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SEMILINEAR GEOMETRIC OPTICS WITH BOUNDARY AMPLIFICATION

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We study weakly stable semilinear hyperbolic boundary value problems with highly oscillatory data. Here weak stability means that exponentially growing modes are absent, but the so-called uniform Lopatinskii condition fails at some boundary frequency β in the hyperbolic region. As a consequence of this degeneracy there is an amplification phenomenon: outgoing waves of amplitude $O(\varepsilon^2)$ and wavelength ε give rise to reflected waves of amplitude $O(\varepsilon)$, so the overall solution has amplitude $O(\varepsilon)$. Moreover, the reflecting waves emanate from a radiating wave that propagates in the boundary along a characteristic of the Lopatinskii determinant.

An approximate solution that displays the qualitative behavior just described is constructed by solving suitable profile equations that exhibit a loss of derivatives, so we solve the profile equations by a Nash–Moser iteration. The exact solution is constructed by solving an associated singular problem involving singular derivatives of the form $\partial_{x'} + \beta\partial_{\theta_0}/\varepsilon$, x' being the tangential variables with respect to the boundary. Tame estimates for the linearization of that problem are proved using a first-order (wavetrain) calculus of singular pseudodifferential operators constructed in a companion article ("Singular pseudodifferential calculus for wavetrains and pulses", arXiv 1201.6202, 2012). These estimates exhibit a loss of one singular derivative and force us to construct the exact solution by a separate Nash–Moser iteration.

The same estimates are used in the error analysis, which shows that the exact and approximate solutions are close in L^∞ on a fixed time interval independent of the (small) wavelength ε. The approach using singular systems allows us to avoid constructing high-order expansions and making small divisor assumptions. Our analysis of the exact singular system applies with no change to the case of pulses, provided one substitutes the pulse calculus from the companion paper for the wavetrain calculus.

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1. Introduction and main results

In this paper we study weakly stable semilinear hyperbolic boundary value problems with oscillatory data. The problems are weakly stable in the sense that exponentially growing modes are absent, but the uniform Lopatinskii condition fails at a boundary frequency β in the hyperbolic region $\mathscr{H}$.[1] As a consequence of this degeneracy in the boundary conditions, there is an amplification phenomenon: boundary data of wavelength ε and amplitude $O(\varepsilon^2)$ in problem (1-1) below gives rise to a response of amplitude $O(\varepsilon)$. In the meantime, resonance may occur between distinct oscillations. In the situation studied below, a resonant quadratic interaction between two incoming waves of amplitude $O(\varepsilon)$ may produce an outgoing wave of amplitude $O(\varepsilon^2)$. When reflected and amplified on the boundary, this oscillation gives rise to incoming waves of amplitude $O(\varepsilon)$. Hence the $O(\varepsilon)$ amplitude regime appears as the natural weakly nonlinear regime.

Let us now introduce some notation. On $\overline{\mathbb{R}}^{d+1}_+ = \{x = (x', x_d) = (t, y, x_d) = (t, x'') : x_d \geq 0\}$, consider the $N \times N$ semilinear hyperbolic boundary problem for $v = v_\varepsilon(x)$, where $\varepsilon > 0$:[2]

$$\begin{aligned} &\text{(a)} \quad L_0(\partial)v + f_0(v) = 0, \\ &\text{(b)} \quad \phi(v) = \varepsilon^2 G\left(x', \frac{x' \cdot \beta}{\varepsilon}\right) \quad \text{on } x_d = 0, \\ &\text{(c)} \quad v = 0 \text{ and } G = 0 \qquad\qquad \text{in } t < 0, \end{aligned} \tag{1-1}$$

where $L_0(\partial) = \partial_t + \sum_{j=1}^d B_j \partial_j$, the matrix B_d is invertible, and both $f_0(v)$ and $\phi(v)$ vanish at $v = 0$. The function $G(x', \theta_0)$ is assumed to be periodic in θ_0, and the frequency $\beta \in \mathbb{R}^d \setminus \{0\}$ is taken to be a boundary frequency at which the so-called uniform Lopatinskii condition fails. A consequence of this failure is that the choice of the factor ε^2 in (1-1)(b) corresponds to the weakly nonlinear regime for this problem. The leading profile is nonlinearly coupled to the next-order profile in the nonlinear system (1-35)–(1-36) derived below. We also refer to Appendix B for a detailed specific example which illustrates the nonlinear feature of the leading profile equation.

Before proceeding, we write the problem in an equivalent form that is better adapted to the boundary. After multiplying (1-1)(a) by $(B_d)^{-1}$, we obtain

$$\begin{aligned} &L(\partial)v + f(v) = 0, \\ &\phi(v) = \varepsilon^2 G\left(x', \frac{x' \cdot \beta}{\varepsilon}\right) \quad \text{on } x_d = 0, \\ &v = 0 \text{ and } G = 0 \qquad\qquad \text{in } t < 0, \end{aligned} \tag{1-2}$$

where we have set

$$L(\partial) = \partial_d + \sum_{j=0}^{d-1} A_j \partial_j \quad \text{with } A_j := B_d^{-1} B_j \text{ for } j = 0, \ldots, d-1.$$

[1] See Definition 1.4 and Assumption 1.6 for precise statements.

[2] We usually suppress the subscript ε.

Setting $v = \varepsilon u$ and writing $f(v) = D(v)v$, $\phi(v) = \psi(v)v$, we get the problem for $u = u_\varepsilon(x)$

$$\begin{aligned} &\text{(a)} \quad L(\partial)u + D(\varepsilon u)u = 0, \\ &\text{(b)} \quad \psi(\varepsilon u)u = \varepsilon G\left(x', \frac{x'\cdot\beta}{\varepsilon}\right) \quad \text{on } x_d = 0, \\ &\text{(c)} \quad u = 0 \qquad\qquad\qquad\qquad \text{in } t < 0. \end{aligned} \tag{1-3}$$

For problem (1-3) we pose the two basic questions of rigorous nonlinear geometric optics:

(1) Does an exact solution u_ε of (1-3) exist for $\varepsilon \in (0, 1]$ on a fixed time interval $[0, T_0]$ independent of ε?

(2) Suppose the answer to the first question is yes. If we let $u_\varepsilon^{\mathrm{app}}$ denote an approximate solution on $[0, T_0]$ constructed by the methods of nonlinear geometric optics (that is, solving eikonal equations for phases and suitable transport equations for profiles), how well does $u_\varepsilon^{\mathrm{app}}$ approximate u_ε for ε small? For example, is it true that[3]

$$\lim_{\varepsilon\to 0} |u_\varepsilon - u_\varepsilon^{\mathrm{app}}|_{L^\infty} \to 0? \tag{1-4}$$

The amplification phenomenon was studied in a formal way for several different *quasilinear* problems [Artola and Majda 1987; Majda and Artola 1988; Majda and Rosales 1983]. The last of these papers studied amplification in connection with Mach stem formation in reacting shock fronts, while [Artola and Majda 1987] explored a connection to the formation of instabilities in compressible vortex sheets. Both papers derived equations for profiles using an ansatz that exhibited amplification; however, neither of the two questions posed above were addressed. The first rigorous amplification results were proved in [Coulombel and Guès 2010] for *linear* problems. That article provided positive answers to the above questions (question (1) is trivial for linear problems) by making use of approximate solutions of high-order, and showed in particular that the limit (1-4) holds.

In this paper we give positive answers to the above questions for the *semilinear* system (1-3). As is typical in nonlinear geometric optics problems involving several phases, difficulties with small divisors rule out the construction of high-order approximate solutions.[4] Instead of constructing the exact solution u_ε as a small perturbation of a high-order approximate solution, we construct u_ε in the form

$$u_\varepsilon(x) = U_\varepsilon(x, \theta_0)|_{\theta_0 = \beta\cdot x'/\varepsilon},$$

where $U_\varepsilon(x, \theta_0)$ is an exact solution of the singular system (1-18). The singular system is solved using symmetrization and diagonalization arguments [Williams 2002], modified and supplemented with methods [Coulombel 2004] for deriving linear estimates for weakly stable hyperbolic boundary problems. In deriving the basic estimate (2-4) for the singular linear problem, a loss of derivatives[5] forces us to use a

[3] Let us observe that by the amplification phenomenon, we expect the solution v to (1-1) to have amplitude $O(\varepsilon)$, so the solution u to (1-3) should have amplitude $O(1)$. Hence the limit (1-4) deals with the difference between two $O(1)$ quantities.

[4] Such difficulties are sometimes avoided by assuming that small divisors do not occur; see, for example, [Joly et al. 1993]. But we do not want to make this assumption.

[5] In fact, the basic L^2 estimate for the singular system (1-18) exhibits loss of a single "singular derivative" $\partial_{x'} + \beta\partial_{\theta_0}/\varepsilon$, which is optimal according to the analysis in [Coulombel and Guès 2010].

new tool, namely, a substantial refinement, given in the companion paper [Coulombel et al. 2012], of the calculus of singular pseudodifferential operators constructed in [Williams 2002]. In the new version of the calculus, residual operators have better smoothing properties than previously realized and can therefore be considered as remainders in our problem. The loss of derivatives in the linear estimate presents a serious difficulty in the application to our semilinear problem. Picard iteration appears to be out of the question, so in Section 5B we use a Nash–Moser iteration scheme adapted to the scale of spaces (1-19) to construct the solution $U_\varepsilon(x, \theta_0)$ to the semilinear singular problem.

If problem (1-3) satisfied the uniform Lopatinskii condition, then, because of the factor ε in the boundary data εG, the equations for the leading profile, $\mathcal{V}^0$ in (1-15), would be linear; and in fact $\mathcal{V}^0$ would vanish. The weakly nonlinear regime would correspond to a source term G (and not εG) in (1-3); see [Williams 1996; 2000]. Under our weak stability assumption, it turns out that $\mathcal{V}^0$ is nonlinearly coupled to the second-order profile $\mathcal{V}^1$ in the profile equations (1-35) and (1-36). To solve these equations, we first isolate a "key subsystem" (1-42) that decouples from the full system. The basic L^2 estimate for the linearization of the key subsystem still exhibits a loss of one derivative, and we are again forced to use Nash–Moser iteration in order to solve this subsystem. Once the key subsystem is solved, the solution of the full profile system (1-35)–(1-36) follows easily. It appears in our analysis that the leading-order amplitude equation shares the weak well-posedness of the original nonlinear problem, but we have not checked whether the loss of derivative for the amplitude equation is optimal (we conjecture that it is).

The error analysis used to answer question (2) above is based on the estimate for the singular system (1-18) (see Proposition 2.2) and is discussed in more detail in Section 1E.

This paper can be read independently of [Coulombel et al. 2012]; for the reader's convenience, we have gathered all the necessary material on the singular calculus in Appendix A. Before discussing this more fully, we provide some definitions, notation, and a precise statement of assumptions.

Remark 1.1. We emphasize that our approach for constructing exact highly oscillating solutions for the system (1-1) can be used without any modification for constructing exact amplified pulses. More precisely, the estimates and well-posedness argument of Sections 2A, 2B, and 2C for the linearized singular system (2-1), and the Nash–Moser argument of Section 5B for the nonlinear singular system (1-18) have all been written so as to carry over verbatim to the case of pulses. Amplification of pulses is treated in [Coulombel and Williams 2013], where we consider a function G in (1-1) that has suitable decay properties with respect to its additional variable $\theta_0 \in \mathbb{R}$ (this functional framework is relevant for applications to lasers). We refer to [Coulombel and Williams 2013] for the precise statements in the pulse case. The main difference between the analysis of wavetrains and pulses lies in the leading-order profile equation and in the construction and estimation of correctors needed in the error analysis. The novelty is that we can get a rate of convergence for (1-4) while this seems out of reach for wavetrains.

1A. ***Assumptions.*** We make the following hyperbolicity assumption on the system (1-1):

Assumption 1.2. There exists an integer $q \geq 1$, some real functions $\lambda_1, \dots, \lambda_q$ that are analytic on $\mathbb{R}^d \setminus \{0\}$ and homogeneous of degree 1, and there exist some positive integers $\nu_1, \dots, \nu_q$ such that

$$\det\left[\tau I + \sum_{j=1}^{d} \xi_j B_j\right] = \prod_{k=1}^{q} (\tau + \lambda_k(\xi))^{\nu_k} \quad \text{for all } \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d \setminus \{0\}.$$

Moreover the eigenvalues $\lambda_1(\xi), \ldots, \lambda_q(\xi)$ are semisimple (their algebraic multiplicity equals their geometric multiplicity) and satisfy $\lambda_1(\xi) < \cdots < \lambda_q(\xi)$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$.

For simplicity, we restrict our analysis to noncharacteristic boundaries, and therefore make the following assumption.

Assumption 1.3. The matrix B_d is invertible and the matrix $B := \psi(0)$ has maximal rank, its rank p being equal to the number of positive eigenvalues of B_d (counted with their multiplicity). Moreover, the integer p satisfies $1 \leq p \leq N-1$.

In the normal modes analysis for (1-3), one first performs a Laplace transform in the time variable t and a Fourier transform in the tangential space variables y. We let $\tau - i\gamma \in \mathbb{C}$ and $\eta \in \mathbb{R}^{d-1}$ denote the dual variables of t and y. We introduce the symbol

$$\mathcal{A}(\zeta) := -i B_d^{-1}\left((\tau - i\gamma) I + \sum_{j=1}^{d-1} \eta_j B_j\right), \quad \zeta := (\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1}.$$

For future use, we also define the following sets of frequencies:

$$\begin{aligned} \Xi &:= \{(\tau - i\gamma, \eta) \in \mathbb{C} \times \mathbb{R}^{d-1} \setminus (0,0) : \gamma \geq 0\}, \qquad &\Sigma &:= \{\zeta \in \Xi : \tau^2 + \gamma^2 + |\eta|^2 = 1\}, \\ \Xi_0 &:= \{(\tau, \eta) \in \mathbb{R} \times \mathbb{R}^{d-1} \setminus (0,0)\} = \Xi \cap \{\gamma = 0\}, \quad &\Sigma_0 &:= \Sigma \cap \Xi_0. \end{aligned}$$

Two key objects in our analysis are the hyperbolic region and the glancing set, defined as follows.

Definition 1.4.
- The hyperbolic region $\mathcal{H}$ is the set of all $(\tau, \eta) \in \Xi_0$ such that the matrix $\mathcal{A}(\tau, \eta)$ is diagonalizable with purely imaginary eigenvalues.
- Let $\boldsymbol{G}$ denote the set of all $(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^d$ such that $\xi \neq 0$ and there exists an integer $k \in \{1, \ldots, q\}$ satisfying
$$\tau + \lambda_k(\xi) = \frac{\partial \lambda_k}{\partial \xi_d}(\xi) = 0.$$
If $\pi(\boldsymbol{G})$ denotes the projection of $\boldsymbol{G}$ on the d first coordinates (that is, $\pi(\tau, \xi) = (\tau, \xi_1, \ldots, \xi_{d-1})$ for all (τ, ξ)), the glancing set $\mathcal{G}$ is $\mathcal{G} := \pi(\boldsymbol{G}) \subset \Xi_0$.

We recall the following result, proved in [Kreiss 1970] in the strictly hyperbolic case (when all integers ν_j in Assumption 1.2 equal 1) and [Métivier 2000] in our more general framework.

Proposition 1.5 [Kreiss 1970; Métivier 2000]. *Let Assumptions 1.2 and 1.3 be satisfied. Then, for all $\zeta \in \Xi \setminus \Xi_0$, the matrix $\mathcal{A}(\zeta)$ has no purely imaginary eigenvalue and its stable subspace $\mathbb{E}^s(\zeta)$ has dimension p. Furthermore, $\mathbb{E}^s$ defines an analytic vector bundle over $\Xi \setminus \Xi_0$ that can be extended as a continuous vector bundle over Ξ.*

For all $(\tau, \eta) \in \Xi_0$, we let $\mathbb{E}^s(\tau, \eta)$ denote the continuous extension of $\mathbb{E}^s$ to the point (τ, η). The analysis in [Métivier 2000] shows that away from the glancing set $\mathcal{G} \subset \Xi_0$, $\mathbb{E}^s(\zeta)$ depends analytically on ζ, and the hyperbolic region $\mathcal{H}$ does not contain any glancing point.

To treat the case when the boundary operator in (1-3)(b) is independent of u, which is to say $\psi(\varepsilon u) \equiv \psi(0) =: B$, we make the following *weak stability assumption* on the problem $(L(\partial), B)$.

Assumption 1.6.

- For all $\zeta \in \Xi \setminus \Xi_0$, $\ker B \cap \mathbb{E}^s(\zeta) = \{0\}$.
- The set $\Upsilon_0 := \{\zeta \in \Sigma_0 : \ker B \cap \mathbb{E}^s(\zeta) \neq \{0\}\}$ is nonempty and included in the hyperbolic region $\mathcal{H}$.
- For all $\underline{\zeta} \in \Upsilon_0$, there exists a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in Σ, a real valued $\mathscr{C}^\infty$ function σ defined on $\mathcal{V}$, a basis $E_1(\zeta), \dots, E_p(\zeta)$ of $\mathbb{E}^s(\zeta)$ that is of class $\mathscr{C}^\infty$ with respect to $\zeta \in \mathcal{V}$, and a matrix $P(\zeta) \in \mathrm{GL}_p(\mathbb{C})$ that is of class $\mathscr{C}^\infty$ with respect to $\zeta \in \mathcal{V}$, such that
$$\text{for all } \zeta \in \mathcal{V}, \quad B\begin{pmatrix} E_1(\zeta) & \cdots & E_p(\zeta) \end{pmatrix} = P(\zeta)\,\mathrm{diag}(\gamma + i\sigma(\zeta), 1, \dots, 1).$$

For comparison and later reference we recall the following definition.

Definition 1.7 [Kreiss 1970]. As before let p be the number of positive eigenvalues of B_d. The problem $(L(\partial), B)$ is said to be *uniformly stable* or to satisfy the *uniform Lopatinskii condition* if

$$B : \mathbb{E}^s(\zeta) \to \mathbb{C}^p$$

is an isomorphism for all $\zeta \in \Sigma$.

Remark 1.8. Observe that if $(L(\partial), B)$ satisfies the uniform Lopatinskii condition, continuity implies that this condition still holds for $(L(\partial), B + \dot{\psi})$, where $\dot{\psi}$ is any sufficiently small perturbation of B. Hence the uniform Lopatinskii condition is a convenient framework for nonlinear perturbation. The analogous statement may not be true when $(L(\partial), B)$ is only weakly stable. Remarkably, weak stability persists under perturbation in the so-called WR class exhibited in [Benzoni-Gavage et al. 2002], and Assumption 1.6 is a convenient equivalent definition of the WR class; see [Coulombel and Guès 2010, Appendix B]. In order to handle general nonlinear boundary conditions as in (1-3), we strengthen Assumption 1.6 in Assumption 1.12.

Boundary and interior phases. We consider a planar real phase ϕ_0 defined on the boundary:

$$\phi_0(t, y) := \underline{\tau} t + \underline{\eta} \cdot y, \quad (\underline{\tau}, \underline{\eta}) \in \Xi_0. \tag{1-5}$$

As follows from earlier works (see, for example, [Majda and Artola 1988]), oscillations on the boundary associated with the phase ϕ_0 give rise to oscillations in the interior associated with some planar phases ϕ_m. These phases are characteristic for the hyperbolic operator $L_0(\partial)$ and their trace on the boundary $\{x_d = 0\}$ equals ϕ_0. For now we make the following assumption.

Assumption 1.9. The phase ϕ_0 defined by (1-5) satisfies $(\underline{\tau}, \underline{\eta}) \in \Upsilon_0$. In particular $(\underline{\tau}, \underline{\eta}) \in \mathcal{H}$.

Thanks to Assumption 1.9, we know that the matrix $\mathscr{A}(\underline{\tau}, \underline{\eta})$ is diagonalizable with purely imaginary eigenvalues. These eigenvalues are denoted by $i\underline{\omega}_1, \dots, i\underline{\omega}_M$, where the $\underline{\omega}_m$s are real and pairwise distinct.

The $\underline{\omega}_m$s are the roots (and all the roots are real) of the dispersion relation

$$\det\left[\underline{\tau} I + \sum_{j=1}^{d-1} \underline{\eta}_j B_j + \omega B_d\right] = 0.$$

To each root $\underline{\omega}_m$ there corresponds a unique integer $k_m \in \{1, \dots, q\}$ such that $\underline{\tau} + \lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) = 0$. We can then define the following real[6] phases and their associated group velocities:

$$\text{for all } m = 1, \dots, M, \quad \phi_m(x) := \phi_0(t, y) + \underline{\omega}_m x_d, \quad \boldsymbol{v}_m := \nabla\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m). \tag{1-6}$$

Let us observe that each group velocity $\boldsymbol{v}_m$ is either incoming or outgoing with respect to the space domain $\mathbb{R}^d_+$: the last coordinate of $\boldsymbol{v}_m$ is nonzero. This property holds because $(\underline{\tau}, \underline{\eta})$ does not belong to the glancing set $\mathscr{G}$. We can therefore adopt the following classification.

Definition 1.10. The phase ϕ_m is incoming when the group velocity $\boldsymbol{v}_m$ is incoming (that is, when $\partial_{\xi_d}\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) > 0$), and it is outgoing when the group velocity $\boldsymbol{v}_m$ is outgoing ($\partial_{\xi_d}\lambda_{k_m}(\underline{\eta}, \underline{\omega}_m) < 0$).

In all that follows, we let $\mathscr{I}$ denote the set of indices $m \in \{1, \dots, M\}$ such that ϕ_m is an incoming phase, and $\mathbb{O}$ denote the set of indices $m \in \{1, \dots, M\}$ such that ϕ_m is an outgoing phase. If $p \geq 1$, $\mathscr{I}$ is nonempty, while if $p \leq N - 1$, $\mathbb{O}$ is nonempty (see Lemma 1.11). We will use the notation

$$L_0(\tau, \xi) := \tau I + \sum_{j=1}^{d} \xi_j B_j, \quad L(\beta, \underline{\omega}_m) := \underline{\omega}_m I + \sum_{k=0}^{d-1} \beta_k A_k,$$
$$\beta := (\underline{\tau}, \underline{\eta}), \quad x' = (t, y), \quad \phi_0(x') = \beta \cdot x'.$$

For each phase ϕ_m, $d\phi_m$ denotes the differential of the function ϕ_m with respect to its argument $x = (t, y, x_d)$. It follows from Assumption 1.2 that the eigenspace of $\mathscr{A}(\beta)$ associated with the eigenvalue $i\underline{\omega}_m$ coincides with the kernel of $L_0(d\phi_m)$ and has dimension ν_{k_m}. The following well-known lemma, whose proof is recalled in [Coulombel and Guès 2010], gives a useful decomposition of $\mathbb{E}^s$ in the hyperbolic region.

Lemma 1.11. *The stable subspace $\mathbb{E}^s(\beta)$ admits the decomposition*

$$\mathbb{E}^s(\beta) = \bigoplus_{m \in \mathscr{I}} \ker L_0(d\phi_m), \tag{1-7}$$

and each vector space in the decomposition (1-7) *admits a basis of real vectors.*

To formulate our last assumption we observe first that for every point $\underline{\zeta} \in \mathscr{H}$ there is a neighborhood $\mathscr{V}$ of $\underline{\zeta}$ in Σ and a C^∞ conjugator $Q_0(\zeta)$ defined on $\mathscr{V}$ such that

$$Q_0(\zeta)\mathscr{A}(\zeta)Q_0^{-1}(\zeta) = \begin{pmatrix} i\omega_1(\zeta)I_{n_1} & & 0 \\ & \ddots & \\ 0 & & i\omega_J(\zeta)I_{n_J} \end{pmatrix} =: -\mathbb{D}_1(\zeta), \tag{1-8}$$

[6] If $(\underline{\tau}, \underline{\eta})$ does not belong to the hyperbolic region $\mathscr{H}$, some of the phases ϕ_m may be complex; see, for example, [Williams 1996; 2000; Lescarret 2007; Marcou 2010]. Moreover, glancing phases introduce a new scale $\sqrt{\varepsilon}$ as well as boundary layers.

where the ω_j are real when $\gamma = 0$ and there is a constant $c > 0$ such that either

$$\mathrm{Re}(i\omega_j) \le -c\gamma \quad \text{or} \quad \mathrm{Re}(i\omega_j) \ge c\gamma \text{ for all } \zeta \in \mathcal{V}.$$

In view of Lemma 1.11, we can choose the first p columns of $Q_0^{-1}(\zeta)$ to be a basis of $\mathbb{E}^s(\zeta)$, and write

$$Q_0^{-1}(\zeta) = [Q_{\mathrm{in}}(\zeta) Q_{\mathrm{out}}(\zeta)].$$

Choose J' so that the first J' blocks of $-\mathbb{D}_1$ lie in the first p columns, and the remaining blocks in the remaining $N - p$ columns. Thus $\mathrm{Re}(i\omega_j) \le -c\gamma$ if and only if $1 \le j \le J'$.

Observing that the linearization of the boundary condition in (1-3) is

$$\dot{u} \mapsto \psi(\varepsilon u)\dot{u} + [d\psi(\varepsilon u)\dot{u}]\varepsilon u,$$

we define the operator

$$\mathcal{B}(v_1, v_2)\dot{u} := \psi(v_1)\dot{u} + [d\psi(v_1)\dot{u}]v_2, \tag{1-9}$$

which appears in Assumption 1.12. For later use we also define

$$\mathcal{D}(v_1, v_2)\dot{u} := D(v_1)\dot{u} + [dD(v_1)\dot{u}]v_2, \tag{1-10}$$

as well as

$$\mathcal{B}(v_1) := \mathcal{B}(v_1, v_1), \mathcal{D}(v_1) := \mathcal{D}(v_1, v_1). \tag{1-11}$$

We now state the weak stability assumption that we make when considering the general case of nonlinear boundary conditions in (1-3).

Assumption 1.12.
- There exists a neighborhood $\mathcal{O}$ of $(0, 0) \in \mathbb{R}^{2N}$ such that for all $(v_1, v_2) \in \mathcal{O}$ and all $\zeta \in \Xi \setminus \Xi_0$, $\ker \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) = \{0\}$. For each $(v_1, v_2) \in \mathcal{O}$, the set

$$\Upsilon(v_1, v_2) := \{\zeta \in \Sigma_0 : \ker \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) \neq \{0\}\}$$

  is nonempty and is included in the hyperbolic region $\mathcal{H}$. Moreover, if we set $\Upsilon := \bigcup_{(v_1,v_2)\in\mathcal{O}} \Upsilon(v_1, v_2)$, $\overline{\Upsilon} \subset \mathcal{H}$ (closure in Σ_0).
- For every $\underline{\zeta} \in \overline{\Upsilon}$, there exists a neighborhood $\mathcal{V}$ of $\underline{\zeta}$ in Σ and a C^∞ function $\sigma(v_1, v_2, \zeta)$ on $\mathcal{O} \times \mathcal{V}$ such that for all $(v_1, v_2, \zeta) \in \mathcal{O} \times \mathcal{V}$ we have $\ker \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) \neq \{0\}$ if and only if $\zeta \in \Sigma_0$ and $\sigma(v_1, v_2, \zeta) = 0$.

  Moreover, there exist matrices $P_i(v_1, v_2, \zeta) \in \mathrm{GL}_p(\mathbb{C})$, $i = 1, 2$, of class C^∞ on $\mathcal{O} \times \mathcal{V}$ such that, for all $(v_1, v_2, \zeta) \in \mathcal{O} \times \mathcal{V}$,

$$P_1(v_1, v_2, \zeta)\mathcal{B}(v_1, v_2) Q_{\mathrm{in}}(\zeta) P_2(v_1, v_2, \zeta) = \mathrm{diag}(\gamma + i\sigma(v_1, v_2, \zeta), 1, \dots, 1). \tag{1-12}$$

For nonlinear boundary conditions, the phase ϕ_0 in (1-5) is assumed to satisfy $(\underline{\tau}, \underline{\eta}) \in \Upsilon(0, 0)$, or, in other words, the intersection $\ker B \cap \mathbb{E}^s(\underline{\tau}, \underline{\eta})$ is not reduced to $\{0\}$ (the set Υ_0 in Assumption 1.6 is a short notation for $\Upsilon(0, 0)$). The phases ϕ_m are still defined by (1-6) and thus only depend on $L(\partial)$ and B, and not on the nonlinear perturbations f_0 and $\psi(\varepsilon u) - \psi(0)$ added in (1-3).

Remark 1.13. (1) The properties stated in Assumption 1.12 are just a convenient description of the requirements for belonging to the WR class of [Benzoni-Gavage et al. 2002]. Like the uniform Lopatinskii condition, Assumption 1.12 can, in practice, be verified by hand via a "constant coefficient" computation. More precisely, for (v_1, v_2) near $(0, 0) \in \mathbb{R}^{2N}$ and $\zeta \in \Sigma$, one can define (see, for example, [Benzoni-Gavage and Serre 2007, chapter 4]) a Lopatinskii determinant $\Delta(v_1, v_2, \zeta)$ that is C^∞ in (v_1, v_2), analytic in $\zeta = (\tau - i\gamma, \eta)$ on $\Sigma \setminus \mathcal{G}$, and satisfies

$$\Delta(v_1, v_2, \zeta) = 0 \quad \text{if and only if } \ker \mathcal{B}(v_1, v_2) \cap \mathbb{E}^s(\zeta) \neq \{0\}.$$

In particular, $\Delta(v_1, v_2, \cdot)$ is real-analytic on $\mathcal{H}$.

Following [Benzoni-Gavage et al. 2002] (see also [Benzoni-Gavage and Serre 2007, chapter 8]), we claim that Assumption 1.12 holds provided

$$\varnothing \neq \{\zeta \in \Sigma : \Delta(0, 0, \zeta) = 0\} \subset \mathcal{H} \quad \text{and} \quad \Delta(0, 0, \underline{\zeta}) = 0 \Rightarrow \partial_\tau \Delta(0, 0, \underline{\zeta}) \neq 0, \tag{1-13}$$

and thus it only involves a weak stability property for the linearized problem at $(v_1, v_2) = (0, 0)$. Indeed, the implicit function theorem then implies that, for (v_1, v_2) near zero and (τ, η) near $\underline{\zeta}$, the set

$$\{(\tau, \eta) \in \Sigma_0 : \Delta(v_1, v_2, \tau, \eta) = 0\}$$

is a real-analytic hypersurface in $\mathcal{H}$. On the other hand, an application of the implicit function theorem to $\Delta(v_1, v_2, z, \eta)$, for $(z, \eta) \in \Sigma$, shows that the real dimension of the manifold

$$\{(z, \eta) \in \Sigma : \Delta(v_1, v_2, z, \eta) = 0\}$$

must be the same, that is, $d - 2$. The two zero sets must then coincide; there are no zeros in $\Sigma \setminus \Sigma_0$. The function σ and the neighborhoods $\mathcal{O}$ and $\mathcal{V}$ arise in a factorization of Δ given by the Weierstrass preparation theorem. The construction of the conjugating matrices P_i, $i = 1, 2$ follows from a construction in [Sablé-Tougeron 1988, Pages 268–270].

Instead of assuming (1-13), we have stated Assumption 1.12 in a form that is more directly applicable to the proof of Proposition 2.2 and to the error analysis of Theorem 4.1.

(2) To prove the basic estimate for the linearized singular system, Proposition 2.2, and to construct the exact solution U_ε to the singular system (1-18), it is enough to require that the analogue of Assumption 1.12 holds when $\mathcal{B}(v_1, v_2)$ is replaced by $\mathcal{B}(v_1) := \mathcal{B}(v_1, v_1)$. However, for the error analysis of Section 4 in the case of nonlinear boundary conditions, we need Assumption 1.12 as stated.

The next lemma, proved in [Coulombel and Guès 2010], gives a useful decomposition of $\mathbb{C}^N$ and introduces projectors needed later for formulating and solving the profile equations.

Lemma 1.14. *The space $\mathbb{C}^N$ admits the decomposition*

$$\mathbb{C}^N = \bigoplus_{m=1}^{M} \ker L_0(d\phi_m), \tag{1-14}$$

and each vector space in (1-14) *admits a basis of real vectors. If we let $P_1, \dots, P_M$ denote the projectors associated with the decomposition* (1-14), *we have* $\operatorname{Im} B_d^{-1} L_0(d\phi_m) = \ker P_m$ *for all* $m = 1, \dots, M$.

1B. ***Main results.*** For each $m \in \{1, \dots, M\}$ we let

$$r_{m,k}, \quad k = 1, \dots, \nu_{k_m},$$

denote a basis of $\ker L_0(d\phi_m)$ consisting of real vectors. In Section 4 we shall construct a "corrected" approximate solution u^c_ε of (1-3) of the form

$$u^c_\varepsilon(x) = \mathcal{V}^0\Big(x, \frac{\phi}{\varepsilon}\Big) + \varepsilon \mathcal{V}^1\Big(x, \frac{\phi}{\varepsilon}\Big) + \varepsilon^2 \mathcal{U}^2_p\Big(x, \frac{\phi_0}{\varepsilon}, \frac{x_d}{\varepsilon}\Big), \tag{1-15}$$

where $\phi := (\phi_1, \dots, \phi_M)$ denotes the collection of all phases,

$$\begin{aligned} \mathcal{V}^0\Big(x, \frac{\phi}{\varepsilon}\Big) &= \sum_{m \in \mathcal{I}} \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}\Big(x, \frac{\phi_m}{\varepsilon}\Big) r_{m,k}, \\ \mathcal{V}^1\Big(x, \frac{\phi}{\varepsilon}\Big) &= \underline{\mathcal{V}}^1(x) + \sum_{m=1}^{M} \sum_{k=1}^{\nu_{k_m}} \tau_{m,k}\Big(x, \frac{\phi_m}{\varepsilon}\Big) r_{m,k} + \mathcal{R}\mathcal{V}^0, \end{aligned} \tag{1-16}$$

and the $\sigma_{m,k}(x, \theta_m)$ and $\tau_{m,k}(x, \theta_m)$ are scalar C^1 functions periodic in θ_m with mean 0 which describe the propagation of oscillations with phase ϕ_m and group velocity $\boldsymbol{v}_m$. Here $\mathcal{R}$ denotes the nonlocal operator

$$\mathcal{R}\mathcal{V}^0 = -R[L(\partial_x)\mathcal{V}^0 + D(0)\mathcal{V}^0]$$

for R defined as in (1-32). The last corrector $\varepsilon^2 \mathcal{U}^2_p(x, \theta_0, \xi_d)$ in (1-15) is a trigonometric polynomial constructed in the error analysis of Section 4.

The next theorem, our main result, is an immediate corollary of the more precise Theorem 4.1. Here we let $\Omega_T := \{(x, \theta_0) = (t, y, x_d, \theta_0) \in \mathbb{R}^{d+1} \times \mathbb{T}^1 : x_d \geq 0, t < T\}$ and $b\Omega_T := \{(t, y, \theta_0) \in \mathbb{R}^d \times \mathbb{T}^1 : t < T\}$. The spaces E^s are defined in (1-19).

Theorem 1.15. *We make Assumptions 1.2, 1.3, 1.6, and 1.9 when the boundary condition in* (1-3) *is linear* ($\psi(\varepsilon u) \equiv \psi(0)$); *in the general case we substitute Assumption 1.12 for Assumption 1.6. Fix* $T > 0$, *set* $M_0 := 3d + 5$, *and let*

$$\mu := [(d+1)/2] + M_0 + 3 \quad \text{and} \quad \tilde{\mu} := 2\mu - [(d+1)/2].$$

Consider the semilinear boundary problem (1-3), *where* $G(t, y, \theta_0) \in H^{\tilde{\mu}}(b\Omega_T)$. *There exists* $\varepsilon_0 > 0$ *such that if* $\langle G \rangle_{H^{\mu+2}(b\Omega_T)}$ *is small enough, there exists a unique function* $U_\varepsilon(x, \theta_0) \in E^{\mu-1}(\Omega_T)$ *satisfying the singular system* (1-18) *on* Ω_T *such that*

$$u_\varepsilon(x) := U_\varepsilon\Big(x, \frac{x' \cdot \beta}{\varepsilon}\Big)$$

is an exact solution of (1-3) *on* $(-\infty, T] \times \overline{\mathbb{R}}^d_+$ *for* $0 < \varepsilon \leq \varepsilon_0$. *In addition there exists a profile* $\mathcal{V}^0(x, \theta)$ *as in* (1-16), *whose components* $\sigma_{m,k}$ *lie in* $H^{\mu-1}(\Omega_T)$, *such that the approximate solution defined by*

$$u^{\mathrm{app}}_\varepsilon := \mathcal{V}^0\Big(x, \frac{\phi}{\varepsilon}\Big)$$

satisfies

$$\lim_{\varepsilon \to 0} |u_\varepsilon - u^{\mathrm{app}}_\varepsilon|_{L^\infty} = 0 \quad \text{on } (-\infty, T] \times \overline{\mathbb{R}}^d_+.$$

Observe that although the boundary data in problem (1-3) is of size $O(\varepsilon)$, the approximate solution $u_\varepsilon^{\mathrm{app}}$ is of size $O(1)$, exhibiting an amplification due to the weak stability at frequency β. The main information provided by Theorem 1.15 is that this amplification does not rule out the existence of a smooth solution on a fixed time interval, that is, it does not trigger a violent instability, at least in this weakly nonlinear regime. As far as we know, the derivation of the leading-order amplitude equation (1-42) is also new in the general framework that we consider. This amplitude equation shares some features of the Burgers equation and we expect that its solutions may develop singularities in finite time; see similar discussions in [Majda and Rosales 1984]. We hope that the analysis developed in this article will be useful in justifying *quasilinear* amplification phenomena such as the Mach stems or kink modes formation [Artola and Majda 1987; Majda and Artola 1988; Majda and Rosales 1983], but there are still many obstacles along the way.

Remark 1.16. (a) In order to avoid some technicalities, we have stated our main result for a problem (1-3) where all data vanish for $t < 0$. This result easily implies a similar result in which outgoing waves defined in $t < 0$ of amplitude $O(\varepsilon)$ and wavelength ε give rise to reflected waves of amplitude $O(1)$. In either formulation, analysis of the profile equations (see Remark 1.28) shows that the waves of amplitude $O(1)$ emanate from a radiating wave that propagates in the boundary along a characteristic of the Lopatinskii determinant.

(b) We have decided to fix $T > 0$ at the start and choose data small enough so that a solution to the nonlinear problem exists up to time T. One can also (as discussed in Remark 3.7) fix the data in the problem (G in (1-3)) at the start, and then choose T small enough so that a solution to the nonlinear problem exists up to time T.

In the remainder of this introduction, we discuss the construction of exact solutions, the construction of the approximate solution $\mathcal{V}^0$, and the error analysis. Complete proofs are given in Sections 2, 3, 4, and 5.

