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BY CONVEX CURVATURE FUNCTIONS**

CYLINDRICAL ESTIMATES FOR HYPERSURFACES MOVING BY CONVEX CURVATURE FUNCTIONS

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We prove a complete family of *cylindrical estimates* for solutions of a class of fully nonlinear curvature flows, generalising the cylindrical estimate of Huisken and Sinestrari [*Invent. Math.* **175**:1 (2009), 1–14, §5] for the mean curvature flow. More precisely, we show, for the class of flows considered, that, at points where the curvature is becoming large, an $(m+1)$ -convex ($0 \leq m \leq n-2$) solution either becomes strictly m -convex or its Weingarten map becomes that of a cylinder $\mathbb{R}^m \times S^{n-m}$. This result complements the convexity estimate we proved with McCoy [*Anal. PDE* **7**:2 (2014), 407–433] for the same class of flows.

1. Introduction

Let M be a smooth, closed manifold of dimension n , and $X_0 : M \rightarrow \mathbb{R}^{n+1}$ a smooth hypersurface immersion. We are interested in smooth families $X : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ of smooth immersions $X(\cdot, t)$ solving the initial value problem

$$\begin{cases} \partial_t X(x, t) = -F(\mathcal{W}(x, t))\nu(x, t), \\ X(\cdot, 0) = X_0, \end{cases} \quad (\text{CF})$$

where ν is the outer normal field of the evolving hypersurface X and \mathcal{W} the corresponding Weingarten curvature. In order that the problem (CF) be well-posed, we require that $F(\mathcal{W})$ be given by a smooth, symmetric function $f : \Gamma \rightarrow \mathbb{R}$ of the principal curvatures κ_i which is monotone increasing in each argument. The symmetry of f ensures that F is a smooth, basis-invariant function of the components of the Weingarten map (or an orthonormal frame-invariant function of the components of the second fundamental form) [Glaeser 1963]. Monotonicity ensures that the flow is (weakly) parabolic. This guarantees local existence of solutions of (CF), as long as the principal curvature n -tuple of the initial data lies in Γ ; see [Langford 2014].

For technical reasons, we require some additional conditions:

Conditions. (i) f is homogeneous of degree one.

(ii) f is convex.

Since the normal points out of the region enclosed by the solution, we may assume, by condition (ii), that $(1, \dots, 1) \in \Gamma$. Thus, by condition (i), we may further assume that f is normalised such that $f(1, \dots, 1) = 1$.

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The additional conditions (i)–(ii) have several consequences. Most importantly, they allow us to obtain a preserved cone $\Gamma_0 \subset \Gamma$ of curvatures for the flow (Lemma 2.2). This allows us to obtain uniform estimates on any degree-zero homogeneous function of curvature along the flow (Lemma 2.3); in particular, we deduce a uniform parabolicity condition (Corollary 2.4). The convexity condition then allows us to apply the second derivative Hölder estimate of [Evans 1982; Krylov 1982] to deduce that the solution exists on a maximal time interval $[0, T)$, $T < \infty$, such that $\max_{M \times \{t\}} F \rightarrow \infty$ as $t \rightarrow T$; see [Andrews et al. 2014a, Proposition 2.6]. Thus, it is of interest to study the behaviour of solutions as $F \rightarrow \infty$. Let us recall the following curvature estimate [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]).

Theorem 1.1 (convexity estimate). *Let $X : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon < \infty$ such that*

$$G(x, t) \leq \varepsilon F(x, t) + C_\varepsilon \quad \text{for all } (x, t) \in M \times [0, T),$$

where G is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures of the evolving hypersurface that vanishes at a point (x, t) if and only if $\mathcal{W}_{(x,t)} \geq 0$.

We remark that the constant C_ε depends only on ε , the dimension n , the choice of speed function f , the preserved curvature cone Γ_0 , and bounds for the initial volume and diameter [Langford 2014].

Theorem 1.1 implies that the ratio of the smallest principal curvature to the speed is almost positive wherever the curvature is large. Combining it with the differential Harnack inequality of [Andrews 1994b] and the strong maximum principle [Hamilton 1986] yields useful information about the geometry of solutions of (CF) near singularities [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]):

Corollary 1.2. *Any blow-up limit of a solution of (CF) is weakly convex. In particular, any type-II blow-up limit about a type-II singularity is an eternal solution of the form $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$, $k \in \{0, 1, \dots, n-1\}$, such that $X_\infty|_{\mathbb{R}^k}$ is flat, and $X_\infty|_{\Gamma^{n-k}}$ is a strictly convex translation solution of the corresponding flow in \mathbb{R}^{n-k+1} .*

Motivated by the surgery construction of [Huisken and Sinestrari 2009, §5] for 2-convex mean curvature flow, we will apply Theorem 1.1 to obtain the following family of cylindrical estimates for solutions of (CF):

Theorem 1.3 (cylindrical estimate). *Let X be a solution of (CF) such that conditions (i)–(ii) hold. Suppose also that X is uniformly $(m+1)$ -convex for some $m \in \{0, 1, \dots, n-2\}$. That is, $\kappa_1 + \dots + \kappa_{m+1} \geq \beta F$ for some $\beta > 0$. Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that*

$$G_m(x, t) \leq \varepsilon F(x, t) + C_\varepsilon \quad \text{for all } (x, t) \in M \times [0, T),$$

where $G_m : M \times [0, T) \rightarrow \mathbb{R}$ is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures that vanishes at a point (x, t) if and only if

$$\kappa_1(x, t) + \dots + \kappa_{m+1}(x, t) \geq \frac{1}{c_m} f(\kappa_1(x, t), \dots, \kappa_n(x, t)),$$

where c_m is the value F takes on the unit radius cylinder $\mathbb{R}^m \times S^{n-m}$.

We note that the constant C_ε will only depend on ε, β, m , the dimension n , the choice of speed function f , the preserved curvature cone Γ_0 , and upper bounds for the initial volume and diameter. Theorem 1.3 implies that the ratio of the quantity

$$K_m := \kappa_1 + \dots + \kappa_{m+1} - \frac{1}{c_m} F$$

to the speed is almost positive wherever the curvature is large. Observe that this quantity is nonnegative on a weakly convex hypersurface Σ only if either Σ is strictly m -convex or $\Sigma = \mathbb{R}^m \times S^{n-m}$. In particular, we find that, whenever $\kappa_1(x, t) + \dots + \kappa_m(x, t)$ is small compared to the speed, the Weingarten curvature is close to that of a thin, round cylinder $\mathbb{R}^m \times S^{n-m}$. We therefore obtain a refinement of Corollary 1.2:

