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CYLINDRICAL ESTIMATES FOR HYPERSURFACES MOVING BY CONVEX CURVATURE FUNCTIONS

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We prove a complete family of *cylindrical estimates* for solutions of a class of fully nonlinear curvature flows, generalising the cylindrical estimate of Huisken and Sinestrari [*Invent. Math.* **175**:1 (2009), 1–14, §5] for the mean curvature flow. More precisely, we show, for the class of flows considered, that, at points where the curvature is becoming large, an $(m+1)$-convex $(0 \le m \le n-2)$ solution either becomes strictly m-convex or its Weingarten map becomes that of a cylinder $\mathbb{R}^m \times S^{n-m}$. This result complements the convexity estimate we proved with McCoy [*Anal. PDE* **7**:2 (2014), 407–433] for the same class of flows.

1. Introduction

Let M be a smooth, closed manifold of dimension n, and $X_0 : M \to \mathbb{R}^{n+1}$ a smooth hypersurface immersion. We are interested in smooth families $X : M \times [0, T) \to \mathbb{R}^{n+1}$ of smooth immersions $X(\cdot\,, t)$ solving the initial value problem

$$\begin{cases} \partial_t X(x,t) = -F(\mathcal{W}(x,t))\nu(x,t), \\ X(\cdot\,,0) = X_0, \end{cases} \tag{CF}$$

where ν is the outer normal field of the evolving hypersurface X and $\mathcal{W}$ the corresponding Weingarten curvature. In order that the problem (CF) be well-posed, we require that $F(\mathcal{W})$ be given by a smooth, symmetric function $f : \Gamma \to \mathbb{R}$ of the principal curvatures κ_i which is monotone increasing in each argument. The symmetry of f ensures that F is a smooth, basis-invariant function of the components of the Weingarten map (or an orthonormal frame-invariant function of the components of the second fundamental form) [Glaeser 1963]. Monotonicity ensures that the flow is (weakly) parabolic. This guarantees local existence of solutions of (CF), as long as the principal curvature n-tuple of the initial data lies in Γ; see [Langford 2014].

For technical reasons, we require some additional conditions:

Conditions. (i) f is homogeneous of degree one.

(ii) f is convex.

Since the normal points out of the region enclosed by the solution, we may assume, by condition (ii), that $(1, \dots, 1) \in \Gamma$. Thus, by condition (i), we may further assume that f is normalised such that $f(1, \dots, 1) = 1$.

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The additional conditions (i)–(ii) have several consequences. Most importantly, they allow us to obtain a preserved cone $\Gamma_0 \subset \Gamma$ of curvatures for the flow (Lemma 2.2). This allows us to obtain uniform estimates on any degree-zero homogeneous function of curvature along the flow (Lemma 2.3); in particular, we deduce a uniform parabolicity condition (Corollary 2.4). The convexity condition then allows us to apply the second derivative Hölder estimate of [Evans 1982; Krylov 1982] to deduce that the solution exists on a maximal time interval $[0, T)$, $T < \infty$, such that $\max_{M\times\{t\}} F \to \infty$ as $t \to T$; see [Andrews et al. 2014a, Proposition 2.6]. Thus, it is of interest to study the behaviour of solutions as $F \to \infty$. Let us recall the following curvature estimate [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]).

Theorem 1.1 (convexity estimate). *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon < \infty$ such that*

$$G(x, t) \le \varepsilon F(x, t) + C_\varepsilon \quad \text{for all } (x, t) \in M \times [0, T),$$

where G is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures of the evolving hypersurface that vanishes at a point (x, t) if and only if $\mathcal{W}_{(x,t)} \ge 0$.

We remark that the constant C_ε depends only on ε, the dimension n, the choice of speed function f, the preserved curvature cone Γ_0, and bounds for the initial volume and diameter [Langford 2014].

Theorem 1.1 implies that the ratio of the smallest principal curvature to the speed is almost positive wherever the curvature is large. Combining it with the differential Harnack inequality of [Andrews 1994b] and the strong maximum principle [Hamilton 1986] yields useful information about the geometry of solutions of (CF) near singularities [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]):

Corollary 1.2. *Any blow-up limit of a solution of (CF) is weakly convex. In particular, any type-II blow-up limit about a type-II singularity is an eternal solution of the form $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \to \mathbb{R}^{n+1}$, $k \in \{0, 1, \dots, n-1\}$, such that $X_\infty|_{\mathbb{R}^k}$ is flat, and $X_\infty|_{\Gamma^{n-k}}$ is a strictly convex translation solution of the corresponding flow in $\mathbb{R}^{n-k+1}$.*

Motivated by the surgery construction of [Huisken and Sinestrari 2009, §5] for 2-convex mean curvature flow, we will apply Theorem 1.1 to obtain the following family of cylindrical estimates for solutions of (CF):

Theorem 1.3 (cylindrical estimate). *Let X be a solution of (CF) such that conditions (i)–(ii) hold. Suppose also that X is uniformly $(m+1)$-convex for some $m \in \{0, 1, \dots, n-2\}$. That is, $\kappa_1 + \cdots + \kappa_{m+1} \ge \beta F$ for some $\beta > 0$. Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that*

$$G_m(x, t) \le \varepsilon F(x, t) + C_\varepsilon \quad \text{for all } (x, t) \in M \times [0, T),$$

where $G_m : M \times [0, T) \to \mathbb{R}$ is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures that vanishes at a point (x, t) if and only if

$$\kappa_1(x, t) + \cdots + \kappa_{m+1}(x, t) \ge \frac{1}{c_m} f(\kappa_1(x, t), \dots, \kappa_n(x, t)),$$

where c_m is the value F takes on the unit radius cylinder $\mathbb{R}^m \times S^{n-m}$.

