + \alpha_N$ and $\alpha! = \alpha_1! \cdots \alpha_N!$ for every multiindex $\alpha \in \mathbb{N}_0^N$.

**Proposition 2.1.** *Let $X$ be a Banach space of cotype $q$, $2 \leq q < \infty$, and*

$$P : \mathbb{C}^N \to X, \quad P(z) = \sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} c_\alpha z^\alpha$$

*be an $m$-homogeneous polynomial. Let*

$$T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \dots, z^{(m)}) = \sum_{i_1, \dots, i_m = 1}^N a_{i_1, \dots, i_m} z_{i_1}^{(1)} \cdots z_{i_m}^{(m)}$$

*be the unique $m$-linear symmetrization of $P$. Then*

$$\Big( \sum_{i_1, \dots, i_m} \|a_{i_1, \dots, i_m}\|_X^q \Big)^{1/q} \leq (C_q(X) K)^m \frac{m^m}{m!} \int_{\mathbb{T}^N} \|P(z)\|_X \, dz.$$

Before we give the proof let us note that [Bombal et al. 2004, Theorem 3.2] is an $m$-linear result that, combined with polarization, gives (with the previous notation)

$$\Big( \sum_{i_1, \dots, i_m} \|a_{i_1, \dots, i_m}\|_X^q \Big)^{1/q} \leq C_q(X)^m \frac{m^m}{m!} \sup_{z \in \mathbb{D}^N} \|P(z)\|.$$

Our result allows us to replace (up to the constant $K$) the $\| \ \|_\infty$ norm with the smaller norm $\| \ \|_1$. We prepare the proof of Proposition 2.1 with three lemmas. The first one is a complex version of [Defant et al. 2010, Lemma 2.2] with essentially the same proof; we include it for the sake of completeness.

**Lemma 2.2.** *Let $X$ be a Banach space of cotype $q$, $2 \le q < \infty$. Then, for every $m$-linear form*

$$T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1,\ldots,i_m=1}^{N} a_{i_1,\ldots,i_m} z^{(1)}_{i_1} \cdots z^{(m)}_{i_m},$$

*we have*

$$\Big( \sum_{i_1,\ldots,i_m=1}^{N} \|a_{i_1,\ldots,i_m}\|_X^q \Big)^{1/q} \le (C_q(X)\, K)^m \int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \|T(z^{(1)}, \ldots, z^{(m)})\|_X \, dz^{(1)} \cdots dz^{(m)}.$$

*Proof.* We prove this result by induction on the degree $m$. For $m = 1$ the result is an immediate consequence of the definition of cotype $q$ and Kahane's inequality. Assume that the result holds for $m-1$. By the continuous Minkowski inequality we then conclude that for every choice of finitely many vectors $a_{i_1,\ldots,i_m} \in X$ with $1 \le i_j \le N$, $1 \le j \le m$ we have

$$\begin{aligned}
\sum_{i_1,\ldots,i_m} \|a_{i_1,\ldots,i_m}\|_X^q &= \sum_{i_1,\ldots,i_{m-1}} \sum_{i_m} \|a_{i_1,\ldots,i_m}\|_X^q \\
&\le C_q(X)^q K^q \bigg( \sum_{i_1,\ldots,i_{m-1}} \Big( \int_{\mathbb{T}^N} \Big\| \sum_{i_m} a_{i_1,\ldots,i_m} z^{(m)}_{i_m} \Big\|_X dz^{(m)} \Big)^q \bigg)^{q/q} \\
&\le C_q(X)^q K^q \bigg( \int_{\mathbb{T}^N} \Big( \sum_{i_1,\ldots,i_{m-1}} \Big\| \sum_{i_m} a_{i_1,\ldots,i_m} z^{(m)}_{i_m} \Big\|_X^q \Big)^{1/q} dz^{(m)} \bigg)^q \\
&\le C_q(X)^{qm} K^{qm} \bigg( \int_{\mathbb{T}^N} \underbrace{\int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N}}_{m-1} \Big\| \sum_{i_1,\ldots,i_{m-1}} a_{i_1,\ldots,i_{m-1}} z^{(1)}_{i_1}, \ldots, z^{(m-1)}_{i_{m-1}} \Big\|_X dz^{(1)} \cdots dz^{(m-1)} \, dz^{(m)} \bigg)^q,
\end{aligned}$$

which is the conclusion. ☐

The following two lemmas are needed to produce a polynomial analog of the preceding result.