Corollary 1.4. *Any blow-up limit of an $(m+1)$ -convex, $0 \leq m \leq n - 2$, solution of (CF) is either strictly m -convex, or a shrinking cylinder $\mathbb{R}^m \times S^{n-m}$. In particular, if the blow-up is of type-II, then this limit is of the form $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \rightarrow \mathbb{R}^{n+1}$ for $k \in \{0, 1, \dots, m - 1\}$, such that $X_\infty|_{\mathbb{R}^k}$ is flat and $X_\infty|_{\Gamma^{n-k}}$ is a strictly convex translation solution of the corresponding flow in \mathbb{R}^{n-k+1} .*

The $m = 0$ case of the cylindrical estimates demonstrates that convex hypersurfaces become umbilic at points where the curvature is blowing up, generalising a result of Huisken [1984, Theorem 5.1] for the mean curvature flow (we note that the convergence result of [Huisken 1984] has been obtained by the first author for the class of flows considered here without the need for such an estimate [Andrews 1994a]). Moreover, Huisken and Sinestrari [2009] have recently obtained the $m = 1$ case of the cylindrical estimates for the mean curvature flow, making spectacular use of it through their surgery program, which yields a classification of 2-convex hypersurfaces. The convexity and cylindrical estimates stated above, in addition to generalising the Huisken–Sinestrari cylindrical estimate to all m in $\{0, \dots, n - 2\}$, constitute a first step towards improving upon such results by allowing a larger class of evolution equations.

2. Preliminaries

We will follow the notation used in [Andrews et al. 2014b]. In particular, we recall that a smooth, symmetric function g of the principal curvatures gives rise to a smooth function G of the components h_i^j of the Weingarten map. Equivalently, G is an orthonormal frame invariant function of the components h_{ij} of the second fundamental form. To simplify notation, we denote $G(x, t) \equiv G(\mathcal{W}(x, t)) = g(\kappa(x, t))$ and use dots to denote derivatives of functions of curvature as follows:

$$\begin{aligned} \dot{g}^k(z)v_k &= \frac{d}{ds} \Big|_{s=0} g(z + sv), & \dot{G}^{kl}(A)B_{kl} &= \frac{d}{ds} \Big|_{s=0} G(A + sB), \\ \ddot{g}^{pq}(z)v_p v_q &= \frac{d^2}{ds^2} \Big|_{s=0} g(z + sv), & \ddot{G}^{pq,rs}(A)B_{pq} B_{rs} &= \frac{d^2}{ds^2} \Big|_{s=0} G(A + sB). \end{aligned}$$

The derivatives of g and G are related in the following way:

Lemma 2.1 [Gerhardt 1996; Andrews 1994a; 2007]. *Let $g : \Gamma \rightarrow \mathbb{R}$ be a smooth, symmetric function. Define the function $G : \mathcal{S}_\Gamma \rightarrow \mathbb{R}$ by $G(A) := g(\lambda(A))$, where $\lambda(A)$ denotes the eigenvalues of A (up to order) and \mathcal{S}_Γ denotes the set of symmetric matrices with eigenvalues in Γ . Then, for any diagonal $A \in \mathcal{S}_\Gamma$,*

$$\dot{G}^{kl}(A) = \dot{g}^k(\lambda(A))\delta^{kl}, \tag{2-1}$$

and, for any diagonal $A \in \mathcal{S}_\Gamma$ with distinct eigenvalues and any symmetric $B \in \text{GL}(n)$,

$$\ddot{G}^{pq,rs}(A)B_{pq}B_{rs} = \ddot{g}^{pq}(\lambda(A))B_{pp}B_{qq} + 2 \sum_{p>q} \frac{\dot{g}^p(\lambda(A)) - \dot{g}^q(\lambda(A))}{\lambda_p(A) - \lambda_q(A)} (B_{pq})^2. \tag{2-2}$$

We note that $\ddot{g} \geq 0$ if and only if $(\dot{g}^p - \dot{g}^q)(z_p - z_q) \geq 0$ for all p, q [Andrews et al. 2014b, Lemma 2.2], so Lemma 2.1 implies that G is convex if and only if g is convex.

The following useful lemma was proved in [Andrews et al. 2014b]:

Lemma 2.2. *Let $f : \Gamma \rightarrow \mathbb{R}$ be a flow speed for (CF) satisfying Conditions (i)–(ii). Then, for any admissible initial datum $X_0 : M \rightarrow \mathbb{R}^{n+1}$ there exists a cone $\Gamma_0 \subset \mathbb{R}^n$ satisfying $\bar{\Gamma}_0 \setminus \{0\} \subset \Gamma$ such that the principal curvatures of the solution $X : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ of the initial value problem (CF) satisfy $\kappa(x, t) := (\kappa_1(x, t), \dots, \kappa_n(x, t)) \in \Gamma_0$ for all $(x, t) \in M \times [0, T)$.*

We refer to such a cone Γ_0 as a *preserved cone* for the solution X . As mentioned in the introduction, the existence of a preserved cone allows us to obtain bounds for homogeneous functions of the curvature:

Lemma 2.3. *Let $X : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Let $g : \Gamma \rightarrow \mathbb{R}$ be a smooth, degree-zero homogeneous symmetric function. Then there exists $c > 0$ (depending only on n, f and M_0) such that*

$$-c \leq g(\kappa_1(x, t), \dots, \kappa_n(x, t)) \leq c \quad \text{for all } (x, t) \in M \times [0, T).$$

If $g > 0$, then there exists $c > 0$ such that

$$\frac{1}{c} \leq g(\kappa_1(x, t), \dots, \kappa_n(x, t)) \leq c.$$

Proof. Let Γ_0 be a preserved cone for the solution X . Then $K := \bar{\Gamma}_0 \cap S^n$ is compact. Since g is continuous, the required bounds hold on K . But these extend to $\bar{\Gamma}_0 \setminus \{0\}$ by homogeneity. The claim follows since $\kappa(x, t) \in \bar{\Gamma}_0 \setminus \{0\}$ for all $(x, t) \in M \times [0, T)$. □

By condition (i), the derivative \dot{f} of f is homogeneous of degree zero. Since $\dot{f}^k > 0$ for each k , we obtain uniform parabolicity of the flow:

Corollary 2.4. *There exists a constant $c > 0$ (depending only on n, f and M_0) such that, for any $v \in T^*M$, it holds that*

$$\frac{1}{c} |v|^2 \leq \dot{F}^{ij} v_i v_j \leq c |v|^2,$$

where $|\cdot|$ is the (time-dependent) norm on M corresponding to the (time-dependent) metric induced by the flow.

We now recall the following evolution equation (see for example [Andrews et al. 2013]).

Lemma 2.5. *Let $X : M \times [0, T) \rightarrow \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Let $G : M \times [0, T) \rightarrow \mathbb{R}$ be given by a smooth, symmetric, degree-one homogeneous function g of the principal curvatures. Then G satisfies the evolution equation*

$$(\partial_t - \mathcal{L})G = (\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs}) \nabla h_{pq} \nabla h_{rs} + G |^q W|_F^2, \tag{2-3}$$

where $\mathcal{L} := \dot{F}^{kl} \nabla_k \nabla_l$ is the linearisation of F , and $|^q W|_F^2 := \dot{F}^{kl} h_k^r h_{rl}$.

