ANALYSIS & PDE

Volume 7 No. 5 2014

msp

Analysis & PDE

msp.org/apde

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

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ANALYSIS AND PDE
Vol. 7, No. 5, 2014
dx.doi.org/10.2140/apde.2014.7.1027

msp

RESONANCE WIDTHS FOR THE MOLECULAR PREDISSOCIATION

ALAIN GRIGIS AND ANDRÉ MARTINEZ

We consider a semiclassical 2×2 matrix Schrödinger operator of the form

$$P = -h^2 \Delta \mathbf{I}_2 + \operatorname{diag}(V_1(x), V_2(x)) + hR(x, hD_x),$$

where V_1, V_2 are real-analytic, V_2 admits a nondegenerate minimum at 0 with $V_2(0) = 0$, V_1 is nontrapping at energy 0, and $R(x, hD_x) = (r_{j,k}(x, hD_x))_{1\le j,k\le 2}$ is a symmetric 2×2 matrix of first-order pseudodifferential operators with analytic symbols. We also assume that $V_1(0) > 0$. Then, denoting by e_1 the first eigenvalue of $-\Delta + \langle V_2''(0)x, x\rangle/2$, and under some ellipticity condition on $r_{1,2}$ and additional generic geometric assumptions, we show that the unique resonance ρ_1 of P such that $\rho_1 = (e_1 + r_{2,2}(0, 0))h + \mathcal{O}(h^2)$ (as $h \to 0_+$) satisfies

$$\operatorname{Im} \rho_1 = -h^{n_0+(1-n_\Gamma)/2} f\left(h, \ln \frac{1}{h}\right) e^{-2S/h},$$

where $f\left(h, \ln \frac{1}{h}\right) \sim \sum_{0\le m\le \ell} f_{\ell,m} h^\ell \left(\ln \frac{1}{h}\right)^m$ is a symbol with $f_{0,0} > 0$, $S > 0$ is the so-called Agmon distance associated with the degenerate metric $\max(0, \min(V_1, V_2))\, dx^2$, between 0 and $\{V_1 \le 0\}$, and $n_0 \ge 1$, $n_\Gamma \ge 0$ are integers that depend on the geometry.

1. Introduction

The theory of predissociation goes back to the very first years of quantum mechanics (see [Kronig 1928; Landau 1932a; 1932b; Zener 1932; Stückelberg 1932], for example). Roughly speaking, it describes the possibility for a molecule to dissociate spontaneously (after a sufficiently large time) into several submolecules, for energies below the crossing of the corresponding energy surfaces of the initial molecule and the final dissociated state. From a physical point of view, one naturally expects that this (typically quantum) phenomenon occurs with extremely small (but nonzero) probability.

Despite the fact that statements concerning this problem are present in the physics literature for more than 70 years, the first mathematically rigorous result is due to M. Klein [1987], where an upper bound on the time of predissociation is given in the framework of the Born–Oppenheimer approximation. More precisely, denoting by h the square root of the ratio of electronic to nuclear mass, Klein proves the existence of resonances ρ with real part below the crossing of the energy surfaces and with exponentially small imaginary part; that is

$$|\operatorname{Im} \rho| = \mathcal{O}(e^{-2(1-\varepsilon)S/h}),$$

Grigis partly supported by the GDR INdAM-CNRS GrefiMefi. Martinez partly supported by Università di Bologna, Funds for Selected Research Topics.
MSC2010: primary 35P15, 35C20; secondary 35S99, 47A75.
Keywords: resonances, Born–Oppenheimer approximation, eigenvalue crossing, microlocal analysis.

where $S > 0$ is a geometric constant, $\varepsilon > 0$ is fixed arbitrarily, and the estimate holds uniformly as h goes to zero.

In terms of probabilities, this result corresponds to an upper bound on the transition probability between the initial molecule and the dissociated state. The purpose of this article is to obtain more complete information on this quantity, in particular, a lower bound. More precisely, under suitable conditions, we prove that the imaginary part of the lowest resonance admits a complete asymptotic expansion of the type

$$\operatorname{Im}\rho_1 = -h^{n_0+(1-n_\Gamma)/2}e^{-2S/h}\sum_{0\le m\le \ell} f_{\ell,m}h^{\ell}\Big(\ln\frac{1}{h}\Big)^m,$$

in the sense that, for any $N \ge 1$, one has

$$\Big|\operatorname{Im}\rho_1 + h^{n_0+(1-n_\Gamma)/2}e^{-2S/h}\sum_{0\le m\le \ell\le N} f_{\ell,m}h^{\ell}\Big(\ln\frac{1}{h}\Big)^m\Big| = \mathbb{O}(h^{n_0+(1-n_\Gamma)/2+N}e^{-2S/h}),$$

where $S>0$, $n_0\ge 1$ and $n_\Gamma\ge 0$ are all geometric constants, and where the leading coefficient $f_{0,0}$ is positive.

As is well-known, the quantity $\operatorname{Im}\rho$ is closely related to the oscillatory behavior of the corresponding resonant state in the unbounded classically allowed region. Hence, the main issue will be to know sufficiently well this behavior.

The strategy of the proof consists in starting from the WKB construction at the bottom of the well and then trying to extend it as much as possible, at least up to the classically allowed unbounded region. This is mainly the same strategy used in [Helffer and Sjöstrand 1986] for the study of shape resonances.

However, from a technical point of view, several new problems are encountered, because of the crossing of the electronic levels.

The first one is that, at the crossing, the only reference on WKB constructions is that of [Pettersson 1997], which has been done for a special type of matrix Schrödinger operators. In particular, it strongly uses the fact that only differential operators are involved. In our case, since our operator comes from a Born–Oppenheimer reduction, it is necessarily of pseudodifferential kind (see [Klein et al. 1992; Martinez and Sordoni 2009], for example). As a consequence, our first step will consist in extending Pettersson's method to pseudodifferential operators. Unfortunately, this extension is far from being straightforward, and needs a specific formal calculus adapted to expressions involving the Weber functions.

The second one is that, after having overcome the crossing, the symbols of the resulting WKB expansions do not anymore satisfy analytic estimates (usually needed in order to resum them, up to exponentially small error terms). In particular, this prevents us from using directly the constructions of [Helffer and Sjöstrand 1986] near the classically allowed unbounded region. Instead, we have to adapt the method of Fujiié, Lahmar-Benbernou and Martinez [Fujiié et al. 2011], which, without analyticity, allows us to extend the WKB constructions into the classically allowed unbounded region up to a distance of order $(h\ln|h|)^{2/3}$ from the barrier. This is not much, but it is enough to have sufficient control in this region on the difference between the true solution and the WKB one. This is actually done by adapting the specific arguments of propagation introduced in [loc. cit.], where the propagation takes place in h-dependent domains.

In the next section, we describe in details the geometrical context and the assumptions.

In Section 3, we state our main result.

Section 4 is devoted to the WKB constructions, starting from the well and proceeding away along some minimal geodesics, until crossing the boundary of the classically forbidden region. It is in this section that we develop a formal pseudodifferential calculus adapted to expressions involving the Weber functions.

Next, in Section 5, we extend the well-known Agmon estimates to our pseudodifferential context. In this case, the main feature is that, since we cannot use general Lipschitz weight functions, we replace them by h-dependent smooth functions with bounded gradient, but with derivatives of higher order that can grow to infinity as $h \to 0$.

In Section 6, we use these estimates in order to obtain a bound for the difference between the WKB solutions and a solution of a modified problem, and this permits us to define an asymptotic solution in a whole neighborhood of the classically forbidden region (but only up to a distance of order $(h \ln |h|)^{1/3}$ from this region).

Section 7 contains the a priori estimates and the propagation arguments that lead to a good control on the difference between the asymptotic solution and the actual one.

Finally, Section 8 makes the link with the width of the resonance. Even if the idea is standard (practically an application of the Green formula; see [Helffer and Sjöstrand 1986], for example), here we have to be careful with the double problem that, on the one hand, we deal with pseudodifferential (not differential) operators and, on the other hand, the magnitude of freedom outside the classically forbidden region is of order $(h \ln |h|)^{1/3}$ as $h \to 0$.

2. Geometrical assumptions

We consider the semiclassical 2×2 matrix Schrödinger operator

$$P = \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} + hR(x, hD_x), \tag{2-1}$$

with

$$P_j := -h^2 \Delta + V_j(x), \quad j = 1, 2,$$

where $x = (x_1, \dots, x_n)$ is the current variable in $\mathbb{R}^n$ $(n \geq 1)$, $h > 0$ denotes the semiclassical parameter, and $R(x, hD_x) = (r_{j,k}(x, hD_x))_{1 \leq j,k \leq 2}$ is a formally self-adjoint 2×2 matrix of first-order semiclassical pseudodifferential operators, in the sense that, for all $\alpha \in \mathbb{N}^{2n}$, $\partial^\alpha r_{j,k}(x, \xi) = \mathcal{O}(1 + |\xi|)$ uniformly on $\mathbb{R}^{2n}$.

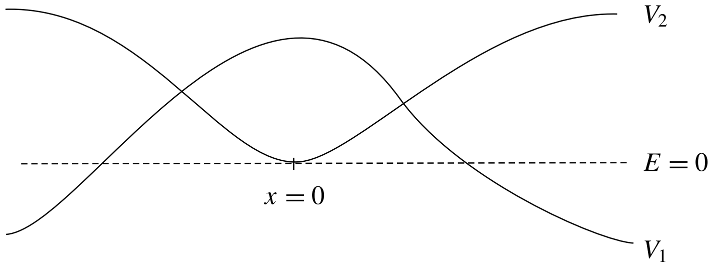
Let us observe that this is typically the kind of operator one obtains in the Born–Oppenheimer approximation, after reduction to an effective Hamiltonian [Klein et al. 1992; Martinez and Sordoni 2009]. In that case, the quantity h^2 stands for the inverse of the mass of the nuclei.

<u>Assumption 1</u>: The potentials V_1 and V_2 are smooth and bounded on $\mathbb{R}^n$, and satisfy:

$$V_1(0) > 0 \text{ and } E = 0 \text{ is a nontrapping energy for } V_1, \tag{2-2}$$

$$V_1 \text{ has a strictly negative limit as } |x| \to \infty, \tag{2-3}$$

$$V_2 \geq 0, \quad V_2^{-1}(0) = \{0\}, \quad \text{Hess } V_2(0) > 0, \quad \liminf_{|x| \to \infty} V_2 > 0. \tag{2-4}$$



In particular, we assume that V_2 has a unique nondegenerate well at $x = 0$. We define the island $\ddot{O}$ as the bounded open set

$$\ddot{O} = \{x \in \mathbb{R}^n : V_1(x) > 0\}, \tag{2-5}$$

and the sea as the set where $V_1(x) < 0$. With (2-2) and (2-4), the well $\{x = 0\}$ for V_2 is included in the island.

The fact that 0 is a nontrapping energy for V_1 means that, for any $(x, \xi) \in p_1^{-1}(0)$, one has that $|\exp tH_{p_1}(x, \xi)| \to +\infty$ as $t \to \infty$, where we let $p_1(x, \xi) := \xi^2 + V_1(x)$ be the symbol of P_1 and $H_{p_1} := (\nabla_\xi p_1, -\nabla_x p_1)$ be the Hamilton field of p_1.

Conditions (2-2)–(2-4) correspond to *molecular predissociation*, as described in [Klein 1987].

Since we plan to study the resonances of P near the energy level $E = 0$, we also assume:

<u>Assumption 2</u>: The potentials V_1 and V_2 extend to bounded holomorphic functions near a complex sector of the form $\mathcal{S}_{R_0,\delta} := \{x \in \mathbb{C}^n : |\mathrm{Re}\, x| \geq R_0, |\mathrm{Im}\, x| \leq \delta|\mathrm{Re}\, x|\}$, with $R_0, \delta > 0$. Moreover V_1 tends to its limit at ∞ in this sector and $\mathrm{Re}\, V_2$ stays away from 0 in this sector.

<u>Assumption 3</u>: The symbols $r_{j,k}(x, \xi)$ for $(j, k) = (1, 1), (1, 2), (2, 2)$ extend to holomorphic functions in (x, ξ) near

$$\widetilde{\mathcal{S}}_{R_0,\delta} := \mathcal{S}_{R_0,\delta} \times \{\xi \in \mathbb{C}^n : |\mathrm{Im}\, \xi| \leq \max(\delta\langle \mathrm{Re}\, x\rangle, \sqrt{M_0})\},$$

with

$$M_0 > \sup_{x \in \mathbb{R}^n} \min(V_1(x), V_2(x)),$$

and, for real x, $r_{j,k}$ is a smooth function of x with values in the set of holomorphic functions of ξ near $\{|\mathrm{Im}\, \xi| \leq \sqrt{M_0}\}$. Moreover we assume that, for any $\alpha \in \mathbb{N}^{2n}$, they satisfy

$$\partial^\alpha r_{j,k}(x, \xi) = \mathcal{O}(\langle \mathrm{Re}\, \xi\rangle) \quad \text{uniformly on } \widetilde{\mathcal{S}}_{R_0,\delta} \cup \big(\mathbb{R}^n \times \{|\mathrm{Im}\, \xi| \leq \sqrt{M_0}\}\big). \tag{2-6}$$

Now we define the cirque Ω as

$$\Omega = \{x \in \mathbb{R}^n : V_2(x) < V_1(x)\}. \tag{2-7}$$

Hence, the well is in the cirque and the cirque is in the island.

We also consider the Agmon distance associated to the pseudometric

$$(\min(V_1, V_2))_+ \, dx^2;$$

see [Pettersson 1997]. There are three places where this metric is not a standard one.

The first is near the well 0, but this case is well-known. It was treated in [loc. cit.] and also in [Helffer and Sjöstrand 1984]. The Agmon distance,

$$\varphi(x) := d(x, 0), \tag{2-8}$$

is smooth at 0. The point $(x, \xi) = (0, 0)$ is a hyperbolic singular point of the Hamilton vector field H_{q_2}, where $q_2 = \xi^2 - V_2(x)$, and the stable and unstable manifold near this point are respectively the Lagrangian manifolds $\{\xi = \nabla\varphi(x)\}$ and $\{\xi = -\nabla\varphi(x)\}$.

Secondly, on $\partial\Omega$, precisely at the points where $V_1 = V_2$. This case has been also considered by Pettersson. At such a point, if one assume that $\nabla V_1 \neq \nabla V_2$, then any geodesic which is transversal to the hypersurface $\{V_1 = V_2\}$ is C^1.

Finally there is the boundary of the island $\partial\ddot{O}$, where $V_1 = 0$. This situation was considered in [Helffer and Sjöstrand 1986]. We will follow this work in the next assumption.

Now we consider the distance from the well to the sea, that is, to $\partial\ddot{O}$:

$$S := d(0, \partial\ddot{O}). \tag{2-9}$$

Setting $B_S := \{x \in \ddot{O} : \varphi(x) < S\}$ and denoting by $\bar{B}_S$ its closure, we also consider the set $\bar{B}_S \cap \partial\ddot{O}$ that consists of the points of the boundary of the island that are joined to the well by a minimal d-geodesic included in the island. These points are called points of type 1 in [loc. cit.], and we denote by G the set of minimal geodesics joining such a point to 0 in $\ddot{O}$.

We make the following assumption:

Assumption 4: For all $\gamma \in G$, γ intersects $\partial\Omega$ at a finite number of points and the intersection is transversal at each of these points. Moreover, $\nabla V_1 \neq \nabla V_2$ on $\gamma \cap \partial\Omega$.

Let us recall that the assumption that 0 is a nontrapping energy for V_1 implies that $\nabla V_1 \neq 0$ on $\partial\ddot{O}$, and therefore that $\partial\ddot{O}$ is a smooth hypersurface.

We define the caustic set $\mathscr{C}$ as the union of the set of points of type 1 and the set of points $x \in \ddot{O}$ with $\varphi(x) = S + d(x, \partial\ddot{O})$. As in [loc. cit.] we assume:

Assumption 5: The points of type 1 form a submanifold Γ, and $\mathscr{C}$ has a contact of order exactly two with $\partial\ddot{O}$ along Γ.

We denote by n_Γ the dimension of Γ. Moreover, for any $\gamma \in G$, we denote by $N_\gamma := \#(\gamma \cap \partial\Omega)$ the number of points where γ crosses the boundary of the cirque, and we set

$$n_0 := \min_{\gamma \in G} N_\gamma, \qquad G_0 := \{\gamma \in G : N_\gamma = n_0\}.$$

Then, we make an assumption that somehow insures that an interaction between the two Schrödinger operators does exist.

Assumption 6: There exists at least one $\gamma \in G_0$ for which the ellipticity condition $r_{12}(x, i\nabla\varphi(x)) \neq 0$ holds at every point $x \in \gamma \cap \partial\Omega$.

3. Main result

Under the previous assumption we plan to study the resonances of the operator P given in (2-1), where $R(x, hD_x)$ is defined as

$$R(x, hD_x) := \begin{pmatrix} \mathrm{Op}_h^L(r_{1,1}) & \mathrm{Op}_h^L(r_{1,2}) \\ \mathrm{Op}_h^R(\overline{r_{1,2}}) & \mathrm{Op}_h^L(r_{2,2}) \end{pmatrix},$$

where for any symbol $a(x, \xi)$ we use the quantizations

$$\mathrm{Op}_h^L(a)u(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\xi/h} a(x, \xi) u(y)\, dy\, d\xi,$$
$$\mathrm{Op}_h^R(a)u(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\xi/h} a(y, \xi) u(y)\, dy\, d\xi.$$

In order to define the resonances we consider the distortion given as follows. Let $F(x) \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ be such that $F(x) = 0$ for $|x| \le R_0$ and $F(x) = x$ for $|x|$ large enough. For $\theta > 0$ small enough, we define the distorted operator P_θ as the value at $\nu = i\theta$ of the extension to the complex numbers of the operator $U_\nu P U_\nu^{-1}$, which is defined for ν real small enough and analytic in ν, where we have set

$$U_\nu \phi(x) = \det(1 + \nu\, dF(x))^{1/2} \phi(x + \nu F(x)). \tag{3-1}$$

Since we have a pseudodifferential operator $R(x, hD)$, the fact that $U_\nu P U_\nu^{-1}$ is analytic in ν is not completely standard but can be done without problem (thanks to Assumption 3), and by using the Weyl perturbation theorem, one can also see that there exists $\varepsilon_0 > 0$ such that for any $\theta > 0$ small enough, the spectrum of P_θ is discrete in $[-\varepsilon_0, \varepsilon_0] - i[0, \varepsilon_0\theta]$. The eigenvalues of P_θ are called the resonances of P [Hunziker 1986; Helffer and Sjöstrand 1986; Helffer and Martinez 1987].

We will need another small parameter $k > 0$ related to the semiclassical parameter $h > 0$, defined as

$$k := h \ln \frac{1}{h}. \tag{3-2}$$

In the sequel, we will study the resonances in the domain $[-\varepsilon_0, Ch] - i[0, Ck]$, where $C > 0$ is arbitrarily large. In this case, we can adapt the WKB constructions near the well made in [Helffer and Sjöstrand 1984] and show that these resonances form a finite set $\{\rho_1, \dots, \rho_m\}$, with asymptotic expansions as $h \to 0$ of the form

$$\rho_j \sim h \sum_{\ell \ge 0} \rho_{j,\ell} h^{\ell/2},$$

where $\rho_{j,\ell} \in \mathbb{R}$ and $\rho_{j,0} = e_j + r_{2,2}(0, 0)$, e_j being the j-th eigenvalue of the harmonic oscillator $-\Delta + \langle V_2''(0)x, x\rangle/2$ (actually, to be more precise, one must also assume that the arbitrarily large constant C does not coincide with one of the e_j).

In this paper we are interested in the imaginary part of these resonances. We have:

Theorem 3.1. *Under Assumptions 1 to 6, the first resonance ρ_1 of P is such that*

$$\mathrm{Im}\, \rho_1 = -h^{n_0 + (1-n_\Gamma)/2} f\Big(h, \ln \frac{1}{h}\Big) e^{-2S/h},$$

where $f\Big(h, \ln\dfrac{1}{h}\Big)$ *admits an asymptotic expansion of the form*

$$f\Big(h, \ln\frac{1}{h}\Big) \sim \sum_{0\le m\le \ell} f_{\ell,m} h^{\ell}\Big(\ln\frac{1}{h}\Big)^m \quad \textit{as } h \to 0,$$

with $f_{0,0} > 0$ *and* $S > 0$ *as defined in* (2-9).

Moreover the other resonances in $[-\varepsilon_0, Ch] - i[0, Ck]$ *verify*

$$\operatorname{Im}\rho_j = \mathcal{O}(h^{\beta_j} e^{-2S/h}),$$

for some real β_j, *uniformly as* $h \to 0$.

4. WKB constructions

In this section, we fix some minimal d-geodesic $\gamma \in G$ and we denote by $x^{(1)}, \dots, x^{(N_\gamma)}$ the sequence of points that constitute $\gamma \cap \partial\Omega$, ordered from the closest to 0 up to the closest to $\ddot{O}$ (note that N_γ is necessarily an odd number). We also denote by $\gamma^{(1)}, \gamma^{(2)}, \dots, \gamma^{(N_\gamma+1)}$ the portions of $\gamma \setminus \partial\Omega$ that are in-between 0 and $x^{(1)}$, $x^{(1)}$ and $x^{(2)}$, ..., $x^{(N_\gamma)}$ and $\ddot{O}$, respectively, in such a way that we have

$$\gamma = \gamma^{(1)} \cup \{x^{(1)}\} \cup \gamma^{(2)} \cup \cdots \cup \{x^{(N_\gamma)}\} \cup \gamma^{(N_\gamma+1)},$$

where the union is disjoint (in particular, by convention we assume that $0 \in \gamma^{(1)}$). Moreover, we start by considering the first resonance ρ_1 only.

In the cirque. As in [Pettersson 1997], the starting point of the construction consists of the WKB asymptotics given near the well $x = 0$ by a method due to Helffer and Sjöstrand [1984]. More precisely, because of the matricial nature of the operator and the fact that p_1 is elliptic above $x = 0$, one finds a formal solution w_1 of $Pw_1 = \rho_1 w_1$ of the form

$$w_1(x; h) = \begin{pmatrix} h a_1(x, h) \\ a_2(x, h) \end{pmatrix} e^{-\varphi(x)/h}, \tag{4-1}$$

where φ is defined in (2-8) and a_j $(j = 1, 2)$ is a classical symbol of order 0 in h, that is, a formal series in h of the form

$$a_j(x, h) = \sum_{k=0}^{\infty} h^k a_{j,k}(x), \tag{4-2}$$

with $a_{j,k}$ smooth near 0 (here no half-powers of h appear since we consider the first resonance ρ_1 only). Moreover, a_2 is elliptic in the sense that $a_{2,0}$ never vanishes. Note that the generalization of the constructions of [Helffer and Sjöstrand 1984] to the case of pseudodifferential operators is done by the use of a so-called *formal semiclassical pseudodifferential calculus*, which in our case is based on the following result.

Lemma 4.1. *Let* $\widetilde{\varphi} = \widetilde{\varphi}(x)$ *be a real bounded* C^∞ *function on* $\mathbb{R}^n$ *and let* $p = p(x, \xi) \in S(1)$ *extend to a bounded function, holomorphic with respect to* ξ *in a neighborhood of the set*

$$\{(x, \xi) \in \operatorname{supp} \nabla\widetilde{\varphi} \times \mathbb{C}^n : |\operatorname{Im}\xi| \le |\nabla\widetilde{\varphi}(x)|\}.$$

Then, denoting by Op_h^L *the left (or standard) semiclassical quantization of symbols, the operator* $e^{\widetilde{\varphi}/h}\,\mathrm{Op}_h^L(p)e^{-\widetilde{\varphi}/h}$ *is uniformly bounded on* $L^2(\mathbb{R}^n)$ *and, for any* $a \in C_0^\infty(\mathbb{R}^n)$ *and* $N \geq 1$, *one has, with* $\Phi(x, y) := \widetilde{\varphi}(x) - \widetilde{\varphi}(y) - (x - y)\nabla\widetilde{\varphi}(x)$,

$$\big(e^{\widetilde{\varphi}/h}\,\mathrm{Op}_h^L(p)e^{-\widetilde{\varphi}/h}a\big)(x;h) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\Big(\frac{h}{i}\Big)^{|\alpha|}\partial_\xi^\alpha p(x, i\nabla\widetilde{\varphi}(x))\partial_y^\alpha\big(a(y)e^{\Phi(x,y)/h}\big)\big|_{y=x} + \mathcal{O}(h^{N/2}), \quad (4\text{-}3)$$

locally uniformly with respect to x, *and uniformly with respect to* h *small enough.*

The proof of this lemma is rather standard, e.g., [Martinez 1987] and we omit it.

Then, the construction can be performed by using the formal series given in (4-3) in order to define the formal action of $R(x, hD_x)$ on w_1. Afterwards, these constructions can be continued along the integral curves of the vector field $\nabla_\xi p_2(x, i\nabla\varphi(x))D_x = 2\nabla\varphi(x).\nabla_x$ (that is, along the minimal geodesic of d starting at 0), as long as $p_1(x, i\nabla\varphi(x))$ does not vanish (that is, as long as these minimal geodesics stay inside the cirque Ω). In that way, after resummation and multiplication by a cutoff function, we obtain a function w_1 of the form (4-1) that satisfies

$$Pw_1 - \rho_1 w_1 = \mathcal{O}(h^\infty e^{-\varphi/h}) \tag{4-4}$$

locally uniformly in $\bigcup \gamma$, where the union is taken over all the minimal d-geodesics γ coming from the well 0 and staying in Ω. In particular, (4-4) is satisfied in a neighborhood $\mathcal{N}_1$ of $\gamma^{(1)}$.

At the boundary of the cirque. Now, we study the situation near the point $x^{(1)} \in \partial\Omega$. By Theorem 2.14 of [Pettersson 1997], we know that there exist a neighborhood $\mathcal{V}_1$ of $x^{(1)}$ and two positive functions $\varphi_1, \varphi_2 \in C^\infty(\mathcal{V}_1)$ such that

$$\begin{aligned}
&\varphi_1 = \varphi && \text{on } \mathcal{V}_1 \cap \{V_1 < V_2\};\\
&\varphi_2 = \varphi && \text{on } \mathcal{V}_1 \cap \{V_2 < V_1\};\\
&|\nabla\varphi_j(x)|^2 = V_j(x), && j = 1, 2;\\
&\varphi_1 = \varphi_2 \ \text{ and } \ \nabla\varphi_1 = \nabla\varphi_2 && \text{on } \mathcal{V}_1 \cap \partial\Omega;\\
&\varphi_2(x) - \varphi_1(x) \sim d(x, \partial\Omega)^2.
\end{aligned}$$

Actually, φ_2 is just $d_2(0, x)$, where d_2 is the Agmon distance associated with the metric $V_2(x)\,dx^2$ and φ_1 is the phase function of the Lagrangian manifold obtained as the flow-out of $\{(x, \nabla\varphi_2(x)) : x \in \mathcal{V}_1 \cap \partial\Omega\}$ under the Hamilton flow of $q_1(x, \xi) := \xi^2 - V_1(x)$.

Then, we set

$$\psi := \tfrac{1}{2}(\varphi_1 + \varphi_2), \tag{4-5}$$

and we consider the smooth function $z(x)$ defined for $x \in \mathcal{V}_1$ by

$$\begin{aligned} z(x)^2 &= 2(\varphi_2(x) - \varphi_1(x))\\ z(x) &< 0 \quad \text{on } \mathcal{V}_1 \cap \{V_2 < V_1\}. \end{aligned} \tag{4-6}$$

In order to extend the WKB construction (4-1) across $\partial\Omega$ near $x^{(1)}$, we follow Pettersson and try a formal ansatz,

$$w_2(x;h) = \sum_{k\geq 0} h^k \left(\alpha_k(x,h) Y_{k,0}\left(\frac{z(x)}{\sqrt{h}}\right) + \sqrt{h}\beta_k(x,h) Y_{k,1}\left(\frac{z(x)}{\sqrt{h}}\right)\right) e^{-\psi(x)/h}, \tag{4-7}$$

where

$$\alpha_k(x,h) = \begin{pmatrix} h\alpha_{k,1}(x,h) \\ \alpha_{k,2}(x,h)\end{pmatrix}, \quad \beta_k(x,h) = \begin{pmatrix} \beta_{k,1}(x,h) \\ h\beta_{k,2}(x,h)\end{pmatrix}. \tag{4-8}$$

Here $\alpha_{k,j}$ and $\beta_{k,j}$ are formal symbols of the type

$$\sum_{l\geq 0} \sum_{m=0}^{l} h^l (\ln h)^m \gamma^{l,m}(x) \tag{4-9}$$

(with $\gamma^{l,m}$ smooth in ω_1) and, for any $k \geq 0$ and $\varepsilon \in \mathbb{C}$, the function $Y_{k,\varepsilon}$ is the so-called Weber function, defined by

$$Y_{k,\varepsilon}(z) = \partial_\varepsilon^k Y_{0,\varepsilon}(z), \tag{4-10}$$

where $Y_{0,\varepsilon}$ is the unique entire function with respect to ε and z that is a solution of the Weber equation,

$$Y''_{0,\varepsilon} + \left(\frac{1}{2} - \varepsilon - \frac{z^2}{4}\right) Y_{0,\varepsilon} = 0, \tag{4-11}$$

such that, for $\varepsilon > 0$, one has

$$Y_{0,\varepsilon}(z) \sim e^{-z^2/4} z^{-\varepsilon} \quad \text{as } z \to -\infty. \tag{4-12}$$

(Then, one also has $Y_{0,\varepsilon}(z) \sim \left(\sqrt{2\pi}/\Gamma(\varepsilon)\right) e^{z^2/4} z^{\varepsilon-1}$ as $z \to +\infty$, by Proposition A.2 of [Pettersson 1997].) As is shown in Pettersson's Theorem 4.3, a resummation of (4-7) is possible up to an error of order $\mathbb{O}(h^\infty e^{-\varphi/h})$.

Now, since φ is not C^∞ (but only C^1) near $x^{(1)}$, we need to find some generalization of Lemma 4.1. For technical reasons, in the rest of this section we prefer to work with the *right* semiclassical quantization of symbols, which we denote by Op_h^R.

For $\nu_0 > 0$ and $g \in C^\infty(\mathbb{R}^{2n}; \mathbb{R}_+)$, we denote by $S_{\nu_0}(g(x,\xi))$ the set of (possibly h-dependent) functions $p \in C^\infty(\mathbb{R}^{2n})$ that extend to holomorphic functions with respect to ξ in the strip

$$\mathcal{A}_{\nu_0} := \{(x,\xi) \in \mathbb{R}^n \times \mathbb{C}^n : |\mathrm{Im}\, \xi| < \nu_0\}$$

such that, for all $\alpha \in \mathbb{N}^{2n}$, one has

$$\partial^\alpha p(x,\xi) = \mathbb{O}(g(x, \mathrm{Re}\, \xi)), \tag{4-13}$$

uniformly with respect to $(x,\xi) \in \mathcal{A}_{\nu_0}$ and $h > 0$ small enough. We also denote by $S_0(g)$ the analogous space of smooth symbols obtained by switching $\mathbb{R}^{2n}$ to $\mathcal{A}_{\nu_0}$ and "smooth" to "holomorphic".

Lemma 4.2. *Let $\nu_0 > 0$, $m \in \mathbb{R}$, $p = p(x,\xi) \in S_{\nu_0}(\langle\xi\rangle^m)$, and let $\phi = \phi(x)$ be a real bounded Lipschitz function on $\mathbb{R}^n$ such that*

$$\|\nabla\phi(x)\|_{L^\infty} < \nu_0.$$

Let also $a = a(x; h) \in C^\infty(\mathbb{R}^n)$ be such that, for all $\alpha \in \mathbb{N}^n$,

$$(hD_x)^\alpha a(x; h) = \mathcal{O}(e^{-\phi(x)/h}),$$

uniformly with respect to h small enough and $x \in \mathbb{R}^n$. Then

$$(\mathrm{Op}_h^R(p)a)(x; h) = \mathcal{O}(e^{-\phi(x)/h})$$

uniformly with respect to h small enough and $x \in \mathbb{R}^n$.

Proof. We write

$$e^{\phi(x)/h}\,\mathrm{Op}_h^R(p)a(x; h) = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h+\phi(x)/h} p(y, \xi)a(y; h)\,dy\,d\xi \tag{4-14}$$

and, following [Sjöstrand 1982], we make the change of contour of integration in ξ,

$$\mathbb{R}^n \ni \xi \mapsto \xi + i\nu_1\frac{x-y}{|x-y|}, \tag{4-15}$$

where $\|\nabla\phi(x)\|_{L^\infty} < \nu_1 < \nu_0$. We obtain

$$e^{\phi(x)/h}\,\mathrm{Op}_h^R(p)a(x; h) = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h} p\Big(y, \xi + i\nu_1\frac{x-y}{|x-y|}\Big)\theta(x, y; h)\,dy\,d\xi, \tag{4-16}$$

with

$$\theta(x, y; h) = a(y; h)e^{(\phi(x)-\nu_1|x-y|)/h} = \mathcal{O}(e^{\phi(x)-\phi(y)-\nu_1|x-y|/h}).$$

Therefore

$$\theta(x, y; h) = \mathcal{O}(e^{-\delta|x-y|/h}), \tag{4-17}$$

with $\delta = \nu_1 - \|\nabla\phi\|_{L^\infty} > 0$.

Then, in the case $m < -n$, the result follows immediately from (4-16)–(4-17) (and standard estimates on oscillatory integrals). In the general case, we just write

$$\mathrm{Op}_h^R(p) = \mathrm{Op}_h^R(p)(2\nu_0 - h^2\Delta_x)^{-k}(2\nu_0 - h^2\Delta_x)^k, \tag{4-18}$$

with k an integer large enough (e.g., $k = 1+|[m]|+n$) and, since $\mathrm{Op}_h^R(p)(2\nu_0-h^2\Delta_x)^{-k}$ is a semiclassical pseudodifferential operator with (h-dependent) symbol in $S_{\nu_0}(\langle\xi\rangle^{m-2k}) \subset S_{\nu_0}(\langle\xi\rangle^{-n-1})$, the result follows by applying the previous case with a replaced by $(2\nu_0 - h^2\Delta_x)^k a$. □

Now, as preparation for defining a formal pseudodifferential calculus acting on expressions such as (4-7), for $j = 1, \dots, n$ and $x \in \omega_1$, we set

$$A_j(x) := \begin{pmatrix} \dfrac{\partial\varphi_2(x)}{\partial x_j} & 0 \\ 0 & \dfrac{\partial\varphi_1(x)}{\partial x_j} \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}). \tag{4-19}$$

Then, for any $k \geq 0$, we have (see [Pettersson 1997, (4.18)])

$$(hD_{x_j} - iA_j(x)) \begin{pmatrix} Y_{k,0}\Big(\dfrac{z(x)}{\sqrt{h}}\Big) \\ Y_{k,1}\Big(\dfrac{z(x)}{\sqrt{h}}\Big) \end{pmatrix} e^{-\psi(x)/h} = \frac{\sqrt{h}}{i}(\partial_{x_j} z(x)) \begin{pmatrix} kY_{k-1,1}\Big(\dfrac{z(x)}{\sqrt{h}}\Big) \\ Y_{k,0}\Big(\dfrac{z(x)}{\sqrt{h}}\Big) \end{pmatrix} e^{-\psi(x)/h}. \tag{4-20}$$

If a and b are (scalar) formal symbols of the type (4-9) and $k \in \mathbb{N}$, we set

$$I_k(a,b)(x;h) = a(x;h)Y_{k,0}\Big(\frac{z(x)}{\sqrt{h}}\Big) + b(x;h)Y_{k,1}\Big(\frac{z(x)}{\sqrt{h}}\Big), \tag{4-21}$$

and we plan to exploit (4-20) in order to define a formal action of a pseudodifferential operator on $I_k(a,b)e^{-\psi/h}$. Using (4-20), we see that we have

$$\begin{aligned} (hD_x - i\nabla\varphi_2(x))(I_k(a,0)e^{-\psi/h}) &= (I_k(hD_x a, 0) + I_{k-1}(0, k\sqrt{h}aD_x z))e^{-\psi/h}, \\ (hD_x - i\nabla\varphi_1(x))(I_k(0,b)e^{-\psi/h}) &= I_k(\sqrt{h}bD_x z, hD_x b)e^{-\psi/h}. \end{aligned} \tag{4-22}$$

Now, for any $p \in S_{\nu_0}(\langle\xi\rangle^m)$, $N \geq 1$ and $j = 1, 2$, Taylor's formula gives

$$p(x,\xi) = \sum_{|\alpha|\leq N} \frac{1}{\alpha!}\partial_\xi^\alpha p(x, i\nabla\varphi_j(x))(\xi - i\nabla\varphi_j(x))^\alpha + \sum_{|\alpha|=N+1} p_{j,\alpha}(x,\xi)(\xi - i\nabla\varphi_j(x))^\alpha, \tag{4-23}$$

where the $p_{j,\alpha}$ are in $S_{\nu_0}(\langle\xi\rangle^m)$. Moreover, we have:

Lemma 4.3. *Let $\nu_0 > \sup_{x\in\omega_1} \min(\sqrt{V_1(x)}, \sqrt{V_2(x)})$ and $m \in \mathbb{R}$. Then, for any $q = q(x,\xi) \in S_{\nu_0}(\langle\xi\rangle^m)$, $k \geq 0$, a in $C_0^\infty(\omega_1)$ and $\alpha \in \mathbb{N}^n$, one has*

$$\begin{aligned} \mathrm{Op}_h^R\big(q(x,\xi)(\xi - i\nabla\varphi_2(x))^\alpha\big)(I_k(a,0)e^{-\psi(x)/h}) &= \mathcal{O}(|\ln h|^k h^{|\alpha|/2} e^{-\varphi(x)/h}), \\ \mathrm{Op}_h^R\big(q(x,\xi)(\xi - i\nabla\varphi_1(x))^\alpha\big)(I_k(0,a)e^{-\psi(x)/h}) &= \mathcal{O}(|\ln h|^k h^{|\alpha|/2} e^{-\varphi(x)/h}), \end{aligned} \tag{4-24}$$

where the estimates hold uniformly for h small enough and $x \in \mathbb{R}^n$.

Proof. We prove both estimates together, by induction on $|\alpha|$. We first notice that, by Lemma 4.6 of [Pettersson 1997], for $\beta \in \mathbb{N}^n$ and $j \in \{0, 1\}$, one has

$$(hD_x)^\beta\bigg(Y_{k,j}\Big(\frac{z(x)}{\sqrt{h}}\Big)e^{-\psi(x)/h}\bigg) = \mathcal{O}(|\ln h|^k e^{-\varphi(x)/h}). \tag{4-25}$$

As a consequence, the result for $\alpha = 0$ follows directly from Lemma 4.2.

Now, assume it is true for $|\alpha| \leq N$ ($N \in \mathbb{N}$ fixed arbitrarily), and let $\gamma \in \mathbb{N}^n$, $|\gamma| = 1$. Using the notation

$$I_{k,2}(a) := I_k(a,0)e^{-\psi/h}, \qquad I_{k,1}(a) := I_k(0,a)e^{-\psi/h}, \tag{4-26}$$

we write (for $|\alpha| \leq N$ and $j = 1, 2$)

$$\begin{aligned} &\mathrm{Op}_h^R\big(q(x,\xi)(\xi - i\nabla\varphi_j(x))^{\alpha+\gamma}\big)I_{k,j}(a)e^{-\psi(x)/h} \\ &\qquad = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h} f_\alpha(y,\xi)(\xi - i\nabla\varphi_j(y))^\gamma I_{k,j}(a)(y)e^{-\psi(y)/h}\,dy\,d\xi, \end{aligned}$$

with $f_\alpha(y,\xi) := q(y,\xi)(\xi - i\nabla\varphi_j(y))^\alpha$. Now, assuming without loss of generality that $\gamma = (1,0,\dots,0)$ and using the fact that

$$\xi_1 e^{i(x-y)\xi/h} = -hD_{y_1}(e^{i(x-y)\xi/h}),$$

we obtain

$$\begin{aligned}\mathrm{Op}_h^R\big(q(x,\xi)(\xi - i\nabla\varphi_j(x))^{\alpha+\gamma}\big)I_{k,j}(a)e^{-\psi(x)/h}\\ = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h}\Big(hD_{y_1} - i\frac{\partial\varphi_j}{\partial x_1}(y)\Big)I_{k,j}(f_\alpha(y,\xi)a(y))e^{-\psi(y)/h}\,dy\,d\xi,\end{aligned}$$

and therefore, by (4-22),

$$\mathrm{Op}_h^R\big(q(x,\xi)(\xi - i\nabla\varphi_j(x))^{\alpha+\gamma}\big)I_{k,j}(a)e^{-\psi(x)/h} = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h}\tilde I_{k,j}(f_\alpha(y,\xi)a(y))e^{-\psi(y)/h}\,dy\,d\xi,$$

with

$$\tilde I_{k,2}(a) := hI_{k,2}(D_{x_1}a) + k\sqrt{h}I_{k-1,1}(aD_{x_1}z), \quad \tilde I_{k,1}(a) := \sqrt{h}I_{k,2}(aD_{x_1}z) + hI_{k,1}(D_{x_1}a).$$

Then, applying the induction hypothesis (and using the fact that $D_{y_1}f_\alpha$ is a sum of terms of the type $g(y,\xi)(\eta - iA(y))^\beta$ with $|\beta| \ge |\alpha| - 1$) this gives

$$\begin{aligned}&\mathrm{Op}_h^R\big(q(x,\xi)(\xi - i\nabla\varphi_j(x))^{\alpha+\gamma}\big)I_{k,j}(a)e^{-\psi(x)/h}\\ &= \mathbb{O}\big(|\ln h|^k h^{1+(|\alpha|-1)/2} + |\ln h|^{k-1}h^{(1+|\alpha|)/2} + |\ln h|^k h^{(1+|\alpha|)/2}\big)e^{-\varphi/h} = \mathbb{O}\big(|\ln h|^k h^{(1+|\alpha|)/2}\big)e^{-\varphi/h},\end{aligned}$$

and the proof is complete. □

Using Lemma 4.3 and (4-23), for any a in $C_0^\infty(\omega_1)$, we obtain (with the notation (4-26))

$$\begin{aligned}&\mathrm{Op}_h^R(p)(I_{k,j}(a)e^{-\psi/h})\\ &= \sum_{|\alpha|\le N}\frac{1}{\alpha!}\mathrm{Op}_h^R\big(\partial_\xi^\alpha p(x,i\nabla\varphi_j(x))(\xi - i\nabla\varphi_j(x))^\alpha\big)(I_{k,j}(a)e^{-\psi(x)/h}) + \mathbb{O}(h^{N/2}e^{-\varphi/h})\\ &= \sum_{\substack{|\alpha|\le N\\ \beta\le\alpha}}\frac{1}{i^{|\beta|}\beta!(\alpha-\beta)!}\mathrm{Op}_h^R(\partial_\xi^\alpha p(x,i\nabla\varphi_j)(\nabla\varphi_j)^\beta\xi^{\alpha-\beta})(I_{k,j}(a)e^{-\psi(x)/h}) + \mathbb{O}(h^{N/2}e^{-\varphi/h}),\end{aligned}$$

and thus, as before, writing down the corresponding oscillatory integral, in the same way we deduce

$$\begin{aligned}&\mathrm{Op}_h^R(p)(I_{k,j}(a)e^{-\psi/h})\\ &= \sum_{\substack{|\alpha|\le N\\ \beta\le\alpha}}\frac{1}{i^{|\beta|}\beta!(\alpha-\beta)!}(hD_x)^{\alpha-\beta}[(\nabla\varphi_j)^\beta\partial_\xi^\alpha p(x,i\nabla\varphi_j)I_{k,j}(a)e^{-\psi/h}] + \mathbb{O}(h^{N/2}e^{-\varphi/h}). \quad (4\text{-}27)\end{aligned}$$

Now, for an integer $M \le 0$ and $\Omega \subset \mathbb{R}^n$ open, we consider the space of sequences of formal symbols,

$$S^M(\omega_1) := \Big\{a = (a_k)_{k\in\mathbb{N}} : a_k(x,h) = \sum_{l=-M}^\infty\sum_{m=0}^l h^l(\ln h)^m\gamma_k^{l,m}(x),\ \gamma_k^{l,m}\in C^\infty(\omega_1)\Big\},$$

and, for $a, b \in S^M(\omega_1)$, we set

$$I(a,b) := \sum_{k\geq 0} h^k I_k(a_k, \sqrt{h} b_k). \tag{4-28}$$

Using (4-22), we see that, for $j = 1, \dots, n$, we have

$$hD_{x_j} I(a,b) e^{-\psi/h} = I((iA_j + L_j)(a,b)) e^{-\psi/h}, \tag{4-29}$$

where $iA_j(a,b) = (i(\partial_{x_j}\varphi_2)a, i(\partial_{x_j}\varphi_1)b)$ and L_j is the operator

$$\begin{aligned} L_j : S^M \times S^M &\to S^{M-1} \times S^{M-1}, \\ (a,b) &\mapsto (\tilde{a}^j, \tilde{b}^j), \end{aligned}$$

defined by, for $k \in \mathbb{N}$,

$$\begin{aligned} \tilde{a}_k^j &:= hD_{x_j} a_k + hb_k D_{x_j} z, \\ \tilde{b}_k^j &:= hD_{x_j} b_k + (k+1)ha_{k+1} D_{x_j} z. \end{aligned} \tag{4-30}$$

In particular, using the notation $L = (L_1, \dots, L_n)$ and $L^\alpha = L_1^{\alpha_1} \cdots L_n^{\alpha_n}$, for all $\alpha \in \mathbb{N}^n$ we have

$$L^\alpha \;\text{ maps }\; S^M(\omega_1) \times S^M(\omega_1) \;\text{ into }\; S^{M-|\alpha|}(\omega_1) \times S^{M-|\alpha|}(\omega_1). \tag{4-31}$$

For any smooth diagonal $\mathcal{M}_2(\mathbb{C})$-valued function $B(x) = \operatorname{diag}(B_1(x), B_2(x))$, we let it act on $S^M \times S^M$ by setting

$$B(a,b) = (B_1 a, B_2 b), \tag{4-32}$$

and we define the formal action of a pseudodifferential operator with symbol $p \in S_{\nu_0}(\langle\xi\rangle^m)$ on expressions of the type $I(a,b)e^{-\psi/h}$ by the formula

$$\operatorname{Op}_h^F(p)(I(a,b)e^{-\psi/h}) := \sum_{\substack{\alpha\in\mathbb{N}^n \\ \beta\leq\alpha}} \frac{1}{i^{|\beta|}\beta!(\alpha-\beta)!} I\big((iA(x)+L)^{\alpha-\beta} A(x)^\beta \partial_\xi^\alpha p(x, iA(x))(a,b)\big) e^{-\psi/h}, \tag{4-33}$$

where we have also set $A := (A_1, \dots, A_n)$ and

$$\partial_\xi^\alpha p(x, iA(x))(a,b) := \big(\partial_\xi^\alpha p(x, i(\nabla\varphi_2))a, \partial_\xi^\alpha p(x, i(\nabla\varphi_1))b\big).$$

Then, in view of Lemma 4.3 and (4-27), we immediately obtain:

Proposition 4.4. *Let $a, b \in S^M(\omega_1)$ and denote by $\tilde{I}(a,b)e^{-\psi/h}$ any resummation of $I(a,b)e^{-\psi/h}$ up to an $\mathbb{O}(h^\infty e^{-\varphi/h})$ error term. Then, for any $\chi \in C_0^\infty(\omega_1)$, the quantity $\operatorname{Op}_h^R(p)(\chi\tilde{I}(a,b)e^{-\psi/h})$ is a resummation of $\operatorname{Op}_h^F(p)(I(\chi a, \chi b)e^{-\psi/h})$, up to an $\mathbb{O}(h^\infty e^{-\varphi/h})$ error term.*

In particular, the operator P naturally acts (up to $\mathbb{O}(h^\infty e^{-\varphi/h})$ error terms) on expressions of the type

$$w_2 = \begin{pmatrix} I(h\alpha_1, \beta_1) \\ I(\alpha_2, h\beta_2) \end{pmatrix} e^{-\psi/h}, \tag{4-34}$$

where $\alpha_j = (\alpha_{j,k})_{k\geq 0}$ and $\beta_j = (\beta_{j,k})_{k\geq 0}$ are in $S^0(\omega_1)$ $(j = 1, 2)$.

Writing down the equation $\widetilde{P} w_2 = \rho_1 w_2$, setting

$$\alpha_{j,k} = \sum_{l \geq 0} \sum_{m=0}^{l} h^l (\ln h)^m \alpha_{j,k}^{l,m}(x),$$

and the analogous formula for $\beta_{j,k}$, and identifying the coefficients of $h^l(\ln h)^m$ for $0 \leq m \leq l \leq 1$, we find (denoting by $p = \begin{pmatrix} p_1+hr_{1,1} & hr_{1,2} \\ hr_{2,1} & p_2+hr_{2,2} \end{pmatrix}$ the right symbol of P),

$$p_1(x, i\nabla\varphi_2)\alpha_{1,0}^{0,0} + r_{1,2}(x, i\nabla\varphi_2)\alpha_{2,0}^{0,0} \\ + \left[\tfrac{1}{i}\nabla_\xi p_1(x, i\nabla\varphi_1)(\nabla z) + \tfrac{1}{2}\langle(\mathrm{Hess}_\xi\, p_1)(x, i\nabla\varphi_1)\nabla z, \nabla(\varphi_2 - \varphi_1)\rangle\right]\beta_{1,0}^{0,0} = 0; \tag{4-35}$$

$$[\partial_\xi p_1(x, i\nabla\varphi_1)D_x - i(\nabla_x \cdot \nabla_\xi p_1)(x, i\nabla\varphi_1) + r_{1,1}(x, i\nabla\varphi_1) - \rho_1]\beta_{1,0}^{0,0} = 0; \tag{4-36}$$

$$p_2(x, i\nabla\varphi_1)\beta_{2,0}^{0,0} + r_{2,1}(x, i\nabla\varphi_1)\beta_{1,0}^{0,0} \\ + \left[\tfrac{1}{i}\partial_\xi p_2(x, i\nabla\varphi_2)(\nabla z) + \tfrac{1}{2}\langle(\mathrm{Hess}_\xi\, p_2)(x, i\nabla\varphi_2)\nabla z, \nabla(\varphi_1 - \varphi_2)\rangle\right]\alpha_{2,1}^{0,0} = 0; \tag{4-37}$$

$$(\partial_\xi p_2(x, i\nabla\varphi_2)D_x - i(\nabla_x \cdot \nabla_\xi p_2)(x, i\nabla\varphi_2) + r_{2,2}(x, i\nabla\varphi_2) - \rho_1)\alpha_{2,0}^{0,0} = 0. \tag{4-38}$$

Here we also have used the fact that $\rho \sim \sum_{k\geq 1} h^k \rho_k$ as $h \to 0$.

Identifying the other coefficients, one obtains a series of equations that (in a way similar to [Pettersson 1997, Section 4]) can be solved in $\mathcal{V}_1$ (possibly after having shrunk it a little bit around $x^{(1)}$), and in such a way that one also has

$$\widetilde{w}_2 - \widetilde{w}_1 = \mathbb{O}(h^\infty e^{-\varphi/h}) \quad \text{locally uniformly in } \mathcal{V}_1 \cap \{V_2 < V_1\}, \tag{4-39}$$

where w_1 is defined in (4-1) and $\widetilde{w}_1$ and $\widetilde{w}_2$ are resummations of w_1 and w_2. Among other things, this implies

$$\alpha_{2,0}^{0,0} = a_{2,0} \quad \text{in } \mathcal{V}_1 \cap \{V_2 < V_1\}. \tag{4-40}$$

Moreover, we see in (4-36) and (4-38) that $\beta_{1,0}^{0,0}$ (respectively $\alpha_{2,0}^{0,0}$) is a solution of a differential equation of order 1 on each integral curve of the real vector field $\nabla\varphi_1(y) \cdot \nabla_y$ (respectively $\nabla\varphi_2(y) \cdot \nabla_y$). In particular, because of the ellipticity of $a_{2,0}$, we deduce from (4-38) and (4-40) that we have that

$$\alpha_{2,0}^{0,0} \quad \text{never vanishes in } \mathcal{V}_1. \tag{4-41}$$

Now, Assumption 6 implies that, if $\gamma \in G_0$, then

$$r_{1,2}(x, i\nabla\varphi_2) \neq 0 \quad \text{on } \mathcal{V}_1. \tag{4-42}$$

Since $p_1(y, i\nabla\varphi_2) = p_1(y, i\nabla\varphi_1) = 0$ on $\omega_1 \cap \partial\Omega$, we deduce from (4-35) and (4-41) that, if $\gamma \in G_0$, then $\beta_{1,0}^{0,0}$ does not vanish on $\omega_1 \cap \partial\Omega$. As before, because of (4-36) (and the fact that $R(x, hD_x)$ is formally self-adjoint), this implies:

$$\text{if } \gamma \in G_0, \text{ then } \beta_{1,0}^{0,0} \text{ never vanishes in } \mathcal{V}_1. \tag{4-43}$$

In the island, outside the cirque. Now, we look at what happens on $\gamma^{(2)}$ and, at first, near $x^{(1)}$. Using the asymptotics of $Y_{k,\varepsilon}(z/\sqrt{h})$ given in [Pettersson 1997, Section 4], one also finds that, in $\mathcal{V}_1 \cap \{V_1 < V_2\}$, w_2 can be formally identified with

$$w_3(x,h) = \sqrt{2\pi h} \begin{pmatrix} b_1(x,h) \\ hb_2(x,h) \end{pmatrix} e^{-\varphi(x)/h}, \tag{4-44}$$

where b_1, b_2 are symbols of the form

$$b_j(x;h) = \sum_{l\geq 0} \sum_{m=0}^{l} h^l (\ln h)^m b_j^{l,m}(x) \quad (j=1,2), \tag{4-45}$$

with $b_j^{l,m} \in C^\infty(\mathcal{V}_1 \cap \{V_1 < V_2\})$, in the sense that, for any resummations $\widetilde{w}_2$ and $\widetilde{w}_3$ of w_2 and w_3,

$$\widetilde{w}_2 - \widetilde{w}_3 = \mathbb{O}(h^\infty e^{-\varphi/h}) \quad \text{locally uniformly in } \Omega \cap \Gamma_+. \tag{4-46}$$

Moreover, one also has

$$b_1^{0,0} = \beta_{1,0}^{0,0}, \tag{4-47}$$

which, by (4-43), shows that, when $\gamma \in G_0$, b_1 is elliptic in $\mathcal{V}_1 \cap \{V_1 < V_2\}$.

Since $p_2(x, i\nabla\varphi(x)) \neq 0$ in $\{V_1 < V_2\}$, we can formally solve the equation $Pw_3 = \rho_1 w_3$, and we see again that b_1 and b_2 can be continued along the integral curves of $\nabla\varphi$, as long as these curves stay inside $\{V_1 < V_2\}$ and φ_1 does not develop caustics. In particular, they can be continued in a neighborhood $\mathcal{N}_2$ of $\gamma^{(2)}$, and the continuation of b_1 remains elliptic in Ω_2.

Clearly, the previous steps can be repeated near $x^{(2)}$, $x^{(3)}$, etc. (in the case $N_\gamma \geq 3$), up to $x^{(N_\gamma+1)}$, obtaining in that way (after having pasted everything in a standard way by using a partition of unity) a function $\mathbf{w}(x,h)$, smooth on a neighborhood $\mathcal{N}(\gamma)$ of γ in $\ddot{O}$, satisfying

$$(P-\rho_1)\boldsymbol{w} = \mathbb{O}(h^\infty e^{-\varphi/h}),$$

locally uniformly in $\mathcal{N}(\gamma)$. Moreover, $\mathcal{N}(\gamma)$ can be decomposed into

$$\mathcal{N}(\gamma) = \mathcal{N}_1 \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_{N_\gamma} \cup \mathcal{N}_{N_\gamma+1},$$

where, for all j, $\mathcal{V}_j$ is a neighborhood of $x^{(j)}$ and $\mathcal{N}_j$ is a neighborhood of $\gamma^{(j)}$, in such a way that, in each $\mathcal{N}_j$, $\boldsymbol{w}$ admits a WKB asymptotics of the form,

$$\boldsymbol{w}(x;h) \sim h^{(j-1)/2} \begin{pmatrix} h^{(1-(-1)^j)/2} a_1^{(j)}(x,h) \\ h^{(1+(-1)^j)/2} a_2^{(j)}(x,h) \end{pmatrix} e^{-\varphi(x)/h}, \tag{4-48}$$

where $a_1^{(j)}$ and $a_2^{(j)}$ are symbols of the same form as in (4-45), and $a_1^{(j)}$ is elliptic if j is even, while $a_2^{(j)}$ is elliptic if j is odd (in particular, $a_1^{(N_\gamma+1)}$ is elliptic). On the other hand, in each $\mathcal{V}_j$, $\boldsymbol{w}$ can be represented by means of the Weber function, in a way similar to that of (4-7).

At and after the boundary of the island. Let us denote by $x_\gamma \in \gamma \cap \partial \ddot{O}$ the point of type 1 where γ touches the boundary of the island. When $x \in \gamma \cap \ddot{O}$ is close enough to x_γ, we know from the previous subsection that the asymptotic solution $\boldsymbol{w}$ is of the form

$$\boldsymbol{w}(x;h) \sim h^{N_\gamma/2}\begin{pmatrix} b_1(x,h) \\ hb_2(x,h)\end{pmatrix} e^{-\varphi(x)/h}, \tag{4-49}$$

where b_1, b_2 are smooth symbols on $\mathcal{N}_{N_\gamma+1}$ of the same form as in (4-45), and b_1 is elliptic. Moreover, as x approaches x_γ, b_1 and b_2 (together with φ) develop singularities on some set $\mathscr{C}$ (called the caustic set). However, following an idea of [Helffer and Sjöstrand 1986], we can represent $h^{-N_\gamma/2}e^{S/h}w$ in the integral (Airy) form

$$I[c_1,c_2](x,h) = h^{-\frac12}\int_{\gamma(x)} \begin{pmatrix} c_1(x',\xi_n,h) \\ hc_2(x',\xi_n,h)\end{pmatrix} e^{-(x_n\xi_n+g(x',\xi_n))/h}\,d\xi_n, \tag{4-50}$$

where we have used local Euclidean coordinates $(x',x_n)\in\mathbb{R}^{n-1}\times\mathbb{R}$ centered at $\gamma\cap\partial\ddot{O}$, such that $V_1(x)=-C_0x_n+\mathbb{O}(x^2)$ near this point. For x in $\ddot{O}$ close to $\gamma\cap\partial\ddot{O}$, the phase function $\xi_n\mapsto x_n\xi_n+g(x',\xi_n)$ admits two real critical points that are close to 0. Then, choosing conveniently the x-dependent interval $\gamma(x)$, the steepest descent method at one of these points gives us the asymptotic expansion of $I[c_1,c_2]$. Comparing this with the symbols b_1 and b_2, one can determine c_1 and c_2 so that the asymptotic expansion of $h^{-N_\gamma/2}e^{S/h}w$ coincides with that of $I[c_1,c_2]$ in $\ddot{O}$. In particular, when $\gamma\in G_0$, one finds that c_1 remains elliptic near 0.