**Lemma 2.3.** *Let $X$ be a Banach space, and $f : \mathbb{C} \to X$ a holomorphic function. Then for $R_1, R_2, R \ge 0$ with $R_1 + R_2 \le R$ we have*

$$\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 \le \int_{\mathbb{T}} \|f(Rz)\|_X \, dz.$$

*Proof.* By the rotation invariance of the normalized Lebesgue measure on $\mathbb{T}$ we get

$$\begin{aligned}
\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 &= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 z_2 + R_2 z_2)\|_X \, dz_1 \, dz_2 \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2(R_1 z_1 + R_2))\|_X \, dz_1 \, dz_2 = \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 |R_1 z_1 + R_2|)\|_X \, dz_2 \, dz_1 \\
&= \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z_2 r(z_1) R)\|_X \, dz_2 \, dz_1 = \int_0^{2\pi} \int_0^{2\pi} \|f(r(e^{is}) R e^{it})\|_X \frac{dt}{2\pi} \frac{ds}{2\pi},
\end{aligned}$$

where $r(z) = (1/R)|R_1 z + R_2|$, $z \in \mathbb{T}$. We know that for each holomorphic function $h : \mathbb{C} \to X$ we have

$$\int_{\mathbb{T}} \|h(z)\|_X \, dz = \sup_{0 \le r \le 1} \int_0^{2\pi} \|h(re^{it})\|_X \, \frac{dt}{2\pi}$$

(see, for example, Blasco and Xu [1991, p. 338]). Define now $h(z) = f\big(Rz\big)$, and note that $0 \le r(z) \le 1$ for all $z \in \mathbb{T}$. Then

$$\begin{aligned}\int_{\mathbb{T}} \int_{\mathbb{T}} \|f(R_1 z_1 + R_2 z_2)\|_X \, dz_1 \, dz_2 &= \int_0^{2\pi} \int_0^{2\pi} \|h(r(e^{is})e^{it})\|_X \, \frac{dt}{2\pi} \, \frac{ds}{2\pi} \\ &\le \int_0^{2\pi} \int_{\mathbb{T}} \|h(z)\|_X \, dz \, \frac{ds}{2\pi} = \int_{\mathbb{T}} \|f(Rz)\|_X \, dz.\end{aligned}$$

This completes the proof. □

A sort of iteration of the preceding result leads to the next:

**Lemma 2.4.** *Let $X$ be a Banach space, and $f : \mathbb{C}^N \to X$ a holomorphic function. Then, for every $m$,*

$$\int_{\mathbb{T}^N} \cdots \int_{\mathbb{T}^N} \|f(z^{(1)} + \cdots + z^{(m)})\|_X \, dz^{(1)} \cdots dz^{(m)} \le \int_{\mathbb{T}^N} \|f(mz)\|_X \, dz.$$

*Proof.* We fix some $m$, and do induction with respect to $N$. For $N = 1$ we obtain from Lemma 2.3 that

$$\begin{aligned}&\underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-2} \int_{\mathbb{T}} \int_{\mathbb{T}} \| \underbrace{f(z^{(1)} + \cdots + z^{(m-2)} + z^{(m-1)} + z^{(m)})}_{=: g_{z^{(1)},\dots,z^{(m-2)}}(z^{(m-1)} + z^{(m)})} \|_X \, dz^{(m-1)} \, dz^{(m)} \, dz^{(1)} \cdots dz^{(m-2)} \\ &\qquad \le \underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-2} \int_{\mathbb{T}} \|g_{z^{(1)},\dots,z^{(m-2)}}(2w)\|_X \, dw \, dz^{(1)} \cdots dz^{(m-2)} \\ &\qquad = \underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-3} \int_{\mathbb{T}} \int_{\mathbb{T}} \|f(z^{(1)} + \cdots + z^{(m-2)} + 2w)\|_X \, dw \, dz^{(m-2)} \, dz^{(1)} \cdots dz^{(m-3)} \\ &\qquad \le \underbrace{\int_{\mathbb{T}} \cdots \int_{\mathbb{T}}}_{m-3} \int_{\mathbb{T}} \|f(z^{(1)} + \cdots + z^{(m-3)} + 3w)\|_X \, dz^{(1)} \cdots dz^{(m-3)} \, dw \\ &\qquad \le \cdots \le \int_{\mathbb{T}} \|f(mz)\|_X \, dz.\end{aligned}$$