1C. ***Exact solutions and singular systems.*** The theory of weakly stable hyperbolic initial boundary value problems fails to provide a solution of the system (1-3) that exists on a fixed time interval independent of ε.[7] In order to obtain such an exact solution to the system (1-3), we adopt the strategy of studying an associated singular problem first used in [Joly et al. 1995] for an initial value problem in free space. We look for a solution of the form

$$u_\varepsilon(x) = U_\varepsilon(x, \theta_0)|_{\theta_0=\phi_0(x')/\varepsilon}, \tag{1-17}$$

where $U_\varepsilon(x, \theta_0)$ is periodic in θ_0 and satisfies the singular system derived by substituting (1-17) into problem (1-3). Recalling that $L(\partial) = \partial_d + \sum_{j=0}^{d-1} A_j \partial_j$ we obtain

$$\begin{aligned} &\partial_d U_\varepsilon + \sum_{j=0}^{d-1} A_j \left(\partial_j + \frac{\beta_j \partial_{\theta_0}}{\varepsilon}\right) U_\varepsilon + D(\varepsilon U_\varepsilon) U_\varepsilon =: \partial_d U_\varepsilon + \mathbb{A}\left(\partial_{x'} + \frac{\beta \partial_{\theta_0}}{\varepsilon}\right) U_\varepsilon + D(\varepsilon U_\varepsilon) U_\varepsilon = 0, \\ &\psi(\varepsilon U_\varepsilon) U_\varepsilon|_{x_d=0} = \varepsilon G(x', \theta_0), \\ &U_\varepsilon = 0 \quad \text{in } t < 0. \end{aligned} \tag{1-18}$$

[7] This would be true even for problems $(L(\partial), B)$ that are uniformly stable in the sense of Definition 1.7.

The special difficulties presented by such singular problems when there is a boundary are described in detail in the introductions to [Williams 1996; 2002; Coulombel et al. 2011]. In particular, we mention:

(a) Symmetry assumptions on the matrices B_j appearing in the problem (1-1) equivalent to (1-3) are generally of no help in obtaining an L^2 estimate for (1-18) (boundary conditions satisfying Assumption 1.6 cannot be maximally dissipative; see [Coulombel and Guès 2010]).

(b) One cannot control L^∞ norms just by estimating tangential derivatives $\partial^\alpha_{(x',\theta_0)} U_\varepsilon$ because (1-18) is not a hyperbolic problem in the x_d direction;[8] moreover, even if one has estimates of tangential derivatives uniform with respect to ε, because of the factors $1/\varepsilon$ in (1-18), one cannot just use the equation to control $\partial_d U_\varepsilon$ and thereby control L^∞ norms.

To deal with these difficulties, Williams [2002] introduced a class of singular pseudodifferential operators, acting on functions $U(x',\theta_0)$ that are 2π-periodic in θ_0 and having the form

$$p_D U(x',\theta_0) = \frac{1}{(2\pi)^d} \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}^d} e^{ix'\cdot\xi' + i\theta_0 k}\, p\left(\varepsilon V(x',\theta_0), \xi' + \frac{k\beta}{\varepsilon}, \gamma\right) \widehat{U}(\xi',k)\, d\xi',\ \gamma \geq 1.$$

Observe that the differential operator $\mathbb{A}$ appearing in (1-18) can be expressed in this form. Kreiss-type symmetrizers $r_s(D_{x',\theta_0})$ in the singular calculus were constructed in [Williams 2002] for (quasilinear systems similar to) (1-18) under the assumption that $(L(\partial), \psi(0))$ is uniformly stable in the sense of Definition 1.7. With these, one can prove $L^2(x_d, H^s(x',\theta_0))$ estimates uniform in ε for (1-18), even when εG is replaced by G in the boundary condition. To progress further and control L^∞ norms, the boundary frequency β is restricted to lie in the complement of the glancing set. With this extra assumption, the singular calculus was used in [Williams 2002] to block-diagonalize the singular operator $\mathbb{A}\left(\varepsilon U_\varepsilon, \partial_{x'} + \beta\partial_{\theta_0}/\varepsilon\right)$ microlocally near the β direction and thereby prove estimates uniform with respect to ε in the spaces

$$E^s := C(x_d, H^s(x',\theta_0)) \cap L^2(x_d, H^{s+1}(x',\theta_0)). \tag{1-19}$$

These spaces are Banach algebras and are contained in L^∞ for $s > (d+1)/2$. For large enough s, as determined by the requirements of the calculus, existence of solutions to (1-18) in E^s on a time interval $[0,T]$ independent of $\varepsilon \in (0,\varepsilon_0]$ follows by Picard iteration in the uniformly stable case.

The singular calculus of [Williams 2002] was used again in [Coulombel et al. 2011] to rigorously justify leading-order geometric optics expansions for the quasilinear analogue of (1-3) in the uniformly stable case (with $\beta \in \mathscr{H}$ and the forcing term G in place of εG in the boundary condition). Under the assumptions made in the present paper, in particular assuming weak stability as in Assumptions 1.6 and 1.12, we face the additional difficulty that the basic L^2 estimate for the problem $(L(\partial), B)$ exhibits a loss of derivatives. A consequence of this is that the singular calculus of [Williams 2002] is no longer adequate for estimating solutions of (1-18). The main reason is that remainders in the calculus of [Williams 2002] are just bounded operators on L^2, while for energy estimates with a loss of derivative, remainders should be smoothing operators. We therefore need to use an improved version of the calculus constructed in

[8] For initial value problems in free space, one *can* control L^∞ norms just by estimating enough derivatives tangent to time slices $t = c$.

[Coulombel et al. 2012] in which residual operators are shown to have better smoothing properties than previously thought. With the improved calculus we are able in Section 2C to estimate solutions of (1-18) in E^s spaces (1-19), but of course there is a loss of one singular derivative in the estimates. This loss forces us in Section 5B to use Nash–Moser iteration on the scale of E^s spaces to obtain an exact solution of the singular system (1-18) on a fixed time interval independent of ε. Observe that one singular derivative costs a factor $1/\varepsilon$ and this is another reason why the scaling εG in (1-18) is crucial.

Remark 1.17. The main idea employed in proving the estimate for the linearized singular problem, Proposition 2.2, is to adapt the techniques of [Coulombel 2004] to the singular pseudodifferential framework. There is however one major obstacle along the way. While the error term in the composition of two zero-order operators (or in the composition of an operator of order -1 (on the left) with an operator of order 1, a $(-1, 1)$ composition) is smoothing of order 1 in the sense of (A-3), the same is unfortunately not true of the error term in $(1, -1)$ compositions (there are counterexamples for that). The properties of the $(1, -1)$ error terms that arise in our proof are described in Lemma 2.6.

1D. ***Derivation of the leading profile equations.*** We now derive the profile equations for the semilinear problem (1-3). We work with profiles $\mathcal{V}^j(x, \theta)$ periodic in $\theta = (\theta_1, \dots, \theta_M)$, where θ_j is a placeholder for ϕ_j/ε. Looking for an approximate solution of (1-3) of the form $u^a = (\mathcal{V}^0 + \varepsilon\mathcal{V}^1 + \varepsilon^2\mathcal{V}^2)|_{\theta=\phi/\varepsilon}$, where $\phi = (\phi_1, \dots, \phi_M)$, we get interior equations

$$\begin{aligned} &\text{(a)} \quad \mathcal{L}(\partial_\theta)\mathcal{V}^0 = 0,\\ &\text{(b)} \quad \mathcal{L}(\partial_\theta)\mathcal{V}^1 + L(\partial)\mathcal{V}^0 + D(0)\mathcal{V}^0 = 0,\\ &\text{(c)} \quad \mathcal{L}(\partial_\theta)\mathcal{V}^2 + L(\partial)\mathcal{V}^1 + D(0)\mathcal{V}^1 + (dD(0)\mathcal{V}^0)\mathcal{V}^0 = 0, \end{aligned} \tag{1-20}$$

by plugging u^a into (1-3)(a) and setting the coefficients of, respectively, ε^{-1}, ε^0, and ε equal to zero. The operator $\mathcal{L}(\partial_\theta)$ is defined by

$$\mathcal{L}(\partial_\theta) := \sum_{j=1}^{M} L(d\phi_j)\partial_{\theta_j}. \tag{1-21}$$

With $B := \psi(0)$, the boundary equations, obtained by plugging u^a into (1-3)(b) and setting the coefficients of ε^0 and ε equal to zero, are

$$\begin{aligned} &B\mathcal{V}^0(x', 0, \theta_0, \dots, \theta_0) = 0,\\ &B\mathcal{V}^1 + (d\psi(0)\mathcal{V}^0)\mathcal{V}^0 = G(x', \theta_0), \end{aligned} \tag{1-22}$$

where θ_0 is a placeholder for ϕ_0/ε. We will see that as a consequence of the weak stability at frequency β, the problem for the leading profile $\mathcal{V}^0$ is nonlinear and nonlocal. (See Appendix B for a concrete example.) Thus, the scaling in (1-2) *is* the weakly nonlinear scaling when the uniform Lopatinskii condition fails at a hyperbolic frequency β. To analyze these equations, we proceed to define appropriate function spaces and a pair of auxiliary operators E and R.

Functions $\mathcal{V}(x, \theta) \in L^2(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$ have Fourier series

$$\mathcal{V}(x, \theta) = \sum_{\alpha \in \mathbb{Z}^M} V_\alpha(x) e^{i\alpha\cdot\theta}. \tag{1-23}$$

Since only quadratic interactions appear in (1-20) and we anticipate that $\mathcal{V}^0$ will have the form in (1-16), for $k = 1, 2$ we let

$$\mathbb{Z}^{M;k} = \{\alpha \in \mathbb{Z}^M : \text{at most } k \text{ components of } \alpha \text{ are nonzero}\},$$

and we consider the subspace $H^{s;k}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) \subset H^s(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$ defined by

$$H^{s;k}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) = \left\{\mathcal{V}(x,\theta) \in H^s(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) : \mathcal{V}(x,\theta) = \sum_{\alpha \in \mathbb{Z}^{M;k}} V_\alpha(x) e^{i\alpha\cdot\theta}\right\}. \tag{1-24}$$

Thus multiplication defines a continuous map

$$H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) \times H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) \to H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) \tag{1-25}$$

for $s > (d+1+2)/2$.

Definition 1.18. Setting $\phi := (\phi_1, \dots, \phi_M)$, we say $\alpha \in \mathbb{Z}^{M;2}$ is a *characteristic mode* and write $\alpha \in \mathcal{C}$ if $\det L(d(\alpha \cdot \phi)) = 0$. Otherwise we call α a *noncharacteristic* mode. We decompose $\mathcal{C}$ as

$$\mathcal{C} = \bigcup_{m=1}^M \mathcal{C}_m, \quad \text{where } \mathcal{C}_m := \{\alpha \in \mathbb{Z}^{M;2} : \alpha\cdot\phi = n_\alpha \phi_m \text{ for some } n_\alpha \in \mathbb{Z}\}.$$

Observe that for $\alpha \in \mathcal{C}_m$, the integer n_α is necessarily equal to $\sum_{k=1}^M \alpha_k$. Since ϕ_i and ϕ_j are linearly independent for $i \neq j$, any $\alpha \in \mathbb{Z}^{M;2} \setminus 0$ belongs to at most one of the sets $\mathcal{C}_m$ and $n_\alpha \neq 0$ if $\alpha \neq 0$.

Elements $\alpha \in \mathcal{C}_m$ with two nonzero components correspond to *resonances*. Resonances are generated in products like $\sigma_{p,k}(x, \phi_p/\varepsilon)\sigma_{r,k'}(x, \phi_r/\varepsilon)$, which arise from the quadratic term in (1-20)(c), whenever there exists a relation of the form

$$n_m\phi_m = n_p\phi_p + n_r\phi_r, \quad \text{where } m \in \{1, \dots, M\} \setminus \{p, r\} \text{ and } n_m, n_p, n_r \in \mathbb{Z}.$$

We then refer to (ϕ_m, ϕ_p, ϕ_r) as a triple of resonant phases. This relation implies, for example, that ϕ_p oscillations interact with ϕ_r oscillations to produce ϕ_m oscillations.

Definition 1.19. We define the continuous projector[9] $E : H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) \to H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$, $s \geq 0$, by

$$E = E_0 + \sum_{m=1}^M E_m, \quad \text{where } E_0\mathcal{V} := V_0 \text{ and } E_m\mathcal{V} := \sum_{\alpha\in\mathcal{C}_m\setminus 0} P_m V_\alpha(x) \mathrm{e}^{in_\alpha\theta_m}, \tag{1-26}$$

for P_m as in Lemma 1.14.

For $\mathcal{L}(\partial_\theta)$ as in (1-21), we have that, for $\mathcal{V}^0 \in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$,

$$E\mathcal{V}^0 = \mathcal{V}^0 \quad \text{if and only if } \mathcal{V}^0 \in H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M) \text{ and } \mathcal{L}(\partial_\theta)\mathcal{V}^0 = 0, \tag{1-27}$$

and (1-27) in turn is equivalent to the property that $\mathcal{V}^0$ has an expansion of the form

$$\mathcal{V}^0 = \underline{v}(x) + \sum_{m=1}^M \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}(x, \theta_m) r_{m,k}, \tag{1-28}$$

[9] The continuity of E is shown in [Coulombel et al. 2011, Remark 2.5].

for some real-valued functions $\sigma_{m,k}$. Moreover, since for any m,

$$L(d\phi_m) = \underline{\omega}_m I + \sum_{j=0}^{d-1} \beta_j A_j = \sum_{k \neq m} (\underline{\omega}_m - \underline{\omega}_k) P_k, \tag{1-29}$$

we have, for $\mathcal{V} \in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$,

$$E\mathcal{L}(\partial_\theta)\mathcal{V} = \mathcal{L}(\partial_\theta)E\mathcal{V} = 0. \tag{1-30}$$

We also need to introduce a partial inverse R for $\mathcal{L}(\partial_\theta)$. We begin by defining

$$R_m := \sum_{k \neq m} \frac{1}{\underline{\omega}_m - \underline{\omega}_k} P_k,$$

which in view of (1-29) satisfies

$$L(d\phi_m) R_m = R_m L(d\phi_m) = I - P_m. \tag{1-31}$$

The operator R is defined formally at first on functions

$$\mathcal{V}(x, \theta) = \sum_{\alpha \in \mathbb{Z}^{M;2}} V_\alpha(x) e^{i\alpha\cdot\theta} \text{ of } H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$$

by

$$R\mathcal{V} := \sum_{\alpha \in \mathbb{Z}^{M;2}} R(\alpha) V_\alpha(x) e^{i\alpha\cdot\theta} \tag{1-32}$$

where

$$R(\alpha) := \begin{cases} R_m/(in_\alpha) & \text{if } \alpha \in \mathscr{C}_m \setminus \{0\}, \\ 0 & \text{if } \alpha = 0, \\ \mathcal{L}(i\alpha)^{-1} & \text{if } \alpha \notin \mathscr{C}, \end{cases} \tag{1-33}$$

and

$$\mathcal{L}(i\alpha) := i \sum_{m=1}^{M} \alpha_m L(d\phi_m) = iL(d(\alpha \cdot \phi)).$$

Remark 1.20. The operator R is well-defined on functions $\mathcal{V} \in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$ whose spectrum contains only finitely many noncharacteristic modes, and then $R\mathcal{V}$ lies in the same space. Otherwise, there can be a problem with small divisors; the possibility of there being infinitely many noncharacteristic modes α for which $\det L(d(\alpha \cdot \phi))$ is close to zero can prevent convergence of (1-32) in $H^{t;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$ for any t.

It follows readily from (1-31) that, for $\mathcal{F} \in H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$, $s > 0$,

$$\mathcal{L}(\partial_\theta) R\mathcal{F} = R\mathcal{L}(\partial_\theta)\mathcal{F} = (I - E)\mathcal{F}. \tag{1-34}$$

Such $\mathcal{F}$ have no noncharacteristic modes. Along with (1-30), (1-34) implies the following.

Proposition 1.21. *Suppose $\mathcal{F} \in H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$, $s \geq 0$. Then the equation $\mathcal{L}(\partial_\theta)\mathcal{V} = \mathcal{F}$ has a solution $\mathcal{V} \in H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$ if and only if $E\mathcal{F} = 0$.*

By applying the operators E and R to the equations (1-20) and using (1-27), (1-30), and (1-34), we obtain

$$\begin{aligned}
&\text{(a)} \quad E\mathcal{V}^0 = \mathcal{V}^0,\\
&\text{(b)} \quad E(L(\partial)\mathcal{V}^0 + D(0)\mathcal{V}^0) = 0,\\
&\text{(c)} \quad B\mathcal{V}^0 = 0 \quad \text{on } x_d = 0, \theta = (\theta_0, \dots, \theta_0),\\
&\text{(d)} \quad \mathcal{V}^0 = 0 \quad \text{in } t < 0
\end{aligned} \tag{1-35}$$

and

$$\begin{aligned}
&\text{(a)} \quad (I - E)\mathcal{V}^1 + R(L(\partial)\mathcal{V}^0 + D(0)\mathcal{V}^0) = 0,\\
&\text{(b)} \quad E\left(L(\partial)\mathcal{V}^1 + D(0)\mathcal{V}^1 + (dD(0)\mathcal{V}^0)\mathcal{V}^0\right) = 0,\\
&\text{(c)} \quad B\mathcal{V}^1 + (d\psi(0)\mathcal{V}^0)\mathcal{V}^0 = G \quad \text{on } x_d = 0, \theta = (\theta_0, \dots, \theta_0),\\
&\text{(d)} \quad \mathcal{V}^1 = 0 \quad \text{in } t < 0.
\end{aligned} \tag{1-36}$$

Remark 1.22. (a) Since $E\mathcal{V}^0 = \mathcal{V}^0$, the function $L(\partial)\mathcal{V}^0 + D(0)\mathcal{V}^0$ in (1-36)(a) has *no* noncharacteristic modes so the action of R on this function is well-defined.

(b) It is easy to check that functions $\mathcal{V}^0$, $\mathcal{V}^1$ belonging to $H^{s;1}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$, $s > (d+3)/2$, and satisfying (1-35) and (1-36)(a) also satisfy (1-20)(a)–(b) and (1-22). Equation (1-36)(b) and Proposition 1.21 suggest that we might obtain a solution of (1-20)(c) by taking

$$(I - E)\mathcal{V}^2 = -R(L(\partial)\mathcal{V}^1 + D(0)\mathcal{V}^1 + (dD(0)\mathcal{V}^0)\mathcal{V}^0).$$

There are two problems with this. First, the quadratic term $(\mathrm{d}D(0)\mathcal{V}^0)\mathcal{V}^0$ generally has *infinitely* many noncharacteristic modes, so one should expect a problem with small divisors. Second, the statement (1-34) and Proposition 1.21 are both *not* true when $\mathcal{F} \in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$, even if $\mathcal{F}$ has finitely many noncharacteristic modes.[10] These difficulties affect the error analysis and are discussed further in Section 1E.

To determine the equations satisfied by the individual profiles $\underline{v}(x)$, $\sigma_{m,k}(x, \theta_m)$ in the expansion (1-28) of $\mathcal{V}^0$, we first refine the decomposition of the projector E in (1-26). For each $m \in \{1, \dots, M\}$ we let

$$\ell_{m,k},\ k = 1, \dots, \nu_{k_m}$$

denote a basis of real vectors for the left eigenspace of the real matrix

$$i\mathcal{A}(\beta) = \underline{\tau} A_0 + \sum_{j=1}^{d-1} \underline{\eta}_j A_j \tag{1-37}$$

[10]This is because of the fact that for any $k \in \mathbb{Z} \setminus \{0\}$, there can be many $\alpha \in (\mathscr{C}_m \setminus 0) \cap \mathbb{Z}^{M;2}$ such that $n_\alpha = k$. See the proof of Proposition 1.29.

associated to the eigenvalue $-\underline{\omega}_m$, chosen to satisfy

$$\ell_{m,k}\cdot r_{m',k'}=\begin{cases}1 & \text{if } m=m' \text{ and } k=k',\\ 0 & \text{otherwise.}\end{cases}$$

For $v\in\mathbb{C}^N$ set

$$P_{m,k}v:=(\ell_{m,k}\cdot v)r_{m,k}\quad\text{(no complex conjugation here).}$$

We can now write

$$E=E_0+\sum_{m=1}^{M}\sum_{k=1}^{\nu_{k_m}}E_{m,k},$$

where $E_{m,k}:=P_{m,k}E_m$. When the multiplicity $k=1$, we write E_m instead of $E_{m,1}$ and do similarly for $\ell_{m,k}$, $r_{m,k}$ and so on.

The following lemma, which is a slight variation on a well-known result [Lax 1957], is included for the sake of completeness.

Lemma 1.23. *Suppose $E\mathcal{V}^0=\mathcal{V}^0$ and that $\mathcal{V}^0$ has the expansion* (1-28). *Then*

$$E_{m,k}(L(\partial)\mathcal{V}^0)=(X_{\phi_m}\sigma_{m,k})r_{m,k}$$

where X_{ϕ_m} is the characteristic vector field associated to ϕ_m:[11]

$$X_{\phi_m}:=\partial_d+\sum_{j=0}^{d-1}-\partial_{\xi_j}\omega_m(\beta)\partial_j.$$

Proof. For $\xi'\in\mathcal{H}$ near β, let $-\omega_m(\xi')$ be the eigenvalues $i\mathcal{A}(\xi')$ — see (1-37) — and let $P_m(\xi')$ be the corresponding projectors; these objects depend smoothly on ξ' near β thanks to the analysis of [Métivier 2000]. Differentiate the equation

$$\left(\omega_m(\xi')I+\sum_{j=0}^{d-1}A_j\xi_j\right)P_m(\xi')=0$$

with respect to ξ_j, evaluate at β, and apply P_m on the left to obtain

$$P_mA_jP_m=-\partial_{\xi_j}\omega_m(\beta)P_m,$$

from which the lemma readily follows. □

By Assumption 1.6 we know that the vector space $\ker B\cap\mathbb{E}^s(\beta)$ is one-dimensional; moreover, it admits a real basis because B has real coefficients and $\mathbb{E}^s(\beta)$ has a real basis. This vector space is therefore spanned by some $e\in\mathbb{R}^N\setminus\{0\}$ that we can decompose in a unique way by using Lemma 1.11:

$$\ker B\cap\mathbb{E}^s(\beta)=\operatorname{Span}\{e\},\quad e=\sum_{m\in\mathcal{I}}e_m,\quad P_me_m=e_m. \tag{1-38}$$

[11]The vector field X_{ϕ_m} is a constant multiple of the vector field $\partial_t+\boldsymbol{v}_m\cdot\nabla_{x''}$ computed by Lax for the Cauchy problem, where $\boldsymbol{v}_m$ is the group velocity defined in Definition 1.10.

Each vector e_m in (1-38) has real components. We also know that the vector space $B\mathbb{E}^s(\beta)$ is $(p-1)$-dimensional. We can therefore write it as the kernel of a real linear form:

$$B\mathbb{E}^s(\beta) = \{X \in \mathbb{C}^p,\ b \cdot X = 0\}, \tag{1-39}$$

for a suitable vector $b \in \mathbb{R}^p \setminus \{0\}$.

Any function $\mathcal{V}(x,\theta) \in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$ can be decomposed:

$$\mathcal{V} = \underline{\mathcal{V}} + \mathcal{V}_{\mathrm{inc}} + \mathcal{V}_{\mathrm{out}} + \mathcal{V}_{\mathrm{nonch}} = \underline{\mathcal{V}} + \mathcal{V}^*,$$

where the terms correspond respectively to the parts of the Fourier series (1-23) with $\alpha = 0$, α incoming, α outgoing, and α noncharacteristic.[12]

Proposition 1.24. *Suppose $\mathcal{V}^0 \in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^M)$, $s \geq 1$, is a solution of* (1-35). *Then*

$$\underline{\mathcal{V}}^0 = 0, \quad \mathcal{V}^0_{\mathrm{out}} = 0, \quad \mathcal{V}^0_{\mathrm{nonch}} = 0, \qquad \textit{and so } \mathcal{V}^0 = \mathcal{V}^0_{\mathrm{inc}} = E\mathcal{V}^0_{\mathrm{inc}},$$
$$\mathcal{V}^0(x', 0, \theta_0, \dots, \theta_0) = a(x', \theta_0)e \quad \textit{for some unknown periodic function } a \textit{ with mean } 0.$$

Proof. Since $E\mathcal{V}^0 = \mathcal{V}^0$, we have $\mathcal{V}^0_{\mathrm{nonch}} = 0$. Applying E_0 to problem (1-35), we find that the mean value $\underline{\mathcal{V}}^0$ satisfies the weakly stable boundary problem

$$\begin{aligned} &L(\partial)\underline{\mathcal{V}}^0 + D(0)\underline{\mathcal{V}}^0 = 0, \\ &B\underline{\mathcal{V}}^0 = 0 \quad \text{on } x_d = 0, \\ &\underline{\mathcal{V}}^0 = 0 \qquad \text{in } t < 0. \end{aligned}$$

By the well-posedness result of [Coulombel 2005] we have $\underline{\mathcal{V}}^0 = 0$.

Lemma 1.23 implies that outgoing profiles $\sigma_{m,k}$, $m \in \mathcal{O}$, in the expansion (1-28) of $\mathcal{V}^0$ satisfy problems of the form

$$X_{\phi_m}\sigma_{m,k} + \sum_{k'=1}^{\nu_{k_m}} (\ell_{m,k} \cdot D(0) r_{m,k'})\sigma_{m,k'} = 0,$$
$$\sigma_{m,k} = 0 \quad \text{in } t < 0,$$

where X_{ϕ_m} is an outgoing vector field. Thus $\sigma_{m,k} = 0$ for all $k = 1, \dots, \nu_{k_m}$.

The last statement of Proposition 1.24 follows immediately from the boundary condition in (1-35) and (1-38). ☐

Since $\mathcal{V}^0 = \mathcal{V}^0_{\mathrm{inc}}$, we obtain from (1-36)(a)

$$(I-E)\mathcal{V}^1 = (I-E)\mathcal{V}^1_{\mathrm{inc}} = -R(L(\partial)\mathcal{V}^0 + D(0)\mathcal{V}^0),$$

so

$$\mathcal{V}^1 = \underline{\mathcal{V}}^1 + \mathcal{V}^1_{\mathrm{inc}} + \mathcal{V}^1_{\mathrm{out}} \in H^{s;1}, \quad \text{where } E\mathcal{V}^1_{\mathrm{out}} = \mathcal{V}^1_{\mathrm{out}}.$$

[12] Here we say α is incoming if $\alpha \in \mathscr{C}_m \setminus 0$ for an index m such that ϕ_m is an incoming phase.

Next decompose the boundary condition (1-36)(c):

$$
\begin{aligned}
BE\mathcal{V}^1_{\mathrm{inc}} &= G^* - [(d\psi(0)\mathcal{V}^0)\mathcal{V}^0)]^* - B\mathcal{V}^1_{\mathrm{out}} - B(I-E)\mathcal{V}^1_{\mathrm{inc}} \\
&= G^* - [(d\psi(0)\mathcal{V}^0)\mathcal{V}^0)]^* - B\mathcal{V}^1_{\mathrm{out}} + BR(L(\partial)\mathcal{V}^0 + D(0)\mathcal{V}^0).
\end{aligned}
\tag{1-40}
$$

Remark 1.25. (a) If $\mathcal{V}^1_{\mathrm{out}}|_{x_d=0,\theta_j=\theta_0}$ were known, one could write down a transport equation for $a(x',\theta_0)$ which is determined by the solvability condition for (1-40) implied by (1-39):

$$
b\cdot(G^* - [(d\psi(0)\mathcal{V}^0)\mathcal{V}^0]^* - B\mathcal{V}^1_{\mathrm{out}} + BR(L(\partial)\mathcal{V}^0 + D(0)\mathcal{V}^0)) = 0. \tag{1-41}
$$

However, the presence of the term $E((dD(0)\mathcal{V}^0)\mathcal{V}^0)$ in (1-36)(b) implies that two incoming modes in $\mathcal{V}^0_{\mathrm{inc}}$ (which is still unknown) can resonate to produce an outgoing mode that will affect $\mathcal{V}^1_{\mathrm{out}}$. Thus we do not know $\mathcal{V}^1_{\mathrm{out}}|_{x_d=0,\theta_j=\theta_0}$, and we see that the nonlinear boundary equation (1-41) is coupled to the nonlinear interior equation (1-36).

(b) If the phases are such that an outgoing mode can never be produced by a product of two incoming modes, $\mathcal{V}^1_{\mathrm{out}}$ can be determined from (1-36) to be 0, and one can proceed as in [Coulombel and Guès 2010] to solve for a without having to use Nash–Moser iteration.

The key subsystem to focus on now is (recalling $\mathcal{V}^0 = E\mathcal{V}^0 = \mathcal{V}^0_{\mathrm{inc}}$ and writing with obvious notation $E = E_0 + E_{\mathrm{inc}} + E_{\mathrm{out}}$)

$$
\begin{aligned}
&\text{(a)}\quad E_{\mathrm{inc}}(L(\partial)\mathcal{V}^0_{\mathrm{inc}} + D(0)\mathcal{V}^0_{\mathrm{inc}}) = 0,\\
&\text{(b)}\quad E_{\mathrm{out}}(L(\partial)\mathcal{V}^1_{\mathrm{out}} + D(0)\mathcal{V}^1_{\mathrm{out}} + (dD(0)\mathcal{V}^0_{\mathrm{inc}})\mathcal{V}^0_{\mathrm{inc}}) = 0,\\
&\text{(c)}\quad b\cdot(G^* - [(d\psi(0)\mathcal{V}^0_{\mathrm{inc}})\mathcal{V}^0_{\mathrm{inc}}]^* - B\mathcal{V}^1_{\mathrm{out}} + BR(L(\partial)\mathcal{V}^0_{\mathrm{inc}} + D(0)\mathcal{V}^0_{\mathrm{inc}})) = 0,\\
&\text{(d)}\quad \mathcal{V}^0_{\mathrm{inc}}(x',0,\theta_0,\dots,\theta_0) = a(x',\theta_0)e,
\end{aligned}
\tag{1-42}
$$

where $\mathcal{V}^0_{\mathrm{inc}}$ and $\mathcal{V}^1_{\mathrm{out}}$ both vanish in $t<0$.

A formula for $\mathcal{V}^0_{\mathrm{inc}}$ in terms of $a(x',\theta_0)$ can be determined by solving transport equations using (1-42)(a), and that formula can be plugged into (1-42)(b) to get $\mathcal{V}^1_{\mathrm{out}}$ in terms of a. Thus the subsystem (1-42) can be expressed as a very complicated nonlinear, nonlocal equation for the single unknown a. This is done in Appendix B for a strictly hyperbolic example with only one resonance. However, that is not the way we solve (1-42); instead we solve the subsystem in its above form by iteration. Picard iteration does not work; there is a loss of derivatives from one iterate to the next (because of R), so we use a Nash–Moser scheme. An essential point is to take advantage of the smoothing property of the interaction integrals that pick out resonances in $E_{\mathrm{out}}((dD(0)\mathcal{V}^0_{\mathrm{inc}})\mathcal{V}^0_{\mathrm{inc}})$;[13] that property allows us to get tame estimates in Section 3.

An important tool in solving the subsystem (1-42) is the following result from [Coulombel and Guès 2010], which will allow us to write the boundary equation (1-42)(c) as a transport equation for $a(x',\theta_0)$.

Proposition 1.26 [Coulombel and Guès 2010, Proposition 3.5]. *Let the vectors b and e_m be as in* (1-39) *and* (1-38), *and let $\sigma(\zeta)$ be the function appearing in Assumption 1.6. There exists a nonzero real number*

[13] Interaction integrals are similar to convolution integrals.

κ such that

$$R_m P_m = 0 \quad \textit{for all } m \in \{1, \dots, M\},$$

$$b \cdot B \sum_{m \in \mathcal{I}} R_m A_0 e_m = \kappa \partial_\tau \sigma(\underline{\tau}, \underline{\eta}) \quad \textit{and} \quad \partial_\tau \sigma(\underline{\tau}, \underline{\eta}) = 1,$$

$$b \cdot B \sum_{m \in \mathcal{I}} R_m A_j e_m = \kappa \partial_{\eta_j} \sigma(\underline{\tau}, \underline{\eta}),\ j = 1, \dots, d-1,$$

and thus

$$b \cdot B \sum_{m \in \mathcal{I}} R_m L(\partial) e_m = \kappa \left(\partial_\tau \sigma(\underline{\tau}, \underline{\eta}) \partial_t + \sum_{j=1}^{d-1} \partial_{\eta_j} \sigma(\underline{\tau}, \underline{\eta}) \partial_{x_j} \right) =: X_{\mathrm{Lop}}.$$

Taking note of the denominator in_α in the definition (1-33) of R, we immediately obtain:

Corollary 1.27. *The boundary term $b \cdot BRL(\partial)\mathcal{V}^0_{\mathrm{inc}}$ in* (1-42) *may be written*

$$b \cdot BRL(\partial)\mathcal{V}^0_{\mathrm{inc}} = X_{\mathrm{Lop}} \mathcal{A},$$

where $\mathcal{A}(x', \theta_0)$ is the unique function with mean 0 *in θ_0 such that $\partial_{\theta_0} \mathcal{A} = a$.*

Remark 1.28. Proposition 1.26 shows that propagation in the boundary, which is described by $a(x', \theta_0)$, is governed by the (x-projection of the) Hamiltonian vector field associated to the Lopatinskii determinant. Since $\mathcal{V}^0(x', 0, \theta_0, \dots, \theta_0) = a(x', \theta_0)e$, this shows that waves of amplitude $O(1)$ emanate from the radiating boundary wave defined by a.

After (1-42) is solved, $\mathcal{V}^0$ is known, so $\underline{\mathcal{V}}^1$, $\mathcal{V}^1_{\mathrm{out}}$, and $(I - E)\mathcal{V}^1_{\mathrm{inc}}$ can now be determined by returning to the full system (1-36). The trace of $E\mathcal{V}^1_{\mathrm{inc}}$ is not yet determined; one should make a *choice* of $E\mathcal{V}^1_{\mathrm{inc}}|_{x_d=0, \theta_j=\theta_0}$ such that (1-40) holds, and then solve for $E\mathcal{V}^1_{\mathrm{inc}}$ using (1-36)(b). A precise description of the regularity of $\mathcal{V}^0$ and $\mathcal{V}^1$ is given in Theorem 5.11. The last piece of the corrected approximate solution, $\varepsilon^2 \mathcal{U}^2_p$ in (1-15), is discussed next.

1E. *Error analysis.* Given a periodic function $f(x, \theta)$, where $\theta = (\theta_1, \dots, \theta_M)$, let us denote

$$f(x, \theta)|_{\theta \to (\theta_0, \xi_d)} := f(x, \theta_0 + \underline{\omega}_1 \xi_d, \dots, \theta_0 + \underline{\omega}_M \xi_d);$$

so we have

$$f(x, \theta)|_{\theta \to (\phi_0/\varepsilon, x_d/\varepsilon)} = f\left(x, \frac{\phi}{\varepsilon}\right).$$

Taking the profiles $\mathcal{V}^0$, $\mathcal{V}^1$ constructed in Theorem 5.11, if we define

$$\mathcal{U}^b_\varepsilon(x, \theta_0) := (\mathcal{V}^0(x, \theta) + \varepsilon \mathcal{V}^1(x, \theta))|_{\theta \to (\theta_0, x_d/\varepsilon)},$$

we find that $\mathcal{U}^b_\varepsilon$ satisfies the singular system

$$\begin{aligned} &\text{(a)} \quad \mathbb{L}_\varepsilon(\mathcal{U}^b_\varepsilon) := \partial_d \mathcal{U}^b_\varepsilon + \mathbb{A}\left(\partial_{x'} + \frac{\beta \partial_{\theta_0}}{\varepsilon}\right) \mathcal{U}^b_\varepsilon + D(\varepsilon \mathcal{U}^b_\varepsilon) \mathcal{U}^b_\varepsilon = O(\varepsilon), \\ &\text{(b)} \quad \psi(\varepsilon \mathcal{U}^b_\varepsilon) \mathcal{U}^b_\varepsilon = \varepsilon G(x', \theta_0) + O(\varepsilon^2) \quad \text{on } x_d = 0, \\ &\text{(c)} \quad \mathcal{U}^b_\varepsilon = 0 \quad \text{in } t < 0, \end{aligned} \tag{1-43}$$

where the error terms refer to norms in E^s and H^t spaces whose orders are made precise in Section 4. For example, (1-43) follows directly from the profile equations (1-20)(a)–(b), together with the identity

$$\mathbb{L}_\varepsilon(f(x,\theta)|_{\theta\to(\theta_0,x_d/\varepsilon)}) = \frac{1}{\varepsilon}(\mathcal{L}(\partial_\theta)f(x,\theta))|_{\theta\to(\theta_0,x_d/\varepsilon)} + (L(\partial)f(x,\theta))|_{\theta\to(\theta_0,x_d/\varepsilon)} + (D(\varepsilon f)f)|_{\theta\to(\theta_0,x_d/\varepsilon)}. \quad (1\text{-}44)$$

Since our basic estimate for the linearized singular system exhibits a loss of one singular derivative (basically, we lose a $1/\varepsilon$ factor), the accuracy in (1-43)(a) is not good enough to conclude that

$$|U_\varepsilon - \mathcal{U}^b_\varepsilon|_{L^\infty(x,\theta_0)}$$

is small (the error terms are only $O(\varepsilon)$). Thus, to improve the accuracy, we construct an additional corrector $\mathcal{U}^2_p(x,\theta_0,\xi_d)$ and replace $\mathcal{U}^b_\varepsilon$ by

$$\mathcal{U}_\varepsilon(x,\theta_0) := (\mathcal{V}^0(x,\theta) + \varepsilon\mathcal{V}^1(x,\theta))|_{\theta\to(\theta_0,x_d/\varepsilon)} + \varepsilon^2\mathcal{U}^2_p\left(x,\theta_0,\frac{x_d}{\varepsilon}\right). \quad (1\text{-}45)$$

In constructing $\mathcal{U}^2_p$, we deal with the first (small divisor) problem described in Remark 1.22(b) by approximating $\mathcal{V}^0$ and $\mathcal{V}^1$ by trigonometric polynomials $\mathcal{V}^0_p$ and $\mathcal{V}^1_p$ to within an accuracy $\delta > 0$ in appropriate Sobolev norms, and seek $\mathcal{U}^2_p$ in the form of a trigonometric polynomial.[14] To deal with the second (solvability) problem, we use the following proposition, which allows us to use the profile equation (1-36)(b) as a solvability condition, in spite of the failure of Proposition 1.21 when $\mathcal{F}\in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+\times\mathbb{T}^M)$. We define

$$\mathcal{L}_0(\partial_{\theta_0},\partial_{\xi_d}) := L(d\phi_0)\partial_{\theta_0} + \partial_{\xi_d}.$$

Proposition 1.29. *Suppose $F(x,\theta)\in H^{s;2}(\overline{\mathbb{R}}^{d+1}_+\times\mathbb{T}^M)$ has a Fourier series which is a finite sum and that $EF=0$. Then there exists a solution of the equation*

$$\mathcal{L}_0(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}(x,\theta_0,\xi_d) = F(x,\theta)|_{\theta\to(\theta_0,\xi_d)} \quad (1\text{-}46)$$

in the form of a trigonometric polynomial in (θ_0,ξ_d) of the form

$$\mathcal{U}(x,\theta_0,\xi_d) = \sum_{(\kappa_0,\kappa_d)\in\mathcal{J}} U_{\kappa_0,\kappa_d}(x)e^{i\kappa_0\theta_0+i\kappa_d\xi_d}, \quad (1\text{-}47)$$

where $\mathcal{J}$ is a finite subset of $\mathbb{Z}\times\mathbb{R}$ and the coefficients U_{κ_0,κ_d} lie in $H^s(\overline{\mathbb{R}}^{d+1}_+)$.