At this point, since we did not assume any analyticity of the potentials near $\ddot{O}$, we have to follow the methods of [Fujiié et al. 2011] — a reference we will henceforth abbreviate as [FLM] — where a similar situation is considered. Indeed, following the constructions of [FLM, Section 4] (that are made in the scalar case, but can be generalized without problem to our vectorial case), we see that there exists a constant $\delta>0$ such that, for any $N\ge 1$, one can construct a (vectorial) function w_N, smooth on the set

$$\mathscr{W}_N(\gamma) := \{|x-x_\gamma|<\varepsilon\}\cap\{\operatorname{dist}(x,\ddot{O})<2(Nk)^{2/3}\} \tag{4-51}$$

with $\varepsilon>0$ small enough (recall from (3-2) that $k=|h\ln h|$), such that [FLM, Propositions 4.5 and 4.6]:

- $(P-\rho_1)w_N=\mathbb{O}(h^{\delta N}e^{-\operatorname{Re}\widetilde{\varphi}_N/h})$ uniformly in $\mathscr{W}_N(\gamma)$.
- For any $\alpha\in\mathbb{Z}_+^n$, there exists $m_\alpha\ge 0$ independent of N such that

$$\partial_x^\alpha w_N = \mathbb{O}(h^{-m_\alpha}e^{-\operatorname{Re}\widetilde{\varphi}_N/h})$$

  uniformly in $\mathscr{W}_N(\gamma)$.
- w_N can be represented by an integral of the form (4-50) (with $\gamma(x)=\gamma_N(x)$ depending on N) in all of $\mathscr{W}_N(\gamma)$.
- $w_N=\boldsymbol{w}$ in $\mathcal{N}_{N_\gamma+1}\cap\mathscr{W}_N(\gamma)$.

- For any large enough L, there exist $C_L > 0$ and $\delta_L > 0$, both independent of N, such that, uniformly in $\mathscr{W}_N(\gamma) \cap \{\mathrm{dist}(x, \ddot{O}) \geq (Nk)^{2/3}\}$, one has

$$w_N(x,h) = h^{N_\gamma/2}\Biggl(\sum_{\substack{\ell=0\\0\leq m\leq \ell}}^{L+[Nk/C_L h]} h^\ell (\ln h)^m \begin{pmatrix} f_{1,N}^{\ell,m}(x)\\ h f_{2,N}^{\ell,m}(x)\end{pmatrix} + \mathcal{O}(h^{\delta_L N} + h^L)\Biggr) e^{-\widetilde{\varphi}_N(x)/h} \quad \text{as } h \to 0, \tag{4-52}$$

with $f_{1,N}^{\ell,m}(x)$, $f_{2,N}^{\ell,m}(x)$ independent of h and of the form

$$\tilde{f}_{j,N}^{\ell,m}(x) = (\mathrm{dist}(x,\mathscr{C}))^{-3\ell/2-1/4}\beta_{j,N}^{\ell,m}(x, \mathrm{dist}(x,\mathscr{C})), \qquad j = 1, 2, \tag{4-53}$$

where $\beta_{j,N}^{\ell,m}$ is smooth near $(x_\gamma, 0)$, and $\beta_1^{\ell,m}(x_\gamma, 0) \neq 0$ in the case $\gamma \in G_0$.

Here, $\widetilde{\varphi}_N$ is a (complex-valued) C^1 function on $\mathscr{W}_N(\gamma)$, smooth on $\mathscr{W}_N(\gamma) \setminus \mathscr{C}$, such that [FLM, Lemma 4.1]:

- $\widetilde{\varphi}_N = \varphi + \mathcal{O}(h^\infty)$ uniformly in $\mathcal{N}_{N_\gamma+1} \cap \mathscr{W}_N(\gamma)$.
- $(\nabla\widetilde{\varphi}_N)^2 = V_1(x) + \mathcal{O}(h^\infty)$ uniformly in $\mathscr{W}_N(\gamma)$.
- There exists $\varepsilon(h) = \mathcal{O}(h^\infty)$ real such that, for $x \in \mathscr{W}_N(\gamma) \setminus \ddot{O}$, one has

$$\mathrm{Re}\,\widetilde{\varphi}_N(x) \geq S - \varepsilon(h). \tag{4-54}$$

- One has

$$\mathrm{Im}\,\nabla\varphi_N(x) = -\nu_N(x)\sqrt{\mathrm{dist}(x,\mathscr{C})}\,\nabla\,\mathrm{dist}(x,\mathscr{C}) + \mathcal{O}(\mathrm{dist}(x,\mathscr{C})),$$

uniformly with respect to $h > 0$ small enough and $x \in \mathscr{W}_N(\gamma) \setminus \ddot{O}$ with $\nu_N(x) \geq \delta$.

The previous results show that we can extend $\boldsymbol{w}$ by taking w_N in $\mathscr{W}_N(\gamma)$, and we obtain in that way a function $\boldsymbol{w}_N$ smooth on $\mathcal{N}(\gamma) \cup \mathscr{W}_N(\gamma)$, such that $(P - \rho_1)\boldsymbol{w}_N = \mathcal{O}(h^{\delta N} e^{-\mathrm{Re}\,\widetilde{\varphi}/h})$ uniformly in $\mathcal{N}(\gamma) \cup \mathscr{W}_N(\gamma)$. Note that, thanks to Assumption 4, the number N_γ is constant on each connected component of Γ.

5. Agmon estimates

Preliminaries. In order to perform Agmon estimates in the same spirit as in [Helffer and Sjöstrand 1984], we need some preliminary results because of the fact that we have to deal with pseudodifferential operators (and not only Schrödinger operators). For this reason, we prefer to work with C^∞ weight functions (instead of Lipschitz ones), and the idea is to take h-dependent regularizations of Lipschitz weights.

At first, we need:

Proposition 5.1. *Let $\nu_0 > 0$, $m \geq 0$, $a = a(x,\xi) \in S_{\nu_0}(\langle\xi\rangle^{2m})$. For $h > 0$ small enough, let also $\Phi_h \in C^\infty(\mathbb{R}^n)$ be real-valued, such that*

$$\sup|\nabla\Phi_h| < \nu_0 \tag{5-1}$$

and, for any multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| \geq 2$,

$$\partial^\alpha\Phi_h(x) = \mathcal{O}(h^{1-|\alpha|}), \tag{5-2}$$

uniformly for $x \in \mathbb{R}^n$ and $h > 0$ small enough. Then, for any $\widetilde{\Sigma} \subset \mathbb{R}^n$ with dist$(\Sigma, \mathbb{R}^n \setminus \widetilde{\Sigma}) > 0$, the operator $e^{\Phi_h/h} A e^{-\Phi_h/h} := e^{\Phi_h/h} \operatorname{Op}_h^W(a) e^{-\Phi_h/h}$ satisfies

$$\|e^{\Phi_h/h} A e^{-\Phi_h/h} u\|_{L^2} \leq C_1 \|\langle hD_x\rangle^m u\|_{L^2}, \tag{5-3}$$

uniformly for all $h > 0$ small enough and $u \in H^m(\mathbb{R}^n)$.

Proof. For $u \in C_0^\infty(\mathbb{R}^n)$, we write

$$e^{\Phi/h} A e^{-\Phi/h} u(x) = \frac{1}{(2\pi h)^n} \int e^{i(x-y)\xi/h + (\Phi(x)-\Phi(y))/h} a\left(\frac{x+y}{2}, \xi\right) u(y)\, dy\, d\xi,$$

and the property (5-1) shows that we can make the change of contour of integration given by

$$\mathbb{R}^n \ni \xi \mapsto \xi + i\Psi(x, y),$$

where $\Psi(x, y) := \int_0^1 \nabla\Phi((1-t)x + ty)\, dt$ (in particular, one has $\Phi(x) - \Phi(y) = (x - y)\Psi(x, y)$). Then, denoting by Op_h the semiclassical quantization of symbols depending on $3n$ variables (see, e.g., [Martinez 2002, Section 2.5]), we obtain

$$e^{\Phi/h} A e^{-\Phi/h} = \operatorname{Op}_h\left(a\left(\frac{x+y}{2}, \xi + i\Psi(x, y)\right)\right),$$

and, using (5-2), we see that, for any $\alpha, \beta, \gamma \in \mathbb{Z}_+^n$, we have

$$\partial_x^\alpha \partial_y^\beta \partial_\xi^\gamma \left(a\left(\frac{x+y}{2}, \xi + i\Psi(x, y)\right)\right) = \mathcal{O}(h^{-|\alpha+\beta|}\langle\xi\rangle^m). \tag{5-4}$$

Then, the result is an easy consequence of the Calderón–Vaillancourt Theorem; see [Martinez 2002, Exercise 2.10.15], for example. □

Proposition 5.2. *Let ϕ and V be two bounded real-valued Lipschitz functions on $\mathbb{R}^n$ with $|\nabla\phi(x)|^2 \leq V(x)$ almost everywhere. Let also $\chi_1 \in C_0^\infty(\mathbb{R}^n; [0, 1])$ be supported in the ball $\{|x| \leq 1\}$, with $\int \chi_1(x)\, dx = 1$. For any $h > 0$, we set $\chi_h(x) = h^{-n}\chi(x/h)$. Then, the smooth function*

$$\phi_h := \chi_h * \phi$$

(*where $*$ stands for the standard convolution*) *satisfies*:

- *$\phi_h = \phi + \mathcal{O}(h)$ uniformly for $h > 0$ small enough and $x \in \mathbb{R}^n$.*
- *For all $x \in \mathbb{R}^n$, one has $|\nabla\phi_h(x)|^2 \leq V(x) + h\|\nabla V\|_{L^\infty}$.*
- *For all $\alpha \in \mathbb{Z}_+^n$ with $|\alpha| \geq 1$, one has $\partial^\alpha \phi_h = \mathcal{O}(h^{1-|\alpha|})$.*

The proof of this proposition is very standard and almost obvious, and we leave it to the reader. Observe that, in particular, ϕ_h satisfies the estimates (5-2).

Agmon estimates. As a corollary of the two previous propositions, we have:

Corollary 5.3. *Let ϕ and ϕ_h be as in Proposition 5.2, with $V = \min(V_1, V_2)_+$. Then one has, for any $u = (u_1, u_2) \in H^2(\mathbb{R}^n) \oplus H^2(\mathbb{R}^n)$,*

$$\begin{aligned}\mathrm{Re}\langle e^{\phi_h/h}Pu, e^{\phi_h/h}u\rangle \\ \geq \|h\nabla(e^{\phi_h/h}u)\|^2 + \sum_{j=1}^{2}\langle (V_j - |\nabla\phi_h|^2)e^{\phi_h/h}u_j, e^{\phi_h/h}u_j\rangle - C_R h(\|e^{\phi_h/h}u\|^2 + \|h\nabla(e^{\phi_h/h}u)\|^2),\end{aligned}$$

where $C_R > 0$ is a constant that depends on $R(x, hD_x)$, χ_1 and $\sup|\nabla\phi|$ only.

Proof. It is standard (and elementary) to show that

$$\mathrm{Re}\langle e^{\phi_h/h}(-h^2\Delta + V_j)u_j, e^{\phi_h/h}u_j\rangle = \|h\nabla(e^{\phi_h/h}u_j)\|^2 + \langle (V_j - |\nabla\phi_h|^2)e^{\phi_h/h}u_j, e^{\phi_h/h}u_j\rangle.$$

Therefore, it is enough to estimate $\langle e^{\phi_h/h}R(x, hD_x)u, e^{\phi_h/h}u\rangle$. Applying Proposition 5.1 , we see that the operator $e^{\phi_h/h}R(x, hD_x)e^{-\phi_h/h}\langle hD_x\rangle^{-1}$ is uniformly bounded on L^2.

Moreover, since the constants appearing in the estimates (5-4) depend on a, α, and on the estimates on the $\partial^\beta\Phi$ only, we see that the norm of $e^{\phi_h/h}R(x, hD_x)e^{-\phi_h/h}\langle hD_x\rangle^{-1}$ depends on r and on estimates on $\partial^\beta(\chi_h * \nabla\phi) = (\partial^\beta\chi_h) * \nabla\phi$ ($|\beta| \leq |\alpha|$) only. Since the latter depend on α, χ_1 and $\sup|\nabla\phi|$ only, the result follows. $\square$

6. Global asymptotic solution

The constructions of Section 4 can be done in a neighborhood of any minimal geodesic $\gamma \in G$, and give rise (after having pasted them together with a partition of unity) to an asymptotic solution (still denoted by $\boldsymbol{w}_N$) on a neighborhood of $\bigcup_{\gamma\in G}\gamma$. Now, we plan to extend this solution to a whole (h-dependent) neighborhood of $\{V_1 \geq 0\}$, by using a modified self-adjoint operator with discrete spectrum near 0.

At first, we fix $\varepsilon_0 > 0$ sufficiently small, and a cutoff function $\chi_0 \in C_0^\infty(\ddot{O}; [0, 1])$ such that

$$\chi_0(x) = 1 \quad \text{if } V_1(x) \geq 2\varepsilon_0, \qquad \chi_0(x) = 0 \quad \text{if } V_1(x) \leq \varepsilon_0,$$

and we set

$$\widetilde{V}_1 := \chi_0 V_1 + \varepsilon_0(1 - \chi_0). \tag{6-1}$$

In particular, $\widetilde{V}_1$ coincides with V_1 on $\{V_1 \geq 2\varepsilon_0\}$, and we have $\widetilde{V}_1 \geq \varepsilon_0$ everywhere. Then, we define $\widetilde{P}_1 := -h^2\Delta + \widetilde{V}_1$, and we consider the self-adjoint operator

$$\widetilde{P} = \begin{pmatrix} \widetilde{P}_1 & 0 \\ 0 & P_2 \end{pmatrix} + hR(x, hD_x). \tag{6-2}$$

By construction, for all $C > 0$ and h small enough, the spectrum of $\widetilde{P}$ is discrete in $[-Ch, Ch]$, and a straightforward adaptation of the arguments used in [Helffer and Sjöstrand 1984] shows that its first eigenvalue E_1 admits the same asymptotics as ρ_1 as $h \to 0_+$. We denote by $\boldsymbol{v}$ its first normalized eigenfunction, and by $\mathcal{N}_0 \subset \{V_1 > 2\varepsilon_0\}$ some fixed neighborhood of $\bigcup_{\gamma\in G} \cap \{V_1 > 2\varepsilon_0\}$ where the asymptotic solution $\boldsymbol{w}_N$ is well-defined. We have:

Proposition 6.1. *There exists $\theta_0 \in \mathbb{R}$ independent of h such that, for any compact subset K of $\mathcal{N}_0$, and for any $\alpha \in \mathbb{Z}_+^n$, one has*

$$\|e^{\varphi/h}\partial^\alpha(e^{i\theta_0}\boldsymbol{v} - h^{n/4}\boldsymbol{w}_N)\|_K = \mathcal{O}(h^\infty).$$

Proof. The existence of θ_0 such that $\partial^\alpha(e^{i\theta_0}\boldsymbol{v} - h^{n/4}\boldsymbol{w}_N) = \mathcal{O}(h^\infty)$ uniformly near 0 is a consequence of [Helffer and Sjöstrand 1984, Proposition 2.5] and standard Sobolev estimates. Let $\chi \in C_0^\infty(\mathcal{N}_0; [0, 1])$, with $\chi = 1$ in a neighborhood of $K \cup \{0\}$. Following [Helffer and Sjöstrand 1984; Pettersson 1997], we plan to apply Corollary 5.3 to $u := \chi(e^{i\theta_0}\boldsymbol{v} - h^{n/4}\boldsymbol{w}_N)$, with a suitable weight function ϕ. Let us first observe that, using Corollary 5.3, for any $\varepsilon > 0$ one has

$$\|e^{(1-\varepsilon)\widetilde{\varphi}/h}\langle hD_x\rangle\boldsymbol{v}\|_{H^1} = \mathcal{O}(1), \tag{6-3}$$

where $\widetilde{\varphi}(x) \geq \varphi(x)$ is the Agmon distance associated with $\min(\widetilde{V}_1, V_2)$ between 0 and x. Now, for $C \geq 1$ arbitrarily large, we define

$$\phi(x) := \min(\phi_1, \phi_2),$$

where

$$\phi_1(x) := \begin{cases} \varphi(x) - Ch\ln\left(\dfrac{\varphi(x)}{h}\right) & \text{if } \varphi(x) \geq Ch, \\ \varphi(x) - Ch\ln C & \text{if } \varphi(x) \leq Ch, \end{cases}$$

$$\phi_2(x) := \begin{cases} \inf\limits_{\chi(y)\neq 1}(1-2\varepsilon)(\varphi(y) + d(y, x)) & \text{if } x \in \operatorname{supp}\chi, \\ (1-2\varepsilon)\varphi(x) & \text{if } x \notin \operatorname{supp}\chi. \end{cases}$$

Here, $\varepsilon > 0$ is taken sufficiently small to have $\phi_2(x) > \varphi(x)$ when $x \in K$. Then, ϕ is Lipschitz continuous, and one has $\phi = \phi_1$ on K and $\phi = \phi_2$ on $\mathbb{R}^n \setminus \{\chi = 1\}$. Moreover, one sees as in the proof of [Pettersson 1997, Theorem 5.5] that, if we set $V := \min(V_1, V_2)$, ϕ satisfies

$$|\nabla\phi|^2 = V \quad \text{in } \{\varphi \leq Ch\}, \qquad |\nabla\phi|^2 \leq V - \delta_0 Ch \quad \text{in } \{\varphi \geq Ch\},$$

where $\delta_0 = \inf_{x\in\operatorname{supp}\chi,\, x\neq 0}(V(x)/\varphi(x)) > 0$. As a consequence, by Proposition 5.2, the regularized ϕ_h of ϕ satisfies

$$|\nabla\phi_h|^2 \leq V + h\|V\|_{L^\infty} \quad \text{in } \{\varphi \leq Ch\}, \qquad |\nabla\phi_h|^2 \leq V - (\delta_0 C - \|V\|_{L^\infty})h \quad \text{in } \{\varphi \geq Ch\}.$$

Then, choosing C sufficiently large and setting $u := \chi(e^{i\theta_0}\boldsymbol{v} - h^{n/4}\boldsymbol{w}_N)$, we see that Corollary 5.3 implies

$$\|h\nabla(e^{\phi_h/h}u)\|^2 + C'h\|e^{\phi_h/h}u\|^2_{\{\varphi\geq Ch\}} \leq \langle e^{\phi_h/h}(\widetilde{P} - E_1)u, e^{\phi_h/h}u\rangle, \tag{6-4}$$

with $C' = C'(C)$ arbitrarily large. Moreover, if $\widetilde{\chi} \in C_0^\infty(\mathcal{N}_0)$ is such that $\widetilde{\chi}\chi = \chi$, we have

$$(\widetilde{P} - E_1)u = [\widetilde{P}, \chi]\widetilde{\chi}u + \mathcal{O}(h^\infty e^{-\varphi/h}),$$

and since $\phi_h = (1-2\varepsilon)\varphi + \mathcal{O}(h)$ on $\operatorname{supp}\nabla\chi$, $\min_{\operatorname{supp}\nabla\chi}\varphi =: \delta_1 > 0$ and, by Proposition 5.1, the operator $e^{\phi_h/h}[R, \chi]e^{-\phi_h/h}$ is uniformly bounded, we obtain, using also (6-3),

$$\langle e^{\phi_h/h}(\widetilde{P} - E_1)u, e^{\phi_h/h}u\rangle = \mathcal{O}(\|e^{(1-\varepsilon)\varphi/h}\langle hD_x\rangle\widetilde{\chi}u\|^2_{\operatorname{supp}\nabla\chi} + h\|e^{\phi_h/h}u\|^2) = \mathcal{O}(e^{-2\varepsilon\delta_1/h} + h\|e^{\phi_h/h}u\|^2).$$

Inserting this estimate into (6-4) and taking C sufficiently large, this permits us to obtain

$$\|h\nabla(e^{\phi_h/h}u)\|^2 + h\|e^{\phi_h/h}u\|^2 = \mathbb{O}(e^{-2\varepsilon\delta_1/h} + \|e^{\phi_h/h}u\|^2_{\{\varphi\le Ch\}}).$$

In particular, since $\phi_h = \phi_1 + \mathbb{O}(h)$ on K and $\phi_h = (1-2\varepsilon)\varphi \le Ch$ on $\{\varphi \le Ch\}$,

$$\|h^C\varphi^{-C}e^{\varphi/h}h\nabla u\|^2_K + \|h^C\varphi^{-C}e^{\varphi/h}u\|^2_K = \mathbb{O}(e^{-2\varepsilon\delta_1/h} + \|u\|^2_{\{\varphi\le Ch\}}).$$

Therefore,

$$\|e^{\varphi/h}\nabla u\|^2_K + \|e^{\varphi/h}u\|^2_K = \mathbb{O}(h^\infty),$$

and the result follows by standard Sobolev estimates. □

Now, following [FLM, Section 4.3], we observe that, if ε_0 has been taken small enough, the asymptotic solution $\boldsymbol{w}_N$ is $\mathbb{O}(h^{\delta N}e^{-S/h})$ uniformly on the set

$$\Big\{\mathrm{dist}\Big(x, \bigcup_{\gamma\in G}\gamma\Big) \ge \varepsilon_0\Big\} \cap \{V_1 \le 2\varepsilon_0\} \cap \Big(\bigcup_{\gamma\in G}\mathcal{N}(\gamma)\cup\mathcal{W}_N(\gamma)\Big)$$

Moreover, by (6-3), the same is true for $\boldsymbol{v}$ on $\big\{\mathrm{dist}(x, \bigcup_{\gamma\in G}\gamma) \ge \varepsilon_0\big\} \cap \{V_1 \le 2\varepsilon_0\}$. Therefore, using also Proposition 6.1, we can paste together $e^{i\theta_0}\boldsymbol{v}$ and $h^{-n/4}\boldsymbol{w}_N$ in order to obtain a function $\boldsymbol{u}_N$ that satisfies the properties of the following proposition; see also [FLM, Proposition 4.6].

Proposition 6.2. *There exists a function $\boldsymbol{u}_N$, smooth on $\ddot{O}_N := \{\mathrm{dist}(x, \ddot{O}) < 2(Nk)^{2/3}\}$, such that*

$$(P-\rho)\boldsymbol{u}_N = \mathbb{O}(h^{\delta N}e^{-\operatorname{Re}\varphi_N/h}), \qquad \partial^\alpha\boldsymbol{u}_N = \mathbb{O}(h^{-m_\alpha}e^{-\operatorname{Re}\varphi_N/h}),$$

uniformly on $\ddot{O}_N$, where $\tilde{\varphi}_N$ is as in (4-54). Moreover, in $\bigcup_{\gamma\in G}\mathcal{W}_N(\gamma)\cap\{\mathrm{dist}(x, \ddot{O}) \ge (Nk)^{2/3}\}$ one can write $\boldsymbol{u}_N$ as in (4-52) (with $\beta_1^{\ell,m}(x_\gamma, 0) \ne 0$), while away from $\bigcup_{\gamma\in G}\mathcal{W}_N(\gamma)\cap\{x \notin \ddot{O}\}$, $\boldsymbol{u}_N$ is $\mathbb{O}(h^{\delta N}e^{-\operatorname{Re}\varphi_N/h})$.

7. Comparison between asymptotic and true solution

A priori estimates. In the same spirit as in [FLM, Theorem 2.2], we start with an a priori estimate for the resonant state of P. From now on, we denote by $\boldsymbol{u}$ the outgoing solution of

$$P\boldsymbol{u} = \rho_1\boldsymbol{u}, \tag{7-1}$$

normalized in the following way: we fix some analytic distorted space (also more recently introduced, in the context of computational physics, under the name of perfectly matched layer; see [Berenger 1994], for example) of the form

$$\widetilde{\mathbb{R}}^n_\theta := \{x + i\theta F(x) : x \in \mathbb{R}^n\}, \tag{7-2}$$

where $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$, $F(x) = 0$ if $|x| \le R_0$, $F(x) = x$ for $|x|$ large enough, and where $\theta > 0$ is sufficiently small and may also tend to 0 with h, but not too rapidly (here, we take $\theta = h|\ln h| = k$). Then, by definition, the fact that ρ_1 is a resonance of P means that (7-1) admits a solution in $L^2(\widetilde{\mathbb{R}}^n_\theta)$, and here we take $\boldsymbol{u}$ such that

$$\|\boldsymbol{u}\|_{L^2(\widetilde{\mathbb{R}}^n_\theta)} = 1. \tag{7-3}$$

As before, d stands for the Agmon distance associated with the pseudometric $\min(V_1, V_2)_+ \, dx^2$, and we denote by $B_d(S) := \{x \in \mathbb{R}^n : d(0, x) < S\}$ the corresponding open ball of radius $S = d(0, \partial \ddot{O})$. Then, we first have:

Proposition 7.1. *For any compact subset $K \subset \mathbb{R}^n$, there exists $N_K \geq 0$ such that*

$$\|e^{s(x)/h} \boldsymbol{u}\|_{H^1(K)} = \mathbb{O}(h^{-N_K}),$$

uniformly as $h \to 0$, where $s(x) = \varphi(x)$ if $x \in B_d(S)$ and $s(x) = S$ otherwise.

Proof. The proof is very similar to that of [FLM, Theorem 2.2], with the only difference that here we have to deal with pseudodifferential operators, forbidding us to use Dirichlet realizations and nonsmooth weight functions. Instead, we modify V_1 in a way similar to (6-1), and we regularize the weights as in Proposition 5.2.

We consider a cutoff function $\hat{\chi}$ (dependent on h) such that

$$\hat{\chi}(x) = 1 \quad \text{if } V_1(x) \geq 2k^{2/3}, \qquad \hat{\chi}(x) = 0 \quad \text{if } V_1(x) \leq k^{2/3}, \qquad \partial^\alpha \hat{\chi} = \mathbb{O}(k^{-2|\alpha|/3}),$$

and we set

$$\hat{V}_1 := \hat{\chi} V_1 + k^{2/3}(1 - \hat{\chi}), \quad \hat{P}_1 := -h^2 \Delta + \hat{V}_1, \quad \hat{P} = \begin{pmatrix} \hat{P}_1 & 0 \\ 0 & P_2 \end{pmatrix} + hR(x, hD_x). \tag{7-4}$$

We denote by $\hat{E}$ the first eigenvalue of $\hat{P}$, and by $\hat{v}$ its first normalized eigenfunction. Moreover, we consider the Agmon distance $\hat{d}$ associated with the pseudometric $(\min(V_1, V_2)_+ - \hat{E})\, dx^2$, and we set $\hat{\varphi}(x) := \hat{d}(0, x)$. Then, the same proof as in [FLM, Lemma 3.1] shows the existence of a constant $C_1 > 0$ such that

$$s(x) - C_1 k \leq \hat{\varphi}(x) \leq \varphi(x) \quad (x \in \mathbb{R}^n). \tag{7-5}$$

Moreover, an adaptation of the proof of [FLM, Lemma 3.2] (obtained by using Proposition 5.2 in order to regularize the Lipschitz weight) gives

$$\|e^{\hat{\varphi}/h} \hat{v}\|_{H^1(\mathbb{R}^n)} = \mathbb{O}(h^{-N_0}), \tag{7-6}$$

for some $N_0 \geq 0$. Then, the result follows by considering the function $\hat{\chi}\hat{v}$ and by observing that, thanks to (7-6), one has [FLM, Lemma 3.3 and Formula (3.20)]

$$\left\| \hat{\chi}\hat{v} - \frac{1}{2i\pi} \int_\gamma (z - P_\theta)^{-1} \hat{\chi}\hat{v}\, dz \right\|_{H^1} = \mathbb{O}(h^{-N_1} e^{-S/h}).$$

Here, γ is the oriented complex circle $\{z \in \mathbb{C} : |z - \hat{E}| = h^2\}$ and P_θ is a convenient distortion of P. The previous estimate actually shows that the distorted $\boldsymbol{u}_\theta$ of $\boldsymbol{u}$ coincides — up to $\mathbb{O}(h^{-N_1} e^{-S/h})$ — with $\mu \hat{\chi} \hat{v}$, where μ is a complex constant satisfying $|\mu| = 1 + \mathbb{O}(e^{-\delta/h})$, for some $\delta > 0$. $\square$

Remark 7.2. The previous proof also gives a global estimate on $\boldsymbol{u}_\theta$,

$$\|e^{s(x)/h} \boldsymbol{u}_\theta\|_{H^1(\mathbb{R}^n)} = \mathbb{O}(h^{-N_1'}),$$

for some constant $N_1' \geq 0$. See [FLM, Lemma 3.3 and Formula (3.20)].

Now, we plan to give an even better a priori estimate on the difference $\boldsymbol{u}-\boldsymbol{u}_N$ near the boundary of the island. Here again, we follow the arguments given in [FLM, Section 5]. For any $N \geq 1$, we set

$$U_N := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \partial \ddot{O}) < 2(Nk)^{2/3}\}.$$

We have [FLM, Propositions 5.1 and 5.2]:

Proposition 7.3. *There exist $N_1 \geq 0$ and $C \geq 1$ such that, for any $N \geq 1$ large enough, one has*

$$\|\boldsymbol{u}-\boldsymbol{u}_{CN}\|_{H^1(U_N)} \leq h^{-N_2} e^{-S/h}.$$

Proof. We just recall the main lines of the proof in [FLM]. At first, thanks to Proposition 7.1 and the particular form of $\boldsymbol{u}_{CN}$, we immediately see that the estimate is true on the set $\{\varphi(x) \geq S-2k\}$. Then, we take a cutoff function $\widetilde{\chi} \in C_0^\infty(\varphi(x) < S-k)$ such that $\widetilde{\chi}=1$ on $\{\varphi(x) \geq S-2k\}$ and $\partial^\alpha \widetilde{\chi} = \mathcal{O}(h^{-N_\alpha})$ for some $N_\alpha \geq 0$. We also consider the Lipschitz weight

$$\phi_N(x) = \min\bigl(\varphi(x) + C_1 Nk + k(S-\varphi(x))^{1/3},\, S + (1-k^{1/3})(S-\varphi(x))\bigr)$$

and, by using Propositions 7.1 and 6.2, we see that, if C is large enough, we have

$$\|e^{\phi_N/h}(P-\rho_1)\widetilde{\chi}(\boldsymbol{u}-\boldsymbol{u}_{CN})\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{-M_1}),$$

for some $M_1 \geq 0$ independent of N. Then, regularizing ϕ_N as in Proposition 5.2, we can perform Agmon estimates as in the proof of [FLM, Proposition 5.1], and we find

$$\|e^{\phi_N/h}\widetilde{\chi}(\boldsymbol{u}-\boldsymbol{u}_{CN})\|_{L^2(\mathbb{R}^n)} = \mathcal{O}(h^{-M_2}),$$

for some $M_1 \geq 0$ independent of N, and the result follows. □

Propagation. Now, we plan to prove (see [FLM, Proposition 6.1]):

Theorem 7.4. *For any $L > 0$ and for any $\alpha \in \mathbb{Z}_+^n$, there exists $N_{L,\alpha} \geq 1$ such that, for any $N \geq N_{L,\alpha}$,*

$$\partial_x^\alpha(\boldsymbol{u}-\boldsymbol{u}_{CN})(x,h) = \mathcal{O}(h^L e^{-S/h}) \quad \text{as } h \to 0, \tag{7-7}$$

uniformly in U_N.

Proof. As in [FLM], the proof relies on three different types of microlocal propagation arguments. We fix some $\hat{x} \in \partial \ddot{O}$ and we define the Fourier–Bors–Iagolnitzer transform T (see [Sjöstrand 1982; Martinez 2002], for example) as

$$Tu(x,\xi;h) := \int_{\mathbb{R}^n} e^{i(x-y)\xi/h-(x-y)^2/2h} u(y)\,dy.$$

(1) *Standard C^∞ propagation.* Since $\boldsymbol{u}$ is outgoing (that is, it becomes L^2 when restricted to the distorted space or the perfectly matched layer defined in (7-2)), one can see as in [FLM, Lemma 6.2] that, if $t_0 > 0$ is large enough, one has

$$T\boldsymbol{u}(x,\xi) = \mathcal{O}(h^\infty e^{-S/h}),$$

uniformly near $\exp(-t_0 H_{p_1})(\hat{x},0)$. Moreover, by Proposition 7.1, we know that $e^{S/h}\boldsymbol{u}$ remains $\mathcal{O}(h^{-N_0})$ (for some $N_0 \geq 0$) on a neighborhood of the x-projection of $\{\exp(-tH_{p_1})(\hat{x},0) : 0 < t \leq t_0\}$.

Then, the standard C^∞ propagation of the frequency set for the solution to a real principal type operator (see, e.g., [Martinez 2002]) shows that the previous estimate remains valid near $\exp(-tH_{p_1})(\hat{x}, 0)$ for any $t > 0$.

(2) *Nonstandard propagation in h-dependent domains.* Thanks to the previous result, we can concentrate our attention on a sufficiently small neighborhood of $\hat{x}$. As before, we choose local Euclidean coordinates $(x', x_n) \in \mathbb{R}^{n-1} \times \mathbb{R}$ centered at $\hat{x}$, such that $V_1(x) = -C_0 x_n + \mathcal{O}(|x-\hat{x}|^2)$. We also set $\mu_N := (Nk)^{-\frac{1}{3}}$, and we consider the modified Fourier–Bors–Iagolnitzer transform T_N defined by

$$T_N u(x, \xi; h) := \int_{\mathbb{R}^n} e^{i(x-y)\xi/h - (x'-y')^2/2h - \mu_N (x_n - y_n)^2/2h} u(y)\, dy. \tag{7-8}$$

Then, using the previous result it is elementary to show that, for any (fixed) $t > 0$ small enough, one has [FLM, Lemma 6.3]

$$T_N \mathbf{1}_{K_1} \boldsymbol{u}(x, \xi) = \mathcal{O}(h^\infty e^{-S/h}),$$

uniformly near $\exp(-tH_{p_1})(\hat{x}, 0)$. Here K_1 is of the form $K_1 = K \setminus B_d(S)$, where K is any compact neighborhood of the closure of $\ddot{O}$. The interest of the latter property is that, as shown in [FLM], it can be propagated up h-dependent times t of order $(Nk)^{1/3}$. More precisely, setting

$$\exp(tH_{p_1})(\hat{x}, 0) = (x'(t), x_n(t); \xi'(t), \xi_n(t)) \quad (t \in \mathbb{R}),$$

we have [FLM, Lemma 6.4]:

Lemma 7.5. *There exists $\delta_0 > 0$ such that, for any $\delta \in (0, \delta_0]$, for all $N \geq 1$ large enough, and for $t_{N,\delta} := \delta^{-1}(Nk)^{1/3}$, one has*

$$\mathbf{T}_N \mathbf{1}_{K_1} \boldsymbol{u} = \mathcal{O}(h^{\delta N} e^{-S/h}) \quad \textit{uniformly in } \mathcal{W}(t_N, h),$$

where

$$\mathcal{W}_\delta(N, h) := \big\{ |x_n - x_n(-t_{N,\delta})| \leq \delta (Nk)^{2/3}, |\xi_n - \xi_n(-t_{N,\delta})| \leq \delta (Nk)^{1/3}, |x' - x'(-t_{N,\delta})| \leq \delta (Nk)^{1/3}, \\ |\xi' - \xi'(-t_{N,\delta})| \leq \delta (Nk)^{1/3} \big\}.$$

Proof. The proof is based on the refined exponential weighted estimates (in the same spirit as in [Martinez 2002]) given in [FLM, Proposition 8.3], which we apply here to the operator P_1. Since the proof is very similar to that of [FLM, Lemma 6.4], we omit the details. □

On the other hand, using the explicit form of $\boldsymbol{u}_{CN}$ given in (4-52), one also sees that, for any L large enough, there exists $\delta_L > 0$ such that, for any $N \geq 1$, one has [FLM, Lemma 6.7]:

$$T_N \mathbf{1}_{K_1} \boldsymbol{u}_{CN} = \mathcal{O}((h^{\delta_L N} + h^L) e^{-S/h}) \quad \text{uniformly in } \mathcal{W}_\delta(N, h).$$

In particular, taking $N = L/\delta_L$ with $L \gg 1$, we obtain a sequence $N = N_L$ along which

$$T_N \mathbf{1}_{K_1} \boldsymbol{u}_{CN} = \mathcal{O}(h^{\delta_L N} e^{-S/h}) \quad \text{uniformly in } \mathcal{W}_\delta(N, h),$$

and with both N and $\delta_L N$ arbitrarily large.

As a consequence, along the same sequence, we also obtain

$$T_N \mathbf{1}_{K_1}(\boldsymbol{u}-\boldsymbol{u}_{CN}) = \mathcal{O}(h^{\delta_L' N} e^{-S/h}) \quad \text{uniformly in } \mathcal{W}_\delta(N,h),$$

with $\delta_L' = \min(\delta, \delta_L)$.

Moreover we see that, when $y \in U_N \cap B_d(S)$ and $x \in \Pi_x \mathcal{W}_\delta(N,h)$ (where Π_x stands for the natural projection onto the x-space), we have

$$\mu_N(x_n - y_n)^2 + s(x) - S \geq C_\delta N k,$$

with $C_\delta > 0$ constant (and actually $C_\delta \to \infty$ as $\delta \to 0$). Therefore, using Proposition 7.3 and the expression (7-8) for T_N, we also obtain

$$T_N \mathbf{1}_{U_N \cap B_d(S)}(\boldsymbol{u}-\boldsymbol{u}_{CN}) = \mathcal{O}(h^{\delta N} e^{-S/h}) \quad \text{uniformly in } \mathcal{W}_\delta(N,h).$$

As a consequence, if we set

$$\chi_N(x) := \chi_0\left(\frac{|x_n - \hat{x}_n|}{(Nk)^{2/3}}\right)\chi_0\left(\frac{|x' - \hat{x}'|}{(Nk)^{1/2}}\right), \tag{7-9}$$

where the function $\chi_0 \in C_0^\infty(\mathbb{R}_+; [0,1])$ satisfies $\chi_0 = 1$ in a sufficiently large neighborhood of 0, and is fixed in such a way that $\chi_N(x) = 1$ in

$$\left\{|x_n - \hat{x}_n| \leq |x_n(-t_N) - \hat{x}_n| + 2\delta(Nk)^{2/3},\ |x' - \hat{x}'| \leq |x'(-t_N) - \hat{x}'| + 2\delta(Nk)^{1/2}\right\}$$

(here, t_N and δ are those of Lemma 7.5), then the function

$$v_N := \chi_N e^{S/h}(\boldsymbol{u}-\boldsymbol{u}_{CN})$$

is such that

$$T_N v_N = \mathcal{O}(h^{\delta_L' N} e^{-S/h}) \quad \text{uniformly in } \mathcal{W}_\delta(N,h). \tag{7-10}$$

Moreover, we have [FLM, Section 6.2]

$$(P - \rho_1)v_N = [P, \chi_N]e^{S/h}(\boldsymbol{u}-\boldsymbol{u}_{CN}) + \mathcal{O}(h^{\delta N}),$$

and thus, on $\{d_N(x, \operatorname{supp}\nabla\chi_N) \geq \varepsilon\} \times \mathbb{R}^n$, where $\varepsilon > 0$ is fixed small enough and d_N is the distance associated with the metric $(Nk)^{-1}(dx')^2 + (Nk)^{-\frac{4}{3}}\, dx_n^2$,

$$T_N(P - \rho_1)v_N = \mathcal{O}(h^{\delta' N}),$$

for some $\delta' = \delta'(\varepsilon) > 0$.

(3) *(Almost) standard analytic propagation.* Although we are in a region where no analytic assumption is made, a rescaling of the problem gives estimates similar to those encountered in the analytic context. Indeed, setting

$$\tilde{h} = \tilde{h}_N := \frac{h}{Nk} = \left(N \ln \frac{1}{h}\right)^{-1},$$

and performing the change of variables (still working in the same coordinates, for which $\hat{x}=0$)

$$x \mapsto \tilde{x} = (\tilde{x}', \tilde{x}_n) := ((Nk)^{-\frac{1}{2}} x', (Nk)^{-\frac{2}{3}} x_n),$$
$$\xi \mapsto \tilde{\xi} = (\tilde{\xi}', \tilde{\xi}_n) := ((Nk)^{-\frac{1}{2}} \xi', (Nk)^{-\frac{2}{3}} \xi_n),$$

we see that the estimate (7-10) implies (see [FLM, Formula (6.43)])

$$T\tilde{v}_N(\tilde{x}, \tilde{\xi}; \tilde{h}_N) = \mathbb{O}(e^{-\delta_L'/2\tilde{h}_N})$$

uniformly in the tubular domain

$$\widetilde{W}(\tilde{h}) := \big\{ |\tilde{x}_n - \tilde{x}_n(-\delta^{-1})| \le \delta,\, |\tilde{\xi}_n - \tilde{\xi}_n(-\delta^{-1})| \le \delta,\, |\tilde{x}' - \tilde{x}'(-\delta^{-1})| \le \delta (Nk)^{-\frac{1}{6}}, \\ |\tilde{\xi}' - \tilde{\xi}'(-\delta^{-1})| \le \delta (Nk)^{-\frac{1}{6}} \big\}, \quad (7\text{-}11)$$

where

$$\begin{aligned} \tilde{v}_N(\tilde{x}) &:= (Nk)^{(n-1)/4+\frac{1}{3}} v_N((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n), \\ (\tilde{x}(\tilde{t}), \tilde{\xi}(\tilde{t})) &:= \exp \tilde{t} H_{\tilde{p}_1}(0,0), \\ \tilde{p}_1(\tilde{x}, \tilde{\xi}) &:= (Nk)^{1/3} |\tilde{\xi}'|^2 + \tilde{\xi}_n^2 + W_1(\tilde{x}, \tilde{h}), \\ W_1(\tilde{x}, \tilde{h}) &:= (Nk)^{-\frac{2}{3}} V_1((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n) - (Nk)^{-\frac{2}{3}} \rho_1. \end{aligned}$$

Moreover, setting

$$\widetilde{P} := -(Nk)^{1/3} \tilde{h}^2 \Delta_{\tilde{x}'} - \tilde{h}^2 \partial_{\tilde{x}_n}^2 + W_1(\tilde{x}),$$

then, for any $N \ge 1$ large enough, we also have

$$T\widetilde{P}\tilde{v}_N(\tilde{x}, \tilde{\xi}; \tilde{h}_N) = \mathbb{O}(e^{-\delta'/2\tilde{h}_N}),$$

uniformly with respect to $h > 0$ small enough and $(\tilde{x}, \tilde{\xi}) \in \mathbb{R}^{2n}$ satisfying

$$d_N\big(((Nk)^{1/2}\tilde{x}', (Nk)^{2/3}\tilde{x}_n), \operatorname{supp} \nabla \chi_N\big) \ge \varepsilon.$$

Finally, by Proposition 7.1 and Proposition 7.3, we have the a priori estimate

$$\|\tilde{v}_N\|_{H^1} = \mathbb{O}(h^{-N_1}) = \mathbb{O}(e^{N_1/(N\tilde{h})}),$$

for some $N_1 \ge 0$ independent of N, and we observe that, for $N = L/\delta_L$, one has $N_1/(\delta_L N) \to 0$ as $L \to +\infty$.

At this point, a small refinement of the propagation of the microsupport (see [FLM, Proposition 6.8]) gives the existence of a constant $\delta_1 > 0$ independent of L such that, for all L large enough and $N = L/\delta_L$, one has

$$T\tilde{v}_N(\tilde{x}, \tilde{\xi}; \tilde{h}) = \mathbb{O}(e^{-\delta_1\delta_L/\tilde{h}}), \quad (7\text{-}12)$$

uniformly in $V(\delta_1) = \{\tilde{x} : |\tilde{x}| \le \delta_1\} \times \big\{\tilde{\xi} : (Nk)^{1/6}|\tilde{\xi}'| + |\tilde{\xi}_n| \le \delta_1\big\}$.

Then, using an ellipticity property of $\tilde{p}_1$ away from $\big\{\tilde{\xi} : (Nk)^{1/6}|\tilde{\xi}'| + |\tilde{\xi}_n| \le \delta_1\big\}$ and reconstructing $\tilde{v}_N$ from $T\tilde{v}_N$, one finally finds

$$\|\tilde{v}_N\|_{H^m(|\tilde{x}| \le \delta_2)} = \mathbb{O}(e^{-\delta_2\delta_L/\tilde{h}}),$$

with $m \geq 0$ arbitrary, $\delta_2 > 0$ independent of L, $N = L/\delta_L$, and L arbitrarily large. Therefore, turning back to the original coordinates x and parameter h and making $\hat{x}$ vary on all of $\partial \ddot{O}$, Theorem 7.4 follows. $\square$

8. Asymptotics of the width

As before, we denote by P_θ the distorted operator obtained from P by means of a complex distortion as in (7-2), with R_0 sufficiently large in order to have $\ddot{O} \subset \{|x| \leq R_0/2\}$. We also denote by $\boldsymbol{u}_\theta$ the corresponding distorted state obtain from $\boldsymbol{u}$ by applying the same distortion (see, e.g., [FLM] for more details).

Let $\psi_0 \in C_0^\infty([0, 2); [0, 1])$ with $\psi_0 = 1$ near $[0, 1]$, and set

$$\psi_N(x) := \psi_0\left(\frac{\operatorname{dist}(x, \ddot{O})}{(Nk)^{2/3}}\right),$$

where, as before, $N = L/\delta_L$ with $L \geq 1$ arbitrarily large.

Then, since $\psi_N \boldsymbol{u} = \psi_N \boldsymbol{u}_\theta$, $P_\theta \mathbf{u}_\theta = \rho_1 \boldsymbol{u}_\theta$ and $\psi_N P_\theta \psi_N \mathbf{u}_\theta = \psi_N P \psi_N \boldsymbol{u}$, we have

$$\operatorname{Im} \rho_1 \|\psi_N \boldsymbol{u}\|^2 = \operatorname{Im}\langle \psi_N P_\theta \mathbf{u}_\theta, \psi_N \boldsymbol{u}\rangle = \operatorname{Im}\langle [\psi_N, P_\theta]\mathbf{u}_\theta, \psi_N \boldsymbol{u}\rangle,$$

and thus

$$\operatorname{Im} \rho_1 = \operatorname{Im} \frac{\bigl\langle 2h^2(\nabla\psi_N)\nabla\mathbf{u} + h^2(\Delta\psi_N)\boldsymbol{u}, \psi_N\boldsymbol{u}\bigr\rangle + h\bigl\langle [\psi_N, R_\theta]\boldsymbol{u}_\theta, \psi_N\boldsymbol{u}\bigr\rangle}{\|\psi_N \boldsymbol{u}\|^2}. \tag{8-1}$$

Moreover, we know that $\|\psi_N \boldsymbol{u}\| = 1 + \mathcal{O}(e^{-\delta/h})$ with $\delta > 0$ and, by Theorem 7.4, on supp $\widetilde{\psi}_N$ we can replace $\boldsymbol{u}$ by $\boldsymbol{u}_{CN}$, up to an error $\mathcal{O}(h^L e^{-S/h})$. Also, using Proposition 7.1 we deduce

$$\operatorname{Im} \rho_1 = \operatorname{Im}\langle 2h^2(\nabla\psi_N)\nabla\boldsymbol{u}_{CN} + h^2(\Delta\psi_N)\boldsymbol{u}, \psi_N\boldsymbol{u}_{CN}\rangle + h\langle [\psi_N, R_\theta]\boldsymbol{u}_\theta, \psi_N\boldsymbol{u}\rangle + \mathcal{O}(h^{L-N_0})e^{-2S/h} \tag{8-2}$$

for some fixed $N_0 \geq 0$ independent of L.

Now, we let $\widetilde{\psi}_0 \in C_0^\infty((1, 2); [0, 1])$ with $\widetilde{\psi}_0 = 1$ near supp $\nabla\psi_0$ and set $\widetilde{\psi}_N(x) = \widetilde{\psi}_0\Bigl(\frac{\operatorname{dist}(x, \ddot{O})}{(Nk)^{2/3}}\Bigr)$.

Lemma 8.1. *One has*

$$\langle [\psi_N, R_\theta]\boldsymbol{u}_\theta, \psi_N\boldsymbol{u}\rangle = \langle \psi_N[\psi_N, R]\widetilde{\psi}_N\boldsymbol{u}, \widetilde{\psi}_N\mathbf{u}\rangle + \mathcal{O}(h^\infty e^{-2S/h}). \tag{8-3}$$

Proof. Thanks to Assumption 3, in $[\psi_N, R_\theta]$, we can make the (complex) change of contour of integration

$$\mathbb{R}^n \ni \xi \mapsto \xi + i\sqrt{M_0}\frac{x-y}{\sqrt{(x-y)^2+h^2}}.$$

We obtain

$$[\psi_N, R_\theta]\boldsymbol{u}_\theta(x) = \frac{1}{(2\pi h)^n}\int e^{i(x-y)\xi/h - \Phi/h}(\psi_N(x) - \psi_N(y))\tilde{r}_\theta \bar{\mathbf{u}}_\theta(y)\,dy\,d\xi,$$

with

$$\Phi := \sqrt{M_0}\frac{(x-y)^2}{\sqrt{(x-y)^2+h^2}}, \qquad \partial_{x,y}^\alpha\partial_\xi^\beta\tilde{r}_\theta(x, y, \xi) = \mathcal{O}(h^{-|\alpha|}\langle\xi\rangle).$$

By construction, on the set

$$A_N := \operatorname{supp}(\psi_N(x) - \psi_N(y)) \cap \{\widetilde{\psi}_N(x) \neq 1 \text{ or } \widetilde{\psi}_N(y) \neq 1\},$$

we have $|x-y| \geq c(Nk)^{2/3}$ for some constant $c > 0$. As a consequence, on this set, the quantity $|x-y|/\sqrt{(x-y)^2+h^2}$ tends to 1 uniformly as $h \to 0$. Moreover, still on this set, we have either $s(x) = S$ or $s(y) = S$, and since $|s(x)-s(y)| \leq \mu|x-y|$ with $0 < \mu < \sqrt{M_0}$, we deduce the existence of a constant $c_0 > 0$ such that for $(x, y) \in A_N$, one has $s(x)+s(y)+\Phi \geq 2S + c_0(Nk)^{2/3}$.

Therefore, by the Calderón–Vaillancourt theorem (and also using Proposition 5.2 in order to regularize the function $s(x)$), we obtain

$$\|e^{-s/h}[\psi_N, R_\theta]e^{-s/h}(1-\widetilde{\psi}_N)\langle hD_n\rangle^{-1}\| + \|(1-\widetilde{\psi}_N)e^{-s/h}[\psi_N, R_\theta]e^{-s/h}\langle hD_n\rangle^{-1}\| = \mathbb{O}(h^\infty e^{-2S/h}).$$

Then, writing

$$\langle[\psi_N, R_\theta]\boldsymbol{u}_\theta, \psi_N\boldsymbol{u}\rangle = \langle e^{-s/h}[\psi_N, R_\theta]e^{-s/h}(e^{s/h}\boldsymbol{u}_\theta), \psi_N e^{s/h}\boldsymbol{u}\rangle$$

and using Proposition 7.1 and Remark 7.2, the result follows. □

Inserting (8-3) into (8-2) and approaching $\widetilde{\psi}_N\boldsymbol{u}$ by $\widetilde{\psi}_N\boldsymbol{u}_{CN}$, we obtain

$$\begin{aligned}\operatorname{Im}\rho_1 = \operatorname{Im}\langle 2h^2(\nabla\psi_N)\nabla\boldsymbol{u}_{CN} \\ + h^2(\Delta\psi_N)\boldsymbol{u}, \psi_N\boldsymbol{u}_{CN}\rangle + h\langle\psi_N[\psi_N, R]\widetilde{\psi}_N\boldsymbol{u}_{CN}, \widetilde{\psi}_N\boldsymbol{u}_{CN}\rangle + \mathbb{O}(h^{L-N_0})e^{-2S/h}.\end{aligned} \quad (8\text{-}4)$$

Finally, using Proposition 6.2 (in particular the expression (4-52) of $\boldsymbol{u}_{CN}$ in $\bigcup_{\gamma\in G}\mathcal{W}_N(\gamma)\cap\operatorname{supp}\widetilde{\psi}_N$), we can perform a stationary-phase expansion in (8-4) (as in [FLM, Section 7]) and, for L large enough, we obtain

$$\operatorname{Im}\rho_1 = -h^{(1-n_\Gamma)/2}\sum_{j=n_0}^{L}\sum_{0\leq m\leq\ell\leq L} f_{j,\ell,m}h^{j+\ell}|\ln h|^m e^{-2S/h} + \mathbb{O}(h^{L/2})e^{-2S/h},$$

with $f_{n_0,0,0} > 0$. In particular, the result for ρ_1 follows.

The result for ρ_j, $j \geq 2$, can be done along the same lines, by using a representation of $\operatorname{Im}\rho_j$ analogous to (8-1) and by approaching $\boldsymbol{u}$ by a linear combination of WKB expressions similar to $\boldsymbol{u}_{CN}$, where the number of terms depends on the asymptotic multiplicity of the resonance; see [Helffer and Sjöstrand 1986, Section 10].

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Received 28 Oct 2011. Revised 23 May 2014. Accepted 30 Jun 2014.

ALAIN GRIGIS: grigis@math.univ-paris13.fr
Departement de Mathematiques, Universite Paris 13 - Institut Galilee, Avenue Jean-Baptiste Clement, 93430 Villetaneuse, France

ANDRÉ MARTINEZ: andre.martinez@unibo.it
Dipartimento di Matematica, Università di Bologna, Piazza di Porta San Donato, I-40127 Bologna, Italy

ANALYSIS AND PDE
Vol. 7, No. 5, 2014
dx.doi.org/10.2140/apde.2014.7.1057

QUASIMODES AND A LOWER BOUND ON THE UNIFORM ENERGY DECAY RATE FOR KERR–ADS SPACETIMES

GUSTAV HOLZEGEL AND JACQUES SMULEVICI

We construct quasimodes for the Klein–Gordon equation on the black hole exterior of Kerr–AdS (anti-de Sitter) spacetimes. Such quasimodes are associated with time-periodic approximate solutions of the Klein–Gordon equation and provide natural candidates to probe the decay of solutions on these backgrounds. They are constructed as the solutions of a semiclassical nonlinear eigenvalue problem arising after separation of variables, with the (inverse of the) angular momentum playing the role of the semiclassical parameter. Our construction results in *exponentially small* errors in the semiclassical parameter. This implies that general solutions to the Klein Gordon equation on Kerr–AdS cannot decay faster than logarithmically. The latter result completes previous work by the authors, where a logarithmic decay rate was established as an upper bound.

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1. Introduction

There is currently a lot of mathematical activity concerning the analysis of waves on the exterior of black hole backgrounds.[1] The main motivation is the black hole stability problem, i.e., the conjectured nonlinear asymptotic stability of the two-parameter family of asymptotically flat Kerr spacetimes $(\mathcal{M}, g_{M,a})$, the latter being stationary solutions of the vacuum Einstein equations $\mathrm{Ric}[g] = 0$. From the point of view of nonlinear partial differential equations, the analysis of linear scalar waves on black holes is a prerequisite to successfully understanding the nonlinear hyperbolic Einstein equations in a neighborhood of the Kerr family.

MSC2010: primary 58J50; secondary 83C57.
Keywords: wave equation, black holes, decay estimates, Kerr – anti-de Sitter.

[1] See [Dafermos and Rodnianski 2013] for an introduction and a review of recent results.

A major insight that crystallized in the last decade [Dafermos and Rodnianski 2005; 2009; 2010a; 2010b; Marzuola et al. 2010; Tataru and Tohaneanu 2011; Andersson and Blue 2009; Dyatlov 2011; 2012] is that the fundamental geometric obstacles to the decay of waves, namely superradiance and trapped null-geodesics, can be overcome by exploiting the normal hyperbolicity of the trapping, the redshift effect near the event horizon and the natural dispersion of waves in asymptotically flat (and asymptotically de Sitter) spacetimes. In particular, polynomial decay rates have been established for solutions to the scalar wave equation on the exterior of any member of the subextremal Kerr family of spacetimes [Dafermos and Rodnianski 2010b].

Changing the black hole geometry can have dramatic effects on the behavior of linear waves through the subtle interplay of the redshift, the superradiance and the trapping. Aretakis [2012; 2013] showed that for *extremal* black holes (whose vanishing surface gravity leads to a degeneration of the redshift effect) the transversal derivatives of general solutions to the wave equation will grow along the event horizon. In [Shlapentokh-Rothman 2014], it is proven that for the *massive* wave equation on a subextremal ($|a| < M$) Kerr spacetime, exponentially growing solutions can be constructed on the exterior, exploiting an amplification of the superradiance caused by the confining properties of the mass term.

In this paper, we shall be interested in a black hole geometry for which a strong trapping phenomenon leads to a very slow (only logarithmic) decay of waves. More precisely, we will study the behavior of solutions to the massive wave equation

$$\left(\Box_g + \frac{\alpha}{l^2}\right)\psi = 0 \tag{1}$$

in the exterior of asymptotically anti-de Sitter (AdS) black holes with spacetime metric g.

Due to their AdS asymptotics, these spacetimes are not globally hyperbolic. Nonetheless, Equation (1) is well-posed in suitably weighted Sobolev spaces, denoted here by H^k_{AdS}, provided α satisfies the Breitenlohner–Friedmann bound $\alpha < \frac{9}{4}$. See [Holzegel 2012] as well as [Vasy 2012; Bachelot 2008; 2011; Ishibashi and Wald 2004; Warnick 2013] for a complete treatment of general boundary conditions.

The *global* properties of solutions to (1) on the exterior of nonsuperradiant[2] Kerr–AdS black holes were studied in [Holzegel 2010; Holzegel and Smulevici 2013a; Holzegel and Warnick 2014]. In particular, boundedness was obtained in [Holzegel 2010; Holzegel and Warnick 2014] and logarithmic decay in time for general H^2_{AdS} solutions in [Holzegel and Smulevici 2013a]. We summarize these results in the following theorem. We refer to Section 2A1 for the precise definitions of the Kerr–AdS spacetimes, the area-radius r_+ of the event horizon and the $\Sigma_{t^\star}$-foliation and to Section 2B for the definitions of the norms and energies used in the statement below. At this point we only remark that $e_1[\psi]$ is an energy density involving all first derivatives of ψ while $e_2[\psi]$ and $\tilde{e}_2[\psi]$ involve all second derivatives (with appropriate weights):

Theorem 1.1. *Let $(g, \mathcal{R})$ denote the black hole exterior of a Kerr–AdS spacetime with mass $M > 0$, angular momentum per unit mass a and cosmological constant $\Lambda = -3/l^2$. Assume that the parameters satisfy $\alpha < \frac{9}{4}$, $|a| < l$. Fix a spacelike slice $\Sigma_{t_0^\star}$ intersecting $\mathcal{H}^+$. Then:*

[2] This means that the parameters of the black hole satisfy $r_+^2 > |a|l$. See Remark 1.3 and Section 2.

1. *Equation (1) is well-posed in CH^k_{AdS} on $(g, \mathcal{R})$ for any $k \geq 2$ for initial data prescribed on $\Sigma_{t_0^\star}$.* (See [Holzegel 2012].)
2. *The solutions of (1) arising from data prescribed on $\Sigma_{t_0^\star}$ remain uniformly bounded on the black hole exterior provided $r_+^2 > |a|l$ holds. In particular,*
$$\int_{\Sigma_{t^*}} e_1[\psi](t^\star)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \lesssim \int_{\Sigma_{t_0^\star}} e_1[\psi](t_0^\star)\, r^2 \sin\theta\, dr\, d\theta\, d\phi. \tag{2}$$
Analogous statements hold for all higher H^k_{AdS}-norms. In particular,
$$\int_{\Sigma_{t^*}} e_2[\psi](t^\star)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \lesssim \int_{\Sigma_{t_0^\star}} e_2[\psi](t_0^\star)\, r^2 \sin\theta\, dr\, d\theta\, d\phi, \tag{3}$$
and the same statement for $\tilde{e}_2[\psi](t^\star)$. (See [Holzegel 2010; Holzegel and Warnick 2014].[3])
3. *If the parameters satisfy $r_+^2 > |a|l$, the solutions of (1) satisfy the global decay estimate*
$$\int_{\Sigma_{t^*}} e_1[\psi](t^\star)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \lesssim \frac{1}{[\log(2+t^\star)]^2} \int_{\Sigma_{t_0^\star}} e_2[\psi](t_0^\star)\, r^2 \sin\theta\, dr\, d\theta\, d\phi \tag{4}$$
for all $t^\star \geq t_0^\star > 0$. (See [Holzegel and Smulevici 2013a].)