We now assume that the conclusion holds for $N - 1$ and write each $z \in \mathbb{T}^N$ as $z = (u, w)$, with $u \in \mathbb{T}^{N-1}$ and $w \in \mathbb{T}$. Then, using the case $N = 1$ in the first inequality and the inductive hypothesis in the second, we have

$$\begin{aligned}
&\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\|f(z^{(1)}+\cdots+z^{(m)})\|_X\,dz^{(1)}\cdots dz^{(m)}\\
&\quad=\int_{\mathbb{T}^{N-1}}\cdots\int_{\mathbb{T}^{N-1}}\left(\int_{\mathbb{T}}\cdots\int_{\mathbb{T}}\|f((u^{(1)},w_1)+\cdots+(u^{(m)},w_m))\|_X\,dw_1\cdots dw_N\right)du^{(1)}\cdots du^{(m)}\\
&\quad\le\int_{\mathbb{T}^{N-1}}\cdots\int_{\mathbb{T}^{N-1}}\left(\int_{\mathbb{T}}\|f((u^{(1)},mw)+\cdots+(u^{(m)},mw))\|_X\,dw\right)du^{(1)}\cdots du^{(m)}\\
&\quad=\int_{\mathbb{T}}\left(\int_{\mathbb{T}^{N-1}}\cdots\int_{\mathbb{T}^{N-1}}\|f((u^{(1)},mw)+\cdots+(u^{(m)},mw))\|_X\,du^{(1)}\cdots du^{(m)}\right)dw\\
&\quad\le\int_{\mathbb{T}}\left(\int_{\mathbb{T}^{N-1}}\|f((mu,mw)+\cdots+(mu,mw))\|_X\,du\right)dw\\
&\quad=\int_{\mathbb{T}^N}\|f(mz)\|_X\,dz,
\end{aligned}$$

as desired. □

*Proof of the inequality from Proposition 2.1.* By the polarization formula we know that for every choice of $z^{(1)},\dots,z^{(m)}\in\mathbb{T}^N$ we have

$$T(z^{(1)},\dots,z^{(m)})=\frac{1}{2^m m!}\sum_{\varepsilon_i=\pm1}\varepsilon_i\cdots\varepsilon_m P\left(\sum_{i=1}^N\varepsilon_i z^{(i)}\right)$$

(see, for example, [Dineen 1999] or [Floret 1997]). Hence we deduce from Lemma 2.4

$$\begin{aligned}
\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\|T(z^{(1)},\dots,z^{(m)})\|_X\,dz^{(1)}\cdots dz^{(m)}&\le\frac{1}{2^m m!}\sum_{\varepsilon_i=\pm1}\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\left\|P\left(\sum_{i=1}^N\varepsilon_i z^{(i)}\right)\right\|_X dz^{(1)}\cdots dz^{(m)}\\
&=\frac{1}{2^m m!}\sum_{\varepsilon_i=\pm1}\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\left\|P\left(\sum_{i=1}^N z^{(i)}\right)\right\|_X dz^{(1)}\cdots dz^{(m)}\\
&=\frac{1}{m!}\int_{\mathbb{T}^N}\cdots\int_{\mathbb{T}^N}\left\|P\left(\sum_{i=1}^N z^{(i)}\right)\right\|_X dz^{(1)}\cdots dz^{(m)}\\
&\le\frac{1}{m!}\int_{\mathbb{T}^N}\|P(mz)\|_X\,dz=\frac{m^m}{m!}\int_{\mathbb{T}^N}\|P(z)\|_X\,dz.
\end{aligned}$$