We note that the constant C_ε will only depend on ε, β, m, the dimension n, the choice of speed function f, the preserved curvature cone Γ_0, and upper bounds for the initial volume and diameter. Theorem 1.3 implies that the ratio of the quantity

$$K_m := \kappa_1 + \cdots + \kappa_{m+1} - \frac{1}{c_m} F$$

to the speed is almost positive wherever the curvature is large. Observe that this quantity is nonnegative on a weakly convex hypersurface Σ only if either Σ is strictly m-convex or $\Sigma = \mathbb{R}^m \times S^{n-m}$. In particular, we find that, whenever $\kappa_1(x,t) + \cdots + \kappa_m(x,t)$ is small compared to the speed, the Weingarten curvature is close to that of a thin, round cylinder $\mathbb{R}^m \times S^{n-m}$. We therefore obtain a refinement of Corollary 1.2:

Corollary 1.4. *Any blow-up limit of an $(m+1)$-convex, $0 \le m \le n-2$, solution of (CF) is either strictly m-convex, or a shrinking cylinder $\mathbb{R}^m \times S^{n-m}$. In particular, if the blow-up is of type-II, then this limit is of the form $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \to \mathbb{R}^{n+1}$ for $k \in \{0, 1, \dots, m-1\}$, such that $X_\infty|_{\mathbb{R}^k}$ is flat and $X_\infty|_{\Gamma^{n-k}}$ is a strictly convex translation solution of the corresponding flow in $\mathbb{R}^{n-k+1}$.*

The $m = 0$ case of the cylindrical estimates demonstrates that convex hypersurfaces become umbilic at points where the curvature is blowing up, generalising a result of Huisken [1984, Theorem 5.1] for the mean curvature flow (we note that the convergence result of [Huisken 1984] has been obtained by the first author for the class of flows considered here without the need for such an estimate [Andrews 1994a]). Moreover, Huisken and Sinestrari [2009] have recently obtained the $m = 1$ case of the cylindrical estimates for the mean curvature flow, making spectacular use of it through their surgery program, which yields a classification of 2-convex hypersurfaces. The convexity and cylindrical estimates stated above, in addition to generalising the Huisken–Sinestrari cylindrical estimate to all m in $\{0, \dots, n-2\}$, constitute a first step towards improving upon such results by allowing a larger class of evolution equations.

2. Preliminaries

We will follow the notation used in [Andrews et al. 2014b]. In particular, we recall that a smooth, symmetric function g of the principal curvatures gives rise to a smooth function G of the components h_i^j of the Weingarten map. Equivalently, G is an orthonormal frame invariant function of the components h_{ij} of the second fundamental form. To simplify notation, we denote $G(x,t) \equiv G(\mathcal{W}(x,t)) = g(\kappa(x,t))$ and use dots to denote derivatives of functions of curvature as follows:

$$\dot{g}^k(z) v_k = \frac{d}{ds}\Big|_{s=0} g(z+sv), \qquad \dot{G}^{kl}(A) B_{kl} = \frac{d}{ds}\Big|_{s=0} G(A+sB),$$

$$\ddot{g}^{pq}(z) v_p v_q = \frac{d^2}{ds^2}\Big|_{s=0} g(z+sv), \quad \ddot{G}^{pq,rs}(A) B_{pq} B_{rs} = \frac{d^2}{ds^2}\Big|_{s=0} G(A+sB).$$

The derivatives of g and G are related in the following way:

Lemma 2.1 [Gerhardt 1996; Andrews 1994a; 2007]. *Let $g : \Gamma \to \mathbb{R}$ be a smooth, symmetric function. Define the function $G : \mathcal{S}_\Gamma :\to \mathbb{R}$ by $G(A) := g(\lambda(A))$, where $\lambda(A)$ denotes the eigenvalues of A (up to order) and $\mathcal{S}_\Gamma$ denotes the set of symmetric matrices with eigenvalues in Γ. Then, for any diagonal $A \in \mathcal{S}_\Gamma$,*

$$\dot{G}^{kl}(A) = \dot{g}^k(\lambda(A))\delta^{kl}, \tag{2-1}$$

and, for any diagonal $A \in \mathcal{S}_\Gamma$ with distinct eigenvalues and any symmetric $B \in \mathrm{GL}(n)$,

$$\ddot{G}^{pq,rs}(A)B_{pq}B_{rs} = \ddot{g}^{pq}(\lambda(A))B_{pp}B_{qq} + 2\sum_{p>q} \frac{\dot{g}^p(\lambda(A)) - \dot{g}^q(\lambda(A))}{\lambda_p(A) - \lambda_q(A)}(B_{pq})^2. \tag{2-2}$$

We note that $\ddot{g} \ge 0$ if and only if $(\dot{g}^p - \dot{g}^q)(z_p - z_q) \ge 0$ for all p, q [Andrews et al. 2014b, Lemma 2.2], so Lemma 2.1 implies that G is convex if and only if g is convex.

The following useful lemma was proved in [Andrews et al. 2014b]:

Lemma 2.2. *Let $f : \Gamma \to \mathbb{R}$ be a flow speed for (CF) satisfying Conditions (i)–(ii). Then, for any admissible initial datum $X_0 : M \to \mathbb{R}^{n+1}$ there exists a cone $\Gamma_0 \subset \mathbb{R}^n$ satisfying $\overline{\Gamma}_0 \setminus \{0\} \subset \Gamma$ such that the principal curvatures of the solution $X : M \times [0, T) \to \mathbb{R}^{n+1}$ of the initial value problem (CF) satisfy $\kappa(x, t) := (\kappa_1(x, t), \dots, \kappa_n(x, t)) \in \Gamma_0$ for all $(x, t) \in M \times [0, T)$.*

We refer to such a cone Γ_0 as a *preserved cone* for the solution X. As mentioned in the introduction, the existence of a preserved cone allows us to obtain bounds for homogeneous functions of the curvature:

Lemma 2.3. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Let $g : \Gamma \to \mathbb{R}$ be a smooth, degree-zero homogeneous symmetric function. Then there exists $c > 0$ (depending only on n, f and M_0) such that*

$$-c \le g(\kappa_1(x, t), \dots, \kappa_n(x, t)) \le c \quad \textit{for all } (x, t) \in M \times [0, T).$$

If $g > 0$, then there exists $c > 0$ such that

$$\frac{1}{c} \le g(\kappa_1(x, t), \dots, \kappa_n(x, t)) \le c.$$

Proof. Let Γ_0 be a preserved cone for the solution X. Then $K := \overline{\Gamma}_0 \cap S^n$ is compact. Since g is continuous, the required bounds hold on K. But these extend to $\overline{\Gamma}_0 \setminus \{0\}$ by homogeneity. The claim follows since $\kappa(x, t) \in \overline{\Gamma}_0 \setminus \{0\}$ for all $(x, t) \in M \times [0, T)$. □

By condition (i), the derivative $\dot{f}$ of f is homogeneous of degree zero. Since $\dot{f}^k > 0$ for each k, we obtain uniform parabolicity of the flow:

Corollary 2.4. *There exists a constant $c > 0$ (depending only on n, f and M_0) such that, for any $v \in T^*M$, it holds that*

$$\frac{1}{c}|v|^2 \le \dot{F}^{ij} v_i v_j \le c|v|^2,$$

where $|\cdot|$ is the (time-dependent) norm on M corresponding to the (time-dependent) metric induced by the flow.

We now recall the following evolution equation (see for example [Andrews et al. 2013]).