Remark 1.2. The constant implicit in the symbol "$\lesssim$" appearing in items 2 and 3 depends only on the fixed parameters M, ℓ, a and α.

Remark 1.3. The condition $r_+^2 > |a|l$ on the parameters in items 2 and 3 guarantees the existence of a globally causal Killing vector field on the black hole exterior, the Hawking–Reall vector field [Hawking and Reall 2000], which explains why such black holes are sometimes referred to as "nonsuperradiant". If one restricts to *axisymmetric* solutions of (1), this condition can be dropped for statements 2 and 3 in Theorem 1.1.

Remark 1.4. The boundedness statement (2) does not lose derivatives while the decay statement (4) does. This is the familiar loss of derivatives caused by the existence of trapped null-geodesics [Ralston 1969; Sbierski 2013].

This logarithmic decay rate (4) was conjectured to be sharp in [Holzegel and Smulevici 2013a] in view of the discovery of a new stable trapping phenomenon, itself a consequence of the coupling between the lack of dispersion at the asymptotic end and the usual (unstable) trapping on black hole exteriors [ibid.].

1A. ***The main results.*** In this paper, we shall prove that the logarithmic decay estimate of Theorem 1.1 is indeed sharp. Recall that for the obstacle problem, it is classical [Ralston 1969] that lower bounds on the rate of energy decay can be obtained from the construction of approximate eigenfunctions, also called *quasimodes*, of the associated elliptic operator, obtained by formally taking the Fourier transform in time

[3]These papers and [Holzegel 2012] are only concerned with the $\tilde{e}_2[\psi]$ energy. It is remarked that by commutation with angular momentum operators one can prove boundedness for the $e_2[\psi]$ energy (which differs from the $\tilde{e}_2[\psi]$ energy through the weights of the angular derivatives). For completeness, we provide in an Appendix an explicit proof of this statement.

of the wave operator. Our main theorem establishes the existence of such quasimodes for Kerr–AdS with exponentially small errors.[4]

The statement of the following theorem will involve the quantity

$$r_{\max} \in \left(r_+, \frac{3M}{1-a^2/l^2}\right],$$

which is determined in Lemma 3.1 as the location of the unique maximum of a simple radial function. For Schwarzschild–AdS, $r_{\max} = 3M$ will be the location of the well-known photon sphere.

Theorem 1.5 (quasimodes for Kerr–AdS). *Let* $(g, \mathcal{R})$ *denote the black hole exterior of a Kerr–AdS spacetime, with mass* $M > 0$, *angular momentum per unit mass* a *and cosmological constant* $\Lambda = -3/l^2$. *Assume that the parameters satisfy* $\alpha < \frac{9}{4}$, $|a| < l$. *Let* (t, r, θ, φ) *denote standard Boyer–Lindquist coordinates on* $\mathcal{R}$. *Then, for* $\delta > 0$ *sufficiently small* (*depending only on the parameters* l, M, a, α), *there exists a family of nonzero functions* $\psi_\ell \in H^k_{\mathrm{AdS}}$ *for any* $k \geq 0$ *satisfying the following conditions*:

(1) $\psi_\ell(t, r, \theta, \varphi) = e^{i\omega_\ell t}\varphi_\ell(r, \theta)$ (*axisymmetric and time-periodic*).

(2) $0 < c < \dfrac{\omega_\ell^2}{\ell(\ell+1)} < C$, *for constants* c *and* C *independent of* ℓ (*uniform bounds on the frequencies*).

(3) *For all* $t^\star \geq t^\star_0$ *and all* $k \geq 0$, *we have* $\left\|\left(\Box_g + \alpha/l^2\right)\psi_\ell\right\|_{H^k_{Ads}(\Sigma_{t^\star})} \leq C_k e^{-C_k \ell}\|\psi_\ell\|_{H^0_{\mathrm{AdS}}(\Sigma_{t^\star_0})}$, *for some* $C_k > 0$ *independent of* ℓ (*approximate solutions to the wave equation*).

(4) *The support of* $F_\ell := (\Box_g + \alpha/l^2)\psi_\ell$ *is contained in* $\{r_{\max} \leq r \leq r_{\max} + \delta\}$ (*spatial localization of the error*).

(5) *The support of* $\varphi_\ell(r, \theta)$ *is contained in* $\{r \geq r_{\max}\}$ (*spatial localization of the solution*).

Note that the ψ_ℓ have constant H^k_{AdS}-norms and hence exhibit no decay. On the other hand, a standard application of Duhamel's formula shows that the ψ_ℓ are good approximations to the solution of $(\Box_g + \alpha/l^2)\psi = 0$ arising from the data induced by ψ_ℓ, at least up to a time $t \sim e^{C_k \ell}$.

Corollary 1.6. *Let* $(\mathcal{R}, g)$ *denote the black hole exterior of a Kerr–AdS spacetime as in Theorem 1.5. Denote by* SCH^2_{AdS} *the set of* CH^2_{AdS} *solutions to* (1) *with* $\alpha < \frac{9}{4}$. *Let* $t^\star_0 \geq 0$ *be fixed and define for any nonzero* ψ *and* $t^\star \geq 0$

$$Q[\psi](t^\star) := \log(2 + t^\star)\left(\frac{\int_{\Sigma_{t^\star}\cap\{r\geq r_{\max}\}} e_1[\psi](t^\star)\, r^2 \sin\theta \, dr\, d\theta\, d\phi}{\int_{\Sigma_{t^\star_0}} e_2[\psi](t^\star_0) r^2 \sin\theta\, dr\, d\theta\, d\phi}\right)^{\frac{1}{2}}.$$

Then there exists a universal (*depending only on* M, α, $|a|$ *and* l) *constant* $C > 0$ *such that*

$$\limsup_{t^\star \to +\infty} \sup_{\substack{\psi \in SCH^2_{\mathrm{AdS}} \\ \psi \neq 0}} Q[\psi](t^\star) > C > 0.$$

Remark 1.7. Corollary 1.6 implies that the semilocal energy in $\{r \geq r_{\max}\}$ cannot decay universally faster than $(\log t^\star)^{-2}$, unless one loses more derivatives.

[4]Note that in order to deduce the sharpness of the logarithmic decay rate from the quasimodes, polynomial errors would a priori not be sufficient.

Remark 1.8. We emphasize that no smallness assumption on the angular momentum a is needed, apart from the condition $|a| < l$ which ensures that the metric is a regular black hole metric.

Remark 1.9. The ψ_ℓ constructed in Theorem 1.5 are axisymmetric, while the decay estimate of Theorem 1.1 holds for general solutions (provided $r_+^2 > |a|l$ holds in the nonaxisymmetric case). Since we are concerned here with a *lower* bound on the uniform decay rate, an analysis within axisymmetry is sufficient. This allows us to drop the nonsuperradiant condition $r_+^2 > |a|l$ in the analysis. On the other hand, for (sufficiently) superradiant black holes $r_+^2 < |a|l$ one can adapt the proof of [Shlapentokh-Rothman 2014] to construct exponentially growing solutions. Hence in this case, the quasimodes we construct are not the "worst" solutions on these backgrounds.

Remark 1.10. Stable trapping occurs only in the region $r \geq r_{\text{max}}$ and is associated to certain frequencies. As a consequence, stronger local energy decay in $r \leq r_{\text{max}}$ or for some frequency projections of solutions is a priori compatible with the results of this paper.

1B. *Related works and discussion.*

1B1. *Nonlinear analysis on asymptotically AdS spacetimes.* In [Holzegel and Smulevici 2012a; 2013b], the nonlinear spherically symmetric Einstein–Klein–Gordon system for asymptotically AdS initial data was studied, and, in particular, asymptotic stability of Schwarzschild–AdS was proven within this model.

For a discussion connecting the logarithmic decay to the nonlinear stability or instability of asymptotically AdS black holes, we refer to Section 1.4 of [Holzegel and Smulevici 2013a]. We also mention the recent heuristic analysis of [Dias et al. 2012] drawing attention to a potential stability mechanism caused by the lack of exact nonlinear resonances in this setting. For AdS itself, instability was conjectured in [Dafermos and Holzegel 2006; Anderson 2006; Dafermos 2006]. More recently, both numerical and additional heuristic evidence has been presented [Bizon and Rostworowski 2011]. Finally, let us note that asymptotically AdS solutions to the Einstein equations have been constructed in [Friedrich 1995].

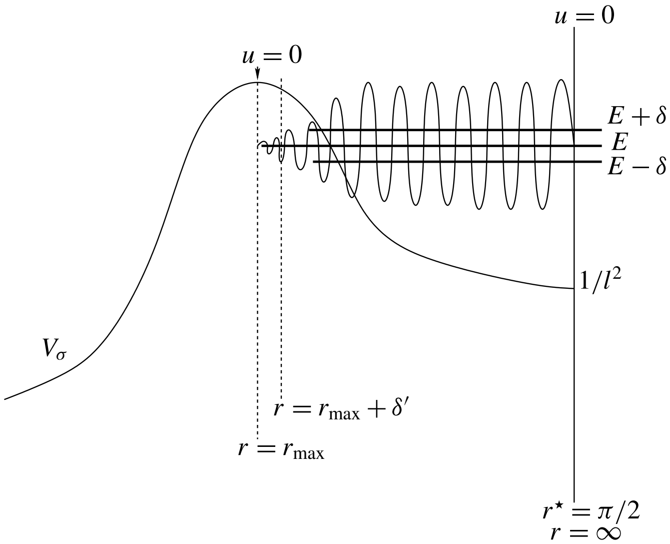
1B2. *Quasinormal modes of the asymptotically AdS black holes.* Quasinormal modes, also called resonances, are complex frequencies generalizing the well-known normal modes to systems which dissipate energy. There is a strong connection between quasimodes and resonances [Tang and Zworski 1998]. One way to mathematically define them is as poles of the meromorphic continuation of a truncated resolvent. In the case of asymptotically de Sitter black holes, this theory has been very successfully developed; see [Bony and Häfner 2008; Dyatlov 2011; 2012]. In a recent paper, Gannot [2014] has established, in the case of Schwarzschild–AdS, the existence of a sequence of quasinormal modes (based on an independent construction of quasimodes), indexed by angular momentum ℓ, and with imaginary parts of size $\mathbb{O}(\exp(-\ell/C))$. In particular, his construction confirms the numerical results of [Festuccia and Liu 2009] and provides an independent proof of Theorem 1.5 and Corollary 1.6, albeit restricted to the Schwarzschild–AdS case. While we restrict ourselves here to the construction of quasimodes, we strongly believe that our results can be used as a basis for the construction of resonances in the Kerr–AdS case for the whole range of parameters satisfying $|a| < l$ and $r_+^2 > |a|l$.

1B3. *Universal minimal decay rates of waves outside stationary black holes.* In the context of the obstacle problem in Minkowski space, a celebrated result of Burq [1998] establishes a logarithmic decay rate for the local energy of waves, independently of the geometry of the obstacle causing the trapping. For waves outside black holes, in view of the results of [Holzegel and Smulevici 2013a], a natural conjecture is: Given any black hole exterior $(\mathcal{R}, g)$ of a stationary spacetime, a logarithmic decay of energy similar to that of [ibid.] will hold, provided a uniform boundedness statement is true for solutions to the wave equation $\Box_g \psi = 0$ on $\mathcal{R}$.

1C. ***Outline and overview of the proof.*** Section 2 introduces the family of Kerr–AdS spacetimes as well as the norms required to state our estimates. In Section 3, we exploit the classical fact that the wave equation separates on Kerr–AdS. An ingredient which considerably simplifies our analysis here is the important observation that we can restrict ourselves to *axisymmetric* solutions. With axisymmetry, the separation of variables leads to relatively simple, one-dimensional, second order ordinary differential equations for the radial functions. In the case of Schwarzschild, they are roughly of the form of the semiclassical problem

$$-u_\ell'' \frac{1}{\ell(\ell+1)} + V_\sigma u_\ell = \frac{\omega^2}{\ell(\ell+1)} u_\ell \tag{5}$$

for a potential $V_\sigma(r)$, whose general form is depicted below.[5] In Section 3C, we shall describe in detail the analytic properties of the potentials appearing in these equations.



To construct the quasimodes, we first construct eigenfunctions for the problem (5) with Dirichlet conditions $u = 0$ imposed on u at $r = r_{\max}$ and $r \to \infty$ (Section 4). In particular, we prove a version of Weyl's law, ensuring that for any energy between $1/l^2 < E < V_{\max} = V_\sigma(r_{\max})$ we can find (lots of) eigenvalues

[5]For the purpose of this exposition, we neglect terms of lower order in $\frac{1}{\ell(\ell+1)}$ in the potential, as well as the mass term. The latter is actually unbounded and needs to be absorbed with a Hardy inequality. We suppress such technical difficulties in the present discussion.

$\omega_\ell^2/(\ell(\ell+1))$ of (5) in a strip $[E-\delta,\ E+\delta]$. In the Kerr case, the eigenvalue problem (5) turns into a problem of the form[6]

$$-u_\ell''\frac{1}{\mu_\ell(a^2\omega^2)}+V_\sigma u_\ell=\frac{\omega^2}{\mu_\ell(a^2\omega^2)}u_\ell, \tag{6}$$

which is *nonlinear* in ω^2. An application of the implicit function theorem together with global estimates on the behavior of the eigenvalues still allow us to conclude the existence of eigenfunctions u_ℓ of (6) with corresponding eigenvalues in the range $\left(1/l^2,\ E+\delta\right]$. *These estimates, together with the analysis of the potential in the Kerr–AdS case, constitute the core of our paper.*

In Section 5, we recall the so-called Agmon estimates, their proof being included to make the paper self-contained. These estimates quantify that the solutions u_ℓ constructed from the above eigenvalue problem decay exponentially in ℓ in a region $[r_{\max},\ r_{\max}+\delta']$.

In Section 6, the quasimodes are constructed by cutting off the solution u_ℓ of the eigenvalue problem in $[r_{\max},\ r_{\max}+\delta']$ so that it vanishes with all derivatives at $r_{\max}$ and then continuing it to be identically zero in $[r_+,\ r_{\max}]$. The function ϕ_ℓ thus constructed will be defined on $[r_+,\ \infty)$ and the corresponding wave function $\psi_\ell=e^{i\omega_\ell}\phi_\ell$ will satisfy the wave equation everywhere except in the small strip $[r_{\max},\ r_{\max}+\delta']$, where the error is exponentially small by the Agmon estimate.

In the last section, we prove Corollary 1.6 using the Duhamel formula. Finally, the Appendix contains a proof of boundedness for the second energy used in this paper. This boundedness statement differs from that obtained in [Holzegel 2010; Holzegel and Warnick 2014] in that it allows for stronger radial weights near infinity for the angular derivatives.

2. Preliminaries

2A. ***The Kerr–AdS family of spacetimes.*** We recall here some basic facts about the family of Kerr–AdS spacetimes required in the paper. We refer the reader to the detailed discussion in our work [Holzegel and Smulevici 2013a].

2A1. *The fixed manifold with boundary $\mathcal{R}$.* Let $\mathcal{R}$ denote the manifold with boundary

$$\mathcal{R}=[0,\ \infty)\times\mathbb{R}\times\mathbb{S}^2.$$

We define standard coordinates y^* for $[0,\ \infty)$, $t^\star$ for $\mathbb{R}$ and $(\theta,\ \phi)$ for $\mathbb{S}^2$. This defines a coordinate system on $\mathcal{R}$ which is global up to the well-known degeneration of the spherical coordinates. We define the event horizon $\mathcal{H}^+$ to be the boundary of $\mathcal{R}$:

$$\mathcal{H}^+=\partial\mathcal{R}=\{y^*=0\}.$$

The manifold $\mathcal{R}$ will coincide with the domain of outer communication of the black hole spacetimes including the future event horizon $\mathcal{H}^+$.

[6]The $\mu_\ell(a^2\omega^2)$ generalize the familiar spherical eigenvalues $\ell(\ell+1)$ of the Schwarzschild case to Kerr. See Section 3A.

2A2. *The parameter space and the radial function.* A Kerr–AdS spacetime is characterized by its mass $M > 0$, its angular momentum per unit mass a and the value of the cosmological constant $\Lambda < 0$. For convenience, we shall use mostly $l = \sqrt{3/|\Lambda|}$, instead of Λ. For the spacetime to be regular, we will require $|a| < l$. Let us thus fix $M, l > 0$ and $|a| < l$. We then define $r_+(M, l, a) > 0$ as the unique real solution of $\Delta_-(x) = 0$, where

$$\Delta_-(x) = \left(1 + \frac{x^2}{l^2}\right)(x^2 + a^2) - 2Mx. \tag{7}$$

We now define a function r on $\mathcal{R}$ as follows. As a function of $(t^\star, y^\star, \varphi, \theta)$, r only depends on $y^\star$ and is a diffeomorphism from $[0, \infty)$ to $[r_+, \infty)$. The collection $(t^\star, r(y^\star), \varphi, \theta)$ then forms a coordinate system on $\mathcal{R}$, global up to the degeneration of the spherical coordinates. Moreover, the horizon $\mathcal{H}^+$ coincides with $\{r = r_+\}$.

2A3. *More coordinates: $r^\star$, t, $\tilde{\phi}$.* Let us define $r^\star$ by

$$\frac{dr^\star}{dr}(r) = \frac{r^2 + a^2}{\Delta_-(r)}, \quad r^\star(r = +\infty) = \frac{\pi}{2},$$

where $\Delta_-(r)$ is given by (7). Note that $r^\star(r_+) = -\infty$.

By a small abuse of notation, we shall often write for functions f and g, $f(r^\star) = g(r)$, instead of $f(r^\star) = g(r(r^\star))$ or $f(r^\star(r)) = g(r)$.

Finally, let $r_{\text{cut}} = r_+ + \frac{1}{2}(r_{\max} - r_+)$, with $r_{\max} > r_+$ defined in Lemma 3.1 depending only on the parameters M, a and l, and let $\chi(r)$ be a smooth cut-off function with the property

$$\chi(r) = \begin{cases} 1 & \text{if } r \in \left[r_+, \frac{1}{2}(r_{\text{cut}} - r_+)\right], \\ 0 & \text{if } r \geq r_{\text{cut}}. \end{cases} \tag{8}$$

We introduce the time coordinate t and another angular coordinate $\tilde{\phi}$ as

$$t = t^\star - A(r)\chi(r) \quad \text{and} \quad \tilde{\phi} = \phi - B(r)\chi(r), \tag{9}$$

where

$$\frac{dA}{dr} = \frac{2Mr}{\Delta_-(1 + r^2/l^2)}, \qquad \frac{dB}{dr} = \frac{a(1 - a^2/l^2)}{\Delta_-},$$

and A and B vanish at infinity.

Note that t, $\tilde{\phi}$ and $r^\star$ are not well-behaved functions at the horizon $r = r_+$. As a consequence, the coordinate systems $(t, r, \theta, \tilde{\phi})$ and $(t, r^\star, \theta, \phi)$ only cover $\text{int}(\mathcal{R})$. Observe also that the two coordinate systems $(t, r, \theta, \tilde{\phi})$ and $(t^\star, r, \theta, \phi)$ are identical for $r \geq r_{\text{cut}}$.

2A4. *The Kerr–AdS metric for fixed (a, M, l).* We may now introduce the Kerr–AdS metric as the unique smooth extension to $\mathcal{R}$ of the tensor given in the Boyer–Lindquist chart by

$$g_{\text{KAdS}} = \frac{\Sigma}{\Delta_-} dr^2 + \frac{\Sigma}{\Delta_\theta} d\theta^2 + \frac{\Delta_\theta(r^2 + a^2)^2 - \Delta_- a^2 \sin^2\theta}{\Xi^2 \Sigma} \sin^2\theta \, d\tilde{\phi}^2 \\ - 2\frac{\Delta_\theta(r^2 + a^2) - \Delta_-}{\Xi \Sigma} a \sin^2\theta \, d\tilde{\phi} dt - \frac{\Delta_- - \Delta_\theta a^2 \sin^2\theta}{\Sigma} dt^2, \tag{10}$$

where Δ_- is defined by (7) and

$$\Sigma = r^2 + a^2\cos^2\theta, \qquad \Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2}. \tag{11}$$

See [Holzegel and Smulevici 2013a] for explicit expressions for the inverse of (10) and its determinant in Boyer–Lindquist coordinates. That the tensor (10) indeed extends to a smooth metric on $\mathcal{R}$ is clear from expressing the metric in $(t^\star, r, \theta, \phi)$ coordinates, which is carried out explicitly in [ibid.]. Note that for $a = 0$ the metric (10) reduces to the well-known Schwarzschild–AdS spacetime

$$g = -\left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right) dt^2 + \left(1 - \frac{2M}{r} + \frac{r^2}{l^2}\right)^{-1} dr^2 + r^2\, d\sigma_{S^2},$$

where $d\sigma_{S^2}$ is the standard metric on the unit sphere.

The Boyer–Lindquist coordinates degenerate at the horizon $r = r_+$, but the importance of these coordinates is that it is in these coordinates that the Klein–Gordon operator separates (see Section 3).

2A5. *The Klein–Gordon operator in Boyer–Lindquist coordinates.* For any Lorentzian metric g and any scalar function ψ, we define as usual

$$\Box_g \psi = \frac{1}{\sqrt{|g|}} \partial_\alpha\big(\sqrt{|g|}\, g^{\alpha\beta} \partial_\beta \psi\big);$$

the Klein–Gordon operator acting on scalar functions in Boyer–Lindquist coordinates is given by

$$\begin{aligned}\left(\Box_g + \frac{\alpha}{l^2}\right)\psi = \left(-\frac{(r^2+a^2)^2}{\Sigma\Delta_-} + \frac{a^2\sin^2\theta}{\Sigma\Delta_\theta}\right)\partial_t^2\psi + \frac{1}{\Sigma}\partial_r(\Delta_-\partial_r\psi) + 2\left(\frac{\Xi(r^2+a^2)a}{\Delta_-\Sigma} - \frac{\Xi a}{\Delta_\theta\Sigma}\right)\partial_t\partial_{\tilde\phi}\psi \\ + \left(\frac{\Xi^2}{\Delta_\theta\Sigma}\frac{1}{\sin^2\theta} - \frac{\Xi^2 a^2}{\Sigma\Delta_-}\right)\partial_{\tilde\phi}\partial_{\tilde\phi}\psi + \frac{1}{\Sigma\sin\theta}\partial_\theta(\Delta_\theta\sin\theta\,\partial_\theta\psi) + \frac{\alpha}{l^2}\psi.\end{aligned} \tag{12}$$

2B. ***The norms.*** Let $\not{g}$ and $\not\nabla$ denote the induced metric and the covariant derivative on the spheres $S^2_{t^\star,r}$ of constant $t^\star$ and r in $\mathcal{R}$.

We write $|\not\nabla\cdots\not\nabla\psi|^2 = \not{g}^{AA'}\cdots\not{g}^{BB'}\not\nabla_A\cdots\not\nabla_B\psi\not\nabla_{A'}\cdots\not\nabla_{B'}\psi$ to denote the induced norms on these spheres. We denote by Ω_i with $i = 1, 2, 3$ the standard basis of angular momentum operators on the unit sphere in θ, ϕ coordinates.

With these conventions, we define the energy densities

$$\begin{aligned} e_1[\psi] &= \frac{1}{r^2}(\partial_{t^\star}\psi)^2 + r^2(\partial_r\psi)^2 + |\not\nabla\psi|^2 + \psi^2, \\ \tilde{e}_2[\psi] &= e_1[\psi] + e_1[\partial_{t^\star}\psi] + r^4(\partial_r\partial_r\psi)^2 + r^2|\partial_r\not\nabla\psi|^2 + |\not\nabla\not\nabla\psi|^2, \\ e_2[\psi] &= \tilde{e}_2[\psi] + \sum_i e_1[\Omega_i\psi]. \end{aligned}$$

Similarly, we define the following energy norms for the scalar field ψ (see [Holzegel 2012]):

$$\|\psi\|^2_{H^{0,s}_{\mathrm{AdS}}(\Sigma_{t^\star})} = \int_{\Sigma_{t^*}} r^s \psi^2 r^2\, dr \sin\theta\, d\theta\, d\phi$$

$$\|\psi\|^2_{H^{1,s}_{\mathrm{AdS}}(\Sigma_{t^\star})} = \int_{\Sigma_{t^*}} r^s \big(r^2(\partial_r\psi)^2 + |\nabla\!\!\!/\,\psi|^2 + \psi^2\big)\, r^2\, dr \sin\theta\, d\theta\, d\phi$$

$$\|\psi\|^2_{H^{2,s}_{\mathrm{AdS}}(\Sigma_{t^\star})} = \|\psi\|^2_{H^{1,s}_{\mathrm{AdS}}(\Sigma_{t^\star})} + \int_{\Sigma_{t^*}} r^s \big(r^4(\partial_r\partial_r\psi)^2 + r^2|\nabla\!\!\!/\,\partial_r\psi|^2 + |\nabla\!\!\!/\,\nabla\!\!\!/\,\psi|^2\big)\, r^2\, dr \sin\theta\, d\theta\, d\phi.$$

Note in particular the relations

$$\int_{\Sigma_{t^*}} e_1[\psi] r^2\, dr \sin\theta\, d\theta\, d\phi = \|\psi\|^2_{H^{1,0}_{\mathrm{AdS}}(\Sigma_{t^\star})} + \|\partial_{t^\star}\psi\|^2_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_{t^\star})}, \tag{13}$$

$$\int_{\Sigma_{t^*}} e_2[\psi] r^2\, dr \sin\theta\, d\theta\, d\phi = \|\Omega_i\psi\|^2_{H^{1,0}_{\mathrm{AdS}}(\Sigma_{t^\star})} + \|\partial_{t^\star}\psi\|^2_{H^{1,0}_{\mathrm{AdS}}(\Sigma_{t^\star})} + \|\psi\|^2_{H^{2,0}_{\mathrm{AdS}}(\Sigma_{t^\star})} + \|\partial_{t^\star}\partial_{t^\star}\psi\|^2_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_{t^\star})}. \tag{14}$$

Higher-order norms may be defined similarly. We denote by $H^{k,s}_{\mathrm{AdS}}(\Sigma_{t^\star})$ the space of functions ψ such that $\nabla^i\psi \in L^2_{loc}(\Sigma_{t^\star})$ for $i = 0, \dots, k$ and $\|\psi\|^2_{H^{k,s}_{\mathrm{AdS}}(\Sigma_{t^\star})} < \infty$. We denote by $CH^{k,s}_{\mathrm{AdS}}$ the set of functions ψ defined on $\mathcal{R}$ such that

$$\psi \in \bigcap_{q=0,\dots,k} C^q\big(\mathbb{R}_{t^\star}; H^{k-q,s_q}_{\mathrm{AdS}}(\Sigma_{t^\star})\big), \quad \text{where } s_k = -2,\ s_{k-1} = 0 \text{ and } s_j = s \text{ for } j = 0, \dots, k-2.$$

When $s = 0$, we will feel free to drop the s in the notation, i.e., $H^{k,0}_{\mathrm{AdS}} := H^k_{\mathrm{AdS}}$ and $CH^k_{\mathrm{AdS}} := CH^{k,0}_{\mathrm{AdS}}$.

2C. ***A final remark.*** In [Holzegel and Smulevici 2013a], the coordinates t and $\tilde{\phi}$ in (9) are defined with the $\chi(r)$ of (8) being *globally* equal to 1. Here, for convenience in the subsequent analysis (which happens mostly away from the horizon, in $r \geq r_{\max}$), we have altered these coordinates *away from the horizon* to agree with the Boyer–Lindquist coordinates. Note that these two coordinate systems are equivalent in the sense that the statement $\|\psi\|^2_{H^{k,s}_{\mathrm{AdS}}(\Sigma_{t^\star})}$ decays logarithmically in $t^\star$ is independent of whether the coordinates (and $\Sigma_{t^\star}$-slices) of our earlier paper or the cut-off coordinates (9) are used.

3. Separation of variables and reduced equations

3A. ***The (modified) oblate spheroidal harmonics.*** For each $\xi \in \mathbb{R}$, define the unbounded $L^2(\sin\theta\, d\theta\, d\tilde{\phi})$-self-adjoint operator $P_{\mathbb{S}^2}(\xi)$ with domain the space of $H^2(\mathbb{S}^2)$-complex valued functions (see Section 7 of [Dafermos and Rodnianski 2010a] for a more detailed discussion), as

$$-P_{\mathbb{S}^2}(\xi) f = \frac{1}{\sin\theta}\partial_\theta(\Delta_\theta \sin\theta\, \partial_\theta f) + \frac{\Xi^2}{\Delta_\theta}\frac{1}{\sin^2\theta}\partial^2_{\tilde{\phi}} f + \Xi\frac{\xi^2}{\Delta_\theta}\cos^2\theta\, f - 2i\xi\frac{\Xi}{\Delta_\theta}\frac{a^2}{l^2}\cos^2\theta\, \partial_{\tilde{\phi}} f. \tag{15}$$

We also define the operator P_α, which is equal to

$$P_{\mathbb{S}^2,\alpha}(\xi) = \begin{cases} P(\xi) + \dfrac{\alpha}{l^2} a^2 \sin^2\theta & \text{if } \alpha > 0, \\[2ex] P(\xi) + \dfrac{|\alpha|}{l^2} a^2 \cos^2\theta & \text{if } \alpha \leq 0. \end{cases} \tag{16}$$

The motivation for introducing these operators is that, formally replacing ξ by $ai\partial_t\psi$, they appear naturally in the Klein–Gordon operator (12) when trying to separate variables; see the next section.

When $l\to\infty$ the operator $P_{\mathbb{S}^2}(\xi)$ reduces to an oblate spheroidal operator on $\mathbb{S}^2$ as considered for instance in [ibid.]. If also $\xi=a=0$ we recover the Laplacian on the round sphere. Both operators $P_{\mathbb{S}^2}(\xi)$ and $P_{\mathbb{S}^2,\alpha}(\xi)$ have discrete spectra. We will use the following notation:

$$P_{\mathbb{S}^2,\alpha}(\xi) \text{ has eigenvalues } \lambda_{m\ell}(\xi) \text{ with eigenfunctions } S_{m\ell}(\xi,\cos\theta)e^{im\tilde{\phi}}.$$

Later we will restrict attention to axisymmetric solutions and hence to the eigenvalues $\lambda_{0\ell}$ of the operators

$$-P_{\theta,\alpha}(\xi^2) := -P_{\mathbb{S}^2,\alpha}(\xi)\big|_{m=0}. \tag{17}$$

In fact, it will be convenient (in view of their manifest positivity) to work with the eigenvalues of the shifted operators, which are, for $\alpha\le 0$,

$$-P_{\theta,\alpha}(\xi^2)-\xi^2=\frac{1}{\sin\theta}\partial_\theta(\Delta_\theta\sin\theta\,\partial_\theta(\,\cdot\,))-\frac{|\alpha|}{l^2}a^2\cos^2\theta-\frac{\sin^2\theta}{\Delta_\theta}\xi^2 \tag{18}$$

($\cos^2$ is replaced by $\sin^2$ in the second term if $\alpha>0$). In view of these considerations, we shall denote the eigenvalues of the operator $P_{\theta,\alpha}(\xi^2)+\xi^2$ by $\mu_\ell(\xi^2)$. By min-max and comparison with the spherical Laplacian [Holzegel and Smulevici 2013a, Lemma 5.1] in the axisymmetric case, we have

$$\mu_\ell(\xi^2)\ge\mu_\ell(0)\ge\Xi\ell(\ell+1)>c_{a,l}\ell(\ell+1). \tag{19}$$

3B. ***The separation of variables.*** We now present the reduced equations obtained after separation of variables. For this purpose, we consider the Klein–Gordon operator in Boyer–Lindquist coordinates (12).

For the *construction* of quasimodes, it would be sufficient to start directly from the reduced equations. However, we will instead derive them from the Klein–Gordon equation (1) to show this relation. Thus, in this section ψ will denote any regular solution to (1). Let us introduce the time-Fourier transform[7]

$$\psi(t,r,\theta,\tilde{\phi})=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}e^{-i\omega t}\hat{\psi}(\omega,r,\theta,\tilde{\phi})\,d\omega. \tag{20}$$

We decompose the $\hat{\psi}$ of (20) as

$$\hat{\psi}(\omega,r,\theta,\tilde{\phi})=\sum_{m\ell}(\hat{\psi})^{(a\omega)}_{m\ell}(r)S_{m\ell}(a\omega,\cos\theta)e^{im\tilde{\phi}}, \tag{21}$$

where $S_{m\ell}$ are the modified spheroidal harmonics introduced above and

$$(\hat{\psi})^{(a\omega)}_{m\ell}(r)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}dt\int_{\mathbb{S}^2(t,r)}d\tilde{\sigma}\,e^{i\omega t}S_{m\ell}(a\omega,\cos\theta)e^{-im\tilde{\phi}}\psi(t,r,\theta,\tilde{\phi}),$$

with $d\tilde{\sigma}=\sin\theta\,d\theta\,d\tilde{\phi}$.

After the renormalization

$$u^{(a\omega)}_{m\ell}(r)=(r^2+a^2)^{1/2}(\hat{\psi})^{(a\omega)}_{m\ell}(r), \tag{22}$$

[7]Note that in general ψ is not an L^2-function in time and therefore, $\hat{\psi}$ is defined only as a tempered distribution. Since here we are merely trying to justify the origin of the reduced equations, it will be sufficient to understand all computations formally.

we finally obtain from (1) and (12) the equation

$$\big(u^{(a\omega)}_{m\ell}(r)\big)'' + \big(\omega^2 - V^{(a\omega)}_{m\ell}(r)\big)u^{(a\omega)}_{m\ell} = 0, \tag{23}$$

where $'$ denotes differentiation with respect to $r^\star$ and where the potential $V^{(a\omega)}_{m\ell}(r)$ is defined as

$$V^{(a\omega)}_{m\ell}(r) = V^{(a\omega)}_{+,m\ell}(r) + V^{(a\omega)}_{0,m\ell}(r) + V_\alpha(r), \tag{24}$$

where

$$V^{(a\omega)}_{+,m\ell}(r) = -\Delta_-^2 \frac{3r^2}{(r^2+a^2)^4} + \Delta_- \frac{5r^4/l^2 + 3r^2(1+a^2/l^2) - 4Mr + a^2}{(r^2+a^2)^3} = (r^2+a^2)^{-1/2}(\sqrt{r^2+a^2})'',$$

$$V^{(a\omega)}_{0,m\ell}(r) = \frac{\Delta_-(\lambda_{m\ell} + \omega^2 a^2) - \Xi^2 a^2 m^2 - 2m\omega a\,\Xi(\Delta_- - (r^2+a^2))}{(r^2+a^2)^2}, \tag{25}$$

$$V_\alpha(r) = -\frac{\alpha}{l^2}\frac{\Delta_-}{(r^2+a^2)^2}(r^2 + \Theta(\alpha)a^2), \tag{26}$$

with $\Theta(\alpha) = 1$ if $\alpha > 0$ and $\Theta(\alpha) = 0$ if $\alpha \le 0$ (recall that the $\lambda_{m\ell}$ also depend on α through (16)). Note that V_+ grows like $2r^2/l^4$ near infinity, while the V_0 part remains bounded.

3C. ***The axisymmetric reduced equations.*** We now look at the axisymmetric case; that is, we consider the above equations under the assumption that ψ is independent of the azimuthal variable $\tilde{\phi}$. The reduced equations are then obtained by setting $m = 0$ in the decomposed equations. Hence, we will consider the following set of equations:

$$\big(u^{(a\omega)}_{0\ell}(r)\big)'' + \big(\omega^2 - V^{(a\omega)}_{0\ell}(r)\big)u^{(a\omega)}_{0\ell}(r) = 0, \tag{27}$$

where the potential $V^{(a\omega)}_{0\ell}(r)$ is defined as

$$V^{(a\omega)}_{0\ell}(r) = V_{\text{junk}}(r) + V_{\text{mass}}(r) + V_\sigma(r)\cdot \mu_\ell(\omega^2 a^2), \tag{28}$$

where

$$V_{\text{junk}}(r) = -\Delta_-^2 \frac{3r^2}{(r^2+a^2)^4} + \Delta_- \frac{5r^4/l^2 + 3r^2(1+a^2/l^2) - 4Mr + a^2}{(r^2+a^2)^3} - \frac{2}{l^2}\frac{\Delta_- r^2}{(r^2+a^2)^2} - \frac{\alpha}{l^2}\frac{\Delta_-}{(r^2+a^2)^2}\Theta(\alpha)a^2,$$

$$V_{\text{mass}}(r) = \frac{2-\alpha}{l^2}\frac{\Delta_- r^2}{(r^2+a^2)^2}, \qquad V_\sigma(r) = \frac{\Delta_-}{(r^2+a^2)^2}. \tag{29}$$

Here we rearranged the terms in the different potentials so that $V_{\text{mass}} = 0$ corresponds to the conformal case $\alpha = 2$. In particular, we have

$$|V_{\text{junk}}(r)| \le C_{M,l,a}\frac{\Delta_-}{(r^2+a^2)^2} \lesssim V_\sigma;$$

hence V_{junk} is uniformly bounded.

Lemma 3.1 (properties of V_σ). *For all $|a| < l$, the potential V_σ enjoys the following properties*:

- $V_\sigma(r^\star) \to 1/l^2$ *as* $r^\star \to \pi/2$.

- *V_σ has a unique local and global maximum $V_{\max}$ at $r^\star_{\max}$ in $[r^\star_+, \pi/2)$. Also, V_σ is monotonically decreasing in $[r^\star_{\max}, \pi/2)$.*
- $V_{\max} = V_\sigma(r^\star_{\max}) \geq \dfrac{1}{l^2} + \dfrac{3M/\Xi + \Xi a^2}{\big((3M/\Xi)^2 + a^2\big)^2}$.

Remark 3.2. In particular, for any $0 < a_0 < l$ and all $|a| \leq a_0$, the size of the interval $[1/l^2, V_\sigma(r^\star_{\max})]$ is bounded uniformly (in a) from below by a strictly positive uniform constant.

Proof. The first claim can be trivially checked. For the second and third claims, let us write

$$V_\sigma(r) = \frac{\Delta_-}{(r^2+a^2)^2} = \frac{(r^2+a^2)(r^2/l^2 + a^2/l^2 + \Xi) - 2Mr}{(r^2+a^2)^2} = \frac{1}{l^2} + \frac{\Xi(r^2+a^2) - 2Mr}{(r^2+a^2)^2}. \tag{30}$$

The r-derivative of V_σ is

$$\begin{aligned}\partial_r V_\sigma(r) &= \frac{(r^2+a^2)^2(2\Xi r - 2M) - 4r(r^2+a^2)(\Xi(r^2+a^2) - 2Mr)}{(r^2+a^2)^4}\\ &= \frac{(r^2+a^2)(2\Xi r - 2M) - 4r(\Xi(r^2+a^2) - 2Mr)}{(r^2+a^2)^3} = \frac{-2\Xi r^3 + 6Mr^2 - 2\Xi a^2 r - 2Ma^2}{(r^2+a^2)^3}. \end{aligned}\tag{31}$$

If $a = 0$, then one has as usual that the only zero in $[r_+, \infty)$ is at $r = 3M$. Assume $a \neq 0$. Observe that the derivative is positive near $-\infty$, negative at $r = 0$, positive at $r = r_+$ and negative near ∞. This tells us that there are three real roots for $\partial_r V_\sigma$. The one of interest to us is the one corresponding to the (unique) maximum of V_σ in the interval $[r_+, \infty)$. Define $r_{\text{guess}} = 3M/\Xi$. We claim $r_+ < r_{\text{guess}} < \infty$. This follows from $r_+ < 2M$, which is in turn a consequence of the fact that $\Delta_- = r(r-2M) + r^4/l^2 + a^2(1 + r^2/l^2)$ is positive for $r \geq 2M$. Finally, at $r_{\text{guess}} = 3M/\Xi$ we have from (30)

$$V_\sigma(r_{\text{guess}}) = \frac{1}{l^2} + \frac{3M/\Xi + \Xi a^2}{(r^2_{\text{guess}} + a^2)^2} > \frac{1}{l^2}. \qquad \square$$

4. Bound states

As proven in [Holzegel and Smulevici 2013a], there exist no periodic solutions of the massive wave equation on Kerr–AdS. In this section, we will introduce an additional boundary, located at $r = r_{\max}$ (the location of the top of the potential V_σ, as defined in Lemma 3.1), enabling us to construct periodic solutions whose associated energies lie below the top of V_σ.

In order to avoid confusion between the mode number ℓ and the real number l determining the cosmological constant, we introduce the semiclassical parameter $h > 0$ by defining

$$h^{-2}(\ell, \omega) := \lambda_{0\ell}(a^2\omega^2) + a^2\omega^2 = \mu_\ell(a^2\omega^2), \tag{32}$$

as well as the shorthand

$$h_0^{-2} = h^{-2}(\ell, 0) = \mu_\ell(0). \tag{33}$$

In the rest of this paper, the notation u_ℓ or u_h will be used when we want to make explicit that a solution u depends on ℓ or h.

Having in mind a semiclassical-type analysis with semiclassical parameter h, we then rewrite (27) as the *nonlinear eigenvalue problem*

$$\begin{gathered} h^2\omega^2 u = P(h)u := -h^2u'' + V_\sigma u + h^2(V_{\text{junk}} + V_{\text{mass}})u, \\ \text{with boundary conditions} \quad u(r^\star_{\max}) = 0 \quad \text{and} \quad \int_{r^\star_{\max}}^{\pi/2} (|u'|^2 + r^2u^2)\,dr^\star < \infty. \end{gathered} \tag{34}$$

Remark 4.1. We will refer to these boundary conditions as Dirichlet conditions. They imply that $u(\pi/2) = 0$. See Remark 4.2 below.

Unless $a = 0$, (34) is a nonlinear eigenvalue problem. Indeed, a solution to the eigenvalue problem

$$\kappa u = P(h)u \tag{35}$$

with Dirichlet boundary conditions is a solution of (34) if and only if $\kappa = h^2\omega^2$. If $a = 0$, (34) reduces to the linear problem (35), since the $P(h)$ operator becomes independent of ω, and, therefore, given any solution to (35), one simply obtains a solution to (34) by defining $\omega^2 = h^{-2}\kappa$.

What we would like to prove is that, given fixed parameters M, a, l and α, we can find, for any sufficiently small h (or, equivalently, any sufficiently large ℓ), an ω_ℓ^2 such that (34) is solved for some u_ℓ, and that we can control the size of this ω_ℓ^2 (see Proposition 4.8 below).

In order to understand (34), it will be useful to first look at the following *linear eigenvalue problem*:

$$\begin{gathered} h_0^2\omega^2\, u = P_{\text{base}}(h_0)u := -h_0^2u'' + V_\sigma u + h_0^2(V_{\text{junk}} + V_{\text{mass}})u, \\ \text{with boundary conditions} \quad u(r^\star_{\max}) = 0 \quad \text{and} \quad \int_{r^\star_{\max}}^{\pi/2} [|u'|^2 + r^2u^2]\,dr^\star < \infty. \end{gathered} \tag{36}$$

As explained above, (36) can be seen as a linear eigenvalue problem because h_0, and therefore $P_{\text{base}}(h_0)$, depends only on ℓ (but not on ω).

Remark 4.2. By Proposition 4 of [Holzegel and Warnick 2014], (36) is a well-posed eigenvalue problem. This is a nontrivial statement because the potential V_{mass} is unbounded on the domain $(r^\star_{\max}, \pi/2i)$ unless $\alpha = 2$. The condition $\int_{r^\star_{\max}}^{\pi/2}[|u'|^2 + r^2u^2]\,dr^\star < \infty$ implies that $\psi = \sqrt{r^2+a^2}\,u \cdot S_{0\ell}(\theta) \in H^1_{\text{AdS}}$ (in particular, $r^{1/2-\epsilon}u(\pi/2) = 0$ for any $\epsilon > 0$) and ensures the existence of a positive discrete spectrum with eigenfunctions in the energy space.[8]

To prove that eigenvalues $h_0^2\omega^2$ for (36) exist in a suitable range, we will perform a semiclassical-type analysis for a semiclassical operator whose principal part should be $-h_0^2u'' + V_\sigma u$. Since $h_0^2 V_{\text{junk}}$ is controlled by $h_0^2 V_\sigma$, this term will be of lower order and hence negligible. On the other hand, unless one considers the conformal case $\alpha = 2$, the a priori lower-order (in powers of h_0) potential term $h_0^2 V_{\text{mass}}$ is unbounded near $r^\star = \pi/2$, so that some care (a Hardy inequality) is required.

Observe finally that if we set $a = 0$ in V_σ, V_{junk} and V_{mass} in (36), then

$$P^{a=0}_{\text{base}}(h_0)u = \kappa \cdot u \tag{37}$$

[8] Using the twisted derivatives of [Holzegel and Warnick 2014], i.e., writing $u'' + V_{\text{mass}}u = r^n(r^{-2n}(r^nu)')' + V_{\text{twist}} \cdot u$ with $n = \frac{1}{2}(1 - \sqrt{9-4\alpha})$, so V_{twist} is uniformly bounded, one could generalize our construction to other boundary conditions.

would be precisely the eigenvalue problem one needs to study for Schwarzschild–AdS. In any case, in the next section we will establish the existence of eigenvalues $\kappa = h_0^2\omega^2$ of the more general eigenvalue problem (36) as the latter is easier to connect to the full problem (34).

4A. ***Weyl's law for the linear eigenvalue problem* (36).** The aim of this section is to prove a Weyl's-law-type result for the linear problem (36). This is a classical problem which we will approach using a slight modification of the usual Dirichlet–Neumann bracketing argument (see for instance [Reed and Simon 1978], Section XIII.15, for an introduction to this method).

For the purpose of this section, it will be convenient to introduce the following notation. For all $c < d$, we define $P_{DD}^{\text{base}}(c, d)$ as the eigenvalue problem

$$P_{\text{base}}(h_0)u = \kappa u,$$

with Dirichlet boundary conditions $u(c) = u(d) = 0$, plus the condition $\int_c^d dr^\star(|u'|^2 + r^2 \cdot u^2) < \infty$ if $d = \pi/2$ (see Remark 4.2); here $P_{\text{base}}(h_0)$ is the operator defined by (36). Similarly, we will write $P_{NN}^{\text{base}}(c, d)$ for the Neumann problem (which will never be considered with $d = \pi/2$). Finally, we write $P_{ND}^{\text{base}}(c, d)$ for the Neumann boundary at c and Dirichlet at d. In the latter case we again impose $\int_c^d dr^\star(|u'|^2 + r^2 \cdot u^2) < \infty$ if $d = \pi/2$ to ensure that all eigenfunctions live in the energy space; see Remark 4.2. Note that $P_{DD}^{\text{base}}(r^\star_{\max}, \pi/2)$ is precisely the linear eigenvalue problem (36).

Proposition 4.3. *Let $\alpha < \frac{9}{4}$, $M > 0$ and $|a| < l$ be fixed, and $E \in (1/l^2, V_{\max})$ be given. Then, for any $\delta > 0$ such that $[E-\delta, E+\delta] \subset (1/l^2, V_{\max})$, there exists an $H_0 > 0$, such that, for any $0 < h_0 \le H_0$, there exists a smooth solution u_{h_0} of the eigenvalue problem $P_{DD}^{\text{base}}(r^\star_{\max}, \pi/2)$, with corresponding eigenvalue κ lying in $[E-\delta, E+\delta]$. In particular, there exists a sequence $((h_0)_n, u_{(h_0)_n})$ such that the associated eigenvalues $\kappa((h_0)_n) \to E$ as $(h_0)_n \to 0$.*

In the rest of this section, (a, M, l, α) are fixed parameters satisfying the assumptions of the proposition.

We shall in fact prove in this section a stronger result than Proposition 4.3, namely a version of Weyl's law adapted to our problem. This is the statement of Lemma 4.5, from which Proposition 4.3 immediately follows. The proof of Lemma 4.5 in turn requires the following auxiliary lemma, which ensures nonexistence of eigenvalues below a certain threshold.

Lemma 4.4. *Let $E > 0$ be given. Then there exists an $H_0 > 0$ so that for all $0 < h_0 \le H_0$, there exists a $r^\star_K(E, h_0, \alpha)$ such that the problems $P_{DD}^{\text{base}}(r^\star_K, \pi/2)$ and $P_{ND}^{\text{base}}(r^\star_K, \pi/2)$ have no solutions with $\kappa \le E$. Moreover,*

$$r_{\max} < r_K \le \frac{C}{h_0} \cdot E, \tag{38}$$

where C depends only on M, a, l and α.

Proof. Assume there was a solution u of $P_{DD}^{\text{base}}(r^\star_K, \pi/2)$ or $P_{ND}^{\text{base}}(r^\star_K, \pi/2)$ with $\kappa \le E$. Then we would have

$$\int_{r^\star_K}^{\pi/2} dr^\star \left[h_0^2|u'|^2 + (V_\sigma + h_0^2 V_{\text{junk}} + h_0^2 V_{\text{mass}} - E)|u|^2\right] \le 0 \tag{39}$$

for this u. On the other hand, since u solves $P_{DD}^{\text{base}}(r^\star_K, \pi/2)$ or $P_{ND}^{\text{base}}(r^\star_K, \pi/2)$, we have $r^{1/2}u(r) = o(1)$, so that the Hardy inequality

$$\int_{r_K^\star}^{\pi/2} dr^\star \frac{\Delta_-}{r^2+a^2}|u|^2 \leq 4l^2 \int_{r_K^\star}^{\pi/2} dr^\star |u'|^2 \tag{40}$$

proven in [Holzegel and Smulevici 2013a] holds for u. This implies that

$$\int_{r_K^\star}^{\pi/2} dr^\star \left[\left(h_0^2 \frac{1}{4l^2}\frac{\Delta_-}{r^2+a^2} + V_\sigma + h_0^2 V_{\text{junk}} + h_0^2 V_{\text{mass}} - E\right)|u|^2\right] \leq 0.$$

The dominant term in the integrand near infinity is $h_0^2(\frac{9}{4}-\alpha)r^2/l^2$, which is positive, while all other terms remain bounded. Hence by choosing $r_K^\star$ sufficiently large ($r_K \geq C/h_0 \cdot E$ for some constant C) we obtain a contradiction as the parenthetical in the integrand eventually becomes positive. □

Consider now the eigenvalue problem $P_{DD}^{\text{base}}(r_{\max}^\star, \pi/2)$ and fix an energy level $\mathscr{E} \in (1/l^2, V_{\max})$. Lemma 4.4 produces an $r_K^\star(\mathscr{E}, h_0, \alpha)$, to which we associate the phase space volume

$$\begin{aligned}\mathscr{P}_{\mathscr{E},h_0,\alpha} &= \mathrm{Vol}\big\{(\xi, r^\star) \in \mathbb{R} \times [r_{\max}^\star, r_K^\star] \;\big|\; \xi^2 + V_\sigma + h_0^2 V_{\text{mass}} + h_0^2 V_{\text{junk}} \leq \mathscr{E}\big\} \\ &= 2\int_{r_{\max}^\star}^{r_K^\star} dr^\star \sqrt{\mathscr{E} - V(r^\star)} \cdot \chi_{\{V(r^\star)\leq \mathscr{E}\}}.\end{aligned} \tag{41}$$

Note that for fixed $\mathscr{E}$ this expression converges uniformly in h_0 as $h_0 \to 0$. This is already immediate for $\alpha \leq 2$: $V(r^\star)$ is then bounded below and hence the integrand itself is obviously uniformly bounded in h_0. For $2 < \alpha < \frac{9}{4}$, the integral (41) also converges uniformly in h_0 since

$$\int_{r_{\max}^\star}^{r_K^\star} dr^\star\, h_0 r \leq C \int_{r_{\max}}^{r_K} dr\, h_0 \frac{1}{r} \leq C h_0 \log\left(\frac{C}{h_0} E\right)$$

goes to zero as $h_0 \to 0$. Here we have used the estimate (38) on r_K.

Finally, to state and prove Weyl's law, we also introduce an expression for the phase space volume between two energy levels, say $[E-\delta, E+\delta] \subset (1/l^2, V_{\max})$:

$$\mathscr{Q}_{E,\alpha} = \lim_{h_0\to 0} \mathscr{P}_{E+\delta,h_0,\alpha} - \lim_{h_0\to 0} \mathscr{P}_{E-\delta,h_0,\alpha} = \mathrm{Vol}\{(\xi, r^\star) \mid E-\delta \leq \xi^2 + V_\sigma \leq E+\delta\}.$$

By an elementary computation, we have a lower bound $\mathscr{Q}_{E,\alpha} \geq C_{E,M,l,\alpha}\delta$ for a constant independent of h_0.

Lemma 4.5. *Consider the eigenvalue problem $P_{DD}^{\text{base}}(r_{\max}^\star, \pi/2)$. Fix an energy level $V_{\max} > E > 1/l^2$ and prescribe a small $\delta > 0$. Then the number of eigenvalues of $P_{DD}^{\text{base}}(r_{\max}^\star, \pi/2)$ lying in the interval $[E-\delta, E+\delta] \subset (1/l^2, V_{\max})$, denoted by $N[E-\delta, E+\delta]$, satisfies Weyl's law*

$$N[E-\delta, E+\delta] \underset{h_0\to 0}{\sim} \frac{1}{2\pi h_0}\mathscr{Q}_{E,\alpha}. \tag{42}$$

Proof. Choose $r_K^\star(E+\delta, h_0)$ such that by Lemma 4.4 there are no eigenvalues below $E+\delta$ of $P_{DD}^{\text{base}}(r_K^\star, \pi/2)$ and $P_{ND}^{\text{base}}(r_K^\star, \pi/2)$. We equipartition the domain $[r_{\max}^\star, r_K^\star(E, h_0)]$ into k intervals of length $\beta = (\pi/2 - r_K^\star)/k$. We then consider two comparison problems:

- The Dirichlet problem $P_{DD}^{\text{base}}(r_K^\star, \pi/2)$ in conjunction with k Dirichlet problems P_D^{i+} $(i = 1, \ldots, k)$ arising as follows: They are the problems $P_{DD}^{\text{base}}(r_{\max}^\star, r_{\max}^\star + \beta), \ldots, P_{DD}^{\text{base}}(r_K^\star - \beta, r_K^\star)$ but with the potential replaced by a constant, which equals the *maximum* of the potential on the interval.
- The mixed problem $P_{ND}^{\text{base}}(r_K^\star, \pi/2)$ in conjunction with k Neumann problems P_N^{i-} $(i = 1, \ldots, k)$ arising as follows: They are the problems $P_{NN}^{\text{base}}(r_{\max}^\star, r_{\max}^\star + \beta), \ldots, P_{NN}^{\text{base}}(r_K^\star - \beta, r_K^\star)$ but with the potential replaced by a constant, which equals the *minimum* of the potential on the interval.

We can estimate the number of eigenvalues of $P_{DD}^{\text{base}}(r_{\max}^\star, \pi/2)$ below a threshold $\mathscr{E}$ by

$$\sum_{i=1}^{k} N_{\le\mathscr{E}}(P_D^{i+}) + N_{\le\mathscr{E}}\big(P_{DD}^{\text{base}}(r_K^\star, \pi/2)\big) \le N_{\le\mathscr{E}}\big(P_{DD}^{\text{base}}(r_{\max}^\star, \pi/2)\big)$$
$$\le \sum_{i=1}^{k} N_{\le\mathscr{E}}(P_N^{i-}) + N_{\le\mathscr{E}}\big(P_{ND}^{\text{base}}(r_K^\star, \pi/2)\big).$$

By our choice of $r_K^\star$, we have $N_{\le\mathscr{E}}\big(P_{DD}^{\text{base}}(r_K^\star, \pi/2)\big) = N_{\le\mathscr{E}}\big(P_{ND}^{\text{base}}(r_K^\star, \pi/2)\big) = 0$ for $\mathscr{E} = E + \delta$. On the other hand, for each P_D^{i+} and each P_N^{i-}, the number of eigenvalues can be estimated directly (as each problem can be solved explicitly). We have

$$\sum_{i=1}^{k} N_{\le\mathscr{E}}(P_D^{i+}) = \sum_{i=1}^{k} \left\lfloor \frac{\beta}{2\pi h_0} \max(0, \mathscr{E} - V_+^i)\sqrt{\mathscr{E} - V_+^i} \right\rfloor = \sum_{i=1}^{k} \frac{\beta}{2\pi h_0} \max(0, \mathscr{E} - V_+^i)\sqrt{\mathscr{E} - V_+^i} + \mathbb{O}(k).$$

The estimate for P_N^{i-} is similar, with the potential replaced by V_-^i and the number of eigenvalues in each cell being

$$\left\lfloor \frac{\beta}{2\pi h_0} \max(0, \mathscr{E} - V_-^i)\sqrt{\mathscr{E} - V_-^i} \right\rfloor + 1.$$

To conclude, let us choose the number of cells k such that $k(h_0)$ tends to ∞ as h_0 goes to 0 and moreover $k(h) = o(1/h_0)$. The sums converge as a Riemann sum and the errors are then of order $o(1/h_0)$. Therefore we get

$$\sum_{i=1}^{k} N_{\le\mathscr{E}}(P_D^{i+}) \underset{h_0\to 0}{\sim} \frac{1}{2\pi h_0} \int_{r_{\max}^\star}^{r_K^\star} dr^\star \sqrt{\mathscr{E} - V(r^\star)} \cdot \chi_{V(r^\star)\le\mathscr{E}}. \tag{43}$$

The statement of the lemma then follows from

$$N[E - \delta,\ E + \delta] = N_{\le E+\delta} - N_{\le E-\delta}$$

using the previous formula with $\mathscr{E} = E \pm \delta$. □

4B. ***Kerr–AdS.*** In the last section we showed that for any fixed given parameters $M > 0$, $|a| < l$, $\alpha < \frac{9}{4}$, the eigenvalue problem (36), $P_{\text{base}}(h_0)u = \kappa \cdot u$ with Dirichlet conditions, admits (lots of) eigenvalues κ in the range $E - \delta \le \kappa = h_0^2\omega^2 < E + \delta$, provided h_0 is chosen sufficiently small (i.e., ℓ large).

As an immediate corollary, we obtain the existence of eigenvalues in the desired range for Schwarzschild–AdS, simply by setting $a = 0$; see (37). For the Kerr–AdS case, we still need to relate the above result to the full problem, which we recall is the nonlinear eigenvalue problem (34) given by

$$P(h)u = \kappa \cdot u, \quad \text{with } \kappa = \omega^2 h^2$$

and boundary conditions $u(r^\star_{\max}) = 0$ as well as $\int_{r^\star_{\max}}^{\pi/2} dr^\star [|u'|^2 + r^2 u^2] < \infty$. To achieve this, consider for fixed $|a| < l$, $M > 0$, $\alpha < \frac{9}{4}$ the two-parameter family of linear eigenvalue problems

$$Q_\ell(b^2, \omega^2)u = \Lambda(b^2, \omega^2)u \tag{44}$$

for the operator

$$Q_\ell(b^2, \omega^2)u := -u'' + \big(V_\sigma\, \mu_\ell(b^2 a^2 \omega^2)u + (V_{\text{junk}} + V_{\text{mass}} - \omega^2)\big)u, \tag{45}$$

complemented by the above boundary conditions. Here $b^2 \in [0, 1]$ is a dimensionless parameter and $\omega^2 \in \mathbb{R}^+$. Our goal is to show that for $b = 1$ there exists an ω^2 such that the above problem has a zero eigenvalue and, moreover, to suitably control the size of this ω^2.

