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THE THEORY OF HAHN-MEROMORPHIC FUNCTIONS, A HOLOMORPHIC FREDHOLM THEOREM, AND ITS APPLICATIONS

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We introduce a class of functions near zero on the logarithmic cover of the complex plane that have convergent expansions into generalized power series. The construction covers cases where noninteger powers of z and also terms containing $\log z$ can appear. We show that, under natural assumptions, some important theorems from complex analysis carry over to this class of functions. In particular, it is possible to define a field of functions that generalize meromorphic functions, and one can formulate an analytic Fredholm theorem in this class. We show that this modified analytic Fredholm theorem can be applied in spectral theory to prove convergent expansions of the resolvent for Bessel type operators and Laplace–Beltrami operators for manifolds that are Euclidean at infinity. These results are important in scattering theory, as they are the key step in establishing analyticity of the scattering matrix and the existence of generalized eigenfunctions at points in the spectrum.

1. Introduction

Asymptotic expansions of the form

$$f(z) \sim \sum_{k,m} a_{k,m} z^{\alpha_k} (-\log z)^{\beta_m} \quad \text{as } z \to 0,$$

with nonintegers α_k and β_m, defined for functions f in some sector centered at 0 in the complex plane, appear frequently in mathematics and mathematical physics. Classical examples are solutions for differential equations (for example, in Frobenius' method) and expansions of algebraic functions at singularities. It has been shown that low energy resolvent expansions in scattering problems are of this form; see, for example, [Jensen and Kato 1979; Jensen and Nenciu 2001] for Schrödinger operators in $\mathbb{R}^n$, [Murata 1982] for operators with constant leading coefficients in $\mathbb{R}^n$, and [Guillarmou and Hassell 2009] for the Laplace operator on a general manifold with a conical end. The resolvent expansion for $|\lambda| \to \infty$ of cone degenerate differential operators leads to similar asymptotics; see, for example, [Gil et al. 2011]. In many of these examples, the expansions can be shown to be convergent under more restrictive assumptions on the structure at infinity of the underlying geometry.

The algebraic theory of generalized power series is well developed and can be found in the literature under the name Hahn series or Maltsev–Neumann series (see, for example, [Hahn 1907; Passman 1977, Chapter 13; Ribenboim 1992]). In this paper we are concerned with the analytic theory of such generalized

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power series. Namely, we will define a ring of functions, the Hahn-holomorphic functions, that have convergent expansions into generalized power series, and we will show that this ring is actually a division ring. We show that the quotient field, the field of Hahn-meromorphic functions, has a nice description in terms of Hahn series, and we generalize the notions of Hahn-holomorphic and Hahn-meromorphic functions to the operator valued case. The theory turns out to be very close to the case of analytic function theory. In particular, one of our main theorems states that an analog of the analytic Fredholm theorem holds in the class of Hahn-holomorphic functions.

The holomorphic Fredholm theorem plays an important role in geometric scattering theory as a tool to prove the existence of a meromorphic continuation of resolvent kernels of elliptic differential operators such as the Laplace operator. The extension is typically from the resolvent set across the continuous spectrum to a branched covering of the complex plane. As soon as such a meromorphic continuation of the resolvent kernel is established, resonances can be defined as poles of its continuation, generalized eigenfunctions may be defined as meromorphic functions of a suitably chosen spectral parameter, and an analytic continuation of the scattering matrix may be constructed. This in many situations leads to a rich mathematical structure that results in functional equations for the scattering matrix and Maass–Selberg relations for the generalized eigenfunctions (see, for example, [Müller 1987] for the case of manifolds with cusps of rank one, [Melrose 1993; Guillopé 1989; Müller and Strohmaier 2010] for manifolds with cylindrical ends, and [Müller 2011] for manifolds with fibered cusps). In particular, the analytic continuation of Eisenstein series may be regarded as a special case of this more general construction.

Often, as for example in the case of $\mathbb{R}^{2n+1}$ on asymptotically hyperbolic manifolds [Mazzeo and Melrose 1987; Guillarmou 2005], geometrically finite hyperbolic manifolds [Guillarmou and Mazzeo 2012], and on globally symmetric spaces of odd rank [Mazzeo and Vasy 2005; Strohmaier 2005], the branch points of the covering of the complex plane are algebraic and can be resolved by a change of variables. In this way, one can make sense of the statement that the resolvent is meromorphic at the branch point. In other examples, as in $\mathbb{R}^{2n}$ on symmetric spaces of even rank [Mazzeo and Vasy 2005; Strohmaier 2005] and on manifolds with generalized cusps [Hunsicker et al. 2014], the branch point is logarithmic, and this statement loses its meaning. The analytic Fredholm theorem can then only be applied away from the branching points. Our philosophy is that, at such branching points, it still makes sense to say when functions are Hahn-holomorphic, that is, have a convergent expansion into more general power series possibly containing log terms. Our Hahn analytic Fredholm theorem therefore allows us to analyze the resolvents at nonalgebraic branching points. Our theorem implies, for example, that the Hahn-meromorphic properties of the resolvent of the Laplace operator on a Riemannian manifold are stable under perturbations of the topology and the metric that are supported in compact regions. The theory can be developed further to establish Hahn-analyticity of the scattering matrix and of the generalized eigenfunctions in this context, but we decided to focus on the theoretical properties of Hahn-meromorphic functions first and keep the presentation self-contained. The applications in scattering theory will be developed elsewhere.

The article is organized as follows. Section 2 deals with the definition and the general theory of Hahn-holomorphic functions and some of their basic properties. In Section 3 we define Hahn-meromorphic functions, and in Section 4 we prove our generalization of the meromorphic Fredholm theorem in the framework of Hahn-holomorphic functions. Sections 5 and 6 deal with two important examples of convergent Hahn series: those that can be expanded into real powers of z and those that have such expansions with additional $\log z$ terms. The theory has a nice application: convergent resolvent expansions for Bessel type operators and Laplace–Beltrami operators on manifolds that are Euclidean at infinity can be shown to be simple consequences of the Hahn-holomorphic Fredholm theorem. These examples are treated in detail in Section 7; the main results here are Theorems 7.6 and 7.9.

We would like to thank the anonymous referee for suggestions leading to considerable simplifications in some of the arguments in Section 7.

2. Hahn-holomorphic functions

Let $(\Gamma, +)$ be a linearly ordered abelian group and let $(G, \cdot)$ be a group. Suppose $e : \Gamma \to G$, $\gamma \mapsto e_\gamma$ is a group homomorphism; in particular

$$e_0 = \mathbf{1} \in G, \quad e_{\gamma_1+\gamma_2} = e_{\gamma_1} \cdot e_{\gamma_2} \quad \text{for all } \gamma_1, \gamma_2 \in \Gamma.$$

The following definition and proposition are due to H. Hahn [1907].

Definition 2.1. Let $\mathcal{R}$ be a ring. A formal series

$$\mathfrak{h} = \sum_{\gamma\in\Gamma} a_\gamma e_\gamma, \quad a_\gamma \in \mathcal{R}$$

is called a *Hahn series* if the support of $\mathfrak{h}$,

$$\operatorname{supp}(\mathfrak{h}) := \{g \in \Gamma \mid a_g \neq 0 \in \mathcal{R}\},$$

is a well-ordered subset of Γ. The set of Hahn series will be denoted by $\mathcal{R}[\![e_\Gamma]\!]$.

Proposition 2.2. *The set of Hahn series $\mathcal{R}[\![e_\Gamma]\!]$ is a ring with multiplication*

$$\Big(\sum_{\alpha\in\Gamma} a_\alpha e_\alpha\Big)\Big(\sum_{\beta\in\Gamma} b_\beta e_\beta\Big) = \sum_{\gamma\in\Gamma} c_\gamma e_\gamma, \quad c_\gamma := \sum_{\substack{(\alpha,\beta)\in\Gamma\times\Gamma\\ \alpha+\beta=\gamma}} a_\alpha b_\beta \tag{1}$$

and addition

$$\sum_{\alpha\in\Gamma} a_\alpha e_\alpha + \sum_{\beta\in\Gamma} b_\beta e_\beta = \sum_{\gamma\in\Gamma} (a_\gamma + b_\gamma) e_\gamma.$$

If $\mathcal{R}$ is a field, so is $\mathcal{R}[\![e_\Gamma]\!]$.

It is well known that if the support of $\mathfrak{h}$ is contained in $\Gamma^+ = \{\gamma \mid \gamma > 0\}$, then $\mathbf{1} - \mathfrak{h}$ is invertible in $\mathcal{R}[\![e_\Gamma]\!]$ and its inverse is given by the Neumann series

$$(\mathbf{1} - \mathfrak{h})^{-1} = \sum_{k=0}^{\infty} \mathfrak{h}^k.$$

This is due to the fact that, for any well-ordered subset W of Γ^+, the semigroup generated by W is also well ordered; see, for example, [Passman 1977, Lemma 2.10]. Here convergence of a sequence

$(\mathfrak{p}_n) \subset \mathcal{R}[\![e_\Gamma]\!]$ to $\mathfrak{p} \in \mathcal{R}[\![e_\Gamma]\!]$ is understood in the sense that, for every element $\alpha \in \Gamma$, there exists an $N > 0$ such that, for all $n > N$, the coefficients of e_α in $\mathfrak{p}$ and $\mathfrak{p}_n$ are equal.

In the following, let $\mathcal{Z}$ be the logarithmic covering surface of the complex plane without the origin. We will use polar coordinates (r, φ) as global coordinates to identify $\mathcal{Z}$ as a set with $\mathbb{R}_+ \times \mathbb{R}$. Adding a single point $\{0\}$ to $\mathcal{Z}$, we obtain a set $\mathcal{Z}_0$ and a projection map $\pi : \mathcal{Z}_0 \to \mathbb{C}$ by extending the covering map $\mathcal{Z} \to \mathbb{C}\backslash\{0\}$ sending $0 \in \mathcal{Z}_0$ to $0 \in \mathbb{C}$. We endow $\mathcal{Z}$ with the covering topology and $\mathcal{Z}_0$ with the topology generated by the open sets in $\mathcal{Z}$ together with the open discs $D_\varepsilon := \{0\} \cup \{(r, \varphi) \mid 0 \le r < \varepsilon\}$. This means a sequence $((r_n, \varphi_n))_n$ converges to zero if and only if $r_n \to 0$. The covering map is continuous with respect to this topology. For a point $z \in \mathcal{Z}_0$, we denote by $|z|$ its r-coordinate and by $\arg z$ its φ coordinate. We will think of the positive real axis as embedded in $\mathcal{Z}$ as the subset $\{z \mid \arg z = 0\}$. In the following, $Y \subset \mathcal{Z}$ will always denote an open subset containing an open interval $(0, \delta)$ for some $\delta > 0$ and such that $0 \notin Y$. The set Y_0 will denote $Y \cup \{0\}$. In the applications we have in mind, the set Y is typically of the form $D_\delta^{[\sigma]}\backslash\{0\}$, where $D_\delta^{[\sigma]} = \{z \in \mathcal{Z}_0 \mid 0 \le |z| < \delta,\ |\varphi| < \sigma\}$. For the discussion and the general theorems, it is not necessary to restrict ourselves to this case.

In the remainder of this article we assume that $G := (\mathrm{Hol}(Y \cap D_\varepsilon), \cdot)^\times$ is a set of nonvanishing holomorphic functions and that the group homomorphism e satisfies the condition

$$\text{for all } \gamma > 0, \quad e_\gamma \text{ is bounded on } Y \text{ and } \lim_{z\to 0} |e_\gamma(z)| = 0. \tag{E1}$$

Definition 2.3. Suppose that $\mathcal{R}$ is a vector space with norm $\|\cdot\|$. A Hahn series $\mathfrak{f} = \sum_{\alpha\in\Gamma} a_\alpha e_\alpha$ is called *normally convergent* in $Y \cap D_\varepsilon$ if its support is countable and

$$\sum_{\alpha\in\Gamma} \|a_\alpha\| \, \|e_\alpha\|_{Y,\varepsilon} < \infty,$$

where $\|e_\alpha\|_{Y,\varepsilon} := \sup_{z\in Y\cap D_\varepsilon} |e_\alpha(z)|$.