Lemma 2.5. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Let $G : M \times [0, T) \to \mathbb{R}$ be given by a smooth, symmetric, degree-one homogeneous function g of the principal curvatures. Then G satisfies the evolution equation*

$$(\partial_t - \mathcal{L})G = (\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla h_{pq}\nabla h_{rs} + G|\mathcal{W}|_F^2, \tag{2-3}$$

where $\mathcal{L} := \dot{F}^{kl}\nabla_k\nabla_l$ is the linearisation of F, and $|\mathcal{W}|_F^2 := \dot{F}^{kl}h_k{}^r h_{rl}$.

In particular, the speed function F satisfies $(\partial_t - \mathcal{L})F = F|\mathcal{W}|_F^2$.

As we shall see, in order to obtain Theorem 1.3, it is crucial to obtain a good upper bound on the term

$$Q(\nabla\mathcal{W}, \nabla\mathcal{W}) := (\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}$$

for the pinching functions G_m which we construct in the following section. The following decomposition of Q is crucial in obtaining this bound.

Lemma 2.6. *For any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, we have*

$$\begin{aligned}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\big|_B T_{kpq}T_{lrs} &= (\dot{g}^k\ddot{f}^{pq} - \dot{f}^k\ddot{g}^{pq})\big|_z T_{kpp}T_{kqq}\\ &\quad + 2\sum_{p>q}\frac{(\dot{f}^p\dot{g}^q - \dot{g}^p\dot{f}^q)\big|_z}{z_p - z_q}((T_{pqq})^2 + (T_{qpp})^2) + 2\sum_{k>p>q}(\vec{g}_{kpq}\times\vec{f}_{kpq})\big|_z\cdot\vec{z}_{kpq}(T_{kpq})^2\end{aligned} \tag{2-4}$$

at any diagonal matrix B with distinct eigenvalues z_i, where "$\times$" and "$\cdot$" are the three-dimensional cross and dot product respectively, and we have defined the vectors

$$\begin{aligned}\vec{f}_{kpq} &:= (\dot{f}^k, \dot{f}^p, \dot{f}^q),\\ \vec{g}_{kpq} &:= (\dot{g}^k, \dot{g}^p, \dot{g}^q),\\ \vec{z}_{kpq} &:= \left(\frac{z_p - z_q}{(z_k - z_p)(z_k - z_q)}, \frac{z_k - z_q}{(z_k - z_p)(z_p - z_q)}, \frac{z_k - z_p}{(z_p - z_q)(z_k - z_q)}\right).\end{aligned}$$

Proof. Since B is diagonal, Lemma 2.1 yields (suppressing the dependence on B)

$$\begin{aligned}&(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs}\\ &\qquad = \sum_{k,p,q}(\dot{g}^k\ddot{f}^{pq} - \dot{f}^k\ddot{g}^{pq})T_{kpp}T_{kqq} + 2\sum_k\sum_{p>q}\left(\dot{g}^k\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{kpq})^2.\end{aligned}$$

We now decompose the second term into the terms satisfying $k = p$, $k = q$, $k > p$, $p > k > q$, and $q > k$ respectively:

$$\begin{aligned}&\sum_k\sum_{p>q}\left(\dot{g}^k\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{kpq})^2\\ &\quad = \sum_{p>q}\left(\dot{g}^p\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^p\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{ppq})^2 + \sum_{p>q}\left(\dot{g}^q\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^q\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{qpq})^2\\ &\qquad + \left(\sum_{k>p>q} + \sum_{p>k>q} + \sum_{p>q>k}\right)\left(\dot{g}^k\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{kpq})^2\end{aligned}$$

$$
\begin{aligned}
&=\sum_{p>q}\frac{\dot f^p\dot g^q-\dot g^p\dot f^q}{z_p-z_q}((T_{pqq})^2+(T_{qpp})^2)\\
&\quad+\sum_{k>p>q}\left(\dot g^k\frac{\dot f^p-\dot f^q}{z_p-z_q}-\dot f^k\frac{\dot g^p-\dot g^q}{z_p-z_q}+\dot g^p\frac{\dot f^k-\dot f^q}{z_k-z_q}-\dot f^p\frac{\dot g^k-\dot g^q}{z_k-z_q}+\dot g^q\frac{\dot f^k-\dot f^p}{z_k-z_p}-\dot f^q\frac{\dot g^k-\dot g^p}{z_k-z_p}\right)(T_{kpq})^2\\
&=\sum_{p>q}\frac{\dot f^p\dot g^q-\dot g^p\dot f^q}{z_p-z_q}((T_{pqq})^2+(T_{qpp})^2)+\sum_{k>p>q}\Bigg[(\dot g^p\dot f^q-\dot f^q\dot g^p)\left(\frac{1}{z_k-z_p}-\frac{1}{z_k-z_q}\right)\\
&\qquad-(\dot g^k\dot f^q-\dot f^k\dot g^q)\left(\frac{1}{z_p-z_q}+\frac{1}{z_k-z_p}\right)+(\dot g^k\dot f^p-\dot f^k\dot g^p)\left(\frac{1}{z_p-z_q}-\frac{1}{z_k-z_q}\right)\Bigg](T_{kpq})^2\\
&=\sum_{p>q}\frac{\dot f^p\dot g^q-\dot g^p\dot f^q}{z_p-z_q}((T_{pqq})^2+(T_{qpp})^2)+\sum_{k>p>q}(\vec g_{kpq}\times\vec f_{kpq})\cdot\vec z_{kpq}(T_{kpq})^2. \qquad\square
\end{aligned}
$$

We complete this section by proving that $(m+1)$-convexity is preserved by the flow (CF), so that this assumption need only be made on initial data:

Proposition 2.7. *Let X be a solution of (CF) such that conditions (i)–(ii) are satisfied. Suppose that there is some $m \in \{1, \dots, n-1\}$ and some $\beta > 0$ such that*

$$\kappa_{\sigma(1)}(x,0)+\cdots+\kappa_{\sigma(m)}(x,0)\geq \beta F(x,0)$$

for all $x \in M$ and all permutations $\sigma \in P_n$. Then this estimate persists at all later times.