The proof is given in Section 4. Observe that $\mathcal{U}$ is periodic in θ_0 but almost periodic in (θ_0,ξ_d). Proposition 1.29 is applied to solve the equation

$$\mathcal{L}_0(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}^2_p = [-(I-E)(L(\partial)\mathcal{V}^1_p + D(0)\mathcal{V}^1_p + (\mathrm{d}D(0)\mathcal{V}^0_p)\mathcal{V}^0_p)]|_{\theta\to(\theta_0,\xi_d)}.$$

[14] Trigonometric polynomial approximations were already used to deal with small divisor problems in the error analysis of [Joly et al. 1995].

With this choice of $\mathcal{U}_p^2$ we show in Section 4 that the new approximate solution $\mathcal{U}_\varepsilon(x, \theta_0)$ in (1-45) satisfies instead of (1-43) the singular system

$$\begin{aligned} &\text{(a)} \quad \mathbb{L}_\varepsilon(\mathcal{U}_\varepsilon) = O(\varepsilon(K\delta + C(\delta)\varepsilon)), \\ &\text{(b)} \quad \psi(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon - \varepsilon G(x', \theta_0) = O(\varepsilon^2 C(\delta)) \quad \text{on } x_d = 0, \\ &\text{(c)} \quad \mathcal{U}_\varepsilon = 0 \quad \text{in } t < 0, \end{aligned} \tag{1-48}$$

where the errors in (1-48)(a)–(b) are measured in appropriate norms. Now one can apply our basic estimate (2-41) for the linearized singular problem to conclude that the difference between exact and approximate solutions of the semilinear singular system (1-18) satisfies, for some constants $C(\delta)$ and K,

$$|U_\varepsilon(x, \theta_0) - \mathcal{U}_\varepsilon(x, \theta_0)|_{E^s} \le K\delta + C(\delta)\varepsilon, \quad \text{for some } s > \frac{d+1}{2}.$$

This estimate clearly implies the conclusion of Theorem 1.15 by choosing first $\delta > 0$ small enough and then letting ε tend to zero (this is the same final argument as in [Joly et al. 1995]).

1F. ***Remarks on quasilinear problems.*** In this article, we are able to rigorously justify a weakly nonlinear regime with amplification for *semilinear* hyperbolic initial boundary value problems. Our assumptions only deal with the principal part of the operators, meaning that we only assume a weak stability property for the problem $(L(\partial), B)$ obtained by linearizing at the origin and dropping the zero-order term in the hyperbolic system. The weak stability is of WR type in the terminology of [Benzoni-Gavage et al. 2002]. Despite the weak regime that we consider ($O(\varepsilon^2)$ source term at the boundary and $O(\varepsilon)$ solution), the leading profile equation displays some *quasilinear* features. We emphasize that the regime that we consider here is exactly one power of ε weaker than the weakly nonlinear regime for the semilinear Cauchy problem or for semilinear uniformly stable boundary value problems. As in [Coulombel and Guès 2010], this power of ε corresponds exactly to the loss of one derivative in the energy estimates.

We believe that the techniques developed here can be extended to give a rigorous justification of weakly nonlinear geometric optics with amplification for *quasilinear* hyperbolic initial boundary value problems of the form

$$\partial_t v + \sum_{j=1}^d B_j(v)\partial_j v + f_0(v) = 0, \tag{1-49}$$

$$\phi(v) = \varepsilon^3 G\left(x', \frac{x' \cdot \beta}{\varepsilon}\right) \qquad \text{on } x_d = 0, \tag{1-50}$$

$$v = 0 \quad \text{and} \quad G = 0 \qquad \text{in } t < 0. \tag{1-51}$$

The corresponding solution v_ε would be of amplitude $O(\varepsilon^2)$. In particular the arguments used in Section 2 to obtain uniform estimates with a loss of one singular derivative for the singular initial boundary value problem might be extended to the corresponding singular quasilinear problem. There are however several new obstacles along the way, one of which is to extend the singular pseudodifferential calculus of [Coulombel et al. 2012] in order to obtain a two-terms expansion of $(1, 0)$ and $(0, 1)$ compositions. The weaker scaling (ε^2 in place of ε) should be sufficient to obtain the appropriate results. Let us observe that, for $O(\varepsilon^2)$ solutions, the principal part of the hyperbolic operator has coefficients that are uniformly bounded in $W^{2,\infty}$, which is precisely the regularity needed in [Coulombel 2004; 2005] to obtain a priori estimates and well-posedness. The leading profile equation obtained in this quasilinear framework is very similar to the one we have derived here, and we thus believe that a weak well-posedness result using Nash–Moser iteration should prove the existence of the leading profile. For all the above reasons, we thus believe that the ε^3 source term on the boundary is the relevant "weakly nonlinear regime with amplification" in the quasilinear case, and we postpone the verification of the many technical details to a future work. Unfortunately, this regime would still be beyond the one considered in [Artola and Majda 1987; Majda and Rosales 1983], so there would still be a new ingredient to incorporate in order to justify the calculations of these papers.

2. Exact oscillatory solutions on a fixed time interval

2A. ***The basic estimate for the linearized singular system.*** In this section, it is our goal to prove Proposition 2.2 and its time-localized version, that is, Proposition 2.9. These propositions provide the a priori estimates for the linearized singular system that form the basis for the Nash–Moser iteration of Section 5B and the error analysis of Section 4.

We begin by gathering some of the notation for spaces and norms that is needed below.

Notation 2.1. Here we take $s \in \mathbb{N} = \{0, 1, 2, \dots\}$.

(a) Let $\Omega := \overline{\mathbb{R}}^{d+1}_+ \times \mathbb{T}^1$, $\Omega_T := \Omega \cap \{-\infty < t < T\}$, $b\Omega := \mathbb{R}^d \times \mathbb{T}^1$, $b\Omega_T := b\Omega \cap \{-\infty < t < T\}$, and set $\omega_T := \overline{\mathbb{R}}^{d+1}_+ \cap \{-\infty < t < T\}$.

(b) Let $H^s \equiv H^s(b\Omega)$, the standard Sobolev space with norm $\langle V(x', \theta_0)\rangle_s$. For $\gamma \geq 1$ we set $H^s_\gamma := e^{\gamma t} H^s$ and $\langle V\rangle_{s,\gamma} := \langle e^{-\gamma t} V\rangle_s$.

(c) $L^2 H^s \equiv L^2(\overline{\mathbb{R}}_+, H^s(b\Omega))$ with norm $|U(x, \theta_0)|_{L^2 H^s} \equiv |U|_{0,s}$ given by

$$|U|^2_{0,s} = \int_0^\infty |U(x', x_d, \theta_0)|^2_{H^s(b\Omega)} dx_d.$$

The corresponding norm on $L^2 H^s_\gamma$ is denoted by $|V|_{0,s,\gamma}$.

(d) $CH^s \equiv C(\overline{\mathbb{R}}_+, H^s(b\Omega))$ denotes the space of continuous bounded functions of x_d with values in $H^s(b\Omega)$, with norm

$$|U(x, \theta_0)|_{CH^s} = |U|_{\infty,s} := \sup_{x_d \geq 0} |U(., x_d, .)|_{H^s(b\Omega_T)}$$

(note that $CH^s \subset L^\infty H^s$). The corresponding norm on CH^s_γ is denoted by $|V|_{\infty,s,\gamma}$.

(e) Let $M_0 := 3d+5$ and define $C^{0,M_0} := C(\overline{\mathbb{R}}_+, C^{M_0}(b\Omega))$ as the space of continuous bounded functions of x_d with values in $C^{M_0}(b\Omega)$, with norm $|U(x, \theta_0)|_{C^{0,M_0}} := |U|_{L^\infty W^{M_0,\infty}}$. Here $L^\infty W^{M_0,\infty}$ denotes the space $L^\infty(\overline{\mathbb{R}}_+; W^{M_0,\infty}(b\Omega))$.[15]

[15] The size of M_0 is determined by the requirements of the singular calculus described in Appendix A.

(f) The corresponding spaces on Ω_T are denoted by $L^2H^s_T$, $L^2H^s_{\gamma,T}$, CH^s_T, $CH^s_{\gamma,T}$ and C^{0,M_0}_T with norms $|U|_{0,s,T}$, $|U|_{0,s,\gamma,T}$, $|U|_{\infty,s,T}$, $|U|_{\infty,s,\gamma,T}$, and $|U|_{C^{0,M_0}_T}$, respectively. On $b\Omega_T$ we use the spaces H^s_T and $H^s_{\gamma,T}$ with norms $\langle U\rangle_{s,T}$ and $\langle U\rangle_{s,\gamma,T}$.

(g) All constants appearing in the estimates below are independent of ε, γ, and T unless such dependence is explicitly noted.

The linearization of the singular problem (1-18) at $U(x,\theta_0)$ has the form

$$\begin{aligned}
&\text{(a)}\quad \partial_d \dot U_\varepsilon + \mathbb{A}\left(\partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right)\dot U_\varepsilon + \mathcal{D}(\varepsilon U)\dot U_\varepsilon = f(x,\theta_0) \quad \text{on } \Omega,\\
&\text{(b)}\quad \mathcal{B}(\varepsilon U)\dot U_\varepsilon|_{x_d=0} = g(x',\theta_0),\\
&\text{(c)}\quad \dot U_\varepsilon = 0 \quad \text{in } t<0,
\end{aligned}\tag{2-1}$$

where the matrices $\mathcal{B}(\varepsilon U)$, $\mathcal{D}(\varepsilon U)$ are defined in (1-11).[16] Instead of (2-1), consider the equivalent problem satisfied by $\dot U^\gamma := e^{-\gamma t}\dot U$:

$$\begin{aligned}
&\partial_d \dot U^\gamma + \mathbb{A}\left((\partial_t+\gamma, \partial_{x''}) + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right)\dot U^\gamma + \mathcal{D}(\varepsilon U)\dot U^\gamma = f^\gamma(x,\theta_0),\\
&\mathcal{B}(\varepsilon U)\dot U^\gamma|_{x_d=0} = g^\gamma(x',\theta_0),\\
&\dot U^\gamma = 0 \quad \text{in } t<0.
\end{aligned}\tag{2-2}$$

Below we let Λ_D denote the singular Fourier multiplier (see (A-2)) associated to the symbol

$$\Lambda(X,\gamma) := \left(\gamma^2 + \left|\xi' + \frac{k\beta}{\varepsilon}\right|^2\right)^{1/2}, \qquad X := \xi' + \frac{k\beta}{\varepsilon}. \tag{2-3}$$

The basic estimate for the linearized singular problem (2-2) is given in the next proposition. Observe that the estimate (2-4) exhibits a loss of one "singular derivative" Λ_D. In view of [Coulombel and Guès 2010, Theorem 4.1], there is strong evidence that the loss below is optimal.

Proposition 2.2 (main L^2 linear estimate). *We make the structural assumptions of Theorem 1.15 and recall $M_0 = 3d+5$. Fix $K>0$ and suppose $|\varepsilon\partial_d U|_{C^{0,M_0-1}} + |U|_{C^{0,M_0}} \le K$ for $\varepsilon\in(0,1]$. There exist positive constants $\varepsilon_0(K)>0$, $C(K)>0$, and $\gamma_0(K)\ge 1$ such that sufficiently smooth solutions $\dot U$ of the linearized singular problem* (2-1) *satisfy*[17]

$$|\dot U^\gamma|_{0,0} + \frac{\langle \dot U^\gamma\rangle_0}{\sqrt{\gamma}} \le C(K)\left(\frac{|\Lambda_D f^\gamma|_{0,0} + |\varepsilon^{-1} f^\gamma|_{0,0}}{\gamma^2} + \frac{\langle\Lambda_D g^\gamma\rangle_0 + \langle\varepsilon^{-1}g^\gamma\rangle_0}{\gamma^{3/2}}\right), \tag{2-4}$$

for $\gamma\ge\gamma_0(K)$, $0<\varepsilon\le\varepsilon_0(K)$.

The same estimate holds if $\mathcal{B}(\varepsilon U)$ in (2-1) *is replaced by $\mathcal{B}(\varepsilon U, \varepsilon\mathcal{U})$ and $\mathcal{D}(\varepsilon U)$ is replaced by $\mathcal{D}(\varepsilon U,\varepsilon\mathcal{U})$, as long as $|\varepsilon\partial_d(U,\mathcal{U})|_{C^{0,M_0-1}} + |U,\mathcal{U}|_{C^{0,M_0}} \le K$ for $\varepsilon\in(0,1]$.*

[16] Here and below we often suppress the subscript ε on $\dot U$.

[17] Note that the norms $|u|_{0,1}$ and $|\Lambda_D u|_{0,0}$ are not equivalent.

Corollary 2.3 (main H^1_{tan} linear estimate). *Under the same assumptions as in Proposition 2.2, smooth enough solutions $\dot{U}$ of the linearized singular problem* (2-1) *satisfy*

$$|\dot{U}^\gamma|_{\infty,0}+|\dot{U}^\gamma|_{0,1}+\frac{\langle\dot{U}^\gamma\rangle_1}{\sqrt{\gamma}}\leq C(K)\left(\frac{|\Lambda_D f^\gamma|_{0,1}+|\varepsilon^{-1}f^\gamma|_{0,1}}{\gamma^2}+\frac{\langle\Lambda_D g^\gamma\rangle_1+\langle\varepsilon^{-1}g^\gamma\rangle_1}{\gamma^{3/2}}\right),\tag{2-5}$$

for $\gamma\geq\gamma_0(K)$, $0<\varepsilon\leq\varepsilon_0(K)$.

Short guide to the proof. The proof of Proposition 2.2 is completed using the next two propositions, each of which has the same hypotheses as Proposition 2.2. In the first step of the proof of Proposition 2.2, we choose a partition of unity defined by frequency cutoffs $\chi_i(\zeta)$, $i=1,\dots,N_1+N_2$, such that for $i=1,\dots,N_1$ the function χ_i is supported near a point of the "bad" set $\overline{\Upsilon}$, while for $i>N_1$ the function χ_i is supported away from $\overline{\Upsilon}$. The estimates of $\chi_{i,D}\dot{U}^\gamma$ for $i>N_1$ are done in Proposition 2.8. For such indices, Kreiss symmetrizers in the singular calculus are used to estimate $\chi_{i,D}\dot{U}^\gamma$ without loss.

Proof of Proposition 2.2. (I): *Partition of unity*. The compactness of $\overline{\Upsilon}$ (see Assumption 1.12) and Σ allows us to choose a finite open covering of Σ, $\mathscr{C}=\{\mathscr{V}_i\}_{i=1,\dots,N_1+N_2}$ such that $\{\mathscr{V}_i\}_{i=1,\dots,N_1}$ covers $\overline{\Upsilon}$ and such that $\bigcup_{N_1+1}^{N_1+N_2}\mathscr{V}_i$ is disjoint from a neighborhood of $\overline{\Upsilon}$. Since $\overline{\Upsilon}\subset\mathscr{H}$, we can arrange so that for each $i\in\{1,\dots,N_1\}$ there is a conjugator $Q_{0,i}(\zeta)$[18] and diagonal matrix $\mathbb{D}_{1,i}(\zeta)$ satisfying (1-8) in $\mathscr{V}_i$. Moreover, we can choose a neighborhood $\mathbb{O}$ of $(0,0)\in\mathbb{R}^{2N}$ such that for each $i\leq N_1$ there are functions σ_i, $P_{i,1}$, and $P_{i,2}$ on $\mathbb{O}\times\mathscr{V}_i$ with the properties described in Assumption 1.12. For these symbols, we shall use the substitution $(v_1,v_2)\to(\varepsilon U(x,\theta_0),\varepsilon U(x,\theta_0))$ to prescribe the space dependence.[19]

We let $\chi_i(\zeta)$, $i=1,\dots,N_1+N_2$ be a smooth partition of unity subordinate to $\mathscr{C}$, and extend the χ_i to all ζ as functions homogeneous of degree zero. We smoothly extend each $Q_{0,i}$ (as a matrix with bounded inverse) first to Σ, and then to all ζ as a function homogenous of degree zero. We take similar extensions in ζ of $P_{i,1}$, $P_{i,2}$, $\mathbb{D}_{1,i}$, and σ_i, but with homogeneity of degree 1 in the cases of $\mathbb{D}_{1,i}$ and σ_i. As with $Q_{0,i}$, the extensions of $P_{i,1}$ and $P_{i,2}$ are taken to have bounded inverses.[20] Of course, for a given $i\leq N_1$, the property (1-12) is satisfied only for $\zeta/|\zeta|\in\mathscr{V}_i$.

(II): *Estimate near the bad set*. The first estimate deals with a piece of $\dot{U}^\gamma$ that is microlocalized near the bad set Υ.

Proposition 2.4. *Fix i such that $1\leq i\leq N_1$, let $\dot{U}_1^\gamma:=\chi_{i,D}\dot{U}^\gamma$ and write*

$$\dot{U}_1^\gamma=\dot{U}_{1,\text{in}}^\gamma+\dot{U}_{1,\text{out}}^\gamma,$$

where[21]

$$\dot{U}_{1,\text{in}}^\gamma:=(Q_D)^{-1}(w_{\text{in}},0)\quad\textit{and}\quad\dot{U}_{1,\text{out}}^\gamma:=(Q_D)^{-1}(0,w_{\text{out}}).$$

[18] Recall the notation $\zeta=(\tau-i\gamma,\eta)$. Sometimes we also write $\zeta=(\xi',\gamma)$ to match the notation of [Coulombel et al. 2012].

[19] The substitution $(v_1,v_2)\to(\varepsilon U(x,\theta_0),\varepsilon\mathscr{U}(x,\theta_0))$ is also used at one point.

[20] Taking such extensions reduces the number of cutoff functions we need later.

[21] Here Q_D, $w_{\text{in}}\in\mathbb{C}^p$, and $w_{\text{out}}\in\mathbb{C}^{N-p}$ are defined by the diagonalization procedure explained in the proof.

Then we have

$$\begin{aligned}|\dot U^\gamma_{1,\mathrm{in}}|_{0,0}+\frac{|\dot U^\gamma_{1,\mathrm{in}}|_{\infty,0}}{\sqrt{\gamma}}+\frac{|(\Lambda_D,\varepsilon^{-1})\dot U^\gamma_{1,\mathrm{out}}|_{0,0}}{\gamma}+\frac{|(\Lambda_D,\varepsilon^{-1})\dot U^\gamma_{1,out}|_{\infty,0}}{\gamma^{3/2}}\\ +|(\varepsilon\Lambda_D)^{-1}\dot U^\gamma_{1,\mathrm{in}}|_{0,0}+\frac{\langle(\varepsilon\Lambda_D)^{-1}\dot U^\gamma_{1,\mathrm{in}}|_{x_d=0}\rangle_0}{\sqrt{\gamma}}\\ \le C\biggl(\frac{|(\Lambda_D,\varepsilon^{-1})f^\gamma|_{0,0}}{\gamma^2}+\frac{\langle(\Lambda_D,\varepsilon^{-1})g^\gamma\rangle_0}{\gamma^{3/2}}+\frac{|\dot U^\gamma|_{0,0}+|(\varepsilon\Lambda_D)^{-1}\dot U^\gamma|_{0,0}}{\gamma^2}\\ +\frac{\langle\dot U^\gamma|_{x_d=0}\rangle_0+\langle(\varepsilon\Lambda_D)^{-1}\dot U^\gamma|_{x_d=0}\rangle_0}{\gamma^{3/2}}\biggr).\end{aligned}\tag{2-6}$$

Proof of Proposition 2.4. The loss of derivatives in the estimate prevents us from treating the zero-order term $\mathcal{D}(\varepsilon U)\dot U^\gamma$ as a forcing term, as we would in a uniformly stable problem. Thus we need to use an argument that simultaneously diagonalizes $\mathbb{A}$ and the lower-order term $\mathcal{D}(\varepsilon U)$.

We now set $\chi_i=\boldsymbol{\chi}$, $v:=\boldsymbol{\chi}_D\dot U^\gamma=\dot U^\gamma_1$, and estimate v. We let $\mathbb{A}(X,\gamma)=-\mathcal{A}(X,\gamma)$ denote the singular symbol such that

$$\mathbb{A}_D=\mathbb{A}\biggl((\partial_t+\gamma,\partial_{x''})+\frac{\beta\partial_{\theta_0}}{\varepsilon}\biggr).$$

Dropping superscripts γ, we see from (2-2) that v satisfies

$$\begin{aligned}\partial_d v+\mathbb{A}_D v+\mathcal{D}(\varepsilon U)v&=\boldsymbol{\chi}_D f+[\mathcal{D}(\varepsilon U),\boldsymbol{\chi}_D]\dot U=\boldsymbol{\chi}_D f+r_{-1,D}\dot U,\\ \mathcal{B}(\varepsilon U)v|_{x_d=0}&=\boldsymbol{\chi}_D g+[\mathcal{B}(\varepsilon U),\boldsymbol{\chi}_D]\dot U|_{x_d=0}=\boldsymbol{\chi}_D g+r_{-1,D}\dot U|_{x_d=0}.\end{aligned}\tag{2-7}$$

Here and below $r_{-1,D}$ denotes a singular operator of order -1 (which can change from one occurrence to the next) computed using the singular calculus. Similarly, $r_{0,D}$ will denote an operator of order 0. In spite of the loss of the factor Λ_D in the estimate (2-4), we are able to treat $r_{-1,D}\dot U$ as a forcing term (see, for example, (2-16) below). A term like $r_{0,D}\dot U/\gamma$ would be too large to absorb.

The first several steps of the proof estimate the terms in the first line of (2-6).

Step 1: *Simultaneous diagonalization.* This diagonalization argument is similar to the one in [Coulombel 2004]. Let $Q_0(\zeta):=Q_{0,i}(\zeta)$ and $\mathbb{D}_1(\zeta):=\mathbb{D}_{1,i}(\zeta)$ be the matrices as in (1-8) such that

$$Q_0(\zeta)\mathbb{A}(\zeta)Q_0^{-1}(\zeta)=\mathbb{D}_1(\zeta)$$

in the conical extension of $\mathcal{V}_i$. We define

$$w:=Q_D v,$$

where $Q=Q_0(X,\gamma)+Q_{-1}(\varepsilon U,X,\gamma)$. Here the matrix $Q_{-1}(\varepsilon U,\zeta)$ is a symbol of order -1 defined for all ζ, but chosen so that, on the conical extension of $\mathcal{V}_i$, the matrix

$$\mathbb{D}_0(\varepsilon U,\zeta):=[Q_{-1}Q_0^{-1},\mathbb{D}_1]+Q_0\mathcal{D}(\varepsilon U)Q_0^{-1}\tag{2-8}$$

is block diagonal, necessarily of order 0, with blocks of the same dimensions $n_1,\dots,n_J$ as those of $\mathbb{D}_1$. Since the eigenvalues associated to the blocks of $\mathbb{D}_1$ are mutually distinct, a direct computation shows that $Q_{-1}Q_0^{-1}$, and thus Q_{-1}, can be chosen so that the commutator cancels the off-diagonal blocks of

$Q_0\mathcal{D}(\varepsilon U)Q_0^{-1}$. (The diagonal blocks of the commutator are all zero blocks and therefore cannot cancel those of $Q_0\mathcal{D}(\varepsilon U)Q_0^{-1}$.) Since $Q_0\mathbb{A}=\mathbb{D}_1Q_0$ on $\mathcal{V}_i$, (2-8) implies the relation

$$Q\mathbb{A}+Q_0\mathcal{D}=\mathbb{D}_1Q+[Q_{-1}Q_0^{-1},\mathbb{D}_1]Q_0+Q_0\mathcal{D}=\mathbb{D}_1Q+\mathbb{D}_0Q_0. \tag{2-9}$$

Remark 2.5. (1) The scalar entries of the matrix $Q_{-1,D}$ can be chosen to have the form

$$(Q_{-1,D})_{i,j}=c(\varepsilon U)a_{-1,D},$$

where $a_{-1}(\zeta)$ is of order -1 and independent of (x,θ), thus giving rise to a Fourier multiplier.

(2) Since $(Q_{0,D})^{-1}Q_{-1,D}$ has norm less than one as an operator on L^2 for γ large, we can define $(Q_D)^{-1}$ as an operator on L^2 using a Neumann series.

Noting that x-dependence is absent in $\mathbb{A}$ and Q_0 and using the commutation property (2-9), we have

$$\begin{aligned}\partial_d w&=Q_D\partial_d v+(\partial_d Q_{-1})_D v=-Q_D(\mathbb{A}+\mathcal{D}(\varepsilon U))_D v+Q_D\chi_D f+r_{-1,D}\dot U+(\partial_d Q_{-1})_D v\\&=-(Q\mathbb{A}+Q_0\mathcal{D}(\varepsilon U))_D v+Q_D\chi_D f+r_{-1,D}\dot U+(\partial_d Q_{-1})_D v\\&=-(\mathbb{D}_1Q+\mathbb{D}_0Q_0)_D v+Q_D\chi_D f+r_{-1,D}\dot U+(\partial_d Q_{-1})_D v\\&=-(\mathbb{D}_1+\mathbb{D}_0)_D w+r_{0,D}f+r_{-1,D}\dot U+R_D^a v.\end{aligned} \tag{2-10}$$

In the final line of (2-10), the operator $r_{-1,D}$ is explicitly given by

$$Q_D[\mathcal{D}(\varepsilon U),\chi_D]\dot U-(Q_{0,D}\mathcal{D}(\varepsilon U)-(Q_0\mathcal{D}(\varepsilon U))_D)\chi_D\dot U-Q_{-1,D}\mathcal{D}(\varepsilon U)\chi_D\dot U+\mathbb{D}_{0,D}Q_{-1,D}\chi_D\dot U, \tag{2-11}$$

and the second remainder term is decomposed as $R_D^a=R_D^b+R_D^c$ with operators R_D^b, R_D^c defined by

$$\begin{aligned}&\text{(a)}\quad R_D^b v:=(\partial_d Q_{-1})_D v,\\&\text{(b)}\quad R_D^c v:=\mathbb{D}_{1,D}(Q_{-1})_D v-(\mathbb{D}_1Q_{-1})_D v.\end{aligned} \tag{2-12}$$

In view of Remark 2.5 the scalar entries of R_D^b and R_D^c have the form

$$(\partial_d c(\varepsilon U))a_{-1,D}\quad\text{and}\quad[\alpha_{1,D},c(\varepsilon U)]a_{-1,D}, \tag{2-13}$$

respectively. In (2-13), $\alpha_1(\zeta)$ denotes one of the diagonal entries of $\mathbb{D}_1(\zeta)$. Here and below $a_{-1,D}$ denotes a singular operator of order -1 associated to a symbol $a_{-1}(\zeta)$ which may change from term to term.

The precise estimate of the above remainder terms is one of the keys to the proof of Proposition 2.4.

Lemma 2.6. *The remainder terms $r_{-1,D}\dot U$ and $R_D^a v$ in the last line of* (2-10) *satisfy estimates of the form*

$$\begin{aligned}&|r_{-1,D}\dot U|_{0,0}\le C(K)|\Lambda_D^{-1}\dot U|_{0,0},\quad &&|R_D^a v|_{0,0}\le C(K)|\Lambda_D^{-1}v|_{0,0},\\&|\Lambda_D r_{-1,D}\dot U|_{0,0}\le C(K)|\dot U|_{0,0},\quad &&|\Lambda_D R_D^a v|_{0,0}\le C(K)(|v|_{0,0}+|(\varepsilon\Lambda_D)^{-1}v|_{0,0}),\end{aligned}$$

with a constant $C(K)$ that is uniform with respect to ε and γ.

Proof of Lemma 2.6. • The estimate of $R_D^a v$ in L^2 comes from the expression (2-13) of the coefficients of R_D^b and R_D^c. For instance, the commutator $[\alpha_{1,D},c(\varepsilon U)]$ is bounded on L^2 uniformly on ε,γ (see

Appendix A), and we can isolate a Fourier multiplier $a_{-1,D}$ on the right. In particular, we obtain the weaker estimate

$$\gamma |R_D^a v|_{0,0} \leq C |v|_{0,0},$$

and we are now going to estimate the singular derivative $(\partial_{x'} + \beta \partial_{\theta_0}/\varepsilon) R_D^a v$. Let us deal with the operator R_D^c (the estimate involving R_D^b is similar). When applying the singular derivative, we need to estimate terms of the form

$$[\alpha_{1,D}, c(\varepsilon U)] \left(\partial_{x_j} + \frac{\beta_j \partial_{\theta_0}}{\varepsilon} \right) a_{-1,D} v + [\alpha_{1,D}, \partial_{x_j} c(\varepsilon U)] a_{-1,D} v + \frac{\beta_j}{\varepsilon} [\alpha_{1,D}, \partial_{\theta_0} c(\varepsilon U)] a_{-1,D} v.$$

The first term is estimated by v in L^2, while the second and, above all, the third term are estimated by $(\varepsilon \Lambda_D)^{-1} v$ in L^2.

- The estimate of $\Lambda_D r_{-1,D} \dot{U}$ is precisely the definition of the notation $r_{-1,D}$ and it follows from the rules of symbolic calculus; see Appendix A. We thus focus on the L^2 estimate of the remainder where we wish to gain a factor Λ_D^{-1} rather than a mere $1/\gamma$. Let us first consider the term $Q_{-1,D} \mathscr{D}(\varepsilon U) \chi_D \dot{U}$ in (2-11). We write

$$Q_{-1,D} \mathscr{D}(\varepsilon U) \chi_D \dot{U} = Q_{-1,D} (\mathscr{D}(\varepsilon U) \Lambda_D) \chi_D \Lambda_D^{-1} \dot{U} = Q_{-1,D} (\mathscr{D}(\varepsilon U) \Lambda)_D \chi_D \Lambda_D^{-1} \dot{U} = r_{0,D} \Lambda_D^{-1} \dot{U},$$

where we have applied the symbolic calculus rule in the end for the $(-1, 1)$ product. Similarly, we can write the first commutator in (2-11) as

$$Q_D [\mathscr{D}(\varepsilon U), \chi_D] \dot{U} = r_{0,D} [(\mathscr{D}(\varepsilon U) \Lambda)_D, \chi_D] \Lambda_D^{-1} \dot{U} = r_{0,D} \Lambda_D^{-1} \dot{U}.$$

We leave to the reader the other two terms in (2-11) that can be treated in an analogous way. Eventually, we can write the term $r_{-1,D} \dot{U}$ in the last line of (2-10) as $r_{0,D} \Lambda_D^{-1} \dot{U}$ and the L^2 estimate follows. □

The estimates of Lemma 2.6 seem to be the best we can hope for in the case of the bad $(1, -1)$ product (2-12)(b), which is the reason for the need to estimate such terms as those on the left of inequality (2-6).

Step 2: *Outgoing modes.* Recall that $-\mathbb{D}_1$ and $-\mathbb{D}_0$ are block diagonal:

$$-\mathbb{D}_1(\zeta) = \begin{pmatrix} i\omega_1(\zeta) I_{n_1} & & 0 \\ & \ddots & \\ 0 & & i\omega_J(\zeta) I_{n_J} \end{pmatrix}, \quad -\mathbb{D}_0(\varepsilon U, \zeta) = \begin{pmatrix} C_1 & & 0 \\ & \ddots & \\ 0 & & C_J \end{pmatrix},$$

so the system (2-10) satisfied by $w = (w_1, \ldots, w_J)$ can be written as a collection of J decoupled transport equations

$$\partial_d w_j = (i\omega_j)_D w_j + C_{j,D} w_j + r_{0,D} f + r_{-1,D} \dot{U} + R_D^a \dot{U} \tag{2-14}$$

with $\mathrm{Re}(i\omega_j) \leq -c\gamma$ for $1 \leq j \leq J'$, and $\mathrm{Re}(i\omega_j) \geq c\gamma$ for $J' + 1 \leq j \leq J$ ($c > 0$ denotes a constant).

Following the strategy of [Coulombel 2004], we now give two preliminary estimates of the outgoing modes w_j, $j \geq J'+1$. Taking the real part of the $L^2(\Omega)$ inner product of (2-14) with $-\Lambda_D^2 w_j$, we obtain

$$\begin{aligned}-\frac{\langle \Lambda_D w_j(0)\rangle_0^2}{2} &= \operatorname{Re}(\Lambda_D (i\omega_j)_D w_j, \Lambda_D w_j)_{L^2(\Omega)} + \operatorname{Re}(\Lambda_D C_{j,D} w_j, \Lambda_D w_j)_{L^2(\Omega)} \\ &\quad + \operatorname{Re}(\Lambda_D r_{0,D} f, \Lambda_D w_j)_{L^2(\Omega)} + \operatorname{Re}(\Lambda_D r_{-1,D} \dot{U}, \Lambda_D w_j)_{L^2(\Omega)} + \operatorname{Re}(\Lambda_D R_D^a \dot{U}, \Lambda_D w_j)_{L^2(\Omega)}.\end{aligned}$$

Since $\operatorname{Re}(i\omega_j) \geq c\gamma$, we get, after absorbing some terms on the left,

$$\gamma |\Lambda_D w_{\text{out}}|_{0,0}^2 + \langle \Lambda_D w_{\text{out}}(0)\rangle_0^2 \leq \frac{C}{\gamma}(|\Lambda_D f|_{0,0}^2 + |\dot{U}|_{0,0}^2 + |(\varepsilon\Lambda_D)^{-1}\dot{U}|_{0,0}^2). \tag{2-15}$$

Here, for example, we have used Young's inequality and Lemma 2.6 and estimated

$$|\operatorname{Re}(\Lambda_D R_D^a \dot{U}, \Lambda_D w_j)_{L^2(\Omega)}| \leq \frac{C_\delta}{\gamma}(|\dot{U}|_{0,0}^2 + |(\varepsilon\Lambda_D)^{-1}\dot{U}|_{0,0}^2) + \delta\gamma|\Lambda_D w_j|_{0,0}^2. \tag{2-16}$$

Taking the real part of the L^2 inner product of (2-14) with w_j on $[x_d, \infty) \times b\Omega$ instead of Ω, we obtain, for all $x_d \geq 0$,

$$\gamma |w_j|_{0,0}^2 + \langle w_j(x_d)\rangle_0^2 \leq \frac{C}{\gamma}\left(|f|_{0,0}^2 + \frac{1}{\gamma^2}|\dot{U}|_{0,0}^2\right). \tag{2-17}$$

Finally, adding to (2-15) the estimate $\gamma^2\times$(2-17) and the estimates we obtain in the same way by pairing (2-14) with w_j/ε^2 (here we use the L^2 estimate of the remainders given in Lemma 2.6), we obtain

$$\gamma|(\Lambda_D, \varepsilon^{-1})w_{\text{out}}|_{0,0}^2 + |(\Lambda_D, \varepsilon^{-1})w_{\text{out}}|_{\infty,0}^2 \leq \frac{C}{\gamma}\big(|(\Lambda_D, \varepsilon^{-1})f|_{0,0}^2 + |\dot{U}|_{0,0}^2 + |(\varepsilon\Lambda_D)^{-1}\dot{U}|_{0,0}^2\big). \tag{2-18}$$

This completes the estimate of the outgoing terms in the first line of (2-6).

Step 3: *Incoming modes* I. Estimating w_j for $j \leq J'$ in a similar way, but now using $\operatorname{Re}(i\omega_j) \leq -c\gamma$ and pairing the corresponding transport equation with w_j, we obtain

$$\gamma^3 |w_{\text{in}}|_{0,0}^2 + \gamma^2 |w_{\text{in}}|_{\infty,0}^2 \leq C\gamma^2 \langle w_{\text{in}}|_{x_d=0}\rangle_0^2 + \frac{C}{\gamma}(|\gamma f|_{0,0}^2 + |\dot{U}|_{0,0}^2). \tag{2-19}$$

This L^2 estimate does not cause any problem because we have a good L^2 control of the remainder $R_D^a v$ appearing on the right of (2-14); see Lemma 2.6 (we have even weakened the estimate of the remainders in Lemma 2.6 by simply estimating them in terms of $|\dot{U}|_{0,0}/\gamma$). Moreover the term $|w_{\text{in}}|_{\infty,0}^2$ was estimated by considering the L^2 pairing on $[0, x_d] \times b\Omega$ instead of Ω.

Step 4: *Boundary estimate.* We observe that v can be expressed in terms of w as

$$v = (Q_0^{-1})_D w + r_{-1,D}\dot{U}.$$

Recalling the boundary condition in (2-7) and using the decomposition $Q_0^{-1}(\zeta) = [Q_{\text{in}}(\zeta) Q_{\text{out}}(\zeta)]$, we accordingly let $w = (w_{\text{in}}, w_{\text{out}})$ and rewrite the boundary condition in (2-7) as

$$\mathcal{B}(\varepsilon U) Q_{\text{in},D} w_{\text{in}}|_{x_d=0} = -\mathcal{B}(\varepsilon U) Q_{\text{out},D} w_{\text{out}}|_{x_d=0} + \boldsymbol{\chi}_D g + r_{-1,D}\dot{U}|_{x_d=0}. \tag{2-20}$$

By (1-12) we have on $\mathscr{V}_i$

$$\mathscr{B}(\varepsilon U)Q_{\text{in}} = P_1^{-1}(P_1\mathscr{B}(\varepsilon U)Q_{\text{in}}P_2)P_2^{-1} = P_1^{-1}\begin{pmatrix}\Lambda^{-1}(\gamma+i\sigma) & \\ & I\end{pmatrix}P_2^{-1},$$

so using the rules of singular calculus, we get

$$\Lambda_D\mathscr{B}(\varepsilon U)Q_{\text{in},D}w_{\text{in}}|_{x_d=0} = (P_1^{-1})_D\begin{pmatrix}\gamma+i\sigma_D & \\ & \Lambda_D I\end{pmatrix}(P_2^{-1})_D w_{\text{in}}|_{x_d=0} + r_{0,D}w_{\text{in}}|_{x_d=0}.$$

With (2-20), this implies

$$\left\langle (P_1^{-1})_D\begin{pmatrix}\gamma+i\sigma_D & \\ & \Lambda_D I\end{pmatrix}(P_2^{-1})_D w_{\text{in}}|_{x_d=0}\right\rangle_0 \le C(\langle\Lambda_D w_{\text{out}}|_{x_d=0}\rangle_0 + \langle\Lambda_D g\rangle_0 + \langle\dot{U}|_{x_d=0}\rangle_0). \quad (2\text{-}21)$$

We have $P_{1,D}(P_1^{-1})_D = I + r_{-1,D}$ so up to choosing γ large (and absorbing the $r_{-1,D}$ term), the estimate (2-21) implies

$$\left\langle \begin{pmatrix}\gamma+i\sigma_D & \\ & \Lambda_D I\end{pmatrix}(P_2^{-1})_D w_{\text{in}}|_{x_d=0}\right\rangle_0 \le C(\langle\Lambda_D w_{\text{out}}|_{x_d=0}\rangle_0 + \langle\Lambda_D g\rangle_0 + \langle\dot{U}|_{x_d=0}\rangle_0). \quad (2\text{-}22)$$

Letting

$$\begin{pmatrix}w_1\\ w'\end{pmatrix} := (P_2^{-1})_D w_{\text{in}}|_{x_d=0},$$

we find, using the fact that σ is real and again choosing γ large enough,

$$\left\langle \begin{pmatrix}\gamma+i\sigma_D & \\ & \Lambda_D I\end{pmatrix}\begin{pmatrix}w_1\\ w'\end{pmatrix}\right\rangle_0^2 \ge \frac{1}{C}(\gamma^2\langle w_1\rangle_0^2 + \langle\Lambda_D w'\rangle_0^2) \ge \frac{\gamma^2}{C}\langle w_1, w'\rangle_0^2.$$

Thus, from (2-22), we may conclude

$$\gamma\langle w_{\text{in}}|_{x_d=0}\rangle_0 \le C(\langle\Lambda_D w_{\text{out}}|_{x_d=0}\rangle_0 + \langle\Lambda_D g\rangle_0 + \langle\dot{U}|_{x_d=0}\rangle_0). \quad (2\text{-}23)$$

Combining the estimates (2-19) and (2-23), we have thus derived the bound

$$\gamma^3|w_{\text{in}}|_{0,0}^2 + \gamma^2|w_{\text{in}}|_{\infty,0}^2 \le \frac{C}{\gamma}(|\gamma f|_{0,0}^2 + |\dot{U}|_{0,0}^2) + C(\langle\Lambda_D g\rangle_0^2 + \langle\dot{U}|_{x_d=0}\rangle_0^2) + C\langle\Lambda_D w_{\text{out}}|_{x_d=0}\rangle_0^2.$$

Together with (2-18) this completes the estimate of the terms in the first line of (2-6).