By the results of the previous section, we know that for $b = 0$ there exists, for any sufficiently large ℓ, an $\omega^2_{0,\ell}$ (satisfying $E - \delta \le \omega^2_{0,\ell}/\mu_\ell(0) \le E + \delta$) such that $Q_\ell(0, \omega^2_{0,\ell})$ admits a zero eigenvalue. Moreover, this eigenvalue is nondegenerate, by standard Sturm–Liouville theory. Listing the eigenvalues of $Q_\ell(0, \omega^2_{0,\ell})$ in ascending order, let us say that it is the n_ℓ-th eigenvalue, which is zero.

The strategy, now, is the following: We will show by an application of the implicit function theorem that for any $b \in [0, 1]$ we can find an $\omega^2_{b,\ell}$ such that the n_ℓ-th eigenvalue of the operator $Q(b^2, \omega^2_{b,\ell})$ is zero. As a second step, we will provide a global estimate on the quotient $\omega^2_{b,\ell}/\mu_\ell(b^2 a^2 \omega^2_{b,\ell})$. For this last step, an important monotonicity will be exploited.

Lemma 4.6. *Fix parameters $|a| < l$, $M > 0$ and $\alpha < \frac{9}{4}$. Suppose we are given parameters $b_0 \in [0, 1]$ and $\omega^2_{b_0,\ell} \in \mathbb{R}^+$ such that the n_ℓ-th eigenvalue of $Q_\ell(b_0^2, \omega^2_{b_0,\ell})$ is zero. Then, there exists an $\varepsilon > 0$ such that:*

1. *For any $b^2 \in (\max(0, b_0^2 - \varepsilon), b_0^2 + \varepsilon)$ one can find an associated $\omega^2_{b,\ell} \in \mathbb{R}^+$ such that the n_ℓ-th eigenvalue of $Q_\ell(b^2, \omega^2_{b,\ell})$ is zero,*
2. *$\omega^2_{b,\ell}$ changes differentiably in $b^2 \in (\max(0, b_0^2 - \varepsilon), b_0^2 + \varepsilon)$, and we have the estimate*

$$0 \le \frac{d\omega^2_{b,\ell}}{db^2} \le C\omega^2_{b,\ell},$$

   for some constant $C > 0$ which is independent of b_0, ℓ and ε.

3. *The $\varepsilon > 0$ can be taken to be independent of b_0 (but may depend on ℓ).*
4. *$\omega^2_{b,\ell}$ satisfies the estimate*

$$c^{-1} \le \frac{\omega^2_{b,\ell}}{\ell(\ell+1)} \le c,$$

   for some $c > 0$ depending only on the parameters a, l, M, α.

Proof. The n-th eigenvalue of $Q_\ell(b^2, \omega_{b,\ell}^2)$, denoted by $\Lambda_n(b^2, \omega^2)$, moves smoothly in the parameters b^2 and ω^2, and we have the formula

$$\Lambda_n(b^2, \omega^2) = \int_{r^\star_{\max}}^{\pi/2} dr^\star \psi_n(b^2, \omega^2) Q_\ell(b^2, \omega^2)\psi_n(b^2, \omega^2) \tag{46}$$

for the eigenvalue, provided we normalize the associated eigenfunctions $\psi_n(b^2, \omega^2)$ by

$$\int_{r^\star_{\max}}^{\pi/2} dr^\star |\psi_n(b^2, \omega^2)|^2 = 1.$$

By assumption, $\Lambda_n(b_0^2, \omega_{b_0,\ell}^2) = 0$. Note that from the normalized condition on the eigenfunctions

$$\int_{r^\star_{\max}}^{\pi/2} dr^* \frac{\partial \psi_n}{\partial \omega^2}(b^2, \omega^2)\psi_n(b^2, \omega^2) = 0,$$

which, combined with the eigenvalue equation $\Lambda_n \psi_n = Q_\ell \psi_n$, leads to

$$\int_{r^\star_{\max}}^{\pi/2} dr^* \frac{\partial \psi_n}{\partial \omega^2}(b^2, \omega^2) Q_\ell(b^2, \omega^2)\psi_n(b^2, \omega^2) = 0.$$

Thus, differentiating (46), we get

$$\frac{\partial \Lambda_n}{\partial \omega^2} = \int_{r^\star_{\max}}^{\pi/2} dr^* \psi_n(b^2, \omega^2)\frac{\partial Q_\ell}{\partial \omega^2}(b^2, \omega^2)\psi_n(b^2, \omega^2),$$

and a similar formula holds replacing $\dfrac{\partial}{\partial \omega^2}$ with $\dfrac{\partial}{\partial b^2}$. Using this, we compute

$$\frac{\partial \Lambda_n}{\partial \omega^2}(b_0^2, \omega_{b_0,\ell}^2) = \int_{r^\star_{\max}}^{\pi/2} dr^\star \psi_n^2(b_0^2, \omega_{b_0,\ell}^2) \cdot \left(V_\sigma \cdot \frac{\partial \mu_\ell}{\partial \omega^2}(b_0^2, \omega_{b_0,\ell}^2) - 1 \right), \tag{47}$$

$$\frac{\partial \Lambda_n}{\partial b^2}(b_0^2, \omega_{b_0,\ell}^2) = \int_{r^\star_{\max}}^{\pi/2} dr^\star \psi_n^2(b_0^2, \omega_{b_0,\ell}^2) \cdot \left(V_\sigma \cdot \frac{\partial \mu_\ell}{\partial b^2}(b_0^2, \omega_{b_0,\ell}^2) \right). \tag{48}$$

The angular eigenvalue $\mu_\ell(b^2, \omega^2)$ is itself a smooth function of the two parameters b^2 and ω^2. We have the formula

$$\mu_\ell(b^2, \omega^2) = \int_0^\pi d\theta \sin\theta \; \phi_\ell(b^2, \omega^2)[P_{\theta,\alpha}(b^2a^2\omega^2) + b^2a^2\omega^2]\phi_\ell(b^2, \omega^2) \tag{49}$$

for the eigenvalues, provided we normalize the associated eigenfunctions $\phi_\ell(b^2, \omega^2)$ by

$$\int_0^\pi d\theta \sin\theta |\phi_\ell(b^2, \omega^2)|^2.$$

Recalling from (18) that

$$P_{\theta,\alpha}(b^2a^2\omega^2) f + b^2a^2\omega^2 f = -\frac{1}{\sin\theta}\partial_\theta(\Delta_\theta \sin\theta \partial_\theta f) + \frac{\sin^2\theta}{\Delta_\theta} b^2a^2\omega^2 f + \frac{|\alpha|}{l^2}a^2 \begin{cases} \cos^2\theta f \\ \sin^2\theta f \end{cases}, \tag{50}$$

to be read with the upper (lower) line if $\alpha < 0$ ($\alpha \geq 0$), we obtain

$$\frac{\partial \mu_\ell}{\partial \omega^2}(b_0^2, \omega_{b_0,\ell}^2) = b_0^2 l^2 \cdot T_x \quad \text{and} \quad \frac{\partial \mu_\ell}{\partial b^2}(b_0^2, \omega_{b_0,\ell}^2) = \omega_{b_0,\ell}^2 l^2 \cdot T_x, \tag{51}$$

where

$$T_x = \int_0^\pi d\theta \sin\theta \frac{\sin^2\theta}{\Delta_\theta}\frac{a^2}{l^2} \, |\phi_\ell(b_0^2, \omega_{b_0,\ell}^2)|^2 \leq \sup_\theta \left(\frac{\sin^2\theta}{\Delta_\theta}\frac{a^2}{l^2}\right) \int_0^\pi d\theta \sin\theta |\phi_\ell(b_0^2, \omega_{b_0,\ell}^2)|^2 \leq \frac{a^2}{l^2}, \tag{52}$$

where we have used the estimate $\sup_{\theta\in[0,\pi)}(\sin^2\theta/\Delta_\theta) \leq 1$, which can be easily checked using (11). Going back to (47) and (48), the implicit function theorem allows us to solve for ω^2 as a (smooth) function of b^2 locally near $(b_0^2, \omega_{b_0,\ell}^2)$, provided that the right-hand side of (47) is nonzero. To achieve this, note that

$$1 - V_\sigma \frac{\partial \mu_\ell}{\partial \omega^2} \geq 1 - b_0^2 a^2 V_\sigma \geq 1 - a^2 V_\sigma,$$

using the estimate obtained for $\frac{\partial \mu_\ell}{\partial \omega^2}$ from (51) and (52). On the other hand, using (30),

$$a^2 V_\sigma = \frac{a^2}{l^2} + \frac{\Xi a^2}{r^2+a^2} = \frac{a^2}{l^2} + \Xi - \frac{\Xi r^2}{r^2+a^2} < 1 - c_{M,l,a},$$

where $c_{M,l,a}$ is a constant depending only on the parameters M, l, a. It then follows that the right-hand side of (47) is bounded away from zero with the lower bound being independent of b_0 (and ℓ), i.e.,

$$-\frac{\partial \Lambda_n}{\partial \omega^2}(b_0^2, \omega_{b_0,\ell}^2) \geq c_{M,l,a}\,. \tag{53}$$

This concludes the proof of the first item of the lemma (with ε a priori depending also on b_0^2).

The implicit function theorem also provides a formula for the derivative of the function $\omega^2(b^2)$ just obtained, namely

$$\frac{d\omega_{b,\ell}^2}{db^2}(b^2) = \frac{\frac{\partial \Lambda_n}{\partial b^2}(b^2, \omega_{b,\ell}^2)}{-\frac{\partial \Lambda_n}{\partial \omega^2}(b^2, \omega_{b,\ell}^2)} \quad \text{for any } b^2 \in (\max(0, b_0^2 - \varepsilon), b_0^2 + \varepsilon). \tag{54}$$

We can now repeat the computations following (47) with b_0^2 replaced by any $b^2 \in (\max(0, b_0^2 - \varepsilon), b_0^2 + \varepsilon)$. Proceeding in this way, we obtain the uniform lower bound on the denominator (53) for any such b^2. The numerator is easily estimated in view of (51), again replacing b_0^2 by b^2. Note moreover that this derivative has a positive sign since μ_ℓ is an increasing function of b and (48). We thus obtain an estimate of the form

$$0 \leq \frac{d\omega_{b,\ell}^2}{db^2}(b^2) \leq C_{M,l,a}\omega_{b,\ell}^2 \quad \text{for any } b^2 \in (\max(0, b_0^2 - \varepsilon), b_0^2 + \varepsilon), \tag{55}$$

establishing item 2 of the lemma.

We can now apply part 1 of the lemma, first with $b_0^2 = 0$, then with $b_0^2 + \delta$, etc. Integrating the differential inequality (55) from $b_0^2 = 0$ to any point in the interval thus produced will provide the uniform bound $\omega_{b,\ell}^2 \leq C_{M,l,a}\omega_{0,\ell}^2$. Now, $\Lambda_n(b^2, \omega^2)$ is a smooth function on the compact set $[0, 1] \times [0, \omega_{\max,\ell}^2]$, where $\omega_{\max,\ell}^2$ denotes an upper bound for $\omega_{b^2,\ell}^2$ (independent of b!). Since we also have the uniform bound

(53) we obtain uniform control of the errors arising in the implicit function theorem and can conclude that the ε in the implicit function theorem does not depend on b_0. This is item 3.

To establish item 4, we observe that the identity $\Lambda_n(\omega^2, b^2) = 0$ implies (using the Hardy inequality (40) as well as $V_\sigma \geq 1/l^2$) that $\omega^2_{b,\ell} \geq (1/2l^2)\mu_\ell(b^2\omega^2) \geq (1/2l^2)\mu_\ell(0) \geq c_{M,l,a}\ell(\ell+1)$ for any $b^2 \in [0, 1]$. Hence the quantity $\omega^2_{b,\ell}$ will stay strictly away from zero and

$$C_{M,l,a} \geq \frac{\omega^2_{0,\ell}}{\ell(\ell+1)} \geq C^{-1}_{M,l,a}\frac{\omega^2_{b,\ell}}{\ell(\ell+1)} \geq c_{M,l,a}, \tag{56}$$

the first inequality following from the analysis of the *linear* eigenvalue problem (36) in Section 4A, and the second from the uniform estimate (55) on $d\omega^2_{b,\ell}/db^2$. □

In view of the fact that ε is uniform in b_0^2, we can apply the implicit function theorem all the way from $b_0 = 0$ to $b = 1$. Note that ε *can* depend on ℓ. However, for each *fixed* ℓ it only takes finitely many applications of the implicit function theorem to reach $b = 1$.

To gain quantitative global control beyond (56) on the behavior of $\omega^2_{b,\ell}(b^2)$, let us look at the quotient

$$E_n(b^2) := \frac{\omega^2_{b,\ell}(b^2)}{\mu_\ell(b^2a^2\omega^2_{b,\ell}(b^2))} \quad \text{with } b^2 \in [0, 1], \tag{57}$$

which by construction is the n_ℓ-th eigenvalue of the semiclassical operator

$$\widetilde{Q}_b(u) = E_n(b^2)u$$

with

$$\widetilde{Q}_b(u) = -u''\big(\mu_\ell(b^2a^2\omega^2_{b,\ell})\big)^{-1} + \big(V_\sigma + [\mu_\ell(b^2a^2\omega^2_{b,\ell})]^{-1}(V_{\text{junk}} + V_{\text{mass}})\big)u.$$

Recall that E is an energy level such that $E \in (1/l^2, V_{\max})$ and that, with the notation just introduced, $E_n(0) \in [E-\delta, E+\delta]$ for some small $\delta > 0$.

Lemma 4.7. *For all $\delta' > 0$, there exists an L such that, for all $\ell > L$, we have*

$$\frac{1}{l^2} \leq E_n(1) \leq E + \delta + \delta'. \tag{58}$$

Proof. We first establish the upper bound. Using the Hardy inequality (40) together with (19) (which implies that $\sigma := 1/\mu_\ell(0) - 1/\mu_\ell(b^2a^2\omega^2_{b,\ell}) \geq 0$ for any $1 \geq b > 0$), we can estimate

$$\begin{aligned}\int_{r^\star_{\max}}^{\pi/2} dr^\star[u(\widetilde{Q}_0 - \widetilde{Q}_b)u] &\geq \sigma\int_{r^\star_{\max}}^{\pi/2} dr^\star|u|^2\left(\frac{1}{4l^2}\frac{\Delta_-}{r^2+a^2} + V_{\text{junk}} + V_{\text{mass}}\right)\\ &\geq \sigma\int_{r^\star_{\max}}^{\pi/2} dr^\star\left(\left(\frac{9}{4}-\alpha\right)\frac{1}{l^2}\frac{\Delta_-}{r^2+a^2}|u|^2 - C_{M,l,a}|u|^2\right),\end{aligned} \tag{59}$$

since V_{junk} is bounded uniformly. In view of $|\sigma| \leq \frac{C_{M,l,a}}{\ell(\ell+1)}$, we conclude that

$$\widetilde{Q}_0 \geq \widetilde{Q}_b - \frac{C_{M,l,a}}{\ell(\ell+1)}$$

holds for any $1 \geq b > 0$. Hence, by min-max, we infer in particular that (independently of the parameters a, M, ℓ and α)

$$E_n(1) \leq E_n(0) + \delta' \leq E + \delta + \delta', \tag{60}$$

where δ' can be chosen arbitrarily small by choosing ℓ sufficiently large.

For the lower bound, we will establish that $\int dr^\star u(\widetilde{Q}_b(u)u) > 1/l^2$ holds for any $u \in H^2_{\mathrm{AdS}}$ and ℓ large. To prove the latter, note first that the Hardy inequality (40) reduces the problem to showing that

$$V_\sigma + h^2\left(\frac{1}{4}\frac{\Delta_-}{r^2+a^2} + V_{\mathrm{junk}} + V_{\mathrm{mass}}\right) > \frac{1}{l^2}. \tag{61}$$

The square bracket is manifestly positive in a region $[r_L^\star, \pi/2)$ for some large $r_L^\star$ close to $\pi/2$ (depending only on the parameters M, $|a| < l$ and $\alpha < \frac{9}{4}$). We fix this $r_L^\star$ and, in view of $V_\sigma \geq \frac{1}{l^2}$, have established (61) in $[r_L^\star, \pi/2)$. In $[r_{\max}^\star, r_L^\star]$, we have the global estimate

$$V_\sigma \geq \frac{1}{l^2} + \frac{Mr}{(r^2+a^2)^2}. \tag{62}$$

The second term on the right will dominate the term $h^2 \cdot (V_{\mathrm{junk}})$ pointwise in $[r_{\max}^\star, r_L^\star]$, provided h is chosen small depending only on M, l, a, α. □

We summarize our results in the following analogue of Proposition 4.3.

Proposition 4.8. *Let $\alpha < \frac{9}{4}$, $M > 0$, $|a| < l$ be fixed and $E \in (1/l^2, V_{\max})$ be given. Then, there exists an $L > 0$ such that for any $\ell > L$ there exist an $\omega_\ell^2 \in \mathbb{R}^+$ and a smooth solution u_ℓ of the axisymmetric reduced equation*

$$-u_\ell'' + \big(V_\sigma \cdot \mu_\ell(a^2\omega_\ell^2) + V_{\mathrm{junk}} + V_{\mathrm{mass}}\big)u_\ell = \omega_\ell^2 \cdot u_\ell \tag{63}$$

satisfying $u_\ell(r_{\max}^\star) = 0$ and $\int_{r^\star}^{\pi/2} \big(|u_\ell'|^2 + |u_\ell|^2 r^2\big)\, dr^\star < \infty$. Moreover, the ω_ℓ^2 satisfy the uniform estimates

$$\frac{1}{l^2} < \frac{\omega_\ell^2}{\mu_\ell(a^2\omega_\ell^2)} \leq E + \frac{V_{\max} - E}{2} \quad \text{and} \quad c_{M,l,a} \leq \frac{\omega_\ell^2}{\ell(\ell+1)} \leq C_{M,l,a}\,. \tag{64}$$

5. Agmon estimates

In this section, we recall the Agmon estimates. These are (well-known) exponential decay estimates for eigenfunctions for Schrödinger-type operators, in the so-called *forbidden* regions.

5A. *Energy inequalities.*

The Agmon estimates will rely on the following identity.

Lemma 5.1 (energy identity for conjugated operator). *Let $r_1^\star > r_0^\star$. Let $h > 0$ and let W, ϕ be smooth real-valued functions on $[r_0^\star, r_1^\star]$. For all smooth functions u defined on $[r_0^\star, r_1^\star]$, we have the identity*

$$\int_{r_0^\star}^{r_1^\star}\left(\left|\frac{d}{dr^\star}(e^{\frac{\phi}{h}}u)\right|^2+h^{-2}\left(W-\left(\frac{d\phi}{dr^\star}\right)^2\right)e^{2\frac{\phi}{h}}|u|^2\right)dr^\star$$
$$=\int_{r_0^\star}^{r_1^\star}\left(-\frac{d^2\bar{u}}{dr^{\star 2}}+h^{-2}W\bar{u}\right)ue^{2\frac{\phi}{h}}\,dr^\star+\int_{x_0}^{x_1}h^{-1}\frac{d\phi}{dr^\star}e^{2\frac{\phi}{h}}2i\Im\left(\bar{u}\frac{du}{dr^\star}\right)dr^\star$$
$$+\left(e^{2\frac{\phi}{h}}\frac{d\bar{u}}{dr^\star}u\right)(r_1^\star)-\left(e^{2\frac{\phi}{h}}\frac{d\bar{u}}{dr^\star}u\right)(r_0^\star).$$

In particular, if u is real-valued and vanishes at $r_0^\star$ and $r_1^\star$, then

$$\int_{r_0^\star}^{r_1^\star}\left(\left|\frac{d}{dr^\star}(e^{\frac{\phi}{h}}u)\right|^2+h^{-2}\left(W-\left(\frac{d\phi}{dr^\star}\right)^2\right)e^{2\frac{\phi}{h}}|u|^2\right)dr^\star=\int_{r_0^\star}^{r_1^\star}\left(-\frac{d^2u}{dr^{\star 2}}+h^{-2}Wu\right)ue^{2\frac{\phi}{h}}\,dr^\star.$$

The same identity holds if instead of assuming that W is smooth on $[r_0^\star, r_1^\star]$, we assume only that $W|u|^2\in L^1(r_0^\star, r_1^\star)$. By density, we may also replace smoothness of u and ϕ by the conditions that $u\in H_0^1[r_0^\star, r_1^\star]$ and ϕ is a Lipschitz function.

Proof. This follows easily from the computations

$$\int_{r_0^\star}^{r_1^\star}\left(-\frac{d^2}{dr^{\star 2}}+h^{-2}W\right)\overline{(e^{\frac{\phi}{h}}u)}ue^{\frac{\phi}{h}}\,dr^\star$$
$$=\int_{r_0^\star}^{r_1^\star}\left|\frac{d}{dr^\star}e^{\frac{\phi}{h}}u\right|^2+h^{-2}We^{2\frac{\phi}{h}}|u|^2\,dr^\star-\left(\overline{\frac{d}{dr^\star}(e^{\frac{\phi}{h}}u)}ue^{\frac{\phi}{h}}\right)(r_1^\star)+\overline{\frac{d}{dr^\star}(e^{\frac{\phi}{h}}u)}ue^{\frac{\phi}{h}}(r_0^\star)$$

and

$$\int_{r_0^\star}^{r_1^\star}\left(-\frac{d^2}{dr^{\star 2}}\right)\overline{(e^{\frac{\phi}{h}}u)}ue^{\frac{\phi}{h}}\,dr^\star$$
$$=\int_{r_0^\star}^{r_1^\star}-\frac{d}{dr^\star}\left(h^{-1}\frac{d\phi}{dr^\star}\bar{u}e^{\frac{\phi}{h}}+\frac{d\bar{u}}{dr^\star}e^{\frac{\phi}{h}}\right)ue^{\frac{\phi}{h}}\,dr^\star$$
$$=\int_{r_0^\star}^{r_1^\star}\left[-h^{-1}\frac{d\phi}{dr^\star}\frac{d\bar{u}}{dr^\star}ue^{2\frac{\phi}{h}}-\frac{d^2\bar{u}}{dr^{\star 2}}ue^{2\frac{\phi}{h}}\right]dr^\star+\int_{r_0^\star}^{r_1^\star}\left[h^{-1}\frac{d\phi}{dr^\star}\frac{du}{dr^\star}\bar{u}e^{2\frac{\phi}{h}}+h^{-2}\left(\frac{d\phi}{dr^\star}\right)^2\bar{u}ue^{2\frac{\phi}{h}}\right]dr^\star$$
$$-h^{-1}\left(\frac{d\phi}{dr^\star}|u|^2e^{2\frac{\phi}{h}}\right)(r_1^\star)+h^{-1}\left(\frac{d\phi}{dr^\star}|u|^2e^{2\frac{\phi}{h}}\right)(r_0^\star)$$
$$=\int_{r_0^\star}^{r_1^\star}\left(h^{-2}\left(\frac{d\phi}{dr^\star}\right)^2|u|^2e^{2\frac{\phi}{h}}-\frac{d^2\bar{u}}{dr^{\star 2}}ue^{2\frac{\phi}{h}}\right)dr^\star+\int_{r_0^\star}^{r_1^\star}h^{-1}\frac{d\phi}{dr^\star}e^{2\frac{\phi}{h}}2i\Im(\bar{u}u_{r^\star})\,dr^\star$$
$$-h^{-1}\left(\frac{d\phi}{dr^\star}|u|^2e^{2\frac{\phi}{h}}\right)(r_1^\star)+h^{-1}\left(\frac{d\phi}{dr^\star}|u|^2e^{2\frac{\phi}{h}}\right)(r_0^\star).\quad\square$$

5B. ***The Agmon distance.*** We will rely on the Agmon distance to establish our exponential decay estimates.[9] Given any energy level $\mathscr{E}>0$ and a potential $V=V(r^\star)$ (which may also depend on a

[9]The Agmon distance is actually typically used to obtain *optimal* exponential decay estimates; see for instance [Fournais and Helffer 2010]. For the main purpose of this paper (the construction of quasimodes), we could have used smooth cut-off constructions to prove slightly weaker exponential decay estimates. However, the Agmon distance (despite leading only to Lipschitz cut-offs) has a nice interpretation, which is why we choose to use it here.

parameter h), we define the Agmon distance d between $r_1^\star$ and $r_2^\star$ as

$$d = d_{(V-\mathscr{E})_+}(r_1^\star, r_2^\star) = \left| \int_{r_1^\star}^{r_2^\star} \chi_{\{V \geq \mathscr{E}\}}(r^\star)(V(r^\star) - \mathscr{E})^{1/2}\, dr^\star \right|,$$

where $\chi_{\{V\geq\mathscr{E}\}}$ is the characteristic function of the set of $r^\star$ satisfying $V(r^\star) \geq \mathscr{E}$. In other words, d is the distance associated to the Agmon metric $(V - \mathscr{E})_+ \, dr^{\star 2}$, where $f_+ = \max(0, f)$ for any function f.

It is easily checked that d satisfies the triangular inequality and that

$$|\nabla_{r^\star} d(r^\star, r_2^\star)|^2 \leq (V - \mathscr{E})_+(r^\star).$$

The distance to a set can also be defined as usual. In particular, we define

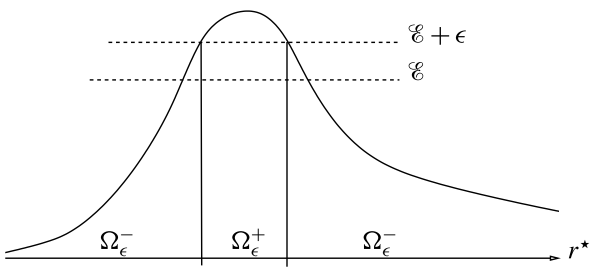
$$d_{\mathscr{E}}(r^\star) := \inf_{r_0^\star \in \{\mathscr{E} \geq V\}} d(r^\star, r_0^\star),$$

which measures the distance to the classical region. We have again

$$|\nabla_{r^\star} d_{\mathscr{E}}(r^\star)|^2 \leq (V - \mathscr{E})_+(r^\star).$$

For a given small $\epsilon \in (0, 1)$ we define the two complementary $r^\star$-regions

$$\Omega_\epsilon^+ = \Omega_\epsilon^+(\mathscr{E}) := \{r^\star \mid V(r^\star) > \mathscr{E} + \epsilon\},$$
$$\Omega_\epsilon^- = \Omega_\epsilon^-(\mathscr{E}) := \{r^\star \mid V(r^\star) \leq \mathscr{E} + \epsilon\}.$$



5C. *The main estimate.* We would like to apply Lemma 5.1 between $r^\star_{\max}$ and $\pi/2$ for u, a solution to the eigenvalue problem (34), and for suitable ϕ.

Lemma 5.2. *Let u be a solution to the eigenvalue problem (34); i.e., $\kappa \cdot u = P(h)u$ for some $\kappa = h^2\omega^2$. Define, for any $\epsilon \in (0, 1)$,*

$$\phi_{\kappa,\epsilon} := (1 - \epsilon) d_\kappa. \tag{65}$$

Then, for all ϵ sufficiently small, u satisfies

$$\int_{r^\star_{\max}}^{\pi/2} h^2 \left| \frac{d}{dr^\star} e^{\phi_{\kappa,\epsilon}/h} u \right|^2 dx \; + \epsilon^2 \int_{\Omega_\epsilon^+} e^{2\phi_{\kappa,\epsilon}/h} |u|^2 \, dr^\star \leq D(\kappa + \epsilon) e^{2a(\epsilon)/h} \|u\|^2_{L^2(r^\star_{\max}, \pi/2)}, \tag{66}$$

where $a(\epsilon) = \sup_{\Omega_\epsilon^-} d_\kappa$ and $D > 0$ is a constant depending only on the parameters a, M, l and α.

Remark 5.3. Note

$$a_\kappa(\epsilon) = \sup_{\Omega_\epsilon^-} d_\kappa \to 0 \quad \text{as } \epsilon \to 0,$$

uniformly in h (and κ) for h sufficiently small. In view of the exponential weight in the second term on the left, the estimate (66) quantifies that u is exponentially small in the forbidden region, provided we can show a uniform lower bound for $\phi_{\kappa,\epsilon}$ in a suitable subset of Ω_ϵ^+. This will be achieved in Lemma 5.4.

Proof. Applying Lemma 5.1 between $r^\star_{\max}$ and $\pi/2$, we get

$$\int_{r^\star_{\max}}^{\pi/2} h^2 \left|\frac{d}{dr^\star} e^{\frac{\phi}{h}} u\right|^2 dr^\star + \int_{\Omega_\epsilon^+} \left(V - \kappa - \left|\frac{d\phi}{dr^\star}\right|^2\right) e^{2\frac{\phi}{h}} |u|^2 \, dr^\star = \int_{\Omega_\epsilon^-} \left(\kappa - V + \left|\frac{d\phi}{dr^\star}\right|^2\right) e^{2\frac{\phi}{h}} |u|^2 \, dr^\star. \quad (67)$$

In view of our choice $\phi = \phi_{\kappa,\epsilon}$, we have in Ω_ϵ^+ the estimate

$$V - \kappa - \left|\frac{d\phi_{\kappa,\epsilon}}{dr^\star}\right|^2 \geq (1 - (1-\epsilon)^2)(V - \kappa) \geq \epsilon^2 \tag{68}$$

for ϵ sufficiently small, which we will use to estimate the left-hand side of (67).

For the right-hand side of (67), we note that if $V \geq 0$ (which occurs if $\alpha \leq 2$), then we immediately obtain

$$\int_{\Omega_\epsilon^-} \left(\kappa - V + \left|\frac{d\phi_{\kappa,\epsilon}}{dr^\star}\right|^2\right) e^{2\phi_{\kappa,\epsilon}/h} |u|^2 \, dr^\star \leq (\kappa + \epsilon) e^{2a(\epsilon)/h} \|u\|^2_{L^2(r^\star_{\max}, \pi/2)}, \tag{69}$$

so that combining (68) and (69) yields (66).

To obtain (66) also in the case $\alpha > 2$ (for which we have $V(r^\star) \to -\infty$ as $r^\star \to \pi/2$), we need once again to appeal to a Hardy-type inequality to absorb the error by the derivative term on the left-hand side of (67). This we do as follows.

Recall that $V = V_\sigma + h^2(V_{\text{junk}} + V_{\text{mass}})$, and the unbounded term is $h^2 V_{\text{mass}} = h^2 \frac{2-\alpha}{l^2} \frac{\Delta_- r^2}{(r^2+a^2)^2} < 0$ for $\alpha > 2$. Note that

$$\frac{\Delta_- r^2}{(r^2 + a^2)^2} = \frac{\Delta_-}{(r^2 + a^2)} - \frac{\Delta_- a^2}{(r^2 + a^2)^2}.$$

The second term is bounded (and in fact will contribute with the right sign if $\alpha \geq 2$) so its contribution can be treated as before. Thus, we only need to estimate

$$\int_{\Omega_\epsilon^-} \frac{\Delta_-}{(r^2 + a^2)} e^{2\phi_{\kappa,\epsilon}/h} |u|^2 \, dr^\star.$$

By [Holzegel and Smulevici 2013a, Lemma 7.2] (see (40)) we have, for any function v in $H^1_0(r^\star_{\max}, \pi/2)$,

$$\int_{R^\star}^{\pi/2} \frac{\Delta_-}{(r^2 + a^2)} |v|^2 \, dr^\star \leq 4l^2 \int_{R^\star}^{\pi/2} \left|\frac{dv}{dr^*}\right|^2 dr^* \quad \text{for any } R^\star \geq r^\star_{\max}. \tag{70}$$

Applying the above Hardy inequality to $v = e^{\phi_{\kappa,\epsilon}/h} u$, we obtain that there exists a uniform constant $C > 0$

such that

$$\int_{r^\star_{\max}}^{\pi/2} h^2\Big(\frac{9}{4}-\alpha\Big)\Big|\frac{d}{dr^\star}e^{\phi_{\kappa,\epsilon}/h}u\Big|^2 dr^\star + \int_{\Omega_\epsilon^+}\Big(V-\kappa-\Big|\frac{d\phi_{\kappa,\epsilon}}{dr^\star}\Big|^2\Big)e^{2\phi_{\kappa,\epsilon}/h}|u|^2\,dr^\star \le C(\kappa+\epsilon)e^{2a(\epsilon)/h}\|u\|^2_{L^2(r^\star_{\max},\pi/2)};$$

i.e., there exists a constant $D>0$ (which degenerates as $\alpha\to\frac{9}{4}$) such that

$$\int_{r^\star_{\max}}^{\pi/2} h^2\Big|\frac{d}{dr^\star}e^{\phi_{\kappa,\epsilon}/h}u\Big|^2 dr^\star + \int_{\Omega_\epsilon^+}\Big(V_\sigma-\kappa-\Big|\frac{d\phi_{\kappa,\epsilon}}{dr^\star}\Big|^2\Big)e^{2\phi_{\kappa,\epsilon}/h}|u|^2\,dr^\star \le D(\kappa+\epsilon)e^{2a(\epsilon)/h}\|u\|^2_{L^2(r^\star_{\max},\pi/2)}. \tag{71}$$

This estimate, when combined with (68), yields again (66) from (67). □

5D. ***Application of the main estimate.*** Before we can exploit (66), we need the following lemma, which quantifies the size of the forbidden region for a given energy level.

Lemma 5.4. *Let $E\in(1/l^2,V_{\max})$ and suppose that $\kappa\in(1/l^2,E+\delta]$ for some $\delta>0$ such that $E+\delta<V_{\max}$. Then there exists $\delta'>0$ and $C>0$, both constants independent of h, such that $V_\sigma-\kappa>2C$, in $[r_{\max},r_{\max}+\delta']$, for all $\kappa\in[E-\delta,\,E+\delta]$.*

Proof. This is a simple consequence of the continuity of V_σ at $r^\star_{\max}$. □

In view of the full potential being $V=V_\sigma+h^2(V_{\text{junk}}+V_{\text{mass}})$ we also obtain:

Corollary 5.5. *For h sufficiently small (depending only on M, l and a) we have $V-\kappa>C$ in $[r_{\max},r_{\max}+\delta']$ for all $\kappa\in[E-\delta,\,E+\delta]$, with both δ' and C depending only on M, l and a.*

With $E\in(1/l^2,V_{\max})$ given, we now fix $\delta'>0$ and $C>0$ as promised by Lemma 5.4. This implies that $\phi_{\kappa,\epsilon}\ge c_{M,l,a}$ in $[r_{\max},r_{\max}+\delta']$ uniformly in ϵ (the constant $c_{M,l,a}$ being of size $C\cdot\delta'$). Next we fix $\epsilon>0$ sufficiently small so that, in particular, $a(\epsilon)\le c_{M,l,a}/2$. We finally conclude from (66) that there exists a $\widetilde{C}>0$ (independent of h) such that

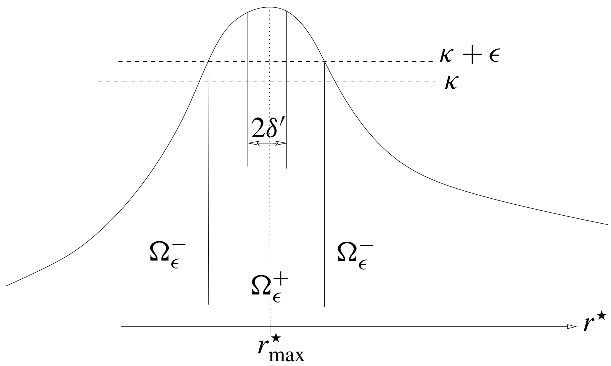
$$\int_{r^\star_{\max}}^{r^\star_{\max}+\delta'} |u|^2\,dr^\star \le \widetilde{C}e^{-\widetilde{C}/h}\|u\|^2_{L^2(r^\star_{\max},\pi/2)}. \tag{72}$$

Turning to the derivative term on the left of (71), we also have

$$\int_{r^\star_{\max}}^{r^\star_{\max}+\delta'} h^2 e^{2\phi_{\kappa,\epsilon}/h}\Big(\frac{1}{h^2}\Big|\frac{d\phi_{\kappa,\epsilon}}{dr^\star}\Big|^2|u|^2+\frac{2}{h}\frac{d\phi_{\kappa,\epsilon}}{dr^\star}u\frac{du}{dr^\star}+\Big|\frac{du}{dr^\star}\Big|^2\Big)dr^\star \lesssim e^{2a(\epsilon)/h}\|u\|^2_{L^2(r^\star_{\max},\pi/2)}.$$

The $|u|^2$ term in the above integral can be ignored since it has the right sign. The cross-term can be absorbed using (72) and $\frac{1}{2}$ of the derivative term. Therefore,

$$\int_{r^\star_{\max}}^{r^\star_{\max}+\delta'} \Big|\frac{du}{dr^\star}\Big|^2 dr^\star \le \widetilde{C}h^{-2}e^{-\widetilde{C}/h}\|u\|^2_{L^2(r^\star_{\max},\pi/2)}.$$



Summarizing these decay estimates, we have proven:

Lemma 5.6. *Let $E \in (1/l^2, V_{\max})$ be fixed and let δ be sufficiently small so that $[E-\delta, E+\delta] \subset (1/l^2, V_{\max})$. Then there exist constants $D, \delta' > 0$, depending only on the parameters M, l, a and α, such that the sequence of eigenfunctions $[u_\ell]_{\ell \geq L}^\infty$ arising from Proposition 4.8 satisfies the estimate*

$$\int_{r^\star_{\max}}^{r^\star_{\max}+\delta'} \left(\left| \frac{du_\ell}{dr^\star} \right|^2 + |u_\ell|^2 \right) dr^\star \leq De^{-D/h} \|u_\ell\|^2_{L^2(r^\star_{\max}, \pi/2)},$$

where $h = (\mu_\ell(a^2\omega_\ell^2))^{-1/2}$ and ω_ℓ are as in Proposition 4.8.

We remark that by reusing once again the equation, we can obtain such exponential decay estimates on all higher-order derivatives, with the constants in the above lemma depending on the order of commutation.

6. The construction of quasimodes

By now we have established the existence (Proposition 4.8) of a sequence of functions $[u_\ell]$ such that for each ℓ the corresponding u_ℓ solves

$$\omega_\ell^2 h^2 u_\ell = P(h) u_\ell,$$

where $h = (\mu_\ell(a^2\omega_\ell^2))^{-1/2} \to 0$ as $\ell \to \infty$, and such that these u_ℓ obey the estimate of Lemma 5.6 with some constants $D, \delta' > 0$ independent of h (or equivalently ℓ).

Now let χ be a smooth function such that $\chi = 1$ on $[r^\star_{\max} + \delta', \pi/2]$ and $\chi = 0$ on $(-\infty, r^\star_{\max}]$. We then define $\psi_\ell(t, r, \theta, \tilde{\phi})$ as

$$\psi_\ell(t, r, \theta, \tilde{\phi}) = e^{i\omega_\ell t} \chi(r^\star(r))(r^2 + a^2)^{-1/2} u_\ell(r^\star(r)) S_{\ell 0}(\theta). \tag{73}$$

Remark 6.1. As defined above, the ψ_ℓ are complex functions but, of course, we could have worked below with $\Re(\psi_\ell)$ or $\Im(\psi_\ell)$.

We now show that the ψ_ℓ satisfy the Klein–Gordon equation up to an exponentially small error.

Lemma 6.2. *For each ℓ and each $k \geq 0$, $\psi_\ell \in CH^k_{\mathrm{AdS}}$. Moreover, there exists $L > 0$ such that we have the following estimates. For all $k \geq 0$, there exists a $C_k > 0$ such that for all $\ell \geq L$ and all $t \geq t_0$,*

$$\left\| \Box_g \psi_\ell + \frac{\alpha}{l^2} \psi_\ell \right\|_{H^k_{Ads}(\Sigma_t)} \leq C_k e^{-C_k \ell} \|\psi_\ell\|_{H^0_{\mathrm{AdS}}(\Sigma_{t_0})}.$$

Moreover, the error $\left|\Box_g\psi_\ell+(\alpha/l^2)\psi_\ell\right|$ *is supported on* $[r_{\max}, r_{\max+\delta'}]$, *for some* $\delta'>0$ *independent of* ℓ. *Finally, all the* H^k_{AdS}*-norms of* ψ_ℓ *and of its time derivatives on each* Σ_t *are constant in* t.

Proof. By standard elliptic estimates, any u_ℓ is smooth on $(r^\star_{\max}, \pi/2)$. Thus, as far as the regularity of ψ_ℓ is concerned, it is sufficient to check that ψ_ℓ and its derivative decay sufficiently fast near $r=\infty$, which is easy and therefore omitted.

Moreover, in view of our construction, we have $(\Box_g+\alpha/l^2)\psi_\ell=0$ in $[r_+, r_{\max}]\cup[r(r^\star_{\max}+\delta'), \infty]$. Hence, the error is supported in a bounded strip in which we have the following naïve estimate: For all (t, r, θ, ϕ) with $r\in[r_{\max}, r_{\max}+\delta']$,

$$\left|\Box_g\psi_\ell+\frac{\alpha}{l^2}\psi_\ell\right|\lesssim\left(\omega^2|u_\ell|+|u''_\ell|+|u'_\ell|+h^{-2}|u_\ell|\right)S_{\ell 0}(\theta),$$

which gives the required estimate for $k=0$ after integration, using the Agmon estimates of the previous section and the equation satisfied by u_ℓ in order to estimate u''_ℓ. For higher k, it suffices to commute the equation and to use the equation for u_ℓ every time two radial derivatives occur, or the equation for $S_{\ell 0}$ every time angular derivatives occur. $\Box$

Note that we finally proved Theorem 1.5. Indeed, the ψ_ℓ are of the form claimed in (1) by construction of (73). The estimate on the ω_ℓ in (2) was obtained as part of Proposition 4.8. The error estimate (3) is the statement of Lemma 6.2, while the localization properties (4) and (5) are obvious from (73) itself.

7. Proof of Corollary 1.6

In this section, we prove Corollary 1.6. Given the quasimodes, the proof is standard, but we include it for the paper to be self-contained.

Let us therefore fix a Kerr–AdS spacetime such that the assumption of Corollary 1.6 is satisfied, and also a Klein–Gordon mass $\alpha<\frac{9}{4}$. For convenience, we set $t^\star_0=0$. Recall also that $t^\star=t$ in $r\geq r_{\max}$.

We shall consider solutions ψ to homogeneous and inhomogeneous Klein–Gordon equations with initial data $\psi|_{\Sigma_t}$ and $\partial_t\psi|_{\Sigma_t}$ given on slices of constant t. We shall avoid completely issues regarding the facts that ∂_t is not always timelike and that the coordinate t breaks down at the horizon by considering only axisymmetric data which is compactly supported away from the horizon.

Thus, given any $t, s\in\mathbb{R}$ and given any smooth, axisymmetric initial data set $w=(\bar\psi, \overline{T\psi})$ whose support is bounded away from the horizon and which decays sufficiently fast near infinity, we will denote by $P(t, s)w$ the unique solution at time t of the homogenous problem

$$\left(\Box_g+\frac{\alpha}{l^2}\right)\psi=0,\quad \psi|_{\Sigma_s}=\bar\psi,\quad \left.\frac{\partial\psi}{\partial t}\right|_{\Sigma_s}=\overline{T\psi}.$$

Given a smooth axisymmetric function F defined on $\mathcal{R}$, compactly supported in r away from the horizon and infinity, we can consider the inhomogeneous problem

$$\left(\Box_g+\frac{\alpha}{l^2}\right)\psi=F,\quad \psi|_{\Sigma_0}=\bar\psi,\quad \left.\frac{\partial\psi}{\partial t}\right|_{\Sigma_0}=\overline{T\psi}.$$

For regular data as above, this problem is well-posed in CH^2_{AdS} and we shall denote its solution by $\psi_F(t)$, suppressing the dependence on r and the angular variables. If the data is axisymmetric, then ψ_F will be axisymmetric and, writing $v(s) = (0, F(s)(g^{tt})^{-1})$, $\psi_F(t)$ is given by the Duhamel formula

$$\psi_F(t) = P(t,0)w + \int_0^t P(t,s)v(s)\,ds.$$

We now consider the family of ψ_ℓ given by Theorem 1.5. For each ℓ, ψ_ℓ provides an initial data set

$$w_\ell = \left(\psi_\ell(t=0),\, \frac{\partial \psi_\ell}{\partial t}(t=0)\right)$$

for (1) on the slice $t = 0$. Moreover, each ψ_ℓ satisfies the inhomogeneous Klein–Gordon equation

$$\left(\Box_g + \frac{\alpha}{l^2}\right)\psi_\ell = F_\ell,$$

for some F_ℓ satisfying $\|F_\ell\|_{H^{k,-2}_{\mathrm{AdS}}} \le C_k e^{-C_k \ell}\|\psi_\ell\|_{H^0_{\mathrm{AdS}}}$.

Note that since, from Lemma 6.2, the error F_ℓ is supported away from $r = +\infty$ independently of ℓ, we can safely ignore all powers of r in the following estimates.

Let $\tilde{\psi}_\ell$ denote the solution of the *homogeneous* problem associated with the *same* initial data w_ℓ, i.e., $\tilde{\psi}_\ell = P(t,0)w_\ell$. From Duhamel's formula, we then get

$$\begin{aligned}
\|\psi_\ell - \tilde{\psi}_\ell\|_{H^1_{\mathrm{AdS}}(\Sigma_t \cap \{r \ge r_{\max}\})} &\le t \sup_{s\in[0,t]} \|P(t,s)(0, F_\ell)(s)\|_{H^1_{\mathrm{AdS}}(\Sigma_t\cap\{r\ge r_{\max}\})} \\
&\le tC\|F_\ell\|_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_0)} \\
&\le tCe^{-C\ell}\|\psi_\ell\|_{H^0_{\mathrm{AdS}}(\Sigma_0\cap\{r\ge r_{\max}\})} \\
&\le tCe^{-C\ell}\|\psi_\ell\|_{H^1_{\mathrm{AdS}}(\Sigma_0\cap\{r\ge r_{\max}\})},
\end{aligned} \tag{74}$$

where we use the boundedness statement of Theorem 1.1 to bound $\|P(t,s)(0,F_\ell)(s)\|_{H^0_{\mathrm{AdS}}(\Sigma_t\cap\{r\ge r_{\max}\})}$ in terms of the data, as well as Lemma 6.2. In particular, since the norms of ψ_ℓ are time-invariant, for any $t \le e^{C\ell}/2C$, the reverse triangle inequality and (74) yield

$$\begin{aligned}
\left(\int_{\Sigma_t\cap\{r\ge r_{\max}\}} e_1[\tilde{\psi}_\ell] r^2\,dr\,\sin\theta\,d\theta\,d\phi\right)^{\frac12} &\ge \|\tilde{\psi}_\ell\|_{H^1_{\mathrm{AdS}}(\Sigma_t\cap\{r\ge r_{\max}\})} \\
&\ge \frac{1}{2}\|\tilde{\psi}_\ell\|_{H^1_{\mathrm{AdS}}(\Sigma_0\cap\{r\ge r_{\max}\})} \\
&\ge \frac{c}{2\ell}\left[\|\Omega_i\tilde{\psi}_\ell\|_{H^1_{\mathrm{AdS}}(\Sigma_0\cap\{r\ge r_{\max}\})} + \|\partial_{t^\star}\tilde{\psi}_\ell\|_{H^1_{\mathrm{AdS}}(\Sigma_0\cap\{r\ge r_{\max}\})}\right] \\
&\ge \frac{c}{2\ell}\left(\int_{\Sigma_0} e_2[\tilde{\psi}_\ell] r^2\,dr\,\sin\theta\,d\theta\,d\phi\right)^{\frac12}.
\end{aligned} \tag{75}$$

Here we have used — in the step from the second to the third line — that the data for $\tilde{\psi}_\ell$ is frequency-localized, which allows us to exchange angular and time derivatives with powers of ℓ using the second item of Theorem 1.5, and radial derivatives by angular and time derivatives using the wave equation the

$\tilde{\psi}_\ell$ satisfy. From the third to the fourth line we exploited the fact that the data is localized in $r \geq r_{\max}$. The constant c depends only on the parameters M, l, a and α.

Finally, setting $t_\ell = e^{C\ell}/2C$, we obtain from (75) a family (t_ℓ, ψ_ℓ) such that, for any sufficiently large ℓ, $Q[\tilde{\psi}_\ell](t_\ell) > C > 0$ holds, which proves the corollary.

Appendix: The improved boundedness statement

The boundedness statement at the H^2-level proven in [Holzegel 2010; 2012] is the estimate (3) for the $\tilde{e}_2[\psi]$-based energies (see Section 2B). It is remarked in the second of these references that stronger norms can be shown to be uniformly bounded using commutation by angular momentum operators leading to the statement (3). Since the latter statement has been used in this paper and also in [Holzegel and Smulevici 2013a], we provide here a sketch of the proof of this well-known (but absent from the literature) argument. We define the energies

$$E_1[\psi](t^\star) = \int_{\Sigma_{t^*}} e_1[\psi](t^\star)\, r^2 \sin\theta \, dr \, d\theta \, d\phi \tag{76}$$

and, with the obvious replacement, $E_2[\psi](t^\star)$ and $\widetilde{E}_2[\psi](t^\star)$. Recall that uniform boundedness for the $\widetilde{E}_2[\psi]$ energy is derived by, in addition to using known techniques near the horizon (cf. the red-shift vector field), commuting the Klein–Gordon equation with ∂_t (which yields (2) with ψ replaced by $\partial_t\psi$) followed by elliptic estimates on spacelike slices, which control the H^2_{AdS} norm.

Let us sketch how to prove boundedness (3) for the $E_2[\psi]$ energy. If we commute the Klein–Gordon equation with angular momentum operators we obtain

$$\Box_g(\Omega_i\psi) + \frac{\alpha}{l^2}(\Omega_i\psi) = 2^{(\Omega_i)}\pi^{\mu\nu}\cdot\nabla_\mu\nabla_\nu\psi + \bigl(2\nabla^\alpha\bigl({}^{(\Omega_i)}\pi_{\alpha\mu}\bigr) - \nabla_\mu\bigl({}^{(\Omega_i)}\pi_\alpha^{\alpha}\bigr)\bigr)\nabla^\mu\psi, \tag{77}$$

with ${}^{(\Omega_i)}\pi$ the (nonvanishing in Kerr!) deformation tensor of Ω_i. The right-hand side decays suitably in r but not in t. More precisely, in view of the fact that there is no integrated decay estimate available, we cannot close the basic energy estimate on its own. Let us instead commute with localized angular momentum operators $\widetilde{\Omega}_i = \chi(r)\Omega_i$, where $\chi(r)$ is equal to 1 for $r \geq 2R$ and equal to 0 for $r \leq R$. Applying the energy estimate for the vector field $\partial_{t^\star}$, we can derive

$$\begin{aligned}&\|\widetilde{\Omega}_i\psi_t\|_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_{t^\star}\cap\{r\geq 2R\})} + \|\widetilde{\Omega}_i\psi\|_{H^1_{\mathrm{AdS}}(\Sigma_{t^\star}\cap\{r\geq 2R\})}\\ &\quad\leq C_{M,l,a,\alpha}\bigl(\|\widetilde{\Omega}_i\psi_t\|_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_{t_0^\star})} + \|\widetilde{\Omega}_i\psi\|_{H^1_{\mathrm{AdS}}(\Sigma_{t_0^\star})}\bigr) + (\tau_2-\tau_1)\bigl(\epsilon \sup_{\tau\in(\tau_1,\tau_2)} E_2[\psi](\tau) + \epsilon\cdot\widetilde{E}_2[\psi](0)\bigr),\end{aligned} \tag{78}$$

where ϵ can be made small by choosing R large. The last term arises from the spacetime error term which decays strongly in r.

The idea is to combine this with an integrated decay estimate for the $\widetilde{\Omega}_i\psi$ which loses linearly in τ. Recall that if $\Box\Psi + (\alpha/l^2)\Psi = f$, then we have the identity

$$\nabla_a\Psi\nabla^a\psi - \frac{\alpha}{l^2}\Psi^2 = \nabla^\mu(\Psi\nabla_\mu\Psi) - g^{t^\star t^\star}\nabla_{t^\star}\Psi\nabla_{t^\star}\Psi - 2g^{t^\star b}\nabla_{t^\star}\Psi\nabla_b\Psi - f\Psi, \tag{79}$$

where a, b run over r, θ, ϕ only. When integrating this identity (with Ψ replaced by $\widetilde{\Omega}_i\psi$ and f the error arising from the commutation in (77)) with the usual spacetime volume we observe that:

- The left-hand side is nonnegative and controls all spatial derivatives after applying the standard Hardy inequality (40).
- The second and third terms on the right are essentially controlled by the $\widetilde{E}_2[\psi]$ energy times the length of the time interval $(\tau_2 - \tau_1)$:

$$\begin{aligned} g^{t^\star t^\star}\nabla_{t^\star}(\widetilde{\Omega}_i\psi)\nabla_{t^\star}(\widetilde{\Omega}_i\psi) &\sim \frac{1}{r^2}r^2|\partial_{t^\star}\nabla\!\!\!/\psi|^2_{g\!\!\!/} = |\partial_{t^\star}\nabla\!\!\!/\psi|^2_{g\!\!\!/} \le e_1[\partial_{t^\star}\psi],\\ g^{t^\star r}\nabla_{t^\star}(\widetilde{\Omega}_i\psi)\nabla_{r}(\widetilde{\Omega}_i\psi) &\sim \frac{1}{r^3}r^2(|\partial_{t^\star}\nabla\!\!\!/\psi|^2_{g\!\!\!/} + |\partial_r\nabla\!\!\!/\psi|^2_{g\!\!\!/}) \le e_1[\partial_{t^\star}\psi],\\ g^{t^\star \phi}\nabla_{t^\star}(\widetilde{\Omega}_i\psi)\nabla_{\phi}(\widetilde{\Omega}_i\psi) &\sim \frac{1}{r^2}r^2\Big(\frac{1}{\epsilon}|\partial_{t^\star}\nabla\!\!\!/\psi|^2_{g\!\!\!/} + \epsilon|\partial_\phi\nabla\!\!\!/\psi|^2_{g\!\!\!/}\Big) \le C_\epsilon\cdot e_1[\partial_{t^\star}\psi] + \epsilon\cdot e_2[\psi], \end{aligned} \tag{80}$$

  and $g^{t^\star\theta} = 0$. (We need to borrow an ϵ of $e_2[\psi]$ because the $t^\star\phi$ coordinates are not optimal near infinity.) The cross-term $g^{t^\star\phi}$ would have much stronger decay in coordinates adapted to the asymptotically AdS end, which would allow us to estimate all terms by the weaker energy $\widetilde{E}_2[\psi]$.
- The first term on the right-hand side is a boundary term, which can be estimated by

$$\left|\int_{\mathcal{D}(\tau_1,\tau_2)} \nabla^\mu(\widetilde{\Omega}_i\psi\nabla_\mu(\widetilde{\Omega}_i\psi))\right| \le \sup_{t^\star}\int_{\Sigma_{t^*}} \tilde{e}_2[\psi]r^2\sin\theta\, dr\, d\theta\, d\phi. \tag{81}$$

- The last term in (79) is controlled as previously by the last line in the energy estimate (78).

It follows that integrating (79) furnishes the estimate

$$\begin{aligned} \int_{\mathcal{D}(\tau_1,\tau_2)\cap\{r\ge 2R\}} & r^2\sin\theta\, dt^\star\, dr\, d\theta\, d\phi\, \big(r^2|\partial_r\Omega_i\psi|^2 + |\nabla\!\!\!/\Omega_i\psi|^2\big)\\ &\le \max(1, \tau_2-\tau_1)\Big(\epsilon \sup_{\tau\in(\tau_1,\tau_2)} E_2[\psi](\tau) + C\cdot \widetilde{E}_2[\psi](0)\Big). \end{aligned} \tag{82}$$

Now note that

$$\widetilde{E}_2[\psi](t^\star) + \|\Omega_i\psi_t\|^2_{H^{0,-2}_{\mathrm{AdS}}(\Sigma_{t^\star}\cap\{r\le 2R\})} + \|\Omega_i\psi\|^2_{H^1_{\mathrm{AdS}}(\Sigma_{t^\star}\cap\{r\le 2R\})} \le C_R\cdot\widetilde{E}_2[\psi](t^\star) \le C_R\cdot\widetilde{E}_2[\psi](0) \tag{83}$$

follows right from the boundedness statement for the $\widetilde{E}_2[\psi]$ energy and estimating the weights away from infinity. We can integrate (83) in time and add it to (82) which yields (first without the boxed terms)

$$\begin{aligned} \boxed{E_2[\psi](\tau_2)} + \int_{\tau_1}^{\tau_2} & E_2[\psi](\tau)\, d\tau\\ &\le \boxed{C_{M,l,a,\alpha}\cdot E_2[\psi](\tau_1)} + \max(1, \tau_2-\tau_1)\Big(\epsilon \sup_{\tau\in(\tau_1,\tau_2)} E_2[\psi](\tau) + C_\epsilon\cdot\widetilde{E}_2[\psi](0)\Big). \end{aligned} \tag{84}$$

The estimate also holds with the boxed terms included, as follows from adding (78) and (83). We claim that (84) implies $E_2[\psi](t^\star) \lesssim E_2[\psi](0)$ provided ϵ is sufficiently small depending only on the parameters (the constant $C_{M,l,a,\alpha}$), and leave the verification to the reader.

Remark. An easier proof is available if one is willing to go to H^3_{AdS}. The Carter operator

$$Q\psi = \Delta_{\mathbb{S}^2}\psi - \partial_\phi^2\psi + (a^2\sin^2\theta)\partial_t^2\psi \tag{85}$$

commutes with the wave operator. Since ∂_ϕ^2 and ∂_t^2 trivially commute, we have

$$\begin{aligned} E_1[\Delta_{\mathbb{S}^2}\psi](t^\star) &\lesssim E_1[Q\psi](t^\star) + E_1[\partial_\phi^2\psi](t^\star) + E_1[\partial_t^2\psi](t^\star) \\ &\lesssim E_1[Q\psi](0) + E_1[\partial_\phi^2\psi](0) + E_1[\partial_t^2\psi](0), \end{aligned} \tag{86}$$

and we can control all derivatives on S^2 from controlling the Laplacian via elliptic estimates. This yields the desired gain, albeit at the level of three derivatives. This is analogous to commuting with angular momentum operators twice.

Acknowledgments

Theorem 1.5 and Corollary 1.6 were announced for the Schwarzschild case at the workshop "Mathematical Aspects of General Relativity" in Oberwolfach, Germany, in August 2012 [Holzegel and Smulevici 2012b]. We thank Matthieu Léautaud for showing us the Agmon estimates and Maciej Zworski for pointing out the work of Gannot [2014]. Holzegel acknowledges support through NSF grant DMS-1161607 and thanks the Department of Mathematics at Orsay for its hospitality.

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Received 3 Jun 2013. Revised 24 Jan 2014. Accepted 20 Apr 2014.