In particular, the speed function F satisfies $(\partial_t - \mathcal{L})F = F |^q W|_F^2$.

As we shall see, in order to obtain Theorem 1.3, it is crucial to obtain a good upper bound on the term

$$Q(\nabla^q W, \nabla^q W) := (\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs}) \nabla_k h_{pq} \nabla_l h_{rs}$$

for the pinching functions G_m which we construct in the following section. The following decomposition of Q is crucial in obtaining this bound.

Lemma 2.6. *For any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, we have*

$$\begin{aligned} (\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs})|_B T_{kpq} T_{lrs} &= (\dot{g}^k \ddot{f}^{pq} - \dot{f}^k \ddot{g}^{pq})|_z T_{kpp} T_{kqq} \\ &+ 2 \sum_{p>q} \frac{(\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q)|_z}{z_p - z_q} ((T_{ppq})^2 + (T_{qqp})^2) + 2 \sum_{k>p>q} (\vec{g}_{kpq} \times \vec{f}_{kpq})|_z \cdot \vec{z}_{kpq} (T_{kpq})^2 \end{aligned} \tag{2-4}$$

at any diagonal matrix B with distinct eigenvalues z_i , where “ \times ” and “ \cdot ” are the three-dimensional cross and dot product respectively, and we have defined the vectors

$$\begin{aligned} \vec{f}_{kpq} &:= (\dot{f}^k, \dot{f}^p, \dot{f}^q), \\ \vec{g}_{kpq} &:= (\dot{g}^k, \dot{g}^p, \dot{g}^q), \\ \vec{z}_{kpq} &:= \left(\frac{z_p - z_q}{(z_k - z_p)(z_k - z_q)}, \frac{z_k - z_q}{(z_k - z_p)(z_p - z_q)}, \frac{z_k - z_p}{(z_p - z_q)(z_k - z_q)} \right). \end{aligned}$$

Proof. Since B is diagonal, Lemma 2.1 yields (suppressing the dependence on B)

$$\begin{aligned} &(\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs}) T_{kpq} T_{lrs} \\ &= \sum_{k,p,q} (\dot{g}^k \ddot{f}^{pq} - \dot{f}^k \ddot{g}^{pq}) T_{kpp} T_{kqq} + 2 \sum_k \sum_{p>q} \left(\dot{g}^k \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{g}^p - \dot{g}^q}{z_p - z_q} \right) (T_{kpq})^2. \end{aligned}$$

We now decompose the second term into the terms satisfying $k = p$, $k = q$, $k > p$, $p > k > q$, and $q > k$ respectively:

$$\begin{aligned} &\sum_k \sum_{p>q} \left(\dot{g}^k \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{g}^p - \dot{g}^q}{z_p - z_q} \right) (T_{kpq})^2 \\ &= \sum_{p>q} \left(\dot{g}^p \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^p \frac{\dot{g}^p - \dot{g}^q}{z_p - z_q} \right) (T_{ppq})^2 + \sum_{p>q} \left(\dot{g}^q \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^q \frac{\dot{g}^p - \dot{g}^q}{z_p - z_q} \right) (T_{qpq})^2 \\ &\quad + \left(\sum_{k>p>q} + \sum_{p>k>q} + \sum_{p>q>k} \right) \left(\dot{g}^k \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{g}^p - \dot{g}^q}{z_p - z_q} \right) (T_{kpq})^2 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} ((T_{pq})^2 + (T_{qp})^2) \\
 &\quad + \sum_{k>p>q} \left(\dot{g}^k \frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k \frac{\dot{g}^p - \dot{g}^q}{z_p - z_q} + \dot{g}^p \frac{\dot{f}^k - \dot{f}^q}{z_k - z_q} - \dot{f}^p \frac{\dot{g}^k - \dot{g}^q}{z_k - z_q} + \dot{g}^q \frac{\dot{f}^k - \dot{f}^p}{z_k - z_p} - \dot{f}^q \frac{\dot{g}^k - \dot{g}^p}{z_k - z_p} \right) (T_{kpq})^2 \\
 &= \sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} ((T_{pq})^2 + (T_{qp})^2) + \sum_{k>p>q} \left[(\dot{g}^p \dot{f}^q - \dot{f}^q \dot{g}^p) \left(\frac{1}{z_k - z_p} - \frac{1}{z_k - z_q} \right) \right. \\
 &\quad \left. - (\dot{g}^k \dot{f}^q - \dot{f}^k \dot{g}^q) \left(\frac{1}{z_p - z_q} + \frac{1}{z_k - z_p} \right) + (\dot{g}^k \dot{f}^p - \dot{f}^k \dot{g}^p) \left(\frac{1}{z_p - z_q} - \frac{1}{z_k - z_q} \right) \right] (T_{kpq})^2 \\
 &= \sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} ((T_{pq})^2 + (T_{qp})^2) + \sum_{k>p>q} (\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} (T_{kpq})^2. \quad \square
 \end{aligned}$$

We complete this section by proving that $(m+1)$ -convexity is preserved by the flow (CF), so that this assumption need only be made on initial data:

Proposition 2.7. *Let X be a solution of (CF) such that conditions (i)–(ii) are satisfied. Suppose that there is some $m \in \{1, \dots, n - 1\}$ and some $\beta > 0$ such that*

$$\kappa_{\sigma(1)}(x, 0) + \dots + \kappa_{\sigma(m)}(x, 0) \geq \beta F(x, 0)$$

for all $x \in M$ and all permutations $\sigma \in P_n$. Then this estimate persists at all later times.

Proof. Denote by SM the unit tangent bundle over $M \times [0, T)$ and consider the function Z defined on $\bigoplus^m SM$ by

$$Z(x, t, \xi_1, \dots, \xi_m) = \sum_{\alpha=1}^m h(\xi_\alpha, \xi_\alpha) - \beta F(x, t).$$