Remark 2.7. At this point we can see the need to estimate the remaining terms on the left in the estimate (2-6) as well as the similar terms on the left in the Kreiss estimate (2-29). We must estimate those terms in order to be able to absorb the terms involving $(\varepsilon\Lambda_D)^{-1}\dot{U}^\gamma$ on the right side of (2-6). Recall that such terms come from the bad $(1, -1)$ product and from the $\partial_d Q_{-1}$ term. This is one of the major differences between our analysis and that in [Coulombel 2004].

Step 5: *Incoming modes* II. Here we begin to estimate the terms in the second line of (2-6). We introduce the functions $\tilde{v} := \Lambda_D^{-1} v$ and $\tilde{v}' := \tilde{v}/\varepsilon$, and see that the function $\tilde{v}'$ satisfies

$$\begin{aligned}\partial_d \tilde{v}' + \mathbb{A}_D \tilde{v}' + \mathcal{D}(\varepsilon U)\tilde{v}' &= \frac{\Lambda_D^{-1}}{\varepsilon}\boldsymbol{\chi}_D f + \frac{\Lambda_D^{-1}}{\varepsilon}[\mathcal{D}(\varepsilon U), \boldsymbol{\chi}_D]\dot{U} + \big(\mathcal{D}(\varepsilon U) - \Lambda_D^{-1}\mathcal{D}(\varepsilon U)\Lambda_D\big)\tilde{v}', \\ \mathcal{B}(\varepsilon U)\tilde{v}'|_{x_d=0} &= \frac{\Lambda_D^{-1}}{\varepsilon}\boldsymbol{\chi}_D g + \frac{\Lambda_D^{-1}}{\varepsilon}[\mathcal{B}(\varepsilon U), \boldsymbol{\chi}_D]\dot{U}|_{x_d=0} + [\mathcal{B}(\varepsilon U), \Lambda_D^{-1}]\frac{v}{\varepsilon}|_{x_d=0}.\end{aligned} \tag{2-24}$$

We can thus diagonalize the problem for $\tilde{v}'$ with the *same* operator $Q_D = Q_{0,D} + Q_{-1,D}$ as before. Introducing the function $\tilde{w}' := Q_D \tilde{v}'$, we find that $\tilde{w}'$ satisfies

$$\begin{aligned}&\text{(a)} \quad \partial_d \tilde{w}' = -(\mathbb{D}_1 + \mathbb{D}_0)_D \tilde{w}' + Q_D \frac{\Lambda_D^{-1}}{\varepsilon}\boldsymbol{\chi}_D f + Q_D \frac{\Lambda_D^{-1}}{\varepsilon}[\mathcal{D}(\varepsilon U), \boldsymbol{\chi}_D]\dot{U} + \frac{1}{\gamma} r_{0,D}\tilde{v}', \\ &\text{(b)} \quad \mathcal{B}(\varepsilon U) Q_{\text{in},D}\tilde{w}'_{\text{in}} = -\mathcal{B}(\varepsilon U) Q_{\text{out},D}\tilde{w}'_{\text{out}} + \frac{\Lambda_D^{-1}}{\varepsilon}\boldsymbol{\chi}_D g + \frac{\Lambda_D^{-1}}{\varepsilon}[\mathcal{B}(\varepsilon U), \boldsymbol{\chi}_D]\dot{U} \\ &\qquad\qquad + \Lambda_D^{-1}[\Lambda_D, \mathcal{B}(\varepsilon U)]\tilde{v}' + r_{-1,D}\tilde{v}',\end{aligned} \tag{2-25}$$

where we have collected several terms into remainders of the form $\gamma^{-1} r_{0,D}\tilde{v}'$. For instance, we have used

$$R_D^a \tilde{v}' = \frac{1}{\gamma} r_{0,D}\tilde{v}', \qquad Q_{-1,D}\mathcal{D}(\varepsilon U)\tilde{v}' = \frac{1}{\gamma} r_{0,D}\tilde{v}'.$$

Next we fix an index $j \in \{1, \dots, J'\}$. Taking the real part of the $L^2(\Omega)$ inner product of (2-25)(a) with $\tilde{w}'_j$, we obtain the standard L^2 estimate for incoming modes:

$$\begin{aligned}&\gamma |\tilde{w}'_{\text{in}}|_{0,0}^2 \\ &\quad \le C\langle \tilde{w}'_{\text{in}}|_{x_d=0}\rangle_0^2 + \frac{C}{\gamma}\Big(|(\varepsilon\Lambda_D)^{-1} f|_{0,0}^2 + \frac{1}{\gamma^2}|(\varepsilon\Lambda_D)^{-1}\dot{U}|_{0,0}^2 + |(\varepsilon\Lambda_D)^{-1}[\mathcal{D}(\varepsilon U), \boldsymbol{\chi}_D]\dot{U}|_{0,0}^2\Big) \\ &\quad \le C\langle \tilde{w}'_{\text{in}}|_{x_d=0}\rangle_0^2 + \frac{C}{\gamma^3}(|\varepsilon^{-1} f|_{0,0}^2 + |(\varepsilon\Lambda_D)^{-1}\dot{U}|_{0,0}^2).\end{aligned} \tag{2-26}$$

We thus wish to control the trace of $\tilde{w}'_{\text{in}}$.

Step 6: *Control of the trace of* $\tilde{w}'_{\text{in}}$. Using (2-25)(b) and arguing as in Step 4, we obtain the boundary estimate

$$\begin{aligned}&\gamma \langle \tilde{w}'_{\text{in}}|_{x_d=0}\rangle_0 \\ &\quad \le C(\langle \Lambda_D \tilde{w}'_{\text{out}}|_{x_d=0}\rangle_0 + \langle \varepsilon^{-1} g\rangle_0 + \langle \varepsilon^{-1}[\mathcal{B}(\varepsilon U), \boldsymbol{\chi}_D]\dot{U}|_{x_d=0}\rangle_0 + \langle \tilde{v}'|_{x_d=0}\rangle_0 + \langle \varepsilon^{-1}[\Lambda_D, \mathcal{B}(\varepsilon U)]\tilde{v}|_{x_d=0}\rangle_0) \\ &\quad \le C(\langle \Lambda_D \tilde{w}'_{\text{out}}|_{x_d=0}\rangle_0 + \langle \varepsilon^{-1} g\rangle_0 + \langle (\varepsilon\Lambda_D)^{-1}\dot{U}|_{x_d=0}\rangle_0).\end{aligned}$$

Combining with (2-26), we have derived

$$\begin{aligned}&\gamma |\tilde{w}'_{\text{in}}|_{0,0}^2 + \langle \tilde{w}'_{\text{in}}|_{x_d=0}\rangle_0^2 \\ &\le \frac{C}{\gamma^2}\langle \Lambda_D \tilde{w}'_{\text{out}}|_{x_d=0}\rangle_0^2 + \frac{C}{\gamma^3}(|\varepsilon^{-1} f|_{0,0}^2 + |(\varepsilon\Lambda_D)^{-1}\dot{U}|_{0,0}^2) + \frac{C}{\gamma^2}(\langle \varepsilon^{-1} g\rangle_0^2 + \langle (\varepsilon\Lambda_D)^{-1}\dot{U}|_{x_d=0}\rangle_0^2).\end{aligned} \tag{2-27}$$

We expect $\Lambda_D \tilde{w}'_{\text{out}}$ to be comparable to $w_{\text{out}}/\varepsilon$ and thus use (2-18); this is checked and made precise in the next and last step of the proof of Proposition 2.4.

Step 7: *Relation between $\Lambda_D \tilde{w}'$ and w/ε, and conclusion.* Using the definitions

$$\tilde{w}' = Q_D \tilde{v}' = Q_D (\varepsilon \Lambda_D)^{-1} v \quad \text{and} \quad w = Q_D v,$$

and the fact that Λ_D commutes with $Q_{0,D}$, we compute

$$\Lambda_D \tilde{w}' = \varepsilon^{-1} Q_D v + r_{0,D} \tilde{v}' = \varepsilon^{-1} w + r_{0,D} \tilde{v}'.$$

We have thus derived the bound from above

$$\frac{1}{\gamma^2} \langle \Lambda_D \tilde{w}'_{\text{out}}|_{x_d=0} \rangle_0^2 \le \frac{C}{\gamma^2} (\langle \varepsilon^{-1} w_{\text{out}}|_{x_d=0} \rangle_0^2 + \langle (\varepsilon \Lambda_D)^{-1} \dot{U}|_{x_d=0} \rangle_0^2),$$

which we combine with (2-27) and (2-18) to obtain

$$\begin{aligned} &\gamma |\tilde{w}'_{\text{in}}|_{0,0}^2 + \langle \tilde{w}'_{\text{in}}|_{x_d=0} \rangle_0^2 \\ &\quad \le \frac{C}{\gamma^3} \big(|(\Lambda_D, \varepsilon^{-1}) f|_{0,0}^2 + |(\varepsilon \Lambda_D)^{-1} \dot{U}|_{0,0}^2 + |\dot{U}|_{0,0}^2 \big) + \frac{C}{\gamma^2} \big(\langle \varepsilon^{-1} g \rangle_0^2 + \langle (\varepsilon \Lambda_D)^{-1} \dot{U}|_{x_d=0} \rangle_0^2 \big). \end{aligned} \tag{2-28}$$

It only remains to derive a bound from below to go from $\tilde{w}'_{\text{in}}$ to $(\varepsilon \Lambda_D)^{-1} \dot{U}^\gamma_{1,\text{in}}$. We first observe that estimating $(\varepsilon \Lambda_D)^{-1} \dot{U}^\gamma_{1,\text{in}}$ as claimed in (2-6) amounts to estimating $Q_D (\varepsilon \Lambda_D)^{-1} \dot{U}^\gamma_{1,\text{in}}$. We use the relation

$$Q_D (\varepsilon \Lambda_D)^{-1} \dot{U}^\gamma_{1,\text{in}} = (\varepsilon \Lambda_D)^{-1} \begin{pmatrix} w_{\text{in}} \\ 0 \end{pmatrix} - [(\varepsilon \Lambda_D)^{-1}, Q_D] \dot{U}^\gamma_{1,\text{in}} = (\varepsilon \Lambda_D)^{-1} \begin{pmatrix} w_{\text{in}} \\ 0 \end{pmatrix} - [(\varepsilon \Lambda_D)^{-1}, Q_{-1,D}] \dot{U}^\gamma_{1,\text{in}},$$

and the special "decoupled" form of the coefficients of Q_{-1} to show that

$$[(\varepsilon \Lambda_D)^{-1}, Q_{-1,D}] = \frac{1}{\gamma^2} r_{0,D} (\varepsilon \Lambda_D)^{-1}.$$

Similarly, taking the "in" component of $\tilde{w}' = Q_D (\varepsilon \Lambda_D)^{-1} \dot{U}^\gamma_1$, we have

$$\tilde{w}'_{\text{in}} = (\varepsilon \Lambda_D)^{-1} w_{\text{in}} + \frac{1}{\gamma^2} r_{0,D} (\varepsilon \Lambda_D)^{-1} \dot{U}^\gamma_1,$$

so we obtain

$$Q_D (\varepsilon \Lambda_D)^{-1} \dot{U}^\gamma_{1,\text{in}} = \begin{pmatrix} \tilde{w}'_{\text{in}} \\ 0 \end{pmatrix} + \frac{1}{\gamma^2} r_{0,D} (\varepsilon \Lambda_D)^{-1} \dot{U}.$$

We have therefore proved that (2-28) implies that the second line in (2-6) is controlled by the terms on the right of (2-6). This finishes the proof of Proposition 2.4. $\square$

(III): *Estimate away from the bad set.* The next proposition provides a Kreiss-type estimate for the terms $\chi_{i,D} \dot{U}^\gamma$, where $i > N_1$.

Proposition 2.8. *Fix i such that $N_1+1 \le i \le N_2$ and let $\dot{U}_2^\gamma := \chi_{i,D}\dot{U}^\gamma$. We have*

$$|\dot{U}_2^\gamma|_{0,0} + \frac{\langle \dot{U}_2^\gamma|_{x_d=0}\rangle_0}{\sqrt{\gamma}} + |(\varepsilon\Lambda_D)^{-1}\dot{U}_2^\gamma|_{0,0} + \frac{\langle (\varepsilon\Lambda_D)^{-1}\dot{U}_2^\gamma|_{x_d=0}\rangle_0}{\sqrt{\gamma}} \\ \le C\biggl(\frac{|f^\gamma|_{0,0}+|(\varepsilon\Lambda_D)^{-1}f^\gamma|_{0,0}}{\gamma} + \frac{\langle g^\gamma\rangle_0 + \langle(\varepsilon\Lambda_D)^{-1}g^\gamma\rangle_0}{\sqrt{\gamma}} \\ + \frac{|\dot{U}^\gamma|_{0,0}+|(\varepsilon\Lambda_D)^{-1}\dot{U}^\gamma|_{0,0}}{\gamma^2} + \frac{\langle \dot{U}^\gamma|_{x_d=0}\rangle_0 + \langle(\varepsilon\Lambda_D)^{-1}\dot{U}^\gamma|_{x_d=0}\rangle_0}{\gamma}\biggr). \tag{2-29}$$

Proof. Step 1: *L^2 estimate.* The first step is to prove the Kreiss-type estimate

$$|\dot{U}_2^\gamma|_{0,0} + \frac{\langle \dot{U}_2^\gamma|_{x_d=0}\rangle_0}{\sqrt{\gamma}} \le C\left(\frac{|f^\gamma|_{0,0}}{\gamma} + \frac{\langle g^\gamma\rangle_0}{\sqrt{\gamma}} + \frac{|\dot{U}^\gamma|_{0,0}}{\gamma^2} + \frac{\langle \dot{U}^\gamma|_{x_d=0}\rangle_0}{\gamma}\right). \tag{2-30}$$

For this we define the good set $G \subset \Sigma$ to be a neighborhood of the closure of $\bigcup_{i=N_1+1}^{N_2}\mathcal{V}_i$ such that G is disjoint from $\overline{\Upsilon}$; here the uniform Lopatinskii condition is satisfied. The classical construction of Kreiss symmetrizers [Kreiss 1970; Chazarain and Piriou 1982] provides us with an $N\times N$ symbol $R(\zeta)$, homogeneous of degree 0, such that, for some positive constants C, c, and $\zeta/|\zeta| \in G$, we have

$$\begin{aligned} &\text{(a)}\quad R(\zeta) = R(\zeta)^*,\\ &\text{(b)}\quad -\mathrm{Re}(R(\zeta)\mathbb{A}(\zeta)) \ge c\gamma I_N,\\ &\text{(c)}\quad R(\zeta)+C\mathcal{B}(0)^*\mathcal{B}(0) \ge cI_N. \end{aligned} \tag{2-31}$$

We take a smooth extension of R to all ζ as a symbol of order 0 such that (2-31)(a) holds. Observe that by continuity (2-31)(c) implies

$$R(\zeta)+C\mathcal{B}(\varepsilon U)^*\mathcal{B}(\varepsilon U) \ge cI_N \quad \text{for } \varepsilon \text{ small enough.} \tag{2-32}$$

As observed in [Williams 2002], we may now use R_D, the singular Fourier multiplier associated to the symbol $R(X,\gamma)$ as a Kreiss symmetrizer for the singular problem. Let $\chi_i = \boldsymbol{\chi}$, $v := \boldsymbol{\chi}_D\dot{U}^\gamma$, and denote by $\langle\cdot\,,\cdot\rangle$ the L^2 inner product on $b\Omega$. Using (2-7) to expand $\partial_d\langle v, R_Dv\rangle$ and integrating in x_d over $[0,\infty)$, we obtain

$$-\langle v|_{x_d=0}, R_Dv|_{x_d=0}\rangle \\ = -2\,\mathrm{Re}(R_D\mathbb{A}_Dv, v) - 2\,\mathrm{Re}(R_D\mathcal{D}(\varepsilon U)v, v) + 2\,\mathrm{Re}(R_D\boldsymbol{\chi}_Df^\gamma, v) + O(|\dot{U}^\gamma|_{0,0}|v|_{0,0}/\gamma).$$

From (2-31)(b), (2-32), and the localized Gårding inequality (Proposition A.9),

$$\mathrm{Re}\langle (R+C\mathcal{B}(\varepsilon U)^*\mathcal{B}(\varepsilon U))_Dv|_{x_d=0}, v|_{x_d=0}\rangle \ge c\langle v|_{x_d=0}\rangle_0^2 - C\frac{\langle \dot{U}^\gamma|_{x_d=0}\rangle_0^2}{\gamma}, \tag{2-33}$$

we easily derive the estimate (2-30).

Step 2: *Estimate of* $(\varepsilon\Lambda_D)^{-1}\dot{U}_2^\gamma$. Set $\tilde{v} := \Lambda_D^{-1}v$ and $\tilde{v}' = \tilde{v}/\varepsilon$. Then $\tilde{v}'$ satisfies the system (2-24), where the truncation function $\boldsymbol{\chi}$ has changed but the forcing terms have exactly the same expression. An argument just like the one that gave the estimate (2-30) yields

$$
\begin{aligned}
|(\varepsilon\Lambda_D)^{-1}\dot{U}_2^\gamma|_{0,0} + \frac{\langle(\varepsilon\Lambda_D)^{-1}\dot{U}_2^\gamma|_{x_d=0}\rangle_0}{\sqrt{\gamma}}
&\le C\bigg(\frac{|(\varepsilon\Lambda_D)^{-1}f^\gamma|_{0,0}}{\gamma} + \frac{\langle(\varepsilon\Lambda_D)^{-1}g^\gamma\rangle_0}{\sqrt{\gamma}} + \frac{|\dot{U}^\gamma|_{0,0} + |(\varepsilon\Lambda_D)^{-1}\dot{U}^\gamma|_{0,0}}{\gamma^2} \\
&\qquad + \frac{\langle\dot{U}^\gamma|_{x_d=0}\rangle_0 + \langle(\varepsilon\Lambda_D)^{-1}\dot{U}^\gamma|_{x_d=0}\rangle_0}{\gamma}\bigg).
\end{aligned}
$$

Here instead of (2-33) we have used

$$
\operatorname{Re}\langle(R + C\mathcal{B}(\varepsilon U)^*\mathcal{B}(\varepsilon U))_D\tilde{v}'|_{x_d=0}, \tilde{v}'|_{x_d=0}\rangle \ge c\langle\tilde{v}'|_{x_d=0}\rangle_0^2 - C\frac{\langle(\varepsilon\Lambda_D)^{-1}\dot{U}^\gamma|_{x_d=0}\rangle_0^2}{\gamma},
$$

to recover the estimate of the trace of $\tilde{v}'$. The L^2 estimates of the forcing terms in the interior and on the boundary are exactly the same as in steps 4 and 5 of the previous proof. □

(IV): *Conclusion.* We use the previous propositions to complete the proof of Proposition 2.2. Summing the estimates (2-6) and (2-29) over $i \in \{1, \dots, N_2\}$ and absorbing error terms from the right by taking γ large, we derive

$$
|\dot{U}^\gamma|_{0,0} + \frac{\langle\dot{U}^\gamma|_{x_d=0}\rangle_0}{\sqrt{\gamma}} + \frac{|\dot{U}_1^\gamma|_{\infty,0}}{\sqrt{\gamma}} \le C(K)\bigg(\frac{|(\Lambda_D, \varepsilon^{-1})f^\gamma|_{0,0}}{\gamma^2} + \frac{\langle(\Lambda_D, \varepsilon^{-1})g^\gamma\rangle_0}{\gamma^{3/2}}\bigg), \tag{2-34}
$$

where we have "forgotten" on the left of the inequality the additional control of $(\varepsilon\Lambda_D)^{-1}\dot{U}^\gamma$ (this term has played its role, meaning that it was used to absorb some bad terms appearing on the right). This gives exactly (2-4) with the additional control of $\dot{U}_1^\gamma$ in $L^\infty(L^2)$. This additional property is used in the proof of Corollary 2.3. □

Proof of Corollary 2.3. It remains to estimate $|\dot{U}^\gamma|_{0,1}$ and $|\dot{U}^\gamma|_{\infty,0}$. We first estimate the first-order tangential derivatives. We can apply the a priori estimate (2-4) to the problem satisfied by $\partial_{(x',\theta_0)}\dot{U}^\gamma$, which is obtained by differentiating (2-2). This yields

$$
|\dot{U}^\gamma|_{0,1} + \frac{\langle\dot{U}^\gamma|_{x_d=0}\rangle_1}{\sqrt{\gamma}} \le C(K)\bigg(\frac{|(\Lambda_D, \varepsilon^{-1})f^\gamma|_{0,1}}{\gamma^2} + \frac{\langle(\Lambda_D, \varepsilon^{-1})g^\gamma\rangle_1}{\gamma^{3/2}}\bigg), \tag{2-35}
$$

which is the same as (2-5), except for the absence of $|\dot{U}^\gamma|_{\infty,0}$ on the left. Here we were able to treat commutators as forcing terms because, for example,

$$
[\mathcal{D}(\varepsilon U), \partial_{(x',\theta_0)}]\dot{U}^\gamma = -(d\mathcal{D}(\varepsilon U)\cdot\varepsilon\partial_{(x',\theta_0)}U)\dot{U}^\gamma,
$$

and the factor of ε coming out from the commutation allows us, for example, to estimate

$$
|\Lambda_D[\mathcal{D}(\varepsilon U), \partial_{(x',\theta_0)}]\dot{U}^\gamma|_{0,0} \le C|\dot{U}^\gamma|_{0,1}.
$$

It thus only remains to estimate the norm $|\dot{U}^\gamma|_{\infty,0}$. For $\delta_2>0$ to be chosen, we take $0<\delta_1<\delta_2$ and consider a symbol of order zero in the extended calculus, $\chi^e(\xi', k\beta/\varepsilon, \gamma)$, such that

$$0\leq\chi^e\leq 1,\qquad \chi^e\left(\xi',\frac{k\beta}{\varepsilon},\gamma\right)=1\quad\text{on}\left\{|\xi',\gamma|\leq\delta_1\frac{|k\beta|}{\varepsilon}\right\},\qquad \operatorname{supp}\chi^e\subset\left\{|\xi',\gamma|\leq\delta_2\frac{|k\beta|}{\varepsilon}\right\}.$$

We then write $\dot{U}^\gamma=\chi^e_D\dot{U}^\gamma+(1-\chi^e_D)\dot{U}^\gamma$ and begin by estimating $|(1-\chi^e_D)\dot{U}^\gamma|_{0,\infty}$ by using the Sobolev-type estimate

$$|(1-\chi^e_D)\dot{U}^\gamma|_{\infty,0}\leq C|(1-\chi^e_D)\partial_d\dot{U}^\gamma|_{0,0}+C|(1-\chi^e_D)\dot{U}^\gamma|_{0,0}\leq C|(1-\chi^e_D)\partial_d\dot{U}^\gamma|_{0,0}+C|\dot{U}^\gamma|_{0,0}. \tag{2-36}$$

Using (2-2) and the fact that

$$|X,\gamma|\left(1-\chi^e\left(\xi',\frac{k\beta}{\varepsilon},\gamma\right)\right)\leq C|\xi',\gamma|,$$

we obtain

$$\begin{aligned}|(1-\chi^e_D)\partial_d\dot{U}^\gamma|_{0,0}&\leq|\mathbb{A}_D(1-\chi^e_D)\dot{U}^\gamma|_{0,0}+|(1-\chi^e_D)\mathscr{D}\dot{U}^\gamma|_{0,0}+|(1-\chi^e_D)f^\gamma|_{0,0}\\&\leq C(|\dot{U}^\gamma|_{0,1}+|f^\gamma|_{0,0})\leq C\left(|\dot{U}^\gamma|_{0,1}+\frac{|\Lambda_D f^\gamma|_{0,1}}{\gamma^2}\right),\end{aligned}$$

where the last inequality follows from $|f^\gamma|_{0,0}\leq C|f^\gamma|_{0,1}/\gamma$. With (2-36) this gives

$$|(1-\chi^e_D)\dot{U}^\gamma|_{\infty,0}\leq C\left(|\dot{U}^\gamma|_{0,1}+\frac{|\Lambda_D(f^\gamma)|_{0,1}}{\gamma^2}\right). \tag{2-37}$$

To estimate $|\chi^e_D\dot{U}^\gamma|_{\infty,0}$ we observe that since $\beta\in\Upsilon$, we have, for $\delta_2>0$ chosen small enough,

$$\chi^e\left(\xi',\frac{k\beta}{\varepsilon},\gamma\right)=\chi^e\left(\xi',\frac{k\beta}{\varepsilon},\gamma\right)\sum_{i=1}^{N_1}\chi_i(X,\gamma),$$

for the χ_i chosen in Step I of the proof of Proposition 2.2. Thus

$$|\chi^e_D\dot{U}^\gamma|_{\infty,0}\leq|\chi^e_D\dot{U}^\gamma_1|_{\infty,0}\leq|\dot{U}^\gamma_1|_{\infty,0},$$

with $\dot{U}^\gamma_1$ defined in Proposition 2.4.[22] We can then apply the a priori estimate (2-34) and obtain

$$|\chi^e_D\dot{U}^\gamma|_{\infty,0}\leq C\left(\frac{|(\Lambda_D,\varepsilon^{-1})f^\gamma|_{0,0}}{\gamma^{3/2}}+\frac{\langle(\Lambda_D,\varepsilon^{-1})g^\gamma\rangle_0}{\gamma}\right).$$

With (2-37) and (2-35), this completes the proof of Corollary 2.3. □

Let us quickly observe that the genuine Gårding inequality was used only once, in the proof of Proposition 2.2, namely in (2-33). In all other cases, we only used Plancherel's theorem for Fourier multipliers. This explains the slight difference between (2-29) and (2-6) for the powers of γ.

[22]More precisely, here $\dot{U}^\gamma_1$ is the sum of the similarly denoted functions in Proposition 2.4.

Next we "localize the estimate" to Ω_T. Since[23]

$$|\Lambda_D f^\gamma|_{0,1} \sim \left|\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) f^\gamma\right|_{0,1} \sim \left|\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) f\right|_{0,1,\gamma},$$

we can rewrite the a priori estimate (2-5) for solutions to the linearized system (2-1) as

$$|\dot U|_{\infty,0,\gamma} + |\dot U|_{0,1,\gamma} + \frac{\langle \dot U|_{x_d=0}\rangle_{1,\gamma}}{\sqrt{\gamma}} \\ \le C(K)\left(\frac{|(\gamma, \partial_{x'} + \beta\partial_{\theta_0}/\varepsilon) f|_{0,1,\gamma} + |f/\varepsilon|_{0,1,\gamma}}{\gamma^2} + \frac{\langle(\gamma, \partial_{x'} + \beta\partial_{\theta_0}/\varepsilon) g\rangle_{1,\gamma} + \langle g/\varepsilon\rangle_{1,\gamma}}{\gamma^{3/2}}\right). \quad (2\text{-}38)$$

Suppose now that the singular problem (2-1) is posed on Ω_T instead of Ω. Given $f \in L^2 H^1_T$, one can define a Seeley extension $\tilde f \in L^2 H^1$ such that

$$\left|\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) \tilde f\right|_{0,1} + |\tilde f/\varepsilon|_{0,1} \le C\left(\left|\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) f\right|_{0,1,T} + |f/\varepsilon|_{0,1,T}\right),$$

where C is independent of γ, ε, and T. It is readily checked that the same extension satisfies

$$\left|\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) \tilde f\right|_{0,1,\gamma} + |\tilde f/\varepsilon|_{0,1,\gamma} \le C\left(\left|\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) f\right|_{0,1,\gamma,T} + |f/\varepsilon|_{0,1,\gamma,T}\right), \quad (2\text{-}39)$$

where again C is independent of γ, ε, and T. We claim that changing f, g, and U in $\{t > T\}$ does not affect the solution of (2-1) in $\{t < T\}$. (This causality principle is discussed further below together with the existence of solutions to the linearized system (2-1).) Hence the estimates (2-38) and (2-39) imply the following estimate for the singular problem on Ω_T:

$$|\dot U|_{\infty,0,\gamma,T} + |\dot U|_{0,1,\gamma,T} + \frac{\langle \dot U|_{x_d=0}\rangle_{1,\gamma,T}}{\sqrt{\gamma}} \\ \le C(K)\left(\frac{|(\gamma, \partial_{x'} + \beta\partial_{\theta_0}/\varepsilon) f|_{0,1,\gamma,T} + |f/\varepsilon|_{0,1,\gamma,T}}{\gamma^2} + \frac{\langle(\gamma, \partial_{x'} + \beta\partial_{\theta_0}/\varepsilon) g\rangle_{1,\gamma,T} + \langle g/\varepsilon\rangle_{1,\gamma,T}}{\gamma^{3/2}}\right).$$

Let us now consider the linearized singular problem (2-1) on Ω_T with data of the form εf and εg instead of f and g. We note that

$$\left|\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) \varepsilon f\right|_{0,1,\gamma,T} \le C|f|_{0,2,\gamma,T} \quad \text{and} \quad \left\langle\left(\gamma, \partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right) \varepsilon g\right\rangle_{1,\gamma,T} \le C\langle g\rangle_{2,\gamma,T}.$$

Let us write the linearized operators on the left sides of (2-1)(a) and (b) as $\mathbb{L}'(\varepsilon U)\dot U$ and $\mathbb{B}'(\varepsilon U)\dot U$, respectively, and define

$$\mathscr{L}'_\varepsilon(U)\dot U := \frac{1}{\varepsilon}\mathbb{L}'(\varepsilon U)\dot U, \quad \mathscr{B}'_\varepsilon(U)\dot U := \frac{1}{\varepsilon}\mathbb{B}'(\varepsilon U)\dot U.$$

We have proved:

[23] Here "$\sim$" denotes equivalence of norms with constants independent of ε and γ.

Proposition 2.9. *Fix $K > 0$ and suppose $|\varepsilon\partial_d U|_{C_T^{0,M_0-1}} + |U|_{C_T^{0,M_0}} \le K$ for $\varepsilon \in (0,1]$. There exist positive constants $\varepsilon_0(K)$, $\gamma_0(K)$ such that solutions of the singular problem*

$$\begin{aligned} &\mathcal{L}'_\varepsilon(U)\dot{U} = f \quad \text{on } \Omega_T,\\ &\mathcal{B}'_\varepsilon(U)\dot{U} = g \quad \text{on } b\Omega_T,\\ &\dot{U} = 0 \qquad\quad\ \text{in } t < 0 \end{aligned} \tag{2-40}$$

satisfy

$$|\dot{U}|_{\infty,0,\gamma,T} + |\dot{U}|_{0,1,\gamma,T} + \frac{\langle \dot{U}|_{x_d=0}\rangle_{1,\gamma,T}}{\sqrt{\gamma}} \le C(K)\left(\frac{|f|_{0,2,\gamma,T}}{\gamma^2} + \frac{\langle g\rangle_{2,\gamma,T}}{\gamma^{3/2}}\right) \tag{2-41}$$

for $0 < \varepsilon \le \varepsilon_0(K)$, $\gamma \ge \gamma_0(K)$, and the constant $C(K)$ only depends on K.

The same estimate holds if $\mathcal{B}(\varepsilon U)$ in (2-1) *is replaced by $\mathcal{B}(\varepsilon U, \varepsilon\mathcal{U})$ given in* (1-9), *and $\mathcal{D}(\varepsilon U)$ is replaced by $\mathcal{D}(\varepsilon U, \varepsilon\mathcal{U})$ given in* (1-10), *as long as $|\varepsilon\partial_d(U, \mathcal{U})|_{C_T^{0,M_0-1}} + |U, \mathcal{U}|_{C_T^{0,M_0}} \le K$ for $\varepsilon \in (0,1]$.*

2B. ***Well-posedness of the linearized singular equations.*** In this short section, we explain why the analysis in [Coulombel 2005] gives existence and uniqueness of a solution to the linearized singular problem (2-40) for which the estimate (2-41) holds. First we define a dual problem for (2-1):

$$\begin{aligned} &\partial_d \dot{U} + \mathbb{A}^*\left(\partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right)\dot{U} + \tilde{\mathcal{D}}(\varepsilon U)\dot{U} = f(x,\theta_0) \quad \text{ on } \Omega,\\ &\mathcal{M}(\varepsilon U)\dot{U}|_{x_d=0} = g(x',\theta_0), \end{aligned} \tag{2-42}$$

where $\mathbb{A}^*$ is obtained from $\mathbb{A}$ by first multiplying the system by the constant matrix B_d, then integrating by parts on Ω, and eventually multiplying by $(B_d^T)^{-1}$. The zero-order term is also changed accordingly. Following the standard procedure described for instance in [Benzoni-Gavage and Serre 2007, Chapter 4.4], the matrix $\mathcal{M}$ giving the adjoint boundary conditions is chosen such that

$$B_d = \mathcal{B}_1(v)^T\mathcal{B}(v) + \mathcal{M}(v)^T\mathcal{M}_1(v)$$

for all v sufficiently close to the origin, where $\mathcal{B}_1(v)$ and $\mathcal{M}_1(v)$ are additional matrices depending smoothly on v.

The expression of $\mathbb{A}^*$ shows that this singular operator coincides with the operator obtained by applying the substitution $\partial_{x'} \to \partial_{x'} + \beta\partial_{\theta_0}/\varepsilon$ to the dual operator

$$\partial_t + \sum_{j=1}^{d} B_j^T\partial_j = -L_0(\partial)^*.$$

It is known from the analysis in [Benzoni-Gavage and Serre 2007, Chapter 8.3] that the latter constant multiplicity hyperbolic operator with boundary conditions given by $\mathcal{M}(v)$ gives rise to a boundary value problem in the "backward" WR class (one just has to replace γ by $-\gamma$ for this dual problem). When we apply the singular transformation $\partial_{x'} \to \partial_{x'} + \beta\partial_{\theta_0}/\varepsilon$ to the boundary value problem defined by $(L_0(\partial)^*, \mathcal{M}(\varepsilon U))$, we can reproduce the analysis of the previous section and show that the same type of a priori estimate as in Proposition 2.2 holds for (2-42).

For all fixed $\varepsilon > 0$ small enough, we have thus proved that both the forward problem (2-1) and its dual problem (2-42) satisfy an a priori estimate with a loss of one tangential derivative. The estimates depend very badly on ε because the singular derivative $\partial_{x'} + \beta\partial_{\theta_0}/\varepsilon$ is estimated by $1/\varepsilon$ times the tangential H^1 norm with respect to (x', θ_0). Nevertheless, we can at this stage reproduce the arguments of [Coulombel 2005] to show the existence and uniqueness of L^2 solutions to (2-1) when the source terms f and g satisfy $f, \partial_{\theta_0} f, \partial_{x'} f \in L^2(\Omega_T)$, $g \in H^1(b\Omega_T)$. The analysis is actually much simpler than in [Coulombel 2005] because most of the technical difficulties there arise from commutations with the hyperbolic operator. Here the hyperbolic operator has constant coefficients so commutation with any scalar Fourier multiplier is exact. The analysis in [Coulombel 2005] also shows that weak solutions are limits of strong solutions when the hyperbolic operator has constant coefficients,[24] so we can show that weak solutions satisfy the energy estimate (2-4) with constants that are *uniform* with respect to the small parameter ε. Such global in time estimates imply the causality principle that "the future does not affect the past" and can be localized to Ω_T by the extension procedure previously described.

2C. ***Tame estimates.*** In this section we prove higher derivative estimates for the linearized singular problem (2-1), first in the "pretame" form of Proposition 2.11, and then in the final, "tame" form of Proposition 2.16, which is suitable for Nash–Moser iteration. Propositions 2.12 and 2.15 give pretame and tame estimates for second derivatives.

Notation 2.10. (a) Let $L^\infty W^{1,\infty} \equiv L^\infty(\overline{\mathbb{R}}_+, W^{1,\infty}(b\Omega))$ with norm $|U|_{L^\infty W^{1,\infty}} := |U|^*$. We also write $|U|_{L^\infty(\Omega)} = |U|_*$, $\langle V\rangle_{L^\infty(b\Omega)} = \langle V\rangle_*$, $\langle V\rangle_{W^{1,\infty}(b\Omega)} = \langle V\rangle^*$, $|U|_{L^\infty(\Omega_T)} = |U|_{*,T}$, etc.

(b) For $k \in \mathbb{N}$, let ∂^k denote the collection of tangential operators $\partial^\alpha_{(x',\theta_0)}$ with $|\alpha| = k$ (α is a multi-index). Sometimes ∂^k is used to denote a particular member of this collection. Set $\partial^0\phi = \phi$.

(c) For $k \in \{1, 2, 3, \dots\}$, denote by $\partial^{\langle k\rangle}\phi$ the set of products of the form $(\partial^{\alpha_1}\phi_{i_1}) \cdots (\partial^{\alpha_r}\phi_{i_r})$ where $1 \le r \le k$, $\alpha_1 + \cdots \alpha_r = k$, $\alpha_i \ge 1$. Set $\partial^{\langle 0\rangle}\phi = 1$.

(d) For $r \ge 0$, let $[r]$ denote the smallest integer greater than r.

Our first goal is to prove the following "pretame" estimate for solutions to (2-40).