Proof. Denote by SM the unit tangent bundle over $M\times[0,T)$ and consider the function Z defined on $\bigoplus^m SM$ by

$$Z(x,t,\xi_1,\dots,\xi_m)=\sum_{\alpha=1}^m h(\xi_\alpha,\xi_\alpha)-\beta F(x,t).$$

Since we have

$$\inf_{\xi_1,\dots,\xi_m\in S_{(x,t)}M} Z(x,t,\xi_1,\dots,\xi_m)=\kappa_{\sigma(1)}(x,t)+\cdots+\kappa_{\sigma(m)}(x,t)-\beta F(x,t)$$

for some $\sigma\in P_n$, it suffices to show that Z remains nonnegative. First fix any $t_1\in[0,T)$ and consider the function $Z_\varepsilon(x,t,\xi_1,\dots,\xi_m):=Z(x,t,\xi_1,\dots,\xi_m)+\varepsilon e^{(1+C)t}$, where $C:=\sup_{M\times[0,t_1]}|{}^{\circ}\mathcal{W}|_F^2$. Note that C is finite since M is compact and $\dot F$ is bounded. Observe that Z_ε is positive when $t=0$. We will show that Z_ε remains positive on $M\times[0,t_1]$ for all $\varepsilon>0$. So suppose to the contrary that Z_ε vanishes at some point $(x_0,t_0,\xi_1^0,\dots,\xi_m^0)$. We may assume that t_0 is the first such time. Now extend the vector $\xi^0:=(\xi_1^0,\dots,\xi_m^0)$ to a field $\xi:=(\xi_1,\dots,\xi_n)$ near (x_0,t_0) by parallel translation in space and solving

$$\frac{\partial\xi_\alpha^i}{\partial t}=F\xi_\alpha^j h_j{}^i.$$

Since the metric evolves according to

$$\partial_t g_{ij}=-2Fh_{ij}$$

the resulting fields have unit length. Now recall (see for example [Andrews 1994a]) the following evolution equation for the second fundamental form:

$$\partial_t h_{ij} = \mathcal{L} h_{ij} + \ddot{F}^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs} + |\mathcal{W}|_F^2 h_{ij} - 2F h_{ij}^2,$$

where $\mathcal{L} := \dot{F}^{kl}\nabla_k\nabla_l$ and $|\mathcal{W}|_F^2 := \dot{F}^{kl}h_{kl}^2$. It follows that

$$\begin{aligned}(\partial_t - \mathcal{L})(Z_\varepsilon(x,t,\xi)) &= \varepsilon(1+C)\mathrm{e}^{(1+C)t} + \sum_{\alpha=1}^{m} \ddot{F}^{pq,rs}\nabla_{\xi_\alpha} h_{pq}\nabla_{\xi_\alpha} h_{rs} + |\mathcal{W}(x,t)|_F^2 Z(x,t,\xi)\\ &\geq \varepsilon(1+C)\mathrm{e}^{(1+C)t} + |\mathcal{W}(x,t)|_F^2 Z(x,t,\xi).\end{aligned}$$

Since the point $(x_0, t_0, \xi_{t=t_0})$ is a minimum of Z_ε, we obtain

$$0 \geq (\partial_t - \mathcal{L})\big|_{(x_0,t_0)}(Z_\varepsilon(x,t,\xi)) \geq \varepsilon(1+C)\mathrm{e}^{(1+C)t_0} - C\varepsilon\mathrm{e}^{(1+C)t_0} = \varepsilon\mathrm{e}^{(1+C)t_0} > 0\,.$$

This is a contradiction, implying that Z_ε cannot vanish at any time in the interval $[0, t_1]$. Since $\varepsilon > 0$ was arbitrary, we find $Z \geq 0$ at all times in the interval $[0, t_1]$. Since $t_1 \in [0, T)$ was arbitrary, we obtain $Z \geq 0$. □

3. Constructing the pinching function

In this section we construct the pinching functions G_m satisfying the conditions in Theorem 1.3. Let us first introduce the *pinching cones*

$$\Gamma_m := \{z \in \Gamma : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} > c_m^{-1} f(z) \text{ for all } \sigma \in H_m\},$$

where H_m is the quotient of P_n, the group of permutations of the set $\{1, \ldots, n\}$, by the equivalence relation

$$\sigma \sim \omega \quad \text{if} \quad \sigma(\{1, \ldots, m+1\}) = \omega(\{1, \ldots, m+1\}).$$

Using the methods of [Huisken 1984], and their adaptations to 2-convex flows in [Huisken and Sinestrari 2009] and fully nonlinear flows in [Andrews et al. 2014b], we will see that, in order to prove Theorem 1.3, it suffices to construct a smooth function $g_m : \Gamma \to \mathbb{R}$ satisfying the following properties.

Properties. (i) $g_m(z) \geq 0$ for all $z \in \Gamma$ with equality if and only if $z \in \overline{\Gamma}_m \cap \Gamma$.

(ii) g_m is smooth and homogeneous of degree one.

(iii) For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$(\dot{G}_m^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_m^{pq,rs})\big|_B T_{kpq}T_{lrs} \leq -c_\varepsilon \frac{|T|^2}{F}$$

for all $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G_m(B) \geq \varepsilon F(B)$ and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, where G_m is the matrix function corresponding to g_m as described in Section 2, and Γ_0 is a preserved cone for the flow.

(iv) For every $\delta > 0$, $\varepsilon > 0$, and $C > 0$, there exist $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$(G_m \dot{F}^{kl} - F\dot{G}_m^{kl})\big|_B B_{kl}^2 \leq -\gamma_1 F^2 (G_m - \delta\gamma_2 F)\big|_B + \gamma_3 C F^2\big|_B$$

for all $(m+1)$-positive $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G_m(B) \geq \varepsilon F(B)$ and

$$\lambda_{\min}(B) \geq -\delta F(B) - C.$$

Our construction of the pinching function g_m will be similar for each choice of m. So let us fix $m \in \{0, 1, \ldots, n-2\}$ and assume that the flow is $(m+1)$-convex. We first consider the preliminary function $g : \Gamma \to \mathbb{R}$ defined by

$$g(z) := f(z) \sum_{\sigma \in H_m} \varphi\left(\frac{\sum_{i=1}^{m+1} z_{\sigma(i)} - c_m^{-1} f(z)}{f(z)}\right), \tag{3-1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth[1] function which is strictly convex and positive, except on $\mathbb{R}_+ \cup \{0\}$ where it vanishes identically. Such a function is readily constructed; for example, we could take

$$\varphi(r) = \begin{cases} r^4 e^{-1/r^2} & \text{if } r < 0, \\ 0 & \text{if } r \geq 0. \end{cases}$$

We note that such a function necessarily satisfies $\varphi(r) - r\varphi'(r) \leq 0$ and $\varphi'(r) \leq 0$ with equality if and only if $r \geq 0$.

Now define the scalar $G : M \times [0, T) \to \mathbb{R}$ by

$$G(x, t) := g(\kappa_1(x, t), \ldots, \kappa_n(x, t)).$$

Then G is a smooth, degree-one homogeneous function of the components of the Weingarten map which is invariant under a change of basis. Moreover, G is nonnegative and vanishes at, and only at, points for which the sum of the smallest $(m+1)$-principal curvatures is not less than $c_m^{-1} F$. Thus properties (i) and (ii) are satisfied by g.