Then by Lemma 2.2 we obtain

$$\begin{aligned}
\left(\sum_{i_1,\dots,i_m}^N\|a_{i_1,\dots,i_m}\|_X^q\right)^{1/q}&\le(C_q(X)K)^m\int_{\mathbb{T}^\infty}\cdots\int_{\mathbb{T}^\infty}\|T(z^{(1)},\dots,z^{(m)})\|_X\,dz^{(1)}\cdots dz^{(m)}\\
&=(C_q(X)K)^m\,\frac{m^m}{m!}\int_{\mathbb{T}^N}\|P(z)\|_X\,dz,
\end{aligned}$$

which completes the proof of Proposition 2.1. □

A second proposition is needed which allows us to reduce the proof of our main result (Theorem 1.1) to the homogeneous case. It is a vector-valued version of a result of [Cole and Gamelin 1986, Theorem 9.2] with a similar proof (here only given for the sake of completeness).

**Proposition 2.5.** *There is a contractive projection*

$$\Phi_m : H_p(\mathbb{T}^N, X) \to H_p(\mathbb{T}^N, X), \quad f \mapsto f_m,$$

*such that, for all $f \in H_p(\mathbb{T}^N, X)$,*

$$\hat{f}(\alpha) = \hat{f}_m(\alpha) \quad \text{for all } \alpha \in \mathbb{N}_0^N \text{ with } |\alpha| = m. \tag{11}$$

*Proof.* Let $\mathcal{P}(\mathbb{C}^N, X) \subset H_p(\mathbb{T}^N, X)$ be the subspace of all finite polynomials $f = \sum_{\alpha\in\Lambda} c_\alpha z^\alpha$; here $\Lambda$ is a finite set of multiindices in $\mathbb{N}_0^N$ and the coefficients $c_\alpha \in X$. Define the linear projection $\Phi_m^0$ on $\mathcal{P}(\mathbb{C}^N, X)$ by

$$\Phi_m^0(f)(z) = f_m(z) = \sum_{\alpha\in\Lambda, |\alpha|=m} \hat{f}(\alpha) z^\alpha ;$$

clearly, we have (11). In order to show that $\Phi_m^0$ is a contraction on $\big(\mathcal{P}(\mathbb{C}^N, X), \|\cdot\|_p\big)$ fix some function $f \in \mathcal{P}(\mathbb{C}^N, X)$ and $z \in \mathbb{T}^N$, and define

$$f(z\,\cdot) : \mathbb{T} \to X, \quad w \mapsto f(zw).$$

Clearly, we have

$$f(zw) = \sum_k f_k(z) w^k,$$

and hence

$$f_m(z) = \int_{\mathbb{T}} f(zw) w^{-m}\, dw.$$

Integration, Hölder's inequality and the rotation invariance of the normalized Lebesgue measure on $\mathbb{T}^N$ give

$$\begin{aligned}\int_{\mathbb{T}^N} \|f_m(z)\|_X^p\, dz &= \int_{\mathbb{T}^N} \left\| \int_{\mathbb{T}} f(zw) w^{-m}\, dw \right\|_X^p dz \\ &\le \int_{\mathbb{T}^N} \left( \int_{\mathbb{T}} \|f(zw)\|_X\, dw \right)^p dz \\ &\le \int_{\mathbb{T}} \int_{\mathbb{T}^N} \|f(zw)\|_X^p\, dz\, dw = \int_{\mathbb{T}^N} \|f(z)\|_X^p\, dz,\end{aligned}$$

which proves that $\Phi_m^0$ is a contraction on $(\mathcal{P}(\mathbb{C}^N, X), \|\cdot\|_p)$. By Fejér's theorem (vector-valued) we know that $\mathcal{P}(\mathbb{C}^N, X)$ is a dense subspace of $H_p(\mathbb{T}^N, X)$. Hence $\Phi_m^0$ extends to a contractive projection $\Phi_m$ on $H_p(\mathbb{T}^N, X)$. This extension $\Phi_m$ still satisfies (11) since the mapping $H_p(\mathbb{T}^N, X) \to X$, $f \mapsto \hat{f}(\alpha)$ is continuous for each multiindex $\alpha$. □