Proposition 2.11. *Fix $K > 0$ and suppose $|\varepsilon\partial_d U|_{C^{0,M_0-1}} + |U|_{C^{0,M_0}} \le K$ for $\varepsilon \in (0, 1]$. For $s \ge 0$ in any fixed finite interval, there exist positive constants $\varepsilon_0(K)$, $\gamma_0(K)$ such that the solution to the linearized singular problem* (2-40) *satisfies*

$$|\dot U|_{\infty,s,\gamma,T} + |\dot U|_{0,s+1,\gamma,T} + \frac{\langle \dot U|_{x_d=0}\rangle_{s+1,\gamma,T}}{\sqrt{\gamma}}$$
$$\le C(K)\biggl(\frac{|f|_{0,s+2,\gamma,T}}{\gamma^2} + \frac{\langle g\rangle_{s+2,\gamma,T}}{\gamma^{3/2}} + \frac{|U|_{0,s+2,\gamma,T}|\dot U|_{*,T}}{\gamma^2} + \frac{\langle U|_{x_d=0}\rangle_{s+2,\gamma,T}\langle \dot U|_{x_d=0}\rangle_{*,T}}{\gamma^{3/2}}\biggr), \quad \text{(2-43)}$$

for $0 < \varepsilon \le \varepsilon_0(K)$ and $\gamma \ge \gamma_0(K)$.

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[24] Weak solutions are only "semistrong" solutions when the hyperbolic operator has variable coefficients.

Proof. The problem satisfied by $\partial^s \dot U$ is

$$\mathcal{L}'_\varepsilon(U)\partial^s \dot U = \partial^s f + \frac{1}{\varepsilon}[\mathfrak{D}(\varepsilon U), \partial^s]\dot U,$$
$$\mathcal{B}'_\varepsilon(U)\partial^s \dot U = \partial^s g + \frac{1}{\varepsilon}[\mathcal{B}(\varepsilon U), \partial^s]\dot U.$$

In applying the estimate (2-41) to this problem, we must, for example, compute $\partial^2([\mathfrak{D}(\varepsilon U), \partial^s]\dot U)$, which is a sum of terms of the form[25]

$$\tilde{\mathfrak{D}}(\varepsilon U)\partial^{\langle j\rangle}(\varepsilon U)\partial^k \dot U, \quad \text{where } j+k=s+2,\ j\ge 1,$$

and $\tilde{\mathfrak{D}}$ is some smooth function of its argument. Since $j \ge 1$, we can rewrite this as

$$\tilde{\mathfrak{D}}(\varepsilon U)\partial^{\langle j-1\rangle}(\varepsilon U)\partial(\varepsilon U)\partial^k \dot U.$$

Using Moser estimates, we obtain

$$\left|\frac{1}{\varepsilon}\tilde{\mathfrak{D}}(\varepsilon U)\partial^{\langle j-1\rangle}(\varepsilon U)\partial(\varepsilon U)\partial^k \dot U\right|_{0,\gamma,T} \le C(K)|\dot U|_{*,T}|U|_{0,s+2,\gamma,T} + C(K)|\dot U|_{0,s+1,\gamma,T}.$$

The contribution from the final term on the right can be absorbed by taking γ large enough; thus this explains the third term on the right in (2-43). The final term on the right in (2-43) arises by the same argument applied to the boundary commutator. □

Next we prove estimates for the second derivatives

$$\mathcal{L}''_\varepsilon(U)(\dot U^a, \dot U^b) = d\mathfrak{D}(\varepsilon U)(\dot U^a, \dot U^b),$$
$$\mathcal{B}''_\varepsilon(U)(\dot U^a, \dot U^b) = d\mathcal{B}(\varepsilon U)(\dot U^a, \dot U^b),$$

where we use the short notation

$$d\mathfrak{D}(\varepsilon U)(\dot U^a, \dot U^b) := (d\mathfrak{D}(\varepsilon U)\dot U^a)\dot U^b.$$

Proposition 2.12. *We have*

(a) $|\mathcal{L}''_\varepsilon(U)(\dot U^a, \dot U^b)|_{\infty,s,\gamma,T}$
$$\le C(|U|_{*,T})(|\dot U^a|_{\infty,s,\gamma,T}|\dot U^b|_{*,T} + |\dot U^b|_{\infty,s,\gamma,T}|\dot U^a|_{*,T} + \varepsilon|U|_{\infty,s,\gamma,T}|\dot U^a|_{*,T}|\dot U^b|_{*,T}),$$

(b) $|\mathcal{L}''_\varepsilon(U)(\dot U^a, \dot U^b)|_{0,s+1,\gamma,T}$
$$\le C(|U|_{*,T})(|\dot U^a|_{0,s+1,\gamma,T}|\dot U^b|_{*,T} + |\dot U^b|_{0,s+1,\gamma,T}|\dot U^a|_{*,T} + \varepsilon|U|_{0,s+1,\gamma,T}|\dot U^a|_{*,T}|\dot U^b|_{*,T}),$$

(c) $\langle\mathcal{B}''_\varepsilon(U)(\dot U^a, \dot U^b)\rangle_{s,\gamma,T}$
$$\le C(\langle U\rangle_{*,T})(\langle\dot U^a\rangle_{s,\gamma,T}\langle\dot U^b\rangle_{*,T} + \langle\dot U^b\rangle_{s,\gamma,T}\langle\dot U^a\rangle_{*,T} + \varepsilon\langle U\rangle_{s,\gamma,T}\langle\dot U^a\rangle_{*,T}\langle\dot U^b\rangle_{*,T}).$$

Proof. For $t \le s$ one computes $\partial^t(\mathcal{L}''_\varepsilon(U)(\dot U^a, \dot U^b))$, which is a sum of terms of the form

$$\tilde{\mathfrak{D}}(\varepsilon U)\partial^{\langle k\rangle}(\varepsilon U)\partial^l \dot U^a \partial^m \dot U^b, \quad \text{where } k+l+m=t.$$

[25] More precisely, each component is a sum of such terms.

Thus, the first estimate follows directly from Moser estimates. The remaining estimates are proved the same way. □

In the iteration scheme of Section 5B we will use H^s_T spaces on the boundary, while in the interior we use the following spaces.

Definition 2.13. For $s \in \{0, 1, 2, \dots\}$ let

$$E^s_T = CH^s_T \cap L^2 H^{s+1}_T \quad \text{with the norm } |U(x,\theta_0)|_{E^s_T} := |U|_{\infty,s,T} + |U|_{0,s+1,T},$$
$$E^s_{\gamma,T} = CH^s_{\gamma,T} \cap L^2 H^{s+1}_{\gamma,T} \quad \text{with the norm } |U(x,\theta_0)|_{E^s_{\gamma,T}} := |U|_{\infty,s,\gamma,T} + |U|_{0,s+1,\gamma,T}.$$

Remark 2.14. By Sobolev embedding we have

$$\begin{aligned} s &\ge [(d+1)/2] &&\Rightarrow E^s_T \subset CH^s_T \subset L^\infty(\Omega_T),\\ s &\ge [(d+1)/2]+1 &&\Rightarrow E^s_T \subset CH^s_T \subset L^\infty(\overline{\mathbb{R}}_+, W^{1,\infty}(b\Omega_T)),\\ s &\ge [(d+1)/2]+M_0 &&\Rightarrow E^s_T \subset CH^s_T \subset C^{0,M_0}_T. \end{aligned}$$

Note that E^s_T is a Banach algebra for $s \ge [(d+1)/2]$.

By Proposition 2.12 and Remark 2.14 we immediately obtain:

Proposition 2.15 (tame estimates for second derivatives). *Let $\mu_0 = [(d+1)/2]$ and suppose $s \ge 0$ lies in some finite interval. Then*

(a) $|\mathscr{L}''_\varepsilon(U)(\dot U^a, \dot U^b)|_{E^s_{\gamma,T}} \le C(|U|_{E^{\mu_0}_T})(|\dot U^a|_{E^s_{\gamma,T}} |\dot U^b|_{E^{\mu_0}_T} + |\dot U^b|_{E^s_{\gamma,T}} |\dot U^a|_{E^{\mu_0}_T} + \varepsilon |U|_{E^s_{\gamma,T}} |\dot U^a|_{E^{\mu_0}_T} |\dot U^b|_{E^{\mu_0}_T}),$

(b) $\langle \mathscr{B}''_\varepsilon(U)(\dot U^a, \dot U^b)\rangle_{s,\gamma,T}$
$$\le C(\langle U\rangle_{\mu_0,T})(\langle \dot U^a\rangle_{s,\gamma,T}\langle \dot U^b\rangle_{\mu_0,T} + \langle \dot U^b\rangle_{s,\gamma,T}\langle \dot U^a\rangle_{\mu_0,T} + \varepsilon\langle U\rangle_{s,\gamma,T}\langle \dot U^a\rangle_{\mu_0,T}\langle \dot U^b\rangle_{\mu_0,T}).$$

In order to obtain a tame estimate for the linearized system suitable for Nash–Moser iteration, we must recast estimate (2-43) without the L^∞ norms of $\dot U$ on the right. First of all, we fix the parameter $K > 0$. For instance, one may take $K = 1$. This choice is arbitrary because we are interested in a small data result.[26] We then choose constants $\varepsilon_0(K)$, $\gamma_0(K)$ as in Proposition 2.11 so that the estimate (2-43) holds for $s \in [0, \tilde\mu]$, where $\tilde\mu$ is defined in (5-57). *For the remainder of Section 2C and in Section 5B, the parameter K is fixed, and γ is also fixed as $\gamma = \gamma_0(K)$.*

Let

$$\kappa := |U|_{0,\mu_0+2,\gamma,T} + \langle U|_{x_d=0}\rangle_{\mu_0+2,\gamma,T}, \quad \text{where } \mu_0 := [(d+1)/2].$$

Applying (2-43) with $s = \mu_0$, we obtain for $0 < \varepsilon \le \varepsilon_0$

$$|\dot U|_{\infty,\mu_0,\gamma,T} + |\dot U|_{0,\mu_0+1,\gamma,T} + \langle \dot U|_{x_d=0}\rangle_{\mu_0+1,\gamma,T} \le C(K,\gamma)(|f|_{0,\mu_0+2,\gamma,T} + \langle g\rangle_{\mu_0+2,\gamma,T} + (|\dot U|_* + \langle \dot U\rangle_*)\kappa). \tag{2-44}$$

[26] If we were interested in a small time result for a given source term G, we would need to fix the constant K in terms of G and the parameters γ, T would be chosen accordingly.

By Remark 2.14 if κ is chosen small enough, we can absorb the last term on the right in (2-44) and obtain, with a new constant C,

$$|\dot{U}|_* + \langle \dot{U}|_{x_d=0}\rangle_* \le C(|f|_{0,\mu_0+2,\gamma,T} + \langle g\rangle_{\mu_0+2,\gamma,T}). \tag{2-45}$$

Substituting (2-45) in (2-43), we obtain:

Proposition 2.16 (tame estimate for the linearized system). *Let K and $\gamma = \gamma(K)$ be fixed as in Proposition 2.11 and suppose $|\varepsilon\partial_d U|_{C^{0,M_0-1}} + |U|_{C^{0,M_0}} \le K$ for $\varepsilon \in (0,1]$. Let $\mu_0 = [(d+1)/2]$ and $s \in [0,\tilde{\mu}]$, where $\tilde{\mu}$ is defined in* (5-57). *There exist positive constants $\kappa_0(\gamma,T)$, ε_0, and C such that if*

$$|U|_{0,\mu_0+2,\gamma,T} + \langle U|_{x_d=0}\rangle_{\mu_0+2,\gamma,T} \le \kappa_0,$$

solutions $\dot{U}$ of the linearized system (2-40) *satisfy, for $0 < \varepsilon \le \varepsilon_0$,*

$$\begin{aligned}|\dot{U}|_{E^s_{\gamma,T}} &+ \langle \dot{U}|_{x_d=0}\rangle_{s+1,\gamma,T} \\ &\le C[|f|_{0,s+2,\gamma,T} + \langle g\rangle_{s+2,\gamma,T} + (|f|_{0,\mu_0+2,\gamma,T} + \langle g\rangle_{\mu_0+2,\gamma,T})(|U|_{0,s+2,\gamma,T} + \langle U|_{x_d=0}\rangle_{s+2,\gamma,T})].\end{aligned}$$

3. Profile equations

3A. ***The key subsystem in the* 3 × 3 *strictly hyperbolic case.*** To simplify the exposition, we first treat the case of a 3×3 strictly hyperbolic system and a boundary frequency β for which there is one single resonance in which two incoming modes interact to produce an outgoing mode. This case already contains the main difficulties and is exactly the one we emphasize in the example of Appendix B. We explain later the relatively small changes needed to treat the general case of systems satisfying the assumptions of Section 1A.

The leading profile is decomposed as

$$\mathcal{V}^0(x,\theta_1,\theta_2,\theta_3) = \sigma_1(x,\theta_1)r_1 + \sigma_3(x,\theta_3)r_3 \tag{3-1}$$

where ϕ_2 is the outgoing phase and the resonant triple $(n_1,n_2,n_3) \in \mathbb{Z}^3 \setminus \{0\}$ satisfies

$$n_1\phi_1 = n_2\phi_2 + n_3\phi_3. \tag{3-2}$$

We can thus write

$$\mathcal{V}^0_{\text{inc}} = \sigma_1(x,\theta_1)r_1 + \sigma_3(x,\theta_3)r_3, \quad \mathcal{V}^1_{\text{out}} = \tau_2(x,\theta_2)r_2. \tag{3-3}$$

Furthermore, we have

$$\mathcal{V}^0_{\text{inc}}|_{x_d=0,\theta_1=\theta_3=\theta_0} = a(x',\theta_0)e = a(x',\theta_0)(e_1+e_3), \tag{3-4}$$

so (recall that $e = e_1 + e_3$, where $e_i \in \operatorname{span}\{r_i\}$ spans $\ker B \cap \mathbb{E}^s(\beta)$)

$$\sigma_i(x',0,\theta_0)r_i = a(x',\theta_0)e_i, \quad i = 1,3, \tag{3-5}$$

which determines the trace of σ_i in terms of a.

Applying the operators E_i for $i=1,3$ to (1-42)(a) and for $i=2$ to (1-42)(b) and using Corollary 1.27 for (1-42)(c), we obtain the following system for the unknowns $(\sigma_1,\tau_2,\sigma_3,a)$, where $\mathscr{A}(x',\theta_0)$ denotes the unique function with mean zero such that $\partial_{\theta_0}\mathscr{A}=a$:

$$\begin{aligned}
&X_{\phi_1}\sigma_1+c_1\sigma_1=0,\\
&X_{\phi_3}\sigma_3+c_3\sigma_3=0,\\
&X_{\phi_2}\tau_2+c_0\tau_2+c_2\int_0^{2\pi}\sigma_{1,n_1}\Big(x,\frac{n_2}{n_1}\theta_2+\frac{n_3}{n_1}\theta_3\Big)\sigma_3(x,\theta_3)\,d\theta_3=0\\
&X_{\mathrm{Lop}}\mathscr{A}+c_4\mathscr{A}+c_5\tau_2|_{x_d=0}+c_6(a^2)^*=-b\cdot G^*\quad\text{on } b\Omega_T,
\end{aligned}\tag{3-6}$$

where the first three equations hold on Ω_T, and the constants c_i are readily computed real constants. Here $\sigma_{1,n_1}(x,\theta_1)$ is the image of the function σ_1 under the *preparation map*

$$\sigma_1(x,\theta_1)=\sum_{k\in\mathbb{Z}}f_k(x)e^{ik\theta_1}\to\sum_{k\in\mathbb{Z}}f_{kn_1}(x)e^{ikn_1\theta_1},\tag{3-7}$$

a map designed so that the integral in (3-6) picks out resonances in the product of σ_1 and σ_3.[27]

Differentiating with respect to θ_0, we rewrite the last equation of (3-6) as

$$X_{\mathrm{Lop}}a+c_4a+c_5\partial_{\theta_0}\tau_2|_{x_d=0}+c_6\partial_{\theta_0}(a^2)=-b\cdot\partial_{\theta_0}G^*=:g\quad\text{on } b\Omega_T.\tag{3-8}$$

We now set $V:=(\sigma_1(x,\theta_1),\sigma_3(x,\theta_3),\tau_2(x,\theta_2),a(x',\theta_0))$ and define the interior and boundary operators for the leading profile system:

$$\begin{aligned}
\mathscr{L}(V)&:=\begin{pmatrix}X_{\phi_1}\sigma_1+c_1\sigma_1\\ X_{\phi_3}\sigma_3+c_3\sigma_3\\ X_{\phi_2}\tau_2+c_0\tau_2+c_2\int_0^{2\pi}\sigma_{1,n_1}\big(x,\frac{n_2}{n_1}\theta_2+\frac{n_3}{n_1}\theta_3\big)\sigma_3(x,\theta_3)\,d\theta_3\end{pmatrix},\\
\mathscr{B}(V)&:=X_{\mathrm{Lop}}a+c_4a+c_5\partial_{\theta_0}\tau_2|_{x_d=0}+c_6\partial_{\theta_0}(a^2).
\end{aligned}\tag{3-9}$$

In this notation the profile subsystem becomes

$$\begin{aligned}
&\mathscr{L}(V)=0 &&\text{in } \Omega_T,\\
&\mathscr{B}(V)=g &&\text{in } b\Omega_T,\\
&V=0 &&\text{in } t\le 0,
\end{aligned}\tag{3-10}$$

where the additional relations (3-5) hold giving the traces of σ_1,σ_3 in terms of a. The following existence result for the key subsystem is proved in Section 5A using the tame estimates derived in Section 3B below.

Proposition 3.1. *Fix $T>0$, let $\nu_0:=[(d+1)/2]+1$, $\nu:=2\nu_0+4$, and $\tilde\nu:=2\nu-\nu_0$, and suppose $g\in H^{\tilde\nu-2}(b\Omega_T)$. Rewrite V as $V=(V',a)$. If $\langle g\rangle_\nu$ is small enough, there exists a solution V of the profile subsystem* (3-10) *with $V'\in H^{\nu-1}(\Omega_T)$, $(V'|_{x_d=0},a)\in H^{\nu-1}(b\Omega_T)$.*

[27] Interaction integrals like the one in (3-6) are discussed further in [Coulombel et al. 2011, Proposition 2.13].

Remark 3.2. (1) Although the original problem is semilinear with a nonlinear zero-order boundary condition, the profile system (3-9) has a *quasilinear* first-order boundary operator and an interior operator that includes a nonlinear, nonlocal, integro-pseudodifferential operator given by the interaction integral. The nonlocality arises both from the $d\theta_3$-integration and from the pseudodifferential operator $\sigma_1 \to \sigma_{1,n_1}$.

(2) Attempts to solve the system (3-10) by a standard Picard iteration lead to a (fatal) loss of a derivative from one iterate to the next. The reason is that σ_1 and σ_3 have the regularity of a (incoming transport equation), and therefore τ_2 has the same regularity as a. However, the equation for a involves the derivative $\partial_{\theta_0}\tau_2|_{x_d=0}$ and this term induces the loss. Thus we shall use Nash–Moser iteration to prove Proposition 3.1.

3B. ***Tame estimates.*** With $V = (\sigma_1, \sigma_3, \tau_2, a)$ and $\dot V = (\dot\sigma_1, \dot\sigma_3, \dot\tau_2, \dot a)$, we compute the first derivatives of $\mathscr{L}$ and $\mathscr{B}$:

$$\text{(a)}\quad \mathscr{L}'(V)\dot V = \begin{pmatrix} X_{\phi_1}\dot\sigma_1 + c_1\dot\sigma_1 \\ X_{\phi_3}\dot\sigma_3 + c_3\dot\sigma_3 \\ X_{\phi_2}\dot\tau_2 + c_0\dot\tau_2 + c_2\int_0^{2\pi}\sigma_{1,n_1}\big(x, \tfrac{n_2}{n_1}\theta_2 + \tfrac{n_3}{n_1}\theta_3\big)\dot\sigma_3(x,\theta_3)\,d\theta_3 \\ + c_2\int_0^{2\pi}\sigma_{3,n_3}\big(x, -\tfrac{n_2}{n_3}\theta_2 + \tfrac{n_3}{n_1}\theta_1\big)\dot\sigma_1(x,\theta_1)\,d\theta_1 \end{pmatrix}, \tag{3-11}$$

$$\text{(b)}\quad \mathscr{B}'(V)\dot V = X_{\mathrm{Lop}}\dot a + c_4\dot a + c_5\partial_{\theta_0}\dot\tau_2|_{x_d=0} + 2c_6(a\partial_{\theta_0}\dot a + \dot a\partial_{\theta_0}a).$$

Here we have used the property

$$\int_0^{2\pi}\sigma_{3,n_3}\left(x, -\frac{n_2}{n_3}\theta_2 + \frac{n_3}{n_1}\theta_1\right)\dot\sigma_1(x,\theta_1)\,d\theta_1 = \int_0^{2\pi}\dot\sigma_{1,n_1}\left(x, \frac{n_2}{n_1}\theta_2 + \frac{n_3}{n_1}\theta_3\right)\sigma_3(x,\theta_3)\,d\theta_3, \tag{3-12}$$

which follows readily by looking at the Fourier series of the factors of the integrand. For the second derivatives we obtain

$$\mathscr{L}''(V)(\dot V^a, \dot V^b) = c_2\begin{pmatrix} 0 \\ 0 \\ \int_0^{2\pi}\dot\sigma^a_{1,n_1}\big(x, \tfrac{n_2}{n_1}\theta_2 + \tfrac{n_3}{n_1}\theta_3\big)\dot\sigma_3^b(x,\theta_3)\,d\theta_3 + \int_0^{2\pi}\dot\sigma^b_{1,n_1}\big(x, \tfrac{n_2}{n_1}\theta_2 + \tfrac{n_3}{n_1}\theta_3\big)\dot\sigma_3^a(x,\theta_3)\,d\theta_3 \end{pmatrix},$$

$$\mathscr{B}''(V)(\dot V^a, \dot V^b) = 2c_6(\dot a^a\partial_{\theta_0}\dot a^b + \dot a^b\partial_{\theta_0}\dot a^a). \tag{3-13}$$

Proposition 3.3 (tame estimates for second derivatives). (a) *Let ν_1 be the smallest integer greater than $(d+2)/2$ and let $s \ge 0$. We have*

$$|\mathscr{L}''(V)(\dot V^a, \dot V^b)|_{s,\gamma} \le C(|\dot V^a|_{s,\gamma}|\dot V^b|_{\nu_1} + |\dot V^b|_{s,\gamma}|\dot V^a|_{\nu_1}), \tag{3-14}$$

where C is independent of V, γ, and T.

(b) *Let ν_2 be the smallest integer greater than $(d+1)/2 + 1$ and let $s \ge 0$. We have*

$$\langle\mathscr{B}''(V)(\dot V^a, \dot V^b)\rangle_{s,\gamma} \le C(\langle\dot V^a\rangle_{s+1,\gamma}\langle\dot V^b\rangle_{\nu_2} + \langle\dot V^b\rangle_{s+1,\gamma}\langle\dot V^a\rangle_{\nu_2}), \tag{3-15}$$

where C is independent of V, γ, and T.

In (3-14) *and* (3-15), *the constant C can be chosen independent of s in any fixed finite interval.*

Proof. (a) Moser estimates imply

$$|\dot\sigma^a_{1,n_1}\left(x, \frac{n_2}{n_1}\theta_2 + \frac{n_3}{n_1}\theta_3\right)\dot\sigma^b_3(x,\theta_3)|_{H^s_\gamma(x,\theta_2)} \le C(|\dot\sigma^a_{1,n_1}|_{s,\gamma}|\dot\sigma^b_3|_{L^\infty} + |\dot\sigma^a_{1,n_1}|_{L^\infty}|\dot\sigma^b_3|_{H^s_\gamma(x)}), \tag{3-16}$$

since $\dot\sigma^b_3$ is independent of θ_2. We have

$$\int_0^{2\pi} |\dot\sigma^b_3(x,\theta_3)|_{H^s_\gamma(x)}\, d\theta_3 \le C|\dot\sigma^b_3|_{L^2(\theta_3, H^s_\gamma(x))} \le C|\dot\sigma^b_3|_{s,\gamma}. \tag{3-17}$$

The estimate (3-14) now follows by Sobolev embedding and the fact that

$$|\dot\sigma^a_{1,n_1}|_{s,\gamma} \le |\dot\sigma^a_1|_{s,\gamma}. \tag{3-18}$$

(b) Again Moser estimates imply

$$\langle \dot a^a \partial_{\theta_0} \dot a^b\rangle_{s,\gamma} \le C(\langle \dot a^a\rangle_{s,\gamma}\langle \partial_{\theta_0}\dot a^b\rangle_{L^\infty} + \langle \dot a^a\rangle_{L^\infty}\langle\partial_{\theta_0}\dot a^b\rangle_{s,\gamma}), \tag{3-19}$$

so the estimate (3-15) follows by Sobolev embedding. □

Next we derive tame energy estimates for the linearized problem

$$\begin{aligned} &\mathscr{L}'(V)\dot V = f && \text{in } \Omega_T,\\ &\mathscr{B}'(V)\dot V = g && \text{on } b\Omega_T,\\ &V = 0 && \text{in } t<0, \end{aligned} \tag{3-20}$$

where f and g vanish in $t<0$. We begin with a simple proposition.

Proposition 3.4. (1) *If the phase ϕ_p is incoming, solutions of*

$$X_{\phi_p}\sigma_p + c_p\sigma_p = h \quad \text{in } \Omega_T, \quad \sigma_p = 0 \quad \text{in } t<0 \tag{3-21}$$

satisfy, for γ large (depending on c_p),

$$\sqrt{\gamma}|\sigma_p|_{s,\gamma} \le C\left(\langle\sigma_p\rangle_{s,\gamma} + \frac{|h|_{s,\gamma}}{\sqrt{\gamma}}\right). \tag{3-22}$$

(2) *If the phase ϕ_p is outgoing, solutions of* (3-21) *satisfy, for γ large (depending on c_p),*

$$\sqrt{\gamma}|\sigma_p|_{s,\gamma} + \langle\sigma_p\rangle_{s,\gamma} \le C\frac{|h|_{s,\gamma}}{\sqrt{\gamma}}. \tag{3-23}$$

(3) *Solutions in ω_T of*

$$X_{\mathrm{Lop}}\dot a + c_4\dot a + 2c_6(a\partial_{\theta_0}\dot a + \dot a\partial_{\theta_0}a) = g,\ \dot a = 0 \quad \text{in } t<0 \tag{3-24}$$

satisfy, for C_K, $\gamma \ge \gamma_K$ (where $K = \langle a\rangle_{W^{1,\infty}}$),

$$\sqrt{\gamma}\langle\dot a\rangle_{s,\gamma} \le \frac{C_K}{\sqrt{\gamma}}\left(\langle g\rangle_{s,\gamma} + \langle a\rangle_{s+1,\gamma}\langle\dot a\rangle_{W^{1,\infty}}\right). \tag{3-25}$$

The second term on the right in (3-25) *does not appear in the $s=0$ estimate.*

Proof. (1) To prove (3-23) with $s=0$, one considers the problem satisfied by $e^{-\gamma t}\sigma_p$, multiplies the equation by $e^{-\gamma t}\sigma_p$, integrates $dx\,d\theta_p$ on Ω_T, and performs obvious integrations by parts. One then applies the L^2 estimate to the problem satisfied by tangential derivatives $\gamma^{s-|\beta|}\partial^\beta_{x',\theta_p}\sigma_p$, $|\beta|\le s$. Normal derivatives are estimated using the equation and the tangential estimates. The proof of (3-23) is similar. We refer to [Benzoni-Gavage and Serre 2007] for a complete discussion of such estimates.

(2) The proof of (3-25) is similar, but in the higher derivative estimates one now has forcing terms that are commutators involving a. The commutators are linear combinations of terms of the form

$$\gamma^{s-|\beta|}(\partial^{\beta_1}_{x',\theta_0}a)(\partial^{\beta_2}_{x',\theta_0}\partial_{\theta_0}\dot{a})\quad\text{where } |\beta_1|+|\beta_2|=|\beta|,\ |\beta_1|\ge 1, \tag{3-26}$$

or linear combinations of terms of the form

$$\gamma^{s-|\beta|}(\partial^{\beta_1}_{x',\theta_0}\dot{a})(\partial^{\beta_2}_{x',\theta_0}\partial_{\theta_0}a)\quad\text{where } |\beta_1|+|\beta_2|=|\beta|,\ |\beta_2|\ge 1. \tag{3-27}$$

Applying Moser estimates to (3-26) after writing $\partial^{\beta_1}a=\partial^{\beta_1'}\partial a$, we obtain

$$\begin{aligned}\langle\gamma^{s-|\beta|}(\partial^{\beta_1}_{x',\theta_0}a)(\partial^{\beta_2}_{x',\theta_0}\partial_{\theta_0}\dot{a})\rangle_{0,\gamma} &\le C(\langle\partial a\rangle_{L^\infty}\langle\partial_{\theta_0}\dot{a}\rangle_{m-1,\gamma}+\langle\partial a\rangle_{m-1,\gamma}\langle\partial_{\theta_0}\dot{a}\rangle_{L^\infty})\\ &\le C(\langle a\rangle_{W^{1,\infty}}\langle\dot{a}\rangle_{s,\gamma}+\langle a\rangle_{s,\gamma}\langle\dot{a}\rangle_{W^{1,\infty}}).\end{aligned} \tag{3-28}$$

The factor $C_K/\sqrt{\gamma}$ on the forcing term in the L^2 estimate allows the first term on the right to be absorbed by taking γ large.

The estimate of (3-27) is similar, but we do not split the ∂^{β_2} derivative, and after absorbing a term we are left with $(C_K/\sqrt{\gamma})\langle\dot{a}\rangle_{L^\infty}\langle a\rangle_{s+1,\gamma}$ on the right. □

We now use Proposition 3.4 to estimate solutions of the linearized problem (3-20) by treating the interaction integrals in (3-11)(a) and the term $c_5\partial_{\theta_0}\tau_2$ in (3-11)(b) as additional forcing terms. Setting

$$V_{\mathrm{inc},n}:=(\sigma_{1,n_1},\sigma_{3,n_3}),\qquad V_{\mathrm{inc}}=(\sigma_1,\sigma_3),\ V_{\mathrm{out}}=\tau_2, \tag{3-29}$$

estimating interaction integrals as in (3-16) and (3-17), and using (3-18), we immediately obtain

$$\begin{aligned}\sqrt{\gamma}|\dot{V}_{\mathrm{out}}|_{s,\gamma}+\langle\dot{V}_{\mathrm{out}}\rangle_{s,\gamma} &\le \frac{C}{\sqrt{\gamma}}(|f|_{s,\gamma}+|V_{\mathrm{inc},n}|_{L^\infty}|\dot{V}_{\mathrm{inc}}|_{s,\gamma}+|V_{\mathrm{inc}}|_{s,\gamma}|\dot{V}_{\mathrm{inc}}|_{L^\infty}),\\ \sqrt{\gamma}|\partial_\theta\dot{V}_{\mathrm{out}}|_{s,\gamma}+\langle\partial_{\theta_0}\dot{V}_{\mathrm{out}}\rangle_{s,\gamma} &\le \frac{C}{\sqrt{\gamma}}(|\partial_\theta f|_{s,\gamma}+|\partial_\theta V_{\mathrm{inc},n}|_{L^\infty}|\dot{V}_{\mathrm{inc}}|_{s,\gamma}+|\partial_\theta V_{\mathrm{inc}}|_{m,\gamma}|\dot{V}_{\mathrm{inc}}|_{L^\infty}),\end{aligned} \tag{3-30}$$

and

$$\begin{aligned}\sqrt{\gamma}|\dot{V}_{\mathrm{inc}}|_{s,\gamma} &\le C\left(\langle\dot{V}_{\mathrm{inc}}\rangle_{s,\gamma}+\frac{|f|_{s,\gamma}}{\sqrt{\gamma}}\right),\\ \sqrt{\gamma}\langle\dot{V}_{\mathrm{inc}}\rangle_{s,\gamma} &\le \frac{C_K}{\sqrt{\gamma}}(\langle g\rangle_{s,\gamma}+\langle\partial_{\theta_0}\dot{V}_{\mathrm{out}}\rangle_{s,\gamma}+\langle V_{\mathrm{inc}}\rangle_{s+1,\gamma}\langle\dot{V}_{\mathrm{inc}}\rangle_{W^{1,\infty}}).\end{aligned} \tag{3-31}$$

This leads to the following “pretame” estimate.

Proposition 3.5. *Let $\mu_0 := [(d+1)/2]$, fix $K_1 > 0$, and suppose $|V_{\mathrm{inc}}|_{\mu_0+2} \le K_1$.*[28] *For $s \ge 0$ in any fixed finite interval, there exist constants $C(K_1)$, $\gamma(K_1)$ such that, for $\gamma \ge \gamma(K_1)$, solutions of the linearized problem* (3-20) *satisfy*

$$\sqrt{\gamma}|\dot{V}_{\mathrm{out}}, \partial_\theta \dot{V}_{\mathrm{out}}, \dot{V}_{\mathrm{inc}}|_{s,\gamma} + \langle \dot{V}_{\mathrm{out}}, \partial_{\theta_0} \dot{V}_{\mathrm{out}}\rangle_{s,\gamma} + \sqrt{\gamma}\langle \dot{V}_{\mathrm{inc}}\rangle_{s,\gamma} \le \frac{C(K_1)}{\sqrt{\gamma}}(|f|_{s+1,\gamma} + \langle g\rangle_{s,\gamma} + |V_{\mathrm{inc}}|_{s+1,\gamma}|\dot{V}_{\mathrm{inc}}|_{L^\infty} + \langle V_{\mathrm{inc}}\rangle_{s+1,\gamma}\langle \dot{V}_{\mathrm{inc}}\rangle_{W^{1,\infty}}). \quad (3\text{-}32)$$

Proof. We add the estimates (3-30) and (3-31) and absorb the terms

$$\frac{C_K}{\sqrt{\gamma}}(\langle \dot{V}_{\mathrm{inc}}\rangle_{s,\gamma} + \langle \partial_{\theta_0} \dot{V}_{\mathrm{out}}\rangle_{s,\gamma} + |V_{\mathrm{inc},n}, \partial_\theta V_{\mathrm{inc},n}|_{L^\infty}|\dot{V}_{\mathrm{inc}}|_{s,\gamma}) \quad (3\text{-}33)$$

by taking γ large, after observing that

$$|V_{\mathrm{inc},n}, \partial_\theta V_{\mathrm{inc},n}|_{L^\infty} \le C|V_{\mathrm{inc},n}|_{\mu_0+2} \le C|V_{\mathrm{inc}}|_{\mu_0+2} \quad (3\text{-}34)$$

and

$$K = \langle V_{\mathrm{inc}}\rangle_{W^{1,\infty}} \le C|V_{\mathrm{inc}}|_{\mu_0+2}. \qquad \square$$

We now set $\tilde{\nu} := 2\nu - \nu_0$ as in Proposition 3.1 and choose constants $C(K_1)$, $\gamma(K_1)$ as in Proposition 3.5 corresponding to the interval $s \in [0, \tilde{\nu}]$.[29] *In the remainder of Section 3B and also in Section 5A, γ is fixed as $\gamma = \gamma(K_1)$.*

To obtain a tame estimate, we need to remove the terms depending on $\dot{V}_{\mathrm{inc}}$ on the right side of (3-32). Let

$$K_2 = |V_{\mathrm{inc}}|_{\mu_0+2,\gamma} + \langle V_{\mathrm{inc}}\rangle_{\mu_0+2,\gamma}. \quad (3\text{-}35)$$

Applying (3-32) with $s = \mu_0 + 1$, we obtain

$$\sqrt{\gamma}|\dot{V}_{\mathrm{inc}}|_{\mu_0+1,\gamma} + \sqrt{\gamma}\langle \dot{V}_{\mathrm{inc}}\rangle_{\mu_0+1,\gamma} \le \frac{C(K_1)}{\sqrt{\gamma}}[|f|_{\mu_0+2,\gamma} + \langle g\rangle_{\mu_0+1,\gamma} + (|\dot{V}_{\mathrm{inc}}|_{L^\infty} + \langle \dot{V}_{\mathrm{inc}}\rangle_{W^{1,\infty}})K_2]. \quad (3\text{-}36)$$

By Sobolev embedding, if $K_2 = K_2(\gamma, T)$ is chosen small enough, we can absorb the last term on the right in (3-36) and obtain, with a new C,

$$|\dot{V}_{\mathrm{inc}}|_{L^\infty} + \langle \dot{V}_{\mathrm{inc}}\rangle_{W^{1,\infty}} \le C(|f|_{\mu_0+2,\gamma} + \langle g\rangle_{\mu_0+1,\gamma}). \quad (3\text{-}37)$$

For γ fixed as above, setting $|U|_{s,\gamma} = |U|_s$ now and substituting (3-37) in (3-32), we obtain the estimate in the following proposition.

Proposition 3.6 (tame estimate for the linearized system). *Let $\mu_0 = [(d+1)/2]$ and $s \in [0, \tilde{\nu}]$. There exists $\kappa_0 = \kappa_0(\gamma, T) > 0$ and a constant C depending on κ_0 such that if*

$$|V_{\mathrm{inc}}|_{\mu_0+2} + \langle V_{\mathrm{inc}}\rangle_{\mu_0+2} \le \kappa_0, \quad (3\text{-}38)$$

solutions of the linearized system (3-20) *satisfy*

$$|\dot{V}|_s + \langle \dot{V}\rangle_s \le C[|f|_{s+1} + \langle g\rangle_s + (|f|_{\mu_0+2} + \langle g\rangle_{\mu_0+1})(|V|_{s+1} + \langle V\rangle_{s+1})]. \quad (3\text{-}39)$$

[28] In this proposition $\mu_0 = [d/2]$ would work, but we make the above choice so as not to have to redefine μ_0 later.

[29] The choice of $\tilde{\nu}$ is explained in Section 5A.

Proof. We have proved the a priori estimate (3-39) for sufficiently smooth solutions of the linearized system. The existence of such solutions now follows by standard arguments, which we summarize here for completeness.

The unknown in the linearized system (3-20) is $(\dot\sigma_1, \dot\sigma_3, \dot\tau_2, \dot a)$. We can solve the linearized system by putting the terms that involve $\partial_{\theta_0}\dot\tau_2$ or $\partial_{\theta_0}\dot a$ on the right and replacing the operator ∂_{θ_0}, when it acts on those terms, by a finite difference operator $\partial^h_{\theta_0}$:

$$\begin{aligned}
&X_{\phi_1}\dot\sigma_1^h + c_1\dot\sigma_1^h = f_1,\\
&X_{\phi_3}\dot\sigma_3^h + c_3\dot\sigma_3^h = f_2,\\
&X_{\phi_2}\dot\tau_2^h + c_0\dot\tau_2^h = f_3 - c_2\int_0^{2\pi}\sigma_{1,n_1}\big(x, \tfrac{n_2}{n_1}\theta_2 + \tfrac{n_3}{n_1}\theta_3\big)\dot\sigma_3^h(x,\theta_3)\,d\theta_3\\
&\qquad\qquad\qquad\qquad - c_2\int_0^{2\pi}\sigma_{3,n_3}\big(x, -\tfrac{n_2}{n_3}\theta_2 + \tfrac{n_3}{n_1}\theta_1\big)\dot\sigma_1^h(x,\theta_1)\,d\theta_1,\\
&X_{\mathrm{Lop}}\dot a^h + c_4\dot a^h + 2c_6\dot a^h\partial_{\theta_0}a = g - c_5\partial^h_{\theta_0}\dot\tau_2^h - 2c_6 a\partial^h_{\theta_0}\dot a^h.
\end{aligned}\tag{3-40}$$

For fixed $h \in (0, 1]$ we can solve this system by Picard iteration, where n-th iterates appear on the right and $(n+1)$-st iterates appear on the left. All iterates are 0 in $t < 0$ and the iterates with index zero are all 0.