We now show that property (iii) is satisfied weakly by g:

Lemma 3.1. *Let G be the matrix function corresponding to the function g defined by (3-1). Then, for any symmetric matrix B and totally symmetric 3-tensor T,*

$$(\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs})\big|_B T_{kpq} T_{lrs} \leq 0.$$

Proof. We will show that each of the terms in the decomposition (2-4) in Lemma 2.6 is nonpositive. Note that, by the invarance properties of G and F, it suffices to prove the claim for diagonal B. In fact, we can also assume that B has distinct eigenvalues, since the result at an arbitrary diagonal matrix B may then be

[1] In fact, φ need only be twice continuously differentiable.

obtained by taking a limit $B^{(k)} \to B$ such that each matrix $B^{(k)}$ has distinct eigenvalues. We first compute

$$\begin{aligned}\dot{g}^k &= \dot{f}^k \sum_{\sigma\in H_m} \varphi(r_\sigma) + \sum_{\sigma\in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1}\left(\delta^k_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^k\right)\\ &= \dot{f}^k \sum_{\sigma\in H_m}\left(\varphi(r_\sigma) - \varphi'(r_\sigma)\frac{\sum_{i=1}^{m+1} z_{\sigma(i)}}{f}\right) + \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\delta^k_{\sigma(i)}\end{aligned}$$

and

$$\begin{aligned}\ddot{g}^{pq} = &\left(\sum_{\sigma\in H_m}\varphi(r_\sigma) - \sum_{\sigma\in H_m}\varphi'(r_\sigma)\frac{\sum_{i=1}^{m+1} z_{\sigma(i)}}{f}\right)\ddot{f}^{pq}\\ &+ \sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^p\right)\sum_{i=1}^{m+1}\left(\delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^q\right),\end{aligned}$$

where we have set

$$r_\sigma(z) := \frac{\sum_{i=1}^{m+1} z_{\sigma(i)} - c_m^{-1} f(z)}{f(z)}.$$

It follows that

$$\dot{g}^k\ddot{f}^{pq} - \dot{f}^k\ddot{g}^{pq} = \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\delta^k_{\sigma(i)}\ddot{f}^{pq} - \dot{f}^k\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^p\right)\sum_{i=1}^{m+1}\left(\delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^q\right).$$

If we fix the index k and set $\xi_p = T_{kpp}$, then, by convexity of φ and positivity of $\dot{f}^k$, we have

$$\begin{aligned}-\dot{f}^k &\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^p\right)\sum_{i=1}^{m+1}\left(\delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^q\right)\xi_p\xi_q\\ &= -\dot{f}^k\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\left(\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot{f}^p\right)\xi_p\right)^2 \le 0.\end{aligned}$$

On the other hand, since φ is monotone nonincreasing, and f is convex, we have

$$\varphi'(r_\sigma)\sum_{i=1}^{m+1}\delta^k_{\sigma(i)}\ddot{f}^{pq}\xi_p\xi_q \le 0$$

for each σ. Since both inequalities hold for all k, we deduce that

$$\sum_{k,p,q}\left(\dot{g}^k\ddot{f}^{pq} - \dot{f}^k\ddot{g}^{pq}\right)T_{kpp}T_{kqq} \le 0.$$

We next consider

$$\dot{f}^p\dot{g}^q - \dot{g}^p\dot{f}^q = \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)(\delta^q_{\sigma(i)}\dot{f}^p - \delta^p_{\sigma(i)}\dot{f}^q) = \sum_{\sigma\in O_q}\varphi'(r_\sigma)\dot{f}^p - \sum_{\sigma\in O_p}\varphi'(r_\sigma)\dot{f}^q.$$

where we have introduced the sets

$$O_a := \big\{\sigma \in H_m : a \in \sigma(\{1, \dots, m+1\})\big\}.$$

If $z_p > z_q$, we obtain

$$\dot f^p \dot g^q - \dot g^p \dot f^q \le \dot f^p \Bigg(\sum_{\sigma \in O_q} \varphi'(r_\sigma) - \sum_{\sigma \in O_p} \varphi'(r_\sigma) \Bigg),$$

We now show that the term in brackets is nonpositive whenever $z_p > z_q$.

Lemma 3.2. *If $z_p \ge z_q$, then*

$$\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) \ge 0\,.$$

Moreover, equality holds only if either $z_p = z_q$ or $r_\sigma(z) \ge 0$ for all $\sigma \in O_{q,p} := O_q \setminus O_p$.

Proof of Lemma 3.2. First note that

$$\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) = \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma)\,,$$

where $O_{a,b} := O_a \setminus O_b$. Next observe that, if $\sigma \in O_{p,q}$, then

$$z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} = z_p + z_{\hat\sigma(i_1)} + \cdots + z_{\hat\sigma(i_m)} \tag{3-2}$$

for some $\hat\sigma \in H_{m-2}(p,q) := P_{n-2}(p,q)/\sim$, where $P_{n-2}(p,q)$ denotes the set of permutations of $\{1, \dots, n\} \setminus \{p, q\}$; $i_1, \dots, i_m$ are m distinct elements of $\{1, \dots, n\} \setminus \{p, q\}$; and $\sim$ is defined by

$$\hat\sigma \sim \hat\omega \quad \text{if} \quad \hat\sigma(\{i_1, \dots, i_m\}) = \hat\omega(\{i_1, \dots, i_m\}).$$

Observe also that the converse holds (that is, (3-2) defines a bijection), so that

$$\sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) = \sum_{\hat\sigma \in H_{m-2}(p,q)} \Bigg(\varphi'\bigg(\frac{z_p + \sum_{k=1}^m z_{\hat\sigma(i_k)} - c_m^{-1} f}{f} \bigg) - \varphi'\bigg(\frac{z_q + \sum_{k=1}^m z_{\hat\sigma(i_k)} - c_m^{-1} f}{f} \bigg) \Bigg).$$

Since $z_p \ge z_q$, the claim follows from (strict) convexity of φ (where it is positive). □

Thus,

$$\sum_{p>q} \frac{\dot f^p \dot g^q - \dot g^p \dot f^q}{z_p - z_q} \big((T_{pqq})^2 + (T_{qpp})^2 \big) \le 0.$$

We now compute

$$\vec g_{kpq} = \bigg(\frac{g}{f} - \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \frac{z_{\sigma(i)}}{f} \bigg) \vec f_{kpq} + \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} (\delta^k_{\sigma(i)}, \delta^p_{\sigma(i)}, \delta^q_{\sigma(i)}),$$

so that

$$
\begin{aligned}
(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} &= \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma)\big[\big(\delta^k_{\sigma(i)}, \delta^p_{\sigma(i)}, \delta^q_{\sigma(i)}\big) \times \vec{f}_{kpq}\big] \cdot \vec{z}_{kpq} \\
&= \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma)\bigg[\frac{(\delta^p_{\sigma(i)} \dot{f}^q - \delta^q_{\sigma(i)} \dot{f}^p)(z_p - z_q)}{(z_k - z_p)(z_k - z_q)} + \frac{(\delta^q_{\sigma(i)} \dot{f}^k - \delta^k_{\sigma(i)} \dot{f}^q)(z_k - z_q)}{(z_k - z_p)(z_p - z_q)} \\
&\qquad\qquad + \frac{(\delta^k_{\sigma(i)} \dot{f}^p - \delta^p_{\sigma(i)} \dot{f}^k)(z_k - z_p)}{(z_k - z_q)(z_p - z_q)}\bigg].
\end{aligned}
$$