Since we have

$$\inf_{\xi_1, \dots, \xi_m \in S(x,t)M} Z(x, t, \xi_1, \dots, \xi_m) = \kappa_{\sigma(1)}(x, t) + \dots + \kappa_{\sigma(m)}(x, t) - \beta F(x, t)$$

for some $\sigma \in P_n$, it suffices to show that Z remains nonnegative. First fix any $t_1 \in [0, T)$ and consider the function $Z_\varepsilon(x, t, \xi_1, \dots, \xi_m) := Z(x, t, \xi_1, \dots, \xi_m) + \varepsilon e^{(1+C)t}$, where $C := \sup_{M \times [0, t_1]} |W|_F^2$. Note that C is finite since M is compact and \dot{F} is bounded. Observe that Z_ε is positive when $t = 0$. We will show that Z_ε remains positive on $M \times [0, t_1]$ for all $\varepsilon > 0$. So suppose to the contrary that Z_ε vanishes at some point $(x_0, t_0, \xi_1^0, \dots, \xi_m^0)$. We may assume that t_0 is the first such time. Now extend the vector $\xi^0 := (\xi_1^0, \dots, \xi_m^0)$ to a field $\xi := (\xi_1, \dots, \xi_n)$ near (x_0, t_0) by parallel translation in space and solving

$$\frac{\partial \xi_\alpha^i}{\partial t} = F \xi_\alpha^j h_j^i.$$

Since the metric evolves according to

$$\partial_t g_{ij} = -2F h_{ij}$$

the resulting fields have unit length. Now recall (see for example [Andrews 1994a]) the following evolution equation for the second fundamental form:

$$\partial_t h_{ij} = \mathcal{L}h_{ij} + \ddot{F}^{pq,rs} \nabla_i h_{pq} \nabla_j h_{rs} + |\mathcal{W}|_F^2 h_{ij} - 2Fh_{ij}^2,$$

where $\mathcal{L} := \dot{F}^{kl} \nabla_k \nabla_l$ and $|\mathcal{W}|_F^2 := \dot{F}^{kl} h_{kl}^2$. It follows that

$$\begin{aligned} (\partial_t - \mathcal{L})(Z_\varepsilon(x, t, \xi)) &= \varepsilon(1 + C)e^{(1+C)t} + \sum_{\alpha=1}^m \ddot{F}^{pq,rs} \nabla_{\xi_\alpha} h_{pq} \nabla_{\xi_\alpha} h_{rs} + |\mathcal{W}(x, t)|_F^2 Z(x, t, \xi) \\ &\geq \varepsilon(1 + C)e^{(1+C)t} + |\mathcal{W}(x, t)|_F^2 Z(x, t, \xi). \end{aligned}$$

Since the point $(x_0, t_0, \xi_{t=t_0})$ is a minimum of Z_ε , we obtain

$$0 \geq (\partial_t - \mathcal{L})|_{(x_0, t_0)}(Z_\varepsilon(x, t, \xi)) \geq \varepsilon(1 + C)e^{(1+C)t_0} - C\varepsilon e^{(1+C)t_0} = \varepsilon e^{(1+C)t_0} > 0.$$

This is a contradiction, implying that Z_ε cannot vanish at any time in the interval $[0, t_1]$. Since $\varepsilon > 0$ was arbitrary, we find $Z \geq 0$ at all times in the interval $[0, t_1]$. Since $t_1 \in [0, T)$ was arbitrary, we obtain $Z \geq 0$. □

3. Constructing the pinching function

In this section we construct the pinching functions G_m satisfying the conditions in Theorem 1.3. Let us first introduce the *pinching cones*

$$\Gamma_m := \{z \in \Gamma : z_{\sigma(1)} + \dots + z_{\sigma(m+1)} > c_m^{-1} f(z) \text{ for all } \sigma \in H_m\},$$

where H_m is the quotient of P_n , the group of permutations of the set $\{1, \dots, n\}$, by the equivalence relation

$$\sigma \sim \omega \quad \text{if} \quad \sigma(\{1, \dots, m+1\}) = \omega(\{1, \dots, m+1\}).$$

Using the methods of [Huisken 1984], and their adaptations to 2-convex flows in [Huisken and Sinestrari 2009] and fully nonlinear flows in [Andrews et al. 2014b], we will see that, in order to prove Theorem 1.3, it suffices to construct a smooth function $g_m : \Gamma \rightarrow \mathbb{R}$ satisfying the following properties.

Properties. (i) $g_m(z) \geq 0$ for all $z \in \Gamma$ with equality if and only if $z \in \bar{\Gamma}_m \cap \Gamma$.

(ii) g_m is smooth and homogeneous of degree one.

(iii) For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$(\dot{G}_m^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}_m^{pq,rs})|_B T_{kpq} T_{lrs} \leq -c_\varepsilon \frac{|T|^2}{F}$$

for all $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G_m(B) \geq \varepsilon F(B)$ and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, where G_m is the matrix function corresponding to g_m as described in Section 2, and Γ_0 is a preserved cone for the flow.

(iv) For every $\delta > 0$, $\varepsilon > 0$, and $C > 0$, there exist $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$(G_m \dot{F}^{kl} - F \dot{G}_m^{kl})|_B B_{kl}^2 \leq -\gamma_1 F^2 (G_m - \delta \gamma_2 F)|_B + \gamma_3 C F^2|_B$$

for all $(m + 1)$ -positive $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G_m(B) \geq \varepsilon F(B)$ and

$$\lambda_{\min}(B) \geq -\delta F(B) - C.$$

Our construction of the pinching function g_m will be similar for each choice of m . So let us fix $m \in \{0, 1, \dots, n - 2\}$ and assume that the flow is $(m+1)$ -convex. We first consider the preliminary function $g : \Gamma \rightarrow \mathbb{R}$ defined by

$$g(z) := f(z) \sum_{\sigma \in H_m} \varphi \left(\frac{\sum_{i=1}^{m+1} z_{\sigma(i)} - c_m^{-1} f(z)}{f(z)} \right), \tag{3-1}$$

where $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is a smooth¹ function which is strictly convex and positive, except on $\mathbb{R}_+ \cup \{0\}$ where it vanishes identically. Such a function is readily constructed; for example, we could take

$$\varphi(r) = \begin{cases} r^4 e^{-1/r^2} & \text{if } r < 0, \\ 0 & \text{if } r \geq 0. \end{cases}$$

We note that such a function necessarily satisfies $\varphi(r) - r\varphi'(r) \leq 0$ and $\varphi'(r) \leq 0$ with equality if and only if $r \geq 0$.

Now define the scalar $G : M \times [0, T) \rightarrow \mathbb{R}$ by

$$G(x, t) := g(\kappa_1(x, t), \dots, \kappa_n(x, t)).$$

Then G is a smooth, degree-one homogeneous function of the components of the Weingarten map which is invariant under a change of basis. Moreover, G is nonnegative and vanishes at, and only at, points for which the sum of the smallest $(m+1)$ -principal curvatures is not less than $c_m^{-1} F$. Thus properties (i) and (ii) are satisfied by g .