We then need an estimate that is uniform in h. This can be done by repeating the existing proof of tame estimates, using the operator $\partial^h_{\theta_0}$ in place of ∂_{θ_0}. This gives an estimate like (3-39):

$$|\dot V^h|_s + \langle\dot V^h\rangle_s \le C[|f|_{s+1} + \langle g\rangle_s + (|f|_{\mu_0+2} + \langle g\rangle_{\mu_0+1})(|V|_{s+1} + \langle V\rangle_{s+1})],\tag{3-41}$$

where $\dot V^h := (\dot\sigma_1^h, \dot\sigma_3^h, \dot\tau_2^h, \dot a^h)$ and C is uniform for $h \in (0, 1]$. Passing to a subsequence, we obtain the desired solution of the linearized system. □

Remark 3.7 (short time, given data). For a given $T > 0$, let K_1 and $\gamma = \gamma(K_1)$ be as in Proposition 3.5. As we saw above, to obtain a tame estimate, we need to take $|\mathcal{V}_{\mathrm{inc}}|_{\mu_0+2} + \langle\mathcal{V}_{\mathrm{inc}}\rangle_{\mu_0+2}$ small. In our formulation of Theorem 1.15, T is fixed ahead of time and we achieve (3-38) by taking G small in an appropriate norm on Ω_T. For a given G as in (1-2) vanishing in $t < 0$, another way to proceed is to shrink T; that is, to work on Ω_{T_1} where $0 < T_1 < T$ is chosen so that $\gamma_1 := 1/T_1 \ge \gamma(K_1)$ and so that

$$|V_{\mathrm{inc}}|_{H^{\mu_0+2}_{\gamma_1}(\Omega_{T_1})} + \langle V_{\mathrm{inc}}\rangle_{H^{\mu_0+2}_{\gamma_1}(\omega_{T_1})}$$

is small enough to absorb the terms involving $\dot V_{\mathrm{inc}}$ on the right in (3-36). One again obtains an estimate of the form (3-39), where now

$$|U|_s := |U|_{H^s_{\gamma_1}(\Omega_{T_1})}.$$

The iteration scheme described in Section 5A applies to this situation with no essential change as well.

3C. ***The key subsystem in the general case.*** Recall that $\{1, \dots, M\} = \mathbb{O} \cup \mathcal{I}$, where $\mathbb{O}$ and $\mathcal{I}$ contain the indices corresponding to outgoing and incoming phases. We further decompose $\mathbb{O} = \mathbb{O}_1 \cup \mathbb{O}_2$, where $\mathbb{O}_1$ consists of indices m such that ϕ_m is part of at least one triple of resonant phases with the property that the other two phases in that triple are incoming. For a given $m \in \mathbb{O}_1$ the phase ϕ_m might belong to more than one such triple.

Now instead of (3-3) we have

$$\mathcal{V}^0_{\mathrm{inc}} = \sum_{m\in\mathcal{I}} \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}(x,\theta_m) r_{m,k} \quad \text{and} \quad \mathcal{V}^1_{\mathrm{out}} = \sum_{m\in\mathcal{O}_1} \sum_{k=1}^{\nu_{k_m}} \tau_{m,k}(x,\theta_m) r_{m,k}, \tag{3-42}$$

since terms $\tau_{m,k}$ in the expansion of $\mathcal{V}^1_{\mathrm{out}}$ vanish if $m \in \mathcal{O}_2$ as a consequence of (1-36) and $\mathcal{V}^0 = \mathcal{V}^0_{\mathrm{inc}}$. Recalling that

$$e = \sum_{m\in\mathcal{I}} \sum_{k=1}^{\nu_{k_m}} e_{m,k}, \quad \text{where } e_{m,k} \in \operatorname{span}\{r_{m,k}\}, \tag{3-43}$$

we see that in place of (3-4) we now have

$$\mathcal{V}^0_{\mathrm{inc}}|_{x_d=0;\theta_m=\theta_0, m\in I} = a(x',\theta_0)e = \sum_{m\in\mathcal{I}} \sum_{k=1}^{\nu_{k_m}} a e_{m,k} = \sum_{m\in\mathcal{I}} \sum_{k=1}^{\nu_{k_m}} \sigma_{m,k}(x',0,\theta_0) r_{m,k}, \tag{3-44}$$

and thus

$$\sigma_{m,k}(x',0,\theta_0) r_{m,k} = a(x',\theta_0) e_{m,k} \quad \text{for } m \in \mathcal{I},\ k = 1,\dots,\nu_{k_m}. \tag{3-45}$$

Next we derive the formulas for $\mathcal{L}(V)$ and $\mathcal{B}(V)$ in the general case. The unknown is now

$$V = (\sigma_{m,k}, m \in \mathcal{I}, k = 1,\dots,\nu_{k_m}; \tau_{m,k}, m \in \mathcal{O}_1, k = 1,\dots,\nu_{k_m}; a). \tag{3-46}$$

Suppose $q \in \mathcal{O}_1$ and that (ϕ_p,ϕ_q,ϕ_s) is a resonant triple such that

$$n_p\phi_p = n_q\phi_q + n_s\phi_s \quad \text{where } p, s \in \mathcal{I} \text{ and } \gcd(n_p,n_q,n_s) = 1. \tag{3-47}$$

Applying the projectors $E_{m,k}$, $m \in \mathcal{I}$, $k = 1,\dots,\nu_{k_m}$ to (1-42)(a) and the projectors $E_{q,l}$, $q \in \mathcal{O}_1$, $l = 1,\dots,\nu_{k_q}$ to (1-42)(b), we obtain

$$\mathcal{L}(V) = \begin{pmatrix} X_{\phi_m}\sigma_{m,k} + c_{m,k}\sigma_{m,k}; \quad m \in \mathcal{I},\ k = 1,\dots,\nu_{k_m} \\ X_{\phi_q}\tau_{q,l} + c_{q,l}\tau_{q,l} + \sum_{k=1}^{\nu_{k_p}} \sum_{k'=1}^{\nu_{k_s}} d^{k,k'}_{q,l} \int_0^{2\pi} (\sigma_{p,k})_{n_p}\big(x, \frac{n_q}{n_p}\theta_q + \frac{n_s}{n_p}\theta_s\big) \sigma_{s,k'}(x,\theta_s)\,d\theta_s + (\text{similar}); \\ \hfill q \in \mathcal{O}_1,\ l = 1,\dots,\nu_{k_q} \end{pmatrix}. \tag{3-48}$$

Here "similar" denotes a finite sum of families (i.e., sums over k and k') of integrals similar to the one explicitly given. One such family corresponds to each distinct resonant triple involving the outgoing phase ϕ_q and two incoming phases.[30] The values of the real constants $c_{m,k}$, $d^{k,k'}_{q,l}$ are not important for our analysis, but, for example, the $d^{k,k'}_{q,l}$ are given by[31]

$$d^{k,k'}_{q,l} = \frac{1}{2\pi} \ell_{q,l} \cdot [dD(0)(r_{p,k}, r_{s,k'}) + dD(0)(r_{s,k'}, r_{p,k})]. \tag{3-49}$$

By a computation similar to the one that produced (3-8), we obtain from (1-42)(c)

[30] We do not distinguish between (ϕ_p,ϕ_q,ϕ_s) and (ϕ_p,ϕ_s,ϕ_q). We do distinguish between (ϕ_p,ϕ_q,ϕ_s) and (ϕ_p,ϕ_q,ϕ_t).

[31] We have suppressed indices r, s on the $d^{k,k'}_{q,l}$.

$$\mathscr{B}(V) = X_{\mathrm{Lop}} a + f_1 a + \sum_{q \in \mathbb{O}_1} \sum_{l=1}^{\nu_{k_q}} f_{q,l} \partial_{\theta_0} \tau_{q,l} + f_2 \partial_{\theta_0}(a^2) \tag{3-50}$$

for some real constants f_1, f_2, $f_{q,l}$. For example, we have $f_2 = -b \cdot [\psi'(0)(e, e)]$. Thus the system (1-42) may be rewritten

$$\begin{aligned} &\mathscr{L}(V) = 0 && \text{in } \Omega_T, \\ &\mathscr{B}(V) = -b \cdot \partial_{\theta_0} G^* := g && \text{on } b\Omega_T, \\ &V = 0 && \text{in } t < 0, \end{aligned} \tag{3-51}$$

where the relations (3-45) hold.

It is now a simple matter to write out the expressions for the first and second derivatives of $\mathscr{L}$ and $\mathscr{B}$. For example, just as the interaction integral in (3-9) gave rise to two integrals in the expression (3-11) for $\mathscr{L}'(V)$ in the 3×3 case, it is clear that each integral in (3-48) will give rise to two integrals in the new expression for $\mathscr{L}'(V)$. The tame estimates for second derivatives are proved exactly as before, and Proposition 3.3 holds verbatim in the general case. Proposition 3.4 is used exactly as before to prove estimates for the linearized system. With the unknown V as given in (3-46) and after defining V_{inc}, V_{out}, $\dot{V}_{\mathrm{inc}}$, $\dot{V}_{\mathrm{out}}$ in the obvious way, we see that the "pretame" estimate of Proposition 3.5 and the tame estimate of Proposition 3.6 hold verbatim in the general case. The iteration scheme of Section 5A depends only on the tame estimates. Thus it applies here without change and Proposition 3.1 holds verbatim in the general case.

Once the key subsystem is solved, we can easily complete the solution of the full profile system (1-35)–(1-36). The precise result for the full system will be proved in Theorem 5.11.

4. Error analysis

Here we carry out the error analysis sketched in Section 1E, beginning with the proof of Proposition 1.29.

Proof of Proposition 1.29. Step 1: *Noncharacteristic modes.* We write

$$F(x, \theta) = F_0(x) + \sum_{\alpha \notin \mathscr{C}} F_\alpha(x) e^{i\alpha\cdot\theta} + \sum_{m=1}^{M} \sum_{\alpha \in \mathscr{C}_m \setminus \{0\}} F_\alpha(x) e^{i\alpha\cdot\theta},$$

and recall that the sums are finite. Set

$$n_\alpha = \sum_{j=1}^{M} \alpha_j \quad \text{and} \quad \underline{\omega} = (\underline{\omega}_1, \dots, \underline{\omega}_M).$$

Since $EF = 0$, we first note that F_0 vanishes. For any α, we have

$$\big(F_\alpha(x) e^{i\alpha\cdot\theta}\big)|_{\theta \to (\theta_0, \xi_d)} = F_\alpha(x) e^{i n_\alpha \theta_0 + i(\alpha\cdot\underline{\omega})\xi_d},$$

and when $\alpha \notin \mathscr{C}$, we look for $U_\alpha(x)$ such that

$$\mathscr{L}_0(\partial_{\theta_0}, \partial_{\xi_d}) U_\alpha(x) e^{i n_\alpha \theta_0 + i(\alpha\cdot\underline{\omega})\xi_d} = F_\alpha(x) e^{i n_\alpha \theta_0 + i(\alpha\cdot\underline{\omega})\xi_d}. \tag{4-1}$$

This holds if and only if

$$iL(n_\alpha\beta, \alpha\cdot\underline{\omega})U_\alpha = F_\alpha.$$

The matrix on the left is invertible, so we obtain a solution of (4-1) for $\alpha \notin \mathscr{C}$.

Step 2: *Characteristic modes.* When $\alpha \in \mathscr{C}_m \setminus \{0\}$, we have $\alpha\cdot\underline{\omega} = n_\alpha\underline{\omega}_m$, so

$$(F_\alpha(x)e^{i\alpha\cdot\theta})|_{\theta\to(\theta_0,\xi_d)} = F_\alpha(x)e^{in_\alpha(\theta_0+\underline{\omega}_m\xi_d)}.$$

We can write

$$\sum_{\alpha\in\mathscr{C}_m\setminus\{0\}} F_\alpha(x)e^{in_\alpha(\theta_0+\underline{\omega}_m\xi_d)} = \sum_{k\in\mathbb{Z}\setminus\{0\}} \mathscr{F}_{m,k}(x)e^{ik(\theta_0+\underline{\omega}_m\xi_d)},$$

where

$$\mathscr{F}_{m,k}(x) := \sum_{\{\alpha\in\mathscr{C}_m\setminus 0,\, n_\alpha=k\}} F_\alpha(x).$$

Since $E_mF = 0$, we have for each $k \in \mathbb{Z}\setminus\{0\}$ that $P_m\mathscr{F}_{m,k}(x) = 0$. So now we look for $U_{m,k}(x)$ such that

$$\mathscr{L}_0(\partial_{\theta_0}, \partial_{\xi_d})U_{m,k}(x)e^{ik(\theta_0+\underline{\omega}_m\xi_d)} = (I-P_m)\mathscr{F}_{m,k}e^{ik(\theta_0+\underline{\omega}_m\xi_d)}.$$

The latter relation holds if and only if

$$iL(k\beta, k\underline{\omega}_m)U_{m,k}(x) = ikL(d\phi_m)U_{m,k}(x) = (I-P_m)\mathscr{F}_{m,k}(x),$$

which is solvable even though $L(d\phi_m)$ is singular. Finally, we take

$$\mathscr{U}(x,\theta_0,\xi_d) = \sum_{\alpha\notin\mathscr{C}} U_\alpha(x)e^{in_\alpha\theta_0+i(\alpha\cdot\underline{\omega})\xi_d} + \sum_{m=1}^{M}\sum_{k\in\mathbb{Z}\setminus\{0\}} U_{m,k}(x)e^{ik(\theta_0+\underline{\omega}_m\xi_d)},$$

which solves (1-46) as claimed. □

The existence theorems for profiles and for the exact solution to the singular system, Theorems 5.11 and 5.13, respectively, are stated and proved in Section 5; we shall only use the statement of these theorems here. In order to formulate the main result of this section we must make some preliminary choices.

Choice of μ and $\tilde{\mu}$. The conditions on the boundary datum $G(x',\theta_0)$ are slightly different in Theorems 5.11 and 5.13. We need to choose μ, $\tilde{\mu}$, and $G(x',\theta_0)$ so that both theorems apply simultaneously. We also need μ large enough so that we can apply Proposition 2.9 in step (4-24) of the error analysis below. These conditions are met if we take[32]

$$\mu = \max(d+9, [(d+1)/2]+M_0+3) = [(d+1)/2]+M_0+3 \quad\text{and}\quad \tilde{\mu} = 2\mu - [(d+1)/2] \tag{4-2}$$

and choose $G \in H^{\tilde{\mu}}(b\Omega_T)$ such that $\langle G\rangle_{H^{\mu+2}(b\Omega_T)}$ is small enough. Applying Theorems 5.11 and 5.13, we now have, for $0 < \varepsilon \le \varepsilon_0$, an exact solution $U_\varepsilon(x,\theta_0) \in E^{\mu-1}(\Omega_T)$ to the singular system (1-18) and profiles $\mathscr{V}^0(x,\theta) \in H^{\mu-1}(\Omega_T)$, $\mathscr{V}^1(x,\theta) \in H^{\mu-2}(\Omega_T)$ satisfying the equations (1-35) and (1-36).

[32]Recall that $M_0 = 3d+5$, $d \ge 2$.

Approximation. Fix $\delta > 0$. Using the Fourier series of $\mathcal{V}^0$ and $\mathcal{V}^1$, we choose trigonometric polynomials $\mathcal{V}^0_p(x,\theta)$ and $\mathcal{V}^1_p(x,\theta)$ such that

$$|\mathcal{V}^0 - \mathcal{V}^0_p|_{H^{\mu-1}(\Omega_T)} < \delta, \quad |\mathcal{V}^1 - \mathcal{V}^1_p|_{H^{\mu-2}(\Omega_T)} < \delta. \tag{4-3}$$

We can smooth the coefficients so that $\mathcal{V}^0_p$ and $\mathcal{V}^1_p$ lie in $H^\infty(\Omega_T)$ and so that (4-3) still holds. Having made these choices, we can now state the main result of this section, which yields the final convergence result of Theorem 1.15 as an immediate corollary.

Theorem 4.1. *We make the same assumptions as in Theorem 1.15 and let μ and $\tilde{\mu}$ be as just chosen. Consider the leading-order approximate solution to the singular semilinear system* (1-18) *given by*

$$\mathcal{U}^0_\varepsilon(x,\theta_0) := \mathcal{V}^0(x,\theta)|_{\theta\to(\theta_0,x_d/\varepsilon)}, \tag{4-4}$$

and let $U_\varepsilon(x,\theta_0) \in E^{\mu-1}(\Omega_T)$ be the exact solution to (1-18) *just obtained. Then*

$$\lim_{\varepsilon\to 0} |U_\varepsilon(x,\theta_0) - \mathcal{U}^0_\varepsilon(x,\theta_0)|_{E^{\mu-3}(\Omega_T)} = 0. \tag{4-5}$$

The following lemma, which is proved in [Coulombel et al. 2011, Lemmas 2.7 and 2.25] by a simple argument based on Fourier series, is an important tool in the proof.

Lemma 4.2 (relation between norms). *For $m \in \mathbb{N}$ suppose $f(x,\theta_j) \in H^{m+1}(\Omega_T)$, and set $f_\varepsilon(x,\theta_0) := f(x,\theta_0 + \underline{\omega}_j x_d/\varepsilon)$. Then*

$$|f_\varepsilon|_{E^m_T} \le C|f|_{H^{m+1}(\Omega_T)}. \tag{4-6}$$

Proof of Theorem 4.1. We shall fill in the sketch provided in Section 1E.

Step 1. First we use Proposition 1.29 to construct $\mathcal{U}^2_p(x,\theta_0,\xi_d)$ satisfying

$$\mathcal{L}_0(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}^2_p = [-(I-E)(L(\partial)\mathcal{V}^1_p + D(0)\mathcal{V}^1_p + dD(0)(\mathcal{V}^0_p,\mathcal{V}^0_p))]|_{\theta\to(\theta_0,\xi_d)}. \tag{4-7}$$

The function $\mathcal{U}^2_p$ is a trigonometric polynomial of the form (1-47) with H^∞ coefficients. We then define the corrected approximate solution

$$\mathcal{U}_\varepsilon(x,\theta_0) := (\mathcal{V}^0(x,\theta) + \varepsilon\mathcal{V}^1(x,\theta))|_{\theta\to(\theta_0,x_d/\varepsilon)} + \varepsilon^2\mathcal{U}^2_p\left(x,\theta_0,\frac{x_d}{\varepsilon}\right). \tag{4-8}$$

Since $\mathcal{V}^1 \in H^{\mu-2}(\Omega_T)$, Lemma 4.2 implies $\mathcal{U}_\varepsilon \in E^{\mu-3}(\Omega_T)$.

Step 2. Next we explain (1-48) and make precise the norms used on the right there. Using the identity (1-44), we compute

$$\begin{aligned}\mathbb{L}_\varepsilon(\mathcal{U}_\varepsilon) = \varepsilon[(\mathcal{L}_0(\partial_{\theta_0},\partial_{\xi_d})\mathcal{U}^2_p)|_{\xi_d=x_d/\varepsilon} + (L(\partial)\mathcal{V}^1 + D(0)\mathcal{V}^1 + dD(0)\mathcal{V}^0\mathcal{V}^0)|_{\theta\to(\theta_0,x_d/\varepsilon)}]\\ + \varepsilon^2(L(\partial)\mathcal{U}^2_p)|_{\xi_d=x_d/\varepsilon} + D(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon - D(0)(\mathcal{V}^0+\varepsilon\mathcal{V}^1) - \varepsilon dD(0)\mathcal{V}^0\mathcal{V}^0,\end{aligned} \tag{4-9}$$

where the second line represents an $O(\varepsilon^2)$ term (see below for a precise estimate). Here the profile equations (1-20)(a)–(b) imply that the terms of order ε^{-1} and ε^0 vanish. Using (4-7), we can rewrite the coefficient of ε in (4-9) as

$$[L(\partial)(\mathcal{V}^1-\mathcal{V}^1_p)+D(0)(\mathcal{V}^1-\mathcal{V}^1_p)+dD(0)(\mathcal{V}^0\mathcal{V}^0-\mathcal{V}^0_p\mathcal{V}^0_p)]|_{\theta\to(\theta_0,x_d/\varepsilon)} \\ +[E(L(\partial)\mathcal{V}^1_p+D(0)\mathcal{V}^1_p+dD(0)\mathcal{V}^0_p\mathcal{V}^0_p)]|_{\theta\to(\theta_0,x_d/\varepsilon)} := A+B. \tag{4-10}$$

Using (4-3), Lemma 4.2, and the fact that $E^s(\Omega_T)$ is a Banach algebra for $s\geq[(d+1)/2]$, we see that

$$|A|_{E^{\mu-4}(\Omega_T)}\leq K\delta. \tag{4-11}$$

To estimate B, let

$$F:=L(\partial)\mathcal{V}^1+D(0)\mathcal{V}^1+dD(0)\mathcal{V}^0\mathcal{V}^0 \quad\text{and}\quad F_p:=L(\partial)\mathcal{V}^1_p+D(0)\mathcal{V}^1_p+dD(0)\mathcal{V}^0_p\mathcal{V}^0_p. \tag{4-12}$$

The profile equation (1-36)(b) implies $EF=0$. Using continuity of the multiplication map (1-25), we see that (4-3) implies[33]

$$|F-F_p|_{H^{\mu-3;2}_T}\leq K\delta. \tag{4-13}$$

From the continuity of $E: H^{s;2}_T\to H^{s;1}_T$ and Lemma 4.2 we then obtain

$$|B|_{E^{\mu-4}(\Omega_T)}=|(EF_p)|_{\theta\to(\theta_0,x_d/\varepsilon)}|_{E^{\mu-4}(\Omega_T)}=|(E(F-F_p))|_{\theta\to(\theta_0,x_d/\varepsilon)}|_{E^{\mu-4}(\Omega_T)}\leq K\delta. \tag{4-14}$$

Step 3. The $O(\varepsilon^2)$ terms in (4-9) consist of

$$|\varepsilon^2(L(\partial)\mathcal{U}^2_p(x,\theta_0,\xi_d))|_{\xi_d\to x_d/\varepsilon}|_{E^{\mu-4}(\Omega_T)}\leq\varepsilon^2C(\delta), \tag{4-15}$$

as well as terms coming from the Taylor expansion of $D(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon$ like $(\varepsilon^2dD(0)\mathcal{V}^0\mathcal{V}^1)|_{\theta\to(\theta_0,x_d/\varepsilon)}$, all of which satisfy an estimate like (4-15). Setting $R_\varepsilon(x,\theta_0):=\mathbb{L}_\varepsilon(\mathcal{U}_\varepsilon)$, we have shown

$$|R_\varepsilon|_{E^{\mu-4}(\Omega_T)}\leq\varepsilon(K\delta+C(\delta)\varepsilon). \tag{4-16}$$

Step 4. The boundary profile equations (1-22) and the fact that the traces of $\mathcal{V}^0$ and $\mathcal{V}^1$ lie in $H^{\mu-1}(b\Omega_T)$ and $H^{\mu-2}(b\Omega_T)$, respectively, imply

$$\langle r_\varepsilon(x',\theta_0)\rangle_{H^{\mu-2}(b\Omega_T)}\leq C(\delta)\varepsilon^2, \quad\text{where } r_\varepsilon:=\psi(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon-\varepsilon G(x',\theta_0). \tag{4-17}$$

Indeed, these $O(\varepsilon^2)$ terms include

$$\langle\varepsilon^2B\mathcal{U}^2_p(x',0,\theta_0,0)\rangle_{H^{\mu-2}(b\Omega_T)}\leq C(\delta)\varepsilon^2, \tag{4-18}$$

and other terms satisfying the same estimate coming from the Taylor expansion of $\psi(\varepsilon\mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon$.

Step 5. Next we consider the singular problem satisfied by the difference $W_\varepsilon:=U_\varepsilon-\mathcal{U}_\varepsilon$:

$$\begin{aligned}\partial_dW_\varepsilon+\mathbb{A}\left(\partial_{x'}+\frac{\beta\partial_{\theta_0}}{\varepsilon}\right)W_\varepsilon+D_2(\varepsilon U_\varepsilon,\varepsilon\mathcal{U}_\varepsilon)W_\varepsilon&=-R_\varepsilon,\\ \psi_2(\varepsilon U_\varepsilon,\varepsilon\mathcal{U}_\varepsilon)W_\varepsilon&=-r_\varepsilon\quad\text{on } x_d=0,\\ W_\varepsilon&=0\quad\text{in } t<0,\end{aligned} \tag{4-19}$$

[33] Here $H^{\mu-3;2}_T$ denotes the space defined in (1-24), but with the obvious restriction on the domain of t.

where

$$\begin{aligned} D_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon &:= D(\varepsilon U_\varepsilon) U_\varepsilon - D(\varepsilon \mathcal{U}_\varepsilon)\mathcal{U}_\varepsilon \\ &= D(\varepsilon U_\varepsilon) W_\varepsilon + \left(\int_0^1 dD(\varepsilon \mathcal{U}_\varepsilon + s\varepsilon(U_\varepsilon - \mathcal{U}_\varepsilon))\, ds\right)(W_\varepsilon, \varepsilon \mathcal{U}_\varepsilon), \end{aligned} \tag{4-20}$$

and $\psi_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon$ is defined similarly. Since $U_\varepsilon \in E^{\mu-1}(\Omega_T)$ and $\mathcal{U}_\varepsilon \in E^{\mu-3}(\Omega_T)$, a short computation shows

$$\psi_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon = \psi(\varepsilon U_\varepsilon) W_\varepsilon + d\psi(\varepsilon U)(W_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) + O(C(\delta)\varepsilon^2) = \mathcal{B}(\varepsilon U, \varepsilon \mathcal{U}) W_\varepsilon + O(C(\delta)\varepsilon^2), \tag{4-21}$$

where the error term is measured in $H^{\mu-3}(b\Omega_T)$ and $\mathcal{B}$ is defined in (1-9). Similarly,

$$D_2(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon = \mathcal{D}(\varepsilon U, \varepsilon \mathcal{U}) W_\varepsilon + O(C(\delta)\varepsilon^2) \quad \text{in } E^{\mu-3}(\Omega_T). \tag{4-22}$$

Thus, using (4-16) and (4-17), we find

$$\begin{gathered} \partial_d W_\varepsilon + \mathbb{A}\left(\partial_{x'} + \frac{\beta \partial_{\theta_0}}{\varepsilon}\right) W_\varepsilon + \mathcal{D}(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon = \varepsilon(K\delta + C(\delta)\varepsilon) \quad \text{in } E^{\mu-4}(\Omega_T), \\ \mathcal{B}(\varepsilon U_\varepsilon, \varepsilon \mathcal{U}_\varepsilon) W_\varepsilon|_{x_d=0} = O(C(\delta)\varepsilon^2) \quad \text{in } H^{\mu-3}(b\Omega_T), \\ W_\varepsilon = 0 \quad \text{in } t < 0. \end{gathered} \tag{4-23}$$

Applying the estimate of Proposition 2.9, we obtain

$$|W_\varepsilon|_{E^0(\Omega_T)} \le K\delta + C(\delta)\varepsilon, \tag{4-24}$$

which implies

$$|U_\varepsilon - \mathcal{U}^0_\varepsilon|_{E^0(\Omega_T)} \le K\delta + C(\delta)\varepsilon. \tag{4-25}$$

Fixing first δ small and then letting $\varepsilon \to 0$, we have shown

$$\lim_{\varepsilon \to 0} |U_\varepsilon - \mathcal{U}^0_\varepsilon|_{E^0(\Omega_T)} = 0. \tag{4-26}$$

The family $U_\varepsilon - \mathcal{U}^0_\varepsilon$, $0 < \varepsilon \le \varepsilon_0$, is bounded in $E^{\mu-2}(\Omega_T)$, so, by interpolation, (4-26) implies

$$\lim_{\varepsilon \to 0} |U_\varepsilon - \mathcal{U}^0_\varepsilon|_{E^{\mu-3}(\Omega_T)} = 0,$$

as required. □

5. Nash–Moser schemes

5A. ***Iteration scheme for profiles.*** A good reference for the Nash–Moser scheme is [Alinhac and Gérard 2007]. The method depends on having a family of smoothing operators with the following properties. For $T > 0$, $s \ge 0$, and $\gamma \ge 1$, we let

$$F^s_\gamma(\Omega_T) := \{u \in H^s_\gamma(\Omega_T),\, u = 0 \text{ for } t < 0\}. \tag{5-1}$$

Lemma 5.1 [Alinhac 1989, Section 4]. *There exists a family of operators* $S_\theta : F^0_\gamma(\Omega_T) \to \bigcap_{\beta\geq 0} F^\beta_\gamma(\Omega_T)$ *such that*

$$
\begin{aligned}
&\text{(a)} \quad |S_\theta u|_\beta \leq C\theta^{(\beta-\alpha)_+}|u|_\alpha && \text{for } \alpha, \beta \geq 0,\\
&\text{(b)} \quad |S_\theta u - u|_\beta \leq C\theta^{(\beta-\alpha)}|u|_\alpha && \text{for } 0 \leq \beta \leq \alpha,\\
&\text{(c)} \quad \left|\frac{d}{d\theta} S_\theta u\right|_\beta \leq C\theta^{(\beta-\alpha-1)}|u|_\alpha && \text{for } \alpha, \beta \geq 0.
\end{aligned}
\tag{5-2}
$$

The constants are uniform for α, β *in a bounded interval.*

There is another family of operators $\tilde{S}_\theta$ *acting on functions defined on the boundary and satisfying the above properties with respect to the norms* $\langle u\rangle_s$ *on* $b\Omega_T$.[34]

Description of the scheme. Our goal is to solve problem (3-10):

$$
\begin{aligned}
&\mathcal{L}(V) = 0 && \text{in } \Omega_T,\\
&\mathcal{B}(V) = g && \text{in } b\Omega_T,\\
&V = 0 && \text{in } t \leq 0.
\end{aligned}
\tag{5-3}
$$

The scheme starts with $V_0 = 0$. Assume that V_k are already given for $k = 1, \dots, n$ and satisfy $V_k = 0$ for $t < 0$. We define

$$V_{n+1} = V_n + \dot{V}_n, \tag{5-4}$$

where the increment $\dot{V}_n$ is specified below. Given $\theta_0 \geq 1$, we set $\theta_n := (\theta_0^2 + n)^{1/2}$ and work with the smoothing operators S_{θ_n}. We write the decomposition

$$\mathcal{L}(V_{n+1}) - \mathcal{L}(V_n) = \mathcal{L}'(V_n)\dot{V}_n + e'_n = \mathcal{L}'(S_{\theta_n}V_n)\dot{V}_n + e'_n + e''_n, \tag{5-5}$$

where e'_n denotes the usual "quadratic error" of Newton's scheme and e''_n the "substitution error". Similarly,

$$
\begin{aligned}
\mathcal{B}((V_{n+1})|_{x_d=0}) - \mathcal{B}((V_n)|_{x_d=0}) &= \mathcal{B}'((V_n)|_{x_d=0}))(\dot{V}_n|_{x_d=0})) + e'_n\\
&= \mathcal{B}'((S_{\theta_n}V_n)|_{x_d=0})(\dot{V}_n|_{x_d=0}) + \tilde{e}'_n + \tilde{e}''_n.
\end{aligned}
\tag{5-6}
$$

The increment $\dot{V}_n$ is computed by solving the linearized problem

$$
\begin{aligned}
&\mathcal{L}'(S_{\theta_n}V_n)\dot{V}_n = f_n,\\
&\mathcal{B}'((S_{\theta_n}V_n)|_{x_d=0})(\dot{V}_n|_{x_d=0}) = g_n,\\
&\dot{V}_n = 0 \quad \text{in } t < 0,
\end{aligned}
\tag{5-7}
$$

where f_n and g_n are computed as we now describe.

We set $e_n := e'_n + e''_n$ and $\tilde{e}_n := \tilde{e}'_n + \tilde{e}''_n$. Given

$$
\begin{gathered}
V_0 := 0, \quad f_0 := 0, \quad g_0 := \tilde{S}_{\theta_0}g, \quad E_0 := 0, \quad \tilde{E}_0 := 0,\\
V_1, \dots, V_n, \quad f_1, \dots, f_{n-1}, \quad g_1, \dots, g_{n-1}, \quad e_0, \dots, e_{n-1}, \quad \tilde{e}_0, \dots, \tilde{e}_{n-1},
\end{gathered}
\tag{5-8}
$$

[34] For u defined on Ω_T, we do not necessarily have equality of $(S_\theta u)|_{x_d=0}$ and $\tilde{S}_\theta(u|_{x_d=0})$.

we first compute for $n \geq 1$ the accumulated errors

$$E_n := \sum_{k=0}^{n-1} e_k, \quad \tilde{E}_n := \sum_{k=0}^{n-1} \tilde{e}_k. \tag{5-9}$$

We then compute f_n and g_n from the equations

$$\sum_{k=0}^{n} f_k + S_{\theta_n} E_n = 0, \quad \sum_{k=0}^{n} g_k + \tilde{S}_{\theta_n} \tilde{E}_n = \tilde{S}_{\theta_n} g, \tag{5-10}$$

solve (5-7) for $\dot{V}_n$, and finally compute V_{n+1} from (5-4).

Next e_n and $\tilde{e}_n$ can be computed from [35]

$$\begin{aligned} &\mathcal{L}(V_{n+1}) - \mathcal{L}(V_n) = f_n + e_n, \\ &\mathcal{B}((V_{n+1})|_{x_d=0}) - \mathcal{B}((V_n)|_{x_d=0}) = g_n + \tilde{e}_n. \end{aligned} \tag{5-11}$$

Thus the order of construction is

$$\cdots \to (e_{n-1}, \tilde{e}_{n-1}) \to (E_n, \tilde{E}_n) \to (f_n, g_n) \to \dot{V}_n \to V_{n+1} \to (e_n, \tilde{e}_n) \to \cdots. \tag{5-12}$$

Adding (5-11) from 0 to n and using (5-10) gives

$$\begin{aligned} &\mathcal{L}(V_{n+1}) = (I - S_{\theta_n}) E_n + e_n, \\ &\mathcal{B}((V_{n+1})|_{x_d=0}) - g = (\tilde{S}_{\theta_n} - I) g + (I - \tilde{S}_{\theta_n}) \tilde{E}_n + \tilde{e}_n. \end{aligned} \tag{5-13}$$

Since $S_{\theta_n} \to I$ and $\tilde{S}_{\theta_n} \to I$ as $n \to \infty$ and we expect $(e_n, \tilde{e}_n) \to 0$, we formally obtain a solution of (5-3) in the limit as $n \to \infty$.

Induction assumption. Let $\Delta_n := \theta_{n+1} - \theta_n$ and observe that

$$\frac{1}{3\theta_n} \leq \Delta_n = \sqrt{\theta_n^2 + 1} - \theta_n \leq \frac{1}{2\theta_n} \quad \text{for all } n \in \mathbb{N}. \tag{5-14}$$

With $\mu_0 = [(d+1)/2]$ as in Proposition 3.6, we now set $\nu_0 := \mu_0 + 1$ and fix a choice of integers $\nu_0 < \nu < \tilde{\nu}$, whose values are explained below:

$$\nu := 2\nu_0 + 4 \quad \text{and} \quad \tilde{\nu} := 2\nu - \nu_0. \tag{5-15}$$

Given $\delta > 0$ our induction assumption is this:

($\mathbf{H}_{n-1}$) For all $k = 0, \ldots, n-1$ and for all $s \in [0, \tilde{\nu}] \cap \mathbb{N}$,

$$|\dot{V}_k|_s + \langle \dot{V}_k \rangle_s \leq \delta \theta_k^{s-\nu-1} \Delta_k. \tag{5-16}$$

The main step in the proof of Theorem 5.11 is to show that, for correctly chosen parameters $\delta > 0$ (small) and $\theta_0 \geq 1$ (large) and for small enough g, ($\mathbf{H}_{n-1}$) implies ($\mathbf{H}_n$). At the end we will verify that ($\mathbf{H}_0$) holds for small enough g.

First we state some easy consequences of ($\mathbf{H}_{n-1}$).

[35] In the estimates of e_n and $\tilde{e}_n$, we instead use (5-20), (5-21) and (5-24).

Lemma 5.2. *If θ_0 is large enough, then, for $k = 0, \dots, n$ and all integers $s \in [0, \tilde{\nu}]$, we have*

$$|V_k|_s + \langle V_k \rangle_s \le \begin{cases} C\delta\theta_k^{(s-\nu)_+}, & \nu \ne s, \\ C\delta \log \theta_k, & \nu = s. \end{cases} \tag{5-17}$$

Proof. This follows by writing $V_k = V_0 + \sum_{j=0}^{k-1} \dot{V}_j$ and using the triangle inequality and an elementary comparison between Riemann sums and integrals. □

Lemma 5.3. *If θ_0 is large enough, then, for $k = 0, \dots, n$ and all integers $s \in [0, \tilde{\nu} + 2]$, we have*

$$|S_{\theta_k} V_k|_s \le \begin{cases} C\delta\theta_k^{(s-\nu)_+}, & \nu \ne s, \\ C\delta \log \theta_k, & \nu = s. \end{cases} \tag{5-18}$$

Moreover, for $k = 0, \dots, n$ and all integers $s \in [0, \tilde{\nu}]$, we have

$$|(I - S_{\theta_k}) V_k|_s \le \begin{cases} C\delta\theta_k^{s-\nu} \log \theta_k, & s \le \nu, \\ C\delta\theta_k^{s-\nu}, & s > \nu. \end{cases} \tag{5-19}$$

Proof. This follows from Lemma 5.2 and the properties of the S_θ. For example, we have

$$\begin{aligned} |(I - S_{\theta_k}) V_k|_s &\le 2|V_k|_s \le C\delta\theta^{s-\nu} && \text{for } s > \nu, \\ |(I - S_{\theta_k}) V_k|_s &\le C\theta^{s-\nu} |V_k|_\nu \le C\delta\theta^{s-\nu} \log \theta_k && \text{for } s \le \nu. \end{aligned}$$

□

Estimate of the quadratic errors. From (5-5) and (5-6) we have

$$e'_k = \mathscr{L}(V_{k+1}) - \mathscr{L}(V_k) - \mathscr{L}'(V_k)\dot{V}_k = \int_0^1 (1-\tau)\mathscr{L}''(V_k + \tau\dot{V}_k)(\dot{V}_k, \dot{V}_k)\, d\tau, \tag{5-20}$$

$$\tilde{e}'_k = \mathscr{B}(V_{k+1}) - \mathscr{B}(V_k) - \mathscr{B}'(V_k)\dot{V}_k = \int_0^1 (1-\tau)\mathscr{B}''(V_k + \tau\dot{V}_k)(\dot{V}_k, \dot{V}_k) d\tau, \tag{5-21}$$

where the arguments in (5-21) are evaluated at $x_d = 0$.

Lemma 5.4. (1) *For large enough θ_0 we have, for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde{\nu}]$,*

$$|e'_k|_s \le C\delta^2\theta_k^{L_1(s)-1}\Delta_k, \tag{5-22}$$

where $L_1(s) = s + \nu_0 - 2\nu - 2$.