Since a normally convergent series converges absolutely and uniformly, the value of the function

$$f(z) = \sum_{\alpha\in\Gamma} a_\alpha e_\alpha(z), \quad z \in Y \cap D_\varepsilon,$$

does not depend on the order of summation and f is holomorphic in $z \neq 0$.

Definition 2.4. Let $S \subset \Gamma_0^+ = \Gamma^+ \cup \{0\}$ be a subset of the nonnegative group elements.

- The family $\{e_\alpha\}_{\alpha\in S}$ is called *weakly monotonic* if there exists an $r_S > 0$ such that, for every $x \in (0, r_S)$, there is a *radius* $\rho(x)$ with $0 < \rho(x) \le x$ and with the property

$$\alpha \in S \implies \|e_\alpha\|_{Y,\rho(x)} \le |e_\alpha(x)|.$$

- The set S is called *admissible for e* (or simply *admissible*) if $\{e_\alpha\}_{\alpha\in S}$ is weakly monotonic, and if, for every $B \subset S$, the family

$$\{e_{\alpha-\min B}\}_{\substack{\alpha\in S\\ \alpha>\min B}}$$

is also weakly monotonic.

Definition 2.5 (Hahn-holomorphic functions). Suppose that $\mathcal{R}$ is a Banach algebra. A continuous function $h : Y_0 \to \mathcal{R}$ which is holomorphic in Y is called (Y, Γ)-Hahn-holomorphic (or simply *Hahn-holomorphic*) if there is a Hahn series

$$\mathfrak{h} = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma, \quad a_\gamma \in \mathcal{R},$$

with countable admissible support, converging normally on $Y \cap D_\delta$ for some $\delta > 0$, and such that

$$h(z) = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma(z), \quad z \in Y \cap D_\delta.$$

We will denote the Hahn series of a Hahn-holomorphic function h by the corresponding "fraktur" letter $\mathfrak{h}$. Note that (E1) together with uniform convergence imply that $\operatorname{supp} \mathfrak{h} \subset \Gamma_0^+$ and $h(0) = a_0$. Of course any normally convergent Hahn series with admissible support gives rise to a Hahn-holomorphic function.

Here is a direct consequence of the support of Hahn-holomorphic functions being admissible:

Lemma 2.6. *Let*

$$h(z) = \sum_{\gamma \in \Gamma} a_\gamma e_\gamma(z), \quad z \in Y \cap D_{2r},$$

be Hahn-holomorphic with $\mathfrak{m} = \min \operatorname{supp}(\mathfrak{h})$. *Then*

$$e_{-\mathfrak{m}}(z) h(z) = \sum_{\gamma \geq \mathfrak{m}} a_\gamma e_{\gamma - \mathfrak{m}}(z)$$

is Hahn-holomorphic.

Proof. Let ρ_1 be the radius for $\{e_\gamma\}$ such that, for all $\gamma \in \operatorname{supp}(\mathfrak{h})$,

$$\|e_\gamma\|_{\rho_1(r)} \leq |e_\gamma(r)|,$$

and similarly let ρ_2 be the radius for $\{e_{\gamma - \mathfrak{m}}\}$. For $\rho(r) = \min\{\rho_1(r), \rho_2(r)\}$,

$$\|e_\mathfrak{m}\|_{\rho(r)} \sum_{\gamma \in \Gamma} \|a_\gamma\| \, \|e_{\gamma - \mathfrak{m}}\|_{\rho(r)} \leq |e_\mathfrak{m}(r)| \sum_{\gamma \in \Gamma} \|a_\gamma\| \, |e_{\gamma - \mathfrak{m}}(r)| = \sum_{\gamma \in \Gamma} \|a_\gamma\| \, |e_\gamma(r)| < \infty.$$

Thus $\sum_{\gamma \in \Gamma} a_\gamma e_{\gamma - \mathfrak{m}}$ converges normally on $D_{\rho(r)}$. □

Proposition 2.7. *Let* $f : Y \to \mathcal{R}$ *be a Hahn-holomorphic function represented by a Hahn series* $\mathfrak{f}$ *on* $Y \cap D_\delta$. *Suppose the zeros of* f *accumulate in* $Y \cup \{0\}$. *Then* $f \equiv 0$ *and* $\mathfrak{f} = 0$. *In particular, the Hahn series of a Hahn-holomorphic function is completely determined by the germ of the function at zero.*

Proof. If the zero set of f has accumulation points in Y, the statement follows from the fact that f is holomorphic in this set. It remains to show that if $f \neq 0$, then 0 can not be an accumulation point of the zero set of f. Let $\mathfrak{f}$ be a Hahn series that represents the function on $Y \cap D_\varepsilon$. Let $f \neq 0$; then $\mathfrak{f} \neq 0$. Let $\mathfrak{m} = \min \operatorname{supp} \mathfrak{f}$. If there is no other element in the support of $\mathfrak{f}$, then $f(z) = a_\mathfrak{m} e_\mathfrak{m}(z)$ and the statement follows from the fact that $e_\mathfrak{m}$ has no zeros in Y. Otherwise, let $\mathfrak{m}_1$ be the smallest element in $\operatorname{supp} \mathfrak{f}$ which is larger than $\mathfrak{m}$. Then

$$f(z) = \sum_\alpha a_\alpha e_\alpha(z) = e_\mathfrak{m}(z) \Big(a_\mathfrak{m} + e_{\mathfrak{m}_1 - \mathfrak{m}}(z) \sum_{\alpha \geq \mathfrak{m}_1} a_\alpha e_{\alpha - \mathfrak{m}_1}(z) \Big) = e_\mathfrak{m}(z)(a_\mathfrak{m} + h(z))$$

with a Hahn-holomorphic function $h(z)$ such that $h(0)=0$. Since h is continuous and $e_{\mathfrak{m}}(z)\neq 0$, this shows that $f(z)\neq 0$ in a neighborhood of 0. □

Now suppose that Y, Γ and the family of functions $(e_\gamma)_{\gamma\in\Gamma}$ are fixed and satisfy (E1).

We want to show that the space of Hahn-holomorphic functions at 0 with values in a Banach algebra $\mathfrak{R}$ is a ring. To that end we need the following.

Lemma 2.8. *Let $A_1, A_2\subset\Gamma^+$ be admissible sets. Then the sets $A_1\cup A_2$, A_1+A_2, and $n\cdot A_1:=A_1+\cdots+A_1$ (n times), $\bigcup_{n=0}^\infty n\cdot A_1$ are admissible.*

Proof. First we show that $A_1\cup A_2$, A_1+A_2 and $n\cdot A_1$ are weakly monotonic. Let ρ_i, $i=1,2$, be the radius for A_i and $\rho(x)=\min\{\rho_1(x),\rho_2(x)\}$. Then ρ is a radius for $A_1\cup A_2$ and for A_1+A_2 as well, because, for $\alpha_i\in A_i$,

$$\begin{aligned}\|e_{\alpha_1+\alpha_2}\|_{\rho(r)} &\leq \|e_{\alpha_1}\|_{\rho(r)}\|e_{\alpha_2}\|_{\rho(r)}\leq \|e_{\alpha_1}\|_{\rho_1(r)}\|e_{\alpha_2}\|_{\rho_2(r)}\\ &\leq |e_{\alpha_1}(r)|\,|e_{\alpha_2}(r)|=|e_{\alpha_1+\alpha_2}(r)|.\end{aligned}$$

The same argument shows that ρ_1 is a radius for $n\cdot A_1$.

Now let $B\subset A:=A_1+A_2$. Then $B=B_1+B_2$ for some $B_i\subset A_i$, $i=1,2$, and $\min B=\min B_1+\min B_2$. Let $\alpha\in A$ with $\alpha=\alpha_1+\alpha_2$, $\alpha_i\in A_i$. Let $\rho_i(r)$ be the radius for $\{e_{\alpha_i-\min B_i}\}$ and $\rho=\min\{\rho_1,\rho_2\}$. The estimate

$$\|e_{\alpha-\min B}\|_{\rho(r)}=\|e_{\alpha_1-\min B_1+\alpha_2-\min B_2}\|_{\rho(r)}\leq \|e_{\alpha_1-\min B_1}\|_{\rho_1(r)}\|e_{\alpha_2-\min B_2}\|_{\rho_2(r)}$$

shows that A_1+A_2 is admissible. The other statements are proven similarly. □

Let $f(z)=\sum_\alpha a_\alpha e_\alpha$ and $g(z)=\sum_\beta b_\beta e_\beta$ be Hahn-holomorphic functions on Y_f and Y_g, respectively. First it is easy to see that $f+g$ is Hahn-holomorphic on $Y=Y_f\cap Y_g$. Since $\mathfrak{f}$ and $\mathfrak{g}$ are Hahn series with support contained in Γ_0^+, we also have $\operatorname{supp}(\mathfrak{f}\cdot\mathfrak{g})\subset\Gamma_0^+$ for the multiplication defined in (1). From Lemma 2.8 we obtain that the support of $\mathfrak{f}\cdot\mathfrak{g}$ is admissible. We claim that $h(z)=f(z)\cdot g(z)$ is represented by the product of Hahn series $\mathfrak{h}=\mathfrak{f}\cdot\mathfrak{g}$ on $Y_f\cap Y_g$. Because f and g are normally convergent,

$$\sum_\gamma\Big\|\sum_{\alpha+\beta=\gamma}a_\alpha b_\beta\Big\|\,\|e_\gamma\|\leq\sum_\gamma\Big(\sum_{\alpha+\beta=\gamma}\|a_\alpha\|\,\|b_\beta\|\Big)\|e_\gamma\|\leq\Big(\sum_\alpha\|a_\alpha\|\,\|e_\alpha\|\Big)\Big(\sum_\beta\|b_\beta\|\,\|e_\beta\|\Big)$$

so that the series $\mathfrak{f}\cdot\mathfrak{g}$ is normally convergent in $Y_f\cap Y_g$. Thus the series $\mathfrak{f}\cdot\mathfrak{g}$ defines a Hahn-holomorphic function on Y with values in $\mathfrak{R}$, and this function equals $h(z)$.