## 3. Proof of the main result

We are now ready to prove Theorem 1.1. Let $1 \le p < \infty$, and recall from (1) that

$$1 - \frac{1}{\operatorname{Cot} X} = S_\infty(X) \le S_p(X);$$

see Remark 3.1 for a direct argument. Hence it suffices to concentrate on the upper estimate in Theorem 1.1: Since we obviously have $S_p(X) \le S_1(X)$, we are going to prove that

$$S_1(X) \le 1 - \frac{1}{\operatorname{Cot} X}. \tag{12}$$

Suppose first that $X$ has no finite cotype, i.e., $\operatorname{Cot} X = \infty$. For $D = \sum_n a_n n^{-s} \in \mathscr{H}_1(X)$ we take $f \in H_1(\mathbb{T}^\infty, X)$ with $D = \mathfrak{B}_X f$. Note that

$$\|\hat{f}(\alpha)\|_X \le \int_{\mathbb{T}^\infty} \|f(w)w^{-\alpha}\|_X \, dw = \|f\|_{L_1(\mathbb{T}^\infty, X)} < \infty;$$

hence, by the definition of $\mathfrak{B}_X$, the coefficients of $D$ are also bounded by $\|f\|_{L_1(\mathbb{T}^\infty,X)}$. As a consequence, for every $\sigma > 1$ we have

$$\sum_{n=1}^\infty \|a_n\|_X \frac{1}{n^\sigma} \le \sum_{n=1}^\infty \|f\|_{L_1(\mathbb{T}^\infty,X)} \frac{1}{n^\sigma} < \infty.$$

This means that $S_1(X) \le 1$ and as a consequence (12) holds.

Now if $X$ has finite cotype, take $q > \operatorname{Cot} X$ and $\varepsilon > 0$, and put $s = (1 - 1/q)(1 + 2\varepsilon)$. Choose an integer $k_0$ such that $p_{k_0}^{\varepsilon/q'} > eC_q(X)K\big(\sum_{j=1}^\infty 1/p_j^{1+\varepsilon}\big)^{1/q'}$ and define

$$\tilde{p} = (\underbrace{p_{k_0}, \dots, p_{k_0}}_{k_0 \text{ times}}, p_{k_0+1}, p_{k_0+2}, \dots).$$

We are going to show that there is a constant $C(q, X, \varepsilon) > 0$ such that for every $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \le C(q, X, \varepsilon)\|f\|_{H_1(\mathbb{T}^\infty,X)}. \tag{13}$$

This finishes the argument: By Remark 1.2 the sequence $1/p^s$ is in mon $H_1(\mathbb{T}^\infty, X)$. But in view of Bohr's transform from (6), this means that for every Dirichlet series $D = \sum_n a_n n^{-s} = \mathfrak{B}_X f \in \mathscr{H}_1(X)$ with $f \in H_1(\mathbb{T}^\infty, X)$ we have

$$\sum_{n=1}^\infty \|a_n\|_X \frac{1}{n^s} = \sum_{\alpha \in \mathbb{N}_0^{(\mathbb{N})}} \|\hat{f}(\alpha)\|_X \frac{1}{p^{s\alpha}} < \infty.$$

Therefore $\sigma_a(D) \le (1 - 1/q)(1 + 2\varepsilon)$ for each such $D$ which, since $\varepsilon > 0$ was arbitrary, is what we wanted to prove.