(2) *For large enough θ_0 we have for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde{\nu} - 1]$*

$$\langle \tilde{e}'_k \rangle_s \le C\delta^2\theta_k^{L_2(s)-1}\Delta_k, \tag{5-23}$$

where $L_2(s) = s + \nu_0 - 2\nu - 1$.

Proof. Using (5-20), Proposition 3.3, and the fact that $\nu_0 > \nu_1$, we have

$$|e'_k|_s \le C|\dot{V}_k|_s|\dot{V}_k|_{\nu_0}.$$

The estimate (5-22) then follows by applying assumption (5-16) and using $\Delta_k \sim 1/\theta_k$. The estimate (5-23) is proved similarly; the restriction $s \in [0, \tilde{\nu} - 1]$ reflects the loss of one derivative in (3-15). □

Estimate of the substitution errors. From (5-5) and (5-6) we have

$$\begin{aligned}
&\text{(a)}\quad e''_k = \int_0^1 \mathscr{L}''(S_{\theta_k}V_k + \tau(V_k - S_{\theta_k}V_k))(\dot V_k, (I - S_{\theta_k})V_k)\,d\tau,\\
&\text{(b)}\quad \tilde e''_k = \int_0^1 \mathscr{B}''(S_{\theta_k}V_k + \tau(V_k - S_{\theta_k}V_k))(\dot V_k, (I - S_{\theta_k})V_k)\,d\tau,
\end{aligned}\tag{5-24}$$

where in (5-24)(b) we have, for example, written $S_{\theta_k}V_k$ for $(S_{\theta_k}V_k)|_{x_d=0}$.

Lemma 5.5. (1) *For large enough θ_0 we have, for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde\nu]$,*

$$|e''_k|_s \le C\delta^2\theta_k^{L_3(s)-1}\Delta_k, \tag{5-25}$$

where $L_3(s) = s + \nu_0 - 2\nu + 1$.

(2) *For large enough θ_0 we have, for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde\nu - 2]$,*

$$\langle \tilde e''_k\rangle_s \le C\delta^2\theta_k^{L_4(s)-1}\Delta_k, \tag{5-26}$$

where $L_4(s) = s + \nu_0 - 2\nu + 3$.

Proof. Using (5-24)(a) and Proposition 3.3, we obtain

$$|e''_k|_s \le C(|\dot V_k|_s|(I - S_{\theta_k})V_k|_{\nu_0} + |(I - S_{\theta_k})V_k|_s|\dot V_k|_{\nu_0}). \tag{5-27}$$

The estimate (5-25) now follows from ($\mathbf{H}_{n-1}$) and Lemma 5.3. The estimate (5-26) is proved the same way, after using the trace estimate

$$\langle (I - S_{\theta_k})V_k\rangle_{s+1} \le C|(I - S_{\theta_k})V_k|_{s+2}. \tag{5-28}$$

The restriction $s \in [0, \tilde\nu - 2]$ reflects the subscript $s + 2$ in (5-28). ☐

Estimate of $(E_n, \tilde E_n)$ and (f_n, g_n). Since $e_k = e'_k + e''_k$ and $\tilde e_k = \tilde e'_k + \tilde e''_k$, we have:

Lemma 5.6. *There exists θ_0 sufficiently large so that*

$$|E_n|_{\tilde\nu} \le C\delta^2\theta_n^{L_3(\tilde\nu)} \quad\text{and}\quad \langle \tilde E_n\rangle_{\tilde\nu-2} \le C\delta^2\theta_n^{L_4(\tilde\nu-2)}. \tag{5-29}$$

Proof. Viewing $E_n = \sum_{k=0}^{n-1} e_k$ as a Riemann sum and using $L_3(\tilde\nu) > 0$,[36] we obtain the estimate of E_n from (5-22) and (5-25). Since $L_4(\tilde\nu - 2) > 0$, the estimate of $\tilde E_n$ is similar. ☐

From (5-10) we have

$$\begin{aligned}
f_n &= -(S_{\theta_n} - S_{\theta_{n-1}})E_{n-1} - S_{\theta_n}e_{n-1},\\
g_n &= (\tilde S_{\theta_n} - \tilde S_{\theta_{n-1}})g - (\tilde S_{\theta_n} - \tilde S_{\theta_{n-1}})\tilde E_{n-1} - \tilde S_{\theta_n}\tilde e_{n-1}.
\end{aligned}\tag{5-30}$$

Lemma 5.7. *There exists θ_0 sufficiently large so that, for $s \in [0, \tilde\nu + 1]$, we have*

$$\begin{aligned}
&\text{(a)}\quad |f_n|_s \le C\delta^2\theta_n^{L_3(s)-1}\Delta_n,\\
&\text{(b)}\quad \langle g_n\rangle_s \le C\delta^2\theta_n^{L_4(s)-1}\Delta_n + C\theta_n^{s-\nu-1}\langle g\rangle_\nu\Delta_n.
\end{aligned}\tag{5-31}$$

[36] This determines $\tilde\nu$ in (5-15).

Proof. Using (5-2)(c), (5-29), and $s-\tilde{\nu}+L_3(\tilde{\nu})=L_3(s)$, we find

$$|(S_{\theta_n}-S_{\theta_{n-1}})E_{n-1}|_s \le C\int_{\theta_{n-1}}^{\theta_n} \theta^{s-\tilde{\nu}-1}|E_{n-1}|_{\tilde{\nu}}\,d\theta \le C\delta^2\theta_{n-1}^{L_3(s)-1}\Delta_n. \tag{5-32}$$

From (5-22), (5-25), and the properties of S_θ, we readily obtain

$$|S_{\theta_n}e_{n-1}|_s \le C\delta^2\theta_n^{L_3(s)-1}\Delta_n, \tag{5-33}$$

and this gives (5-31)(a).

The first term on the right in (5-31)(b) arises similarly. With

$$\langle(\tilde{S}_{\theta_n}-\tilde{S}_{\theta_{n-1}})g\rangle_s \le C\int_{\theta_{n-1}}^{\theta_n}\theta^{s-\nu-1}\langle g\rangle_\nu\,d\theta \le C\theta_n^{s-\nu-1}\langle g\rangle_\nu\Delta_n, \tag{5-34}$$

we obtain (5-31)(b). □

Induction step. We claim that, for $\delta>0$ sufficiently small, the estimate for the linearized system (3-39) applies to (5-7) and gives for $s\in[0,\tilde{\nu}]$

$$|\dot{V}_n|_s+\langle\dot{V}_n\rangle_s \le C[|f_n|_{s+1}+\langle g_n\rangle_s+(|f_n|_{\nu_0+1}+\langle g_n\rangle_{\nu_0})(|S_{\theta_n}V_n|_{s+1}+\langle S_{\theta_n}V_n\rangle_{s+1})]. \tag{5-35}$$

Indeed, (5-18) and $\nu>\nu_0+2$ imply that, for $\delta>0$ small enough, the requirement (3-38) holds.[37] For the terms involving f_n and g_n, except $\langle g_n\rangle_{\nu_0}$, we substitute directly into (5-35) the corresponding estimates from Lemma 5.7. For $\langle g_n\rangle_{\nu_0}$ we have

$$\langle g_n\rangle_{\nu_0} \le C(\delta^2\theta_n^{L_4(\nu_0)-1}\Delta_n+\theta_n^{-\nu-2}\langle g\rangle_{\nu_0+\nu+1}\Delta_n), \tag{5-36}$$

where the last term arises from (5-34) with $s=\nu_0$ and ν replaced by $\nu+\nu_0+1$. We also use

$$\langle S_{\theta_n}V_n\rangle_{s+1} \le |S_{\theta_n}V_n|_{s+2} \le C\delta\theta_n^{(s+2-\nu)_++1}, \tag{5-37}$$

and a similar estimate for $|S_{\theta_n}V_n|_{s+1}$, which follow directly from (5-18).

Since $L_4(s)>L_3(s+1)$, this gives for $s\in[0,\tilde{\nu}]$

$$\begin{aligned}&|\dot{V}_n|_s+\langle\dot{V}_n\rangle_s\\ &\quad\le C[\delta^2\theta_n^{L_4(s)-1}\Delta_n+\theta_n^{s-\nu-1}\langle g\rangle_\nu\Delta_n+(\delta^2\theta_n^{L_4(\nu_0)-1}\Delta_n+\theta_n^{-\nu-2}\langle g\rangle_{\nu_0+\nu+1}\Delta_n)\delta\theta_n^{(s+2-\nu)_++1}].\end{aligned} \tag{5-38}$$

For $s\in[0,\tilde{\nu}]$ the parameters ν_0 and ν (recall (5-15)) satisfy

$$\begin{aligned}&L_4(s)\le s-\nu,\\ &L_4(\nu_0)+(s+2-\nu)_++1\le s-\nu,\\ &(s+2-\nu)_+<s.\end{aligned} \tag{5-39}$$

Thus we have proved $(\mathbf{H}_n)$, which is the content of the following lemma.

[37] We use a trace estimate like (5-37) here as well.

Lemma 5.8 ($\mathbf{H}_n$). *If $\delta > 0$, $\langle g\rangle_\nu/\delta$ are sufficiently small, and θ_0 sufficiently large, we have*

$$|\dot{V}_n|_s + \langle \dot{V}_n\rangle_s \le \delta\theta_n^{s-\nu-1}\Delta_n \quad \textit{for all integers } s \in [0, \tilde{\nu}]. \tag{5-40}$$

Still assuming ($\mathbf{H}_{n-1}$), we now show the following.

Lemma 5.9. *Suppose $n \ge 1$. If $\delta > 0$ is sufficiently small and θ_0 sufficiently large, we have*

$$\begin{aligned} &\text{(a)} \quad |\mathscr{L}(V_n)|_s \le \delta\theta_n^{s-\nu-1} && \textit{for all integers } s \in [0, \tilde{\nu}],\\ &\text{(b)} \quad \langle \mathscr{B}(V_n) - g\rangle_s \le \delta\theta_n^{s-\nu-1} && \textit{for all integers } s \in [0, \tilde{\nu}-2]. \end{aligned} \tag{5-41}$$

Proof. From (5-13) we have

$$\begin{aligned} &\text{(a)} \quad |\mathscr{L}(V_n)|_s \le |(I - S_{\theta_{n-1}})E_{n-1}|_s + |e_{n-1}|_s,\\ &\text{(b)} \quad \langle B(V_n) - g\rangle_s \le \langle (\tilde{S}_{\theta_{n-1}} - I)g\rangle_s + \langle (I - \tilde{S}_{\theta_{n-1}})\tilde{E}_{n-1}\rangle_s + \langle \tilde{e}_{n-1}\rangle_s. \end{aligned} \tag{5-42}$$

Using (5-2) and the above estimates of E_{n-1} and e_{n-1}, we find

$$\begin{aligned} |(I - S_{\theta_{n-1}})E_{n-1}|_s &\le C\theta_n^{s-\tilde{\nu}}|E_{n-1}|_{\tilde{\nu}} \le C\delta^2\theta_n^{(s-\nu-1)+(\nu_0+2-\nu)},\\ |e_{n-1}|_s &\le C\delta^2\theta_n^{L_3(s)-1}\Delta_n, \end{aligned} \tag{5-43}$$

which imply (5-41)(a) since $\nu_0 + 2 - \nu < 0$ and $L_3(s) < s - \nu$.

The last two terms on the right in (5-42)(b) are estimated similarly. To finish, we use

$$\langle (\tilde{S}_{\theta_{n-1}} - I)g\rangle_s \le C\theta_n^{s-(\tilde{\nu}-2)}\langle g\rangle_{\tilde{\nu}-2} \quad \text{for } s \le \tilde{\nu} - 2 \tag{5-44}$$

and observe that $s - \tilde{\nu} + 2 < s - \nu - 1$. □

We now fix δ and θ_0 as above and check ($\mathbf{H}_0$).

Lemma 5.10. *If $\langle g\rangle_\nu$ is small enough*, ($\mathbf{H}_0$) *holds.*

Proof. Applying the estimate for the linearized system to

$$\begin{aligned} \mathscr{L}'(0)\dot{V}_0 &= 0,\\ \mathscr{B}'(0)\dot{V}_0 &= S_{\theta_0}g, \end{aligned} \tag{5-45}$$

we obtain for integer $s \in [0, \tilde{\nu}]$

$$|\dot{V}_0|_s + \langle \dot{V}_0\rangle_s \le C\langle S_{\theta_0}g\rangle_s \le C\begin{cases} \theta_0^{s-\nu}\langle g\rangle_\nu, & s \ge \nu,\\ \langle g\rangle_\nu, & s < \nu. \end{cases} \tag{5-46}$$

Thus, ($\mathbf{H}_0$) holds if $\langle g\rangle_\nu$ is small enough. □

Proof of Proposition 3.1. We have

$$V_n = V_{n-1} + \dot{V}_{n-1} = \sum_{k=0}^{n-1} \dot{V}_k. \tag{5-47}$$

Let $\nu' := \nu - 1$. Since $\theta_k \sim \sqrt{k}$ we have by ($\mathbf{H}_n$)

$$\sum_{k=0}^{\infty} |\dot{V}_k|_{\nu'} + \sum_{k=0}^{\infty} \langle \dot{V}_k\rangle_{\nu'} \le \delta\sum_k \theta_k^{-2}\Delta_k \le C\sum_k k^{-3/2} < \infty. \tag{5-48}$$

Thus, for some V as described in Proposition 3.1, $V_k \to V$ in $H^{\nu'}(\Omega_T)$ and $V_k|_{x_d=0} \to V|_{x_d=0}$ in $H^{\nu'}(\Omega_T)$. This implies

$$\mathcal{L}(V_k) \to \mathcal{L}(V) \text{ in } H^{\nu'-1}(\Omega_T) \quad \text{and} \quad \mathcal{B}(V_k|_{x_d=0}) \to \mathcal{B}(V|_{x_d=0}) \text{ in } H^{\nu'-1}(b\Omega_T).$$

Applying Lemma 5.9 with $s = \nu' - 1$, we conclude that V is a solution of the profile system (3-10). □

Having solved the key subsystem we can now easily complete the solution of the full profile system (1-35)–(1-36) and obtain the following result.

Theorem 5.11. *Fix $T > 0$, let $\nu_0 = [(d+1)/2] + 1$, $\nu = 2\nu_0 + 4$, and $\tilde{\nu} = 2\nu - \nu_0$, and suppose $G \in H^{\tilde{\nu}-1}(\Omega_T)$. If $\langle G \rangle_{\nu+1}$ is small enough, there exist solutions*

$$\mathcal{V}^0 = \mathcal{V}^0_{\text{inc}} \in H^{\nu-1}(\Omega_T), \quad \mathcal{V}^1 = \underline{\mathcal{V}}^1 + \mathcal{V}^1_{\text{inc}} + \mathcal{V}^1_{\text{out}} \in H^{\nu-2}(\Omega_T)$$

of the full profile system (1-35)–(1-36) *satisfying*[38]

$$\begin{aligned}
&\mathcal{V}^0 = E\mathcal{V}^0 \in H^{\nu-1}(\Omega_T), \quad \mathcal{V}^0_{\text{inc}}|_{x_d=0,\theta_j=\theta_0} \in H^{\nu-1}(b\Omega_T),\\
&\mathcal{V}^1_{\text{out}} = E\mathcal{V}^1_{\text{out}} \in H^{\nu-1}(\Omega_T), \quad (E\mathcal{V}^1_{\text{out}})|_{x_d=0,\theta_j=\theta_0} \in H^{\nu-1}(b\Omega_T),\\
&\underline{\mathcal{V}}^1 \in H^{\nu-2}(\Omega_T), \quad (I-E)\mathcal{V}^1_{\text{inc}} \in H^{\nu-2}(\Omega_T), \quad E\mathcal{V}^1_{\text{inc}} \in H^{\nu-2}(\Omega_T).
\end{aligned} \tag{5-49}$$

These statements remain true if ν is increased and if $\tilde{\nu} \geq 2\nu - \nu_0$.

Proof. After the subsystem (1-42) is solved, we know $\mathcal{V}^0 = \mathcal{V}^0_{\text{inc}} = E\mathcal{V}^0_{\text{inc}}$, $\mathcal{V}^1_{\text{out}} = E\mathcal{V}^1_{\text{out}}$, and these functions have the regularity described in Proposition 3.1. Taking the mean of equations (1-36)(b)–(d), using the fact that the mean of the quadratic term in (1-36)(b) lies in $H^{\nu-1}(\Omega_T)$, and applying the result of [Coulombel 2005] to the resulting weakly stable system, we conclude $\underline{\mathcal{V}}^1 \in H^{\nu-2}(\Omega_T)$. From (1-36)(a) we find

$$(I-E)\mathcal{V}^1 = (I-E)\mathcal{V}^1_{\text{inc}} \in H^{\nu-2}(\Omega_T). \tag{5-50}$$

It remains to determine $E\mathcal{V}^1_{\text{inc}}$. Since the solvability condition (1-41) holds, we can make a choice of $E\mathcal{V}^1_{\text{inc}}|_{x_d=0,\theta_j=\theta_0} \in H^{\nu-2}(b\Omega_T)$ satisfying the boundary equation (1-40), whose right side is now known and lies in $H^{\nu-2}(b\Omega_T)$.[39] Finally, we determine the components of $E\mathcal{V}_{\text{inc}}$ by solving the transport equations determined by (1-36)(b), the choice of initial data, and the initial condition (1-36)(d). Observe that the interaction integrals corresponding to the quadratic term in (1-36)(b) lie in $H^{\nu-1}(\Omega_T)$. □

5B. ***Iteration scheme for the exact solution.*** The Nash–Moser scheme for the exact solution will use the scale of spaces $E^s_{\gamma,T}$ on Ω_T and $H^s_{\gamma,T}$ on $b\Omega_T$. Since T was fixed at the start and γ was fixed in Section 2C, we now drop these subscripts in the notation for norms and function spaces. For $s \geq 0$ we let

$$\mathbb{F}^s := \{u(x,\theta_0) \in E^s, u = 0 \text{ for } t < 0\}. \tag{5-51}$$

Moreover, we shall now denote E^s norms simply by $|U|_s$ and H^s norms by $\langle U \rangle_s$.

[38] Here when we write $\mathcal{V}^0_{\text{inc}} \in H^{\nu-1}(\Omega_T)$, for example, we mean that the individual components of $\mathcal{V}^0_{\text{inc}}$ lie in that space.

[39] All terms on the right in (1-40) lie in $H^{\nu-1}(b\Omega_T)$, except the term involving $L(\partial)$. That term is actually more regular than $H^{\nu-2}(b\Omega_T)$, but we do not wish to introduce more refined spaces to capture this.

Lemma 5.12. *There exists a family of operators* $S_\theta : \mathbb{F}^0 \to \bigcap_{\beta\geq 0} \mathbb{F}^\beta$ *such that*

$$\begin{aligned} &\text{(a)} \quad |S_\theta u|_\beta \leq C\theta^{(\beta-\alpha)_+}|u|_\alpha && \textit{for } \alpha, \beta \geq 0,\\ &\text{(b)} \quad |S_\theta u - u|_\beta \leq C\theta^{(\beta-\alpha)}|u|_\alpha && \textit{for } 0 \leq \beta \leq \alpha,\\ &\text{(c)} \quad \left|\frac{d}{d\theta} S_\theta u\right|_\beta \leq C\theta^{(\beta-\alpha-1)}|u|_\alpha && \textit{for } \alpha, \beta \geq 0. \end{aligned} \tag{5-52}$$

The constants are uniform for α, β *in a bounded interval.*

There is a family of operators $\tilde{S}_\theta$ *acting on functions defined on the boundary and satisfying the above properties with respect to the norms* $\langle u\rangle_s$ *on* $b\Omega_T$, *and we have*

$$(S_\theta u)|_{x_d=0} = \tilde{S}_\theta(u|_{x_d=0}). \tag{5-53}$$

Proof. Let $\tilde{S}_\theta$ be a standard family of smoothing operators, for example, as in [Alinhac 1989], acting in the (x', θ_0) variables on the scale of spaces H^s. For $U \in E^s$ simply treat x_d as a parameter and define

$$S_\theta U = \tilde{S}_\theta U(\,\cdot\,, x_d, \cdot\,). \tag{5-54}$$

The properties (5-52) then follow immediately from the corresponding properties of the operators $\tilde{S}_\theta$. □

To avoid excessive repetition, we use the notation and arguments of Section 5A as much as possible, and just point out where changes are needed. Thus, we now denote the solution to the semilinear singular problem (1-18) by V instead of U, and rewrite (1-18) as

$$\begin{aligned} &\mathscr{L}(V) = 0 && \text{on } \Omega_T,\\ &\mathscr{B}(V) = G && \text{on } b\Omega_T,\\ &V = 0 && \text{in } T < 0, \end{aligned} \tag{5-55}$$

where

$$\begin{aligned} \mathscr{L}(V) &:= \frac{1}{\varepsilon}\left(\partial_d V + \mathbb{A}\left(\partial_{x'} + \frac{\beta\partial_{\theta_0}}{\varepsilon}\right)V + D(\varepsilon V)V\right),\\ \mathscr{B}(V) &:= \frac{1}{\varepsilon}(\psi(\varepsilon V)V). \end{aligned} \tag{5-56}$$

We now let[40]

$$\mu_0 := [(d+1)/2], \quad \mu_1 := \mu_0 + M_0, \quad \mu := \max(2\mu_0+3, \mu_1+1) = \mu_1+1, \quad \tilde{\mu} := 2\mu - \mu_0. \tag{5-57}$$

We now state the main result of this section.

Theorem 5.13. *Fix* $T > 0$, *define* μ_0, μ_1, μ, *and* $\tilde{\mu}$ *as in* (5-57), *and suppose* $G \in H^{\tilde{\mu}}$. *There exists* $\varepsilon_0 > 0$ *such that if* $\langle G\rangle_{\mu+2}$ *is small enough, there exists a solution* V *of the system* (5-55) *on* Ω_T *for* $0 < \varepsilon \leq \varepsilon_0$ *with* $V \in E^{\mu-1}$, $V|_{x_d=0} \in H^\mu$. *Thus* $U_\varepsilon = V$ *is a solution of the singular system* (1-18) *on* Ω_T *for* $0 < \varepsilon \leq \varepsilon_0$. *These statements remain true if* μ *is increased and if* $\tilde{\mu} \geq 2\mu - \mu_0$.

[40] The parameter $\tilde{\mu}$ is determined so that $L_2(\tilde{\mu}) > 0$ for $L_2(s)$ as in Lemma 5.18. The definition of μ is chosen so that $\mu_1 < \mu$ and the conditions (5-76) hold.

The linearized singular problem (2-40) is now written

$$\begin{aligned} &\mathscr{L}'(V)\dot{V} = f \quad \text{on } \Omega_T, \\ &\mathscr{B}'(V)\dot{V} = g \quad \text{on } b\Omega_T, \\ &\dot{V} = 0 \qquad\qquad \text{in } t < 0. \end{aligned} \tag{5-58}$$

With this notation the description of the scheme in Section 5A starting at line (5-3) applies here word for word down to line (5-14).

Remark 5.14. (a) In order to apply the tame estimate of Proposition 2.16 to the linearized system (5-58), by Sobolev embedding (Remark 2.14), it suffices to have

$$|\varepsilon\partial_d V|_{\mu_1-1} + |V|_{\mu_1} \le K' \text{ for } \varepsilon \in (0,1] \quad \text{and} \quad |V|_{\mu_0+2} \le \kappa_0, \tag{5-59}$$

for some constant K' depending on K and κ_0 as in Proposition 2.16. In fact, we use the slightly weaker (because we use E^s norms on the right) estimate for $s \in [0, \tilde{\mu}]$:

$$|\dot{V}|_s + \langle \dot{V}\rangle_{s+1} \le C[|f|_{s+1} + \langle g\rangle_{s+2} + (|f|_{\mu_0+1} + \langle g\rangle_{\mu_0+2})(|U|_{s+1} + \langle U\rangle_{s+2})]. \tag{5-60}$$

(b) By Proposition 2.15, when $|V|_{\mu_0} \le K'$, the tame estimates for second derivatives now take the form

$$\begin{aligned} &\text{(a)} \quad |\mathscr{L}''(V)(\dot{V}^a, \dot{V}^b)|_s \le C(|\dot{V}^a|_s|\dot{V}^b|_{\mu_0} + |\dot{V}^b|_s|\dot{V}^a|_{\mu_0} + \varepsilon|V|_s|\dot{V}^a|_{\mu_0}|\dot{V}^b|_{\mu_0}), \\ &\text{(b)} \quad \langle \mathscr{B}''(V)(\dot{V}^a, \dot{V}^b)\rangle_s \le C(\langle \dot{V}^a\rangle_s\langle \dot{V}^b\rangle_{\mu_0} + \langle \dot{V}^b\rangle_s\langle \dot{V}^a\rangle_{\mu_0} + \varepsilon\langle V\rangle_s\langle \dot{V}^a\rangle_{\mu_0}\langle \dot{V}^b\rangle_{\mu_0}). \end{aligned} \tag{5-61}$$

With μ and $\tilde{\mu}$ redefined as in (5-57), for a given $\delta > 0$, the induction hypothesis ($\mathbf{H}_{n-1}$) is now as follows.

($\mathbf{H}_{n-1}$) For all $k = 0, \dots, n-1$ and for all $s \in [0, \tilde{\mu}] \cap \mathbb{N}$

$$|\dot{V}_k|_s + \langle \dot{V}_k\rangle_{s+1} \le \delta\theta_k^{s-\mu-1}\Delta_k. \tag{5-62}$$

Lemmas 5.2 and 5.3 are now replaced, with no real change in the proofs, by the following two lemmas.

Lemma 5.15. *If θ_0 is large enough, then, for $k = 0, \dots, n$ and all integers $s \in [0, \tilde{\mu}]$, we have*

$$|V_k|_s + \langle V_k\rangle_{s+1} \le \begin{cases} C\delta\theta_k^{(s-\mu)_+}, & \mu \ne s, \\ C\delta\log\theta_k, & \mu = s. \end{cases} \tag{5-63}$$

Lemma 5.16. *If θ_0 is large enough, then, for $k = 0, \dots, n$ and all integers $s \in [0, \tilde{\mu}+2]$, we have*

$$|S_{\theta_k}V_k|_s + \langle S_{\theta_k}V_k\rangle_{s+1} \le \begin{cases} C\delta\theta_k^{(s-\mu)_+}, & \mu \ne s, \\ C\delta\log\theta_k, & \mu = s. \end{cases} \tag{5-64}$$

For $k = 0, \dots, n$ and all integers $s \in [0, \tilde{\mu}]$, we have

$$|(I - S_{\theta_k})V_k|_s + \langle (I - S_{\theta_k})V_k\rangle_{s+1} \le \begin{cases} C\delta\theta_k^{s-\mu}\log\theta_k, & s \le \mu, \\ C\delta\theta_k^{s-\mu}, & s > \mu. \end{cases} \tag{5-65}$$

We have used (5-54) for the estimate on traces in Lemma 5.16. In place of Lemma 5.4 we now have:

Lemma 5.17. (1) *For large enough θ_0 and small enough δ we have, for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde{\mu}]$,*

$$|e'_k|_s \le C\delta^2 \theta_k^{L_1(s)-1} \Delta_k, \tag{5-66}$$

where $L_1(s) := \max(s + \mu_0 - 2\mu - 2, (s-\mu)_+ + 2\mu_0 - 2\mu - 1)$.

(2) *For large enough θ_0 and small enough δ we have, for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde{\mu}]$,*

$$\langle \tilde{e}'_k \rangle_{s+1} \le C\delta^2 \theta_k^{L_1(s)-1} \Delta_k. \tag{5-67}$$

Proof. Again we use (5-20) and (5-21). By Lemma 5.15 and ($\mathbf{H}_{n-1}$), we see that, for δ small enough, $|V_k + \tau \dot{V}_k|_{\mu_0} \le K'$, so we can apply the estimates (5-61). The new definition of $L_1(s)$ reflects the third term on the right in the estimates (5-61). □

In place of Lemma 5.5, the estimate of substitution errors, we now have:

Lemma 5.18. (1) *For large enough θ_0 and small enough δ we have, for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde{\mu}]$,*

$$|e''_k|_s \le C\delta^2 \theta_k^{L_2(s)-1} \Delta_k, \tag{5-68}$$

where $L_2(s) := \max(s + \mu_0 - 2\mu + 1, (s-\mu)_+ + 2\mu_0 - 2\mu + 2)$.

(2) *For large enough θ_0 and small enough δ we have, for all $k = 0, \dots, n-1$ and all integers $s \in [0, \tilde{\mu}]$,*

$$\langle \tilde{e}''_k \rangle_{s+1} \le C\delta^2 \theta_k^{L_2(s)-1} \Delta_k. \tag{5-69}$$

Proof. Again we use the formulas (5-24). By Lemma 5.3 we have $|S_{\theta_k} V_k + \tau(I - S_{\theta_k})V_k|_{\mu_0} \le K'$ for δ small enough, so we can apply the estimates (5-61). When estimating the right sides of (5-61), we use, for example,

$$|(I - S_{\theta_k})V_k|_s \le C\delta\theta_k^{s-\mu+1}.$$

□

In place of Lemma 5.6, the estimate of accumulated errors, we now have:

Lemma 5.19. *There exist θ_0 sufficiently large and δ_0 sufficiently small so that, for $0 < \delta \le \delta_0$,*

$$|E_n|_{\tilde{\mu}} \le C\delta^2 \theta_n^{L_2(\tilde{\mu})} \quad \textit{and} \quad \langle \tilde{E}_n \rangle_{\tilde{\mu}+1} \le C\delta^2 \theta_n^{L_2(\tilde{\mu})}. \tag{5-70}$$

Proof. Since $\tilde{\mu} = 2\mu - \mu_0$, we have $L_2(\tilde{\mu}) > 0$, so the proof is the same as that of Lemma 5.6. □

The new version of Lemma 5.7, the estimate of f_n and g_n, is this:

Lemma 5.20. *There exist θ_0 sufficiently large and δ_0 sufficiently small so that, for $s \in [0, \tilde{\mu}+1]$, $0 < \delta \le \delta_0$, we have*

$$\begin{aligned} &\text{(a)} \quad |f_n|_s \le C\delta^2 \theta_n^{L_2(s)-1} \Delta_n, \\ &\text{(b)} \quad \langle g_n \rangle_{s+1} \le C\delta^2 \theta_n^{L_2(s)-1} \Delta_n + C\theta_n^{s-\mu-2} \langle G \rangle_{\mu+2} \Delta_n. \end{aligned} \tag{5-71}$$

Proof. Since $s - \tilde{\mu} + L_2(\tilde{\mu}) \le L_2(s)$, the proof of Lemma 5.7 can be repeated here. □

Induction step. For $\delta > 0$ sufficiently small, the estimate (5-60) for the linearized system applies to (5-7) and gives for $s \in [0, \tilde{\mu}]$

$$|\dot{V}_n|_s + \langle \dot{V}_n \rangle_{s+1} \le C[|f_n|_{s+1} + \langle g_n \rangle_{s+2} + (|f_n|_{\mu_0+1} + \langle g_n \rangle_{\mu_0+2})(|S_{\theta_n} V_n|_{s+1} + \langle S_{\theta_n} V_n \rangle_{s+2})]. \tag{5-72}$$

Indeed, (5-64) implies that for $\delta > 0$ small enough, $S_{\theta_n} V_n$ satisfies the requirement (5-59).[41] For the terms involving f_n and g_n, except $\langle g_n \rangle_{\mu_0+2}$, we substitute directly into (5-72) the corresponding estimates from Lemma 5.20. For $\langle g_n \rangle_{\mu_0+2}$, we have

$$\langle g_n \rangle_{\mu_0+2} \le C(\delta^2 \theta_n^{L_2(\mu_0+1)-1} \Delta_n + \theta_n^{-\mu-2} \langle G \rangle_{\mu_0+\mu+3} \Delta_n), \tag{5-73}$$

where the last term arises from an estimate like (5-34) with $s = \mu_0 + 2$ and μ replaced by $\mu + \mu_0 + 3$. We also use

$$\langle S_{\theta_n} V_n \rangle_{s+2} \le C\delta \theta_n^{(s+1-\mu)_+ +1} \tag{5-74}$$

and a similar estimate for $|S_{\theta_n} V_n|_{s+1}$, which follow directly from (5-64).

Making these substitutions in (5-72) gives, for $s \in [0, \tilde{\mu}]$,

$$\begin{aligned} |\dot{V}_n|_s + \langle \dot{V}_n \rangle_{s+1} \le C[&\delta^2 \theta_n^{L_2(s+1)-1} \Delta_n + \theta_n^{s-\mu-1} \langle G \rangle_{\mu+2} \Delta_n \\ &+ (\delta^2 \theta_n^{L_2(\mu_0+1)-1} \Delta_n + \theta_n^{-\mu-2} \langle G \rangle_{\mu_0+\mu+3} \Delta_n) \delta \theta_n^{(s+1-\mu)_+ +1}]. \end{aligned} \tag{5-75}$$

For $s \in [0, \tilde{\mu}]$ the parameters μ_0 and μ (recall (5-57)) satisfy

$$\begin{aligned} &\text{(a)} \quad L_2(s+1) \le s - \mu, \\ &\text{(b)} \quad L_2(\mu_0+1) + (s+1-\mu)_+ + 1 \le s - \mu, \\ &\text{(c)} \quad (s+1-\mu)_+ < s. \end{aligned} \tag{5-76}$$

Thus, we have proved $(\mathbf{H}_n)$, which is the content of the following lemma.

Lemma 5.21 $(\mathbf{H}_n)$. *If $\delta > 0$ and $\langle G \rangle_{\mu+2}/\delta$ are sufficiently small and θ_0 is sufficiently large, we have*

$$|\dot{V}_n|_s + \langle \dot{V}_n \rangle_{s+1} \le \delta \theta_n^{s-\mu-1} \Delta_n \quad \text{for all integers } s \in [0, \tilde{\mu}]. \tag{5-77}$$

Still assuming $(\mathbf{H}_{n-1})$ we now show:

Lemma 5.22. *Suppose $n \ge 1$. If $\delta > 0$ is sufficiently small and θ_0 sufficiently large, we have*

$$\begin{aligned} &\text{(a)} \quad |\mathcal{L}(V_n)|_s \le \delta \theta_n^{s-\mu-1}, \\ &\text{(b)} \quad \langle \mathcal{B}(V_n) - G \rangle_{s+1} \le \delta \theta_n^{s-\mu-1}, \end{aligned} \tag{5-78}$$

for all integers $s \in [0, \tilde{\mu}]$.

Proof. From (5-13) we have

$$\begin{aligned} &\text{(a)} \quad |\mathcal{L}(V_n)|_s \le |(I - S_{\theta_{n-1}}) E_{n-1}|_s + |e_{n-1}|_s, \\ &\text{(b)} \quad \langle B(V_n) - G \rangle_{s+1} \le \langle (\tilde{S}_{\theta_{n-1}} - I) G \rangle_{s+1} + \langle (I - \tilde{S}_{\theta_{n-1}}) \tilde{E}_{n-1} \rangle_{s+1} + \langle \tilde{e}_{n-1} \rangle_{s+1}. \end{aligned} \tag{5-79}$$

Using (5-52) and the above estimates of E_{n-1} and e_{n-1}, we find

$$\begin{aligned} |(I - S_{\theta_{n-1}}) E_{n-1}|_s \le C\theta_n^{s-\tilde{\mu}} |E_{n-1}|_{\tilde{\mu}} \le C\delta^2 \theta_n^{(s-\mu-1)+(\mu_0-\mu)}, \\ |e_{n-1}|_s \le C\delta^2 \theta_n^{L_2(s)-1} \Delta_n, \end{aligned} \tag{5-80}$$

which imply (5-78)(a) since $\mu_0 - \mu < 0$ and $L_2(s) < s - \mu$.

[41] Here we use $\mu_1 < \mu$. Also, the term $|\varepsilon \partial_d (S_{\theta_n} V_n)|_{\mu_1 - 1}$ is estimated using (5-7).

The last two terms on the right in (5-79)(b) are estimated similarly. To finish, we use

$$\langle(\tilde{S}_{\theta_{n-1}} - I)G\rangle_{s+1} \le C\theta_n^{s+1-\tilde{\mu}}\langle G\rangle_{\tilde{\mu}} \quad \text{for } s \le \tilde{\mu} - 1, \tag{5-81}$$

and observe that $s - \tilde{\mu} + 1 < s - \mu - 1$. ☐

We now fix δ and θ_0 as above and check $(\mathbf{H}_0)$.

Lemma 5.23. *If $\langle G\rangle_{\mu+2}$ is small enough*, $(\mathbf{H}_0)$ *holds.*

Proof. Applying the estimate for the linearized system to

$$\begin{aligned} \mathscr{L}'(0)\dot{V}_0 &= 0, \\ \mathscr{B}'(0)\dot{V}_0 &= S_{\theta_0}G, \end{aligned} \tag{5-82}$$

we obtain, for integers $s \in [0, \tilde{\mu}]$,

$$|\dot{V}_0|_s + \langle\dot{V}_0\rangle_{s+1} \le C\langle S_{\theta_0}G\rangle_{s+2} \le C\begin{cases}\theta_0^{s-\mu}\langle G\rangle_{\mu+2}, & s \ge \mu, \\ \langle G\rangle_{\mu+2}, & s < \mu.\end{cases} \tag{5-83}$$

Thus $(\mathbf{H}_0)$ holds if $\langle G\rangle_{\mu+2}$ is small enough. ☐

Proof of Theorem 5.13. We have

$$V_n = V_{n-1} + \dot{V}_{n-1}\sum_{k=0}^{n-1}\dot{V}_k.$$

Let $\nu := \mu - 1$. Since $\theta_k \sim \sqrt{k}$ we have by $(\mathbf{H}_n)$

$$\sum_{k=0}^{\infty}|\dot{V}_k|_\nu + \sum_{k=0}^{\infty}\langle\dot{V}_k\rangle_{\nu+1} \le \delta\sum_k\theta_k^{-2}\Delta_k \le C\sum_k k^{-3/2} < \infty.$$

Thus, for some V as described in Theorem 5.13, $V_k \to V$ in E^ν and $V_k|_{x_d=0} \to V|_{x_d=0}$ in $H^{\nu+1}$ (in fact, uniformly for $0 < \varepsilon \le \varepsilon_0$). Lemma 5.22 applied with $s = \nu - 1$ now implies that V is a solution of the semilinear system (5-55). ☐

Appendix A: A calculus of singular pseudodifferential operators

Here we summarize the parts of the singular calculus constructed in [Coulombel et al. 2012] that are needed in this article.

Symbols. Our singular symbols are built from the following sets of classical symbols.

Definition A.1. Let $\mathcal{O} \subset \mathbb{R}^N$ be an open subset that contains the origin. For $m \in \mathbb{R}$ we let $\boldsymbol{S}^m(\mathcal{O})$ denote the class of all functions $\sigma : \mathcal{O} \times \mathbb{R}^d \times [1, \infty) \to \mathbb{C}^{N\times N}$, $N \ge 1$, such that σ is C^∞ on $\mathcal{O} \times \mathbb{R}^d$ and, for all compact sets $K \subset \mathcal{O}$,

$$\sup_{v\in K}\sup_{\xi'\in\mathbb{R}^d}\sup_{\gamma\ge 1}(\gamma^2 + |\xi'|^2)^{-(m-|\nu|)/2}|\partial_v^\alpha\partial_{\xi'}^\nu\sigma(v, \xi', \gamma)| \le C_{\alpha,\nu,K}.$$

Let $\mathscr{C}_b^k(\mathbb{R}^d \times \mathbb{T})$, $k \in \mathbb{N}$, denote the space of continuous and bounded functions on $\mathbb{R}^d \times \mathbb{R}$ that are 2π-periodic in their last argument, and whose derivatives up to order k are continuous and bounded.