Altogether we have this:

Proposition 2.9. *Let $\mathfrak{R}$ be a Banach algebra. The Hahn-holomorphic functions with values in $\mathfrak{R}$ on Y form a ring under usual addition and multiplication, and the map $\psi_{\mathfrak{R}}: f\mapsto\mathfrak{f}$ is a ring isomorphism onto its image in $\mathfrak{R}[\![e_\gamma]\!]$.*

Corollary 2.10. *The ring of Hahn-holomorphic functions on Y with values in an integral domain $\mathfrak{R}$ is an integral domain.*

Proof. By looking at the coefficient c_γ with $\gamma = \min \operatorname{supp} \mathfrak{f}$ in (1), we observe that $\mathfrak{R}[\![e_\Gamma]\!]$ is an integral domain if $\mathfrak{R}$ is an integral domain. Because $\psi_{\mathfrak{R}}$ is an isomorphism, the Hahn-holomorphic functions must be an integral domain. □

Theorem 2.11. *Let $\mathfrak{R}$ be a Banach algebra and suppose $f : Y_0 \to \mathfrak{R}$ is Hahn-holomorphic and $f(z)$ is invertible for all $z \in Y_0$. Then $f(z)^{-1}$ is also Hahn-holomorphic on Y_0.*

Proof. Since $1/f$ is holomorphic in Y, we only have to show that there is a Hahn series for $f(z)^{-1}$ that converges normally on some $Y_0 \cap D_\varepsilon$. Since $f(z)^{-1} = f(0)^{-1}(f(z)f(0)^{-1})^{-1}$, we can assume without loss of generality that $f(0) = \mathrm{Id}$. Thus we can write $f(z) = \mathrm{Id} - h(z)$, where $\mathfrak{m} := \min \operatorname{supp}(\mathfrak{h}) > 0$. By assumption, the series $\mathfrak{h} := \sum_{\alpha\in\Gamma} a_\alpha e_\alpha$ defining $h(z)$ converges normally on the set $Y_0 \cap D_{\delta_0}$ for some $\delta_0 > 0$. The function $\tilde{h}$ defined by

$$\tilde{h}(t) = \sum_{\alpha\in\Gamma} \|a_\alpha\| \, \|e_\alpha\|_{Y_0,t} \le \|e_{\mathfrak{m}}\|_{Y_0,t} \sum_{\alpha\ge\mathfrak{m}} \|a_\alpha\| \, \|e_{\alpha-\mathfrak{m}}\|_{Y_0,t}$$

converges to 0 for $t \to 0$ due to (E1) and Lemma 2.6. Therefore we can choose $\delta > 0$ so small that $\tilde{h} := \tilde{h}(\delta) < 1/2$. Because $|h(z)| \le \tilde{h}$ for $z \in Y_0 \cap D_\delta$, the geometric series

$$f(z)^{-1} = \sum_{n=0}^{\infty} h(z)^n$$

then converges normally on $Y_0 \cap D_\delta$. But we also know that $\mathfrak{f}$ is invertible:

$$\mathfrak{f}^{-1} = \sum_{n=0}^{\infty} \mathfrak{h}^n =: \sum_{\alpha\in\mathcal{S}} b_\alpha e_\alpha \quad \text{with } \operatorname{supp}(\mathfrak{f}^{-1}) \subset \mathcal{S} := \bigcup_{n\ge 0} \operatorname{supp}(\mathfrak{h}^n).$$

From Lemma 2.8 we obtain that $\mathcal{S}$ is admissible. It remains to show that $\sum\limits_{\alpha\in\mathcal{S}} b_\alpha e_\alpha(z)$ is normally convergent on $Y_0 \cap D_\delta$ and represents $f(z)^{-1}$. We have the implication

$$\sum_{n=0}^{N} \mathfrak{h}^n = \sum_{\alpha\in\mathcal{S}} c_\alpha(N) e_\alpha \quad \Longrightarrow \quad \sum_{\alpha\in\mathcal{S}} \|c_\alpha(N)\| \, \|e_\alpha\| \le \sum_{n=0}^{N} \tilde{h}^n \quad \text{in } Y_0 \cap D_\delta,$$

as a simple consequence of the triangle inequality. For every fixed finite set $A \subset \mathcal{S}$, there exists an $N_A > 0$ such that, for all $N \ge N_A$,

$$\mathfrak{f}^{-1} - \sum_{n=0}^{N} \mathfrak{h}^n = \sum_{\alpha\in\mathcal{S}\setminus A} (b_\alpha - c_\alpha(N)) e_\alpha$$

has support away from A. In particular, $c_\alpha(N) = b_\alpha$ for $\alpha \in A$ and $N \ge N_A$. Therefore, for $N > N_A$,

$$\sum_{\alpha\in A} \|b_\alpha\| \, \|e_\alpha\| \le \sum_{\alpha\in\mathcal{S}} \|c_\alpha(N)\| \, \|e_\alpha\| \le \sum_{n=0}^{N} \tilde{h}^n < \frac{1}{1-\tilde{h}},$$

and this proves convergence, since this bound is independent of A. In particular, $\sum_{\alpha\in\mathcal{S}} b_\alpha e_\alpha(z)$ converges absolutely in $\mathfrak{R}$, hence it converges and the value does not depend on the order of summation. After reordering,

$$\sum_{\alpha\in\mathcal{S}} b_\alpha e_\alpha(z) = \sum_{n=0}^{\infty} h(z)^n = f(z)^{-1}. \qquad \square$$

Because of Lemma 2.6, every complex valued Hahn-holomorphic f that is not identically 0 can be inverted away from its zeros. Let $\mathfrak{m} := \min \operatorname{supp}(\mathfrak{f}) \geq 0$. Then

$$f^{-1}(z) = a_{\mathfrak{m}}^{-1} e_{-\mathfrak{m}}(z) \sum_{n=0}^{\infty} (1 - a_{\mathfrak{m}}^{-1} e_{-\mathfrak{m}}(z) f(z))^n.$$

Theorem 2.12. *Suppose that $f : Y_0 \to \mathbb{C}$ is a Hahn-holomorphic function with Hahn series $\mathfrak{f}$. Suppose that U is an open neighborhood of $f(0)$ and $h : U \to \mathbb{C}$ is holomorphic. Then $h \circ f$ is Hahn-holomorphic on its domain.*

Proof. Since holomorphicity away from zero is obvious, it is enough to show that $h \circ f$ has a normally convergent expansion into a Hahn series. Replacing $f(z)$ by $f(z) - f(0)$ and $h(z)$ by $h(z - f(0))$, we can assume without loss of generality that $f(0) = 0$ and thus $\operatorname{supp}(\mathfrak{f}) \subset \Gamma^+$. Since h is holomorphic near $f(0)$, it has a uniformly and absolutely convergent expansion

$$h(z) = \sum_{k=0}^{\infty} a_k (z - f(0))^k.$$

Thus

$$h \circ f(z) = \sum_{k=0}^{\infty} a_k (f(z))^k.$$

Note that $\sum_{k=0}^{\infty} a_k \mathfrak{f}^k$ is a Hahn series. A similar argument as in the proof of Theorem 2.11 shows that this Hahn series is normally convergent and represents $h \circ f(z)$. $\square$

3. Hahn-meromorphic functions

Definition 3.1. A meromorphic function $h : Y \to \mathbb{C}$ is called *Hahn-meromorphic* if h is represented by a Hahn series $\mathfrak{h}$ in $Y \cap D_\varepsilon$ for some $\varepsilon > 0$ and there exist Hahn-holomorphic functions $f, g \not\equiv 0$ on $Y_0 \cap D_\varepsilon$ such that $\mathfrak{h} \cdot \mathfrak{g} = \mathfrak{f}$.

In this sense, a Hahn-meromorphic function can be written as a quotient $h = f/g$ of Hahn-holomorphic functions in a neighborhood of 0.

Remark 3.2. Since $\mathbb{C}$-valued Hahn-holomorphic functions form an integral domain, Hahn-meromorphic functions form a field. More generally, let $\mathcal{R}$ be a (commutative) integral domain. From Corollary 2.10 we know that Hahn-holomorphic functions with coefficients in $\mathcal{R}$ are a commutative integral domain whose quotient field is defined. Furthermore, the map $f \mapsto \mathfrak{f}$ induces an injective morphism from the quotient field of Hahn-holomorphic functions to the quotient field $\mathcal{R}((e_\Gamma))$ of Hahn series $\mathcal{R}[\![e_\Gamma]\!]$. Note that $\mathcal{R}((e_\Gamma)) = \mathcal{R}[\![e_\Gamma]\!]$ if $\mathcal{R}$ is a field.

An important difference with usual meromorphic functions is that Hahn-meromorphic functions may have infinitely many negative exponents. For example, the function

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} z^{1-1/n}$$

is Hahn-holomorphic and therefore

$$\sum_{n=1}^{\infty} \frac{1}{n^2} z^{-1/n-1} = \frac{f(z)}{z^2}$$

is Hahn-meromorphic.

It follows from our analysis for Hahn-holomorphic functions that every $\mathbb{C}$-valued Hahn-meromorphic function h can be written as

$$h(z) = e_{\min \operatorname{supp} \mathfrak{h}}(z) f(z),$$

where f is Hahn-holomorphic. Moreover, if $h \neq 0$, then $f(0) \neq 0$. In particular, this implies that Hahn-meromorphic functions bounded on $(0, \delta)$ are Hahn-holomorphic in some neighborhood of 0.

We can also define Hahn-meromorphic functions with values in a Banach algebra.

Definition 3.3. Let $\mathcal{R}$ be a Banach algebra. A function $h : Y \to \mathcal{R}$ is called *Hahn-meromorphic* if it is meromorphic on Y and there exists a $\delta > 0$ and a nonzero Hahn-holomorphic function f on $Y_0 \cap D_\delta$ such that $f(z)h(z)$ is a Hahn-holomorphic function on $Y_0 \cap D_\delta$ with values in $\mathcal{R}$.

Remark 3.4. Let $R > 0$ and $\sigma > 0$. If there exists one nonzero Hahn-holomorphic function on $Y \cap D_R^{[\sigma]}$ that vanishes with positive order at 0, then one can use the Weierstrass product theorem together with Theorem 2.12 to show that the set of complex valued Hahn-meromorphic functions on $Y \cap D_R^{[\sigma]}$ can be identified with the quotient field of the division ring of Hahn-holomorphic functions on $Y \cap D_R^{[\sigma]}$.

4. A Hahn-holomorphic Fredholm theorem

Let $\mathcal{H}$ be a complex Hilbert space and denote by $\mathcal{K}(\mathcal{H})$ the space of compact operators on $\mathcal{H}$.

Theorem 4.1. *Suppose $Y_0 \subset \mathfrak{X}$ is connected and let $f : Y \to \mathcal{K}(\mathcal{H})$ be either Hahn-holomorphic or Hahn-meromorphic such that all coefficients of e_γ with $\gamma < 0$ and all Laurent coefficients in the principal part away from the point $z = 0$ have range in a common finite-dimensional subspace $\mathcal{H}_0 \subset \mathcal{H}$.*

Then either $(\mathrm{Id} - f(z)) \in \mathcal{B}(\mathcal{H})$ *is invertible nowhere in Y_0 or its inverse* $(\mathrm{Id} - f(z))^{-1}$ *exists everywhere except at a discrete set of points in Y_0 and defines a Hahn-meromorphic function. Moreover, in the Hahn series of* $(\mathrm{Id} - f(z))^{-1}$, *the coefficients of e_γ with $\gamma < 0$ are finite-rank operators, and the coefficients in the principal part of its Laurent expansion away from $z = 0$ are finite-rank operators too.*

Proof. The proof generalizes that of [Reed and Simon 1980, Theorem VI.14]. The assumptions imply that there exists a Hahn-meromorphic function $B(z)$ with range in $\mathcal{H}_0$, a finite-rank operator A, and a $\delta > 0$ such that $f(z) - A - B(z)$ is Hahn-holomorphic and $\|f(z) - A - B(z)\| < 1$ for all $z \in U^{[\sigma]} := D_\delta^{[\sigma]} \cap Y$. Thus $(\mathrm{Id} - f(z) + A + B(z))^{-1}$ exists and is Hahn-holomorphic by Theorem 2.11. Consequently, $g(z) = (A + B(z))(\mathrm{Id} - f(z) + A + B(z))^{-1}$ is a Hahn-meromorphic function on $U^{[\sigma]}$ with values in the

Banach space $\mathcal{B}(\mathcal{H}, V)$, where V is the finite-dimensional subspace of $\mathcal{H}$ spanned by $\mathcal{H}_0$ and $\operatorname{rg}(A)$. It is easy to see that

$$(\mathrm{Id} - f(z))^{-1} = (\mathrm{Id} - f(z) + A + B(z))^{-1}(\mathrm{Id} - g(z))^{-1}, \tag{2}$$

where equality means here that the left hand side exists if and only if the right hand side exists. Now let P be the orthogonal projection onto V and let $G(z)$ be the endomorphisms of V defined by restricting $g(z)$ to V, that is, $G(z) = g(z) \circ P$. Invertibility of $\mathrm{Id} - g(z)$ in $\mathcal{B}(\mathcal{H})$ is equivalent to invertibility of

$$P(\mathrm{Id} - g(z))P : V \to V,$$

and this is equivalent to $\det(\mathrm{Id}_V - G(z)) \neq 0$. Moreover, a straightforward computation shows

$$(\mathrm{Id} - g(z))^{-1} = (P(\mathrm{Id} - g(z))P)^{-1}(P + g(z)(\mathrm{Id} - P)) + (\mathrm{Id} - P). \tag{3}$$