GUSTAV HOLZEGEL: g.holzegel@imperial.ac.uk
Department of Mathematics, Imperial College London, South Kensington Campus, London SW7 2AZ, United Kingdom

JACQUES SMULEVICI: jacques.smulevici@math.u-psud.fr
Laboratoire de Mathématiques, Université Paris Sud 11, bât. 425, 91405 Orsay, France

ANALYSIS AND PDE
Vol. 7, No. 5, 2014
dx.doi.org/10.2140/apde.2014.7.1091

msp

CYLINDRICAL ESTIMATES FOR HYPERSURFACES MOVING BY CONVEX CURVATURE FUNCTIONS

BEN ANDREWS AND MAT LANGFORD

We prove a complete family of *cylindrical estimates* for solutions of a class of fully nonlinear curvature flows, generalising the cylindrical estimate of Huisken and Sinestrari [*Invent. Math.* **175**:1 (2009), 1–14, §5] for the mean curvature flow. More precisely, we show, for the class of flows considered, that, at points where the curvature is becoming large, an $(m+1)$-convex $(0 \le m \le n-2)$ solution either becomes strictly m-convex or its Weingarten map becomes that of a cylinder $\mathbb{R}^m \times S^{n-m}$. This result complements the convexity estimate we proved with McCoy [*Anal. PDE* **7**:2 (2014), 407–433] for the same class of flows.

1. Introduction

Let M be a smooth, closed manifold of dimension n, and $X_0 : M \to \mathbb{R}^{n+1}$ a smooth hypersurface immersion. We are interested in smooth families $X : M \times [0, T) \to \mathbb{R}^{n+1}$ of smooth immersions $X(\,\cdot\,, t)$ solving the initial value problem

$$\begin{cases} \partial_t X(x,t) = -F(\mathcal{W}(x,t))\nu(x,t), \\ X(\,\cdot\,,0) = X_0, \end{cases} \tag{CF}$$

where ν is the outer normal field of the evolving hypersurface X and $\mathcal{W}$ the corresponding Weingarten curvature. In order that the problem (CF) be well-posed, we require that $F(\mathcal{W})$ be given by a smooth, symmetric function $f : \Gamma \to \mathbb{R}$ of the principal curvatures κ_i which is monotone increasing in each argument. The symmetry of f ensures that F is a smooth, basis-invariant function of the components of the Weingarten map (or an orthonormal frame-invariant function of the components of the second fundamental form) [Glaeser 1963]. Monotonicity ensures that the flow is (weakly) parabolic. This guarantees local existence of solutions of (CF), as long as the principal curvature n-tuple of the initial data lies in Γ; see [Langford 2014].

For technical reasons, we require some additional conditions:

Conditions. (i) f is homogeneous of degree one.

(ii) f is convex.

Since the normal points out of the region enclosed by the solution, we may assume, by condition (ii), that $(1, \dots, 1) \in \Gamma$. Thus, by condition (i), we may further assume that f is normalised such that $f(1, \dots, 1) = 1$.

Research partially supported by Discovery grant DP120100097 of the Australian Research Council. Langford gratefully acknowledges the support of an Australian Postgraduate Award during the completion of this work.
MSC2010: 53C44, 35K55, 58J35.
Keywords: curvature flows, cylindrical estimates, fully nonlinear, convexity estimates.

The additional conditions (i)–(ii) have several consequences. Most importantly, they allow us to obtain a preserved cone $\Gamma_0 \subset \Gamma$ of curvatures for the flow (Lemma 2.2). This allows us to obtain uniform estimates on any degree-zero homogeneous function of curvature along the flow (Lemma 2.3); in particular, we deduce a uniform parabolicity condition (Corollary 2.4). The convexity condition then allows us to apply the second derivative Hölder estimate of [Evans 1982; Krylov 1982] to deduce that the solution exists on a maximal time interval $[0, T)$, $T < \infty$, such that $\max_{M\times\{t\}} F \to \infty$ as $t \to T$; see [Andrews et al. 2014a, Proposition 2.6]. Thus, it is of interest to study the behaviour of solutions as $F \to \infty$. Let us recall the following curvature estimate [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]).

Theorem 1.1 (convexity estimate). *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon < \infty$ such that*

$$G(x,t) \le \varepsilon F(x,t) + C_\varepsilon \quad \text{for all } (x,t) \in M \times [0, T),$$

where G is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures of the evolving hypersurface that vanishes at a point (x,t) if and only if $\mathcal{W}_{(x,t)} \ge 0$.

We remark that the constant C_ε depends only on ε, the dimension n, the choice of speed function f, the preserved curvature cone Γ_0, and bounds for the initial volume and diameter [Langford 2014].

Theorem 1.1 implies that the ratio of the smallest principal curvature to the speed is almost positive wherever the curvature is large. Combining it with the differential Harnack inequality of [Andrews 1994b] and the strong maximum principle [Hamilton 1986] yields useful information about the geometry of solutions of (CF) near singularities [Andrews et al. 2014b] (cf. [Huisken and Sinestrari 1999a; 1999b]):

Corollary 1.2. *Any blow-up limit of a solution of (CF) is weakly convex. In particular, any type-II blow-up limit about a type-II singularity is an eternal solution of the form $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \to \mathbb{R}^{n+1}$, $k \in \{0, 1, \dots, n-1\}$, such that $X_\infty|_{\mathbb{R}^k}$ is flat, and $X_\infty|_{\Gamma^{n-k}}$ is a strictly convex translation solution of the corresponding flow in $\mathbb{R}^{n-k+1}$.*

Motivated by the surgery construction of [Huisken and Sinestrari 2009, §5] for 2-convex mean curvature flow, we will apply Theorem 1.1 to obtain the following family of cylindrical estimates for solutions of (CF):

Theorem 1.3 (cylindrical estimate). *Let X be a solution of (CF) such that conditions (i)–(ii) hold. Suppose also that X is uniformly $(m+1)$-convex for some $m \in \{0, 1, \dots, n-2\}$. That is, $\kappa_1 + \cdots + \kappa_{m+1} \ge \beta F$ for some $\beta > 0$. Then, for all $\varepsilon > 0$, there is a constant $C_\varepsilon > 0$ such that*

$$G_m(x,t) \le \varepsilon F(x,t) + C_\varepsilon \quad \text{for all } (x,t) \in M \times [0, T),$$

where $G_m : M \times [0, T) \to \mathbb{R}$ is given by a smooth, nonnegative, degree-one homogeneous function of the principal curvatures that vanishes at a point (x,t) if and only if

$$\kappa_1(x,t) + \cdots + \kappa_{m+1}(x,t) \ge \frac{1}{c_m} f(\kappa_1(x,t), \dots, \kappa_n(x,t)),$$

where c_m is the value F takes on the unit radius cylinder $\mathbb{R}^m \times S^{n-m}$.

We note that the constant C_ε will only depend on ε, β, m, the dimension n, the choice of speed function f, the preserved curvature cone Γ_0, and upper bounds for the initial volume and diameter. Theorem 1.3 implies that the ratio of the quantity

$$K_m := \kappa_1 + \cdots + \kappa_{m+1} - \frac{1}{c_m} F$$

to the speed is almost positive wherever the curvature is large. Observe that this quantity is nonnegative on a weakly convex hypersurface Σ only if either Σ is strictly m-convex or $\Sigma = \mathbb{R}^m \times S^{n-m}$. In particular, we find that, whenever $\kappa_1(x,t) + \cdots + \kappa_m(x,t)$ is small compared to the speed, the Weingarten curvature is close to that of a thin, round cylinder $\mathbb{R}^m \times S^{n-m}$. We therefore obtain a refinement of Corollary 1.2:

Corollary 1.4. *Any blow-up limit of an* $(m+1)$*-convex*, $0 \le m \le n-2$, *solution of* (CF) *is either strictly* m*-convex, or a shrinking cylinder* $\mathbb{R}^m \times S^{n-m}$. *In particular, if the blow-up is of type-II, then this limit is of the form* $X_\infty : (\mathbb{R}^k \times \Gamma^{n-k}) \times \mathbb{R} \to \mathbb{R}^{n+1}$ *for* $k \in \{0, 1, \dots, m-1\}$, *such that* $X_\infty|_{\mathbb{R}^k}$ *is flat and* $X_\infty|_{\Gamma^{n-k}}$ *is a strictly convex translation solution of the corresponding flow in* $\mathbb{R}^{n-k+1}$.

The $m = 0$ case of the cylindrical estimates demonstrates that convex hypersurfaces become umbilic at points where the curvature is blowing up, generalising a result of Huisken [1984, Theorem 5.1] for the mean curvature flow (we note that the convergence result of [Huisken 1984] has been obtained by the first author for the class of flows considered here without the need for such an estimate [Andrews 1994a]). Moreover, Huisken and Sinestrari [2009] have recently obtained the $m = 1$ case of the cylindrical estimates for the mean curvature flow, making spectacular use of it through their surgery program, which yields a classification of 2-convex hypersurfaces. The convexity and cylindrical estimates stated above, in addition to generalising the Huisken–Sinestrari cylindrical estimate to all m in $\{0, \dots, n-2\}$, constitute a first step towards improving upon such results by allowing a larger class of evolution equations.

2. Preliminaries

We will follow the notation used in [Andrews et al. 2014b]. In particular, we recall that a smooth, symmetric function g of the principal curvatures gives rise to a smooth function G of the components h_i^j of the Weingarten map. Equivalently, G is an orthonormal frame invariant function of the components h_{ij} of the second fundamental form. To simplify notation, we denote $G(x,t) \equiv G(\mathcal{W}(x,t)) = g(\kappa(x,t))$ and use dots to denote derivatives of functions of curvature as follows:

$$\dot{g}^k(z) v_k = \frac{d}{ds}\Big|_{s=0} g(z+sv), \qquad \dot{G}^{kl}(A) B_{kl} = \frac{d}{ds}\Big|_{s=0} G(A+sB),$$
$$\ddot{g}^{pq}(z) v_p v_q = \frac{d^2}{ds^2}\Big|_{s=0} g(z+sv), \quad \ddot{G}^{pq,rs}(A) B_{pq} B_{rs} = \frac{d^2}{ds^2}\Big|_{s=0} G(A+sB).$$

The derivatives of g and G are related in the following way:

Lemma 2.1 [Gerhardt 1996; Andrews 1994a; 2007]. *Let* $g : \Gamma \to \mathbb{R}$ *be a smooth, symmetric function. Define the function* $G : \mathcal{S}_\Gamma :\to \mathbb{R}$ *by* $G(A) := g(\lambda(A))$, *where* $\lambda(A)$ *denotes the eigenvalues of* A *(up to order) and* $\mathcal{S}_\Gamma$ *denotes the set of symmetric matrices with eigenvalues in* Γ. *Then, for any diagonal* $A \in \mathcal{S}_\Gamma$,

$$\dot{G}^{kl}(A) = \dot{g}^k(\lambda(A))\delta^{kl}, \tag{2-1}$$

and, for any diagonal $A \in \mathcal{S}_\Gamma$ with distinct eigenvalues and any symmetric $B \in \mathrm{GL}(n)$,

$$\ddot{G}^{pq,rs}(A)B_{pq}B_{rs} = \ddot{g}^{pq}(\lambda(A))B_{pp}B_{qq} + 2\sum_{p>q} \frac{\dot{g}^p(\lambda(A)) - \dot{g}^q(\lambda(A))}{\lambda_p(A) - \lambda_q(A)}(B_{pq})^2. \tag{2-2}$$

We note that $\ddot{g} \geq 0$ if and only if $(\dot{g}^p - \dot{g}^q)(z_p - z_q) \geq 0$ for all p, q [Andrews et al. 2014b, Lemma 2.2], so Lemma 2.1 implies that G is convex if and only if g is convex.

The following useful lemma was proved in [Andrews et al. 2014b]:

Lemma 2.2. *Let $f : \Gamma \to \mathbb{R}$ be a flow speed for (CF) satisfying Conditions (i)–(ii). Then, for any admissible initial datum $X_0 : M \to \mathbb{R}^{n+1}$ there exists a cone $\Gamma_0 \subset \mathbb{R}^n$ satisfying $\overline{\Gamma}_0 \setminus \{0\} \subset \Gamma$ such that the principal curvatures of the solution $X : M \times [0, T) \to \mathbb{R}^{n+1}$ of the initial value problem (CF) satisfy $\kappa(x,t) := (\kappa_1(x,t), \dots, \kappa_n(x,t)) \in \Gamma_0$ for all $(x,t) \in M \times [0,T)$.*

We refer to such a cone Γ_0 as a *preserved cone* for the solution X. As mentioned in the introduction, the existence of a preserved cone allows us to obtain bounds for homogeneous functions of the curvature:

Lemma 2.3. *Let $X : M \times [0,T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Let $g : \Gamma \to \mathbb{R}$ be a smooth, degree-zero homogeneous symmetric function. Then there exists $c > 0$ (depending only on n, f and M_0) such that*

$$-c \leq g(\kappa_1(x,t), \dots, \kappa_n(x,t)) \leq c \quad \textit{for all } (x,t) \in M \times [0,T).$$

If $g > 0$, then there exists $c > 0$ such that

$$\frac{1}{c} \leq g(\kappa_1(x,t), \dots, \kappa_n(x,t)) \leq c.$$

Proof. Let Γ_0 be a preserved cone for the solution X. Then $K := \overline{\Gamma}_0 \cap S^n$ is compact. Since g is continuous, the required bounds hold on K. But these extend to $\overline{\Gamma}_0 \setminus \{0\}$ by homogeneity. The claim follows since $\kappa(x,t) \in \overline{\Gamma}_0 \setminus \{0\}$ for all $(x,t) \in M \times [0,T)$. □

By condition (i), the derivative $\dot{f}$ of f is homogeneous of degree zero. Since $\dot{f}^k > 0$ for each k, we obtain uniform parabolicity of the flow:

Corollary 2.4. *There exists a constant $c > 0$ (depending only on n, f and M_0) such that, for any $v \in T^*M$, it holds that*

$$\frac{1}{c}|v|^2 \leq \dot{F}^{ij} v_i v_j \leq c|v|^2,$$

where $|\cdot|$ is the (time-dependent) norm on M corresponding to the (time-dependent) metric induced by the flow.

We now recall the following evolution equation (see for example [Andrews et al. 2013]).

Lemma 2.5. *Let $X : M \times [0, T) \to \mathbb{R}^{n+1}$ be a solution of (CF) such that f satisfies conditions (i)–(ii). Let $G : M \times [0, T) \to \mathbb{R}$ be given by a smooth, symmetric, degree-one homogeneous function g of the principal curvatures. Then G satisfies the evolution equation*

$$(\partial_t - \mathcal{L})G = (\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla h_{pq}\nabla h_{rs} + G|\mathcal{W}|^2_F, \tag{2-3}$$

where $\mathcal{L} := \dot{F}^{kl}\nabla_k\nabla_l$ is the linearisation of F, and $|\mathcal{W}|^2_F := \dot{F}^{kl}h_k{}^r h_{rl}$.

In particular, the speed function F satisfies $(\partial_t - \mathcal{L})F = F|\mathcal{W}|^2_F$.

As we shall see, in order to obtain Theorem 1.3, it is crucial to obtain a good upper bound on the term

$$Q(\nabla\mathcal{W}, \nabla\mathcal{W}) := (\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}$$

for the pinching functions G_m which we construct in the following section. The following decomposition of Q is crucial in obtaining this bound.

Lemma 2.6. *For any totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, we have*

$$\begin{aligned}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\big|_B T_{kpq}T_{lrs} &= (\dot{g}^k\ddot{f}^{pq} - \dot{f}^k\ddot{g}^{pq})\big|_z T_{kpp}T_{kqq} \\ &\quad + 2\sum_{p>q}\frac{(\dot{f}^p\dot{g}^q - \dot{g}^p\dot{f}^q)\big|_z}{z_p - z_q}((T_{pqq})^2 + (T_{qpp})^2) + 2\sum_{k>p>q}(\vec{g}_{kpq}\times\vec{f}_{kpq})\big|_z\cdot\vec{z}_{kpq}(T_{kpq})^2\end{aligned} \tag{2-4}$$

at any diagonal matrix B with distinct eigenvalues z_i, where "$\times$" and "$\cdot$" are the three-dimensional cross and dot product respectively, and we have defined the vectors

$$\begin{aligned}\vec{f}_{kpq} &:= (\dot{f}^k, \dot{f}^p, \dot{f}^q),\\ \vec{g}_{kpq} &:= (\dot{g}^k, \dot{g}^p, \dot{g}^q),\\ \vec{z}_{kpq} &:= \left(\frac{z_p - z_q}{(z_k - z_p)(z_k - z_q)}, \frac{z_k - z_q}{(z_k - z_p)(z_p - z_q)}, \frac{z_k - z_p}{(z_p - z_q)(z_k - z_q)}\right).\end{aligned}$$

Proof. Since B is diagonal, Lemma 2.1 yields (suppressing the dependence on B)

$$\begin{aligned}&(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs}\\ &\qquad = \sum_{k,p,q}(\dot{g}^k\ddot{f}^{pq} - \dot{f}^k\ddot{g}^{pq})T_{kpp}T_{kqq} + 2\sum_k\sum_{p>q}\left(\dot{g}^k\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{kpq})^2.\end{aligned}$$

We now decompose the second term into the terms satisfying $k = p$, $k = q$, $k > p$, $p > k > q$, and $q > k$ respectively:

$$\begin{aligned}&\sum_k\sum_{p>q}\left(\dot{g}^k\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{kpq})^2\\ &\quad = \sum_{p>q}\left(\dot{g}^p\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^p\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{ppq})^2 + \sum_{p>q}\left(\dot{g}^q\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^q\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{qpq})^2\\ &\qquad + \left(\sum_{k>p>q} + \sum_{p>k>q} + \sum_{p>q>k}\right)\left(\dot{g}^k\frac{\dot{f}^p - \dot{f}^q}{z_p - z_q} - \dot{f}^k\frac{\dot{g}^p - \dot{g}^q}{z_p - z_q}\right)(T_{kpq})^2\end{aligned}$$

$$
\begin{aligned}
&= \sum_{p>q} \frac{\dot f^p \dot g^q - \dot g^p \dot f^q}{z_p - z_q}((T_{pqq})^2 + (T_{qpp})^2)\\
&\quad + \sum_{k>p>q}\left(\dot g^k \frac{\dot f^p - \dot f^q}{z_p - z_q} - \dot f^k \frac{\dot g^p - \dot g^q}{z_p - z_q} + \dot g^p \frac{\dot f^k - \dot f^q}{z_k - z_q} - \dot f^p \frac{\dot g^k - \dot g^q}{z_k - z_q} + \dot g^q \frac{\dot f^k - \dot f^p}{z_k - z_p} - \dot f^q \frac{\dot g^k - \dot g^p}{z_k - z_p}\right)(T_{kpq})^2\\
&= \sum_{p>q} \frac{\dot f^p \dot g^q - \dot g^p \dot f^q}{z_p - z_q}((T_{pqq})^2 + (T_{qpp})^2) + \sum_{k>p>q}\bigg[(\dot g^p \dot f^q - \dot f^q \dot g^p)\left(\frac{1}{z_k - z_p} - \frac{1}{z_k - z_q}\right)\\
&\qquad - (\dot g^k \dot f^q - \dot f^k \dot g^q)\left(\frac{1}{z_p - z_q} + \frac{1}{z_k - z_p}\right) + (\dot g^k \dot f^p - \dot f^k \dot g^p)\left(\frac{1}{z_p - z_q} - \frac{1}{z_k - z_q}\right)\bigg](T_{kpq})^2\\
&= \sum_{p>q} \frac{\dot f^p \dot g^q - \dot g^p \dot f^q}{z_p - z_q}((T_{pqq})^2 + (T_{qpp})^2) + \sum_{k>p>q} (\vec g_{kpq} \times \vec f_{kpq}) \cdot \vec z_{kpq} (T_{kpq})^2. \qquad \square
\end{aligned}
$$

We complete this section by proving that $(m+1)$-convexity is preserved by the flow (CF), so that this assumption need only be made on initial data:

Proposition 2.7. *Let X be a solution of (CF) such that conditions (i)–(ii) are satisfied. Suppose that there is some $m \in \{1, \dots, n-1\}$ and some $\beta > 0$ such that*

$$\kappa_{\sigma(1)}(x, 0) + \cdots + \kappa_{\sigma(m)}(x, 0) \geq \beta F(x, 0)$$

for all $x \in M$ and all permutations $\sigma \in P_n$. Then this estimate persists at all later times.

Proof. Denote by SM the unit tangent bundle over $M \times [0, T)$ and consider the function Z defined on $\bigoplus^m SM$ by

$$Z(x, t, \xi_1, \dots, \xi_m) = \sum_{\alpha=1}^{m} h(\xi_\alpha, \xi_\alpha) - \beta F(x, t).$$

Since we have

$$\inf_{\xi_1,\dots,\xi_m \in S_{(x,t)}M} Z(x, t, \xi_1, \dots, \xi_m) = \kappa_{\sigma(1)}(x, t) + \cdots + \kappa_{\sigma(m)}(x, t) - \beta F(x, t)$$

for some $\sigma \in P_n$, it suffices to show that Z remains nonnegative. First fix any $t_1 \in [0, T)$ and consider the function $Z_\varepsilon(x, t, \xi_1, \dots, \xi_m) := Z(x, t, \xi_1, \dots, \xi_m) + \varepsilon \mathrm{e}^{(1+C)t}$, where $C := \sup_{M\times[0,t_1]} |\mathcal{W}|_F^2$. Note that C is finite since M is compact and $\dot F$ is bounded. Observe that Z_ε is positive when $t = 0$. We will show that Z_ε remains positive on $M \times [0, t_1]$ for all $\varepsilon > 0$. So suppose to the contrary that Z_ε vanishes at some point $(x_0, t_0, \xi_1^0, \dots, \xi_m^0)$. We may assume that t_0 is the first such time. Now extend the vector $\xi^0 := (\xi_1^0, \dots, \xi_m^0)$ to a field $\xi := (\xi_1, \dots, \xi_n)$ near (x_0, t_0) by parallel translation in space and solving

$$\frac{\partial \xi_\alpha^i}{\partial t} = F \xi_\alpha^j h_j{}^i.$$

Since the metric evolves according to

$$\partial_t g_{ij} = -2F h_{ij}$$

the resulting fields have unit length. Now recall (see for example [Andrews 1994a]) the following evolution equation for the second fundamental form:

$$\partial_t h_{ij} = \mathscr{L} h_{ij} + \ddot{F}^{pq,rs}\nabla_i h_{pq}\nabla_j h_{rs} + |\mathscr{W}|_F^2 h_{ij} - 2F h_{ij}^2,$$

where $\mathscr{L} := \dot{F}^{kl}\nabla_k\nabla_l$ and $|\mathscr{W}|_F^2 := \dot{F}^{kl}h_{kl}^2$. It follows that

$$\begin{aligned}(\partial_t - \mathscr{L})(Z_\varepsilon(x,t,\xi)) &= \varepsilon(1+C)\mathrm{e}^{(1+C)t} + \sum_{\alpha=1}^{m} \ddot{F}^{pq,rs}\nabla_{\xi_\alpha} h_{pq}\nabla_{\xi_\alpha} h_{rs} + |\mathscr{W}(x,t)|_F^2 Z(x,t,\xi)\\ &\geq \varepsilon(1+C)\mathrm{e}^{(1+C)t} + |\mathscr{W}(x,t)|_F^2 Z(x,t,\xi).\end{aligned}$$

Since the point $(x_0, t_0, \xi_{t=t_0})$ is a minimum of Z_ε, we obtain

$$0 \geq (\partial_t - \mathscr{L})\big|_{(x_0,t_0)}(Z_\varepsilon(x,t,\xi)) \geq \varepsilon(1+C)\mathrm{e}^{(1+C)t_0} - C\varepsilon\mathrm{e}^{(1+C)t_0} = \varepsilon\mathrm{e}^{(1+C)t_0} > 0\,.$$

This is a contradiction, implying that Z_ε cannot vanish at any time in the interval $[0, t_1]$. Since $\varepsilon > 0$ was arbitrary, we find $Z \geq 0$ at all times in the interval $[0, t_1]$. Since $t_1 \in [0, T)$ was arbitrary, we obtain $Z \geq 0$. □

3. Constructing the pinching function

In this section we construct the pinching functions G_m satisfying the conditions in Theorem 1.3. Let us first introduce the *pinching cones*

$$\Gamma_m := \{z \in \Gamma : z_{\sigma(1)} + \cdots + z_{\sigma(m+1)} > c_m^{-1} f(z) \text{ for all } \sigma \in H_m\},$$

where H_m is the quotient of P_n, the group of permutations of the set $\{1, \dots, n\}$, by the equivalence relation

$$\sigma \sim \omega \quad \text{if} \quad \sigma(\{1,\dots,m+1\}) = \omega(\{1,\dots,m+1\}).$$

Using the methods of [Huisken 1984], and their adaptations to 2-convex flows in [Huisken and Sinestrari 2009] and fully nonlinear flows in [Andrews et al. 2014b], we will see that, in order to prove Theorem 1.3, it suffices to construct a smooth function $g_m : \Gamma \to \mathbb{R}$ satisfying the following properties.

Properties. (i) $g_m(z) \geq 0$ for all $z \in \Gamma$ with equality if and only if $z \in \overline{\Gamma}_m \cap \Gamma$.

(ii) g_m is smooth and homogeneous of degree one.

(iii) For every $\varepsilon > 0$ there exists $c_\varepsilon > 0$ such that

$$(\dot{G}_m^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}_m^{pq,rs})\big|_B T_{kpq}T_{lrs} \leq -c_\varepsilon\frac{|T|^2}{F}$$

for all $B \in \mathscr{S}_{\Gamma_0}$ satisfying $G_m(B) \geq \varepsilon F(B)$ and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$, where G_m is the matrix function corresponding to g_m as described in Section 2, and Γ_0 is a preserved cone for the flow.

(iv) For every $\delta > 0$, $\varepsilon > 0$, and $C > 0$, there exist $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ such that

$$(G_m \dot{F}^{kl} - F \dot{G}_m^{kl})\big|_B B_{kl}^2 \le -\gamma_1 F^2 (G_m - \delta\gamma_2 F)\big|_B + \gamma_3 C F^2\big|_B$$

for all $(m+1)$-positive $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G_m(B) \ge \varepsilon F(B)$ and

$$\lambda_{\min}(B) \ge -\delta F(B) - C.$$

Our construction of the pinching function g_m will be similar for each choice of m. So let us fix $m \in \{0, 1, \dots, n-2\}$ and assume that the flow is $(m+1)$-convex. We first consider the preliminary function $g : \Gamma \to \mathbb{R}$ defined by

$$g(z) := f(z) \sum_{\sigma \in H_m} \varphi\left(\frac{\sum_{i=1}^{m+1} z_{\sigma(i)} - c_m^{-1} f(z)}{f(z)} \right), \tag{3-1}$$

where $\varphi : \mathbb{R} \to \mathbb{R}$ is a smooth[1] function which is strictly convex and positive, except on $\mathbb{R}_+ \cup \{0\}$ where it vanishes identically. Such a function is readily constructed; for example, we could take

$$\varphi(r) = \begin{cases} r^4 e^{-1/r^2} & \text{if } r < 0, \\ 0 & \text{if } r \ge 0. \end{cases}$$

We note that such a function necessarily satisfies $\varphi(r) - r\varphi'(r) \le 0$ and $\varphi'(r) \le 0$ with equality if and only if $r \ge 0$.

Now define the scalar $G : M \times [0, T) \to \mathbb{R}$ by

$$G(x, t) := g(\kappa_1(x, t), \dots, \kappa_n(x, t)).$$

Then G is a smooth, degree-one homogeneous function of the components of the Weingarten map which is invariant under a change of basis. Moreover, G is nonnegative and vanishes at, and only at, points for which the sum of the smallest $(m+1)$-principal curvatures is not less than $c_m^{-1} F$. Thus properties (i) and (ii) are satisfied by g.

We now show that property (iii) is satisfied weakly by g:

Lemma 3.1. *Let G be the matrix function corresponding to the function g defined by (3-1). Then, for any symmetric matrix B and totally symmetric 3-tensor T,*

$$(\dot{G}^{kl} \ddot{F}^{pq,rs} - \dot{F}^{kl} \ddot{G}^{pq,rs})\big|_B T_{kpq} T_{lrs} \le 0.$$

Proof. We will show that each of the terms in the decomposition (2-4) in Lemma 2.6 is nonpositive. Note that, by the invarance properties of G and F, it suffices to prove the claim for diagonal B. In fact, we can also assume that B has distinct eigenvalues, since the result at an arbitrary diagonal matrix B may then be

[1] In fact, φ need only be twice continuously differentiable.

obtained by taking a limit $B^{(k)} \to B$ such that each matrix $B^{(k)}$ has distinct eigenvalues. We first compute

$$\begin{aligned}\dot g^k &= \dot f^k \sum_{\sigma\in H_m} \varphi(r_\sigma) + \sum_{\sigma\in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1}\left(\delta^k_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^k\right)\\ &= \dot f^k \sum_{\sigma\in H_m}\left(\varphi(r_\sigma) - \varphi'(r_\sigma)\frac{\sum_{i=1}^{m+1} z_{\sigma(i)}}{f}\right) + \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\delta^k_{\sigma(i)}\end{aligned}$$

and

$$\begin{aligned}\ddot g^{pq} = &\left(\sum_{\sigma\in H_m}\varphi(r_\sigma) - \sum_{\sigma\in H_m}\varphi'(r_\sigma)\frac{\sum_{i=1}^{m+1} z_{\sigma(i)}}{f}\right)\ddot f^{pq}\\ &+ \sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^p\right)\sum_{i=1}^{m+1}\left(\delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^q\right),\end{aligned}$$

where we have set

$$r_\sigma(z) := \frac{\sum_{i=1}^{m+1} z_{\sigma(i)} - c_m^{-1} f(z)}{f(z)}.$$

It follows that

$$\dot g^k \ddot f^{pq} - \dot f^k \ddot g^{pq} = \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)\delta^k_{\sigma(i)}\ddot f^{pq} - \dot f^k\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^p\right)\sum_{i=1}^{m+1}\left(\delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^q\right).$$

If we fix the index k and set $\xi_p = T_{kpp}$, then, by convexity of φ and positivity of $\dot f^k$, we have

$$\begin{aligned}-\dot f^k \sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^p\right)\sum_{i=1}^{m+1}\left(\delta^q_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^q\right)\xi_p\xi_q\\ = -\dot f^k\sum_{\sigma\in H_m}\frac{\varphi''(r_\sigma)}{f}\left(\sum_{i=1}^{m+1}\left(\delta^p_{\sigma(i)} - \frac{z_{\sigma(i)}}{f}\dot f^p\right)\xi_p\right)^2 \le 0.\end{aligned}$$

On the other hand, since φ is monotone nonincreasing, and f is convex, we have

$$\varphi'(r_\sigma)\sum_{i=1}^{m+1}\delta^k_{\sigma(i)}\ddot f^{pq}\xi_p\xi_q \le 0$$

for each σ. Since both inequalities hold for all k, we deduce that

$$\sum_{k,p,q}\left(\dot g^k \ddot f^{pq} - \dot f^k \ddot g^{pq}\right)T_{kpp}T_{kqq} \le 0.$$

We next consider

$$\dot f^p \dot g^q - \dot g^p \dot f^q = \sum_{\sigma\in H_m}\sum_{i=1}^{m+1}\varphi'(r_\sigma)(\delta^q_{\sigma(i)}\dot f^p - \delta^p_{\sigma(i)}\dot f^q) = \sum_{\sigma\in O_q}\varphi'(r_\sigma)\dot f^p - \sum_{\sigma\in O_p}\varphi'(r_\sigma)\dot f^q.$$

where we have introduced the sets

$$O_a := \{\sigma \in H_m : a \in \sigma(\{1, \dots, m+1\})\}.$$

If $z_p > z_q$, we obtain

$$\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q \le \dot{f}^p \biggl(\sum_{\sigma \in O_q} \varphi'(r_\sigma) - \sum_{\sigma \in O_p} \varphi'(r_\sigma) \biggr),$$

We now show that the term in brackets is nonpositive whenever $z_p > z_q$.

Lemma 3.2. *If $z_p \ge z_q$, then*

$$\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) \ge 0 .$$

Moreover, equality holds only if either $z_p = z_q$ or $r_\sigma(z) \ge 0$ for all $\sigma \in O_{q,p} := O_q \setminus O_p$.

Proof of Lemma 3.2. First note that

$$\sum_{\sigma \in O_p} \varphi'(r_\sigma) - \sum_{\sigma \in O_q} \varphi'(r_\sigma) = \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma) ,$$

where $O_{a,b} := O_a \setminus O_b$. Next observe that, if $\sigma \in O_{p,q}$, then

$$z_{\sigma(1)} + \dots + z_{\sigma(m+1)} = z_p + z_{\hat{\sigma}(i_1)} + \dots + z_{\hat{\sigma}(i_m)} \tag{3-2}$$

for some $\hat{\sigma} \in H_{m-2}(p,q) := P_{n-2}(p,q)/\sim$, where $P_{n-2}(p,q)$ denotes the set of permutations of $\{1, \dots, n\} \setminus \{p,q\}$; $i_1, \dots, i_m$ are m distinct elements of $\{1, \dots, n\} \setminus \{p,q\}$; and $\sim$ is defined by

$$\hat{\sigma} \sim \hat{\omega} \quad \text{if} \quad \hat{\sigma}(\{i_1, \dots, i_m\}) = \hat{\omega}(\{i_1, \dots, i_m\}).$$

Observe also that the converse holds (that is, (3-2) defines a bijection), so that

$$\sum_{\sigma \in O_{q,p}} \varphi'(r_\sigma) - \sum_{\sigma \in O_{p,q}} \varphi'(r_\sigma) = \sum_{\hat{\sigma} \in H_{m-2}(p,q)} \biggl(\varphi'\biggl(\frac{z_p + \sum_{k=1}^m z_{\hat{\sigma}(i_k)} - c_m^{-1} f}{f} \biggr) - \varphi'\biggl(\frac{z_q + \sum_{k=1}^m z_{\hat{\sigma}(i_k)} - c_m^{-1} f}{f} \biggr) \biggr).$$

Since $z_p \ge z_q$, the claim follows from (strict) convexity of φ (where it is positive). □

Thus,

$$\sum_{p>q} \frac{\dot{f}^p \dot{g}^q - \dot{g}^p \dot{f}^q}{z_p - z_q} \bigl((T_{pqq})^2 + (T_{qpp})^2 \bigr) \le 0.$$

We now compute

$$\vec{g}_{kpq} = \biggl(\frac{g}{f} - \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} \frac{z_{\sigma(i)}}{f} \biggr) \vec{f}_{kpq} + \sum_{\sigma \in H_m} \varphi'(r_\sigma) \sum_{i=1}^{m+1} (\delta^k_{\sigma(i)}, \delta^p_{\sigma(i)}, \delta^q_{\sigma(i)}),$$

so that

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} = \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) \big[\big(\delta^k_{\sigma(i)}, \delta^p_{\sigma(i)}, \delta^q_{\sigma(i)} \big) \times \vec{f}_{kpq} \big] \cdot \vec{z}_{kpq}$$
$$= \sum_{\sigma \in H_m} \sum_{i=1}^{m+1} \varphi'(r_\sigma) \bigg[\frac{(\delta^p_{\sigma(i)} \dot{f}^q - \delta^q_{\sigma(i)} \dot{f}^p)(z_p - z_q)}{(z_k - z_p)(z_k - z_q)} + \frac{(\delta^q_{\sigma(i)} \dot{f}^k - \delta^k_{\sigma(i)} \dot{f}^q)(z_k - z_q)}{(z_k - z_p)(z_p - z_q)} + \frac{(\delta^k_{\sigma(i)} \dot{f}^p - \delta^p_{\sigma(i)} \dot{f}^k)(z_k - z_p)}{(z_k - z_q)(z_p - z_q)} \bigg].$$

Removing the positive factor $\alpha_{kpq} := [(z_k - z_p)(z_k - z_q)(z_p - z_q)]^{-1}$ and setting

$$P_a := \sum_{\sigma \in O_a} \varphi'(r_\sigma),$$

we obtain

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} = \alpha_{kpq} \big[(P_p \dot{f}^q - P_q \dot{f}^p)(z_p - z_q)^2 + (P_q \dot{f}^k - P_k \dot{f}^q)(z_k - z_q)^2 + (P_k \dot{f}^p - P_p \dot{f}^k)(z_k - z_p)^2 \big].$$

Applying Lemma 3.2 yields

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \le \alpha_{kpq} (P_q \dot{f}^k - P_k \dot{f}^q)[(z_k - z_q)^2 - (z_k - z_p)^2 - (z_p - z_q)^2].$$

Since the term in square brackets is nonnegative, applying Lemma 3.2 once more yields

$$(\vec{g}_{kpq} \times \vec{f}_{kpq}) \cdot \vec{z}_{kpq} \le 0.$$

This completes the proof of the lemma. □

Corollary 3.3. *There exists $C < \infty$ (depending only on n, f and M_0) such that $G/F \le C$ along the flow.*

Proof. In view of Lemma 3.1 and the evolution equation (2-3), this is a simple application of the maximum principle. □

In order to obtain the uniform estimate required by property (iii), we modify G in order to obtain a function with a strict convexity property. A well-known trick (cf. [Andrews 1994b, Lemma 7.10; Huisken and Sinestrari 1999a, Theorem 2.14; Andrews et al. 2014b, Lemma 3.3]) then allows us to extract the required uniform estimate. First, we relabel the preliminary pinching function $g \to g_1$ ($G \to G_1$), and consider the new pinching function g defined by

$$g := K(g_1, g_2) := \frac{g_1^2}{g_2}, \tag{3-3}$$

where $g_2(z) = M \sum_{i=1}^n z_i - |z|$ for some large constant $M \gg 1$, for which g_2 is positive along the flow. That there is such a constant follows from applying the maximum principle to the evolution equation (2-3) for the function $G_2(x,t) := g_2(\kappa(x,t))$ as in [Andrews et al. 2014b, Lemma 3.1]. Note that $\dot{K}^1 > 0$, $\dot{K}^2 < 0$ and $\ddot{K} > 0$ wherever $g_1 > 0$.

Observe that properties (i) and (ii) are not harmed in the transition from g_1 to g. We now show that the estimates listed in properties (iii) and (iv) are satisfied by the curvature function defined in (3-3).

Proposition 3.4. *Let g be the pinching function defined by (3-3) and G its corresponding matrix function. Then, for every $\varepsilon > 0$, there exists $c_\varepsilon > 0$ (depending only on ε, n, f and Γ_0) such that*

$$(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\big|_B T_{kpq}T_{lrs} \le -c_\varepsilon \frac{|T|^2}{F}$$

for all $B \in \mathcal{S}_{\Gamma_0}$ satisfying $G(B) \ge \varepsilon F(B)$ and all totally symmetric $T \in \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n$.

Proof. First note that (suppressing dependence on B)

$$\begin{aligned}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})T_{kpq}T_{lrs} &= \dot{K}^\alpha(\dot{G}^{kl}_\alpha\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}_\alpha)T_{kpq}T_{lrs} - \dot{F}^{kl}\ddot{K}^{\alpha\beta}\dot{G}^{pq}_\alpha\dot{G}^{rs}_\beta T_{kpq}T_{lrs}\\ &\le \dot{K}^2(\dot{G}^{kl}_2\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs}_2)T_{kpq}T_{lrs}\\ &\le -\dot{K}^2\dot{F}^{kl}\ddot{G}^{pq,rs}_2 T_{kpq}T_{lrs},\end{aligned}$$

where we used Lemma 3.1, convexity of K, and the inequalities $\dot{K}^1 \ge 0$ and $\dot{F} \ge 0$ in the first inequality, and the inequalities $\dot{G}_2 \ge 0$ and $\dot{K}^2 \le 0$, and convexity of F in the second. Since $\dot{K}^2 < 0$ whenever $G_1 > 0$ and G_2 is strictly concave in nonradial directions, the claim follows exactly as in [Andrews et al. 2014b, Lemma 3.3]. □

The uniform estimate of Proposition 3.4 yields a good bound for the term $Q(\nabla\mathcal{W}, \nabla\mathcal{W})$ in the evolution equations for the pinching functions. This is a crucial component in obtaining the L^p-estimates of the following section. This is the starting point for the Stampacchia–de Giorgi iteration argument. The second crucial estimate is the Poincaré-type inequality, Lemma 4.2 (see also [Huisken and Sinestrari 2009, §§4–5; in particular, Lemma 5.5]), which we can obtain with the help of property (iv). This estimate (corresponding to [Huisken and Sinestrari 2009, Lemma 5.2]) provides an estimate on the zero order term that occurs in contracting the Simons-type identity for $\dot{F}^{pq}\nabla_p\nabla_q h_{ij}$ with $\dot{G}^{ij}$ (see [Andrews et al. 2014b, Proposition 4.4]).

Proposition 3.5. *Let g be the pinching function defined by (3-3) and G its corresponding matrix function. ,Then for every $\delta > 0$, $\varepsilon > 0$, and $C > 0$ there exist $\gamma_1 > 0$, $\gamma_2 > 0$ and $\gamma_3 > 0$ (depending only on δ, $\varepsilon > 0$, C, n, m, f and Γ_0) such that*

$$Z(B) := (F\dot{G}^{kl} - G\dot{F}^{kl})\big|_B B^2_{kl} \ge \gamma_1 F^2(G - \delta\gamma_2 F)\big|_B - \gamma_3 F^2\big|_B$$

for all symmetric, $(m+1)$-positive matrices B satisfying $\lambda(B) \in \Gamma_0$, $\lambda_{\min}(B) \ge -\delta F(B) - C$, and $G_m(B) \ge \varepsilon F(B)$.

Proof. From the definition of G we have

$$Z = \dot{K}^1 Z_1 + \dot{K}^2 Z_2,$$

where

$$Z_i(B) := (F\dot{G}^{kl}_i - G_i\dot{F}^{kl})\big|_B B^2_{kl}.$$

Thus, since $\dot{K}^2 = 2g_1/g_2$ is uniformly bounded below when $g \ge \varepsilon f$, it suffices to prove the estimate for Z_1.

So let B be a symmetric, $(m+1)$-positive matrix with eigenvalues $z_1 \le \cdots \le z_n$. Then

$$Z_1(B) = f\dot{g}_1^p z_p^2 - g_1 \dot{f}^p z_p^2 = \sum_{p>q}\big(\dot{g}_1^p \dot{f}^q - \dot{g}_1^q \dot{f}^p\big) z_p z_q (z_p - z_q) = \sum_{p>q}\big(P_p \dot{f}^q - P_q \dot{f}^p\big) z_p z_q (z_p - z_q)$$

$$= \bigg(\sum_{p>q>l} + \sum_{p>l\ge q} + \sum_{l\ge p>q}\bigg)\big(P_p \dot{f}^q - P_q \dot{f}^p\big) z_p z_q (z_p - z_q),$$

where we recall the notation $P_a := \sum_{\sigma\in O_a} \varphi'(r_\sigma)$ and we have defined $l \le m$ as the number of nonpositive eigenvalues z_i. Recalling that $P_p \dot{f}^q - P_q \dot{f}^p \ge 0$ whenever $z_p \ge z_q$, we discard the final sum and part of the first to obtain

$$Z_1(B) \ge \sum_{p=m+2}^{n}\sum_{q=l+1}^{m+1}(P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) + \sum_{p=l+1}^{n}\sum_{q=1}^{l}(P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q)$$

$$= \sum_{p=m+2}^{n}\sum_{q=l+1}^{m+1}(P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q) - f^2 \sum_{i=1}^{l} z_i$$

$$+ f^2 \sum_{i=1}^{l} z_i + \sum_{p=l+1}^{n}\sum_{q=1}^{l}(P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q).$$

So consider the term

$$S_1(z) := \sum_{p=m+2}^{n}\sum_{q=l+1}^{m+1}(P_p(z) \dot{f}^q(z) - P_q(z) \dot{f}^p(z)) z_p z_q (z_p - z_q) - f(z)^2 \sum_{i=1}^{l} z_i.$$

Observe that $S_1 \ge 0$. We claim that $S_1(z) > 0$ for all z in the cone

$$\Gamma_{\varepsilon,l} := \{z \in \Gamma_0 : g(z) \ge \varepsilon f(z),\ z_1 \le \cdots \le z_l \le 0 < z_{l+1} \le \cdots \le z_n\}.$$

Suppose, to the contrary, that $S_1(z) = 0$ for some $z \in \Gamma_{\varepsilon,l}$. Then $z_1 = \cdots = z_l = 0$ and, for all $p > m+1 \ge q > l$, $(P_p(z)\dot{f}^q(z) - P_q(z)\dot{f}^p(z)) z_p z_q (z_p - z_q) = 0$. But, by Lemma 3.2, the latter implies that, for all $p > m+1 \ge q > l$, either $z_p = z_q$, or $r_\sigma(\lambda) \ge 0$ for all $\sigma \in O_{q,p}$. Note that the latter case cannot occur: since $p > m+1 \ge q$, there is a permutation $\sigma \in O_{q,p}$ such that $0 \le r_\sigma(z) = (z_1 + \cdots + z_{m+1} - c_m^{-1} f(z))/f(z)$, which implies $g_1(z) = 0$, contradicting $z \in \Gamma_{\varepsilon,l}$. On the other hand, if $z_p = z_q$ for all $p > m+1 \ge q > l$, then we again obtain the contradiction $g_1(z) = 0$. Thus, $S_1 > 0$ on $\Gamma_{\varepsilon,l}$. Since S_1 is homogeneous of degree three, it follows that

$$S_1 \ge c_1 f^2 g$$

on $\Gamma_{\varepsilon,l}$, where $c_1 := \min_l \min_{\Gamma_{\varepsilon,l}} \dfrac{S_1}{f^2 g} > 0$.

Now consider

$$S_2 := f^2 \sum_{i=1}^{l} \lambda_i + \sum_{p=l+1}^{n}\sum_{q=1}^{l}(P_p \dot{f}^q - P_q \dot{f}^p) z_p z_q (z_p - z_q).$$

Note that, by homogeneity, $c_2 := \sup\{P_p(z)\dot{f}^q(z) - P_q(z)\dot{f}^p(z) : z \in \Gamma_0,\ 1 \le p,\ q \le n\} < \infty$. Thus, S_2 is easily controlled using the "convexity estimate" $\lambda_1 \ge -\delta f - C$:

$$S_2 \ge -lf^2(\delta f + C) + (n-l)c_2 z_n \sum_{q=1}^{l} z_q(z_n - z_q) \ge -nf^2(\delta f + C) + 2nc_2c_3^2 f^2 \sum_{q=1}^{l} z_q$$
$$\ge -nf^2(\delta f + C) - 2nc_2c_3^2 f^2(\delta F + C) \ge -n(1 + 2c_2c_3^2) f^2(\delta f + C),$$

where $c_3 := \max\{|z_i|/f(z) : z \in \Gamma_0, 1 \le i \le n\}$.

The claim follows. □

We note that the above estimate is only useful in the presence of the convexity estimate Theorem 1.1, since then, for any $\delta > 0$, there is a constant $C_\delta > 0$ for which $\Gamma_{\delta, C_\delta} := \{z \in \Gamma_0 : z_i > -\delta f(z) - C_\delta \text{ for all } i\}$ is preserved by the flow.

4. Proof of Theorem 1.3

In order to prove Theorem 1.3, it suffices to obtain, for any $\varepsilon > 0$, an upper bound on the function

$$G_{\varepsilon,\sigma} := \left(\frac{G}{F} - \varepsilon\right) F^\sigma$$

for some $\sigma > 0$. We will use the estimates of Propositions 3.5 and 3.4 to obtain bounds on the spacetime L^p-norms of the positive part of $G_{\varepsilon,\sigma}$, so long as p is sufficiently large and σ sufficiently small, just as in [Huisken and Sinestrari 1999b; 1999a; 2009] (see also [Andrews et al. 2014b] where these techniques are applied in the fully nonlinear setting). A Stampacchia–de Giorgi iteration procedure similar to that used in [Huisken 1984] (see also [Huisken and Sinestrari 1999b; Andrews et al. 2014b]) then allows us to extract a supremum bound on $G_{\varepsilon,\sigma}$.

We begin with an evolution equation for $G_{\varepsilon,\sigma}$:

Lemma 4.1 [Andrews et al. 2014b]. *The function $G_{\varepsilon,\sigma}$ satisfies the evolution equation*

$$(\partial_t - \mathcal{L})G_{\varepsilon,\sigma} = F^{\sigma-1}(\dot{G}^{kl}\ddot{F}^{pq,rs} - \dot{F}^{kl}\ddot{G}^{pq,rs})\nabla_k h_{pq}\nabla_l h_{rs}$$
$$+ \frac{2(1-\sigma)}{F}\langle \nabla G_{\varepsilon,\sigma}, \nabla F\rangle_F - \frac{\sigma(1-\sigma)}{F^2}|\nabla F|_F^2 + \sigma G_{\varepsilon,\sigma}|\mathcal{W}|_F^2, \quad (4\text{-}1)$$

where $\langle u, v\rangle_F := \dot{F}^{kl}u_k u_l$.

Now set $E := \max\{G_{\varepsilon,\sigma}, 0\}$. We need to obtain spacetime L^p-estimates for E. Let us first observe that integration by parts and application of Young's inequality, in conjunction with Lemma 2.3 and Proposition 3.4, yields the estimate (cf. [Andrews et al. 2014b])

$$\frac{d}{dt}\int E^p\,d\mu \le -\left(A_1 p(p-1) - A_2 p^{\frac{3}{2}}\right)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu$$
$$-\left(B_1 p - B_2 p^{\frac{1}{2}}\right)\int E^p \frac{|\nabla \mathcal{W}|^2}{F^2}\,d\mu + C_1\sigma p\int E^p|\mathcal{W}|^2 d\mu \quad (4\text{-}2)$$

for some positive constants A_1, A_2, B_1, B_2, C_1 (which depend only on ε, n, m, f and M_0).

To estimate the final term, we make use of Proposition 3.5 in a similar manner to [Huisken and Sinestrari 2009, §5]. We first observe:

Lemma 4.2. *There are positive constants A_3, A_4, A_5, B_3, B_4, C_2, independent of p and σ, such that*

$$\int E^p \frac{Z(\mathcal{W})}{F}\,d\mu \le \big(A_3 p^{\frac{3}{2}} + A_4 p^{\frac{1}{2}} + A_5\big)\int E^{p-2}|\nabla G_{\varepsilon,\sigma}|^2\,d\mu + \big(B_3 p^{\frac{1}{2}} + B_4\big)\int E^p \frac{|\nabla \mathcal{W}|^2}{F^2}\,d\mu.$$

Proof. As in [Andrews et al. 2014b, §4], contraction of the commutation formula for $\nabla^2\mathcal{W}$ with $\dot{F}$ and $\dot{G}$ yields the identity

$$\begin{aligned}\mathcal{L}G_{\varepsilon,\sigma} = -F^{\sigma-1}Q(\nabla\mathcal{W},\nabla\mathcal{W}) + F^{\sigma-1}Z(\mathcal{W}) + F^{\sigma-2}(F\dot{G}^{kl} - G\dot{F}^{kl})\nabla_k\nabla_l F \\ + \frac{\sigma}{F}G_{\varepsilon,\sigma}\mathcal{L}F - 2\frac{(1-\sigma)}{F}\langle\nabla F, \nabla G_{\varepsilon,\sigma}\rangle_F + \frac{\sigma(1-\sigma)}{F^2}G_{\varepsilon,\sigma}|\nabla F|_F^2.\end{aligned}$$

The claim is now proved using integration by parts and Young's inequality, with the help of Lemma 2.3 and Proposition 3.4 (see [Andrews et al. 2014b, Lemma 4.2]). □

Corollary 4.3. *For all $\varepsilon > 0$ there exist constants $\ell > 0$ and $L < \infty$ (depending only on ε, n, m, f and M_0) such that for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$ there is a constant $K = K_{\varepsilon,\sigma,p}$ (depending only on ε, n, m, f, M_0, σ and p) for which the following estimate holds:*

$$\int (G_{\varepsilon,\sigma})_+^p\,d\mu \le \int (G_{\varepsilon,\sigma}(\,\cdot\,,0))_+^p\,d\mu_0 + tK\mu_0(M),$$

where μ_0 is the measure induced on M by the initial immersion.

Proof. Recall Proposition 3.5. Setting $\delta = \varepsilon/(2\gamma_2)$ and applying the convexity estimate, we obtain

$$\frac{Z(\mathcal{W})}{F} \ge \frac{\varepsilon}{2}\gamma_1 F^2 - \gamma_3 C_{\varepsilon/(2\gamma_2)} F \tag{4-3}$$

whenever $G - \varepsilon F > 0$. We now use Young's inequality to obtain (cf. [Huisken and Sinestrari 2009, §5])

$$F = F^{-\sigma p}F^{1+\sigma p} \le F^{-\sigma p}\left(\frac{b^q}{q}F^{q(1+\sigma p)} + \frac{b^{-q'}}{q'}\right)$$

for any $b > 0$ and $q > 0$, where q' is the Hölder conjugate of q: $\dfrac{1}{q} + \dfrac{1}{q'} = 1$. Choosing $q = \dfrac{2+\sigma p}{1+\sigma p}$, so that $q' = 2 + \sigma p$, we obtain

$$F \le b^{(2+\sigma p)/(1+\sigma p)}\frac{1+\sigma p}{2+\sigma p}F^2 + \frac{b^{-(2+\sigma p)}}{2+\sigma p}F^{-\sigma p} \le b^{(2+\sigma p)/(1+\sigma p)}F^2 + b^{-(2+\sigma p)}F^{-\sigma p}.$$

Now choose $b := \left(\dfrac{\varepsilon\gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}}\right)^{\frac{1+\sigma p}{2+\sigma p}}$, so that

$$\gamma_3 C_{\varepsilon/(2\gamma_2)} F \le \frac{\varepsilon\gamma_1}{4}F^2 + KF^{-\sigma p},$$

where

$$K := \gamma_3 C_{\varepsilon/(2\gamma_2)}\left(\frac{\varepsilon\gamma_1}{4\gamma_3 C_{\varepsilon/(2\gamma_2)}}\right)^{-(1+\sigma p)}.$$

Returning to Equation (4-3), we find

$$\frac{\varepsilon\gamma_1}{4}F^2 \le K F^{-\sigma p} + \frac{Z(\mathcal{W})}{F}.$$

Estimating $G_{\varepsilon,\sigma} \le c_1 F^{\sigma}$ and $|\mathcal{W}|^2 \le c_2 F^2$, we obtain

$$E^p|\mathcal{W}|^2 \le \widetilde{K} + c_3 E^p \frac{Z(\mathcal{W})}{F}$$

for some constants $\widetilde{K} > 0$ (depending on F, M_0, ε, σ and p) and $c_3 > 0$ (depending on F, M_0, and ε).

Combining Lemma 4.2 and inequality (4-2) now yields

$$\frac{d}{dt}\int E^p\,d\mu \le K_{\varepsilon,\sigma,p}\mu_0(M) - \left(\alpha_0 p^2 - \alpha_1\sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right)\int E^{p-2}|G_{\varepsilon,\sigma}|^2\,d\mu \\ - \left(\beta_0 p - \beta_1\sigma p^{\frac{3}{2}} - \beta_2\sigma p - \beta_3 p^{\frac{1}{2}}\right)\int E^p \frac{|\nabla\mathcal{W}|^2}{F^2}\,d\mu$$

for some positive constants α_i and β_i, which depend on ε but not on σ or p, and $K_{\varepsilon,\sigma,p}$, which depends on ε, σ and p.

It is clear that $L > 0$ and $\ell > 0$ may be chosen such that

$$\left(\alpha_0 p^2 - \alpha_1\sigma p^{\frac{5}{2}} - \alpha_2 p^{\frac{3}{2}} - \alpha_3 p\right) \ge 0 \quad \text{and} \quad \left(\beta_0 p - \beta_1\sigma p^{\frac{3}{2}} - \beta_2\sigma p - \beta_3 p^{\frac{1}{2}}\right) \ge 0$$

for all $p > L$ and $0 < \sigma < \ell p^{-\frac{1}{2}}$. The claim then follows by integrating with respect to the time variable. □

The proof of Theorem 1.3 is completed by proceeding with Huisken's Stampacchia–de Giorgi iteration argument. We omit these details as the arguments required already appear in [Andrews et al. 2014b, §5] with no significant changes necessary.

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Received 8 Oct 2013. Accepted 30 Jun 2014.

BEN ANDREWS: ben.andrews@anu.edu.au
Mathematical Sciences Institute, Australian National University, ACT 0200, Australia
and
Mathematical Sciences Center, Tsinghua University, Beijing 100084, China

MAT LANGFORD: mathew.langford@uni-konstanz.de
Mathematical Sciences Institute, Australian National University, ACT 0200, Australia
and
Fachbereich Mathematik und Statistik, Universität Konstanz, 78457 Konstanz, Germany

ANALYSIS AND PDE
Vol. 7, No. 5, 2014
dx.doi.org/10.2140/apde.2014.7.1109

UNIFORM L^p-IMPROVING FOR WEIGHTED AVERAGES ON CURVES

BETSY STOVALL

We define variable parameter analogues of the affine arclength measure on curves and prove near-optimal L^p-improving estimates for associated multilinear generalized Radon transforms. Some of our results are new even in the convolution case.

1. Introduction

We consider weighted versions of multilinear generalized Radon transforms of the form

$$M_0(f_1, \dots, f_k) := \int_{\mathbb{R}^d} \prod_{i=1}^{k} f_i \circ \pi_i(x) a(x)\, dx, \tag{1-1}$$

where a is a continuous cutoff function and the $\pi_i : \mathbb{R}^d \to \mathbb{R}^{d-1}$ are smooth submersions.

In [Tao and Wright 2003; Stovall 2011], near endpoint estimates of the form

$$|M_0(f_1, \dots, f_k)| \le C \prod_{i=1}^{k} \|f_i\|_{L^{p_i}(\mathbb{R}^{d-1})}, \tag{1-2}$$

with $C = C(\pi_1, \dots, \pi_k, p_1, \dots, p_k)$, were established for M_0 under the assumption that the π_i satisfy a certain finite-type condition on the support of a. In particular, it was found that the exponents on the right in (1-2) depend on this type. These results are nearly sharp in the sense that if the type of the π_i degenerates anywhere on the set where $a \neq 0$, then the corresponding near endpoint estimates also fail. It is not, however, known in general what happens when the type degenerates at some point where $a \neq 0$ (for instance, on the boundary of the support) or the rate at which the constants in (1-2) blow up as the type degenerates.

Our goal is to quantify and counteract the failure of (1-2) in such situations by replacing M_0 by an appropriately weighted operator, for which we will establish near-optimal Lebesgue space bounds. The exponents (though not the implicit constants) in these bounds will be independent of the choice of $\pi_1, \dots, \pi_k$ and the cutoff function a. Further, the weights we employ transform naturally under changes of coordinates, so they may reasonably be viewed as generalizations of the affine arclength measure on curves in $\mathbb{R}^d$. A number of recent articles (such as [Bak et al. 2009; Dendrinos et al. 2009; Dendrinos and Müller 2013; Dendrinos and Stovall 2012; Dendrinos and Wright 2010; Drury and Marshall 1987; Oberlin 2002; 2003; 2010; Sjölin 1974; Stovall 2010]) have been devoted to establishing uniform estimates for

MSC2010: 42B99.
Keywords: generalized Radon transform, affine arclength.

operators weighted by affine arclength measure, and these results provide much of the motivation for this article.