We now show that property (iii) is satisfied weakly by g :

Lemma 3.1. *Let G be the matrix function corresponding to the function g defined by (3-1). Then, for any symmetric matrix B and totally symmetric 3-tensor T ,*

$$(\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs})|_B T_{kpq} T_{lrs} \leq 0.$$

Proof. We will show that each of the terms in the decomposition (2-4) in Lemma 2.6 is nonpositive. Note that, by the invariance properties of G and F , it suffices to prove the claim for diagonal B . In fact, we can also assume that B has distinct eigenvalues, since the result at an arbitrary diagonal matrix B may then be

¹In fact, φ need only be twice continuously differentiable.

obtained by taking a limit $B^{(k)} \rightarrow B$ such that each matrix $B^{(k)}$ has distinct eigenvalues. We first compute

$$\begin{aligned} \dot{g}^k &= \dot{f}^k \sum_{\sigma \in H_m} \varphi(r_\sigma) + \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^k - \frac{z_{\sigma(i)}}{f} \dot{f}^k \right) \\ &= \dot{f}^k \sum_{\sigma \in H_m} \left(\varphi(r_\sigma) - \varphi'(r_\sigma) \frac{\sum_{i=1}^{m+1} z_{\sigma(i)}}{f} \right) + \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) \delta_{\sigma(i)}^k \end{aligned}$$

and

$$\begin{aligned} \ddot{g}^{pq} &= \left(\sum_{\sigma \in H_m} \varphi(r_\sigma) - \sum_{\sigma \in H_m} \varphi'(r_\sigma) \frac{\sum_{i=1}^{m+1} z_{\sigma(i)}}{f} \right) \ddot{f}^{pq} \\ &\quad + \sum_{\sigma \in H_m} \frac{\varphi''(r_\sigma)}{f} \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^p - \frac{z_{\sigma(i)}}{f} \dot{f}^p \right) \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^q - \frac{z_{\sigma(i)}}{f} \dot{f}^q \right), \end{aligned}$$

where we have set

$$r_\sigma(z) := \frac{\sum_{i=1}^{m+1} z_{\sigma(i)} - c_m^{-1} f(z)}{f(z)}.$$

It follows that

$$\dot{g}^k \ddot{f}^{pq} - \dot{f}^k \ddot{g}^{pq} = \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) \delta_{\sigma(i)}^k \ddot{f}^{pq} - \dot{f}^k \sum_{\sigma \in H_m} \frac{\varphi''(r_\sigma)}{f} \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^p - \frac{z_{\sigma(i)}}{f} \dot{f}^p \right) \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^q - \frac{z_{\sigma(i)}}{f} \dot{f}^q \right).$$

If we fix the index k and set $\xi_p = T_{kpp}$, then, by convexity of φ and positivity of \dot{f}^k , we have

$$\begin{aligned} -\dot{f}^k \sum_{\sigma \in H_m} \frac{\varphi''(r_\sigma)}{f} \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^p - \frac{z_{\sigma(i)}}{f} \dot{f}^p \right) \sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^q - \frac{z_{\sigma(i)}}{f} \dot{f}^q \right) \xi_p \xi_q \\ = -\dot{f}^k \sum_{\sigma \in H_m} \frac{\varphi''(r_\sigma)}{f} \left(\sum_{i=1}^{m+1} \left(\delta_{\sigma(i)}^p - \frac{z_{\sigma(i)}}{f} \dot{f}^p \right) \xi_p \right)^2 \leq 0. \end{aligned}$$

On the other hand, since φ is monotone nonincreasing, and f is convex, we have

$$\varphi'(r_\sigma) \sum_{i=1}^{m+1} \delta_{\sigma(i)}^k \dot{f}^{pq} \xi_p \xi_q \leq 0$$

for each σ . Since both inequalities hold for all k , we deduce that

$$\sum_{k,p,q} (\dot{g}^k \ddot{f}^{pq} - \dot{f}^k \ddot{g}^{pq}) T_{kpp} T_{kqq} \leq 0.$$

We next consider

$$\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q = \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) (\delta_{\sigma(i)}^q \dot{f}^p - \delta_{\sigma(i)}^p \dot{f}^q) = \sum_{\sigma \in O_q} \varphi'(r_\sigma) \dot{f}^p - \sum_{\sigma \in O_p} \varphi'(r_\sigma) \dot{f}^q.$$

where we have introduced the sets

$$O_a := \{\sigma \in H_m : a \in \sigma(\{1, \dots, m+1\})\}.$$

If $z_p > z_q$, we obtain

$$\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q \leq \dot{f}^p \left(\sum_{\sigma \in O_q} \varphi'(r_\sigma) - \sum_{\sigma \in O_p} \varphi'(r_\sigma) \right),$$

We now show that the term in brackets is nonpositive whenever $z_p > z_q$.

Lemma 3.2. *If $z_p \geq z_q$, then*

$$\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) \geq 0.$$

Moreover, equality holds only if either $z_p = z_q$ or $r_\sigma(z) \geq 0$ for all $\sigma \in O_{q,p} := O_q \setminus O_p$.

Proof of Lemma 3.2. First note that

$$\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) = \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma),$$

where $O_{a,b} := O_a \setminus O_b$. Next observe that, if $\sigma \in O_{p,q}$, then

$$z_{\sigma(1)} + \dots + z_{\sigma(m+1)} = z_p + z_{\hat{\sigma}(i_1)} + \dots + z_{\hat{\sigma}(i_m)} \tag{3-2}$$

for some $\hat{\sigma} \in H_{m-2}(p, q) := P_{n-2}(p, q) / \sim$, where $P_{n-2}(p, q)$ denotes the set of permutations of $\{1, \dots, n\} \setminus \{p, q\}$; i_1, \dots, i_m are m distinct elements of $\{1, \dots, n\} \setminus \{p, q\}$; and \sim is defined by

$$\hat{\sigma} \sim \hat{\omega} \quad \text{if} \quad \hat{\sigma}(\{i_1, \dots, i_m\}) = \hat{\omega}(\{i_1, \dots, i_m\}).$$

Observe also that the converse holds (that is, (3-2) defines a bijection), so that

$$\sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) = \sum_{\hat{\sigma} \in H_{m-2}(p,q)} \left(\varphi' \left(\frac{z_p + \sum_{k=1}^m z_{\hat{\sigma}(i_k)} - c_m^{-1} f}{f} \right) - \varphi' \left(\frac{z_q + \sum_{k=1}^m z_{\hat{\sigma}(i_k)} - c_m^{-1} f}{f} \right) \right).$$

Since $z_p \geq z_q$, the claim follows from (strict) convexity of φ (where it is positive). □

Thus,

$$\sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} ((T_{pq})^2 + (T_{qp})^2) \leq 0.$$

We now compute

$$\vec{g}_{kpq} = \left(\frac{g}{f} - \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \frac{z_{\sigma(i)}}{f} \right) \vec{f}_{kpq} + \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} (\delta_{\sigma(i)}^k, \delta_{\sigma(i)}^p, \delta_{\sigma(i)}^q),$$

so that

$$\begin{aligned}
 (\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} &= \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) [(\delta_{\sigma(i)}^k, \delta_{\sigma(i)}^p, \delta_{\sigma(i)}^q) \times \vec{f}_{kpq}] \cdot \vec{z}_{kpq} \\
 &= \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) \left[\frac{(\delta_{\sigma(i)}^p \dot{f}^q - \delta_{\sigma(i)}^q \dot{f}^p)(z_p - z_q)}{(z_k - z_p)(z_k - z_q)} + \frac{(\delta_{\sigma(i)}^q \dot{f}^k - \delta_{\sigma(i)}^k \dot{f}^q)(z_k - z_q)}{(z_k - z_p)(z_p - z_q)} \right. \\
 &\qquad \qquad \qquad \left. + \frac{(\delta_{\sigma(i)}^k \dot{f}^p - \delta_{\sigma(i)}^p \dot{f}^k)(z_k - z_p)}{(z_k - z_q)(z_p - z_q)} \right].
 \end{aligned}$$