Now note that $G(z)$ is a Hahn-holomorphic family of endomorphisms of V. In particular, $\det(\mathrm{Id} - G(z))$ is a Hahn-meromorphic $\mathbb{C}$-valued function. As such, it is meromorphic in $U^{[\sigma]} \setminus \{0\}$, and together with Proposition 2.7, this shows that the set

$$S = \{z \in U^{[\sigma]} \mid \det(\mathrm{Id} - G(z)) = 0\}$$

is either discrete in $U^{[\sigma]}$ or $S = U^{[\sigma]}$. If $\det(\mathrm{Id} - G(z)) \neq 0$, then, after a choice of basis of V, the inverse $(\mathrm{Id} - G(z))^{-1}$ can be computed with Cramer's rule, showing that, with respect to this basis,

$$\det(\mathrm{Id} - G(z))(\mathrm{Id} - G(z))^{-1} \in \operatorname{Mat}(\dim V, \mathbb{C}[\![e_\Gamma]\!])$$

is represented by a matrix with Hahn-meromorphic entries. After the identification

$$\operatorname{Mat}(\dim V, \mathbb{C}[\![e_\Gamma]\!]) = \operatorname{Mat}(\dim V, \mathbb{C})[\![e_\Gamma]\!],$$

we see that the function $(\mathrm{Id} - G(z))^{-1}$ is Hahn-meromorphic with coefficients in $\operatorname{End}(V)$ if there is only a single point in $U^{[\sigma]}$ for which it exists. Consequently, due to (2) and (3), $(\mathrm{Id} - f(z))^{-1}$ is Hahn-meromorphic with all coefficients of $e_\gamma(z)$ with $\gamma < 0$ being of finite rank if there is only a single point in $U^{[\sigma]}$ for which $\mathrm{Id} - f(z)$ is invertible. So far we have proved the statement in $U^{[\sigma]}$. By the usual analytic Fredholm theorem, invertibility of $\mathrm{Id} - f(z)$ at a single point in Y implies that the inverse exists as a meromorphic function on Y. Conversely, we have seen that invertibility of $\mathrm{Id} - f(z)$ at a single point in $U^{[\sigma]}$ implies that $(\mathrm{Id} - f(z))^{-1}$ exists as a Hahn-meromorphic function on $U^{[\sigma]}$. By the usual meromorphic Fredholm theorem, it then exists as a Hahn-meromorphic function on Y. □

5. z-Hahn-holomorphic functions

The prominent class of Hahn-holomorphic functions is defined by convergent power series with noninteger powers.

Let $\Gamma \subset \mathbb{R}$ be a subgroup with order inherited from the standard ordering of $\mathbb{R}$. As the group G we will take the group generated by the set of functions

$$e_\alpha(z) := z^\alpha, \quad \alpha \in \Gamma,\ z \in D_r^{[\sigma]} \setminus \{0\}.$$

In this definition we choose the principal branch of the logarithm with $|\mathrm{Im}\log z| < \pi$ for $z \in \mathbb{C} \setminus (-\infty, 0]$, and, as usual, we set $\log(re^{\mathrm{i}\varphi}) = \log r + \mathrm{i}\varphi$, $|\varphi| < \sigma$, and $z^\alpha := e^{\alpha \log z}$.

A z-Hahn-holomorphic function f with values $\mathbb{C}$ then is a holomorphic function on $D_r^{[\sigma]} \setminus \{0\}$ such that the generalized power series

$$f(z) = \sum_\gamma a_\gamma z^\gamma, \quad a_\gamma \in \mathbb{C},$$

is normally convergent in $Y \cap D_\delta^{[\sigma]}$ for some $\delta > 0$.

Note that every well-ordered subset of $W \subset \Gamma^+$ is admissible for e, because for every $\alpha \in W$,

$$|z^\alpha| = |z|^\alpha \le |z|^{\min W}, \quad z \in D_{1/2}^{[\sigma]}. \tag{4}$$

Example 5.1. If $\Gamma = \mathbb{Z}$ and $e_k(z) = z^k$, the set of Hahn series corresponds to the formal power series and the set of z-Hahn-holomorphic functions can be identified with the set of functions that are holomorphic on the disc of radius $\delta > 0$ centered at the origin.

Example 5.2. The series

$$z^\pi \sum_{k=0}^\infty \frac{z^{2k}}{(2k)!}$$

converges normally on D_r for any $r > 0$ and defines a z-Hahn-holomorphic function for $\Gamma = \pi\mathbb{Z} + 2\mathbb{Z}$.

Example 5.3. Puiseux series and Levi-Civita series, as defined in, for example, [Ribenboim 1992], are special cases of Hahn series with certain $\Gamma \subset \mathbb{Q}$. When they are normally convergent, they define z-Hahn-holomorphic functions.

In the following, let $D_R = D_R^{[\infty]} \setminus \{0\}$ be the pointed disk of radius R in the logarithmic covering of the complex plane. The next result is in analogy with complex analysis, where series expansions converge normally on the maximal disc embedded in the domain of holomorphicity.

Theorem 5.4. *Let $\mathcal{R}$ be a Banach algebra, and suppose f is z-Hahn-holomorphic. Suppose further that f is bounded on $D_{\tilde{R}R+\varepsilon}$ for some ε, $\tilde{R} > 0$, and let*

$$f(z) = \sum_{\alpha \in \operatorname{supp} f} a_\alpha z^\alpha$$

be its expansion (which we do not assume converges normally on $D_{\tilde{R}}$).

Then, for all R with $0 < R < \tilde{R}$,

$$\sum_{\alpha \in \operatorname{supp} f} \|a_\alpha\| R^\alpha \le \sup_{|z|=R} \|f(z)\| \sum_{\alpha \in \operatorname{supp} f} (R/\tilde{R})^\alpha.$$

In particular, if $\sum_\alpha (R/\tilde{R})^\alpha < \infty$, the Hahn series converges normally on D_R.

Proof. As a Hahn-holomorphic function, f converges normally on $D_{2\delta}$ for some $\delta > 0$ and is holomorphic in $D_{\tilde{R}}$. Let $\Lambda_{R,L}$ be the averaging operator

$$\Lambda_{R,L}(f) = \frac{1}{2\pi \mathrm{i} L} \int_{S_R^{(L)}} \frac{f(z)}{z}\, dz,$$

where $S_R^{(L)}(t) = Re^{\mathrm{i}\pi t}$, $t \in (-L, L]$, is the L-fold cover of the circle with radius R. Certainly

$$\|\Lambda_{R,L}(f)\| \le \sup_{|z|=R} \|f(z)\|.$$

Since f is holomorphic for $0 < |z| < R$, we have

$$\frac{1}{2\pi\mathrm{i}L}\int_{S_R^{(L)}} \frac{f(z)}{z}\,dz = \frac{1}{2\pi\mathrm{i}L}\int_{S_\delta^{(L)}} \frac{f(z)}{z}\,dz + O(L^{-1}).$$

This shows that

$$\Lambda_R(f) := \lim_{L\to\infty} \Lambda_{R,L}(f) = \lim_{L\to\infty} \sum_{\alpha\in\operatorname{supp} f} \frac{a_\alpha}{2\pi\mathrm{i}L}\int_{S_\delta^{(L)}} z^{\alpha-1}\,dz = a_0.$$

Suppose (I_k) is a family of finite subsets of $\operatorname{supp} f$ such that

$$I_1 \subset I_2 \subset \cdots \quad \text{and} \quad \bigcup_k I_k = \operatorname{supp} f.$$

For $z \in D_{\tilde{R}}$, let $g_k(z) = \sum_{\alpha\in I_k} \lambda_\alpha z^{-\alpha} R^\alpha$, where $\lambda_\alpha \in \mathcal{R}^* := \mathcal{B}(\mathcal{R}, \mathbb{R})$ are chosen such that

$$\|\lambda_\alpha\| = 1, \quad \lambda_\alpha(a_\alpha) = \|a_\alpha\|.$$

Such λ_α exist by the Hahn–Banach theorem.

Then g_k is holomorphic in $D_{\tilde{R}}$ and $\|g_k(z)\| \le \sum_{\alpha\in I_k} |z|^{-\alpha} R^\alpha$. Moreover,

$$\langle g_k, f\rangle(z) = \sum_{\substack{\alpha\in I_k\\ \gamma=-\alpha+\beta}} \lambda_\alpha(a_\beta) R^\alpha z^\gamma,$$

and the constant term of this function is

$$\sum_{\alpha\in I_k} \|a_\alpha\| R^\alpha = \Lambda_\delta(\langle g_k, f\rangle) = \Lambda_{\tilde{R}}(\langle g_k, f\rangle).$$

Therefore

$$\sum_{\alpha\in I_k} \|a_\alpha\| R^\alpha \le \sup_{|z|=R} |\langle g_k, f\rangle(z)| \le \sup_{|z|=R} \|g_k(z)\| \|f(z)\| \le \sup_{|z|=R} \|f(z)\| \sum_{\alpha\in\operatorname{supp} f} (R/\tilde{R})^\alpha,$$

and the theorem follows by letting $k \to \infty$. □

Theorem 5.5. *Let $R > 0$, and assume $f_k : D_R \to V$ is a sequence of bounded z-Hahn-holomorphic functions that converge uniformly to a bounded function $f : D_R \to V$. Suppose that there exist constants $C > 0$, $\hat{\varepsilon} > 0$ such that, for each $k \in \mathbb{N}$,*

$$\sum_{\alpha\in\operatorname{supp} f_k} \hat{\varepsilon}^\alpha < C.$$

Suppose furthermore that there exists $I \subset \mathbb{R}$ such that $\operatorname{supp} f_k \to I$ *in the following sense. For each compact subset $K \Subset \mathbb{R}$, there exists $N > 0$ such that* $\operatorname{supp} f_k \cap K = I \cap K$ *for all $k \ge N$.*

Then f is Hahn-holomorphic on D_R with $\operatorname{supp} f \subset I$.

Proof. First, I is well ordered because $\operatorname{supp} f_k \to I$. Let $f_k(z) = \sum_{\alpha\in\operatorname{supp} f_k} a_\alpha^{(k)} z^\alpha$ be the expansion of f_k.