A motivating example. Stating the main results of this article, or even the results of [Tao and Wright 2003; Stovall 2011] requires some notation, so we postpone this until the next section. By way of background and motivation, we will spend the remainder of the introduction describing a concrete case about which much is known, and which provides the inspiration for the more general operators considered in this article. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a smooth curve and a a continuous cutoff function. Consider the operator

$$T_0 f(x) := \int_{\mathbb{R}} f(x-\gamma(t))a(t)\,dt, \qquad f \in C_0^0(\mathbb{R}^d).$$

By duality, $T_0 : L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ if and only if, for all $f \in L^p(\mathbb{R}^d)$ and $g \in L^q(\mathbb{R}^d)$,

$$\left| \int_{\mathbb{R}^d} \int_{\mathbb{R}} f(x-\gamma(t))g(x)a(t)\,dt \right| \le C(\gamma, p, q)\|f\|_{L^p(\mathbb{R}^d)}\|g\|_{L^{q'}(\mathbb{R}^d)};$$

this may be compared with (1-2).

The curve γ is said to be of type (at most) N when $\det(\gamma'(t), \dots, \gamma^{(d)}(t))$ vanishes to order at most N at any point. The results of [Dendrinos and Stovall 2014] imply that if γ is of type N on the support of a, $\|T_0\|_{L^p \to L^q} < \infty$ if (p^{-1}, q^{-1}) lies in the trapezoid with vertices

$$(0,0), \quad (1,1), \quad (p_N^{-1}, q_N^{-1}) := \left(\frac{d}{N+d(d+1)/2}, \frac{d-1}{N+d(d+1)/2} \right), \quad (1-q_N^{-1}, 1-p_N^{-1}). \tag{1-3}$$

(The nonendpoint result was due to Tao and Wright [2003].) Further, if N is the maximal type of T_0 on $\{t : a(t) \neq 0\}$, this is sharp. If γ is not of finite type, T_0 satisfies no $L^p(\mathbb{R}^d) \to L^q(\mathbb{R}^d)$ estimates off the line $\{p = q\}$.

It was first noticed in [Sjölin 1974] and [Drury and Marshall 1985] that affine, as opposed to Euclidean, arclength has a uniformizing effect on the bounds for convolution and Fourier restriction operators associated to possibly degenerate curves. It is now known that, for a polynomial curve γ, the convolution operator with affine arclength measure on γ,

$$Tf(x) := \int_{\mathbb{R}} f(x-\gamma(t))|\det(\gamma'(t), \dots, \gamma^{(d)}(t))|^{2/(d(d+1))}\,dt,$$

maps $L^p(\mathbb{R}^d)$ boundedly into $L^q(\mathbb{R}^d)$ if and only if (p^{-1}, q^{-1}) lies on the line segment joining (p_0^{-1}, q_0^{-1}), $(1-q_0^{-1}, 1-p_0^{-1})$, with p_0, q_0 defined as above (provided $T \not\equiv 0$) [Oberlin 2002; Dendrinos et al. 2009; Stovall 2010]. Further, the operator norms these papers established depend only on the degree of the polynomial; for this, it is crucial that the affine arclength transforms nicely under reparametrizations and affine transformations. Further investigations have been carried out in [Oberlin 2010; Dendrinos and Stovall 2014] in the nonpolynomial case. The above mentioned results are essentially optimal, both in terms of the exponents involved and in terms of pointwise estimates on the weight [Oberlin 2003] (see Proposition 2.2). Analogous results are also known for the restricted X-ray transform [Dendrinos and Stovall 2012; 2014]. There have also been a number of recent articles aimed at establishing uniform estimates for Fourier restriction to curves with affine arclength measure, for instance [Bak et al. 2009; Dendrinos and Müller 2013; Dendrinos and Wright 2010; Stovall 2014].

Our goal in this article is to address the gap between the general results of [Tao and Wright 2003; Stovall 2011] and the type-independent results of [Dendrinos et al. 2009; Dendrinos and Stovall 2012; Oberlin 2002; Stovall 2010] by introducing a generalization of the affine arclength measure, well-suited to (1-1). We will also prove near endpoint bounds for the weighted operator and, in particular, will generalize the results of [Tao and Wright 2003; Stovall 2011] to the case when the π_i completely fail to be of finite type on the support of a. Some of our results are new even in the translation-invariant case.

2. Basic notions and statements of the main results

Notation. Throughout the article, we will use the now-standard notation $A \lesssim B$ to mean that $A \le CB$ for some innocuous implicit constant C. The value of this constant will be allowed to change from line to line. The meaning of "innocuous" will be specified at the beginning of most sections, though in this section it will be specified *in situ*, and in the next it does not arise. Additionally, $A \gtrsim B$ if $B \lesssim A$, and $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We denote the nonnegative integers by $\mathbb{Z}_0$. If ℓ is any integer, δ is an ℓ-tuple of real numbers and $\beta \in \mathbb{Z}_0^\ell$ is a multiindex, we denote by δ^β the quantity $\delta_1^{\beta_1} \cdots \delta_\ell^{\beta_\ell}$.

We will also use some less standard notation. We consider the partial order $\preceq$ on $\mathbb{Z}_0^k$ defined by $b_1 \preceq b_2$ if $b_1^i \le b_2^i$ for $1 \le i \le k$. We say $b_1 \prec b_2$ if at least one of these inequalities is strict. If $\mathscr{B} \subseteq \mathbb{Z}_0^k$ is any set, we define a polytope

$$\mathscr{P}(\mathscr{B}) := \mathrm{ch} \bigcup_{b \in \mathscr{B}} ([0, \infty)^k + \{b\}),$$

where "ch" denotes the convex hull.

Fix a dimension d and an integer $k \ge 2$; k may exceed d. We will consider vector fields $X_1, \dots, X_k$, defined and smooth on the closure of an open set U. A word w is an element of $\mathscr{W} := \bigcup_{n=1}^\infty \{1, \dots, k\}^n$. To each word is associated a vector field X_w, defined recursively by $X_{(i)} := X_i$ for $1 \le i \le k$ and $X_{(w,i)} := [X_w, X_i]$ for $w \in \mathscr{W}$ and $1 \le i \le k$. The degree of $w \in \mathscr{W}$ is the k-tuple, $\deg w$, whose i-th entry is the number of occurrences of i in w.

All brackets of such vector fields lie in the span of the X_w: if $w, w' \in \mathscr{W}$,

$$[X_w, X_{w'}] = \sum_{\deg \widetilde{w} = \deg w + \deg w'} C_{w,w'}^{\widetilde{w}} X_{\widetilde{w}}, \tag{2-1}$$

where $C_{w,w'}^{\widetilde{w}}$ is an integer. Indeed, by the Jacobi identity,

$$[X_w, [X_{w'}, X_i]] = [[X_w, X_{w'}], X_i] - [X_{(w,i)}, X_{w'}],$$

so (2-1) is easily obtained by inducting on $\|\deg w'\|_{\ell^1}$ [Hörmander 1967]. We note that for each $b \in \mathbb{N}^k$ there are only finitely many words w with $\deg w = b$, so the sum in (2-1) is finite.

If $I = (w_1, \dots, w_d)$ is a d-tuple of words, we define $\deg I := \sum_{i=1}^d \deg w_i$ and

$$\lambda_I := \det(X_{w_1}, \dots, X_{w_d}).$$

The Newton polytope of the vector fields $X_1, \dots, X_k$ at the point $x_0 \in U$ is defined to be

$$\mathcal{P}_{x_0} := \mathcal{P}(\{\deg I : I \text{ is a } d\text{-tuple of words satisfying } \lambda_I(x_0) \neq 0\}),$$

and we define the Newton polytope of a set $A \subseteq U$ to be

$$\mathcal{P}_A := \mathrm{ch}\Big(\bigcup_{x \in A} \mathcal{P}_x\Big).$$

The Hörmander condition is the statement that $\mathcal{P}_{x_0} \neq \varnothing$ for each $x_0 \in U$. When the X_i are nonvanishing vector fields tangent to the fibers of the π_i, this is the finite-type hypothesis in [Tao and Wright 2003; Stovall 2011].

Results. Let $U \subseteq \mathbb{R}^d$ be an open set and let $\pi_1, \dots, \pi_k : \overline{U} \to \mathbb{R}^{d-1}$ be smooth submersions (i.e., they have surjective differentials). Letting $\star$ denote the composition of the Hodge-star operator, which maps $(d-1)$-forms to 1-forms, with the natural identification of 1-forms with vectors via the Euclidean metric, we define vector fields

$$X_j := \star(d\pi_j^1 \wedge \cdots \wedge d\pi_j^{d-1}), \quad 1 \leq j \leq k. \tag{2-2}$$

Let a be a continuous function with compact support contained in U.

Fix a d-tuple of words $I_0 = (w_1, \dots, w_d)$ and define the generalized affine arclength

$$\rho = \rho_{I_0} := |\det(X_{w_1}, \dots, X_{w_d})|^{1/(|\deg I_0|_1 - 1)}, \tag{2-3}$$

where $|b|_1$ denotes the ℓ_1-norm. Define a k-linear form $M : [C^0(\mathbb{R}^d)]^k \to \mathbb{C}$ by

$$M(f_1, \dots, f_k) := \int_{\mathbb{R}^d} \prod_{j=1}^k f_j \circ \pi_j(x) \rho(x) a(x)\, dx. \tag{2-4}$$

For $b \in \mathbb{R}^k$ with $|b|_1 > 1$, define

$$\boldsymbol{q}(b) := \frac{b}{|b|_1 - 1}. \tag{2-5}$$

It is easy to check that $\boldsymbol{q}$ equals its own inverse. The following is our main theorem.

Theorem 2.1. *Assume that* $\deg I_0$ *is an extreme point of* $\mathcal{P}_{\operatorname{supp} a}$. *Then, for all* $\boldsymbol{p} \in [1, \infty]^k$ *satisfying* $(p_1^{-1}, \dots, p_k^{-1}) \preceq \boldsymbol{q}(b)$ *and* $p_j^{-1} < q_j(b)$ *when* $(\deg I_0)_j \neq 0$, *we have the estimate*

$$|M(f_1, \dots, f_k)| \lesssim \prod_{j=1}^k \|f_j\|_{L^{p_j}(\mathbb{R}^{d-1})}, \tag{2-6}$$

for all continuous $f_1, \dots, f_k$. *The implicit constant depends on the* π_j, a, $\boldsymbol{p}$ *and* b_0, *but not on the* f_j. *Thus* M *extends to a bounded* k*-linear form on* $\prod_{j=1}^k L^{p_j}(\mathbb{R}^{d-1})$.

The extremality hypothesis seems natural by analogy with the translation-invariant case; it also leads to certain invariants of the weight, as we will discuss below. However, we ultimately prove a more general result, Theorem 6.1, which does not require extremality. (We postpone stating the latter because it requires more notation.)

With the given weight, the above theorem is nearly sharp. Indeed, under the hypotheses and notation above, we have the following.

Proposition 2.2. *Let μ be a nonnegative Borel measure whose support is contained in U, and assume that the bound*

$$M_\mu(\chi_{E_1}, \dots, \chi_{E_k}) := \int_{\mathbb{R}^d} \prod_{j=1}^{k} \chi_{E_j} \circ \pi_j \, d\mu \le A(\mu) \prod_{j=1}^{k} |E_j|^{1/p_j} \tag{2-7}$$

holds for all Borel sets $E_1, \dots, E_k \subseteq \mathbb{R}^{d-1}$ and some constant $A(\mu) < \infty$. If $\mu \not\equiv 0$, $(p_1, \dots, p_k) \in [1, \infty]^k$. If $\sum_j p_j^{-1} > 1$, let $b_p := \boldsymbol{q}(p_1^{-1}, \dots, p_k^{-1})$. Then $\mu(\{x : b_p \notin \mathcal{P}_x\}) = 0$. If in addition b_p is an extreme point of $\mathcal{P}_{\operatorname{supp}\mu}$, then μ is absolutely continuous with respect to Lebesgue measure and its Radon–Nikodym derivative satisfies

$$\frac{d\mu}{dx} \lesssim A(\mu) \sum_{\deg I = b_p} |\lambda_I|^{1/(|b_p|_1 - 1)}. \tag{2-8}$$

The implicit constant in (2-8) may be chosen to depend only on d and p; $A(\mu)$ has the same value in (2-7) and (2-8).

In the translation-invariant case, a similar result is due to Oberlin [2003] (see [Dendrinos and Stovall 2012] for the restricted X-ray transform). The final statement in the proposition only applies in the endpoint case, which is not otherwise addressed in this article. The endpoint version of Theorem 2.1 is known to fail without further assumptions on the X_i than those made here, as can be seen by considering the example of convolution with affine arclength on $\gamma(t) = (t, e^{-1/t} \sin(1/t^k))$, $t > 0$, for k sufficiently large [Sjölin 1974].

The proofs of Theorem 2.1 and Proposition 2.2 will rely on a more general result about smooth vector fields $X_1, \dots, X_k$ on $\mathbb{R}^d$. To state this result, we need some additional terminology.

Let $J \in \{1, \dots, k\}^d$. We define $\deg J$ to be the k-tuple whose i-th entry is the number of occurrences of i in J. If $\alpha \in \mathbb{Z}_0^d$ is a multiindex, we define $\deg_J \alpha$ to be the k-tuple whose i-th entry is $\sum_{\ell : J_\ell = i} \alpha_\ell$. We define

$$\Psi_{x_0}^J(t_1, \dots, t_d) := \exp(t_d X_{J_d}) \circ \cdots \circ \exp(t_1 X_{J_1})(x_0). \tag{2-9}$$

We define another polytope,

$$\widetilde{\mathcal{P}}_{x_0} := \mathcal{P}\big(\{\deg J + \deg_J \alpha : J \in \{1, \dots, k\}^d \text{ and } \alpha \in (\mathbb{Z}_0)^d \text{ satisfy } \partial_t^\alpha \det D\Psi_{x_0}^J(0) \ne 0\}\big).$$

Proposition 2.3. *For each $x_0 \in U$, $\widetilde{\mathcal{P}}_{x_0} = \mathcal{P}_{x_0}$. Furthermore, for each extreme point b_0 of $\mathcal{P}_{x_0}$,*

$$\sum_{\deg I = b_0} |\lambda_I(x_0)| \sim \sum_{J \in \{1,\dots,k\}^d} \sum_{\substack{\alpha \in (\mathbb{Z}_0)^d : \\ \deg J + \deg_J \alpha = b_0}} |\partial_t^\alpha \det D\Psi_{x_0}^J(0)|. \tag{2-10}$$

The implicit constants may be taken to depend only on d and b_0, and in particular may be chosen to be independent of the X_i.

Examples. We take a moment to discuss a few concrete cases where these results apply.

The translation-invariant case. Let $\gamma : \mathbb{R} \to \mathbb{R}^d$ be a smooth map and for $(t, x) \in \mathbb{R}^{1+d}$ define $\pi_1(t, x) = x$, $\pi_2(t, x) = x - \gamma(t)$. Thus the unweighted operator M_0 in (1-1) is essentially convolution with Euclidean arclength measure on γ, paired with a test function.

Using the definition above, $X_1 = \partial_t$, $X_2 = \partial_t + \gamma' \cdot \nabla_x$. If w is any word of length $n \geq 2$ and if the first two letters of w are 1 and 2, $X_w(t, x) = \gamma^{(n)}(t)$. If $d \geq 2$, the Hörmander condition is equivalent to the statement that the torsion of γ does not vanish to infinite order at any point. We note in particular that

$$|\det(X_1, X_2, X_{(1,2)}, \dots, X_{(1,\dots,1,2)})| = |\det(X_1, X_2, X_{(2,1)}, \dots, X_{(2,\dots,2,1)})| = |\det(\gamma', \dots, \gamma^{(d)})|$$

and, if U is any open set, the only extreme points of $\mathcal{P}_U$ (unless $\mathcal{P}_U$ is empty) are

$$\left(\tfrac{1}{2}d(d-1)+1, d\right), \quad \left(d, \tfrac{1}{2}d(d-1)+1\right).$$

Thus the affine arclength in this case is defined in the usual way:

$$\rho(t, x) = |\det(\gamma'(t), \dots, \gamma^{(d)}(t))|^{2/(d(d+1))}.$$

By Theorem 2.1, for any smooth $\gamma : \mathbb{R} \to \mathbb{R}^d$ and any continuous cutoff function a, the convolution operator

$$Tf(x) = \int f(x - \gamma(t)) |\det(\gamma'(t), \dots, \gamma^{(d)}(t))|^{2/(d(d+1))} a(t)\, dt$$

maps $L^p(\mathbb{R}^d)$ into $L^q(\mathbb{R}^d)$ whenever (p^{-1}, q^{-1}) lies in the interior of the trapezoid with vertices as in (1-3) in the case $N = 0$. For general smooth curves this result is new but, as mentioned in the introduction, even stronger results are known in some special cases.

Restricted X-ray transforms. Let $\gamma : \mathbb{R} \to \mathbb{R}^{d-1}$ be a smooth map and, for $(s, t, x) \in \mathbb{R}^{1+1+d-1}$, define $\pi_1(s, t, x) := (t, x)$, $\pi_2(s, t, x) := (s, x - s\gamma(t))$. Then the operator M_0 in (1-1) is the restricted X-ray transform

$$Xf(t, x) = \int_{\mathbb{R}} f(s, x - s\gamma(t)) a(s, t)\, ds,$$

paired with a test function. Using the above definition,

$$X_1 = \partial_s, \quad X_2 = \partial_t + s\gamma'(t) \cdot \nabla_x.$$

If $d \geq 3$, the only $(d+1)$-tuples of words $(w_1, \dots, w_{d+1})$ with $\det(X_{w_1}, \dots, X_{w_{d+1}}) \not\equiv 0$ are, after reordering, those satisfying

$$w_1 = 1, \qquad w_2 = 2, \qquad w_i = (1, 2, \dots, 2), \quad 3 \leq i \leq d+1.$$

Thus the only extreme point of the Newton polytope is $(d, 1 + \frac{1}{2}d(d-1))$, and

$$\rho(s, t, x) = |\det(\gamma'(t), \dots, \gamma^{(d-1)}(t))|^{2/(d(d+1))},$$

which is a power of the usual affine arclength. Theorem 2.1 thus gives a partial generalization of the results of [Dendrinos and Stovall 2012], wherein a sharp strong-type bound for the X-ray transform restricted to polynomial curves with affine arclength was established.

Generalized Loomis–Whitney. Let $\pi_1, \dots, \pi_d : \mathbb{R}^d \to \mathbb{R}^{d-1}$ be smooth submersions. The point $(1, \dots, 1)$ is always extreme or in the exterior of the Newton polytope, so for $\varepsilon > 0$

$$\left| \int_{\mathbb{R}^d} \prod_{i=1}^{d} f_i \circ \pi_i(x) |\det(X_1, \dots, X_d)(x)|^{1/(d-1)} a(x)\, dx \right| \lesssim \prod_{i=1}^{d} \| f_i \|_{L^{d-1+\varepsilon}(\mathbb{R}^{d-1})},$$

with the implicit constant depending on the π_i and ε. In the case when the X_i do span at every point of the support of a, the endpoint estimate was proved in [Bennett et al. 2005]. (The classical Loomis–Whitney inequality is the endpoint estimate when the π_i are linear and $a \equiv 1$.)

Outline. In Section 3, we show that the weights we employ satisfy certain natural invariants; this makes them reasonable generalizations of the usual affine arclength measure. In Section 4, we prove Proposition 2.3 by employing the results of [Street 2011] and a compactness argument; we also use a combinatorial lemma, whose proof is postponed to the Appendix. In Section 5, we prove the optimality result, Proposition 2.2. Finally, in Section 6, we prove a more general result, Theorem 6.1, which implies Theorem 2.1. Our techniques for the proof of the main theorem are essentially those of [Christ 2008; Tao and Wright 2003; Stovall 2011], with some modifications to handle the potential failure of the Hörmander condition.

3. Invariants of the affine arclengths

Let $U, \pi_1, \dots, \pi_k$, and $X_1, \dots, X_k$ be as defined above. For $1 \le j \le k$, let $V_j := \pi_j(U)$. Fix a d-tuple of words I_0, and assume that $b_0 := \deg I_0$ is minimal in the sense that if $\deg I' \prec \deg I_0$, then $\lambda_I \equiv 0$. (This minimality is essential.) Define ρ as in (2-3).

Proposition 3.1. *Let $F : U \to \mathbb{R}^d$ and $G_j : V_j \to \mathbb{R}^{d-1}$, $1 \le j \le k$, be smooth maps. Define $\widetilde{\pi}_j := G_j \circ \pi_j \circ F$ for $1 \le j \le k$, and let $\widetilde{X}_j$, $\tilde{\rho}$ be defined as in (2-2), (2-3), with tildes inserted. Then*

$$\tilde{\rho} = \left(\prod_{j=1}^{k} |(\det DG_j) \circ \pi_j|^{q_j(b_0)} \right) |\det DF| \rho \circ F, \tag{3-1}$$

where $\boldsymbol{q}$ is defined as in (2-5).

In the notation above, let a be a continuous, compactly supported function with $\operatorname{supp} a \subseteq U$, and define

$$\widetilde{M}(f_1, \dots, f_k) := \int_U \prod_{j=1}^{k} f_j \circ \widetilde{\pi}_j(x) \tilde{\rho}(x) a \circ F(x)\, dx.$$

Proposition 3.1 implies that if each G_j is equal to the identity and F is one-to-one, then

$$\widetilde{M}(f_1, \dots, f_k) = M(f_1, \dots, f_k).$$

If we simply assume that F and all of the G_j are one-to-one, the proposition implies that

$$\sup_{f_1,\dots,f_k\neq 0} \frac{\widetilde{M}(f_1,\dots,f_k)}{\prod_{j=1}^k \|f_j\|_{L^{p_j}(\mathbb{R}^{d-1})}} = \sup_{f_1,\dots,f_k\neq 0} \frac{M(f_1,\dots,f_k)}{\prod_{j=1}^k \|f_j\|_{L^{p_j}(\mathbb{R}^{d-1})}} \quad \text{for } (p_1^{-1},\dots,p_k^{-1}) := \boldsymbol{q}(b_0).$$

We stress, however, that our theorem covers only the nonendpoint cases satisfying $(p_1^{-1},\dots,p_k^{-1}) \neq \boldsymbol{q}(b_0)$ and b_0 extreme, so it is not known that either side is finite except in certain cases; see [Bennett et al. 2005; Dendrinos et al. 2009; Dendrinos and Stovall 2012; Oberlin 2002; Stovall 2010].

If we fix j, we may consider the family of curves $\gamma_j^{\underline{x}}(t) := \pi_j(\underline{x},t)$. For any smooth one-to-one function $\phi : \mathbb{R} \to \mathbb{R}$, $(\underline{x},t) \mapsto (\underline{x},\phi(t))$ is also smooth and one-to-one and has Jacobian determinant $\phi'(t)$. Thus we obtain:

Corollary 3.2. *The generalized affine arclength defines a parametrization-invariant measure on each of the curves* $\gamma_j^{\underline{x}} = \pi_j(\underline{x},t)$.

Proof of Proposition 3.1. We will prove the proposition first when the G_j are equal to the identity and then when F is. The general case follows by taking compositions.

In the first case, it suffices by simple approximation arguments to prove the identity when $\det DF \neq 0$. In this case, careful computations reveal that

$$\widetilde{X}_j = (\det DF) F^* X_j,$$

where F^* is the pullback by F, given by

$$F^* X := (DF)^{-1} X \circ F. \tag{3-2}$$

For $1 \le i \le k$, let $Y_i = F^* X_i$. Then, by naturality of the Lie bracket, $Y_w = F^* X_w$ for $w \in \mathcal{W}$. By induction (with base case $w = (j)$), the coordinate expression for the Lie bracket $[X, X'] = X(X') - X'(X)$, and the product rule, for each $w \in \mathcal{W}$,

$$\widetilde{X}_w = (\det DF)^{|\deg w|_1} Y_w + \sum_{\deg w' \prec \deg w} f_{w,w'} Y_{w'}, \tag{3-3}$$

where the $f_{w,w'}$ are smooth functions.

By (3-3), (3-2) and our minimality assumption,

$$\begin{aligned}\det(\widetilde{X}_{w_1},\dots,\widetilde{X}_{w_d}) &= (\det DF)^{|b_0|_1} \det(Y_{w_1},\dots,Y_{w_d}) + \sum_{b' \prec b_0} \sum_{\deg I' = b'} f_{I,I'} \det(Y_{w'_1},\dots,Y_{w'_d}) \\ &= (\det DF)^{|b_0|_1 - 1} \det(X_{w_1},\dots,X_{w_d}) \circ F + 0.\end{aligned}$$

This completes the proof in the first case.

In the second case, when F is the identity, it is easy to compute $\widetilde{X}_j = [(\det DG_j) \circ \pi_j] X_j$, and it can be shown using the product rule and minimality of b_0 (as above) that

$$\det(\widetilde{X}_{w_1},\dots,\widetilde{X}_{w_d}) = \prod_{j=1}^k [(\det DG_j) \circ \pi_j]^{b_0^j} \det(X_{w_1},\dots,X_{w_d}),$$

which implies (3-1). □

4. Equivalence of the two polytopes: the proof of Proposition 2.3

Fix a point $b_0 \in [0, \infty)^k$. We say that an object (such as a constant, vector, or set) is admissible if it may be chosen from a finite collection, depending only on b_0 and d, of such objects. In particular, all implicit constants in this section will be admissible.

The proof of Proposition 2.3 will rely on a compactness result about polytopes with vertices in $\mathbb{Z}_0^k$:

Proposition 4.1. *Let $\mathcal{B} \subseteq \mathbb{Z}_0^k$ and assume that $b_0 \notin \mathcal{P}(\mathcal{B})$. There exist*

(i) *$\varepsilon > 0$ and $v_0 \in (\varepsilon, 1]^k$ such that $v_0 \cdot b_0 + \varepsilon < v_0 \cdot p$ for every $p \in \mathcal{P}(\mathcal{B})$, and*

(ii) *a finite set $\mathcal{A} \subseteq \mathbb{Z}_0^k$ such that $b_0 \notin \mathcal{P}(\mathcal{A})$ and $\mathcal{P}(\mathcal{B}) \subseteq \mathcal{P}(\mathcal{A})$.*

Moreover, ε, v_0, $\mathcal{A}$ are admissible.

Note that this also applies when b_0 is an extreme point of $\mathcal{P}(\mathcal{B})$, since in this case $b_0 \notin \mathcal{P}(\mathcal{B} \setminus \{b_0\})$.

Assuming the validity of Proposition 4.1 for now (it will be proved in the Appendix), we devote the remainder of the section to the proof of Proposition 2.3.

We may of course assume that $x_0 = 0$ and that U is a bounded neighborhood of 0. Furthermore, we may assume that $k > d$ and $X_i = \partial_i$, $1 \le i \le d$. Indeed, if the proposition holds under this assumption, it holds for $\partial_1, \dots, \partial_d, X_1, \dots, X_k$, with $k + d$ replacing k. We may then transfer the result back to $X_1, \dots, X_k$ by restricting to those $b \in [0, \infty)^{k+d}$ with $b^1 = \cdots = b^d = 0$. By this assumption, $\mathcal{P}_0 \ne \varnothing$, and it suffices to prove that if b_0 is an extreme point of $\mathcal{P}_{x_0}$ then (2-10) holds, and if $b_0 \notin \mathcal{P}_{x_0}$ then $b_0 \notin \widetilde{\mathcal{P}}_{x_0}$.

We begin with the case when b_0 is an extreme point of $\mathcal{P}_0$. Fix a neighborhood V of 0, sufficiently small for later purposes, with $\overline{V} \subseteq U$. Choose a d-tuple $I_0 = (w_1, \dots, w_d) \in \mathcal{W}^d$ with $\deg I_0 = b_0$ and

$$|\lambda_{I_0}(0)| = \max_{\deg I = b_0} |\lambda_I(0)|. \tag{4-1}$$

(Note that I_0 is admissible, since only finitely many d-tuples of words give rise to this degree.) By smoothness of the X_j, we may assume that V is so small that

$$\tfrac{1}{4}|\lambda_{I_0}(0)| \le \tfrac{1}{2} \max_{\deg I = b_0} |\lambda_I(x)| \le |\lambda_{I_0}(x)| \le 2|\lambda_{I_0}(0)|, \quad \text{for all } x \in V.$$

By Proposition 4.1, we may choose admissible $v_0 = (v_0^1, \dots, v_0^k) \in (0, 1]^k$ and $\varepsilon > 0$ such that $v_0 \cdot b_0 + \varepsilon < v_0 \cdot p$ for every $p \in \mathcal{P}_0 \cap \mathbb{Z}_0^k \setminus \{b_0\}$.

Lemma 4.2. *For each $m \ge 1$, there exists $\delta(m) > 0$, depending on m, b_0, $X_1, \dots, X_k$ such that, for all $0 < \delta < \delta(m)$, the map*

$$\Phi^\delta(y_1, \dots, y_d) := \exp\bigl(y_1 \delta^{v_0 \cdot \deg w_1} X_{w_1} + \cdots + y_d \delta^{v_0 \cdot w_d} X_{w_d}\bigr)(0) \tag{4-2}$$

and the pullbacks

$$Y_j^\delta := (\Phi^\delta)^* \delta^{v_0^j} X_j = (D\Phi^\delta)^{-1} \delta^{v_0^j} X_j \circ \Phi^\delta \tag{4-3}$$

satisfy these properties: Φ^δ is a diffeomorphism of the unit ball $B(1)$ onto a neighborhood of 0 in V,

$$|\det D\Phi^\delta(y)| \sim \delta^{v_0 \cdot b_0} |\lambda_{I_0}(0)|, \quad y \in B(1), \tag{4-4}$$

$$\|Y_j^\delta\|_{C^m(B(1))} \lesssim 1, \quad 1 \le j \le k, \tag{4-5}$$

$$|\det(Y_{w_1}^\delta(y), \dots, Y_{w_d}^\delta(y))| \sim 1, \quad y \in B(1). \tag{4-6}$$

Proof. Recall that $\mathcal{W}$ is the set of all words. Let

$$\mathcal{W}_0 := \{w \in \mathcal{W} : \deg w \cdot v_0 \le d\} \quad \text{and} \quad \mathcal{W}_1 := \{w \in \mathcal{W} : d < \deg w \cdot v_0 \le 2d\}. \tag{4-7}$$

Since v_0 is an admissible element of $(0,1]^k$, these are admissible, finite sets, and $\mathcal{W}_0$ contains the one-letter words $(1), (2), \dots, (k)$. Furthermore, $\mathcal{W}_0$ contains b_0 since our choice of v_0 and assumption that $X_j = \partial_j$ for $1 \le j \le d$ imply that

$$v_0 \cdot b_0 \le v_0 \cdot (1, \dots, 1, 0, \dots, 0) = (v_0)_1 + \dots + (v_0)_d \le d.$$

The vector fields X_w are all smooth, $\mathcal{W}_0 \cup \mathcal{W}_1$ is a finite set, and each coefficient of v_0 is positive. Thus for each $M \ge 0$, for all sufficiently small $\delta > 0$ and all $w \in \mathcal{W}_0 \cup \mathcal{W}_1$,

$$\|\delta^{v_0 \cdot \deg w} X_w\|_{C^0(V)} \le \frac{1}{d} \operatorname{dist}(0, \partial V), \quad \|\delta^{v_0 \cdot \deg w} X_w\|_{C^M(V)} \le 1. \tag{4-8}$$

Additionally, by our choice of v_0 and ε,

$$|\delta^{v_0 \cdot \deg I} \lambda_I(0)| < \delta^\varepsilon |\delta^{v_0 \cdot b_0} \lambda_{I_0}(0)|, \quad I \in (\mathcal{W}_0 \cup \mathcal{W}_1)^d, \quad \deg I \ne b_0. \tag{4-9}$$

By the Jacobi identity, if $w, w' \in \mathcal{W}_0$,

$$[\delta^{v_0 \cdot \deg w} X_w, \delta^{v_0 \cdot \deg w'} X_{w'}] = \sum_{\deg \widetilde{w} = \deg w + \deg w'} C_{w,w'}^{\widetilde{w}} (\delta^{v_0 \cdot \deg \widetilde{w}} X_{\widetilde{w}}), \tag{4-10}$$

for constants $C_{w,w'}^{\widetilde{w}}$ that are admissible because $\mathcal{W}_0$ is. If $v_0 \cdot (\deg w + \deg w') \le d$, each $\widetilde{w}$ in the sum is an element of $\mathcal{W}_0$. If not, each $\widetilde{w}$ is in $\mathcal{W}_1$, and we can expand

$$\delta^{v_0 \cdot \deg \widetilde{w}} X_{\widetilde{w}} = \sum_{j=1}^d \delta^{v_0 \cdot \deg \widetilde{w}} X_{\widetilde{w}}^j \partial_j = \sum_{j=1}^d (\delta^{v_0 \cdot \deg \widetilde{w} - v_0^j} X_{\widetilde{w}}^j)(\delta^{v_0^j} X_j).$$

Note that $v_0 \cdot \deg \widetilde{w} - v_0^j > 0$ for $\widetilde{w} \in \mathcal{W}_1$. Using (4-10) to put the pieces back together, for sufficiently small $\delta > 0$ and any $w, w' \in \mathcal{W}_0$,

$$[\delta^{v_0 \cdot \deg w} X_w, \delta^{v_0 \cdot \deg w'} X_{w'}] = \sum_{\widetilde{w} \in \mathcal{W}_0} c_{w,w'}^{\widetilde{w},\delta} \delta^{v_0 \cdot \deg \widetilde{w}} X_{\widetilde{w}},$$

with

$$\|c_{w,w'}^{\widetilde{w},\delta}\|_{C^M(V)} \lesssim 1. \tag{4-11}$$

The conclusion of the lemma is now a direct application of [Street 2011, Theorem 5.3], whose (lengthy) proof uses compactness arguments and Gronwall's inequality, among other tools. For the convenience of the reader wishing to verify this, we provide a short dictionary to translate the notation. Let M be sufficiently large (depending on m, d, I_0) and choose $\delta(m) > 0$ sufficiently small that (4-8), (4-9) and (4-11) all hold. Then the terms

$$\{X_1, \dots, X_q\},\ \{d_1, \dots, d_q\},\ \mathscr{A},\ (\delta^d X),\ n_0(x, \delta)$$

from [Street 2011] are, in our notation,

$$\{X_w\}_{w\in\mathscr{W}_0},\ \{\deg w\}_{w\in\mathscr{W}_0},\ \{(\delta^{v_0^1}, \dots, \delta^{v_0^k}) : 0 < \delta \le \delta(m)\},\ (\delta^{v_0\cdot\deg w} X_w)_{w\in\mathscr{W}_0},\ d.$$

A priori, the results of [Street 2011] only guarantee that for each $m \ge 0$ there exists an admissible constant $\eta > 0$ such that the conclusions hold on $B(\eta)$. We want $\eta = 1$, but this is just a matter of rescaling. Define

$$D^\eta_{v_0,I_0}(t_1, \dots, t_d) := (\eta^{v_0\cdot\deg w_1} t_1, \dots, \eta^{v_0\cdot\deg w_d} t_d);$$

then

$$\Phi^{\eta\delta} = \Phi^\delta \circ D^\eta_{v_0,I_0}, \qquad Y^{\eta\delta}_w = (D^\eta_{v_0,I_0})^{-1} \eta^{v_0\cdot\deg w} Y_w \circ D^\eta_{v_0,I_0}.$$

Thus the lemma holds with a slightly smaller (η times the original) value of $\delta(M)$. □

Lemma 4.3. *Let m be a sufficiently large admissible integer, and let $Y_1, \dots, Y_k$ be vector fields with the properties that*

$$\|Y_j\|_{C^m(B(1))} \lesssim 1, \tag{4-12}$$

$$|\det(Y_{w_1}, \dots, Y_{w_d})| \sim 1 \quad \textit{on } B(1); \tag{4-13}$$

here we recall that $(w_1, \dots, w_d) = I_0$. For $J \in \{1, \dots, k\}^d$, define

$$\widetilde{\Psi}^J(t_1, \dots, t_d) := e^{t_d Y_{J_d}} \circ \cdots \circ e^{t_1 Y_{J_1}}(0).$$

Then

$$\max_{J\in\{1,\dots,k\}^d} \|\det D\widetilde{\Psi}^J\|_{C^0(B(c_0))} \sim 1 \tag{4-14}$$

for some admissible constant $c_0 > 0$; in particular, $\widetilde{\Psi}^J$ is defined on the ball $B(c_0)$.

Proof. There are similar results in [Christ 2008; Christ et al. 1999; Stovall 2011; Tao and Wright 2003], but without the uniformity, so we give a complete proof.

The upper bound $\|\det D\widetilde{\Psi}^J\|_{C^0(B(c_0))} \sim 1$ is an immediate consequence of (4-12) for $m \ge 2$, by Picard's existence theorem.

For the lower bound, we first show that if $m \ge |b_0|_1 + 2$, the left side of (4-14) is nonzero. For $1 \le i \le d$ and $J \in \{1, \dots, k\}^i$, define

$$\widetilde{\Psi}^J_i(t_1, \dots, t_i) := e^{t_i Y_{J_i}} \circ \cdots \circ e^{t_1 Y_{J_1}}(0);$$

$\widetilde{\Psi}^J_i \in C^{m+1}(B(c_0))$ for admissible $c_0 > 0$ by standard ODE existence results. Supposing that the left side of (4-14) is zero, there exists some minimal $i \in \{0, \dots, d-1\}$ such that

$$\max_{J\in\{1,\dots,k\}^{i+1}} \|\partial_{t_1}\widetilde{\Psi}^J_{i+1} \wedge \cdots \wedge \partial_{t_{i+1}}\widetilde{\Psi}^J_{i+1}\|_{C^0(B(c_0))} = 0.$$

By (4-13), the Y_j cannot all vanish at zero, so this i is at least 1.

By minimality of i, there exist $J \in \{1, \dots, k\}^i$, $t_0 \in \mathbb{R}^i$ with $|t_0| < c_0$, and $\varepsilon > 0$ such that $\widetilde{\Psi}_i^J$ is an injective immersion on $\{t \in \mathbb{R}^i : |t - t_0| < \varepsilon\} =: B_{t_0}(\varepsilon)$. Our assumption and the definition of exponentiation imply that, for all $1 \le j \le k$ and $(t_1, \dots, t_i) \in B(c_0)$,

$$0 = \big(\partial_{t_1} \widetilde{\Psi}_{i+1}^{(J,j)} \wedge \cdots \wedge \partial_{t_{i+1}} \widetilde{\Psi}_{i+1}^{(J,j)}\big)(t_1, \dots, t_i, 0) = \big(\partial_{t_1} \widetilde{\Psi}_i^J \wedge \cdots \wedge \partial_{t_i} \widetilde{\Psi}_i^J\big)(t_1, \dots, t_i) \wedge Y_j\big(\widetilde{\Psi}_i^J(t_1, \dots, t_i)\big).$$

Therefore $Y_1, \dots, Y_k$ are tangent to $\widetilde{\Psi}_i^J(B_{c_0}(\varepsilon))$, as must be any Lie brackets that are defined, in particular all of those up to order m. Since $m \ge |b_0|_1$, this contradicts (4-13). Tracing back, we see that we must have $\det \widetilde{\Psi}^J \not\equiv 0$ on $B(c_0)$ for some $J \in \{1, \dots, k\}^d$.

Now we prove that there is a uniform lower bound for $m := |b_0|_1 + 3$. If not, there exists a sequence $\big(Y_1^{(n)}, \dots, Y_k^{(n)}\big)$ satisfying hypotheses (4-12) and (4-13), but with

$$\max_{J \in \{1, \dots, k\}^d} \|\det D\widetilde{\Psi}^{(n),J}\|_{C^0(B(c_0))} \to 0,$$

where $\widetilde{\Psi}^{(n),J}(t_1, \dots, t_d) := \exp\big(t_d Y_{J_d}^{(n)}\big) \circ \cdots \circ \exp\big(t_1 Y_{J_1}^{(n)}\big)(0)$. By Arzelà–Ascoli, after passing to a subsequence, each $\big(Y_j^{(n)}\big)$ converges in $C^{m-1}(B(1))$ to some vector field Y_j. Thus for $|\deg w|_1 \le m - 1$, $Y_w^{(n)} \to Y_w$, and by standard ODE results, for each J, the sequence $(\widetilde{\Psi}^{(n),J})$ converges to $\widetilde{\Psi}^J$ in $C^m(B(c_0))$. So $Y_1, \dots, Y_k$ satisfy hypotheses (4-12) and (4-13) (the former with $m = |b_0|_1 + 2$), but $\det D\widetilde{\Psi}^J \equiv 0$ on $B(c_0)$ for all $J \in \{1, \dots, k\}^d$. This is impossible, so the lower bound in (4-14) must hold. □

We return to a consideration of the vector fields $X_1, \dots, X_k$ in the next lemma, where we transfer the inequality in Lemma 4.3 from $\widetilde{\Psi}^J$ to Ψ^J.

Lemma 4.4. *For $J \in \{1, \dots, k\}^d$ and $\alpha \in \mathbb{Z}_0^d$, if $v_0 \cdot (\deg J + \deg_J \alpha) < v_0 \cdot b_0$, then $\partial^\alpha \det D\Psi^J(0) = 0$. Furthermore,*

$$\sum_{J \in \{1, \dots, k\}^d} \sum_{\substack{\alpha \in (\mathbb{Z}_0)^d \\ v_0 \cdot (\deg J + \deg_J \alpha) = v_0 \cdot b_0}} |\partial^\alpha \det D\Psi^J(0)| \sim |\lambda_{I_0}(0)|. \tag{4-15}$$

Proof. For $J \in \{1, \dots, k\}^d$, let

$$\Psi^{J,\delta} := \Psi^J \circ D_J^\delta, \quad \text{where } D_J^\delta(t_1, \dots, t_d) := (\delta^{v_0^{J_1}} t_1, \dots, \delta^{v_0^{J_d}} t_d),$$
$$\widetilde{\Psi}^{J,\delta} := e^{t_d Y_{J_d}^\delta} \circ \cdots \circ e^{t_1 Y_{J_1}^\delta}(0),$$

with $Y_1^\delta, \dots, Y_k^\delta$ as in (4-3). By naturality of exponentiation, $\Psi^{J,\delta} = \Phi^\delta \circ \widetilde{\Psi}^{J,\delta}$, where Φ^δ is defined in (4-2). Hence by Lemmas 4.2 and 4.3,

$$\max_{J \in \{1, \dots, k\}^d} \|\det D\Psi^{J,\delta}\|_{C^0(B(c_0))} \sim \delta^{v_0 \cdot b_0} |\lambda_{I_0}(0)|, \quad 0 < \delta < \delta(m), \tag{4-16}$$

where $m = m(b_0, d)$ is sufficiently large and $\delta(m)$ is the (inadmissible) constant from Lemma 4.2. As we will see, the lemma follows by sending $\delta \searrow 0$.

Let $M = M(b_0, d)$ be a sufficiently large integer, let $J \in \{1, \dots, k\}^d$, and let $P^{J,\delta}$ be the degree M Taylor polynomial of $\det D\Psi^{J,\delta}$, centered at 0. Then

$$\begin{aligned}\|P^{J,\delta} - \det D\Psi^{J,\delta}\|_{C^0(B(c_0))} &= \Big(\frac{\delta}{\delta(m)}\Big)^{v_0\cdot\deg J} \|P^{J,\delta(m)} - \det D\Psi^{J,\delta(m)}\|_{C^0(D^{\delta/\delta(m)}B(c_0))} \\ &\lesssim i\Big(\frac{\delta}{\delta(m)}\Big)^{v_0\cdot\deg J+(M+1)\min_i v_0^i} \|\det D\Psi^{J,\delta(m)}\|_{C^0(D^{\delta/\delta(m)}B(c_0))} \\ &\lesssim \Big(\frac{\delta}{\delta(m)}\Big)^{v_0\cdot\deg J+(M+1)\min_i v_0^i}, \end{aligned} \tag{4-17}$$

where the first inequality is by Taylor's theorem and admissibility of M, and the second is from (4-8), if m is sufficiently large, depending on M. Motivated by this inequality, we assume that $v_0 \cdot b_0 < M \min_i v_0^i$.

By the equivalence of all norms on the space of polynomials of d variables of degree at most M,

$$\|P^{J,\delta}\|_{C^0(B(c_0))} \sim \sum_{|\alpha|_1\le M} |\partial^\alpha P^{J,\delta}(0)| = \sum_{|\alpha|_1\le M} \delta^{v_0\cdot(\deg J+\deg_J\alpha)}|\partial^\alpha \det D\Psi^J(0)|. \tag{4-18}$$

If $\alpha \in \mathbb{Z}_0^d$ and $v_0\cdot(\deg J + \deg_J \alpha) \le v_0 \cdot b_0$, then $|\alpha|_1 \le (v_0\cdot\deg_J\alpha)/\min_i v_0^i \le M$, and

$$\begin{aligned}\delta^{v_0\cdot(\deg J+\deg_J\alpha)}|\partial^\alpha \det D\Psi^J(0)| = |\partial^\alpha P^{J,\delta}(0)| &\lesssim \|P^{J,\delta}\|_{C^0(B(c_0))} \\ &\lesssim \|\det D\Psi^{J,\delta}\|_{C^0(B(c_0))} + \Big(\frac{\delta}{\delta(m)}\Big)^{v_0\cdot\deg J+(M+1)\min_i v_0^i} \\ &\lesssim \delta^{v_0\cdot b_0}|\lambda_{I_0}(0)| + \Big(\frac{\delta}{\delta(m)}\Big)^{v_0\cdot\deg J+(M+1)\min_i v_0^i}.\end{aligned}$$

Sending $\delta \searrow 0$, we see that

$$\partial^\alpha \det D\Psi^J(0) = 0 \quad \text{whenever } v_0\cdot(\deg J + \deg_J\alpha) < v_0\cdot b_0, \tag{4-19}$$

$$|\partial^\alpha \det D\Psi^J(0)| \lesssim |\lambda_{I_0}(0)| \quad \text{if } v_0\cdot(\deg J+\deg_J\alpha) = v_0\cdot b_0. \tag{4-20}$$

Now for the lower bound. By (4-16) and the fact that there are only finitely many choices for J, there exist $J \in \{1, \dots, k\}^d$ and a sequence $\delta_n \searrow 0$ such that

$$\|\det D\Psi^{J,\delta_n}\|_{C^0(B(c_0))} \gtrsim \delta_n^{v_0\cdot b_0}|\lambda_{I_0}(0)|. \tag{4-21}$$

Since $M\min_i v_0^i > v_0\cdot b_0$ and $\lambda_{I_0}(0) \neq 0$, (4-21), (4-17) and (4-18) imply that for δ_n sufficiently (inadmissibly) small,

$$\delta_n^{v_0\cdot b_0}|\lambda_{I_0}(0)| \lesssim \|P^{J,\delta_n}\|_{C^0(B(c_0))} \lesssim \sum_{|\alpha|_1\le M} \delta_n^{v_0\cdot(\deg J+\deg_J\alpha)}|\partial^\alpha \det D\Psi^J(0)|.$$

Applying (4-19) and letting $n \to \infty$,

$$|\lambda_{I_0}(0)| \lesssim \sum_{v_0\cdot(\deg J+\deg_J\alpha)=v_0\cdot b_0} |\partial^\alpha \det D\Psi^J(0)|.$$

This completes the proof of (4-15), and thus of Lemma 4.4. □

By our choice of v_0, (4-15) is just (2-10), so to complete the proof of Proposition 2.3, it suffices to prove the following.

Lemma 4.5. $\mathcal{P}_0 = \widetilde{\mathcal{P}}_0$.

Proof. By (2-10), $\widetilde{\mathcal{P}}_0$ contains the extreme points of $\mathcal{P}_0$, so $\mathcal{P}_0 \subseteq \widetilde{\mathcal{P}}_0$. Now suppose that $b_0 \notin \mathcal{P}_0$. Then there exist $v_0 \in (0, 1]^k$ and $\varepsilon > 0$ such that $v_0 \cdot b_0 + \varepsilon < v_0 \cdot p$, for all $p \in \mathcal{P}_0$. At least one extreme point b of $\mathcal{P}_0$ satisfies $v_0 \cdot b = \max_{p \in \mathcal{P}_0} v_0 \cdot p$; perturbing v_0 slightly, we may assume that there exists $b_1 \in \mathcal{P}_0$ such that

$$v_0 \cdot b_0 < v_0 \cdot b_1 < v_0 \cdot p, \quad \text{for all } p \in \mathcal{P}_0 \text{ with } p \neq b_1.$$

By Lemma 4.4, $\partial^\alpha \det D\Psi^J(0) = 0$ whenever $(\deg J + \deg_J \alpha) \cdot v_0 < v_0 \cdot b_1$, so $b_0 \notin \widetilde{\mathcal{P}}_0$. Thus $\mathcal{P}_0 \subseteq \widetilde{\mathcal{P}}_0$, and we are done. □

Remarks. A more direct argument, using the Baker–Campbell–Hausdorff formula, should be possible, but the author has not been able to carry this out. Let $k = d$ and consider vector fields $X_1, \dots, X_d$. Using the approximation $\exp(tX) = \sum_{n=0}^N (t^n/n!) X^{n-1}(X) + O(|t|^N)$ [Christ et al. 1999], the formula for the Lie derivative of a determinant of d vector fields, and somewhat tedious computations, one can show that

$$\partial_t^\alpha|_{t=0} \det D_t\bigl(e^{t_d X_d} \circ \cdots \circ e^{t_1 X_1}\bigr)(x_0) = \pm \sum_{w_1,\dots,w_d}^{*} \prod_{i=1}^d \binom{\alpha_i}{\deg_i w_{i+1}, \dots, \deg_i w_d} \det(X_{w_1}, X_{w_2}, \dots, X_{w_d}),$$

where $*$ indicates the sum is over those words $w_i = (w_i^1, \dots, w_i^{n_i})$ that satisfy $\sum_i \deg w_i = \alpha + (1, \dots, 1)$ and $w_i^1 = i > w_i^2 \geq \cdots \geq w_i^{n_i}$ (in particular, $w_1 = (1)$). Replacing X_i above with X_{J_i} gives an alternative proof that the right (Jacobian) side of (2-10) is bounded by the left (determinant) side, but using this formula to bound the left of (2-10) by the right seems nontrivial.

The estimate (2-10) may fail if b is not extreme (even if it is minimal). To see this, let $\gamma(t) := (t, \dots, t^d)$ and define $X_0 := \partial_t$, $X_i := \partial_t - \gamma'(t) \cdot \nabla_x$, $1 \leq i \leq d$, and take $b := (1 + \frac{1}{2}d(d-1), 1, \dots, 1)$. In this case, the only I with $\deg I = b$ and $\lambda_I \not\equiv 0$ are those of the form

$$I = ((1), (j_1), (1, j_2), \dots, (1, \dots, 1, j_d)),$$

with the j_i distinct. Thus the left side of (2-10) is a nonzero dimensional constant. On the other hand, simple combinatorial considerations show that the right side of (2-10) must be identically zero.

Less uniform versions of (2-10) may be found in [Christ et al. 1999; Stovall 2011; Tao and Wright 2003]. Let $X_1, \dots, X_k$ be smooth vector fields and assume that there exists a d-tuple $I = (w_1, \dots, w_d)$ such that $|\lambda_I| \geq 1$ on U. Let $\delta_1, \dots, \delta_k$ be scalars satisfying the smallness and weak comparability conditions

$$\delta_i \leq K, \quad \delta_i \leq K\delta_j^\varepsilon, \quad 1 \leq i, j \leq k.$$

Then [Tao and Wright 2003; Stovall 2011] prove that there exist $N \geq |\deg I|_1$ and N' (depending on I) such that

$$\sum_{|\deg I|_1 \leq N} \Bigl(\prod_{i=1}^k \delta_i^{(\deg I)_i}\Bigr) |\lambda_I(x_0)| \sim \sum_{J \in \{1,\dots,k\}^d} \sum_{\substack{\alpha \in (\mathbb{Z}_0)^d \\ \deg J + \deg_J \alpha \leq N'}} \Bigl(\prod_{i=1}^k \delta_i^{\deg J + \deg_J \alpha}\Bigr) |\partial_t^\alpha \det D_t \Psi_{x_0}^J(0)|, \quad x_0 \in U,$$

with inadmissible implicit constants. It is not shown, however, how to remove the dependence of the implicit constant on ε, K, or the X_i, or, in particular, how to remove the assumption that the Hörmander condition holds uniformly.

5. Proof of the optimality result: Proposition 2.2

The entirety of this section will be devoted to the proof of Proposition 2.2. It suffices to prove the proposition when $\operatorname{supp}\mu \subseteq V$, and V and W are bounded open subsets of U with $\overline{V} \subseteq W$, $\overline{W} \subseteq U$. (Recall that U is the set on which the π_i, and hence the X_i, are defined.) By (2-7) with $E_i = \pi_i(V)$ for $1 \le i \le k$, $\mu(V) < \infty$.

Throughout this section, an object will be said to be admissible if it depends (or it is taken from a finite set depending) only on d and $p = (p_1, \dots, p_k)$. All implicit constants will be admissible. The constant $A(\mu)$ will always represent precisely the quantity in (2-7), and in particular will not be allowed to change from line to line.

First suppose that $p_{j_0} < 1$. Without loss of generality, $j_0 = 1$. We may cover $\pi_1(V)$ by $C_{V,\pi_1}\varepsilon^{-(d-1)}$ balls B_i of radius ε, so

$$\mu(V) \le \sum_i \int \chi_{B_1} \circ \pi_1 \prod_{j=2}^{k} \chi_{\pi_j(V)} \circ \pi_j \, d\mu \le A(\mu) \sum_i |B_1|^{1/p_1} \prod_{j=2}^{k} |\pi_j(V)|^{1/p_j}$$
$$\le C(\mu, d, p, V, \pi_2, \dots, \pi_k)\varepsilon^{(d-1)(1/p_1 - 1)}.$$

Letting $\varepsilon \to 0$, we see that $\mu \equiv 0$.

We now turn to the case when $\sum_j p_j^{-1} > 1$. Replacing $\{X_1, \dots, X_k\}$ with $\{\partial_1, \dots, \partial_d, X_1, \dots, X_k\}$, $(p_1, \dots, p_k)$ with $(\infty, \dots, \infty, p_1, \dots, p_k)$, and k with $d + k$ if necessary, we may assume that $X_i = \partial_i$, $1 \le i \le d$, without affecting either of the sets

$$Z := \{x \in V : b_p \notin \mathcal{P}_x\}, \quad \Omega := \{x \in V : b_p \text{ is an extreme point of } \mathcal{P}_x\},$$

or the quantity on the right of (2-8).

The proposition will follow from the next two lemmas.

Lemma 5.1. $\mu(Z) = 0$.

Lemma 5.2. *If* $\rho := \sum_{\deg I = b_p} |\lambda_I|^{1/(|b_p|_1 - 1)}$ *and*

$$\Omega_n := \{x \in \Omega : 2^n \le \rho(x) \le 2^{n+1}\}, \quad n \in \mathbb{Z},$$

then $\mu(\Omega') \lesssim A(\mu) 2^n |\Omega'|$ *for any Borel set* $\Omega' \subseteq \Omega_n$.

Proof of Lemma 5.1. By Proposition 4.1, there exist admissible, finite sets $\mathcal{A}_i$, $i = 1, \dots, C_{p,d}$ such that $b_p \notin \mathcal{P}(\mathcal{A}_i)$ for any i and, for each $x \in Z$, there exists an i such that $\mathcal{P}_x \subseteq \mathcal{P}(\mathcal{A}_i)$. For the remainder of the proof of the lemma, we let $\mathcal{A} = \mathcal{A}_i$ be fixed and define

$$Z' := \{x \in Z : \mathcal{P}_x \subseteq \mathcal{P}(\mathcal{A})\}.$$

It suffices to show that $\mu(Z') = 0$.

Choose admissible $\varepsilon > 0$ and $v \in (\varepsilon, 1]^k$ such that

$$v \cdot b_p + \varepsilon < v \cdot b, \quad \text{for } b \in \mathcal{P}(\mathcal{A}).$$

Define

$$\mathcal{W}_0 := \{w \in \mathcal{W} : v \cdot \deg w \le d\}.$$

Let $N = N_{d,p}$ be an integer whose size will be determined in a moment and which is, in particular, larger than d/ε. Since $\overline{W}$ is compact and contained in U, the X_i are smooth on U and $\{X_w : w \in \mathcal{W}_0\}$ contains the coordinate vector fields, it follows that there exists $\delta_0 > 0$, depending on the π_i, p and W, such that for all $0 < \delta \le \delta_0$, $I \in \mathcal{W}_0^d$ satisfying $\deg I \in \mathcal{P}(\mathcal{A})$, $x \in W$, and $w, w' \in \mathcal{W}_0$,

$$|\delta^{v\cdot \deg I}\lambda_I(x)| < \delta^{\varepsilon}\delta^{v\cdot b_p}, \tag{5-1}$$

$$\|\delta^{v\cdot\deg w}X_w\|_{C^0(W)} \le \frac{1}{d}\operatorname{dist}(V, \partial W), \quad \|\delta^{v\cdot\deg w}X_w\|_{C^N(W)} \le 1, \tag{5-2}$$

$$[\delta^{v\cdot\deg w}X_w, \delta^{v\cdot\deg w'}X_{w'}] = \sum_{\widetilde{w}\in\mathcal{W}_0} c_{w,w'}^{\widetilde{w},\delta}\delta^{v\cdot\deg\widetilde{w}}X_{\widetilde{w}},$$

with

$$\|c_{w,w'}^{\widetilde{w},\delta}\|_{C^N(W)} \lesssim 1.$$

We omit the details since they are essentially the same as arguments found in the proof of Lemma 4.2.

For $x \in Z'$ and $0 < \delta \le \delta_0$, choose $I_x^\delta \in \mathcal{W}_0^d$ such that

$$\delta^{v\cdot\deg I_x^\delta}|\lambda_{I_x^\delta}(x)| = \max_{I\in\mathcal{W}_0^d}\delta^{v\cdot\deg I}|\lambda_I(x)|.$$

Let

$$\begin{aligned}\Phi_x^\delta(t_1, \dots, t_d) &:= \exp\bigl(t_1\delta^{v\cdot\deg w_1}X_{w_1} + \dots + t_d\delta^{v\cdot\deg w_d}X_{w_d}\bigr)(x),\\ B(x,\delta) &:= \{\Phi_x^\delta(t) : |t| < 1\},\end{aligned} \tag{5-3}$$

where $I_x^\delta = (w_1, \dots, w_d)$. Then $B(x,\delta) \subseteq W$ by (5-2) and the fact that $x \in Z' \subseteq V$.