Removing the positive factor $\alpha_{kpq} := [(z_k - z_p)(z_k - z_q)(z_p - z_q)]^{-1}$ and setting

$$P_a := \sum_{\sigma \in O_a} \varphi'(r_\sigma),$$

we obtain

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} = \alpha_{kpq}\big[(P_p \dot{f}^q - P_q \dot{f}^p)(z_p - z_q)^2 + (P_q \dot{f}^k - P_k \dot{f}^q)(z_k - z_q)^2 + (P_k \dot{f}^p - P_p \dot{f}^k)(z_k - z_p)^2\big].$$

Applying Lemma 3.2 yields

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \le \alpha_{kpq}(P_q \dot{f}^k - P_k \dot{f}^q)[(z_k - z_q)^2 - (z_k - z_p)^2 - (z_p - z_q)^2].$$

Since the term in square brackets is nonnegative, applying Lemma 3.2 once more yields

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \le 0.$$

This completes the proof of the lemma. □

Corollary 3.3. *There exists $C < \infty$ (depending only on n, f and M_0) such that $G/F \le C$ along the flow.*

Proof. In view of Lemma 3.1 and the evolution equation (2-3), this is a simple application of the maximum principle. □

In order to obtain the uniform estimate required by property (iii), we modify G in order to obtain a function with a strict convexity property. A well-known trick (cf. [Andrews 1994b, Lemma 7.10; Huisken and Sinestrari 1999a, Theorem 2.14; Andrews et al. 2014b, Lemma 3.3]) then allows us to extract the required uniform estimate. First, we relabel the preliminary pinching function $g \to g_1$ ($G \to G_1$), and consider the new pinching function g defined by

$$g := K(g_1, g_2) := \frac{g_1^2}{g_2}, \tag{3-3}$$

where $g_2(z) = M \sum_{i=1}^n z_i - |z|$ for some large constant $M \gg 1$, for which g_2 is positive along the flow. That there is such a constant follows from applying the maximum principle to the evolution equation (2-3) for the function $G_2(x, t) := g_2(\kappa(x, t))$ as in [Andrews et al. 2014b, Lemma 3.1]. Note that $\dot{K}^1 > 0$, $\dot{K}^2 < 0$ and $\ddot{K} > 0$ wherever $g_1 > 0$.

Observe that properties (i) and (ii) are not harmed in the transition from g_1 to g. We now show that the estimates listed in properties (iii) and (iv) are satisfied by the curvature function defined in (3-3).

Proposition 3.4. *Let g be the pinching function defined by (3-3) and G its corresponding matrix function. Then, for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ (depending only on ε, n, f and Γ_0) such that*

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\big|_B T_{kpq}T_{lrs} \le -c_\varepsilon \frac{|T|^2}{F}$$

for all $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G(B) \ge \varepsilon F(B)$ and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. First note that (suppressing dependence on B)

$$\begin{aligned}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs} &= \dot{K}^\alpha(\dot{G}^{kl}_\alpha\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}_\alpha)T_{kpq}T_{lrs} - \dot{F}^{kl}\ddot{K}^{\alpha\beta}\dot{G}^{pq}_\alpha\dot{G}^{rs}_\beta T_{kpq}T_{lrs} \\ &\le \dot{K}^2(\dot{G}^{kl}_2\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}_2)T_{kpq}T_{lrs} \\ &\le -\dot{K}^2\dot{F}^{kl}\ddot{G}^{pq,rs}_2 T_{kpq}T_{lrs},\end{aligned}$$

where we used Lemma 3.1, convexity of K, and the inequalities $\dot{K}^1 \ge 0$ and $\dot{F} \ge 0$ in the first inequality, and the inequalities $\dot{G}_2 \ge 0$ and $\dot{K}^2 \le 0$, and convexity of F in the second. Since $\dot{K}^2 < 0$ whenever $G_1 > 0$ and G_2 is strictly concave in nonradial directions, the claim follows exactly as in [Andrews et al. 2014b, Lemma 3.3]. □

The uniform estimate of Proposition 3.4 yields a good bound for the term $Q(\nabla\mathcal{W}, \nabla\mathcal{W})$ in the evolution equations for the pinching functions. This is a crucial component in obtaining the L^p-estimates of the following section. This is the starting point for the Stampacchia–de Giorgi iteration argument. The second crucial estimate is the Poincaré-type inequality, Lemma 4.2 (see also [Huisken and Sinestrari 2009, §§4–5; in particular, Lemma 5.5]), which we can obtain with the help of property (iv). This estimate (corresponding to [Huisken and Sinestrari 2009, Lemma 5.2]) provides an estimate on the zero order term that occurs in contracting the Simons-type identity for $\dot{F}^{pq}\nabla_p\nabla_q h_{ij}$ with $\dot{G}^{ij}$ (see [Andrews et al. 2014b, Proposition 4.4]).

Proposition 3.5. *Let g be the pinching function defined by (3-3) and G its corresponding matrix function. ,Then for every $\delta > 0$, $\varepsilon > 0$, and $C > 0$ there exist $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ (depending only on δ, $\varepsilon > 0$, C, n, m, f and Γ_0) such that*

$$Z(B) := (F\dot{G}^{kl} - G\dot{F}^{kl})\big|_B B^2_{kl} \ge \gamma_1 F^2(G - \delta\gamma_2 F)\big|_B - \gamma_3 F^2\big|_B$$

for all symmetric, $(m+1)$-positive matrices B satisfying $\lambda(B) \in \Gamma_0$, $\lambda_{\min}(B) \ge -\delta F(B) - C$, and $G_m(B) \ge \varepsilon F(B)$.

Proof. From the definition of G we have

$$Z = \dot{K}^1 Z_1 + \dot{K}^2 Z_2,$$

where

$$Z_i(B) := (F\dot{G}^{kl}_i - G_i\dot{F}^{kl})\big|_B B^2_{kl}.$$

Thus, since $\dot{K}^2 = 2g_1/g_2$ is uniformly bounded below when $g \geq \varepsilon f$, it suffices to prove the estimate for Z_1.