Let $\varepsilon > 0$. Then there exists $N_1 > 0$ such that $\|f_\ell(z) - f_k(z)\| < \varepsilon$ for all $k, \ell > N_1$ and all $z \in D_R$. Given a finite subset $\tilde{I} \subset I$, we can choose $N > N_1$ such that $\tilde{I} \cap \operatorname{supp} f_k = \tilde{I}$ for all $k > N$. Theorem 5.4 then shows that, for all $k, \ell > N$ and $\tilde{R} < \hat{\varepsilon} R$,

$$\sum_{\alpha\in\tilde{I}} \|a_\alpha^{(\ell)} - a_\alpha^{(k)}\| \cdot \tilde{R}^\alpha \;\le\; \sup_{|z|=\tilde{R}} \|f_\ell(z) - f_k(z)\| \sum_{\alpha\in\operatorname{supp} f_k \cup \operatorname{supp} f_\ell} \hat{\varepsilon}^\alpha < 2\,C\,\varepsilon.$$

It follows that $(a_\alpha^{(k)})_k$ is a Cauchy sequence for each α. Let $a_\alpha := \lim_{k\to\infty} a_\alpha^{(k)}$. Given a finite subset $\tilde{I} \subset I$ and $\varepsilon > 0$, we can find N such that $\|a_\alpha^{(k)} - a_\alpha\| < \varepsilon$ for all $k > N$, $\alpha \in \tilde{I}$. Then, for $|z| < \tilde{R} < \hat{\varepsilon}$,

$$\left\| \sum_{\alpha\in\tilde{I}} a_\alpha^{(k)} z^\alpha - \sum_{\alpha\in\tilde{I}} a_\alpha z^\alpha \right\| < \sum_{\alpha\in\tilde{I}} \|a_\alpha^{(k)} - a_\alpha\| \tilde{R}^\alpha < C\,\varepsilon.$$

This shows that $\sum_{\alpha\in I} a_\alpha z^\alpha$ is a Hahn series for f. By the uniform convergence of (f_k), f is analytic in $D_R^{[\sigma]} \setminus \{0\}$. Its Hahn series converges normally on $D_{\tilde{R}}$ because

$$\begin{aligned}\sum_{\alpha\in\tilde{I}} \|a_\alpha\| \tilde{R}^\alpha &\le \sum_{\alpha\in\tilde{I}} \|a_\alpha^{(\ell)}\| \tilde{R}^\alpha + \sum_{\alpha\in\tilde{I}} \|a_\alpha^{(k)} - a_\alpha\| \tilde{R}^\alpha + \sum_{\alpha} \|a_\alpha^{(k)} - a_\alpha^{(\ell)}\| \tilde{R}^\alpha \\ &\le \sum_{\alpha\in\operatorname{supp} f_\ell} \|a_\alpha^{(\ell)}\| \tilde{R}^\alpha + C\varepsilon + 2C\varepsilon < \infty\end{aligned}$$

for all finite $\tilde{I} \subset I$, ℓ sufficiently large, and $k \gg \ell$ depending on $\tilde{I}$. □

6. $z \log z$-Hahn-holomorphic functions

In the following, let $\mathbb{R}^2$ be equipped with the lexicographical order and let $\Gamma \subset \mathbb{R}^2$ be a subgroup with order inherited from that of $\mathbb{R}^2$. Let $Y = D_{1/2}^{[\sigma]}$ for fixed $\sigma > 0$. The group G will be generated by

$$e_{(\alpha,\beta)}(z) := z^\alpha (-\log z)^{-\beta}, \quad (\alpha, \beta) \in \Gamma,\ |z| < 1.$$

With the inclusion $\mathbb{R} \times \{0\} \subset \mathbb{R}^2$, this comprises the power functions z^α from Section 5. Note that

$$\lim_{z\to 0} e_{(\alpha,\beta)}(z) = 0 \iff \alpha > 0 \vee (\alpha = 0 \wedge \beta > 0),$$

which is equivalent to $(\alpha, \beta) > (0, 0)$ in the lexicographical ordering of $\mathbb{R}^2$. The monotonicity (4) of power functions z^α has to be replaced by the following "weak monotonicity" property.

Lemma 6.1. *Let $\mathcal{S} \subset \Gamma^+ = \{\gamma \in \Gamma \mid \gamma > 0\}$ be a set such that there exists an $N \in \mathbb{N}_0$ with*

$$-\beta \le N\alpha \quad \textit{for all } (\alpha, \beta) \in \mathcal{S}. \tag{$*$}$$

(a) *There exists $r_N < 1$ such that, for $(\alpha, \beta) \in \mathcal{S}$ and $|\theta| < \sigma$, the function*

$$r \mapsto |re^{\mathrm{i}\theta}|^\alpha |\log(re^{\mathrm{i}\theta})|^{-\beta}$$

is monotonically increasing on $[0, r_N)$.

(b) *Given x with $0 < x < r_N$, there exists $\rho_N(x) \le x$ such that, for all z with $0 \le |z| \le \rho_N(x)$ and $|\arg z| < \sigma$, we have*

$$(\alpha, \beta) \in \mathcal{S} \implies |e_{(\alpha,\beta)}(z)| \le e_{(\alpha,\beta)}(x)$$

The proof is elementary and will be omitted.

It is not difficult to see that if $\mathcal{S}$ satisfies $(*)$, a similar inequality holds for the set $(\mathcal{S} - A) \cap \Gamma^+$, where $A \subset \mathcal{S}$ and the constant N depends on A. Thus a set $\mathcal{S}$ with $(*)$ is admissible for e.

Now the assumptions from Section 2 are all satisfied and we can consider Hahn-holomorphic and meromorphic functions: A *$z \log z$-Hahn-holomorphic function* with values in a Banach algebra $\mathcal{R}$ is defined by a normally convergent series

$$f(z) = \sum_{(\alpha,\beta)\in\Gamma} a_{(\alpha,\beta)} z^{\alpha} (-\log z)^{-\beta}, \quad a_{(\alpha,\beta)} \in \mathcal{R},\ z \in D_{1/2}^{[\sigma]},$$

such that $\operatorname{supp}(f)$ is contained in a set $\mathcal{S} \cup \{(0, 0)\}$ with $\mathcal{S}$ as in Lemma 6.1.

Note that the property $(*)$ is invariant under addition and multiplication of Hahn-holomorphic functions, so that $z \log z$-Hahn-holomorphic functions indeed are a ring, and all results from Section 2 apply.

Example 6.2. The series

$$\sum_{n=0}^{\infty} z^n (-\log z)^n = (1 + z \log z)^{-1}$$

is a Hahn series in $\Gamma = \mathbb{Z} \times \mathbb{Z}$ with support $\{(n, -n) \mid n \in \mathbb{N}_0\}$. It converges normally on the set $\left\{z \in \mathcal{Z} \;\middle|\; |z \log z| < \frac{1}{2}\right\}$ and therefore defines a $z \log z$-Hahn-holomorphic function on $D_r^{[\sigma]}$ for any $\sigma > 0$ and sufficiently small $r = r(\sigma)$.

Example 6.3. The formal series

$$\sum_{n=0}^{\infty} \frac{1}{n!} z (-\log z)^n$$

is *not* a Hahn series for $\Gamma = \mathbb{Z} \times \mathbb{Z}$, because the support

$$\{(1, -n) \mid n \in \mathbb{N}_0\}$$

is not a well-ordered subset of Γ.

Example 6.4. The logarithm $\log z = \dfrac{z \log z}{z}$ is Hahn-meromorphic for $\Gamma \subset \mathbb{Z} \times \mathbb{Z}$.

Example 6.5. The series

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{m^2} z^n (-\log z)^{2n-1+1/m}$$

defines a $z \log z$-Hahn-holomorphic function in a neighborhood $D_{\varepsilon}^{[\sigma]}$ of 0 for any $\sigma > 0$ and for small enough $\varepsilon = \varepsilon(\sigma)$ with $\Gamma = \mathbb{Z} \times \mathbb{Q}$. Its support is

$$\{(n, 1 - 2n - 1/m) \mid n, m \in \mathbb{N}\}.$$

7. Applications: Hahn-meromorphic continuation of resolvent kernels

7A. Suppose that $\nu \geq 0$. The differential operator B_ν associated to the Bessel differential equation in its Liouville normal form,

$$B_\nu := -\frac{\partial^2}{\partial x^2} + \frac{\nu^2 - \frac{1}{4}}{x^2}\mathrm{Id}, \tag{5}$$

is a nonnegative symmetric operator on the space $C_c^\infty((0,\infty))$ equipped with the inner product inherited from $L^2((0,\infty), dx)$. We will denote the Friedrichs extension of B_ν by the same symbol B_ν.

The kernel $r_\lambda^{(\nu)}$ of the resolvent $(B_\nu - \lambda^2)^{-1}$ can be constructed directly out of the fundamental system of the corresponding Sturm–Liouville equation and this results in (see, for example, [Brüning and Seeley 1987, p. 371])

$$r_\lambda^{(\nu)}(x, y) = \frac{\mathrm{i}\pi}{2}\sqrt{xy} \cdot J_\nu(\lambda \min(x, y)) H_\nu^{(1)}(\lambda \max(x, y)), \quad 0 < x, y < \infty, \tag{6}$$

where $H_\nu^{(1)}$ is the Hankel function of order ν of the first kind and J_ν is the Bessel function.

The proof of the following lemma uses the well-known expansion of Bessel and Hankel functions, and will be given at the end of this section.

Lemma 7.1. *For every $\nu > 0$ and $(x, y) \in (0,\infty) \times (0,\infty)$, the kernel $\lambda \mapsto r_\lambda^{(\nu)}(x, y)$ defines a $z \log z$-Hahn-holomorphic function.*

(a) *For $\nu \in \mathbb{R}_+ \setminus \mathbb{N}_0$, we have*

$$r_\lambda^{(\nu)}(x, y) = \lambda^{2\nu} f_1^{(\nu)}(x, y)(\lambda) + f_2^{(\nu)}(x, y)(\lambda),$$

where the maps $\lambda \mapsto f_j^{(\nu)}(x, y)(\lambda)$ are even and entire. In particular, $r_\lambda^{(\nu)}(x, y)$ is z-Hahn-holomorphic with support contained in $2\mathbb{Z} + 2\nu\mathbb{Z}$.

Let $a_{j;2k}^{(\nu)}(x, y)$ be the coefficient of λ^{2k} in the Taylor series expansion of $f_j^{(\nu)}(x, y)$.

There is a constant C_1 such that, for $0 \leq x \leq y$,

$$|a_{1;2k}^{(\nu)}(x, y)| \leq R^{-2k} C_1(\nu)(xy)^{\nu+1/2} e^{R(x+y)}, \quad R > 0.$$

For $c > 0$ and every $r_0 > 0$, there is a constant C_2 such that, for all $y \geq x \geq c$,

$$|a_{2;2k}^{(\nu)}(x, y)| \leq R^{-2k} \frac{C_2(\nu, r_0)}{\sqrt{R}} \sqrt{x}(x/y)^\nu e^{R(x+y)}, \quad R \geq r_0.$$

(b) *The kernel $\lambda \mapsto r_\lambda^{(\nu)}(x, y)$ is a $z \log z$-Hahn-holomorphic function with support contained in $2\mathbb{Z} \times \mathbb{Z}$ if $\nu = n \in \mathbb{N}$:*

$$r_\lambda^{(n)}(x, y) = \log(\lambda) g_1^{(n)}(x, y)(\lambda) + g_2^{(n)}(x, y)(\lambda),$$

where the maps $\lambda \mapsto g_j^{(\nu)}(x, y)(\lambda)$ are even and entire.

The coefficients $b_{j;2k}^{(n)}(x, y)$ in its Hahn-series expansion can be estimated by

$$|b^{(n)}_{1;2k}(x,y)| \le R^{-2k}\sqrt{xy}\,\frac{(R/2)^{2n}}{(n!)^2}e^{R(x+y)}, \qquad R>0,$$
$$|b^{(n)}_{2;2k}(x,y)| \le R^{-2k}e^{R(x+y)}\left(\hat{c}_1 x^{n+1}y^{n+1/2}\frac{(R/2)^{2n}}{n!\,(n-1)!}+c_2\right), \quad R>0.$$

Remark 7.2. For $\nu=0$, the expansion (11) below gives

$$r^{(0)}_\lambda(x,y) = -\sqrt{xy}\log\frac{\lambda y}{2} + \underline{h}(x,y)(\lambda)$$

with a Hahn-holomorphic function $\underline{h}(x,y)$. In particular, $\lambda \mapsto r^{(0)}_\lambda(x,y)$ is $z\log z$-Hahn-meromorphic.