By the results of [Street 2011], provided $N = N_{d,p}$ is sufficiently large, these balls are doubling in the sense that $|B(x,\delta)| \sim |B(x, 2\delta)|$, for all $x \in Z'$ and $0 < \delta \le \delta_0$. (Here we are using the fact that ε and v are admissible.) Furthermore, for $x \in V$,

$$|B(x,\delta)| \sim \delta^{v\cdot\deg I_x^\delta}|\lambda_{I_x^\delta}(x)|, \tag{5-4}$$

$$\exp(tX_i)(y) \in B(x, C\delta) \quad \text{whenever } y \in B(x,\delta),\ |t| < \delta^{v^i}, \tag{5-5}$$

where $C = C_{d,p}$. By the doubling property, the change of variables formula and (5-5), if $\sigma_i : \pi_i(W) \to \mathbb{R}^d$ is any smooth section of π_i (i.e., $\sigma_i \circ \pi_i$ is the identity) with $\sigma_i(\pi_i(V)) \subseteq W$, then

$$\begin{aligned}|B(x,\delta)| \sim |B(x, C\delta)| &= \int_{\pi_i(B(x,C\delta))}\int_{\mathbb{R}} \chi_{B(x,C\delta)}(e^{tX_i}(\sigma_i(y)))\,dt\,dy\\ &\ge \int_{\pi_i(B(x,\delta/2))}\int_{\mathbb{R}} \chi_{B(x,C\delta)}(e^{tX_i}(\sigma_i(y)))\,dt\,dy \gtrsim \delta^{v^i}|\pi_i(B(x,\delta))|.\end{aligned} \tag{5-6}$$

By the Vitali covering lemma (as stated in [Stein 1993], for instance), for each $0 < \delta \le \delta_0$ there exists a collection of points $\{x_j\}_{j=1}^{M_\delta} \subseteq Z'$ such that $Z' \subseteq \bigcup_{j=1}^{M_\delta} B(x_j, \delta)$ and such that the balls $B(x_j, C^{-1}\delta)$ are pairwise disjoint. By this, (2-7) and the fact that $\chi_{B(x_j,\delta)} \le \prod_{i=1}^k \chi_{\pi_i(B(x_j,\delta))} \circ \pi_i$, (5-6), (5-4) and the definition of b_p, the doubling property and (5-1), and, finally, disjointness of the $B(x_j, \delta)$,

$$\begin{aligned}\mu(Z') &\le \sum_{j=1}^{M_\delta} \mu(B(x_j,\delta)) \le A(\mu) \sum_j \prod_{i=1}^k |\pi_i(B(x_j,\delta))|^{1/p_i} \\ &\lesssim A(\mu) \sum_j |B(x_j, C\delta)|^{\sum_i 1/p_i} \prod_i \delta^{-v^i/p_i} \\ &\sim A(\mu) \sum_j |B(x_j, C\delta)| (\delta^{v\cdot \deg I^\delta_{x_j} - v\cdot b_p} |\lambda_{I^\delta_{x_j}}(x_j)|)^{\sum_i 1/p_i - 1} \\ &\lesssim A(\mu) \sum_j |B(x_j, C^{-1}\delta)| \delta^{\varepsilon \sum_i 1/p_i - 1} \le A(\mu)|W|\delta^{\varepsilon \sum_i 1/p_i - 1}.\end{aligned}$$

The lemma follows by sending δ to 0. $\square$

Proof of Lemma 5.2. The proof is similar to that of Lemma 5.1. Fix n and $\Omega' \subseteq \Omega_n$. Let $x \in \Omega'$. Since $\Omega' \subseteq \Omega$, b_p is an extreme point of $\mathcal{P}_x$. By the definition of ρ, $\max_{\deg I = b_p} |\lambda_I(x)| \sim 2^{n(|b_p|_1 - 1)}$.

By Proposition 4.1 and a covering argument, we may assume that there exists a finite set $\mathcal{A} \subseteq \mathbb{Z}_0^k$ such that $b_p \notin \mathcal{P}(\mathcal{A})$ and for each $x \in \Omega'$, $\mathcal{P}_x \subseteq \mathcal{P}(\mathcal{A} \cup \{b_p\})$. Choose $\varepsilon > 0$, $v \in (\varepsilon, 1]^k$ such that $v \cdot b_p + \varepsilon < v \cdot b$ for each $b \in \mathcal{P}(\mathcal{A} \cup \{b_p\}) \cap \mathbb{Z}_0^k \setminus \{b_p\}$, and let

$$\mathcal{W}_0 := \{w \in \mathcal{W} : v \cdot \deg w \le d\}.$$

Since $(1, \dots, 1, 0, \dots, 0) \in \mathcal{P}_x$ for each $x \in U$, $(1, \dots, 1, 0, \dots, 0) \in \mathcal{P}(\mathcal{A} \cup \{b_p\})$. Therefore we have $v \cdot b_p \le \sum_{i=1}^d v^i \le d$, so $\deg I = b_p$ implies that $I \in \mathcal{W}_0^d$.

Let $N = N_{d,p}$ be a large integer. As before, there exists $\delta_n > 0$, which depends on n, the π_i and p, such that for all $0 < \delta \le \delta_n$, $x \in \Omega'$, $I \in \mathcal{W}_0^d$ with $\deg I \ne b_p$, and $w, w' \in \mathcal{W}_0$,

$$|\delta^{v\cdot \deg I} \lambda_I(x)| < \delta^\varepsilon \max_{\deg I' = b_p} \delta^{v \cdot \deg I'} |\lambda_{I'}(x)|,$$

$$\|\delta^{v\cdot \deg w} X_w\|_{C^0(W)} \le \frac{1}{d} \operatorname{dist}(V, \partial W), \qquad \|\delta^{v\cdot\deg w} X_w\|_{C^N(W)} \le 1,$$

$$[\delta^{v\cdot \deg w} X_w, \delta^{v\cdot\deg w'} X_{w'}] = \sum_{\widetilde{w} \in \mathcal{W}_0} c_{w,w'}^{\widetilde{w},\delta} \delta^{v\cdot \deg \widetilde{w}} X_{\widetilde{w}},$$

with

$$\|c_{w,w'}^{\widetilde{w},\delta}\|_{C^N(W)} \le C_{d,p},$$

for all $w, w' \in \mathcal{W}_0$. In particular, we may choose δ_n sufficiently small that for each $x \in \Omega'$ and $0 < \delta \le \delta_n$, there exists a d-tuple $I_x^\delta \in \mathcal{W}_0^d$ such that $\deg I_x^\delta = b_p$ and

$$\delta^{v\cdot \deg I_x^\delta} |\lambda_{I_x^\delta}(x)| = \max_{I \in \mathcal{W}_0^d} \delta^{v\cdot \deg I} |\lambda_I(x)| \sim \delta^{v\cdot b_p} 2^{n(|b_p|_1 - 1)}.$$

Thus, considering the balls $B(x,\delta)$ (defined in (5-3)) for $x \in \Omega'$ and $0 < \delta \le \delta_n$,

$$|B(x,\delta)| \sim 2^{n(|b_p|_1-1)}\delta^{v\cdot b_p} = 2^{n/(\sum_i 1/p_i - 1)}\delta^{v\cdot b_p}.$$

Since the balls $B(x,\delta)$ are doubling, for each $\eta > 0$ there exist a collection $\{x_j\}_{j=1}^{M_\delta} \subseteq \Omega'$ and a parameter $0 < \delta \le \delta_n$ such that

$$\Omega' \subseteq \bigcup_{j=1}^{M_\delta} B(x_j,\delta), \qquad \biggl|\bigcup_{j=1}^{M_\delta} B(x_j,\delta)\biggr| \le |\Omega'| + \eta,$$

and such that the $B(x_j, C^{-1}\delta)$ are pairwise disjoint.

Arguing as in the proof of Lemma 5.1,

$$\begin{aligned}\mu(\Omega') \le \sum_{j=1}^{M_\delta} \mu(B(x_j,\delta)) &\lesssim A(\mu)\sum_j |B(x_j,\delta)||B(x_j,\delta)|^{\sum_i 1/p_i - 1}\delta^{-v\cdot b_p(\sum_i 1/p_i - 1)}\\ &\sim A(\mu)\sum_j |B(x_j,\delta)|2^n \lesssim A(\mu)2^n(|\Omega'|+\eta).\end{aligned}$$

Letting $\eta \to 0$ completes the proof. □

Remarks. The pointwise upper bound (2-8) is false if no assumptions are made on b_p. Indeed, if b_p lies in the interior of $\mathcal{P}_{x_0}$, then for some $\theta < 1$, $b_{\theta p}$ lies in the interior of $\mathcal{P}_{x_0}$, where $\theta p = (\theta p_1, \dots, \theta p_k)$. Thus for some neighborhood U of x_0, $b_{\theta p}$ lies in the interior of $\mathcal{P}_x$ for every $x \in U$. Hence by the main result in [Stovall 2011], if a is continuous with compact support in U,

$$\biggl|\int \prod_{j=1}^k f_j \circ \pi_j(x) a(x)\,dx\biggr| \lesssim \prod_{j=1}^k \|f_j\|_{L^{\theta p_j}}.$$

Additionally,

$$\biggl|\int \prod_{j=1}^k f_j \circ \pi_j(x)\bigl|\log|x-x_0|\bigr|a(x)\,dx\biggr| \lesssim \prod_{j=1}^k \|f_j\|_{L^\infty}.$$

Thus by interpolation,

$$\biggl|\int \prod_{j=1}^k f_j \circ \pi_j(x)\bigl|\log|x-x_0|\bigr|^{1-\theta}a(x)\,dx\biggr| \lesssim \prod_{j=1}^k \|f_j\|_{L^{p_j}}.$$

For the unweighted bilinear operator in the “polynomial-like” case, the endpoint-restricted weak-type bounds are known and are due to Gressman [2009]; in the multilinear case, the corresponding estimates follow by combining his techniques with arguments in [Stovall 2011]. The deduction of endpoint bounds from the arguments in [Gressman 2009] does not seem to be immediate in the weighted case, and so these questions remain open except for certain special configurations (such as convolution or restricted X-ray transform along polynomial curves).

6. Proof of the main theorem: Theorem 2.1

In this section, undecorated constants and implicit constants (C, c, $\lesssim$, $\gtrsim$, $\sim$) will be allowed to depend on a cutoff function a (specifically, on upper bounds for $\operatorname{diam}(\operatorname{supp} a)$ and $\|a\|_{L^\infty}$), a point $b_0 \in \mathbb{Z}_0^k$, and exponents $p_1, \dots, p_k$ (all of which will be given in a moment), as well as the π_j. Other parameters (namely ε, δ, N) that depend on $b_0, p_1, \dots, p_k$ will arise later on, so implicit constants may depend on these quantities as well. Unless otherwise stated, decorated constants and implicit constants (c_d, $\lesssim_{N,d}$, etc.) will only be allowed to depend on the objects in their subscripts.

Let $J_0 \in \{1, \dots, k\}^d$ and for $x \in U$ define $\Psi_x^{J_0}(t)$ as in (2-9). Let β_0 be a multiindex and define $b_0 := \deg J_0 + \deg_{J_0} \beta_0$. Let

$$\tilde{\rho}(x) := \left|\partial_t^{\beta_0}|_{t=0} \det D_t \Psi_x^{J_0}(t)\right|^{1/(|b_0|_1-1)}. \tag{6-1}$$

Let a be continuous and compactly supported in U, and define the multilinear form

$$\widetilde{M}(f_1, \dots, f_k) := \int_{\mathbb{R}^d} \prod_{j=1}^{k} f_j \circ \pi_j(x) \tilde{\rho}(x) a(x)\, dx.$$

In light of Proposition 2.3, the following more general result (we need not assume that b_0 is extreme) implies Theorem 2.1.

Theorem 6.1. *Let $(p_1, \dots, p_k) \in [1, \infty)^k$ satisfy $(p_1^{-1}, \dots, p_k^{-1}) \prec \boldsymbol{q}(b_0)$, with $p_i^{-1} < \boldsymbol{q}_i(b_0)$ when $b_0^i \neq 0$. Then*

$$|\widetilde{M}(f_1, \dots, f_k)| \lesssim \prod_{j=1}^{k} \|f_j\|_{L^{p_j}}, \tag{6-2}$$

for all continuous $f_1, \dots, f_k$.

Since J_0 and β_0 are fixed, we will henceforth drop the tildes from our notation, with the understanding that we are using (6-1) instead of (2-3) to define ρ.

It suffices to prove (6-2) when the f_j are nonnegative. Suppose that $b_j = 0$ for some j. Then π_j plays no role in the definition of ρ, and $p_j = \infty$ so, by Hölder's inequality, we may ignore f_j entirely. Thus we may assume that $b_j \neq 0$ for each j. In fact, we may assume that, for each j, $p_j < \infty$, since $\|f_j\|_{L^{p_j}(\pi_j(\operatorname{supp} a))} \lesssim \|f_j\|_{L^\infty}$, by the compact support of a.

We only claim a nonendpoint result, so by real interpolation with the trivial (by Hölder) inequalities of the form

$$M(f_1, \dots, f_k) \lesssim \prod_{j=1}^{k} \|f_j\|_{L^{\tilde{p}_j}}, \quad \sum_{j=1}^{k} p_j^{-1} \leq 1,$$

it suffices to prove that, for all Borel sets $E_1, \dots, E_k$ and some sufficiently small $\varepsilon > 0$,

$$\int_{\mathbb{R}^d} \prod_{j=1}^{k} \chi_{E_j} \circ \pi_j(x) \rho(x) a(x)\, dx \lesssim \prod_{j=1}^{k} |E_j|^{\boldsymbol{q}_j(b_0)-\varepsilon}. \tag{6-3}$$

Letting $\Omega := \operatorname{supp} a \cap \bigcap_{j=1}^{k} \pi_j^{-1}(E_j)$, (6-3) will follow from

$$\rho(\Omega) \lesssim \prod_{j=1}^{k} |\pi_j(\Omega)|^{\boldsymbol{q}_j(b_0)-\varepsilon}. \tag{6-4}$$

If we define

$$\alpha_j := \frac{\rho(\Omega)}{|\pi_j(\Omega)|}, \tag{6-5}$$

a bit of arithmetic shows that (6-4) is equivalent to

$$\prod_{j=1}^{k} \alpha_j^{\boldsymbol{q}_j(\boldsymbol{q}(b_0)-(\varepsilon,\dots,\varepsilon))} \lesssim \rho(\Omega),$$

which in turn would be implied by

$$\prod_{j=1}^{k} \alpha_j^{b_0^j+\varepsilon} \lesssim \rho(\Omega), \tag{6-6}$$

with a slightly smaller ε. (We recall that $\boldsymbol{q}$ equals its own inverse.)

By the coarea formula,

$$\alpha_j = |\pi_j(\Omega)|^{-1} \int_{\pi_j(\Omega)} \int_{\pi_j^{-1}\{y\}} \chi_\Omega(x)\rho(x) \frac{1}{|X_j(x)|}\, d\mathcal{H}^1(x)\, dy. \tag{6-7}$$

Since π_j is a submersion, $|X_j| \gtrsim 1$ and $\mathcal{H}^1(\pi_j^{-1}\{y\}) \lesssim 1$ for all $y \in \pi_j(\Omega)$. Since $\rho \lesssim 1$ by smoothness of the π_j, (6-7) implies that

$$\alpha_j \lesssim \operatorname{diam}(\Omega) \le \operatorname{diam}(\operatorname{supp} a). \tag{6-8}$$

By taking a partition of unity, we may assume that the α_j are as small as we like, in particular, that they are smaller than $\frac{1}{2}$. Reordering if necessary, $\alpha_1 \le \cdots \le \alpha_k$.

For $n \in \mathbb{Z}$, let $\Omega_n = \{x \in \Omega : 2^n \le \rho(x) < 2^{n+1}\}$. Then for C sufficiently large, $\Omega_n = \varnothing$ for all $n > C$. On the other hand, since π_1 is a submersion and $\operatorname{supp} a$ is compact,

$$\sum_{n \le \log \alpha_1 - C} \rho(\Omega_n) \lesssim \sum_{n \le \log \alpha_1 - C} 2^n |\pi_1(\Omega)| \lesssim 2^{-C} \alpha_1 |\pi_1(\Omega)| = 2^{-C} \rho(\Omega).$$

Thus, for C sufficiently large,

$$\rho\Big(\bigcup_{n \le \log \alpha_1 - C} \Omega_n\Big) < \tfrac{1}{2}\alpha_1 |\pi_1(\Omega)| = \tfrac{1}{2}\rho(\Omega).$$

By pigeonholing, there exists n with $\log \alpha_1 - C \le n \le C$ such that

$$\rho(\Omega_n) \ge \big(2(|\log \alpha_1| + 2C)\big)^{-1} \rho(\Omega) \gtrsim \alpha_1^{\varepsilon} \rho(\Omega). \tag{6-9}$$

Define

$$\alpha_{n,j} := \frac{\rho(\Omega_n)}{|\pi_j(\Omega_n)|}, \qquad j = 1, \dots, k.$$

By (6-9) and the triviality $\rho(\Omega_n) \le \rho(\Omega)$, together with the proof of (6-8) and the small diameter of $\operatorname{supp} a$,

$$\alpha_1^{\varepsilon} \alpha_j \lesssim \alpha_{n,j} \le \tfrac{1}{2}.$$

Therefore (6-6) follows from

$$\rho(\Omega_n) \gtrsim \prod_{j=1}^{k} (\alpha_{n,j})^{b_0^j+\varepsilon}, \tag{6-10}$$

with a slightly smaller value of ε. Henceforth, we let $\rho_0 := 2^n$ (for this value of n) and drop the n from the notation in (6-10). We note that $\rho(\Omega) \sim \rho_0|\Omega|$. Reordering again, we may continue to assume that $\alpha_1 \le \cdots \le \alpha_k$.

Let $\delta > 0$ be a small constant (depending on ε, b_0, d), which will be determined later on. Cover Ω by $c_d\alpha_1^{-\delta d}$ balls of radius α_1^{δ}. By pigeonholing, there exists $\Omega' \subseteq \Omega$ with

$$\rho(\Omega') \gtrsim \alpha_1^{\delta d}\rho(\Omega).$$

Arguing as above, the parameters $\alpha'_j := |\pi_j(\Omega')|^{-1}\rho(\Omega')$ satisfy

$$\alpha_1^{1+\delta d} \le \alpha_1^{\delta d}\alpha_j \lesssim \alpha'_j \lesssim \operatorname{diam}(\Omega') \le \alpha_1^{\delta}. \tag{6-11}$$

Thus, for δ sufficiently small, (6-10) would follow from

$$\rho(\Omega') \gtrsim \prod_{j=1}^{k} (\alpha'_j)^{b_0^j+\varepsilon},$$

with a slightly smaller value of ε.

Since $\alpha'_j \lesssim \operatorname{diam}(\operatorname{supp} a)$, we may assume that the α'_j are as small as we like (depending on the π_j, ε and δ). Thus (6-11) implies that, for each $1 \le j \le k$,

$$\operatorname{diam}(\Omega') \le c(\alpha'_j)^{\delta},$$

for some slightly smaller value of δ and with c as small as we like. By the same argument as for (6-8),

$$\alpha'_j \lesssim \rho_0 \operatorname{diam}(\Omega') \lesssim \rho_0(\alpha'_j)^{\delta},$$

whence $\rho_0 \ge c^{-1}(\alpha'_j)^{1-\delta}$, again with a slightly smaller value of δ.

In summary, to complete the proof of Theorem 6.1 (and thereby that of Theorem 2.1) it suffices to prove the following.

Lemma 6.2. *Let $\varepsilon > 0$ be sufficiently small depending on b_0 and $\delta > 0$ be sufficiently small depending on ε, b_0. Let $\Omega \subseteq \operatorname{supp} a$ be a Borel set, and define $\alpha_1, \dots, \alpha_k$ as in (6-5). Assume that $\alpha_1 \le \dots \le \alpha_k$, that*

$$\rho_0 \le \rho(x) \le 2\rho_0 \quad \text{for all } x \in \Omega,$$

and that

$$\alpha_k < c, \quad \rho_0 \ge c^{-1}\alpha_k^{1-\delta}, \quad \operatorname{diam}(\Omega) \le c\alpha_1^{\delta}. \tag{6-12}$$

Then for c sufficiently small, depending on the π_j, b_0, ε, δ, we have

$$\prod_{j=1}^{k} \alpha_j^{b_0^j+\varepsilon} \lesssim \rho(\Omega). \tag{6-13}$$

We note in particular that all constants and implicit constants are independent of ρ_0, Ω, and the α_j.

We devote the remainder of this section to the proof of Lemma 6.2. We use the method of refinements, which originated in [Christ 1998] and was further developed in similar contexts in [Christ 2008; Tao and Wright 2003].

Recalling (6-1),

$$|\partial^{\beta_0} \det D\Psi_{x_0}^{J_0}(0)| \sim \rho_0^{|b_0|_1-1} =: \lambda_0, \quad \text{for } x_0 \in \Omega. \tag{6-14}$$

As in [Tao and Wright 2003], for $w > 0$, we say that a set $S \subseteq [-w, w]$ is a central set of width w if, for any interval $I \subseteq [-w, w]$,

$$|I \cap S| \lesssim \left(\frac{|I|}{w}\right)^{\varepsilon} |S|.$$

Lemma 6.3. *For each subset $\Omega' \subseteq \Omega$ with $\rho(\Omega') \gtrsim \alpha_1^{C\varepsilon}\rho(\Omega)$ and each $1 \le j \le k$, there exists a refinement $\langle\Omega'\rangle_j \subseteq \Omega'$ with $\rho(\langle\Omega'\rangle_j) \gtrsim \alpha_1^{2C\varepsilon}\rho(\Omega')$ such that, for each $x \in \langle\Omega'\rangle_j$, there is a central set*

$$\mathcal{F}_j(x, \langle\Omega'\rangle_j) \subseteq \{t : |t| \lesssim \alpha_1^{\delta} \text{ and } e^{tX_j}(x) \in \langle\Omega'\rangle_j\} \tag{6-15}$$

whose width w_j and measure satisfy

$$\rho_0^{-1}\alpha_1^{2C\varepsilon}\alpha_j \lesssim w_j \le c\alpha_1^{\delta} \quad \text{and} \quad |\mathcal{F}_j(x, \langle\Omega'\rangle_j)| \gtrsim \rho_0^{-1}\alpha_1^{2C\varepsilon}\alpha_j. \tag{6-16}$$

This lemma has essentially the same proof as [Tao and Wright 2003, Lemma 8.2], but we sketch the argument for the convenience of the reader.

Sketch proof of Lemma 6.3. First we discard shorter-than-average π_j fibers in Ω', leaving a subset $\Omega'' \subseteq \Omega'$ with $\rho(\Omega'') \gtrsim \rho(\Omega')$ such that, for each $x \in \Omega''$,

$$\left|\{t : |t| \lesssim \alpha_1^{\delta} \text{ and } e^{tX_j}(x) \in \Omega''\}\right| \gtrsim \frac{|\Omega'|}{|\pi_j(\Omega')|} \gtrsim \alpha_1^{C\varepsilon}\rho_0^{-1}\alpha_j.$$

Next, if $S \subseteq [-c\alpha_1^{\delta}, c\alpha_1^{\delta}]$ is a measurable set, it contains a translate S' of a central set of measure at least $|S|^{1+2\varepsilon}$ and width at most $c\alpha_1^{\delta}$. Indeed, take $S' = S \cap I'$, where I' is a minimal length dyadic interval with $|S \cap I'| \ge (|I'|/\alpha_1^{\delta})^{\varepsilon}|S|$.

Using the exponential map, each π_j fiber in Ω'' is naturally associated to a set $S \subseteq [-c\alpha_1^{\delta}, c\alpha_1^{\delta}]$; S can be refined to a translate S' of a central set, and S' is then a fiber of the set $\langle\Omega'\rangle_j$. By the definition of exponentiation, for $x \in \langle\Omega'\rangle_j$ the set $\mathcal{F}_j(x, \langle\Omega'\rangle_j)$ in (6-15) contains 0, and it is easy to see that a 0-containing translate of a central set of width w is a central set of width $2w$. Finally, by pigeonholing, we can select only those fibers having the most popular dyadic width (there are at most $\log\alpha_1$ options). ☐

Write $J_0 = (j_1, \dots, j_d)$. With $\Omega_0 := \Omega$, for $1 \le i \le d$ we define

$$\Omega_i := \langle\Omega_{i-1}\rangle_{j_{d-i+1}}.$$

By Lemma 6.3, for each i, $\rho(\Omega_i) \gtrsim \alpha_1^{C\varepsilon}\rho(\Omega)$.

Fix $x_0 \in \Omega_d$. Let

$$F_1 := \mathcal{F}_{j_1}(x_0, \Omega_d), \quad x_1(t) := e^{tX_{j_1}}(x_0),$$

and for $2 \le i \le d$, let

$$F_i := \{(t_1, \dots, t_i) : (t_1, \dots, t_{i-1}) \in F_{i-1},\ t_i \in \mathscr{F}_{j_i}(x_{i-1}(t_1, \dots, t_{i-1}),\ \Omega_{d-i+1})\}$$
$$x_i(t_1, \dots, t_i) := e^{t_i X_{j_i}} x_{i-1}(t_1, \dots, t_{i-1}).$$

By construction, for each i and each $(t_1, \dots, t_i) \in F_i$,

$$x_i(t_1, \dots, t_i) \in \Omega_{d-i+1} \subseteq \Omega_{d-i},$$

so $\mathscr{F}_{j_{i+1}}(x_i(t_1, \dots, t_i), \Omega_{d-i})$ is a central set whose width and measure satisfy (6-16) (with j_{i+1} in place of j). Furthermore,

$$\Psi^{J_0}_{x_0}(F_d) \subseteq \Omega \quad \text{and} \quad |F_d| \gtrsim \rho_0^{-d} \alpha_1^{C\varepsilon} \alpha^{\deg J_0}; \tag{6-17}$$

here we recall that $\deg J$ is the k-tuple whose i-th entry is the number of appearances of i in the d-tuple J.

Let $\Psi^N_{x_0}$ be the degree N Taylor polynomial of $\Psi^{J_0}_{x_0}$, where $N \ge |b_0|_1 + 1$ is a large integer to be chosen later. Let $Q_w = \prod_{i=1}^d [-w_i, w_i]$ and let $Q_1 = Q_{(1,\dots,1)}$. By scaling, the equivalence of all norms on the degree N polynomials in d variables, and (6-14),

$$\begin{aligned} \|\det D\Psi^N_{x_0}\|_{C^0(Q_w)} = \sup_{t \in Q_1} |\det D\Psi^N_{x_0}(w_1 t_1, \dots, w_d t_d)| &\sim_{N,d} \sum_\beta w^\beta |\partial^\beta \det D\Psi^N_{x_0}(0)| \\ &\ge w^{\beta_0} |\partial^{\beta_0} \det D\Psi^N_{x_0}(0)| \sim w^{\beta_0} \lambda_0. \end{aligned}$$

Thus, by (6-16), the definition of λ_0, and some arithmetic,

$$\|\det D\Psi^N_{x_0}\|_{C^0(Q_w)} \gtrsim \rho_0^{d-1} \alpha_1^{C\varepsilon} \alpha^{\deg_{J_0} \beta_0}. \tag{6-18}$$

(We recall that $\deg_J \beta$ is the k-tuple whose i-th entry equals $\sum_{\ell : J_\ell = i} \beta_\ell$.)

Lemma 6.4. *If P is any degree N polynomial on $\mathbb{R}^d$, there is a subset $F'_d \subseteq F_d$ such that $|F'_d| \gtrsim_{N,\varepsilon,d} |F_d|$ and*

$$|P(t)| \gtrsim_{N,\varepsilon,d} \|P\|_{C^0(Q_w)} \quad \text{for } t \in F'_d.$$

The lemma follows from [Christ 2008, Lemma 6.2] or [Tao and Wright 2003, Lemma 7.3]. Roughly, if S is a central set of width w_0 and p is a degree N polynomial, p is close to $\|p\|_{C^0([-w_0,w_0])}$ on most of S. This is because the set where p is small is the union of at most N small intervals. Recalling how our set F_d was constructed (from a "tower" of central sets), it is possible to iterate d times to obtain the lemma.

Now we use $\Psi^N_{x_0}$ to control $\Psi^{J_0}_{x_0}$ via the following lemma, which just paraphrases [Christ 2008, Lemma 7.1]. We recall that Q_1 is the unit cube.

Lemma 6.5. *Let $N, C_1, c_2, c_3 > 0$. There exists a constant $c_0 > 0$, depending on C_1, c_2, c_3, N and d, such that the following holds. Let $\Psi : Q_1 \to \mathbb{R}^d$ be twice continuously differentiable and let $\Psi^N : \mathbb{R}^d \to \mathbb{R}^d$ be a degree N polynomial. Set $J_\Psi := \|\det D\Psi\|_{C^0(Q_1)}$ and assume that*

$$\|\Psi\|_{C^0(Q_1)} \le C_1, \quad \|\Psi - \Psi^N\|_{C^2(Q_1)} \le c_0 \mathscr{J}_\Psi^2. \tag{6-19}$$

Let $G \subseteq Q_1$ be a Borel set with the property that, for any degree N^d polynomial $P : \mathbb{R}^d \to \mathbb{R}$,

$$\big|\{t \in G : |P(t)| \geq c_2 \|P\|_{C^0(Q_1)}\}\big| \geq c_3 |G|. \tag{6-20}$$

Then

$$|\Psi(G)| \geq c_0 |G| \|\det D\Psi^N\|_{C^0(Q_1)}.$$

For the complete details, see [Christ 2008]. We give a quick sketch of that argument here.

Sketch proof of Lemma 6.5. Let $P = \det D\Psi^N$ and let G' denote the set on the left of (6-20). By (6-19),

$$|\det D\Psi(t)| \sim |P(t)| \sim \|P\|_{C^0(Q_1)} \sim \mathcal{J}_\Psi, \text{ for } t \in G', \quad \text{and} \quad \|\Psi^N\|_{C^2(Q_1)} \leq 2C_1. \tag{6-21}$$

This first series of inequalities above imply that

$$\int_{G'} |\det D\Psi| \geq c_0^{1/2} |G| \|\det D\Psi^N\|_{C^0(Q_1)}.$$

It remains to show that Ψ is finite-to-one on G', so that $|\Psi(G')| \gtrsim \int_{G'} |\det D\Psi|$.

First the local case. For c_0 sufficiently small and B any ball with radius $c_0^{1/2}\mathcal{J}_\Psi$ and center in G', Ψ, Ψ^N may be shown to be one-to-one on $10B$ and to satisfy

$$|\det D\Psi(t)| \sim |P(t)| \sim \mathcal{J}_\Psi, \quad t \in 10B. \tag{6-22}$$

We cover G' by a finitely overlapping collection of such balls B.

Globally, we know (it is an application of Bezout's theorem) that Ψ^N is at most $C_{N,d}$-to-one on G'. Thus a point $x \in \mathbb{R}^d$ lies in $\Psi^N(10B)$ for at most $C_{N,d}$ balls $B \in \mathcal{B}$. We are done if we can show that $\Psi(B) \subseteq \Psi^N(10B)$. By the mean value theorem (applied to $(\Psi^N)^{-1}$), then Cramer's rule, (6-21) and (6-22),

$$\operatorname{dist}\big(\Psi^N(B), (\Psi^N(10B))^c\big) \geq \operatorname{dist}(B, (10B)^c) \|(D\Psi^N)^{-1}\|_{C^0(10B)}^{-1} > c_0^{1/2} \mathcal{J}_\Psi \operatorname{diam}(B).$$

The right side is just $c_0 \mathcal{J}_\Psi^2 \geq \operatorname{dist}(\Psi(B), \Psi^N(B))$, so we are done. □

Let D_w denote the dilation $D_w(t_1, \dots, t_d) = (w_1 t_1, \dots, w_d t_d)$. We will apply Lemma 6.5 with $\Psi = \Psi_{x_0}^{J_0} \circ D_w$, $\Psi^N = \Psi_{x_0}^N \circ D_w$ and $G = D_w F_d$. By Lemma 6.4, we just need to verify (6-19).

Since $w_j \leq 1$ for each j, $\|\Psi\|_{C^2(Q_1)} \leq \|\Psi_{x_0}^{J_0}\|_{C^2(Q_w)} \lesssim 1$. For the error bound,

$$\|\Psi_{x_0}^{J_0} - \Psi_{x_0}^N\|_{C^2(Q_w)} \lesssim \max_i w_i^{N-1} \|\Psi_{x_0}^{J_0}\|_{C^{N+1}(Q_w)} \lesssim (c\alpha_1^\delta)^N, \tag{6-23}$$

where c is as in (6-12). (Recall that implicit constants do not depend on c.) We choose N larger than $\delta^{-1}(10 \deg_{J_0} \beta_0 + 10d)$ and then choose c sufficiently small. Combining (6-23), (6-12) and (6-18),

$$\|\Psi_{x_0}^{J_0} - \Psi_{x_0}^N\|_{C^2(Q_w)} \leq c_0 \bigg(\prod_j w_j\bigg)^2 \|\det D\Psi_{x_0}^N\|_{C^0(Q_w)}^2.$$

For c_0 sufficiently small, this implies that

$$\|\det D\Psi_{x_0}^{J_0} - \det D\Psi_{x_0}^N\|_{C^0(Q_w)} < \tfrac{1}{2} \|\det D\Psi_{x_0}^N\|_{C^0(Q_w)},$$

so $\|\det D\Psi_{x_0}^{J_0}\|_{C^0(Q_w)} \geq \frac{1}{2}\|\det D\Psi_{x_0}^{N}\|_{C^0(Q_w)}$. Rescaling gives us (6-19).

Applying Lemma 6.5, inequality (6-18), and $b_0 = \deg J_0 + \deg_{J_0} \beta_0$,

$$|\Omega| \geq |\Psi_{x_0}^{J_0}(F_d)| \gtrsim |F_d|\rho_0^{d-1}\alpha_1^{C\varepsilon}\alpha^{\deg_{J_0}\beta_0} \gtrsim \rho_0^{-1}\alpha_1^{2C\varepsilon}\alpha^{b_0}.$$

The proof of Theorem 2.1 is finally complete.

Appendix: proof of Proposition 4.1

In this section we prove Proposition 4.1, which was used in proving Propositions 2.2 and 2.3. We fix, for the remainder of this section, a point $b_0 \in [0, \infty)^k$. An object is admissible if it may be chosen from a finite collection, depending only on b_0, of such objects, and all implicit constants will be admissible (i.e., depending only on b_0).

The following two lemmas show that conclusions (i) and (ii) of Proposition 4.1 are equivalent.

Lemma A.1. *If $\mathscr{A} \subseteq \mathbb{Z}_0^k$ is a finite set and $b_0 \notin \mathcal{P}(\mathscr{A})$, there exist $\varepsilon > 0$ and $v_0 \in (\varepsilon, 1]^k$ such that $v_0 \cdot b_0 + \varepsilon < v_0 \cdot p$ for every $p \in \mathcal{P}(\mathscr{A})$.*

Lemma A.2. *If $v_0 \in (0, 1]^k$, there exists a finite set $\mathscr{A} \subseteq \mathbb{Z}_0^k$ such that $b_0 \notin \mathcal{P}(\mathscr{A})$ and*

$$\{b \in \mathbb{Z}_0^k : v_0 \cdot b_0 < v_0 \cdot b\} \subseteq \mathcal{P}(\mathscr{A}).$$

Proof of Lemma A.1. We may assume that $b_0 \neq (0, \dots, 0)$ and $\mathscr{A} \neq \varnothing$; otherwise, the result is trivial. Since $b_0 \notin \mathcal{P}(\mathscr{A})$, there exists $v_1 \in \mathbb{R}^k$ such that $v_1 \cdot b_0 < v_1 \cdot p$ for every $p \in \mathcal{P}(\mathscr{A})$. Since $\mathcal{P}(\mathscr{A})$ contains a translate of $[0, \infty)^k$, $v_1 \in [0, \infty)^k$. We may assume that $v_1 \in [0, 1]^k$. Let

$$\delta := \tfrac{1}{2}|b_0|_1^{-1} \min_{b \in \mathscr{A}} v_1 \cdot (b - b_0).$$

Since $\mathscr{A}$ is finite, $\delta > 0$. Let $v_2 := v_1 + (\delta, \dots, \delta)$. Then $v_2 \in [\delta, 1 + \delta]^k$. If $b \in \mathscr{A}$,

$$b \cdot v_2 = v_1 \cdot b_0 + v_1 \cdot (b - b_0) + \delta|b|_1 \geq v_2 \cdot b_0 + \delta|b_0|_1 \geq v_2 \cdot b_0 + \delta.$$

The conclusion thus holds with $\varepsilon := \frac{1}{2}\delta/(1+\delta)$, $v_0 := v_2/(1+\delta)$. □

Proof of Lemma A.2. Let $\varepsilon := \min_i v_0^i$ and let $N := \lceil k\varepsilon^{-1}(b_0 \cdot v_0 + 1)\rceil$. If $p \in \mathbb{Z}_0^k$ and $|p|_1 \geq N$,

$$v_0 \cdot p \geq \min_j v_0^j \max_i p^i \geq \varepsilon\left(\frac{N}{k}\right) \geq b_0 \cdot v_0 + 1,$$

so the conclusion holds with

$$\mathscr{A} := \{b \in \mathbb{Z}_0^k : |b|_1 \leq N \text{ and } v_0 \cdot b > v_0 \cdot b_0\}.$$ □

The following lemma implies that the conclusions of Proposition 4.1 hold whenever $\mathscr{B}$ is a finite set with $\#\mathscr{B} \leq k+1$.

Lemma A.3. *Let $\mathscr{B} \subseteq \mathbb{Z}_0^k$ be a finite set. Assume that $\#\mathscr{B} \leq k+1$ and that $b_0 \notin \mathcal{P}(\mathscr{B})$. Then there exist admissible $\varepsilon > 0$ and $v_0 \in (\varepsilon, 1]^k$ such that $b \cdot v_0 > b_0 \cdot v_0 + \varepsilon$ for every $p \in \mathcal{P}(\mathscr{B})$.*

The same proof shows that, for any finite $\mathcal{B}$ with $b_0 \notin \mathcal{P}(\mathcal{B})$, there exist $\varepsilon > 0$ and $v_0 \in (\varepsilon, 1]^k$, taken from a finite list that depends only on b_0 and m, such that $b \cdot v_0 > b_0 \cdot v_0 + \varepsilon$ for every $p \in \mathcal{P}(\mathcal{B})$, but for simplicity we only prove the version that we use.

Proof. The conclusion is trivial if $\mathcal{B} = \varnothing$, so we write $\mathcal{B} = \{b_1, \dots, b_m\}$ with $m \leq k+1$. By Lemma A.1, the conclusion is trivial if $\{b_1, \dots, b_m\}$ is admissible; we will reduce to this case.

If $|b_i|_1 > |b_0|_1$, $1 \leq i \leq m$, the conclusion holds with $v_0 = (1, \dots, 1)$, $\varepsilon = \frac{1}{2}(\lceil |b_0|_1 + 1\rceil - 1)$. Reindexing if necessary, we may assume that $|b_1|_1 \leq |b_0|_1$, in which case $\{b_1\}$ is admissible.

Assume that for some $j < m$, $\{b_1, \dots, b_j\}$ is admissible. By assumption, $b_0 \notin \mathcal{P}(\{b_1, \dots, b_j\})$, so by Lemma A.1 there exist admissible $\varepsilon_j > 0$, $v_j \in (\varepsilon_j, 1]^k$ such that $v_j \cdot b_0 + \varepsilon_j < v_j \cdot b_i$ for $1 \leq i \leq j$. If $v_j \cdot b_0 + \varepsilon_j < v_j \cdot b_i$ for every i, the conclusion of the lemma holds with $\varepsilon = \varepsilon_j$, $v_0 = v_j$. Otherwise, after reindexing, we may assume that $v_j \cdot b_{j+1} \leq v_j \cdot b_0$. Therefore b_{j+1} is admissible, and hence $\{b_1, \dots, b_{j+1}\}$ is admissible as well. The procedure must terminate after at most $m \leq k+1$ steps, and so the lemma is proved. $\square$

Lemma A.3 has the following corollary.

Lemma A.4. *Under the hypotheses of Lemma A.3, there exists an admissible $\varepsilon > 0$ such that if*

$$b(\theta) := \sum_{i=1}^{m} \theta_i b_i$$

is any convex combination of $b_1, \dots, b_m$, there exists an i, $1 \leq i \leq k$ such that $b^i(\theta) \geq b_0^i + \varepsilon$.

Proof. By Lemma A.3, there exist admissible $\varepsilon > 0$, $v_0 \in (\varepsilon, 1]^k$ such that

$$\varepsilon < (b(\theta) - b_0) \cdot v_0 \leq \left(\sum_{i=1}^{k} v_0^i\right) \max_{1 \leq i \leq k} (b^i(\theta) - b_0^i) \leq \max_{1 \leq i \leq k} (b^i(\theta) - b_0^i). \qquad \square$$

Finally, we are ready to complete the proof of Proposition 4.1.

Proof of Proposition 4.1. Let $C > |b_0|_1$ be a large constant, to be determined (admissibly) in a moment. Define $\mathcal{A} := \mathcal{B}' \cup \mathcal{B}''$, where

$$\mathcal{B}' := \{b \in \mathcal{B} : |b|_1 \leq C\},$$
$$\mathcal{B}'' := \{C e_i : 1 \leq i \leq k\}.$$

Here e_i denotes the i-th standard basis vector. Then, since $\mathcal{P}(\mathcal{B}'') = \mathcal{P}(\{b \in \mathbb{Z}_0^k : |b|_1 \geq C\})$, $\mathcal{P}(\mathcal{B}) \subseteq \mathcal{P}(\mathcal{A})$. It remains to show that, for C sufficiently large, $b_0 \notin \mathcal{P}(\mathcal{A})$.

Assume that $b_0 \in \mathcal{P}(\mathcal{A})$. By Carathéodory's theorem from combinatorics (see, for instance, [Ziegler 1995, p. 46]), $b_0 \succeq \sum_{l=1}^{k+1} \theta_l a_l$, for some $a_1, \dots, a_{k+1} \in \mathcal{A}$ and $0 \leq \theta_l \leq 1$ satisfying $\sum_l \theta_l = 1$. Reindexing if necessary,

$$b_0 \succeq \sum_{l=1}^{j} \theta_l C e_{i_l} + \sum_{l=j+1}^{k+1} \theta_l b_l, \tag{A-1}$$

where $b_{j+1}, \dots, b_{k+1} \in \mathcal{B}'$. Since $C > |b_0|_1$, $\sum_{l=j+1}^{k+1} \theta_l > 0$ and, since $b_0 \notin \mathcal{P}(\mathcal{B}') \subseteq \mathcal{P}(\mathcal{B})$, $\sum_{l=1}^{j} \theta_l > 0$.

Let

$$b(\theta) := \Bigl(\sum_{l=j+1}^{k+1} \theta_l\Bigr)^{-1} \sum_{l=j+1}^{k+1} \theta_l b_l.$$

By Lemma A.4, there exists an i, $1 \le i \le k+1$ such that $b^i(\theta) \ge b_0^i + \varepsilon$, where $\varepsilon > 0$ depends only on b_0 (crucially, not on C). By (A-1),

$$b_0 \succeq \Bigl(\sum_{l=j+1}^{k+1} \theta_j\Bigr) b(\theta),$$

so, comparing the i-th coordinates, we see that

$$\sum_{l=j+1}^{k+1} \theta_j \le \frac{b_0^i}{b_0^i + \varepsilon} \le \frac{|b_0|_\infty}{|b_0|_\infty + \varepsilon},$$

so

$$\sum_{l=1}^{j} \theta_j \ge 1 - \frac{|b_0|_\infty}{|b_0|_\infty + \varepsilon} = \frac{\varepsilon}{|b_0|_\infty + \varepsilon}. \tag{A-2}$$

On the other hand, by (A-1) and the fact that all coordinates of the b_i are nonnegative, $\sum_{l=1}^{j} \theta_j \le |b_0|_1/C$. For $C = C(\varepsilon, b_0)$ sufficiently large (admissible since ε is), this contradicts (A-2), and the proof of Proposition 4.1 is complete. □

Acknowledgements

The idea for this project came from a conversation with Michael Christ, and Terence Tao provided many valuable suggestions during the early stages of this work. The author is very grateful for these discussions. She would also like to thank the anonymous referee for a detailed and enormously helpful report. This project was supported in part by NSF DMS-0902667 and 1266336.

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Received 27 Nov 2013. Accepted 9 Jun 2014.

BETSY STOVALL: stovall@math.wisc.edu
Department of Mathematics, University of Wisconsin, 480 Lincoln Dr., Madison, WI 53706-1325, United States

ANALYSIS AND PDE
Vol. 7, No. 5, 2014
dx.doi.org/10.2140/apde.2014.7.1137

PROPAGATION OF SINGULARITIES FOR ROUGH METRICS

HART F. SMITH

We use a wave packet transform and weighted norm estimates in phase space to establish propagation of singularities for solutions to time-dependent scalar hyperbolic equations that have coefficients of limited regularity. It is assumed that the second order derivatives of the principal coefficients belong to $L^1_t L^\infty_x$, and that u is a solution to the homogeneous equation of global Sobolev regularity $s_0 = 0$ or 1. It is then proven that the H^{s_0+1} wavefront set of u is a union of maximally extended null bicharacteristic curves.

1. Introduction

In this paper we establish a propagation of singularities theorem for second-order, scalar hyperbolic operators of $(t, x) \in (-T, T) \times \mathbb{R}^n$ of the form

$$L = D_t^2 - 2b^j(t,x)D_j D_t - c^{ij}(t,x)D_i D_j + d^0(t,x)D_t + d^j(t,x)D_j, \quad 1 \le i, j \le n,$$

where summation notation is used, and $D_t = -i\partial_t$, $D_j = -i\partial_{x_j}$ for $1 \le j \le n$. Under the assumption that the second derivatives of the principal coefficients belong to $L^1_t L^\infty_x$, we establish the following.

Theorem 1.1. *Suppose that $s_0 \in \{0, 1\}$. Suppose that $Lu = 0$ and that*

$$u \in C^0((-T,T), H^{s_0}(\mathbb{R}^n)), \quad D_t u \in C^0((-T,T), H^{s_0-1}(\mathbb{R}^n)). \tag{1-1}$$

If $\gamma(t)$ is a null bicharacteristic curve of L and $\gamma(t_0) \notin \mathrm{WF}_{s_0+1}(u)$ for some $t_0 \in (-T, T)$, then $\gamma \cap \mathrm{WF}_{s_0+1}(u) = \varnothing$.

The improvement of this paper over prior results for twice-differentiable coefficients is the gain of 1 derivative over the background regularity, which we show by example to be the best possible in the setting we consider. Also, we assume integrability in t of the second order derivatives, as opposed to uniform bounds, which by a limiting argument will show that the theorem holds for piecewise regular coefficients. By Theorem B.2, the assumptions on u imply that $\mathrm{WF}_{s_0+1}(u)$ is contained in the characteristic set of L, so the restriction to null bicharacteristics is natural. Theorem 1.1 can be localized in x; see Remark B.7. Also, for $s_0 = 1$, the regularity assumption on u may be reduced to $u \in H^1((-T,T) \times \mathbb{R}^n)$ by Theorem B.6. For $s_0 = 0$, it is not clear how to interpret Lu in case $u \in L^2((-T,T) \times \mathbb{R}^n)$. However, if L is of divergence form, or if the regularity assumption on the coefficients of L is increased to $b^j, c^{ij} \in C^{1,1}((-T,T) \times \mathbb{R}^n)$, and $d^0, d^j \in C^{0,1}((-T,T) \times \mathbb{R}^n)$, then Remark B.8 shows that Theorem 1.1 holds for L^2 solutions.

This material is based upon work supported by the National Science Foundation under grant DMS-1161283. This work was partially supported by a grant from the Simons Foundation (# 266371 to Hart Smith).
MSC2010: 35A21, 35L10.
Keywords: wavefront set, wave equation, propagation of singularities.

Hörmander's theorem [1971] on propagation of singularities for operators of real principal type shows that if the coefficients of L are C^∞, then the conclusion of Theorem 1.1 holds for all s_0, with no global regularity condition required of u. Propagation of singularities theorems in the setting of nonsmooth nonlinear equations were obtained by Bony [1981], where the method of paradifferential operators was introduced. In that case a local regularity assumption is required of u. Related work on nonlinear equations includes [Rauch and Reed 1980; Beals and Reed 1984]. More closely related to this paper, Taylor [2000] used the positive commutator method and paradifferential theory to establish propagation of singularities for linear differential equations, including results for coefficients of Hölder regularity less than 2. In the case of $C^{1,1}$ coefficients, [Taylor 2000, Proposition 11.4, Chapter 3] implies invariance of the H^{s_0+1} wavefront set if $u \in H^{s_0+\epsilon}$, any $\epsilon > 0$, for $s_0 \in [-1, 1]$. In [de Hoop et al. 2012], the authors studied reflection of the H^{s_0} wavefront set off conormal singularities of metrics with singularities of Hölder regularity $C^{1,\alpha}$, where $0 < \alpha \le 1$. The limiting result in [de Hoop et al. 2012] for $\alpha = 1$ would be a gain of $1/2$ derivative relative to the assumed background regularity of u. For C^2 metrics in domains with C^3 boundary, Burq [1997] established the propagation result for microlocal defect measures. In the setting of [Burq 1997], as well as in that of [de Hoop et al. 2012], there may be multiple generalized bicharacteristics passing through a given initial point in phase space.

We now make more precise the regularity conditions that we place on the coefficients. We assume that the coefficient functions b^j and c^{ij} are real, and that the equation is uniformly hyperbolic in t:

$$\sum_{i,j=1}^{n} c^{ij}(t,x)\xi_i\xi_j \ge c_0|\xi|^2, \qquad c_0 > 0. \tag{1-2}$$

The b^j and c^{ij} are assumed continuously differentiable, with uniform bounds

$$\sup_{|t|<T,\, x\in\mathbb{R}^n} \sum_{|\gamma|\le 1} \big(|\partial_{t,x}^\gamma b^j(t,x)| + |\partial_{t,x}^\gamma c^{ij}(t,x)|\big) \le C_0. \tag{1-3}$$

In addition, we assume that the second-order derivatives of b^j and c^{ij} belong to L^1L^∞. Precisely, we assume that their distributional derivatives of second-order are locally integrable functions of (t, x), and that there exists a function $\alpha(t) \in L^1((-T, T))$ such that

$$\sup_{x\in\mathbb{R}^n} \sum_{|\gamma|=2} \big(|\partial_{t,x}^\gamma b^j(t,x)| + |\partial_{t,x}^\gamma c^{ij}(t,x)|\big) \le \alpha(t). \tag{1-4}$$

This condition in fact implies that the coefficients are continuously differentiable functions of (t, x), so that the assumption of C^1 coefficients (as opposed to Lipschitz) is redundant. It also follows from (1-4) that

$$\|c^{ij}(t,\cdot\,) - c^{ij}(s,\cdot\,)\|_{C^1(\mathbb{R}^n)} \le \int_s^t \alpha(r)\,dr, \tag{1-5}$$

so the map $s \to c^{ij}(s,\cdot\,)$ is continuous from $(-T, T)$ into $C^1(\mathbb{R}^n)$, similarly for b^j.

The coefficients d^0 and d^j are assumed to have the same regularity as the first order derivatives of b^j and c^{ij}; that is, d^0 and d^j are assumed to be continuous functions of (t, x) with uniform upper bounds, with first order derivatives in $L^1 L^\infty$. Precisely, for d denoting either d^0 or d^j, we assume bounds with $\alpha(t)$ as above:

$$\sup_{|t|<T, x\in\mathbb{R}^n} |d(t,x)| \le C_0, \qquad \sup_{x\in\mathbb{R}^n} \sum_{|\gamma|=1} |\partial_{t,x}^\gamma d(t,x)| \le \alpha(t). \tag{1-6}$$

The coefficients of L all admit extensions to $\mathbb{R}\times\mathbb{R}^n$ with the same regularity. For example, consider $c(t,x)$ defined on $t>0$ with second order derivatives belonging to $L^1L^\infty(\mathbb{R}_+\times\mathbb{R}^n)$. By (1-5), $c(t,x)$ extends to a C^1 function on $t\ge 0$. For $t<0$ define

$$c(t,x) = 3c(-t,x) - 2c(-2t,x). \tag{1-7}$$

It is then easily verified that all second order distributional derivatives of c belong to $L^1L^\infty(\mathbb{R}^{1+n})$, and that the same extension preserves the first order regularity of d^0, d^j. For convenience, we will thus assume when needed that all coefficients of L have been extended to $\mathbb{R}\times\mathbb{R}^n$, and, in addition, that L equals the standard wave operator for $|t|\ge T+1$.

We note that the product of functions satisfying (1-3) and (1-4) is of the same type, hence there is no loss of generality in our assumption that the coefficient of D_t^2 is 1. Such an L can also be written in divergence form:

$$L = D_t^2 - 2D_j b^j(t,x) D_t - D_i c^{ij}(t,x) D_j + \tilde d^0(t,x) D_t + \tilde d^j(t,x) D_j$$

for $\tilde d$ satisfying (1-6). This form will be more convenient for certain proofs.

Consider the principal symbol of L, where (τ,ξ) are the phase space coordinates dual to (t,x),

$$H(t,x,\tau,\xi) = \tau^2 - 2\sum_{j=1}^n b^j(t,x)\xi_j\tau - \sum_{i,j=1}^n c^{ij}\xi_i\xi_j.$$

This factors as

$$H(t,x,\tau,\xi) = (\tau - p_+(t,x,\xi))(\tau + p_-(t,x,\xi)),$$

where

$$\begin{aligned} p_\pm(t,x,\xi) &= p(t,x,\xi) \pm b^j(t,x)\xi_j, \\ p(t,x,\xi) &= (c^{ij}(t,x)\xi_i\xi_j + (b^j(t,x)\xi_j)^2)^{\frac12}. \end{aligned} \tag{1-8}$$

We modify $p(t,x,\xi)$ near $\xi=0$ so that it is smooth in ξ and homogeneous of degree 1 for $|\xi|>1$. The symbols p and $p_\pm$ are continuously differentiable in (t,x) and satisfy

$$\begin{aligned} &\sup_{|\xi|=1}\sup_{|\beta|\le 1} |\partial_\xi^\gamma \partial_{t,x}^\beta p(t,x,\xi)| \le C_\gamma, \\ &\sup_{|\xi|=1}\sup_{|\beta|=2} |\partial_\xi^\gamma \partial_{t,x}^\beta p(t,x,\xi)| \le C_\gamma \alpha(t), \end{aligned} \tag{1-9}$$

and similarly for $p_\pm$. As a consequence, the Hamiltonian flow of $\pm p_\pm$,

$$\frac{dx_t}{dt} = \pm d_\xi p_\pm(t,x_t,\xi_t), \qquad \frac{d\xi_t}{dt} = \mp d_x p_\pm(t,x_t,\xi_t),$$

is well-posed and induces a bilipschitz homeomorphism on $\mathbb{R}^{2n}_{x,\xi}$, since the Lipschitz norm of $p_\pm$ with respect to (x, ξ) is bounded by $\alpha(t) \in L^1((-T, T))$. The null bicharacteristics of $H(t, x, \tau, \xi)$ are, after reparametrization, curves of the form

$$\gamma(t) = (t, x_t, \pm p_\pm(t, x_t, \xi_t), \xi_t),$$

where (x_t, ξ_t) is, respectively, an integral curve of $\pm p_\pm$. We will refer to such curves $\gamma(t)$ as the null bicharacteristic curves of L.

The outline of this paper is as follows. In Section 2 we reduce the proof of Theorem 1.1 to an analogous result for a first order pseudodifferential equation, which requires a careful factorization of L. In Section 3 we construct the evolution groups for the first order factors of L as one-parameter families of operators on the appropriate range of Sobolev spaces, through the use of wave packet transform methods. In Section 4 we establish spatial-wavefront mapping properties (pseudolocality) for the evolution operators at fixed time. This is the heart of the paper, where pseudolocality is established via weighted-norm estimates on the fixed-time evolution operators expressed in the wave-packet frame. In Section 5 we deduce the space-time wavefront propagation of Theorem 1.1 from the fixed time result. In Section 6 we show that Theorem 1.1 applies, through a limiting process, to coefficients that satisfy the above regularity assumptions on the elements of a partition of $(-T, T) \times \mathbb{R}^n$ into time slices, with matching assumptions at the endpoints. We then produce an example of such a metric showing that the assumption of H^{s_0} regularity on u cannot be relaxed when establishing propagation of H^{s_0+1} singularities. Appendix A contains the various commutator and paraproduct estimates that are used throughout the paper. Some of these results are standard in paraproduct theory, but we collect them here for reference. Appendix B contains energy estimates and well-posedness results for the operators considered in this paper.

Notation. We use the following notation for function spaces. For $1 \le p \le \infty$, and $s \in \mathbb{R}$, L^pH^s denotes functions for which $\|u(t)\|_{H^s(\mathbb{R}^n)}$ belongs to $L^p((-T, T))$ with norm

$$\|u\|_{L^pH^s} = \left(\int_{-T}^{T} \|u(t)\|_{H^s}^p \, dt \right)^{1/p},$$

with the obvious modification if $p = \infty$, and where we write L^2 instead of H^0. Here and throughout this paper, $u(t)$ denotes the function $x \to u(t, x)$. The L^pL^q norm is similarly defined as $\|u\|_{L^p((-T,T),L^q(\mathbb{R}^n))}$.

The space $C^{k,1}$, for nonnegative integer k, consists of functions whose k-th derivatives satisfy a Lipschitz condition,

$$\|f\|_{C^{0,1}} = \sup_{x \ne y} \frac{|f(x) - f(y)|}{|x - y|},$$

$$\|f\|_{C^{k,1}} = \sup_{|\alpha| \le k} \|\partial_x^\alpha f\|_{C^{0,1}}.$$

For k a nonnegative integer, C^kH^s denotes the space of u such that $t \to u(t)$ is a C^k map of $(-T, T) \to H^s(\mathbb{R}^n)$ with the norm

$$\|u\|_{C^kH^s} = \sup_{t \in (-T,T)} \sup_{j \le k} \|\partial_t^j u(t)\|_{H^s}.$$

The notation $\|f\|_{H^s}$ denotes the norm in the Sobolev space $H^s(\mathbb{R}^n)$. In case we use the norm in $H^s((-T,T)\times\mathbb{R}^n)$ or $H^s(\mathbb{R}^{1+n})$, we write the domain explicitly unless it is obvious from the context; in the first case, s will be a nonnegative integer.

For a sequence of functions $f=\{f_k\}_{k=0}^{\infty}$,

$$\|f\|_{\ell^2L^2}=\left(\sum_{k=0}^{\infty}\|f_k\|_{L^2}^2\right)^{\frac{1}{2}},\qquad \|f\|_{2^{k\sigma}\ell^2L^2}=\left(\sum_{k=0}^{\infty}2^{2k\sigma}\|f_k\|_{L^2}^2\right)^{\frac{1}{2}}.$$

The space $S^m\subset C^\infty(\mathbb{R}^n)$ denotes smooth symbols satisfying the standard multiplier condition; that is, for all multi-indices α,

$$|\partial_\xi^\alpha q(\xi)|\le C_\alpha(1+|\xi|)^{m-|\alpha|}.$$

The space $S^m_{\mathrm{cl}}\subset S^m$ denotes symbols that are homogeneous of degree m on $|\xi|\ge 1$,

$$q(r\xi)=r^m q(\xi),\qquad r\ge 1,\ |\xi|\ge 1.$$

Given two positive functions f and g, we say that $f\lesssim g$, respectively $f\approx g$, if there is a constant $C<\infty$ such that

$$f\le Cg,\qquad \text{respectively } C^{-1}g\le f\le Cg.$$

2. Reduction to a first order operator

In this section we reduce Theorem 1.1 to results for a first order pseudodifferential equation through a factorization of the operator L. We introduce the notation

$$P=p(t,x,D),\qquad P^{\pm}=p_{\pm}(t,x,D)=P\pm b^jD_j \tag{2-1}$$

with p and $p_\pm$ defined by (1-8).

Throughout, $D=(D_1,\dots,D_n)=-i\partial_x$, and always $1\le i,j\le n$. The operator $P(t)$, respectively $P^{\pm}(t)$, will denote the corresponding pseudodifferential operator acting on functions of x, obtained by freezing the t variable.

We start with a factorization of L of the form

$$L=(D_t+P^-+d^0)(D_t-P^+)+R_1^+, \tag{2-2}$$

where $R_1^+(t)$ is a one-parameter family of first-order operators acting on functions of x, with the precise form of $R_1^+(t)$ stated below.