Definition A.2 (singular symbols). Let $m \in \mathbb{R}$, $n \in \mathbb{N}$, and fix $\beta \in \mathbb{R}^d \setminus 0$. We let S^m_n denote the family of functions $(a_{\varepsilon,\gamma})_{\varepsilon \in (0,1], \gamma \geq 1}$ that are constructed as follows:

$$\text{for all } (x', \theta_0, \xi', k) \in \mathbb{R}^d \times \mathbb{T} \times \mathbb{R}^d \times \mathbb{Z}, \quad a_{\varepsilon,\gamma}(x', \theta_0, \xi', k) := \sigma\left(\varepsilon V(x', \theta_0), \xi' + \frac{k\beta}{\varepsilon}, \gamma\right), \tag{A-1}$$

where $\sigma \in \boldsymbol{S}^m(\mathcal{O})$ and $V \in \mathscr{C}^n_b(\mathbb{R}^d \times \mathbb{T})$. Below and in the main text we often set

$$X := \xi' + \frac{k\beta}{\varepsilon}.$$

All results below extend to the case where in place of a function V that is independent of ε, the representation (A-1) is considered with a function V_ε that is indexed by ε, provided that we assume that all functions V_ε take values in a *fixed* convex compact subset K of $\mathcal{O}$ that contains the origin, and $(V_\varepsilon)_{\varepsilon \in (0,1]}$ is a bounded family of $\mathscr{C}^n_b(\mathbb{R}^d \times \mathbb{T})$. Such singular symbols with a function V_ε are exactly the kind of symbols that we manipulated in the construction of exact solutions to the singular system (1-18).

Singular pseudodifferential operators. To each symbol $a_{\varepsilon,\gamma}$ as in (A-1), we associate a singular pseudodifferential operator $\mathrm{Op}^{\varepsilon,\gamma}(a)$ whose action on Schwartz class functions $u \in \mathscr{S}(\mathbb{R}^d \times \mathbb{T} : \mathbb{C}^N)$ is defined by

$$\mathrm{Op}^{\varepsilon,\gamma}(a)u(x', \theta_0) := \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d} e^{ix' \cdot \xi' + ik\theta_0} \sigma\left(\varepsilon V(x', \theta_0), \xi' + \frac{k\beta}{\varepsilon}, \gamma\right) \hat{u}(\xi', k)\, d\xi', \tag{A-2}$$

where $\hat{u}(\xi', k)$ denotes the Fourier transform at ξ' of the k-th Fourier coefficient of u with respect to θ_0. When $a_{\varepsilon,\gamma}$ is defined as in (A-1), below and in the main text of the article, we often write $\sigma(\varepsilon V(x, \theta_0), X, \gamma)$ in place of $a_{\varepsilon,\gamma}(x', \theta_0, \xi', k)$, and σ_D in place of $\mathrm{Op}^{\varepsilon,\gamma}(a)$. In particular, we let Λ_D denote the singular Fourier multiplier associated to the function

$$\Lambda(X, \gamma) := (\gamma^2 + |X|^2)^{1/2}.$$

When $V(x', x_d, \theta_0)$ depends also on a normal variable $x_d \geq 0$, we define the associated family of operators depending on the parameter x_d in the obvious way. The pseudodifferential calculus takes place only in the tangential directions (x', θ_0). To discuss mapping properties, we first define "singular" Sobolev spaces as follows.

Definition A.3. We let

$$H^{s,\varepsilon}(\mathbb{R}^d \times \mathbb{T}) := \left\{ u \in \mathscr{S}'(\mathbb{R}^d \times \mathbb{T}) : \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d} (1 + |X|^2)^s |\hat{u}(\xi', k)|^2 \mathrm{d}\xi' < \infty \right\}.$$

This space is equipped with the family of norms[42]

$$|u|^2_{H^{s,\varepsilon},\gamma} := \frac{1}{(2\pi)^d} \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^d} (\gamma^2 + |X|^2)^s |\hat{u}(\xi', k)|^2\, d\xi'.$$

[42]In this appendix we use $|\cdot|$ instead of $\langle \cdot \rangle$ in the notation for norms on $\mathbb{R}^d \times \mathbb{T}$, but otherwise we retain notation from the main text.

Observe that, for s fixed, the space $H^{s,\varepsilon}$ depends on ε with no obvious inclusion if $\varepsilon_1 < \varepsilon_2$. However, for fixed $\varepsilon > 0$, the norms $|\cdot|_{H^{s,\varepsilon},\gamma_1}$ and $|\cdot|_{H^{s,\varepsilon},\gamma_2}$ are equivalent.

The next proposition describes some of the mapping properties of these operators. Detailed proofs can be found in [Coulombel et al. 2012]. The constant C is always independent of $\varepsilon \in (0, 1]$ and $\gamma \geq 1$, and we denote the $L^2(\mathbb{R}^d \times \mathbb{T})$ norm by $|\cdot|_0$ (which corresponds to $s = 0$ in Definition A.3).

Proposition A.4 (mapping properties). (a) *Suppose $\sigma(\varepsilon V(x,\theta_0), X, \gamma) \in S^m_n$, where $n \geq d+1$ and $m \leq 0$. Then $\sigma_D : L^2(\mathbb{R}^d \times \mathbb{T}) \to L^2(\mathbb{R}^d \times \mathbb{T})$ and*

$$|\sigma_D u|_0 \leq \frac{C}{\gamma^{|m|}} |u|_0.$$

(b) *Suppose $\sigma(\varepsilon V(x,\theta_0), X, \gamma) \in S^m_n$, where $n \geq d+1$ and $m > 0$. Then $\sigma_D : H^{m,\varepsilon}(\mathbb{R}^d \times \mathbb{T}) \to L^2(\mathbb{R}^d \times \mathbb{T})$ and*

$$|\sigma_D u|_0 \leq C |u|_{H^{m,\varepsilon},\gamma}.$$

(c) (*Smoothing property*) *Suppose $\sigma(\varepsilon V(x,\theta_0), X, \gamma) \in S^{-1}_n$, where $n \geq d+2$. Then*

$$\sigma_D : L^2(\mathbb{R}^d \times \mathbb{T}) \to H^{1,\varepsilon}(\mathbb{R}^d \times \mathbb{T})$$

and

$$|\sigma_D u|_{H^{1,\varepsilon},\gamma} \leq C |u|_0.$$

(d) *Suppose $\sigma(\varepsilon V(x,\theta_0), X, \gamma) \in S^0_n$, where $n \geq d+2$. Then $\sigma_D : H^{1,\varepsilon}(\mathbb{R}^d \times \mathbb{T}) \to H^{1,\varepsilon}(\mathbb{R}^d \times \mathbb{T})$ and*

$$|\sigma_D u|_{H^{1,\varepsilon},\gamma} \leq C |u|_{H^{1,\varepsilon},\gamma}.$$

Residual operators. We sometimes denote by $r_{0,D}$ an operator that maps $L^2(\mathbb{R}^d \times \mathbb{T}) \to L^2(\mathbb{R}^d \times \mathbb{T})$ and satisfies a uniform operator bound

$$|r_{0,D} u|_0 \leq C |u|_0,$$

even when $r_{0,D}$ is not necessarily defined by a symbol in some class S^0_n. Similarly, we sometimes let $r_{-1,D}$ denote an operator not necessarily associated to a symbol in S^{-1}_n such that

$$|r_{-1,D} u|_{H^{1,\varepsilon},\gamma} \leq C |u|_0. \tag{A-3}$$

For example, the composition $\sigma_{-1,D} \tau_{0,D} = r_{-1,D}$ of an operator of order -1 (case (c) in Proposition A.4) with an operator of order 0 (case (a) when $m = 0$) is of this latter type.

Remark A.5. Observe that a composition of the form $r_{0,D} r_{-1,D}$ is not necessarily an operator of type $r_{-1,D}$, a fact that is a source of difficulty in the proof of the main linear estimate, Proposition 2.2. This is the case, for example, if $r_{0,D}$ is the operator of multiplication by $V(x', \theta_0) \in \mathscr{C}^1_b(\mathbb{R}^d \times \mathbb{T})$. On the other hand we have

$$\varepsilon V(x', \theta_0) r_{-1,D} = r_{-1,D},$$

and, more generally, Proposition A.4(d) implies that if $\sigma \in S^0_n$, $n \geq d+2$, we have

$$\sigma_D r_{-1,D} = r_{-1,D}.$$

Adjoints and products. In spite of the fact that singular symbols and their derivatives fail to decay in the classical way in $\langle \xi', k, \gamma \rangle$, it is possible to construct a crude calculus of singular pseudodifferential operators with useful formulas for adjoints and products, which, in particular, permit Gårding inequalities to be proved. This calculus was used repeatedly in the proof of the main linear estimate, Proposition 2.2. Detailed proofs can be found in [Coulombel et al. 2012].

In the next proposition, σ^* denotes the conjugate transpose of the $N \times N$ matrix valued symbol σ, while $(\sigma_D)^*$ denotes the adjoint operator for the L^2 scalar product.

Proposition A.6 (adjoints). (a) *Let $\sigma \in S^0_n$, where $n \geq 2d+3$. Then $(\sigma_D)^* - (\sigma^*)_D = r_{-1,D}$.*

(b) *Let $\sigma \in S^1_n$, where $n \geq 3d+4$. Then $(\sigma_D)^* - (\sigma^*)_D = r_{0,D}$.*

Proposition A.7 (products). (a) *Suppose σ and τ lie in S^0_n, where $n \geq 2d+3$. Then*

$$\sigma_D \tau_D - (\sigma\tau)_D = r_{-1,D}.$$

(b) *Suppose $\sigma \in S^1_n$, $\tau \in S^0_n$ or $\sigma \in S^0_n$, $\tau \in S^1_n$, where $n \geq 3d+4$. Then*

$$\sigma_D \tau_D - (\sigma\tau)_D = r_{0,D}.$$

(c) *Suppose $\sigma \in S^{-1}_n$, $\tau \in S^1_n$, where $n \geq 3d+4$. Then*

$$\sigma_D \tau_D - (\sigma\tau)_D = r_{-1,D}. \tag{A-4}$$

Remark A.8. Observe that when $\tau = \tau(X, \gamma)$ is independent of $\varepsilon V(x, \theta_0)$ and thus gives rise to a Fourier multiplier, the composition $\sigma_D \tau_D = (\sigma\tau)_D$ is exact, a fact that has been used several times in the proof of Proposition 2.2.

The equality (A-4) does not hold in general when $\sigma \in S^1_n$ and $\tau \in S^{-1}_n$, and this is one of the main difficulties in the proof of Proposition 2.4.

In the proof of Proposition 2.2 we use the following localized Gårding inequality for zero-order operators. As before, we write $\zeta = (\xi', \gamma)$.

Proposition A.9 (Gårding inequality). *Let $\sigma(v, \zeta) \in \boldsymbol{S}^0(\mathbb{O})$ and $\chi(v, \zeta) \in \boldsymbol{S}^0(\mathbb{O})$ be such that*

$$\operatorname{Re} \sigma(v, \zeta) \geq c > 0$$

on a conic neighborhood of supp χ. *Provided the corresponding singular symbols lie in S^0_n, $n \geq 2d+2$, we have*

$$\operatorname{Re}(\sigma_D \chi_D u, \chi_D u) \geq \frac{c}{2} |\chi_D u|_0^2 - \frac{C}{\gamma} |u|_0^2.$$

Extended calculus. In the proof of Corollary 2.3 we use a slight extension of the singular calculus. For given parameters $0 < \delta_1 < \delta_2 < 1$, we choose a cutoff $\chi^e(\xi', k\beta/\varepsilon, \gamma)$ such that

$$0 \leq \chi^e \leq 1, \quad \chi^e\left(\xi', \frac{k\beta}{\varepsilon}, \gamma\right) = 1 \text{ on } \left\{(\gamma^2 + |\xi'|^2)^{1/2} \leq \delta_1 \left|\frac{k\beta}{\varepsilon}\right|\right\}, \quad \operatorname{supp} \chi^e \subset \left\{(\gamma^2 + |\xi'|^2)^{1/2} \leq \delta_2 \left|\frac{k\beta}{\varepsilon}\right|\right\},$$

and define a corresponding Fourier multiplier χ_D in the extended calculus by the formula (A-2) with $\chi^e(\xi', k\beta/\varepsilon, \gamma)$ in place of $\sigma(\varepsilon V, X, \gamma)$. Composition laws involving such operators are proved in

[Coulombel et al. 2012], but here we need only the fact that part (a) of Proposition A.7 holds when either σ or τ is replaced by an extended cutoff χ^e.

Appendix B: An example derived from the Euler equations

In this appendix we explain in a particular example how one can derive a single nonlocal nonlinear equation that governs the evolution of the amplitude function a, which itself determines the leading profile $\mathcal{V}^0$; see Proposition 1.24. In the process, we provide explicit constructions of a number of the objects that appeared in our earlier discussion of approximate solutions.

As in [Coulombel and Guès 2010], we consider the linearized Euler equations in two space dimensions to which we add a nonlinear zero-order term (we slightly change notation compared with the introduction). More precisely, we consider the system

$$\begin{cases} \partial_t V^\varepsilon + A_1 \partial_{x_1} V^\varepsilon + A_2 \partial_{x_2} V^\varepsilon + \boldsymbol{D}(V^\varepsilon, V^\varepsilon) = 0, & (t, x_1, x_2) \in (-\infty, T] \times \mathbb{R}^2_+, \\ B V^\varepsilon|_{x_2=0} + \Psi(V^\varepsilon, V^\varepsilon)|_{x_2=0} = \varepsilon^2 G(t, x_1, \phi_0(t, x_1)/\varepsilon), & (t, x_1) \in (-\infty, T] \times \mathbb{R}, \\ V^\varepsilon|_{t<0} = 0, \end{cases} \tag{B-1}$$

where the 3×3 matrices A_1, A_2 are given by

$$A_1 := \begin{pmatrix} 0 & -v & 0 \\ -c^2/v & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 := \begin{pmatrix} u & 0 & -v \\ 0 & u & 0 \\ -c^2/v & 0 & u \end{pmatrix},$$

and the parameters v, u, c are chosen so that

$$v > 0, \quad 0 < u < c.$$

The latter assumption corresponds to the linearization of the Euler equations at a given specific volume v with corresponding sound speed c, and a *subsonic incoming* velocity $(0, u)$ (observe the difference with [Coulombel and Guès 2010]). We also assume that $\boldsymbol{D}$ in (B-1) is a symmetric bilinear operator from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^3$, and that Ψ is a bilinear operator from $\mathbb{R}^3 \times \mathbb{R}^3$ into $\mathbb{R}^2$ (why we choose $\mathbb{R}^2$ is explained below).

For such parameters, the operator $\partial_t + A_1 \partial_{x_1} + A_2 \partial_{x_2}$ in (B-1) is strictly hyperbolic with three characteristic speeds:

$$\lambda_1(\xi_1, \xi_2) := u\xi_2 - c\sqrt{\xi_1^2 + \xi_2^2}, \quad \lambda_2(\xi_1, \xi_2) := u\xi_2, \quad \lambda_3(\xi_1, \xi_2) := u\xi_2 + c\sqrt{\xi_1^2 + \xi_2^2}.$$

There are two incoming characteristics and one outgoing characteristic, so B should be a 2×3 matrix of maximal rank. The choice of B is made precise below. Of course, the source term G in (B-1) is valued in $\mathbb{R}^2$. We assume moreover that G is 1-periodic and has mean zero with respect to its third variable θ_0. We choose a planar phase ϕ_0 for the oscillations of the boundary source term in (B-1):

$$\phi_0(t, x_1) := \underline{\tau} t + \underline{\eta} x_1, \quad (\underline{\tau}, \underline{\eta}) \neq (0, 0).$$

The hyperbolic region $\mathcal{H}$ can be explicitly computed and is given by

$$\mathcal{H} = \{(\tau, \eta) \in \mathbb{R} \times \mathbb{R} / |\tau| > \sqrt{c^2 - u^2}|\eta|\}.$$

For concreteness, we fix from now on parameters $(\underline{\tau}, \underline{\eta})$ such that $\underline{\eta} > 0$ and $\underline{\tau} = c\underline{\eta}$. In this way, we have $(\underline{\tau}, \underline{\eta}) \in \mathcal{H}$.

We determine the planar characteristic phases whose trace on $\{x_2 = 0\}$ equals ϕ_0. This amounts to finding the roots ω of the dispersion relation

$$\det[\underline{\tau} I + \underline{\eta} A_1 + \omega A_2] = 0.$$

We obtain three real roots that are given by

$$\underline{\omega}_1 := \frac{2M}{1 - M^2}\underline{\eta}, \quad \underline{\omega}_2 := 0, \quad \underline{\omega}_3 := -\frac{1}{M}\underline{\eta}, \quad M := \frac{u}{c} \in (0, 1).$$

The associated (real) phases are $\phi_i(t, x) := \phi_0(t, x_1) + \underline{\omega}_i x_2$, $i = 1, 2, 3$. The relations

$$\underline{\tau} + \lambda_1(\underline{\eta}, \underline{\omega}_1) = \underline{\tau} + \lambda_1(\underline{\eta}, \underline{\omega}_2) = \underline{\tau} + \lambda_2(\underline{\eta}, \underline{\omega}_3) = 0$$

yield the group velocity $\boldsymbol{v}_i$ associated with each phase ϕ_i:

$$\boldsymbol{v}_1 := \frac{1 - M^2}{1 + M^2}\begin{pmatrix} -c \\ -u \end{pmatrix}, \quad \boldsymbol{v}_2 := \begin{pmatrix} -c \\ u \end{pmatrix}, \quad \boldsymbol{v}_3 := \begin{pmatrix} 0 \\ u \end{pmatrix}.$$

Hence the phase ϕ_1 is outgoing while ϕ_2, ϕ_3 are incoming. With the notation of the introduction, we can also compute

$$r_1 := \begin{pmatrix} \dfrac{1+M^2}{1-M^2} v \\ c \\ \dfrac{2Mc}{1-M^2} \end{pmatrix}, \qquad r_2 := \begin{pmatrix} v \\ c \\ 0 \end{pmatrix}, \qquad r_3 := \begin{pmatrix} 0 \\ c \\ u \end{pmatrix},$$

$$\ell_1 := \frac{1 - M^2}{2(1 + M^2)}\begin{pmatrix} 1/v \\ -1/c \\ 1/u \end{pmatrix}, \quad \ell_2 := \frac{1}{2}\begin{pmatrix} 1/v \\ 1/c \\ -1/u \end{pmatrix}, \quad \ell_3 := \frac{1}{1 + M^2}\begin{pmatrix} -1/v \\ 1/c \\ M/c \end{pmatrix},$$

from which one can obtain the expression of the projectors P_1, P_2, P_3 as well as the expression of the partial inverses R_1, R_2, R_3. The stable subspace at the frequency $(\underline{\tau}, \underline{\eta})$ is spanned by the vectors r_2, r_3. The matrix B in (B-1) is chosen as

$$B := \begin{pmatrix} 0 & v & 0 \\ u & 0 & v \end{pmatrix},$$

so that we can choose $e := r_2 - r_3$ as the vector that spans $\ker B \cap \mathbb{E}^s(\underline{\tau}, \underline{\eta})$. The reader can check that all our weak stability assumptions are satisfied with this particular choice of boundary conditions. (We skip the details, which are just slightly more complicated than those in [Coulombel and Guès 2010].) The one-dimensional space $B\mathbb{E}^s(\underline{\tau}, \underline{\eta})$ can be written as the orthogonal of the vector $b := (u, -c)^T$.

The leading profile $\mathcal{V}^0$ and the corrector $\mathcal{V}^1$ satisfy (see Proposition 1.24)

$$\mathcal{V}^0 = \mathcal{V}^0_{\text{inc}} = \sigma_2(t,x,\theta_2)r_2 + \sigma_3(t,x,\theta_3)r_3, \quad \mathcal{V}^1_{\text{out}} = \tau_1(t,x,\theta_1)r_1.$$

Moreover, we have

$$\mathcal{V}^0(t,x_1,0,\theta_0,\theta_0,\theta_0) = a(t,x_1,\theta_0)e = a(t,x_1,\theta_0)(r_2 - r_3),$$

where the scalar function a is 1-periodic with respect to θ_0 and has mean 0. The Fourier coefficients of a are denoted by a_k, $k \in \mathbb{Z}$, where a_0 equals 0 for all time t. Since the functions σ_2, σ_3 satisfy the transport equations[43]

$$\partial_t \sigma_2 + \boldsymbol{v}_2 \cdot \nabla_x \sigma_2 = \partial_t \sigma_3 + \boldsymbol{v}_3 \cdot \nabla_x \sigma_3 = 0,$$

and vanish for $t < 0$, we obtain the expressions

$$\sigma_2(t,x,\theta_2) = a\left(t - \frac{x_2}{u}, x_1 + \frac{x_2}{M}, \theta_2\right), \quad \sigma_3(t,x,\theta_3) = -a\left(t - \frac{x_2}{u}, x_1, \theta_3\right). \tag{B-2}$$

To compute $\mathcal{V}^1_{\text{out}}$, we must solve

$$E_{\text{out}}(L(\partial)\mathcal{V}^1_{\text{out}} + A_2^{-1}\boldsymbol{D}(\mathcal{V}^0_{\text{inc}}, \mathcal{V}^0_{\text{inc}})) = 0 \quad (\text{here } E_{\text{out}} = E_1), \tag{B-3}$$

and we thus need to determine the resonances between the phases. A simple calculation shows that there is a nontrivial $n \in \mathbb{Z}^3$ satisfying $n_1\phi_1 = n_2\phi_2 + n_3\phi_3$ if and only if M^2 is a rational number. We thus assume this to be the case from now on. The resonance between the phases reads

$$n_1 := q, \quad n_2 := p + q, \quad n_3 := -p, \quad \text{with } \frac{2M^2}{1 - M^2} = \frac{p}{q},$$

and it is understood that p, q are both positive and have no common divisor (for instance $p = q = 1$ when M equals $1/\sqrt{3}$). Expanding the quadratic term $\boldsymbol{D}(\mathcal{V}^0_{\text{inc}}, \mathcal{V}^0_{\text{inc}})$ in Fourier series, and using the relation

$$\mathcal{C}_1 = \mathbb{Z}\begin{pmatrix}1\\0\\0\end{pmatrix} \cup \mathbb{Z}\begin{pmatrix}0\\n_2\\n_3\end{pmatrix},$$

we obtain (using the expressions (B-2))

$$E_1(A_2^{-1}\boldsymbol{D}(\mathcal{V}^0_{\text{inc}}, \mathcal{V}^0_{\text{inc}})) = -2\sum_{k \in \mathbb{Z}} a_{k(p+q)}\left(t - \frac{x_2}{u}, x_1 + \frac{x_2}{M}\right) a_{-kp}\left(t - \frac{x_2}{u}, x_1\right) e^{2i\pi kq\theta_1} P_1 A_2^{-1}\boldsymbol{D}(r_2, r_3).$$

In terms of the interaction integral, we obtain the expression

$$\begin{aligned} &E_1(A_2^{-1}\boldsymbol{D}(\mathcal{V}^0_{\text{inc}}, \mathcal{V}^0_{\text{inc}})) \\ &\qquad = -2\int_0^1 (a)_{n_2}\left(t - \frac{x_2}{u}, x_1 + \frac{x_2}{M}, \frac{n_1}{n_2}\theta_1 - \frac{n_3}{n_2}\theta_3\right) a\left(t - \frac{x_2}{u}, x_1, \theta_3\right) d\theta_3 \, P_1 A_2^{-1}\boldsymbol{D}(r_2, r_3), \end{aligned}$$

[43]Observe that there is no zero-order term in the transport equations because the zero-order term in (B-1) has only a quadratic part. This choice has been made for the sake of simplicity.

where $(a)_{n_2}$ still denotes the action of a under the preparation map that retains only Fourier coefficients that are multiples of n_2. Consequently, (B-3) reads

$$\left(\partial_t - \frac{1-M^2}{1+M^2} c\,\partial_{x_1} - \frac{1-M^2}{1+M^2} u\,\partial_{x_2}\right)\tau_1 \\ = \boldsymbol{d} \int_0^1 (a)_{n_2}\left(t-\frac{x_2}{u}, x_1+\frac{x_2}{M}, \frac{n_1}{n_2}\theta_1 - \frac{n_3}{n_2}\theta_3\right) a\left(t-\frac{x_2}{u}, x_1, \theta_3\right) d\theta_3, \quad \text{(B-4)}$$

with

$$\boldsymbol{d} := -2u\frac{1-M^2}{1+M^2}\ell_1 \cdot A_2^{-1}\boldsymbol{D}(r_2, r_3).$$

The transport equation (B-4) is solved by integrating along the characteristics, and we obtain the expression

$$\tau_1(t, x_1, 0, \theta_1) = \boldsymbol{d} \int_0^t \int_0^1 (a)_{n_2}\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + 2c\frac{1-M^2}{1+M^2}(t-s), \frac{n_1}{n_2}\theta_1 - \frac{n_3}{n_2}\theta_3\right) \\ \times a\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + c\frac{1-M^2}{1+M^2}(t-s), \theta_3\right) d\theta_3\, ds. \quad \text{(B-5)}$$

The Fourier series expansion of τ_1 reads

$$\tau_1(t, x_1, 0, \theta_1) = \boldsymbol{d} \sum_{k\in\mathbb{Z}} \int_0^t a_{k(p+q)}\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + 2c\frac{1-M^2}{1+M^2}(t-s)\right) \\ \times a_{-kp}\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + c\frac{1-M^2}{1+M^2}(t-s)\right) ds\, e^{2i\pi kq\theta_1}.$$

The equation governing the amplitude a reads

$$b \cdot ((a^2)^*\Psi(e,e) + \tau_1|_{x_2=0} B r_1 - BR(L(\partial)\mathcal{V}^0_{\text{inc}})|_{x_2=0}) = b \cdot G,$$

where functions are evaluated at $x_2 = 0$ and $\theta_1 = \theta_2 = \theta_3 = \theta_0$. Since we already have the expression of τ_1 in terms of a, the only task left is to compute the trace of the term $BR(L(\partial)\mathcal{V}^0_{\text{inc}})$. Recalling that $R_2 r_2 = R_3 r_3 = 0$, we have

$$BR(L(\partial)\mathcal{V}^0_{\text{inc}})|_{x_2=0} = (BR_2A_2^{-1}r_2 + BR_3A_2^{-1}r_3)\partial_t\mathfrak{a} + (BR_2A_2^{-1}A_1r_2 + BR_3A_2^{-1}A_1r_3)\partial_{x_1}\mathfrak{a},$$

with $\mathfrak{a}$ the unique primitive function of a with zero mean. Using the expressions of R_2, R_3 in terms of the projectors P_1, P_2, P_3, which themselves can be obtained from the vectors r_i, ℓ_i, we get

$$b \cdot (BR_2A_2^{-1}r_2 + BR_3A_2^{-1}r_3) = -\frac{uv(1+M^2)}{M^2\underline{\eta}},$$

$$b \cdot (BR_2A_2^{-1}A_1r_2 + BR_3A_2^{-1}A_1r_3) = \frac{ucv(1+M^2)}{M^2\underline{\eta}}.$$

The fact that both quantities are proportional to each other with a factor $-c$ comes from a general fact; see [Coulombel and Guès 2010, Lemma 5.1].

The function a should therefore satisfy the amplitude equation

$$\frac{uv(1+M^2)}{M^2\underline{\eta}}(\partial_t \mathfrak{a} - c\partial_{x_1}\mathfrak{a}) + b\cdot\Psi(e,e)(a^2)^* + b\cdot B r_1 \tau_1|_{x_2=0} = b\cdot G,$$

or, equivalently,

$$\frac{uv(1+M^2)}{M^2\underline{\eta}}(\partial_t a - c\partial_{x_1} a) + b\cdot\Psi(e,e)\partial_{\theta_0}(a^2) + b\cdot B r_1 \partial_{\theta_0}\tau_1|_{x_d=0} = b\cdot \partial_{\theta_0} G. \tag{B-6}$$

Let us define the two constants

$$\alpha_1 := \frac{M^2\underline{\eta}}{uv(1+M^2)} b\cdot\Psi(e,e), \quad \alpha_2 := \frac{4ucM^2\underline{\eta}}{1+M^2}\ell_1\cdot A_2^{-1}\boldsymbol{D}(r_2,r_3).$$

Then (B-6) reads

$$\partial_t a - c\partial_{x_1} a + \alpha_1\partial_{\theta_0}(a^2) + \alpha_2\partial_{\theta_0}\frac{\tau_1}{\boldsymbol{d}}|_{x_2=0} = \frac{M^2\underline{\eta}}{uv(1+M^2)} b\cdot\partial_{\theta_0} G,$$

where the derivative $\partial_{\theta_0}\tau_1/\boldsymbol{d}|_{x_2=0}$ is computed from the relation (B-5):

$$\partial_{\theta_0}\frac{\tau_1}{\boldsymbol{d}}\Big|_{x_2=0} = \frac{n_1}{n_2}\int_0^t\int_0^1 (\partial_{\theta_0} a)_{n_2}\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + 2c\frac{1-M^2}{1+M^2}(t-s), \frac{n_1}{n_2}\theta_0 - \frac{n_3}{n_2}\Theta\right) \\ \times a\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + c\frac{1-M^2}{1+M^2}(t-s), \Theta\right) d\Theta\, ds.$$

In terms of the Fourier coefficients a_k, the latter equation is seen to be equivalent to the infinite system of transport equations

$$\partial_t a_k - c\partial_{x_1} a_k + 2i\pi k\alpha_1 \sum_{k'\in\mathbb{Z}} a_{k'}a_{k-k'} = 2i\pi k\frac{M^2\underline{\eta}}{uv(1+M^2)} b\cdot G_k, \quad k\notin q\mathbb{Z},$$

and

$$\partial_t a_{kq} - c\partial_{x_1} a_{kq} + 2i\pi kq\alpha_1 \sum_{k'\in\mathbb{Z}} a_{k'}a_{kq-k'} + 2i\pi kq\alpha_2 \\ \times \int_0^t a_{k(p+q)}\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + 2c\frac{1-M^2}{1+M^2}(t-s)\right) a_{-kp}\left(\frac{2s-(1-M^2)t}{1+M^2}, x_1 + c\frac{1-M^2}{1+M^2}(t-s)\right) ds \\ = 2i\pi kq\frac{M^2\underline{\eta}}{uv(1+M^2)} b\cdot G_{kq}.$$

We recall that the coefficient a_0 vanishes.

In the special case $M = 1/\sqrt{3}$, the above system reduces to

$$\partial_t a_k - c\partial_{x_1} a_k + 2i\pi k\alpha_1 \sum_{k'\in\mathbb{Z}} a_{k'}a_{k-k'} + 2i\pi k\alpha_2 \int_0^t a_{2k}\left(\frac{3s-t}{2}, x_1 + c(t-s)\right) a_{-k}\left(\frac{3s-t}{2}, x_1 + \frac{c}{2}(t-s)\right) ds \\ = 2i\pi k\frac{\eta}{4uv} b\cdot G_k, \quad k\in\mathbb{Z},$$

with parameters α_1, α_2 computed from the nonlinearities $\boldsymbol{D}$, Ψ in (B-1):

$$\alpha_1 := \frac{\underline{\eta}}{4uv} b \cdot \Psi(e, e), \quad \alpha_2 := uc\underline{\eta}\ell_1 \cdot A_2^{-1} \boldsymbol{D}(r_2, r_3).$$

References

[Alinhac 1989] S. Alinhac, "Existence d'ondes de raréfaction pour des systèmes quasi-linéaires hyperboliques multidimensionnels", *Comm. Partial Differential Equations* **14**:2 (1989), 173–230. MR 90h:35147b Zbl 0692.35063

[Alinhac and Gérard 2007] S. Alinhac and P. Gérard, *Pseudo-differential operators and the Nash–Moser theorem*, Graduate Studies in Mathematics **82**, Amer. Math. Soc., Providence, 2007. MR 2007m:35001 Zbl 1121.47033

[Artola and Majda 1987] M. Artola and A. J. Majda, "Nonlinear development of instabilities in supersonic vortex sheets, I: The basic kink modes", *Phys. D* **28**:3 (1987), 253–281. MR 88i:76025 Zbl 0632.76074

[Benzoni-Gavage and Serre 2007] S. Benzoni-Gavage and D. Serre, *Multidimensional hyperbolic partial differential equations: first-order systems and applications*, Oxford University Press, 2007. MR 2008k:35002 Zbl 1113.35001

[Benzoni-Gavage et al. 2002] S. Benzoni-Gavage, F. Rousset, D. Serre, and K. Zumbrun, "Generic types and transitions in hyperbolic initial-boundary-value problems", *Proc. Roy. Soc. Edinburgh Sect. A* **132**:5 (2002), 1073–1104. MR 2003j:35200 Zbl 1029.35165

[Chazarain and Piriou 1982] J. Chazarain and A. Piriou, *Introduction to the theory of linear partial differential equations*, Studies in Mathematics and its Applications **14**, North-Holland, Amsterdam, 1982. MR 83j:35001 Zbl 0487.35002

[Coulombel 2004] J.-F. Coulombel, "Weakly stable multidimensional shocks", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **21**:4 (2004), 401–443. MR 2005e:35157 Zbl 1072.35120

[Coulombel 2005] J.-F. Coulombel, "Well-posedness of hyperbolic initial boundary value problems", *J. Math. Pures Appl.* (9) **84**:6 (2005), 786–818. MR 2006h:35166 Zbl 1078.35066

[Coulombel and Guès 2010] J.-F. Coulombel and O. Guès, "Geometric optics expansions with amplification for hyperbolic boundary value problems: linear problems", *Ann. Inst. Fourier* (*Grenoble*) **60**:6 (2010), 2183–2233. MR 2012d:35209 Zbl 1218.35137

[Coulombel and Williams 2013] J.-F. Coulombel and M. Williams, "Amplification of pulses in nonlinear geometric optics", preprint, 2013. To appear in J. Hyp. Differential Equations. arXiv 1308.6686

[Coulombel et al. 2011] J.-F. Coulombel, O. Guès, and M. Williams, "Resonant leading order geometric optics expansions for quasilinear hyperbolic fixed and free boundary problems", *Comm. Partial Differential Equations* **36**:10 (2011), 1797–1859. MR 2012k:35315 Zbl 1241.35131

[Coulombel et al. 2012] J.-F. Coulombel, O. Guès, and M. Williams, "Singular pseudodifferential calculus for wavetrains and pulses", Preprint, 2012. To appear in *Bull. Soc. Math. France*. arXiv 1201.6202

[Joly et al. 1993] J.-L. Joly, G. Métivier, and J. Rauch, "Generic rigorous asymptotic expansions for weakly nonlinear multidimensional oscillatory waves", *Duke Math. J.* **70**:2 (1993), 373–404. MR 94c:35048 Zbl 0815.35066

[Joly et al. 1995] J.-L. Joly, G. Métivier, and J. Rauch, "Coherent and focusing multidimensional nonlinear geometric optics", *Ann. Sci. École Norm. Sup.* (4) **28**:1 (1995), 51–113. MR 95k:35035 Zbl 0836.35087

[Kreiss 1970] H.-O. Kreiss, "Initial boundary value problems for hyperbolic systems", *Comm. Pure Appl. Math.* **23** (1970), 277–298. MR 55 #10862 Zbl 0193.06902

[Lax 1957] P. D. Lax, "Asymptotic solutions of oscillatory initial value problems", *Duke Math. J.* **24** (1957), 627–646. MR 20 #4096 Zbl 0083.31801

[Lescarret 2007] V. Lescarret, "Wave transmission in dispersive media", *Math. Models Methods Appl. Sci.* **17**:4 (2007), 485–535. MR 2008c:78037 Zbl 1220.35170

[Majda and Artola 1988] A. J. Majda and M. Artola, "Nonlinear geometric optics for hyperbolic mixed problems", pp. 319–356 in *Analyse mathématique et applications*, Gauthier-Villars, Montrouge, 1988. MR 90c:35152 Zbl 0674.35057

[Majda and Rosales 1983] A. Majda and R. Rosales, "A theory for spontaneous Mach stem formation in reacting shock fronts, I: The basic perturbation analysis", *SIAM J. Appl. Math.* **43**:6 (1983), 1310–1334. MR 84i:76051 Zbl 0544.76135

[Majda and Rosales 1984] A. Majda and R. Rosales, “A theory for spontaneous Mach-stem formation in reacting shock fronts, II: Steady-wave bifurcations and the evidence for breakdown”, *Stud. Appl. Math.* **71**:2 (1984), 117–148. MR 86b:35133

[Marcou 2010] A. Marcou, “Rigorous weakly nonlinear geometric optics for surface waves”, *Asymptot. Anal.* **69**:3-4 (2010), 125–174. MR 2011m:35370 Zbl 1222.35118

[Métivier 2000] G. Métivier, “The block structure condition for symmetric hyperbolic systems”, *Bull. London Math. Soc.* **32**:6 (2000), 689–702. MR 2001i:35198 Zbl 1073.35525

[Sablé-Tougeron 1988] M. Sablé-Tougeron, “Existence pour un problème de l’élastodynamique Neumann non linéaire en dimension 2”, *Arch. Rational Mech. Anal.* **101**:3 (1988), 261–292. MR 89f:35191 Zbl 0652.73019

[Williams 1996] M. Williams, “Nonlinear geometric optics for hyperbolic boundary problems”, *Comm. Partial Differential Equations* **21**:11-12 (1996), 1829–1895. MR 98d:35136 Zbl 0881.35068

[Williams 2000] M. Williams, “Boundary layers and glancing blow-up in nonlinear geometric optics”, *Ann. Sci. École Norm. Sup.* (4) **33**:3 (2000), 383–432. MR 2001f:35392 Zbl 0962.35118

[Williams 2002] M. Williams, “Singular pseudodifferential operators, symmetrizers, and oscillatory multidimensional shocks”, *J. Funct. Anal.* **191**:1 (2002), 132–209. MR 2003e:35334 Zbl 1028.35174

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JEAN-FRANCOIS COULOMBEL: jean-francois.coulombel@univ-nantes.fr
CNRS, Laboratoire de mathématiques Jean Leray (UMR CNRS 6629), Université de Nantes, 2 rue de la Houssinière, BP 92208, 44322 Nantes, France

OLIVIER GUÈS: gues@cmi.univ-mrs.fr
Laboratoire d’Analyse, Topologie et Probabilités (UMR CNRS 6632), Université de Provence,
Technopôle Château-Gombert, 39 rue F. Joliot Curie, 13453 Marseille 13, France

MARK WILLIAMS: williams@email.unc.edu
Mathematics Department, University of North Carolina, CB 3250, Phillips Hall, Chapel Hill, NC 27599, United States

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