For $c>0$, let $\chi_c : [0,\infty) \to \mathbb{R}_+$ be a smooth cutoff function with

$$\chi_c(x) = \begin{cases} 0 & \text{if } x \le c, \\ 1 & \text{if } x \ge 2c. \end{cases}$$

Multiplication by this function defines a bounded operator on $L^2((0,\infty))$. The "restricted resolvent" $\chi_c(B_\nu-\lambda^2)^{-1}\chi_c$ then is the bounded operator on $L^2((0,\infty))$ with integral kernel

$$(\chi_c \circ r^{(\nu)}_\lambda)(x,y) := \chi_c(x)\cdot r^{(\nu)}_\lambda(x,y)\cdot \chi_c(y).$$

Proposition 7.3. *Let $I=(0,\infty)$, $\nu>0$, and $c>0$. For any $\kappa>0$ and $\sigma>0$, the restricted resolvent $\chi_c(B_\nu-\lambda^2)^{-1}\chi_c$ extends, as a function of λ, to a $z\log z$-Hahn-holomorphic function on some neighborhood $D^{[\sigma]}_r$ of 0 with values in the compact operators*

$$\mathcal{K}\big(L^2(I,e^{\kappa x}\,dx),\,L^2(I,e^{-\kappa x}\,dx)\big).$$

Proof. First let $\nu\notin\mathbb{N}_0$. In Lemma 7.1(a), let $r_0=R=\kappa/3$. Using

$$\int_c^\infty\int_c^\infty \min(x,y)\left(\frac{\min(x,y)}{\max(x,y)}\right)^{2\nu} e^{(2R-\kappa)(x+y)}\,dx\,dy \le C(\kappa) \tag{7a}$$

and

$$\int_c^\infty\int_c^\infty (xy)^{2\nu+1}e^{(2R-\kappa)(x+y)}\,dx\,dy \le \left(\frac{\Gamma(2+2\nu)}{(\kappa/3)^{2+2\nu}}\right)^2, \tag{7b}$$

it is easy to see that the coefficients $a^{(\nu)}_{j;2k}(x,y)$ of the Hahn series expansion of $r^{(\nu)}$ satisfy

$$|\chi_c\circ a^{(\nu)}_{j;2k}| \in L^2(I\times I, e^{-\kappa(x+y)}\,dx\otimes dy), \quad j=1,2.$$

Therefore the kernels $\{\chi_c\circ a^{(\nu)}_{j;2k}\}_k$ define Hilbert–Schmidt operators

$$A^{(\nu)}_{j;2k} : L^2(I,e^{\kappa x}\,dx) \to L^2(I,e^{-\kappa x}\,dx) =: \mathcal{H}_\kappa$$

with norm bounded from above by

$$\|A^{(\nu)}_{j;2k}\| \le \|\chi_c\circ a^{(\nu)}_{j;2k}\|_{\mathcal{H}_\kappa\times\mathcal{H}_\kappa} \le R^{-2k}C(\nu,\kappa), \tag{8}$$

where $C(\nu,\kappa)$ can be obtained from (10a), (7a), (10b), (7b). But then the series

$$\lambda^{2\nu}\sum_{k=0}^{\infty}\|A^{(\nu)}_{1;2k}\|\,|\lambda|^{2k}+\sum_{k=0}^{\infty}\|A^{(\nu)}_{2;2k}\|\,|\lambda|^{2k}$$

converges normally in some neighborhood $U\subset D_r^{[\sigma]}$ of 0 and the kernel $r_\lambda^{(\nu)}$ defines a z-Hahn-holomorphic family of Hilbert–Schmidt operators in

$$\mathcal{K}\big(L^2(I,e^{\kappa x}\,dx),\,L^2(I,e^{-\kappa x}\,dx)\big).$$

For integral $\nu=n\in\mathbb{N}$, we can argue similarly, using Lemma 7.1(b). $\square$

Proof of Lemma 7.1. First let $\nu\notin\mathbb{N}_0$. Recall that $J_\nu(z)=\left(\frac{z}{2}\right)^{\nu}h_\nu(z)$, where

$$h_\nu(z)=\sum_{k=0}^{\infty}a_k^{(\nu)}z^{2k}\quad\text{with } a_k^{(\nu)}=\frac{(-1)^k}{4^k\,k!\,\Gamma(k+\nu+1)}.$$

The function h_ν is entire. We have

$$H_\nu^{(1)}(z)=\frac{\mathrm{i}}{\sin\nu\pi}\big(J_\nu(z)e^{-\mathrm{i}\nu\pi}-J_{-\nu}(z)\big),\qquad H_n^{(1)}(z)=\lim_{\nu\to n}H_\nu^{(1)}(z),\qquad n\in\mathbb{Z}.$$

(a) Let $x\le y$. Then

$$-\frac{2\mathrm{i}}{\pi}r_\lambda^{(\nu)}(x,y)=\sqrt{xy}\,J_\nu(\lambda x)H_\nu^{(1)}(\lambda y)=\lambda^{2\nu}f_1^{(\nu)}(x,y)(\lambda)+f_2^{(\nu)}(x,y)(\lambda)$$

with even, analytic functions in λ:

$$f_1^{(\nu)}(x,y)(\lambda)=\frac{\mathrm{i}e^{-\mathrm{i}\nu\pi}}{4^\nu\sin\nu\pi}(xy)^{\nu+1/2}h_\nu(\lambda x)h_\nu(\lambda y)$$
$$f_2^{(\nu)}(x,y)(\lambda)=\frac{-\mathrm{i}}{\sin\nu\pi}\sqrt{xy}\left(\frac{x}{y}\right)^{\nu}h_\nu(\lambda x)h_{-\nu}(\lambda y)$$

Due to Cauchy's integral formula,

$$|a^{(\nu)}_{j,2k}(x,y)|\le R^{-2k}\sup_{|\lambda|=R}|f_j^{(\nu)}(x,y)|,\qquad j=1,2.$$

We know from, for example, [Olver and Maximon 2011, (10.14.4)] that, for $\nu\ge 0$,

$$|h_\nu(z)|\le\frac{e^{|\operatorname{Im} z|}}{\Gamma(\nu+1)}.\tag{9}$$

Using $J_\nu=\frac12(H_\nu^{(1)}+H_\nu^{(2)})$, $|H_{-\nu}^{(*)}|=|H_\nu^{(*)}|$, and that $h_{-\nu}$ is a holomorphic and even function,

$$(R/2)^\nu\sup_{|z|=R}|h_{-\nu}(z)|=\sup_{\substack{|z|=R\\ \operatorname{Re} z>0}}|J_{-\nu}(z)|\le\sup_{\substack{|z|=R\\ \operatorname{Re} z>0}}\tfrac12\big(|H_\nu^{(1)}(z)|+|H_\nu^{(2)}(z)|\big).$$

But from [Olver and Maximon 2011, (10.17.13)], for $-\pi/2<\arg z<\pi/2$,

$$|H_\nu^{(1;2)}(z)|\le\sqrt{\frac{2}{\pi|z|}}\,e^{\mp\operatorname{Im} z}\big(1+\tau_\nu(|z|)e^{\tau_\nu(|z|)}\big),\qquad \tau_\nu(s):=\frac{\pi}{2}\left|\nu^2-\tfrac14\right|\cdot s^{-1}.$$

Thus there exists a constant $C_1 > 0$ with

$$\sup_{\substack{|\lambda|=R\\ \operatorname{Re}\lambda>0}} |H_\nu^{(1;2)}(\lambda y)| \le C_1 \frac{e^{Ry}}{\sqrt{Ry}}(1+\tau(R)e^{\tau(R)}), \quad y \ge c, \quad \tau(R) := \frac{\pi}{2}\frac{\left|\nu^2-\frac{1}{4}\right|}{Rc}.$$

This shows that, for every $r_0 > 0$, there is a constant C such that, for every $R \ge r_0$,

$$|a_{2;2k}^{(\nu)}(x, y)| \le R^{-2k-\nu}\frac{C_\nu}{\sqrt{R}}\sqrt{x}(x/y)^\nu e^{R(x+y)}, \quad C_\nu := \frac{C\cdot 2^\nu(1+\tau(r_0)e^{\tau(r_0)})}{|\sin(\nu\pi)|\Gamma(\nu+1)}. \tag{10a}$$

Also from (9),

$$|a_{1;2k}^{(\nu)}(x, y)| \le R^{-2k}\frac{(xy)^{\nu+1/2}e^{R(x+y)}}{4^\nu|\sin\nu\pi|\,\Gamma(\nu+1)^2}. \tag{10b}$$

(b) Let $\nu = n \in \mathbb{N}$ and $x \le y$. Then, from $H_n^{(1)} = J_n + \mathrm{i}Y_n$ and [Olver and Maximon 2011, (10.8.1)],

$$\begin{aligned}\frac{-2\mathrm{i}}{\pi\sqrt{xy}}r_\lambda^{(n)}(x, y) = \frac{2\mathrm{i}}{\pi}\Big(\log\lambda+\log\frac{y}{2}\Big)J_n(\lambda x)J_n(\lambda y)+J_n(\lambda x)J_n(\lambda y)-\frac{\mathrm{i}}{\pi}h_n(\lambda x)\sum_{k=0}^{n-1}\frac{(n-k-1)!}{k!}\Big(\frac{\lambda y}{2}\Big)^{2k}\\ -J_n(\lambda x)\Big(\frac{\lambda y}{2}\Big)^n\frac{1}{\pi}\sum_{k=0}^{\infty}\frac{\psi(k+1)+\psi(n+k+1)}{k!\,(n+k)!}(-1)^k(\lambda y/2)^{2k}\end{aligned} \tag{11}$$

with $\psi(x) = \Gamma'(x)/\Gamma(x)$. The only logarithmic terms in the Hahn-series expansion of $r^{(n)}(x, y)$ are $e_{(2k,-1)}(\lambda)$, $k \ge n$. Because of (9), the coefficient of $e_{(2k,-1)}$ is bounded by

$$R^{-2k}\sqrt{xy}\frac{(R/2)^{2n}}{(n!)^2}e^{R(x+y)}, \quad R > 0.$$

From Stirling's inequalities for Γ, we obtain, for $0 \le k \to \infty$,

$$1 \ge \sqrt{k+1}\frac{(2k)!}{4^k(k!)^2} \sim \frac{1}{\sqrt{\pi}}; \tag{12}$$

hence

$$\frac{(n-k-1)!\,(2k)!}{4^k k! n!} = \frac{1}{(n-k)\binom{n}{k}}\cdot\frac{(2k)!}{4^k(k!)^2} \le \frac{1}{n}, \quad 0 \le k < n.$$

Because $|z|^{2k} \le (2k)!\,e^{|z|}$ and because of (9), the norm of the sum in the second line of (11) can be bounded by $(e^{|\operatorname{Im}\lambda|x}/\pi)e^{|\lambda y|}$.

For the last line in (11), we first note that the polygamma function is monotonically increasing and $\psi(k) \lesssim \log k$, $k > 0$, and estimate as above

$$\begin{aligned}\sum_{k=0}^{\infty}\left|\frac{\psi(k+1)+\psi(n+k+1)}{k!\,(n+k)!4^k}\right||\lambda y|^{2k} &\le e^{|\lambda y|}\sum_{k=0}^{\infty}\frac{1}{\sqrt{k+1}}\frac{2\log(n+k+1)}{(k+n)\cdot(k+n-1)\cdots(k+1)}\\ &\le 2e^{|\lambda y|}\sum_{k=0}^{\infty}\frac{1}{\sqrt{k+1}(k+n)^{2/3}}\frac{\log(n+k+1)}{(k+n)^{1/3}}\frac{1}{(n-1)!} \le \frac{c_1\,e^{|\lambda y|}}{(n-1)!},\end{aligned}$$

where the constant c_1 can be obtained from $\zeta\left(\frac{7}{6}\right)$ and $x^{-1/3}\log(x+1) < \frac{3}{2}$ for $x > 0$.