Removing the positive factor $\alpha_{kpq} := [(z_k - z_p)(z_k - z_q)(z_p - z_q)]^{-1}$ and setting

$$P_a := \sum_{\sigma \in O_a} \varphi'(r_\sigma),$$

we obtain

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} = \alpha_{kpq} [(P_p \dot{f}^q - P_q \dot{f}^p)(z_p - z_q)^2 + (P_q \dot{f}^k - P_k \dot{f}^q)(z_k - z_q)^2 + (P_k \dot{f}^p - P_p \dot{f}^k)(z_k - z_p)^2].$$

Applying Lemma 3.2 yields

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \leq \alpha_{kpq} (P_q \dot{f}^k - P_k \dot{f}^q) [(z_k - z_q)^2 - (z_k - z_p)^2 - (z_p - z_q)^2].$$

Since the term in square brackets is nonnegative, applying Lemma 3.2 once more yields

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \leq 0.$$

This completes the proof of the lemma. □

Corollary 3.3. *There exists $C < \infty$ (depending only on n, f and M_0) such that $G/F \leq C$ along the flow.*

Proof. In view of Lemma 3.1 and the evolution equation (2-3), this is a simple application of the maximum principle. □

In order to obtain the uniform estimate required by property (iii), we modify G in order to obtain a function with a strict convexity property. A well-known trick (cf. [Andrews 1994b, Lemma 7.10; Huisken and Sinestrari 1999a, Theorem 2.14; Andrews et al. 2014b, Lemma 3.3]) then allows us to extract the required uniform estimate. First, we relabel the preliminary pinching function $g \rightarrow g_1$ ($G \rightarrow G_1$), and consider the new pinching function g defined by

$$g := K(g_1, g_2) := \frac{g_1^2}{g_2}, \tag{3-3}$$

where $g_2(z) = M \sum_{i=1}^n z_i - |z|$ for some large constant $M \gg 1$, for which g_2 is positive along the flow. That there is such a constant follows from applying the maximum principle to the evolution equation (2-3) for the function $G_2(x, t) := g_2(\kappa(x, t))$ as in [Andrews et al. 2014b, Lemma 3.1]. Note that $\dot{K}^1 > 0$, $\dot{K}^2 < 0$ and $\ddot{K} > 0$ wherever $g_1 > 0$.

Observe that properties (i) and (ii) are not harmed in the transition from g_1 to g . We now show that the estimates listed in properties (iii) and (iv) are satisfied by the curvature function defined in (3-3).

Proposition 3.4. *Let g be the pinching function defined by (3-3) and G its corresponding matrix function. Then, for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ (depending only on ε, n, f and Γ_0) such that*

$$(\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs})|_B T_{kpq} T_{lrs} \leq -c_\varepsilon \frac{|T|^2}{F}$$

for all $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G(B) \geq \varepsilon F(B)$ and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. First note that (suppressing dependence on B)

$$\begin{aligned} (\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs}) T_{kpq} T_{lrs} &= \dot{K}^\alpha (\dot{G}_\alpha^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}_\alpha^{pq,rs}) T_{kpq} T_{lrs} - \dot{F}^{kl} \ddot{K}^{\alpha\beta} \dot{G}_\alpha^{pq} \dot{G}_\beta^{rs} T_{kpq} T_{lrs} \\ &\leq \dot{K}^2 (\dot{G}_2^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}_2^{pq,rs}) T_{kpq} T_{lrs} \\ &\leq -\dot{K}^2 \dot{F}^{kl} \ddot{G}_2^{pq,rs} T_{kpq} T_{lrs}, \end{aligned}$$

where we used Lemma 3.1, convexity of K , and the inequalities $\dot{K}^1 \geq 0$ and $\dot{F} \geq 0$ in the first inequality, and the inequalities $\dot{G}_2 \geq 0$ and $\dot{K}^2 \leq 0$, and convexity of F in the second. Since $\dot{K}^2 < 0$ whenever $G_1 > 0$ and G_2 is strictly concave in nonradial directions, the claim follows exactly as in [Andrews et al. 2014b, Lemma 3.3]. \square

The uniform estimate of Proposition 3.4 yields a good bound for the term $Q(\nabla^q W, \nabla^q W)$ in the evolution equations for the pinching functions. This is a crucial component in obtaining the L^p -estimates of the following section. This is the starting point for the Stampacchia–de Giorgi iteration argument. The second crucial estimate is the Poincaré-type inequality, Lemma 4.2 (see also [Huisken and Sinestrari 2009, §§4–5; in particular, Lemma 5.5]), which we can obtain with the help of property (iv). This estimate (corresponding to [Huisken and Sinestrari 2009, Lemma 5.2]) provides an estimate on the zero order term that occurs in contracting the Simons-type identity for $\dot{F}^{pq} \nabla_p \nabla_q h_{ij}$ with \dot{G}^{ij} (see [Andrews et al. 2014b, Proposition 4.4]).

Proposition 3.5. *Let g be the pinching function defined by (3-3) and G its corresponding matrix function. Then for every $\delta > 0, \varepsilon > 0$, and $C > 0$ there exist $\gamma_1 > 0, \gamma_2 > 0$ and $\gamma_3 > 0$ (depending only on $\delta, \varepsilon > 0, C, n, m, f$ and Γ_0) such that*

$$Z(B) := (F \dot{G}^{kl} - G \dot{F}^{kl})|_B B_{kl}^2 \geq \gamma_1 F^2 (G - \delta \gamma_2 F)|_B - \gamma_3 F^2|_B$$

for all symmetric, $(m+1)$ -positive matrices B satisfying $\lambda(B) \in \Gamma_0, \lambda_{\min}(B) \geq -\delta F(B) - C$, and $G_m(B) \geq \varepsilon F(B)$.

Proof. From the definition of G we have

$$Z = \dot{K}^1 Z_1 + \dot{K}^2 Z_2,$$

where

$$Z_i(B) := (F \dot{G}_i^{kl} - G_i \dot{F}^{kl})|_B B_{kl}^2.$$

Thus, since $\dot{K}^2 = 2g_1/g_2$ is uniformly bounded below when $g \geq \varepsilon f$, it suffices to prove the estimate for Z_1 .