It remains to check (13); the idea is to show first that (13) holds for all $X$-valued $H_1$-functions which only depend on $N$ variables: There is a constant $C(q, X, \varepsilon) > 0$ such that for all $N$ and every

$f \in H_1(\mathbb{T}^N, X)$ we have

$$\sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \leq C(q, X, \varepsilon) \|f\|_{H_1(\mathbb{T}^N, X)}. \tag{14}$$

In order to understand that (14) implies (13) (and hence the conclusion), assume that (14) holds and take some $f \in H_1(\mathbb{T}^\infty, X)$. Given an arbitrary $N$, define

$$f_N : \mathbb{T}^N \to X, \quad f_N(w) = \int_{\mathbb{T}^\infty} f(w, \tilde{w})\, d\tilde{w}.$$

Then it can be easily shown that $f_N \in L_1(\mathbb{T}^N, X)$, $\|f_N\|_1 \leq \|f\|_1$, and $\hat{f}_N(\alpha) = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{Z}^N$. If we now apply (14) to this $f_N$, we get

$$\sum_{\alpha \in \mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \leq C(q, X, \varepsilon) \|f\|_{H_1(\mathbb{T}^\infty, X)},$$

which, after taking the supremum over all possible $N$ on the left side, leads to (13).

We turn to the proof of (14), and here in a first step will show the following: For every $N$, every $m$-homogeneous polynomial $P : \mathbb{C}^N \to X$ and every $u \in \ell_{q'}$ we have

$$\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} \|\hat{P}(\alpha) u^\alpha\|_X \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X \, dz \left( \sum_{j=1}^\infty |u_j|^{q'} \right)^{m/q'}. \tag{15}$$

Indeed, take such a polynomial $P(z) = \sum_{\alpha \in \mathbb{N}_0^N, |\alpha| = m} \hat{P}(\alpha) z^\alpha$, $z \in \mathbb{T}^N$, and look at its unique $m$-linear symmetrization

$$T : \mathbb{C}^N \times \cdots \times \mathbb{C}^N \to X, \quad T(z^{(1)}, \ldots, z^{(m)}) = \sum_{i_1, \ldots, i_m = 1}^N a_{i_1, \ldots, i_m} z_{i_1}^{(1)}, \ldots, z_{i_m}^{(m)}.$$

Then we know from Proposition 2.1 that

$$\left( \sum_{i_1, \ldots, i_m = 1}^N \|a_{i_1, \ldots, i_m}\|_X^q \right)^{1/q} \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X \, dz.$$

Hence (15) follows by Hölder's inequality:

$$\sum_{\substack{\alpha \in \mathbb{N}_0^N \\ |\alpha| = m}} \|\hat{P}(\alpha) u^\alpha\|_X = \sum_{i_1, \ldots, i_m = 1}^N \|a_{i_1, \ldots, i_m}\|_X |u_{i_1} \cdots u_{i_N}| \leq (eC_q(X)K)^m \int_{\mathbb{T}^N} \|P(z)\|_X \, dz \left( \sum_{j=1}^\infty |u_j|^{q'} \right)^{m/q'}.$$

We finally give the proof of (14): Take $f \in H_1(\mathbb{T}^N, X)$, and recall from Proposition 2.5 that for each integer $m$ there is an $m$-homogeneous polynomial $P_m : \mathbb{C}^N \to X$ such that $\|P_m\|_{H_1(\mathbb{T}^N, X)} \leq \|f\|_{H_1(\mathbb{T}^N, X)}$

and $\hat{P}_m(\alpha) = \hat{f}(\alpha)$ for all $\alpha \in \mathbb{N}_0^N$ with $|\alpha| = m$. From (15), the definition of $s$, and the fact that $\max\{p_{k_0}, p_j\} \le \tilde{p}_j$ for all $j$ we have

$$\begin{aligned}
\sum_{\alpha\in\mathbb{N}_0^N} \|\hat{f}(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} &= \sum_{m=1}^{\infty} \sum_{\substack{\alpha\in\mathbb{N}_0^N \\ |\alpha|=m}} \|\hat{P}_m(\alpha)\|_X \frac{1}{\tilde{p}^{s\alpha}} \\
&\le \sum_{m=1}^{\infty} (eC_q(X)K)^m \|P_m\|_{H_1(\mathbb{T}^N,X)} \Big(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{sq'}}\Big)^{m/q'} \\
&= \sum_{m=1}^{\infty} (eC_q(X)K)^m \|f\|_{H_1(\mathbb{T}^N,X)} \Big(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+2\varepsilon}}\Big)^{m/q'} \\
&= \sum_{m=1}^{\infty} (eC_q(X)K)^m \|f\|_{H_1(\mathbb{T}^N,X)} \Big(\sum_{j=1}^{\infty} \frac{1}{\tilde{p}_j^{1+\varepsilon}} \frac{1}{\tilde{p}_j^{\varepsilon}}\Big)^{m/q'} \\
&\le \|f\|_{H_1(\mathbb{T}^N,X)} \sum_{m=1}^{\infty} \Bigg(\underbrace{\frac{eC_q(X)K\big(\sum_{j=1}^{\infty} p_j^{-(1+\varepsilon)}\big)^{1/q'}}{p_{k_0}^{\varepsilon/q'}}}_{<1}\Bigg)^m.
\end{aligned}$$