So let B be a symmetric, $(m+1)$-positive matrix with eigenvalues $z_1 \leq \cdots \leq z_n$. Then

$$Z_1(B) = f\dot{g}_1^p z_p^2 - g_1 \dot{f}^p z_p^2 = \sum_{p>q} \big(\dot{g}_1^p \dot{f}^q - \dot{g}_1^q \dot{f}^p\big) z_p z_q (z_p - z_q) = \sum_{p>q} \big(P_p \dot{f}^q - P_q \dot{f}^p\big) z_p z_q (z_p - z_q)$$

$$= \Bigg(\sum_{p>q>l} + \sum_{p>l\geq q} + \sum_{l\geq p>q} \Bigg) \big(P_p \dot{f}^q - P_q \dot{f}^p\big) z_p z_q (z_p - z_q),$$

where we recall the notation $P_a := \sum_{\sigma \in O_a} \varphi'(r_\sigma)$ and we have defined $l \leq m$ as the number of nonpositive eigenvalues z_i. Recalling that $P_p \dot{f}^q - P_q \dot{f}^p \geq 0$ whenever $z_p \geq z_q$, we discard the final sum and part of the first to obtain

$$Z_1(B) \geq \sum_{p=m+2}^{n} \sum_{q=l+1}^{m+1} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) + \sum_{p=l+1}^{n} \sum_{q=1}^{l} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q)$$

$$= \sum_{p=m+2}^{n} \sum_{q=l+1}^{m+1} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) - f^2 \sum_{i=1}^{l} z_i$$

$$+ f^2 \sum_{i=1}^{l} z_i + \sum_{p=l+1}^{n} \sum_{q=1}^{l} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q).$$

So consider the term

$$S_1(z) := \sum_{p=m+2}^{n} \sum_{q=l+1}^{m+1} (P_p(z) \dot{f}^q(z) - P_q(z) \dot{f}^p(z)) z_p z_q (z_p - z_q) - f(z)^2 \sum_{i=1}^{l} z_i.$$

Observe that $S_1 \geq 0$. We claim that $S_1(z) > 0$ for all z in the cone

$$\Gamma_{\varepsilon,l} := \{z \in \Gamma_0 : g(z) \geq \varepsilon f(z),\ z_1 \leq \cdots \leq z_l \leq 0 < z_{l+1} \leq \cdots \leq z_n\}.$$

Suppose, to the contrary, that $S_1(z) = 0$ for some $z \in \Gamma_{\varepsilon,l}$. Then $z_1 = \cdots = z_l = 0$ and, for all $p > m+1 \geq q > l$, $(P_p(z) \dot{f}^q(z) - P_q(z) \dot{f}^p(z)) z_p z_q (z_p - z_q) = 0$. But, by Lemma 3.2, the latter implies that, for all $p > m+1 \geq q > l$, either $z_p = z_q$, or $r_\sigma(\lambda) \geq 0$ for all $\sigma \in O_{q,p}$. Note that the latter case cannot occur: since $p > m+1 \geq q$, there is a permutation $\sigma \in O_{q,p}$ such that $0 \leq r_\sigma(z) = (z_1 + \cdots + z_{m+1} - c_m^{-1} f(z))/f(z)$, which implies $g_1(z) = 0$, contradicting $z \in \Gamma_{\varepsilon,l}$. On the other hand, if $z_p = z_q$ for all $p > m+1 \geq q > l$, then we again obtain the contradiction $g_1(z) = 0$. Thus, $S_1 > 0$ on $\Gamma_{\varepsilon,l}$. Since S_1 is homogeneous of degree three, it follows that

$$S_1 \geq c_1 f^2 g$$

on $\Gamma_{\varepsilon,l}$, where $c_1 := \min_l \min_{\Gamma_{\varepsilon,l}} \dfrac{S_1}{f^2 g} > 0$.

Now consider

$$S_2 := f^2 \sum_{i=1}^{l} \lambda_i + \sum_{p=l+1}^{n} \sum_{q=1}^{l} (P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q).$$

Note that, by homogeneity, $c_2 := \sup\{P_p(z)\dot f^q(z) - P_q(z)\dot f^p(z) : z \in \Gamma_0,\ 1 \le p,\ q \le n\} < \infty$. Thus, S_2 is easily controlled using the "convexity estimate" $\lambda_1 \ge -\delta f - C$:

$$S_2 \ge -lf^2(\delta f + C) + (n-l)c_2 z_n \sum_{q=1}^{l} z_q(z_n - z_q) \ge -nf^2(\delta f + C) + 2nc_2c_3^2 f^2 \sum_{q=1}^{l} z_q$$
$$\ge -nf^2(\delta f + C) - 2nc_2c_3^2 f^2(\delta F + C) \ge -n(1+2c_2c_3^2)f^2(\delta f + C),$$

where $c_3 := \max\{|z_i|/f(z) : z \in \Gamma_0, 1 \le i \le n\}$.

The claim follows. □

We note that the above estimate is only useful in the presence of the convexity estimate Theorem 1.1, since then, for any $\delta > 0$, there is a constant $C_\delta > 0$ for which $\Gamma_{\delta,C_\delta} := \{z \in \Gamma_0 : z_i > -\delta f(z) - C_\delta \text{ for all } i\}$ is preserved by the flow.

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, it suffices to obtain, for any $\varepsilon > 0$, an upper bound on the function

$$G_{\varepsilon,\sigma} := \left(\frac{G}{F} - \varepsilon\right) F^\sigma$$

for some $\sigma > 0$. We will use the estimates of Propositions 3.5 and 3.4 to obtain bounds on the spacetime L^p-norms of the positive part of $G_{\varepsilon,\sigma}$, so long as p is sufficiently large and σ sufficiently small, just as in [Huisken and Sinestrari 1999b; 1999a; 2009] (see also [Andrews et al. 2014b] where these techniques are applied in the fully nonlinear setting). A Stampacchia–de Giorgi iteration procedure similar to that used in [Huisken 1984] (see also [Huisken and Sinestrari 1999b; Andrews et al. 2014b]) then allows us to extract a supremum bound on $G_{\varepsilon,\sigma}$.

We begin with an evolution equation for $G_{\varepsilon,\sigma}$:

Lemma 4.1 [Andrews et al. 2014b]. *The function $G_{\varepsilon,\sigma}$ satisfies the evolution equation*

$$(\partial_t - \mathcal{L})G_{\varepsilon,\sigma} = F^{\sigma-1}(\dot G^{kl}\ddot F^{pq,rs} - \dot F^{kl}\ddot G^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs} + \frac{2(1-\sigma)}{F}\langle \nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}|\nabla F|_F^2 + \sigma G_{\varepsilon,\sigma}|\mathcal{W}|_F^2, \quad (4\text{-}1)$$

where $\langle u, v\rangle_F := \dot F^{kl} u_k u_l$.