Altogether, this shows that the coefficient of λ^{2k} in $r_\lambda^{(\nu)}(x, y)$ is bounded by

$$CR^{-2k}\sqrt{xy}e^{R(x+y)}\left(\left(\log\frac{y}{2}+1\right)\frac{(xy)^n(R/2)^{2n}}{(n!)^2}+\frac{1}{\pi}+c_1(xy)^n\frac{(R/2)^{2n}}{n!\,(n-1)!}\right),\quad R>0,$$

and, for $y \geq x \geq c$, this is smaller than

$$R^{-2k}e^{R(x+y)}\left(\hat{c}_1 x^{n+1}y^{n+1/2}\frac{(R/2)^{2n}}{n!\,(n-1)!}+c_2\right),\quad R>0. \qquad \square$$

7B. ***The resolvent of the Laplace operator on cones.*** Let $Z = (0, \infty) \times M$ be equipped with the cone metric $g^Z = dx^2 + x^2 g^M$, where (M, g^M) is a compact n-dimensional Riemannian manifold (without boundary); we will call Z a *cone*. We consider the Friedrichs extension Δ of the Laplace operator on compactly supported functions $C_0^\infty(Z)$ to $L^2(Z, g^Z)$. Under the isometry

$$\Psi : L^2(Z, dx^2 + g^M) \to L^2(Z, g^Z), \quad f(x, p) \mapsto x^{-n/2} f(x, p),$$

this Laplacian becomes

$$\widetilde{\Delta} := \Psi^{-1}\circ\Delta\circ\Psi = -\frac{\partial^2}{\partial x^2}+\frac{1}{x^2}\left(\Delta_M+\frac{n}{2}\left(\frac{n}{2}-1\right)\right).$$

Let $\{\mu_k\}$ be the eigenvalues of the Laplace operator Δ_M on $L^2(M)$, and define $\nu_k := \nu(\mu_k)$ as the positive solution of $\nu_k^2 - \frac{1}{4} = (n/2)(n/2 - 1) + \mu_k$. Let V be the set of these solutions and let $\{\phi_\nu\}_{\nu\in V}$ be the corresponding orthonormal Hilbert space basis of $L^2(M)$ consisting of eigenfunctions of Δ_M such that $\Delta_M\phi_{\nu(\mu)} = \mu\phi_{\nu(\mu)}$.

For a smooth function $f(x, p) = \sum_{\nu\in V} f_\nu(x)\phi_\nu(p) \in L^2(Z)$, we obtain

$$\Psi^{-1}(\Delta-\lambda^2)\Psi f(x,p) = \sum_{\nu\in V}((B_\nu-\lambda^2)f_\nu)(x)\phi_\nu(p),$$

where B_ν is the Bessel operator defined in (5).

Let $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda > 0$; in particular, λ^2 lies in the resolvent set of Δ. Then the integral kernel of the resolvent $(\widetilde{\Delta} - \lambda^2)^{-1}$ is given by

$$K((x,p),(y,q),\lambda) = \sum_{\nu\in V} r^{(\nu)}(x,y)(\lambda)\phi_\nu(q)\otimes\phi_\nu(p), \tag{13}$$

where $r^{(\nu)}$ is defined in (6). Recall from Lemma 7.1 that $r^{(\nu)}$ is a $z\log z$-Hahn-holomorphic function,

$$r^{(\nu)}(x,y)(\lambda) = \sum_{\gamma\in\tilde{S}_\nu\subset\mathbb{R}^2} a_\gamma^{(\nu)}(x,y)e_\gamma(\lambda),\quad e_{(\alpha,\beta)}(\lambda) := \lambda^\alpha(-\log\lambda)^{-\beta},$$

where $\tilde{S}_\nu$ is the Hahn series support of $r^{(\nu)}$. In this expansion, logarithmic terms occur only for $\nu \in \mathbb{N}$.

Take $\mathscr{G} \subset \mathbb{R}^2$ to be the group generated by $\bigcup_\nu S_\nu$ for $S_\nu := \bigcup_{x,y\in(0,\infty)} \tilde{S}_\nu(x, y)$. Then it is clear that the resolvent kernel is a Hahn series with support in $\mathcal{G} \subset \mathbb{R}^2$:

$$\begin{aligned} K((x,p),(y,q),\lambda) &:= \sum_{\nu\in V} r^{(\nu)}(x,y)(\lambda)\phi_\nu(q)\otimes\phi_\nu(p) = \sum_{\nu\in V}\sum_{\gamma\in S_\nu} a_\gamma^{(\nu)}(x,y)e_\gamma(\lambda)\phi_\nu(q)\otimes\phi_\nu(p) \\ &= \sum_{\gamma\in\mathcal{G}} e_\gamma(\lambda)\Bigg(\sum_{\nu\in V:\gamma\in S_\nu} a_\gamma^{(\nu)}(x,y)\phi_\nu(q)\otimes\phi_\nu(p)\Bigg) \\ &=: \sum_{\gamma\in\mathcal{G}} \tilde{a}_\gamma((x,p),(y,q))e_\gamma(\lambda), \end{aligned} \tag{14}$$

where we have set $\tilde{a}_\gamma = 0$ if $\gamma \notin \bigcup_{\nu\in V} S_\nu$.

To show normal convergence of the operator-valued series defined by (14), we will make the additional assumption that each $\nu \in V$ either is an integer, or is not "too close" to an integer in the following sense.

Definition 7.4. For $\kappa > 0$, a family of orders $V \subset \mathbb{R}_{\geq 0}$ is called *κ-suitable* if the set

$$\left\{ \frac{1}{(2\kappa)^\nu \sin(\nu\pi)\Gamma(\nu+1)} \;\middle|\; \nu \in V \setminus \mathbb{N} \right\} \tag{15}$$

is bounded.

Example 7.5. For $M = S^n$, $n \geq 1$, the n-sphere equipped with the standard metric, it is well known (see, for example, [Shubin 2001, Section 22]) that the eigenvalues of the Laplace operator on functions are $\mu_k = k(k+n-1)$, $k \in \mathbb{N}_0$ with multiplicity

$$m_k := \binom{n+k}{n} - \binom{n+k-2}{n}.$$

Then $\nu_k := \nu(\mu_k) := (n-1)/2 + k$ is (half-)integral for odd (even) n and $V = (\nu_0, \nu_1, \ldots, \nu_1, \nu_2, \ldots)$, where each ν_j appears m_j times. For $n \geq 2$, all $\nu(\mu_k)$ are positive.

In Section 7A we defined a smooth cutoff function χ, which can be extended to a bounded operator χ on $L^2((0,\infty)\times M, dx)$ by setting

$$\chi(f)(x,p) = \chi(x)f(x,p) \quad \text{for } f \in C_0^\infty(Z),\ x \in (0,\infty),\ p \in M,$$

and taking the closure.

Here is the main result of this section:

Theorem 7.6. *Let $c, \sigma, \kappa > 0$ and assume that the family $V = \{\nu\}$ of orders is κ-suitable. Then the restricted resolvent $\chi_c(\widetilde{\Delta} - \lambda^2)^{-1}\chi_c$ extends, as a function of λ, to a $z\log z$-Hahn-meromorphic function on some $D_r^{[\sigma]}$ with values in*

$$\mathcal{K}\big(L^2(Z, e^{\kappa x^2}\,dx\otimes \mathrm{vol}_M), L^2(Z, e^{-\kappa x^2}\,dx\otimes\mathrm{vol}_M)\big), \tag{16}$$

where the only term $\lambda^\alpha(-\log\lambda)^{-\beta}$ in its Hahn-series expansion with $(\alpha,\beta) < 0$ that possibly has a nonzero coefficient is the one with $(\alpha,\beta) = (0,-1)$, and its coefficient has finite rank. If V does not contain $\nu = 0$, then, in a (possibly smaller) neighborhood of zero, this function is Hahn-holomorphic.

Proof. Let A_γ be the operator on $L^2(Z)$ defined by the "restricted kernel"

$$(\chi_c \circ \tilde{a}_\gamma)((x,q),(y,q)) := \chi_c(x)\tilde{a}_\gamma((x,p),(y,q))\chi_c(y)$$

with $\tilde{a}$ from (14), so that $\sum_{\gamma\in\mathscr{G}} A_\gamma e_\gamma(\lambda)$ is the Hahn series of the restricted resolvent.

As in (8), in the proof of Proposition 7.3, we can estimate

$$\|\chi_c \circ \tilde{a}_\gamma^{(\nu)}\| \le R^{-k(\gamma)} C(\nu,\kappa);$$

now instead of (7b) we choose $R < c\kappa/4$ and use

$$\int_c^\infty x^{2\nu+1} e^{2Rx-\kappa x^2}\,dx \le \int_c^\infty x^{2\nu+1} e^{-(\kappa/2)x^2}\,dx \le \frac{\Gamma(\nu+1)}{2(\kappa/2)^{\nu+1}}. \tag{17}$$

Because the family V is κ-suitable, the constants $C(\nu,\kappa)$ are bounded in ν. Thus the kernel A_γ defines a Hilbert–Schmidt operator

$$A_\gamma : L^2(Z, e^{\kappa x^2}\,dx \otimes \mathrm{vol}_M) \to L^2(Z, e^{-\kappa x^2}\,dx \otimes \mathrm{vol}_M)$$

between weighted L^2-spaces, with norm bounded by

$$\|A_\gamma\| \le \|\chi_c \circ \tilde{a}_\gamma\| \le \sup_{\nu:\gamma\in\operatorname{supp} r_\nu} \|\chi_c \circ a_\gamma^{(\nu)}\| \le C$$

for all $\gamma \in \bigcup_\nu S_\nu$.

Therefore the Hahn series $\sum_{\gamma\in\mathscr{G}} A_\gamma e_\gamma(\lambda)$ is normally convergent in $D_\delta^{[\sigma]}$ for some $\delta > 0$, provided that

$$\sum_{\gamma\in\mathscr{S}} \|e_\gamma\|_\delta < \infty.$$

Due to Lemma 7.1, the support $\mathscr{S}$ is given by

$$\mathscr{S} = \bigcup_\nu \operatorname{supp} r_\nu = \mathscr{S}_\mathsf{r} \cup \mathscr{S}_\mathsf{i} \subset \mathbb{R}_+ \times (-\mathbb{N}_0), \tag{18}$$

where $\mathscr{S}_\mathsf{i}$ and $\mathscr{S}_\mathsf{r}$ correspond to integer and noninteger real coefficients ν. Furthermore, elements in $\mathscr{S}_\mathsf{i}$ are of the form $(2sn+2\ell, -s)$ with $\ell \in \mathbb{N}_0$, $s \in \{0,1\}$, and those in $\mathscr{S}_\mathsf{i}$ have the form $(2s\nu+2\ell, 0)$, $\ell \in \mathbb{N}_0$, $s \in \{0,1\}$ for ν noninteger.

For $0 < |\lambda| < \delta < 1$ and $\nu \in V \setminus \mathbb{N}_0$,

$$\sum_{\gamma\in\mathscr{S}_\mathsf{r}} |e_\gamma(\lambda)| \le \sum_{\ell\in\mathbb{N}_0} |\lambda^{2\ell}| + \sum_\nu \sum_{\ell\in\mathbb{N}_0} |\lambda^{2\ell+2\nu}| \le \frac{1}{1-\delta^2}\left(1 + \sum_\nu |\lambda^{2\nu}|\right).$$

Now, from Weyl's formula, we obtain that there exists an $R > 0$ such that $\sum_{\nu\in V} R^\nu < \infty$. This shows that, for $|\lambda| < \min(\delta, \sqrt{R})$, the partial series $\sum_{\gamma\in\mathscr{S}_\mathsf{r}} |e_\gamma(\lambda)|$ converges absolutely.

Finally, for $\nu = n \in \mathbb{N}$, we use $|\log\lambda|\,|\lambda|^{2k} \le C_\sigma |\lambda|^{2k-1}$ to estimate $\sum_{\gamma\in\mathscr{S}_\mathsf{i}} |e_\gamma(\lambda)|$ by the geometric series.