So let B be a symmetric, $(m+1)$ -positive matrix with eigenvalues $z_1 \leq \dots \leq z_n$. Then

$$\begin{aligned} Z_1(B) &= f \dot{g}_1^p z_p^2 - g_1 \dot{f}^p z_p^2 = \sum_{p>q} (\dot{g}_1^p \dot{f}^q - \dot{g}_1^q \dot{f}^p) z_p z_q (z_p - z_q) = \sum_{p>q} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) \\ &= \left(\sum_{p>q>l} + \sum_{p>l>q} + \sum_{l \geq p>q} \right) (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q), \end{aligned}$$

where we recall the notation $P_a := \sum_{\sigma \in O_a} \varphi'(r_\sigma)$ and we have defined $l \leq m$ as the number of nonpositive eigenvalues z_i . Recalling that $P_p \dot{f}^q - P_q \dot{f}^p \geq 0$ whenever $z_p \geq z_q$, we discard the final sum and part of the first to obtain

$$\begin{aligned} Z_1(B) &\geq \sum_{p=m+2}^n \sum_{q=l+1}^{m+1} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) + \sum_{p=l+1}^n \sum_{q=1}^l (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) \\ &= \sum_{p=m+2}^n \sum_{q=l+1}^{m+1} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) - f^2 \sum_{i=1}^l z_i \\ &\quad + f^2 \sum_{i=1}^l z_i + \sum_{p=l+1}^n \sum_{q=1}^l (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q). \end{aligned}$$

So consider the term

$$S_1(z) := \sum_{p=m+2}^n \sum_{q=l+1}^{m+1} (P_p(z) \dot{f}^q(z) - P_q(z) \dot{f}^p(z)) z_p z_q (z_p - z_q) - f(z)^2 \sum_{i=1}^l z_i.$$

Observe that $S_1 \geq 0$. We claim that $S_1(z) > 0$ for all z in the cone

$$\Gamma_{\varepsilon,l} := \{z \in \Gamma_0 : g(z) \geq \varepsilon f(z), z_1 \leq \dots \leq z_l \leq 0 < z_{l+1} \leq \dots \leq z_n\}.$$

Suppose, to the contrary, that $S_1(z) = 0$ for some $z \in \Gamma_{\varepsilon,l}$. Then $z_1 = \dots = z_l = 0$ and, for all $p > m+1 \geq q > l$, $(P_p(z) \dot{f}^q(z) - P_q(z) \dot{f}^p(z)) z_p z_q (z_p - z_q) = 0$. But, by Lemma 3.2, the latter implies that, for all $p > m+1 \geq q > l$, either $z_p = z_q$, or $r_\sigma(\lambda) \geq 0$ for all $\sigma \in O_{q,p}$. Note that the latter case cannot occur: since $p > m+1 \geq q$, there is a permutation $\sigma \in O_{q,p}$ such that $0 \leq r_\sigma(z) = (z_1 + \dots + z_{m+1} - c_m^{-1} f(z))/f(z)$, which implies $g_1(z) = 0$, contradicting $z \in \Gamma_{\varepsilon,l}$. On the other hand, if $z_p = z_q$ for all $p > m+1 \geq q > l$, then we again obtain the contradiction $g_1(z) = 0$. Thus, $S_1 > 0$ on $\Gamma_{\varepsilon,l}$. Since S_1 is homogeneous of degree three, it follows that

$$S_1 \geq c_1 f^2 g$$

on $\Gamma_{\varepsilon,l}$, where $c_1 := \min_l \min_{\Gamma_{\varepsilon,l}} \frac{S_1}{f^2 g} > 0$.

Now consider

$$S_2 := f^2 \sum_{i=1}^l \lambda_i + \sum_{p=l+1}^n \sum_{q=1}^l (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q).$$

Note that, by homogeneity, $c_2 := \sup\{P_p(z)\dot{f}^q(z) - P_q(z)\dot{f}^p(z) : z \in \Gamma_0, 1 \leq p, q \leq n\} < \infty$. Thus, S_2 is easily controlled using the “convexity estimate” $\lambda_1 \geq -\delta f - C$:

$$\begin{aligned} S_2 &\geq -lf^2(\delta f + C) + (n - l)c_2z_n \sum_{q=1}^l z_q(z_n - z_q) \geq -nf^2(\delta f + C) + 2nc_2c_3^2f^2 \sum_{q=1}^l z_q \\ &\geq -nf^2(\delta f + C) - 2nc_2c_3^2f^2(\delta F + C) \geq -n(1 + 2c_2c_3^2)f^2(\delta f + C), \end{aligned}$$

where $c_3 := \max\{|z_i|/f(z) : z \in \Gamma_0, 1 \leq i \leq n\}$.

The claim follows. □

We note that the above estimate is only useful in the presence of the convexity estimate Theorem 1.1, since then, for any $\delta > 0$, there is a constant $C_\delta > 0$ for which $\Gamma_{\delta, C_\delta} := \{z \in \Gamma_0 : z_i > -\delta f(z) - C_\delta \text{ for all } i\}$ is preserved by the flow.

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, it suffices to obtain, for any $\varepsilon > 0$, an upper bound on the function

$$G_{\varepsilon, \sigma} := \left(\frac{G}{F} - \varepsilon\right)F^\sigma$$

for some $\sigma > 0$. We will use the estimates of Propositions 3.5 and 3.4 to obtain bounds on the spacetime L^p -norms of the positive part of $G_{\varepsilon, \sigma}$, so long as p is sufficiently large and σ sufficiently small, just as in [Huisken and Sinestrari 1999b; 1999a; 2009] (see also [Andrews et al. 2014b] where these techniques are applied in the fully nonlinear setting). A Stampacchia–de Giorgi iteration procedure similar to that used in [Huisken 1984] (see also [Huisken and Sinestrari 1999b; Andrews et al. 2014b]) then allows us to extract a supremum bound on $G_{\varepsilon, \sigma}$.

We begin with an evolution equation for $G_{\varepsilon, \sigma}$:

Lemma 4.1 [Andrews et al. 2014b]. *The function $G_{\varepsilon, \sigma}$ satisfies the evolution equation*

$$\begin{aligned} (\partial_t - \mathcal{L})G_{\varepsilon, \sigma} &= F^{\sigma-1}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} \\ &\quad + \frac{2(1 - \sigma)}{F}\langle \nabla G_{\varepsilon, \sigma}, \nabla F \rangle_F - \frac{\sigma(1 - \sigma)}{F^2}|\nabla F|_F^2 + \sigma G_{\varepsilon, \sigma}|\mathcal{W}|_F^2, \end{aligned} \tag{4-1}$$

where $\langle u, v \rangle_F := \dot{F}^{kl}u_k v_l$.