Now set $E := \max\{G_{\varepsilon,\sigma}, 0\}$. We need to obtain spacetime L^p-estimates for E. Let us first observe that integration by parts and application of Young's inequality, in conjunction with Lemma 2.3 and Proposition 3.4, yields the estimate (cf. [Andrews et al. 2014b])

$$\frac{d}{dt}\int E^p\,d\mu \le -\big(A_1 p(p-1) - A_2 p^{\frac32}\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu - \big(B_1 p - B_2 p^{\frac12}\big)\int E^p \frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu + C_1\sigma p\int E^p|\mathcal{W}|^2 d\mu \quad (4\text{-}2)$$

for some positive constants A_1, A_2, B_1, B_2, C_1 (which depend only on ε, n, m, f and M_0).

To estimate the final term, we make use of Proposition 3.5 in a similar manner to [Huisken and Sinestrari 2009, §5]. We first observe:

Lemma 4.2. *There are positive constants A_3, A_4, A_5, B_3, B_4, C_2, independent of p and σ, such that*

$$\int E^p \frac{Z(\mathcal{W})}{F}\,d\mu \le \big(A_3 p^{\frac{3}{2}} + A_4 p^{\frac{1}{2}} + A_5\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu + \big(B_3 p^{\frac{1}{2}} + B_4\big)\int E^p \frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu.$$

Proof. As in [Andrews et al. 2014b, §4], contraction of the commutation formula for $\nabla^2\mathcal{W}$ with $\dot{F}$ and $\dot{G}$ yields the identity

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} = -F^{\sigma-1}Q(\nabla\mathcal{W},\nabla\mathcal{W}) + F^{\sigma-1}Z(\mathcal{W}) + F^{\sigma-2}(F\dot{G}^{kl} - G\dot{F}^{kl})\nabla_k\nabla_l F\\ + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F - 2\frac{(1-\sigma)}{F}\langle\nabla F,\nabla G_{\varepsilon,\sigma}\rangle_F + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2.\end{aligned}$$

The claim is now proved using integration by parts and Young's inequality, with the help of Lemma 2.3 and Proposition 3.4 (see [Andrews et al. 2014b, Lemma 4.2]). □

Corollary 4.3. *For all $\varepsilon > 0$ there exist constants $\ell > 0$ and $L < \infty$ (depending only on ε, n, m, f and M_0) such that for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$ there is a constant $K = K_{\varepsilon,\sigma,p}$ (depending only on ε, n, m, f, M_0, σ and p) for which the following estimate holds*:

$$\int (G_{\varepsilon,\sigma})_+^p\,d\mu \le \int (G_{\varepsilon,\sigma}(\,\cdot\,,0))_+^p\,d\mu_0 + tK\mu_0(M),$$

where μ_0 is the measure induced on M by the initial immersion.

Proof. Recall Proposition 3.5. Setting $\delta = \varepsilon/(2\gamma_2)$ and applying the convexity estimate, we obtain

$$\frac{Z(\mathcal{W})}{F} \ge \frac{\varepsilon}{2}\gamma_1 F^2 - \gamma_3 C_{\varepsilon/(2\gamma_2)}F \tag{4-3}$$

whenever $G - \varepsilon F > 0$. We now use Young's inequality to obtain (cf. [Huisken and Sinestrari 2009, §5])

$$F = F^{-\sigma p}F^{1+\sigma p} \le F^{-\sigma p}\left(\frac{b^q}{q}F^{q(1+\sigma p)} + \frac{b^{-q'}}{q'}\right)$$

for any $b > 0$ and $q > 0$, where q' is the Hölder conjugate of q: $\frac{1}{q} + \frac{1}{q'} = 1$. Choosing $q = \frac{2+\sigma p}{1+\sigma p}$, so that $q' = 2 + \sigma p$, we obtain

$$F \le b^{(2+\sigma p)/(1+\sigma p)}\frac{1+\sigma p}{2+\sigma p}F^2 + \frac{b^{-(2+\sigma p)}}{2+\sigma p}F^{-\sigma p} \le b^{(2+\sigma p)/(1+\sigma p)}F^2 + b^{-(2+\sigma p)}F^{-\sigma p}.$$

Now choose $b := \left(\frac{\varepsilon\gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}}\right)^{\frac{1+\sigma p}{2+\sigma p}}$, so that

$$\gamma_3 C_{\varepsilon/(2\gamma_2)}F \le \frac{\varepsilon\gamma_1}{4}F^2 + KF^{-\sigma p},$$

where

$$K := \gamma_3 C_{\varepsilon/(2\gamma_2)}\left(\frac{\varepsilon\gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}}\right)^{-(1+\sigma p)}.$$

Returning to Equation (4-3), we find

$$\frac{\varepsilon\gamma_1}{4}F^2 \le K F^{-\sigma p} + \frac{Z(\mathcal{W})}{F}.$$

Estimating $G_{\varepsilon,\sigma} \le c_1 F^\sigma$ and $|\mathcal{W}|^2 \le c_2 F^2$, we obtain

$$E^p|\mathcal{W}|^2 \le \widetilde{K} + c_3 E^p \frac{Z(\mathcal{W})}{F}$$

for some constants $\widetilde{K} > 0$ (depending on F, M_0, ε, σ and p) and $c_3 > 0$ (depending on F, M_0, and ε).

Combining Lemma 4.2 and inequality (4-2) now yields

$$\frac{d}{dt}\int E^p\,d\mu \le K_{\varepsilon,\sigma,p}\mu_0(M) - \left(\alpha_0 p^2 - \alpha_1\sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right)\int E^{p-2}|G_{\varepsilon,\sigma}|^2\,d\mu \\ - \left(\beta_0 p - \beta_1\sigma p^{\frac{3}{2}} - \beta_2\sigma p - \beta_3 p^{\frac{1}{2}}\right)\int E^p\frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu$$

for some positive constants α_i and β_i, which depend on ε but not on σ or p, and $K_{\varepsilon,\sigma,p}$, which depends on ε, σ and p.

It is clear that $L > 0$ and $\ell > 0$ may be chosen such that

$$\left(\alpha_0 p^2 - \alpha_1\sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right) \ge 0 \quad\text{and}\quad \left(\beta_0 p - \beta_1\sigma p^{\frac{3}{2}} - \beta_2\sigma p - \beta_3 p^{\frac{1}{2}}\right) \ge 0$$

for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$. The claim then follows by integrating with respect to the time variable. □

The proof of Theorem 1.3 is completed by proceeding with Huisken's Stampacchia–de Giorgi iteration argument. We omit these details as the arguments required already appear in [Andrews et al. 2014b, §5] with no significant changes necessary.

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