Note that the only term that gives rise to a nonzero coefficient of e_γ with $\gamma < 0$ is the order $\nu = 0$. □

Remark 7.7. From the proof of Theorem 7.6, it is clear that a similar statement holds if the weights in (16) are replaced by $e^{\pm\kappa x^{1+\varepsilon}}$ for any $\varepsilon > 0$.

For the Laplace operator on differential forms $L^2(S^n, \Lambda^* T^* S^n)$, the eigenvalues μ are integers (cf. [Ikeda and Taniguchi 1978, Theorem 4.2]), and the corresponding $\nu = \nu(\mu_k, p)$ are square roots of integers. In this case we have:

Lemma 7.8. *Any family $V = (\sqrt{q_i})_i$ with $q_i \in \mathbb{N}_0$ is κ-suitable for every $\kappa > 0$.*

Proof. First we show for $n \in \mathbb{N}_0$ and q with $n^2 < q < (n+1)^2$ that

$$\min\{\sqrt{q} - n, (n+1) - \sqrt{q}\} > \frac{1}{2(n+1)}.$$

We then use $|\sin x\pi| > 2|x|$ for $0 < |x| < \frac{1}{2}$ to prove that, for $\nu = \sqrt{q}$,

$$\frac{1}{(\nu+1)|\sin \nu\pi|} < 1, \quad \frac{1}{\nu|\sin \nu\pi|} < \frac{3}{2}.$$

Together with Stirling's formula for the asymptotics of $\Gamma(\nu)$, this shows the boundedness of (15). □

Therefore Theorem 7.6 has a straightforward extension to differential forms. A similar statement can be proven for the Laplacian acting on differential forms on $(0, \infty) \times P^n(\mathbb{C})$, where $P^n(\mathbb{C})$ is equipped with the Fubini–Study metric. The eigenvalues for the Laplace operator on sections of $\Lambda^p T^* P^n(\mathbb{C})$ have been computed in [Ikeda and Taniguchi 1978, Theorem 5.2], they are integers.

7C. ***The resolvent of the Laplace operator on compact perturbations of $\mathbb{R}^n$ or conic spaces.*** Set $Z = (0, \infty) \times M$ and let (Z, g^Z) be a cone as defined in the previous section. Let X be a Riemannian manifold that is isometric to Z away from a compact set. This means that, for some $a > 0$, we can identify X with $X = X_a \cup_{M_a} Z_a$, where $Z_a = [a, \infty) \times M$, $M_a = \{a\} \times M$, and X_a is a compact Riemannian manifold with boundary M_a.

In this section we denote by Δ_0 the self-adjoint operator on the cone that is obtained from the Friedrichs extension of the Laplace operator on $C_0^\infty(Z)$. Let Δ be the Laplace operator acting on compactly supported functions on X, and let L be a formally self-adjoint first order differential operator that is compactly supported in X_a for some $a > 0$. Then, of course $P := \Delta + L$ is of Laplace type and therefore essentially self-adjoint on compactly supported smooth functions. We will denote its self-adjoint extension by the same symbol P whenever there is no danger of confusion. It follows from standard results in perturbation theory that the essential spectrum of P equals the essential spectrum of the Laplace operator on the cone, namely, $[0, \infty)$. Moreover, it is well known that the distributional kernel of the resolvent $(P - \lambda^2)^{-1}$ has a meromorphic continuation across the spectrum away from the point $\lambda = 0$. Now Theorem 4.1 allows us to refine this statement and show that the resolvent kernel is Hahn-meromorphic at $\lambda = 0$ if this is true for the (restricted) kernel of $(\Delta_0 - \lambda^2)^{-1}$. The precise statement is formulated in the following theorem.

Theorem 7.9. *Let $a > 0$, $\kappa > 0$ and suppose that, for some $\sigma > 0$, the restricted resolvent $\chi_a(\Delta_0 - \lambda^2)^{-1}\chi_a$ extends, as a function of λ, to a $z \log z$-Hahn-meromorphic function on $D_r^{[\sigma]}$ with values in*

$$\mathcal{K}\big(L^2(Z, e^{\kappa x^2}\, dx \otimes \mathrm{vol}_M), L^2(Z, e^{-\kappa x^2}\, dx \otimes \mathrm{vol}_M)\big)$$

for a group $\Gamma \subset \mathbb{Z} \times \mathbb{Z}$, such that the range of the coefficients of e_γ with $\gamma < 0$ of its Hahn series are finite-rank operators with range contained in a fixed finite-dimensional subspace. Let $\widetilde{\Gamma}$ be the subgroup of $\mathbb{Z} \times \mathbb{Z}$ generated by Γ and $2\mathbb{Z} \times \{0\}$. Then $(P - \lambda^2)^{-1}$ has an extension, as a function of λ, to a $z \log z$ Hahn-meromorphic function on $D_r^{[\sigma]}$ for the group $\widetilde{\Gamma}$ with values in

$$\mathcal{K}\big(L^2(X, w(x)\mathrm{vol}_X), L^2(X, w(x)^{-1}\mathrm{vol}_X)\big),$$

where $w(x)$ is any positive function on X such that $w(x) = e^{\kappa x^2}$ on Z_a. Moreover, in the Hahn series expansion of this extension, the coefficients of e_γ with $\gamma < 0$ are finite-rank operators.

Proof. The proof is identical to the standard proof that the meromorphic properties of the resolvent do not change under compactly supported topological or metric perturbations. The only difference is that we apply our Hahn-meromorphic Fredholm theorem. For the sake of completeness, we give the full argument here. By assumption, we can choose $b > a > 0$ such that the operators Δ_0 and P agree on $C_0^\infty(Z_a)$. Suppose $\psi_1, \psi_2, \phi_1, \phi_2$ are smooth functions on X such that $\operatorname{supp} \phi_1 \subset X_b$ and $\operatorname{supp} \psi_1 \subset X_a$ and such that

$$\psi_1 + \psi_2 = 1, \quad \psi_1\phi_1 + \psi_2\phi_2 = 1, \quad \operatorname{dist}(\operatorname{supp} d\phi_i, \operatorname{supp} \psi_i) > 0.$$

Now denote by P_0 the self-adjoint operator obtained from P by imposing Dirichlet boundary conditions at M_b. Since P is an elliptic operator and the boundary conditions are elliptic, P_0 has compact resolvent and therefore $Q_1(\Lambda) := (P_0 - \lambda^2)^{-1}$ is a meromorphic function with values in $\mathcal{B}(L^2(X_b))$ and the residues of its poles are finite-rank operators. Let us denote by $Q_2(\lambda)$ the Hahn-meromorphic extension of $(\Delta_0 - \lambda^2)^{-1}$ that exists by assumption. Then

$$Q(\lambda) := \phi_1 Q_1(\lambda)\psi_1 + \phi_2 Q_2(\lambda)\psi_2$$

is a Hahn-meromorphic family with values in

$$\mathcal{K}\big(L^2(X, w(x)\mathrm{vol}_X), L^2(X, w(x)^{-1}\mathrm{vol}_X)\big)$$

with respect to the group $\widetilde{\Gamma}$ and the coefficients of e_γ with $\gamma < 0$ of its Hahn series are finite-rank operators with range contained in a fixed finite-dimensional subspace. By construction, for $\lambda \in D_r^{[\sigma]}$, $\operatorname{Im} \lambda > 0$,

$$Q(\lambda)(P - \lambda^2) = \mathrm{Id} + K(\lambda)$$

with

$$K(\lambda) = K_1(\lambda) + K_2(\lambda), \quad K_i(\lambda) := \phi_i Q_i(\lambda)(\Delta\psi_i - 2\nabla_{\operatorname{grad} \psi_i}).$$

Since the integral kernels of Q_i are smooth off the diagonal, the operator $K(\lambda)$ is smoothing. Moreover, its integral kernel has compact support in the second variable.

Given the previous remarks, since $Q_1(\lambda)$ is meromorphic and $Q_2(\lambda)$ is Hahn-meromorphic, $K(\lambda)$ is a Hahn-meromorphic family with values in $\mathcal{K}\big(L^2(X, w(x)^{-1}\mathrm{vol}_X)\big)$ for the group $\widetilde{\Gamma}$, and the coefficients of the e_γ in its Hahn series, for $\gamma < 0$, are finite-rank operators with range contained in a fixed finite-dimensional subspace. Furthermore, for $\lambda = \mathrm{i}r$ purely imaginary, one derives $\|K_i(\mathrm{i}r)\| \le c/r$ for $r > 1$. Therefore, for a sufficiently large r, the operator $\mathrm{Id} + K(\mathrm{i}r)$ is invertible. By the meromorphic Fredholm theory and Theorem 4.1, $(\mathrm{Id}+K(\lambda))^{-1}$ is a family of operators in $\mathcal{K}\big(L^2(X, w(x)^{-1}\mathrm{vol}_X)\big)$ which is meromorphic away from zero with finite-rank negative Laurent coefficients at its nonzero poles and finite-rank coefficients of e_γ with $\gamma < 0$. It is Hahn-meromorphic at zero for the group $\widetilde{\Gamma}$. Hence we have

$$(\mathrm{Id}+K(\lambda))^{-1}Q(\lambda)(P-\lambda^2)=\mathrm{Id},$$

and $(\mathrm{Id}+K(\lambda))^{-1}Q(\lambda)$ extends the resolvent of P to a Hahn-meromorphic function with the desired properties, as claimed. □

Combining Theorems 7.6 and 7.9, we obtain:

Corollary 7.10. *Let M be a Riemannian manifold that is isometric to $\mathbb{R}^n \backslash B_R$, $n \geq 2$, outside a compact set for some sufficiently large $R > 0$. Let P be a compactly supported perturbation of the Laplace operator in the sense of Theorem 7.9, and let $w(x)$ be as in that theorem. Then the resolvent $\lambda \mapsto (P-\lambda^2)^{-1}$, as a map*

$$\{\mathrm{Im}\,\lambda > 0\} \to \mathcal{K}\big(L^2(X, w(x)\mathrm{vol}_X), L^2(X, w(x)^{-1}\mathrm{vol}_X)\big),$$

has a continuation to a function in λ that is $z\log z$-Hahn-meromorphic for the group $\mathbb{Z}\times\mathbb{Z}$.

When n is odd, from Example 7.5 we conclude that $\Gamma = \mathbb{Z}\times\{0\}$. In this case, Theorem 7.9 and its corollary are well known and follow from the usual meromorphic Fredholm theorem. Similar convergent expansions in the case of two-dimensional potential scattering with suitable decay at infinity were obtained in [Bollé et al. 1988]. For example, in [Bollé et al. 1988, Theorem 3.3], it was shown by more direct methods that the transition operator $T(k)$ in $L^2(\mathbb{R}^2)$ has a convergent expansion in powers of k and $\log k$.

Remark 7.11. Set $Z = [1,\infty)\times N$. Let X be a Riemannian manifold with an end isometric to

$$\big(Z, dx^2 + x^{-2a}g^N\big), \quad a > 0,$$

for some closed Riemannian manifold (N, g^N). The spectral theory of the Laplace operator on X is examined in detail in [Hunsicker et al. 2014]. There the authors show that the spectral decomposition of the Laplace operator on differential forms on Z can also be described with the Weber transform. The same arguments as in Section 7A together with the proof of Theorem 7.9 then implies that the resolvent of the Laplace operator on X is $z\log z$-Hahn-meromorphic, provided that the eigenvalues of the Laplace operator on N lead to suitable ν.

Remark 7.12. Our method may also be applied to noncompactly supported perturbations of the Laplace operator on $\mathbb{R}^n$, such as for example potential perturbations that have a suitable decay rate at infinity. This is in line with the well-known result in the odd-dimensional case that uniform exponential decay of the potential at infinity guarantees the existence of an analytic continuation of the resolvent into a neighborhood of the spectrum.

References

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