Now set $E := \max\{G_{\varepsilon, \sigma}, 0\}$. We need to obtain spacetime L^p -estimates for E . Let us first observe that integration by parts and application of Young’s inequality, in conjunction with Lemma 2.3 and Proposition 3.4, yields the estimate (cf. [Andrews et al. 2014b])

$$\begin{aligned} \frac{d}{dt} \int E^p d\mu &\leq -(A_1 p(p - 1) - A_2 p^{\frac{3}{2}}) \int E^{p-2} |\nabla G_{\varepsilon, \sigma}|^2 d\mu \\ &\quad - (B_1 p - B_2 p^{\frac{1}{2}}) \int E^p \frac{|\nabla \mathcal{W}|^2}{F^2} d\mu + C_1 \sigma p \int E^p |\mathcal{W}|^2 d\mu \end{aligned} \tag{4-2}$$

for some positive constants A_1, A_2, B_1, B_2, C_1 (which depend only on ε, n, m, f and M_0).

To estimate the final term, we make use of Proposition 3.5 in a similar manner to [Huisken and Sinestrari 2009, §5]. We first observe:

Lemma 4.2. *There are positive constants $A_3, A_4, A_5, B_3, B_4, C_2$, independent of p and σ , such that*

$$\int E^p \frac{Z(W)}{F} d\mu \leq (A_3 p^{\frac{3}{2}} + A_4 p^{\frac{1}{2}} + A_5) \int E^{p-2} |\nabla G_{\varepsilon,\sigma}|^2 d\mu + (B_3 p^{\frac{1}{2}} + B_4) \int E^p \frac{|\nabla^q W|^2}{F^2} d\mu.$$

Proof. As in [Andrews et al. 2014b, §4], contraction of the commutation formula for $\nabla^2 W$ with \dot{F} and \dot{G} yields the identity

$$\begin{aligned} \mathcal{L}G_{\varepsilon,\sigma} = & -F^{\sigma-1} Q(\nabla^q W, \nabla^q W) + F^{\sigma-1} Z(W) + F^{\sigma-2} (F \dot{G}^{kl} - G \dot{F}^{kl}) \nabla_k \nabla_l F \\ & + \frac{\sigma}{F} G_{\varepsilon,\sigma} \mathcal{L}F - 2 \frac{(1-\sigma)}{F} \langle \nabla F, \nabla G_{\varepsilon,\sigma} \rangle_F + \frac{\sigma(1-\sigma)}{F^2} G_{\varepsilon,\sigma} |\nabla F|_F^2. \end{aligned}$$

The claim is now proved using integration by parts and Young’s inequality, with the help of Lemma 2.3 and Proposition 3.4 (see [Andrews et al. 2014b, Lemma 4.2]). □

Corollary 4.3. *For all $\varepsilon > 0$ there exist constants $\ell > 0$ and $L < \infty$ (depending only on ε, n, m, f and M_0) such that for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$ there is a constant $K = K_{\varepsilon,\sigma,p}$ (depending only on $\varepsilon, n, m, f, M_0, \sigma$ and p) for which the following estimate holds:*

$$\int (G_{\varepsilon,\sigma})_+^p d\mu \leq \int (G_{\varepsilon,\sigma}(\cdot, 0))_+^p d\mu_0 + tK\mu_0(M),$$

where μ_0 is the measure induced on M by the initial immersion.

Proof. Recall Proposition 3.5. Setting $\delta = \varepsilon/(2\gamma_2)$ and applying the convexity estimate, we obtain

$$\frac{Z(W)}{F} \geq \frac{\varepsilon}{2} \gamma_1 F^2 - \gamma_3 C_{\varepsilon/(2\gamma_2)} F \tag{4-3}$$

whenever $G - \varepsilon F > 0$. We now use Young’s inequality to obtain (cf. [Huisken and Sinestrari 2009, §5])

$$F = F^{-\sigma p} F^{1+\sigma p} \leq F^{-\sigma p} \left(\frac{b^q}{q} F^{q(1+\sigma p)} + \frac{b^{-q'}}{q'} \right)$$

for any $b > 0$ and $q > 0$, where q' is the Hölder conjugate of q : $\frac{1}{q} + \frac{1}{q'} = 1$. Choosing $q = \frac{2+\sigma p}{1+\sigma p}$, so that $q' = 2 + \sigma p$, we obtain

$$F \leq b^{(2+\sigma p)/(1+\sigma p)} \frac{1+\sigma p}{2+\sigma p} F^2 + \frac{b^{-(2+\sigma p)}}{2+\sigma p} F^{-\sigma p} \leq b^{(2+\sigma p)/(1+\sigma p)} F^2 + b^{-(2+\sigma p)} F^{-\sigma p}.$$

Now choose $b := \left(\frac{\varepsilon \gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}} \right)^{\frac{1+\sigma p}{2+\sigma p}}$, so that

$$\gamma_3 C_{\varepsilon/(2\gamma_2)} F \leq \frac{\varepsilon \gamma_1}{4} F^2 + K F^{-\sigma p},$$

where

$$K := \gamma_3 C_{\varepsilon/(2\gamma_2)} \left(\frac{\varepsilon \gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}} \right)^{-(1+\sigma p)}.$$

Returning to Equation (4-3), we find

$$\frac{\varepsilon\gamma_1}{4}F^2 \leq KF^{-\sigma p} + \frac{Z({}^qW)}{F}.$$

Estimating $G_{\varepsilon,\sigma} \leq c_1 F^\sigma$ and $|{}^qW|^2 \leq c_2 F^2$, we obtain

$$E^p |{}^qW|^2 \leq \tilde{K} + c_3 E^p \frac{Z({}^qW)}{F}$$

for some constants $\tilde{K} > 0$ (depending on F , M_0 , ε , σ and p) and $c_3 > 0$ (depending on F , M_0 , and ε).

Combining Lemma 4.2 and inequality (4-2) now yields

$$\begin{aligned} \frac{d}{dt} \int E^p d\mu &\leq K_{\varepsilon,\sigma,p} \mu_0(M) - (\alpha_0 p^2 - \alpha_1 \sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p) \int E^{p-2} |G_{\varepsilon,\sigma}|^2 d\mu \\ &\quad - (\beta_0 p - \beta_1 \sigma p^{\frac{3}{2}} - \beta_2 \sigma p - \beta_3 p^{\frac{1}{2}}) \int E^p \frac{|\nabla {}^qW|^2}{F^2} d\mu \end{aligned}$$

for some positive constants α_i and β_i , which depend on ε but not on σ or p , and $K_{\varepsilon,\sigma,p}$, which depends on ε , σ and p .

It is clear that $L > 0$ and $\ell > 0$ may be chosen such that

$$(\alpha_0 p^2 - \alpha_1 \sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p) \geq 0 \quad \text{and} \quad (\beta_0 p - \beta_1 \sigma p^{\frac{3}{2}} - \beta_2 \sigma p - \beta_3 p^{\frac{1}{2}}) \geq 0$$

for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$. The claim then follows by integrating with respect to the time variable. \square

The proof of Theorem 1.3 is completed by proceeding with Huisken's Stampacchia-de Giorgi iteration argument. We omit these details as the arguments required already appear in [Andrews et al. 2014b, §5] with no significant changes necessary.

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