This completes the proof of Theorem 1.1. □

**Remark 3.1.** We end this note with a direct proof of the fact

$$1 - \frac{1}{\operatorname{Cot} X} \le S_p(X), \quad 1 \le p < \infty, \tag{16}$$

in which we do not use the inequality

$$1 - \frac{1}{\operatorname{Cot} X} \le S_\infty(X) \tag{17}$$

from [Defant et al. 2008] (here repeated in (1)). The proof of (17) given in that reference shows in a first step that $1 - 1/\Pi(X) \le S_\infty(X)$ where

$$\Pi(X) = \inf\{r \ge 2 \mid \mathrm{id}_X \text{ is } (r, 1)\text{-summing}\},$$

and then, in a second step, applies a fundamental theorem of Maurey and Pisier stating that $\Pi(X) = \operatorname{Cot} X$.

The following argument for (16) is very similar to the original one from [Defant et al. 2008] but does not use the Maurey–Pisier theorem (since we here consider $\mathcal{H}_p(X)$, $1 \le p < \infty$ instead of $\mathcal{H}_\infty(X)$): By the proof of Corollary 1.3, inequality (16) is equivalent to

$$M_p(X) \le \frac{\operatorname{Cot} X}{\operatorname{Cot} X - 1}.$$

Take $r < M_p(X)$, so that $\ell_r \cap B_{c_0} \subset \operatorname{mon} H_p(\mathbb{T}^\infty, X)$. Let $H_p^1(\mathbb{T}^\infty, X)$ be the subspace of $H_p(\mathbb{T}^\infty, X)$ formed by all 1-homogeneous polynomials (i.e., linear operators). We can define a bilinear operator

$\ell_r \times H_p^1(\mathbb{T}^\infty, X) \to \ell_1(X)$ by $(z, f) \mapsto (z_j f(e_j))_j$ which, by a closed graph argument, is continuous. Therefore, there is a constant $M$ such that for all $z \in \ell_r$ and all $f \in H_p^1(\mathbb{T}^\infty, X)$ we have

$$\sum_j |z_j| \|f(e_j)\|_X \le M \|z\|_{\ell_r} \|f\|_{H_p(\mathbb{T}^\infty, X)}.$$

Taking the supremum over all $z \in B_{\ell_r}$ we obtain for all $f \in H_p^1(\mathbb{T}^\infty, X)$

$$\Big(\sum_j \|f(e_j)\|_X^{r'}\Big)^{1/r'} \le M \|f\|_{H_p(\mathbb{T}^\infty, X)}.$$

Now, take $x_1, \dots, x_N \in X$, define $f \in H_p^1(\mathbb{T}^\infty, X)$ by

$$f(e_j) = \begin{cases} x_j & \text{if } 1 \le j \le N, \\ 0 & \text{if } j > N \end{cases}$$

and extend it by linearity. By the previous inequality and Proposition 2.5 we have

$$\Big(\sum_{j=1}^N \|x_j\|_X^{r'}\Big)^{1/r'} \le M \Big(\int_{\mathbb{T}^N} \Big\|\sum_{j=1}^N x_j z_j\Big\|_X^{r'} dz\Big)^{1/r'}.$$

By Kahane's inequality, $X$ has cotype $r'$, which means that $r' > \operatorname{Cot} X$ or, equivalently, $r < \dfrac{\operatorname{Cot} X}{\operatorname{Cot} X - 1}$. Since $r < M_p(X)$ was arbitrary, we obtain (16).

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