Since $P^{\pm}=P\pm b^jD_j$, the product of parentheses on the right-hand side expands to

$$D_t^2-2b^jD_jD_t+b^ib^jD_iD_j+d^0D_t-P^2+R,$$

where

$$R=-((D_tb^j)-b^i(D_ib^j))D_j-[D_t,P]+[b^jD_j,P]-d^0P^+.$$

Using a symbol expansion of the homogeneous symbol $p(t,x,\xi)$ as in (A-1), we see that R is a convergent sum of terms of the form

$$d(t,x)q_0(D) \quad \text{and} \quad a_1(t,x)[a_2(t,x),q_1(D)]q_2(D), \tag{2-3}$$

where d satisfies (1-6), each a_j satisfies the regularity conditions (1-3) and (1-4), and each $q_j(\xi)\in S^1_{\mathrm{cl}}(\mathbb{R}^n)$.

Next, observe that

$$p(t,x,D)^2 = p^2(t,x,D)+R = c^{ij}D_iD_j + b^ib^jD_iD_j + R,$$

with R again of the form (2-3), as seen by using the symbol expansion (A-1) of p. Thus, (2-2) holds with R_1^+ a convergent sum of terms of the form (2-3). By (1-3) and (1-4), and Theorem A.1, we have the following bounds, for each $t\in(-T,T)$:

$$\|R_1^+(t)f\|_{L^2} \le C\|f\|_{H^1}, \tag{2-4}$$

$$\|R_1^+(t)f\|_{H^s} \le C\alpha(t)\|f\|_{H^{s+1}}, \qquad -1\le s\le 1, \tag{2-5}$$

$$\|D_tR_1^+(t)f\|_{L^2} + \|[q_1(D),R_1^+(t)]f\|_{L^2} \le C\alpha(t)\|f\|_{H^1}, \tag{2-6}$$

whenever $q_1(D)$ is an order 0 multiplier in the x-variable. Additionally, by (2-6) or (1-5), we have the following norm-continuity of $R_1^+(t)$ with respect to t:

$$\|R_1^+(t)f - R_1^+(s)f\|_{L^2} \le C\biggl(\int_s^t \alpha(r)\,dr\biggr)\|f\|_{H^1}. \tag{2-7}$$

We now fix $s_0\in\{0,1\}$ and produce a factorization of L modulo order 0 terms,

$$L = (D_t + P^- + d^0 + Q^+)(D_t - P^+ - Q^+) + R_0^+, \tag{2-8}$$

where $Q^+=Q^+(t)$ will be a uniformly bounded family of operators on $H^{s_0}(\mathbb{R}^n)$, depending on the parameter t, and where the form of Q^+ will depend on the choice of $s_0\in\{0,1\}$. Here, $R_0^+(t)$ is a one-parameter family of operators on $H^{s_0}(\mathbb{R}^n)$, and we construct $Q^+(t)$ such that

$$\|R_0^+(t)f\|_{H^{s_0}} \le C\alpha(t)\|f\|_{H^{s_0}}, \tag{2-9}$$

and such that

$$\begin{aligned} \|Q^+(t)f\|_{H^s} &\le C\alpha(t)\|f\|_{H^s}, \qquad s_0-1\le s\le s_0+1,\\ \|Q^+(t)f\|_{H^{s_0}} &\le C\|f\|_{H^{s_0}},\\ \|Q^+(t)f - Q^+(s)f\|_{H^{s_0}} &\le C\biggl(\int_s^t \alpha(r)\,dr\biggr)\|f\|_{H^{s_0}}. \end{aligned} \tag{2-10}$$

In particular, $Q^+(t)$ is a continuous function of t in the H^{s_0} operator norm.

Expanding the product of parentheses in (2-8) leads to

$$L - R_1^+(t) - P(t)Q^+(t) - Q^+(t)P(t) - R(t),$$

where

$$R = [D_t,Q^+] - [b^j,Q^+]D_j - b^j[D_j,Q^+] + (Q^+)^2 + d^0Q^+.$$

Assuming (2-10), and since $\|d^0(t,\cdot)\|_{C^{0,1}} \le C\alpha(t)$, the last two terms satisfy the bound in (2-9), and hence can be absorbed into the error $R_0^+(t)$. The estimates (2-12) below will imply that the first three terms also satisfy the bound in (2-9).

So, given R_1^+ of the form (2-3), it suffices to construct $Q^+(t)$ solving

$$P(t)Q^+(t) + Q^+(t)P(t) - R_1^+(t) = R_0^+(t), \tag{2-11}$$

with $R_0^+(t)$ satisfying (2-9) and $Q^+(t)$ satisfying the conditions

$$\begin{aligned} \|D_t Q^+(t) f\|_{H^{s_0}} + \|[Q^+(t), q(D)] f\|_{H^{s_0}} &\le C\alpha(t)\|f\|_{H^{s_0}}, \\ \|[b, Q^+(t)] f\|_{H^{s_0}} &\le C\alpha(t)\|b\|_{C^{0,1}}\|f\|_{H^{s_0-1}}, \end{aligned} \tag{2-12}$$

where $q(D)$ denotes a general $S^1_{\mathrm{cl}}(\mathbb{R}^n)$ multiplier in the x-variable, and $b(x)$ a general Lipschitz function of x. An immediate corollary of (2-12) is that

$$\|[Q^+(t), q_0(D)] f\|_{H^{s+1}} \le C\alpha(t)\|f\|_{H^s}, \quad s_0 - 1 \le s \le s_0, \tag{2-13}$$

whenever $q_0 \in S^0_{\mathrm{cl}}(\mathbb{R}^n)$, as is seen by interpolation and writing

$$\begin{aligned} [Q^+(t), q_0(D)]D &= [Q^+(t), q_0(D)D] - q_0(D)[Q^+(t), D], \\ D[Q^+(t), q_0(D)] &= [Q^+(t), q_0(D)D] + [Q^+(t), D]q_0(D). \end{aligned}$$

After adding a harmless constant to $p(t,x,\xi)$, by Lemma A.10 the operator $P(t)$ is invertible for every t, and with uniform bounds over $t \in (-T, T)$,

$$\|P(t)^{-1} f\|_{H^s} \le C\|f\|_{H^{s-1}}, \quad 0 \le s \le 2.$$

For the case $s_0 = 1$, we define

$$Q^+(t) = \tfrac{1}{2} P(t)^{-1} R_1^+(t). \tag{2-14}$$

Then (2-11) holds with

$$R_0^+(t) = \tfrac{1}{2} P(t)^{-1}[R_1^+(t), P(t)].$$

Lemma 2.1 below will show that

$$\|R_0^+(t) f\|_{H^1} \le C\alpha(t)\|f\|_{H^1}.$$

Thus (2-9) holds with $s_0 = 1$. Furthermore, the operator

$$(D_t Q^+)(t) = \tfrac{1}{2} P(t)^{-1}(D_t P)(t) P(t)^{-1} R_1^+(t) + \tfrac{1}{2} P(t)^{-1}(D_t R_1^+)(t)$$

has the same mapping properties by (2-6). This also holds for $[D_j, Q^+(t)]$. Finally, if $b \in C^{0,1}$, then

$$2[b, Q^+(t)] = P(t)^{-1}[b, P(t)]P(t)^{-1} R_1^+(t) + P(t)^{-1}[b, R_1^+(t)].$$

The first term on the right maps L^2 to H^1 with norm $\lesssim \alpha(t)\|b\|_{C^{0,1}}$, which follows by Theorem A.1 together with (2-5) for $s = -1$. For the second term, we apply Lemma 2.1 below to see that it satisfies similar bounds on H^1. Thus, (2-12) holds with $s_0 = 1$.

For the case $s_0 = 0$, we set

$$Q^+(t) = \tfrac{1}{2} R_1^+(t) P(t)^{-1}. \tag{2-15}$$

Then (2-11) holds with

$$R_0^+(t) = \tfrac{1}{2}[P(t), R_1^+(t)] P(t)^{-1}.$$

Hence, by Lemma 2.1,

$$\| R_0^+(t) f \|_{L^2} \le C\alpha(t) \| f \|_{L^2}.$$

Furthermore, (2-10) and (2-12) hold with $s_0 = 0$, so that the other terms in $R_0^+(t)$ have the same mapping property. Hence (2-9) holds with $s_0 = 0$ for this choice of $Q^+(t)$.

Lemma 2.1. *Assuming $R_1^+(t)$ is a convergent sum of terms of the form* (2-3),

$$\| [P(t), R_1^+(t)] f \|_{L^2} \le C\alpha(t) \| f \|_{H^1},$$

and, for $b \in C^{0,1}(\mathbb{R}^n)$,

$$\| [R_1^+(t), b] f \|_{L^2} \le C\alpha(t) \| b \|_{C^{0,1}} \| f \|_{L^2}.$$

Proof. In these estimates, the type of terms in $R_1^+(t)$ of the form $d(t,x)q(D)$ lead to commutators that are easily handled, so we replace $R_1^+(t)$ in the statement by an operator of the form $a_1[a_2, q_1(D)]q_2(D)$.

For the first estimate, we take the symbol expansion (A-1) of $p(t,x,\xi)$ and consider a term of the form

$$\begin{aligned}[a_0 q_0(D), a_1[a_2, q_1(D)]q_2(D)] &= a_0[q_0(D), a_1][a_2, q_1(D)]q_2(D) + a_0 a_1 [q_0(D), [a_2, q_1(D)]] q_2(D) \\ &\quad + a_1[a_0, [a_2, q_1(D)]] q_2(D) q_0(D) + a_1 [a_2, q_1(D)][a_0, q_2(D)] q_0(D),\end{aligned}$$

where each $q_j \in S^1_{\mathrm{cl}}(\mathbb{R}^n)$. Each term on the right satisfies the desired bound by Theorem A.1 and Lemma A.7.

For the second estimate, we need consider

$$[[a_2, q_1(D)]q_2(D), b] = [a_2, q_1(D)][q_2(D), b] + [b, [a_2, q_1(D)]] q_2(D),$$

which is handled similarly. □

The same calculation also constructs one-parameter families of operators $R_1^-(t)$, $Q^-(t)$, and $R_0^-(t)$ satisfying the above conditions, such that

$$L = (D_t - P^+ + d^0)(D_t + P^-) + R_1^- \tag{2-16}$$

and

$$L = (D_t - P^+ + d^0 - Q^-)(D_t + P^- + Q^-) + R_0^-. \tag{2-17}$$

Suppose now that we are given s_0 from Theorem 1.1, and construct the corresponding $Q^{\pm}(t)$ as above. In the next section we construct evolution groups $E_{\pm}(t, t_0)$, for $t, t_0 \in (-T, T)$, satisfying

$$D_t E_{\pm}(t, t_0) = \pm (P^{\pm}(t) + Q^{\pm}(t)) E_{\pm}(t, t_0), \qquad E_{\pm}(t_0, t_0) = I. \tag{2-18}$$

Precisely, $E_\pm(t,t_0)$ is a bounded family of maps on $H^s(\mathbb{R}^n)$ for $s_0 \le s \le s_0+1$, which is strongly continuous in t and t_0 such that if $f \in H^s(\mathbb{R}^n)$ and $t_0 \in (-T,T)$, then

$$E_\pm(t,t_0)f \in C^0H^s \cap C^1H^{s-1}, \quad s_0 \le s \le s_0+1,$$

and such that

$$D_tE_\pm(t,t_0)f = \pm(P^\pm(t)+Q^\pm(t))E_\pm(t,t_0)f, \quad E(t_0,t_0)f = f.$$

Then the above factorizations show that

$$LE_\pm(t,t_0)f = R_0^\pm(t)E_\pm(t,t_0)f.$$

Given the $E_\pm$ and a $u \in C^0H^{s_0} \cap C^1H^{s_0-1}$ that solves the Cauchy problem

$$Lu = 0, \quad u(t_0) = u_0, \quad D_tu(t_0) = u_1, \quad (u_0,u_1) \in H^{s_0} \times H^{s_0-1}$$

for some given $t_0 \in (-T,T)$, we can write u in the form

$$u = v + \sum_\pm E_\pm(t,t_0)f_\pm, \quad \text{where } f_\pm \in H^{s_0},\ \mathrm{WF}_{s_0+1}(v) \cap \mathrm{char}(L) = \varnothing.$$

To see this, we impose the conditions

$$f_+ + f_- = u_0, \quad (P^+(t_0)+Q^+(t_0))f_+ - (P^-(t_0)+Q^-(t_0))f_- = u_1,$$

which is solved by

$$f_\pm = (2P(t_0)+Q^-(t_0)+Q^+(t_0))^{-1}(P^\mp(t_0)u_0 + Q^\mp(t_0)u_0 \pm u_1) \in H^{s_0}(\mathbb{R}^n),$$

where the inverse exists by Lemma A.10, after adding a harmless constant to p. We then write

$$L\Big(u - \sum_\pm E_\pm f_\pm\Big) = -\sum_\pm R_0^\pm E_\pm f_\pm \in L^1H^{s_0}.$$

Also,

$$\Big(u - \sum_\pm E_\pm f_\pm\Big)\Big|_{t=t_0} = 0, \quad D_t\Big(u - \sum_\pm E_\pm f_\pm\Big)\Big|_{t=t_0} = 0.$$

Thus, by Theorem B.6,

$$v = u - \sum_\pm E_\pm f_\pm \in C^0H^{s_0+1} \cap C^1H^{s_0},$$

and, in particular, $\mathrm{WF}_{s_0+1}(v) \cap \mathrm{char}(L) = \varnothing$, since $\mathrm{char}(L) \subset \{\xi \neq 0\}$.

We can thus reduce Theorem 1.1 to a result about the functions $E_\pm(t,t_0)f_\pm$. Suppose for simplicity that the γ in the statement of Theorem 1.1 is contained in the forward cone $\tau = p_+(t,x,\xi)$. By Theorem B.4,

$$\gamma \cap \mathrm{WF}_{s_0+1}(E_-(t,t_0)f_-) = \varnothing.$$

Since $\gamma \cap \mathrm{WF}_{s_0+1}(v) = \varnothing$, Theorem 1.1 is reduced to the following.

Theorem 2.2. *Suppose that $s_0 \in \{0, 1\}$ and that $E_+(t, t_0)$ is the wave group constructed in Section 3 for*

$$D_t E_+(t, t_0) = (P^+(t) + Q^+(t))E_+(t, t_0), \quad E(t_0, t_0) = I,$$

where $Q^+(t)$ is given by (2-14) or (2-15), if $s_0 = 1$ or $s_0 = 0$, respectively.

Let $\gamma(t) = (t, x_t, p_+(t, x_t, \xi_t), \xi_t)$ be a null bicharacteristic curve of L. If $f \in H^{s_0}$, and for some $t_0 \in (-T, T)$, we have $\gamma(t_0) \notin \mathrm{WF}_{s_0+1}(E_+(t, t_0) f)$. It follows that

$$\gamma \cap \mathrm{WF}_{s_0+1}(E_+(t, t_0) f) = \varnothing.$$

The analogous result holds for the wave group $E_-(t, t_0)$.

3. The wave packet transform and construction of the wave group

In this section we construct the wave groups $E_\pm(t, t_0)$. For simplicity we drop the superscripts $+$ and $-$, and let $P(t)$ be either $P^\pm(t)$. Given $s_0 \in \{0, 1\}$, we let $Q(t)$ denote either $Q^\pm(t)$, given by (2-14) or (2-15) if $s_0 = 1$ or $s_0 = 0$, respectively, where $R_1(t)$ is a convergent sum of expressions of the form (2-3). There is a minor inconsistency in that the $P(t)$ in (2-14) and (2-15) refers to the original $p(t, x, D)$ as in (2-1), but this is unimportant as all three symbols p and $p_\pm$ have the same regularity.

We construct $E(t, t_0) : H^{s_0} \to C^0 H^{s_0}$ such that

$$D_t E(t, t_0) f = (P(t) + Q(t))E(t, t_0) f, \quad E(t_0, t_0) f = f, \quad f \in H^{s_0}.$$

By Theorem B.5, the evolution group $E(t, t_0)$ is uniquely determined, although in the proof of Theorem 1.1 the existence of $E(t, t_0)$ with the desired properties is all that is used. Our construction will show that $E(t, t_0)$ is also uniformly bounded on H^s for $s_0 \le s \le s_0 + 1$ and is strongly continuous in both t and t_0 on each such H^s. It follows from (2-10) that if $f \in H^s$ for some $s_0 \le s \le s_0 + 1$, then

$$Q(t)E(t, t_0) f \in C^0 H^{s_0} \subset C^0 H^{s-1}.$$

The same holds for $P(t)E(t, t_0) f$. Thus, for $s_0 \le s \le s_0 + 1$,

$$E(t, t_0) f \in C^0 H^s \cap C^1 H^{s-1}, \quad \text{if } f \in H^s. \tag{3-1}$$

Since the proof below works equally well if $Q \equiv 0$, it will also construct the evolution groups $E_{0,\pm}(t, t_0)$ for the equation

$$D_t E_{0,\pm}(t, t_0) f = \pm P^\pm(t) E_{0,\pm}(t, t_0) f, \quad E_0(t_0, t_0) f = f, \tag{3-2}$$

and (3-1) holds for $f \in H^s$ if $0 \le s \le 2$.

Following [Smith 2006], we work with a scaled wave-packet transform, similar to the FBI transform used in [Tataru 2002], but based on a Schwartz function with Fourier transform of compact support instead of a Gaussian.

We fix a real, even Schwartz function $h(x) \in \mathcal{S}(\mathbb{R}^n)$ with $\|h\|_{L^2} = (2\pi)^{-n/2}$ and assume that its Fourier transform $\hat{h}(\xi)$ is supported in the unit ball $\{|\xi| \le 1\}$. For $k \ge 0$, we define $T_k : \mathcal{S}'(\mathbb{R}^n) \to C^\infty(\mathbb{R}^{2n})$ by the rule

$$(T_k g)(x,\xi) = 2^{nk/4} \int e^{-i\langle \xi, z-x\rangle} h(2^{k/2}(z-x)) g(z)\, dz.$$

A simple calculation shows that

$$g(y) = 2^{nk/4} \int e^{i\langle \xi, y-x\rangle} h(2^{k/2}(y-x)) (T_k g)(x,\xi)\, dx\, d\xi,$$

so that $T_k^* T_k = I$. In particular,

$$\|T_k g\|_{L^2(\mathbb{R}^{2n})} = \|g\|_{L^2}.$$

The following shows that the L^2-continuity of T_k holds under relaxed conditions.

Lemma 3.1 [Smith 2006, Lemma 3.1]. *Suppose that $h_{x,\xi}(z)$ is a family of Schwartz functions on $\mathbb{R}^n$ depending on the parameters x and ξ, with uniform bounds over x and ξ on each Schwartz seminorm of h. Then the operator*

$$(R_k g)(x,\xi) = 2^{nk/4} \int e^{-i\langle \xi, z-x\rangle} h_{x,\xi}(2^{k/2}(z-x)) g(z)\, dz$$

satisfies the bound

$$\|R_k g\|_{L^2(\mathbb{R}^{2n})} \le C\|g\|_{L^2}.$$

We will apply T_k to the localization of u at frequency k. We introduce a nonnegative function $\beta(s) \in C_c^\infty(\mathbb{R})$, supported in the interval $[2^{-\delta}, 2^{1+\delta}]$, where $\delta > 0$ will be taken sufficiently small. With $\beta_k(\xi) = \beta(2^{-k}|\xi|)$ if $k \ge 1$, and β_0 an appropriate compactly supported function on $\mathbb{R}^n$, we assume that

$$\sum_{k=0}^{\infty} \beta_k(\xi)^2 = 1. \tag{3-3}$$

Now define $T : L^2(\mathbb{R}^n) \to \ell^2(\mathbb{N}, L^2(\mathbb{R}^{2n}))$ by

$$Tg \equiv \tilde{g} \equiv \{\tilde{g}_k\}_{k=0}^{\infty}, \quad \tilde{g}_k = T_k \beta_k(D) g.$$

Then T is a norm isomorphism, hence $T^*T = I$. Furthermore, for k large enough so that $2^{-k/2} \le 2^{-\delta}(1-2^{-\delta})$, $\tilde{g}_k$ is supported in the set $\{2^{k-2\delta} \le |\xi| \le 2^{k+1+2\delta}\}$. It follows that, for $\sigma \in \mathbb{R}$,

$$\|g\|_{H^\sigma} \approx \left(\sum_{k=0}^{\infty} 2^{2k\sigma} \|\tilde{g}_k\|^2_{L^2(\mathbb{R}^{2n})} \right)^{\frac{1}{2}}. \tag{3-4}$$

We obtain $E(t,t_0)$ by constructing its lift $\tilde{E}(t,t_0)$ to $\ell^2(\mathbb{N}, L^2(\mathbb{R}^{2n}))$ via the wave packet transform T:

$$\tilde{E}(t,t_0) f = T E(t,t_0) T^* f.$$

The group $\tilde{E}(t,t_0)$ will be constructed in a manner similar to that used in [Smith 2006], approximating the lifted equation by the Hamiltonian flow of an appropriately mollified p and obtaining $\tilde{E}(t,t_0)$ by a convergent iteration from the Hamiltonian flow group.

For $k \in \mathbb{N}$, we introduce the spatial regularization of p,

$$p_k(t, x, \xi) = \phi(2^{-k/2} D) p(t, x, \xi),$$

which regularizes the symbol in x to frequencies of magnitude $\leq c2^{k/2}$, some small fixed $c > 0$. We let $P_k(t) = p_k(t, x, D)$. We remark that in [Smith 2006] the symbol regularization was over both t and x variables, but that is unimportant for [Smith 2006, Lemmas 3.2 and 3.3], the specific results that we use in this paper.

Let $V_k = V_k(t, x, \xi, \partial_x, \partial_\xi)$ denote the real, linear first-order differential operator

$$V_k f = d_\xi p_k(t, x, \xi) \cdot d_x f - d_x p_k(t, x, \xi) \cdot d_\xi f,$$

This vector field is Lipschitz regular in (x, ξ) provided $|\xi|$ is bounded above, with Lipschitz constant $\alpha(t) \in L^1((-T, T))$. Hence the associated flow group is well-posed.

Let $\Theta^k_{s,t}$ denote the associated time $t \to s$ flow map on $\mathbb{R}^{2n}$,

$$\partial_t f(\Theta^k_{s,t}(x, \xi)) = V_k f(\Theta^k_{s,t}(x, \xi)),$$

which is the Hamiltonian flow induced by p_k. Also let $\Theta_{s,t}$ denote the $t \to s$ Hamiltonian flow map for p. By a simple extension of [Smith 1998, Lemma 3.6], if (x_t, ξ_t) is the flow out of (x_0, ξ_0) through p, (x^k_t, ξ^k_t) is the flow out of (x_0, ξ_0) through p_k, and $|\xi_0| \approx 1$, then

$$|x^k_t - x_t| + |\xi^k_t - \xi_t| \lesssim 2^{-k/2}.$$

Also $\Theta_{s,t}$, and each $\Theta^k_{s,t}$, are biLipschitz measure preserving maps on $\mathbb{R}^{2n}$, homogeneous of degree 1 in ξ, and, by homogeneity, it holds that $|\xi_t| \approx |\xi_0|$, and similarly $|\xi^k_t| \approx |\xi_0|$.

We define a unitary evolution group $W(t, s)$ on $\ell^2(\mathbb{N}, L^2(\mathbb{R}^{2n}))$ by evolving each f_k along V_k. Thus, for $f = \{f_k(x, \xi)\}_{k=0}^\infty \in \ell^2(\mathbb{N}, L^2(\mathbb{R}^{2n}))$, we set

$$(W(t, s) f)_k = f_k \circ \Theta^k_{s,t}.$$

Suppose that $\tilde{u}(t) = T(u(t))$. Then the equation $D_t u - P(t)u = Q(t)u$ is equivalent to the collection of equations for $k \in \mathbb{N}$,

$$-i(\partial_t - V_k)\tilde{u}_k = (T_k P_k + i V_k T_k)\beta_k(D)u + T_k[\beta_k(D), P_k]u + T_k\beta_k(D)(P - P_k)u + T_k\beta_k(D)Qu. \tag{3-5}$$

Inserting $u = T^*\tilde{u}$, we can write this as a series of equations

$$(\partial_t - V_k)\tilde{u}_k(t) = (B(t)\tilde{u}(t))_k, \tag{3-6}$$

where $(B\tilde{u})_k$ is the right-hand side of (3-5) applied to $u = T^*\tilde{u}$. Note that $(B\tilde{u})_k$ is supported where $|\xi| \in [2^{k-3\delta}, 2^{k+1+3\delta}]$, by the frequency localization of P_k and T_k.

We will show that

$$\|B(t) f\|_{2^{k\sigma}\ell^2 L^2} \leq C\alpha(t)\|f\|_{2^{k\sigma}\ell^2 L^2}, \quad s_0 - 1 \leq \sigma \leq s_0 + 1, \tag{3-7}$$

where the norm denotes the one on the right-hand side in (3-4). We can then obtain the solution to (3-6) with given initial condition $\tilde{u}(t_0)$ by solving the integral equation

$$\tilde{u}(t) = W(t,t_0)\tilde{u}(t_0) + \int_{t_0}^{t} W(t,s)B(s)\tilde{u}(s)\,ds. \tag{3-8}$$

Indeed, for $u(t_0) \in H^\sigma(\mathbb{R}^n)$, $s_0 - 1 \le \sigma \le s_0 + 1$, the integral equation (3-8) admits a series solution $\tilde{u} = \sum_{j=0}^{\infty} \tilde{u}^{(j)}$, convergent in $C^0((-T,T), 2^{k\sigma}\ell^2L^2)$, where

$$\tilde{u}^{(0)}(t) = W(t,t_0)\tilde{u}(t_0), \qquad \tilde{u}^{(j+1)}(t) = \int_{t_0}^{t} W(t,s)B(s)\tilde{u}^{(j)}(s)\,ds.$$

We express the solution as $\tilde{u}(t) = \tilde{E}(t,t_0)u(t_0)$, which by uniqueness determines an evolution group $\tilde{E}(t,t_0)$. Note that each $\tilde{u}_k^{(j+1)}$ is supported where $C^{-1}2^k \le |\xi| \le C2^k$, for some fixed C, by the localization of $(B\tilde{u})_k$ and homogeneity of $W(t,s)$.

It is easily seen from its construction that the group is strongly continuous in the $2^{k\sigma}\ell^2L^2$ norm, as a function of the parameters $(t,t_0) \in (-T,T)^2$. Since (3-6) is obtained by lifting the equation $D_t u - P(t)u = Q(t)u$, it follows that $\tilde{E}(t,t_0)$ preserves the range of T, and thus is of the form $T E(t,t_0)T^*$, where $E(t,t_0) = T^*\tilde{E}(t,t_0)T$ is consequently strongly continuous on H^σ in both parameters. It follows from (3-8) that $E(t,t_0)u(t_0)$ is a distribution solution of the equation $D_t u - P(t)u = Q(t)u$, which, as noted before, belongs to $C^0H^\sigma \cap C^1H^{\sigma-1}$ provided that $s_0 \le \sigma \le s_0+1$.

It remains to establish (3-7). Let $B_{kj}(t)$ denote the kj component of $B(t)$, so $(B\tilde{u})_k(t) = \sum_j B_{kj}(t)\tilde{u}_j(t)$. By the above, B_{kj} is the sum of four terms:

$$\begin{aligned} B_{kj} &= (T_kP_k + iV_kT_k)\beta_k(D)\beta_j(D)T_j^* + T_k\big[\beta_k(D), P_k\big]\beta_j(D)T_j^* \\ &\qquad\qquad + T_k\beta_k(D)(P-P_k)\beta_j(D)T_j^* + T_k\beta_k(D)Q\beta_j(D)T_j^* \\ &\equiv {}_1B_{kj} + {}_2B_{kj} + {}_3B_{kj} + {}_4B_{kj}. \end{aligned}$$

The bounds in (3-7) are satisfied by the operator ${}_4B = TQ(t)T^*$ by (3-4) and (2-10), so we focus on the first three components of $B(t)$. The terms ${}_1B_{kj}$ and ${}_2B_{kj}$ vanish unless $|j-k| \le 1$. Thus, it suffices to prove that each is bounded on $L^2(\mathbb{R}^{2n})$ with norm $\lesssim \alpha(t)$, uniformly over j and k. For ${}_2B_{jk}$, this follows by Theorem A.1 (or indeed the $S_{1,1/2}$ pseudodifferential calculus). For ${}_1B_{jk}$, it follows by [Smith 2006, Lemmas 3.1 and 3.2]. In the next section we will prove even stronger estimates for these terms.

To handle the term ${}_3B$, we take the symbol expansion (A-1) of $p(t,x,\xi)$ to reduce matters to considering $p(t,x,D) = a(t,x)q(D)$. For $|j-k| \le 1$, uniform boundedness of ${}_3B_{jk}$ follows, since $\|a - a_k\|_{L^\infty} \lesssim 2^{-k}\alpha(t)$.

If $|j-k| \ge 2$, then, after this substitution,

$${}_3B_{kj} = T_k\beta_k(D)a(t,x)q(D)\beta_j(D)T_j^*, \qquad |j-k| \ge 2.$$

These off-diagonal terms give an operator which is in fact smoothing of order 1, as we now show.

Set $P(t) = a(t,x)q(D)$. If $2 \le |j-k| \le 3$, since $\|a(t,\cdot)\|_{C^{1,1}} \le \alpha(t)$ and $q(D)$ is of order 1, we have

$$\|T_k\beta_k(D)a(t,x)q(D)\beta_j(D)T_j^*\|_{L^2\to L^2} \lesssim C\alpha(t)2^{-k}. \tag{3-9}$$

If $|j-k|\ge 4$, using the Littlewood–Paley partition of unity given by $\psi_j=\beta_j^2$,

$$ {}_3B_{kj}=\sum_{|l-m|\ge 2} T_k\beta_k(D)\psi_l(D)a(t,x)q(D)\psi_m(D)\beta_j(D)T_j^*, $$

and hence

$$ T R_{a(t)}q(D)T^* - \sum_{|j-k|\ge 4} {}_3B_{kj}(t) = \sum_{\substack{|j-k|\le 3\\ |l-m|\ge 2}} T_k\beta_k(D)\psi_l(D)a(t,x)q(D)\psi_m(D)\beta_j(D)T_j^*, $$

where R_a is defined as in Lemma A.4. In the latter sum, j,k,l,m differ by at most 5, and each term satisfies the bound in (3-9). Combined with Lemma A.4, we see that

$$ \Bigl\| \sum_{|j-k|\ge 2} {}_3B_{kj}(t)f_j \Bigr\|_{2^{k(\sigma+1)}\ell^2L^2} \le C\alpha(t)\|f\|_{2^{k\sigma}\ell^2L^2}, \quad -1\le\sigma\le 1. \tag{3-10} $$

As a consequence, the operator ${}_3B$ satisfies the bound in (3-7) on the range $-1\le s\le 2$, which contains $s_0-1\le s\le s_0+1$ for $s_0\in\{0,1\}$. This concludes the proof of (3-7), and hence the existence of $E(t,t_0)$. If $Q\equiv 0$, then (3-7) holds on the union of the ranges, $-1\le\sigma\le 2$, hence the wave group $E_0(t,t_0)$ exists on the range $-1\le s\le 2$.

We summarize the results of this section.

Theorem 3.2. *Suppose that $s_0\in\{0,1\}$ and that $Q^\pm(t)$ is respectively given by (2-15) or (2-14). Then an evolution group $E_\pm(t,t_0)$ for Equation (2-18) exists as a family of bounded maps on $H^s(\mathbb{R}^n)$ for $s_0-1\le s\le s_0+1$ and is strongly continuous in both t and t_0. Additionally, for $s_0\le s\le s_0+1$,*

$$ E_\pm(t,t_0)f\in C^0H^s\cap C^1H^{s-1} \quad \textit{when } f\in H^s. $$

The evolution group $E_{0,\pm}(t,t_0)$ for the equation

$$ D_tE_{0,\pm}(t,t_0)=\pm P^\pm(t)E_{0,\pm}(t,t_0), \qquad E_{0,\pm}(t_0,t_0)=I $$

similarly exists, is strongly continuous in both variables on H^s for $-1\le s\le 2$, and if $0\le s\le 2$, we have

$$ E_{0,\pm}(t,t_0)f\in C^0H^s\cap C^1H^{s-1} \quad \textit{when } f\in H^s. $$

4. Weighted estimates for the wave group

The null bicharacteristics of $\tau\mp p_\pm(t,x,\xi)$ are in one-to-one correspondence with the Hamiltonian curves (x_t,ξ_t) for $\pm p_\pm(t,x,\xi)$. In this section we prove the following result about wavefront mapping properties on $\mathbb{R}^n$ for the fixed time wave groups $E_\pm(t,t_0)$ constructed in the previous section, and in Section 5 we derive Theorem 2.2 as a corollary.

Theorem 4.1. *Given $s_0\in\{0,1\}$, let $E_\pm(t,t_0)$ be the wave group constructed in Section 3. Let (x_t,ξ_t) be the Hamiltonian curve of the corresponding $\pm p_\pm(t,x,\xi)$ that passes through (x_0,ξ_0) at $t=t_0$.*

Then, given $f\in H^{s_0}$, if $(x_0,\xi_0)\notin \mathrm{WF}_{s_0+1}(f)$, it follows that

$$ (x_t,\xi_t)\notin \mathrm{WF}_{s_0+1}(E_\pm(t,t_0)f), \quad t\in(-T,T). $$

Furthermore, if $T < \infty$, there is a constant $c > 0$ such that if $\chi_t(x)$ is a $C_c^\infty(\mathbb{R}^n)$-bounded family of cutoffs supported in the ball of radius c about x_t, and if $\Gamma_t(\xi)$ is an $S^0(\mathbb{R}^n)$-bounded family of conic cutoffs supported in the cone of angle c about ξ_t, then, with uniform bounds over $t \in (-T, T)$,

$$\Gamma_t(D)\chi_t(x)(E_\pm(t, t_0) f) \in H^{s_0+1}.$$

We consider the case of $E_+(t, t_0)$ and denote the wavegroup simply by $E(t, t_0)$. We prove Theorem 4.1 through weighted-norm estimates on the lifted evolution group $\tilde{E}(t, s) = T E(t, s)T^*$, where the weights are time-dependent functions of (x, ξ). It suffices to consider the case $t \geq t_0$ in Theorem 4.1, which we will assume in the rest of this section. Also, by making a smooth, t-dependent change of variables in x, we will from now on assume that ξ_t remains within a small cone about the positive ξ_1 axis.

Suppose that $M(t, x, \xi)$ is a family of strictly positive functions on $(-T, T) \times \mathbb{R}^{2n}$, continuous in all parameters, such that, for some $C < \infty$,

$$C^{-1}\langle\xi\rangle^{s_0} \leq M(t, x, \xi) \leq C\langle\xi\rangle^{s_0+1}.$$

Assume that the following holds, where $B(t)$ and $W(t, s)$ are as in Section 3:

$$\|M(s, x, \xi)B(s) f\|_{\ell^2 L^2} \leq C\alpha(s)\|M(s, x, \xi) f\|_{\ell^2 L^2}. \tag{4-1}$$

In addition, for $t_0 \leq s \leq t \leq T$, assume that

$$\|M(t, x, \xi)W(t, s) f\|_{\ell^2 L^2} \leq C\|M(s, x, \xi) f\|_{\ell^2 L^2}. \tag{4-2}$$

It follows from (3-8) that

$$\|M(t, x, \xi)\tilde{u}^{(j+1)}(t)\|_{\ell^2 L^2} \leq C\|M(t_0, x, \xi)\tilde{u}(t_0)\|_{\ell^2 L^2} + C\int_0^t \alpha(s)\|M(s, x, \xi)\tilde{u}^{(j)}(s)\|_{\ell^2 L^2}\, ds.$$

Since $\alpha \in L^1((-T, T))$, the sum of the $\tilde{u}^{(j)}$ converges to $\tilde{u}$ in the weighted norms, and we conclude that

$$\sup_{t\in(-T,T)} \|M(t, x, \xi)\tilde{u}(t)\|_{\ell^2 L^2} \leq C \exp\biggl(C\int_{t_0}^t \alpha(s)\, ds\biggr)\|M(t_0, x, \xi)\tilde{u}(t_0)\|_{\ell^2 L^2}. \tag{4-3}$$

With data $u(t_0) \in H^{s_0}(\mathbb{R}^n)$, we thus need to construct $M(t, x, \xi)$ such that the right-hand side is finite if $(x_0, \xi_0) \notin \mathrm{WF}_{s_0+1}(u(t_0))$, and such that finiteness of the left-hand side implies $(x_t, \xi_t) \notin \mathrm{WF}_{s_0+1}(u(t))$. The weight $M(t, x, \xi)$ we construct will be of size $\langle\xi\rangle^{s_0+1}$ on some locally uniform conic set about (x_t, ξ_t), so the statement about uniformity of the neighborhoods in Theorem 4.1 will be a consequence of the following arguments, and we thus focus on the fixed time estimates.

We start by equating weighted $L^2(\mathbb{R}^{2n})$ estimates on $(Tg)(x, \xi)$ to multiplier estimates on $g(x)$.

Lemma 4.2. *Suppose that $w(\xi) \in C(\mathbb{R}^n)$ is a strictly positive function for which there is a constant $m < \infty$ such that, if $k \geq 0$ and $2^{k-1} \leq |\xi|, |\eta| \leq 2^{k+2}$, we have*

$$w(\eta) \leq C w(\xi)(1 + 2^{-k/2}|\xi - \eta|)^m.$$

Assume also that $C^{-1}\langle\xi\rangle^{-N} \le w(\xi) \le C\langle\xi\rangle^{N}$ *for some* N *and* $C < \infty$. *If* $g \in H^s(\mathbb{R}^n)$ *for some* $s \in \mathbb{R}$, *then*

$$\|w(D)^{\pm 1} g\|_{L^2} \approx \|w(\xi)^{\pm 1} T g\|_{\ell^2 L^2},$$

and consequently

$$\|w(D) T^* f\|_{L^2} \lesssim \|w(\xi) f\|_{\ell^2 L^2}.$$

Proof. If $\chi \in C_c^\infty$ is supported in the cube $[-.6, .6]^n$ such that $\sum_{j\in\mathbb{Z}^n} \chi(\xi - j) = 1$, then, on the set $2^{k-1} \le |\xi| \le 2^{k+2}$,

$$w(\xi) \approx \sum_{j\in\mathbb{Z}^n} w(2^{k/2} j)\chi(2^{-k/2}\xi - j).$$

If we replace w by the right-hand side, then, for $|\xi|, |\eta| \approx 2^k$,

$$|\partial_\eta^\alpha w(\eta)^{\pm 1}| \le C_\alpha 2^{-k|\alpha|/2} w(\xi)^{\pm 1}(1 + 2^{-k/2}|\xi - \eta|)^m. \tag{4-4}$$

Smoothing out w in this way on each component of w with respect to a Littlewood–Paley decomposition, we may assume that (4-4) is satisfied whenever $2^{k-1} \le |\xi|, |\eta| \le 2^{k+2}$.

Since the conditions on w are symmetric in w and w^{-1}, it suffices to show that

$$\|w(\xi) T w(D)^{-1} g\|^2_{\ell^2 L^2} \lesssim \|g\|_{L^2},$$

as writing $g = T^* T g$ and using the adjoint bound with w replaced by w^{-1} implies the reverse inequality. Let $g_k = \beta_k(D) g$, and write

$$\begin{aligned} w(\xi) T_k w(D)^{-1} g_k &= 2^{-nk/4} \int e^{i\langle\zeta,x\rangle} w(\xi) w(\zeta)^{-1} \hat{h}(2^{-k/2}(\zeta - \xi))\hat{g}_k(\zeta)\, d\zeta \\ &= 2^{-nk/4} \int e^{i\langle\zeta,x\rangle} \hat{h}_\xi(2^{-k/2}(\zeta - \xi))\hat{g}_k(\zeta)\, d\zeta, \end{aligned}$$

where

$$\hat{h}_\xi(\eta) = w(\xi) w(\xi - 2^{k/2}\eta)^{-1}\hat{h}(\eta).$$

Here, $|\xi| \approx 2^k$ and $|\eta| \le 1$, so by (4-4) it follows that the function $h_\xi(z)$ is a smooth function of z with Schwartz seminorms bounded uniformly over ξ. By Lemma 3.1,

$$\|w(\xi) T_k w(D)^{-1} g_k\|_{L^2} \lesssim \|g_k\|_{L^2},$$

and the result follows. □

For weights in x, the analogue is the following result, which holds by a similar proof.

Lemma 4.3. *Suppose that* $w_k(x) \in C^0(\mathbb{R}^n)$ *is a strictly positive function such that, for some* $m < \infty$,

$$w_k(x) \le C w_k(y)(1 + 2^{k/2}|x - y|)^m.$$

Then

$$\|w_k(x)^{\pm 1} T_k g\|_{L^2(\mathbb{R}^{2n})} \approx \|w_k(x)^{\pm 1} g\|_{L^2},$$

and consequently

$$\|w_k(x)T_k^* f\|_{L^2} \lesssim \|w_k(x) f\|_{L^2(\mathbb{R}^{2n})}.$$

Furthermore, the constants in the bounds are independent of k.

We can now use weighted estimates to characterize the H^σ-wavefront set of g.

Lemma 4.4. *Suppose $g \in H^s(\mathbb{R}^n)$ for some s. Then $(x_0, \xi_0) \notin \mathrm{WF}_\sigma(g)$ if and only if there exists an open ball Ω centered on x_0 and an open conic set $\Gamma \subset \mathbb{R}^n$ centered on ξ_0, such that*

$$\sum_{k=0}^{\infty} \int_{\Omega\times\Gamma} \langle\xi\rangle^{2\sigma} |\tilde{g}_k(x,\xi)|^2 \, dx \, d\xi < \infty. \tag{4-5}$$

Proof. Suppose that $(x_0, \xi_0) \notin \mathrm{WF}_\sigma(g)$. For $\chi(x) \in C_c^\infty(\mathbb{R}^n)$, and $q(\xi) \in S^\sigma_{\mathrm{cl}}$ real and homogeneous of degree σ for $|\xi| \geq 1$, we consider

$$\int \chi(x) q(\xi)^2 |\tilde{g}_k(x,\xi)|^2 \, dx \, d\xi = \int e^{i\langle x, \eta-\zeta\rangle} \chi(x) b_k(\zeta,\eta) \hat{g}(\eta) \overline{\hat{g}(\zeta)} \, d\eta \, dx \, d\zeta,$$

where

$$b_k(\zeta,\eta) = 2^{-nk/2} \int q(\xi)^2 \hat{h}(2^{-k/2}(\eta-\xi)) \hat{h}(2^{-k/2}(\zeta-\xi)) \beta_k(\eta)\beta_k(\zeta) d\xi.$$

Since $\hat{h}$ is supported in the unit ball, $b_k(\zeta,\eta)$ vanishes unless

$$2^{k-\delta} \leq |\eta| \leq 2^{k+1+\delta}, \quad \operatorname{dist}(\eta, \operatorname{supp}(q)) \leq 2^{-k/2},$$

and the same condition holds with η replaced by ζ. In particular, if Γ' is an open cone containing the support of q, then $b_k(\zeta,\eta)$ is supported in $\Gamma' \times \Gamma'$ for k sufficiently large. Additionally, a simple calculation shows that

$$|\partial_\zeta^\alpha \partial_\eta^\beta b_k(\zeta,\eta)| \leq C_{\alpha,\beta} 2^{2k\sigma-(k/2)(|\alpha|+|\beta|)}.$$

Hence, the compound symbol $a(\zeta,x,\eta) = \chi(x)\sum_{k=0}^\infty b_k(\zeta,\eta)$ is of type $S^{\sigma,\sigma}_{1/2,1/2,0}$. If the support of $\chi(x)q(\xi)$ is contained in a small conic neighborhood of (x_0,ξ_0), then standard pseudodifferential calculus arguments show that

$$\int \overline{g(x)} a(D,x,D) g(x) \, dx < \infty.$$

The bound (4-5) follows by taking a sufficiently small conic neighborhood $\Omega\times\Gamma$ of (x_0,ξ_0) with $\chi(x)q(\xi)$ equal to one on $\Omega\times\Gamma$.

Conversely, suppose (4-5) holds and $g \in H^s(\mathbb{R}^n)$. Let $q(\xi) \in S^\sigma_{\mathrm{cl}}$, and write

$$\chi(y)(q(D)g)(y) = \sum_{k=0}^\infty 2^{-nk/4} e^{i\langle y-x,\eta\rangle} \chi(y) q(\eta) \beta_k(\eta) \hat{h}(2^{-k/2}(\eta-\xi)) \tilde{g}_k(x,\xi) \, dx \, d\xi \, d\eta.$$

Let

$$\begin{aligned} &K_{j,k}(x',\xi';x,\xi) \\ &= 2^{-n(j+k)/4} \int e^{i\langle y-x,\eta\rangle - i\langle y-x',\zeta\rangle} \chi^2(y) \times q(\zeta)\beta_j(\zeta)\hat{h}(2^{-j/2}(\zeta-\xi'))q(\eta)\beta_k(\eta)\hat{h}(2^{-k/2}(\eta-\xi)) \, d\eta \, dy \, d\zeta. \end{aligned}$$

Then $K_{j,k}$ vanishes unless ξ and ξ' both lie in a small conic neighborhood of the support of q, and $|\xi| \approx 2^k$, $|\xi'| \approx 2^j$. Additionally, for all N,

$$|K_{j,k}(x',\xi';x,\xi)| \le C_N 2^{\sigma(j+k)} 2^{-N|j-k|} (1+2^{\min(j,k)/2}|x-x'|)^{-N} (1+2^{-\max(j,k)/2}|\xi-\xi'|)^{-N} \\ \times (1+2^{k/2}\operatorname{dist}(x,\operatorname{supp}(\chi)))^{-N} (1+2^{j/2}\operatorname{dist}(x',\operatorname{supp}(\chi)))^{-N}.$$

An application of the Schur test and the Schwarz inequality then show that, if χ is supported inside Ω and q is supported inside the open cone Γ, we have

$$\|\chi(x)q(D)g\|_{L^2}^2 \lesssim \sum_{k=0}^{\infty} \int_{\Omega\times\Gamma} \langle\xi\rangle^{2\sigma} |\tilde g_k(x,\xi)|^2 \, dx\, d\xi + \sum_{k=0}^{\infty} \int_{\mathbb{R}^{2n}} \langle\xi\rangle^{2s} |\tilde g_k(x,\xi)|^2 \, dx\, d\xi,$$

hence $(x_0,\xi_0) \notin \mathrm{WF}_\sigma(g)$ by elliptic regularity. □

Suppose that $(x_0,\xi_0) \notin \mathrm{WF}_{s_0+1}(u(t_0))$. Given $\Omega\times\Gamma$ as in Lemma 4.4, we will produce a family of t-dependent weight functions $M(t,x,\xi)$ for $t \ge t_0$, such that

$$\begin{aligned} C^{-1}\langle\xi\rangle^{s_0} \le M(t,x,\xi) \le C\langle\xi\rangle^{s_0+1}&, \\ M(t_0,x,\xi) \le C\langle\xi\rangle^{s_0} \qquad &\text{for } (x,\xi)\notin\Omega\times\Gamma. \end{aligned} \tag{4-6}$$

Also, for some $c_t > 0$, if

$$\Omega_t = \{x : |x-x_t| < c_t\}, \quad \Gamma_t = \left\{\xi : \left|\frac{\xi}{|\xi|} - \frac{\xi_t}{|\xi_t|}\right| < c_t\right\},$$

then

$$M(t,x,\xi) \ge C^{-1}\langle\xi\rangle^{s_0+1} \quad \text{for } (x,\xi)\in\Omega_t\times\Gamma_t. \tag{4-7}$$

In addition, we will show that (4-1) and (4-2) hold. Theorem 4.1 then follows immediately from Lemma 4.4 and (4-3).

The weight function. For $c_t > 0$ and ξ_t close to the positive ξ_1 axis, we take $M(t,x,\xi)$ to be the weight function

$$\langle\xi\rangle^{s_0+1}\Big(1 + |\xi|\min\big(1, \operatorname{dist}^2(x,\Omega_{c_t}(x_t))\big) + |\xi|\operatorname{dist}^2\Big(\frac{\xi}{|\xi|}, K_{c_t}(\xi_t)\Big)\Big)^{-1},$$

where $\Omega_{c_t}(x_t)$ is the ball of radius c_t centered on x_t, and $K_{c_t}(\xi_t)$ is the closed conic set contained in the half-space $\xi_1 > 0$ whose intersection with the set $\xi_1 = 1$ is the cube of side length $2c_t$ centered on $\xi_t/(\xi_t)_1$, with sides parallel to the ξ_j axes. The time-dependent number c_t is given in Lemma 4.5 below, where c_{t_0} is chosen as follows.

Provided that $\Omega_{2c_{t_0}}(x_0)\subset\Omega$ and $K_{2c_{t_0}}(\xi_0)\subset\Gamma$, condition (4-6) is seen to hold. Thus, if $(x_0,\xi_0)\notin \mathrm{WF}_{s_0+1}\big(u(t_0)\big)$, we can choose c_{t_0} small so that

$$\|M(t_0,x,\xi)\tilde u(t_0)\|_{\ell^2L^2} < \infty.$$

Also, (4-7) holds (with the same c_t), since $\Gamma_t \subset K_{c_t}(\xi_t)$. It thus remains to verify the mapping bounds (4-1) and (4-2) for $t_0 \le s \le t \le T$.

We start with the proof of (4-2), which reduces to showing that, uniformly in k, t, s,

$$\int M(t,x,\xi)^2 |f \circ \Theta^k_{s,t}|^2(x,\xi)\,dx\,d\xi \le C \int M(s,x,\xi)^2 |f|^2(x,\xi)\,dx\,d\xi, \quad s \le t.$$

Since each $\Theta^k_{s,t}$ is a volume preserving diffeomorphism, this is equivalent to the bound

$$M(t, \Theta_{t,s}(x,\xi)) \le M(s,x,\xi), \quad s \le t. \tag{4-8}$$

The map $\Theta^k_{s,t}$ is homogeneous of degree 1 in ξ, and preserves $|\xi|$ up to a uniform multiple, so the factor $\langle\xi\rangle^{s_0+1}$ can be ignored. Furthermore, the projective map induced by $\Theta^k_{s,t}$ on the cosphere bundle is a bilipschitz map with uniform bounds over k, s, t. Thus, (4-8) holds as a consequence of the following.

Lemma 4.5. *For $c_0 > 0$, let*

$$c_t = c_0 \exp\biggl(-C \int_{t_0}^t \alpha(r)\,dr\biggr).$$

Then, for c_0 sufficiently small and C given below,

$$\Theta_{t,s}(\Omega_{c_s}(x_s) \times K_{c_s}(\xi_s)) \supset \Omega_{c_t}(x_t) \times K_{c_t}(\xi_t), \quad s \le t.$$

Proof. Write $\xi = (\xi_1, \xi')$, and consider the projection $(x_t, \xi_t) \to (x_t, 1, (\xi_t)_1^{-1}\xi'_t)$ of a Hamiltonian curve onto the set $\xi_1 = 1$. Let $\zeta_t = (\xi_t)_1^{-1}\xi'_t$. Then, by homogeneity of p_k,

$$\dot{x}_t = d_\xi p_k(x_t, 1, \zeta_t), \quad \dot{\zeta}_t = -d_{x'} p_k(x_t, 1, \zeta_t) + d_{x_1} p_k(x_t, 1, \zeta_t)\zeta_t.$$

On the set $|\zeta| \le 10$, the right-hand side is Lipschitz in (x, ζ) with Lipschitz constant $C\alpha(t)$. Hence, if we let

$$Q_c(x^0, \zeta^0) = \{(x,\zeta) : |x - x^0| + \sup_{2 \le i \le n} |\zeta_i - \zeta_i^0| \le c\},$$

then for $t > s$ the image of $Q_{c_t}(x_t, \zeta_t)$ under the reverse-time projected flow is contained in $Q_{c_s}(x_s, \zeta_s)$, where c_s is as in the statement. Since $K_{c_t}(\xi_t)$ is the conic subset of $\mathbb{R}^n \cap \{\xi_1 > 0\}$, whose intersection with $\{\xi_1 = 1\}$ equals $Q_{c_t}(x_t, \zeta_t)$, then, by homogeneity of the Hamiltonian flow, for $t > s$,

$$\Theta_{t,s}(\Omega_{c_s}(x_s) \times K_{c_s}(\xi_s)) \supset \Omega_{c_t}(x_t) \times K_{c_t}(\xi_t),$$

provided we choose c_{t_0} small enough so that $Q_{c_t}(x_t, \zeta_t)$ remains within the set $|\zeta| < 10$. Here we use that ζ_t remains in the set $|\zeta_t| < 1$ by the assumption that ξ_t lies in a cone of small angle about the ξ_1 axis. □

Fixed time weight bounds for $B(t)$. We now turn to the proof of (4-1). The operator $B(t) : \ell^2 L^2 \to \ell^2 L^2$ is a sum of four terms, $B = {}_1B + {}_2B + {}_3B + {}_4B$. As before, we let $B_{kj}(t) : L^2(\mathbb{R}^{2n}) \to L^2(\mathbb{R}^{2n})$ denote the $j \to k$ component of $B(t)$, which we recall is localized dyadically in ξ on each side.

We start by considering the terms ${}_mB$ for $m = 1, 2, 3$. Recall that ${}_1B_{kj}$ and ${}_2B_{kj}$ vanish unless $|j-k| \le 1$. For ${}_3B_{kj}$ we may also restrict attention to $|j-k| \le 1$, since (4-1) holds for the sum over $|j-k| \ge 2$ by (3-10) and (4-6).

Consider then the terms for $|j-k|\le 1$. By finite overlap in ξ, these are almost orthogonal in k, hence we are reduced to establishing, for $m=1,2,3$, that uniformly for j,k with $|j-k|\le 1$ and $f\in L^2(\mathbb{R}^{2n})$,

$$\|M(s,x,\xi)_m B_{kj}(s)f\|_{L^2(\mathbb{R}^{2n})}\le C\alpha(s)\|M(s,x,\xi)f\|_{L^2(\mathbb{R}^{2n})}. \tag{4-9}$$

We will consider the case $j=k$, as the terms with $j=k\pm 1$ are handled the same way. We ignore the factor $\langle\xi\rangle^{s_0+1}$ in the definition of M, since it introduces the same factor of $2^{k(s_0+1)}$ on both sides.

The terms ${}_1B_{kk}$ and ${}_2B_{kk}$ are the simplest to handle. The operator ${}_1B_{kk}$ is, by [Smith 2006, Lemmas 3.1 and 3.2], represented by an integral kernel operator K satisfying

$$|K(x,\xi;y,\eta)|\le C_N\alpha(s)(1+2^{k/2}|x-y|+2^{-k/2}|\xi-\eta|)^{-N}.$$

On the other hand, if $|\xi|\approx|\eta|\approx 2^k$, then

$$\left[\frac{M(s,x,\xi)}{M(s,y,\eta)}\right]^{\pm 1}\le C(1+2^{k/2}|x-y|+2^{-k/2}|\xi-\eta|)^2,$$

so the term ${}_1B_{kk}$ is seen by the Schur test to satisfy the desired weighted L^2 bound (4-9). The operator ${}_2B_{kk}$ is represented by a similar kernel; this follows from the fact that $\alpha(s)^{-1}[\beta_k(D),a_k(s,x)]q(D)$ is an $S^0_{1,1/2}$ pseudodifferential operator in x, dyadically localized to $|\xi|\approx 2^k$.

For the term ${}_3B_{kk}$, after substituting $p(t,x,D)=a(t,x)q(D)$, freezing t, and replacing $q(D)\beta_k(D)$ by $2^k\beta_k(D)$ (since the exact form of β is unimportant), we can assume that

$${}_3B_{kk}=T_k\beta_k(D)2^k(a(x)-a_k(x))\beta_k(D)T_k^*,$$

and we need to show that (4-9) holds with $\alpha(s)=1$ if $\|D^2a\|_{L^\infty}\le 1$. The adjoint operator ${}_3B_{kk}^*$ then has the same form as ${}_3B_{kk}$, so that in the estimate (4-9) we may replace $M(s,x,\xi)$ by $M(s,x,\xi)^{-1}$. Letting $\Omega=\Omega_{c_s}(x_s)$, $K=K_{c_s}(\xi_s)$, since the estimate is over the region $|\xi|\approx 2^k$, we may thus work with the weight

$$M(x,\xi)=1+2^k\min(1,\operatorname{dist}^2(x,\Omega))+2^{-k}\operatorname{dist}^2(\xi,K),$$

and show that the analogue of (4-9) holds for ${}_3B_{kk}$.

The conic set K is obtained by intersecting $2n-2$ distinct half-spaces. Let $\{\omega_j\}_{j=1}^{2n-1}$ be the collection of their outer normals, together with the vector $-e_1$ pointing on the negative ξ_1 axis. We let

$$\langle\omega_j,\xi\rangle_+=\max(\langle\omega_j,\xi\rangle,0),$$

and claim that

$$\operatorname{dist}^2(\xi,K)\approx\sum_{j=1}^{2n-1}\langle\omega_j,\xi\rangle_+^2. \tag{4-10}$$

To see this, we first note that each term on the right vanishes on K, so the right side is dominated by the left. To prove the converse, we make an affine transformation preserving ξ_1 so that K is centered on the ξ_1 axis. The collection of $\langle\omega_j,\xi\rangle$ are then equivalent to the collection of $\pm\xi_j-c\xi_1$ and $-\xi_1$. In the case where $\xi_1 \le 0$, we have $\langle -e_1, \xi\rangle_+ = |\xi_1|$. Since $\xi_1 \le 0$,

$$|\xi_j| \le \sum_{\pm} (\pm\xi_j - c\xi_1)_+,$$

so the right-hand side of (4-10) dominates $|\xi|^2 \ge \operatorname{dist}^2(\xi, K)$. If $\xi_1 > 0$, let η be the point in K closest to ξ. If $|\xi_j| \le c\xi_1$, then $\eta_j = \xi_j$, so by reducing dimension and multiplying ξ_j by -1 if needed, we may assume that $\xi_j > c\xi_1$ for each j. Then

$$\sum_{j=1}^{2n-1} \langle \omega_j, \xi\rangle_+^2 = \sum_{j=2}^{n} |\xi_j - c\xi_1|^2 = \operatorname{dist}^2(\xi, (\xi_1, c\xi_1, \dots, c\xi_1))^2 \ge \operatorname{dist}^2(\xi, K).$$

Including the spatial weight, we can thus replace $M(x,\xi)$ by a sum of $2n$ weights, and it suffices to establish the analogue of (4-9) separately for each. Precisely, by Lemmas 4.2 and 4.3, it suffices to show that multiplication by $2^k(a(x) - a_k(x))$ preserves the spaces with norms

$$\left\| \left(1 + 2^k \min(1, \operatorname{dist}^2(x, \Omega))\right) g(x) \right\|_{L^2(dx)}, \quad \|(1 + 2^{-k}\langle \omega, \xi\rangle_+^2)\hat{g}(\xi)\|_{L^2(d\xi)}$$

for a general unit vector ω.

Boundedness in the first norm is immediate, since $\|a - a_k\|_{L^\infty} \le 2^{-k}\|D^2 a\|_{L^\infty}$. For the second norm, we make a rotation to reduce to the case $\omega = (1, 0, \dots, 0)$. Let $b(x) = 2^k(a - a_k)(2^{-k/2}x)$. Then

$$\|b\|_{L^\infty} + \|Db\|_{L^\infty} + \|D^2 b\|_{L^\infty} \lesssim 1.$$

Thus, after scaling $x \to 2^{k/2}x$, we need to show that

$$\|(1 + (\xi_1)_+^2)\widehat{bf}\|_{L^2(d\xi)} \lesssim \|b\|_{C^{1,1}} \|(1 + (\xi_1)_+^2)\hat{f}\|_{L^2(d\xi)}.$$

Since the weight is a function of ξ_1 only and $C^{1,1}(\mathbb{R}^n) \subset L^\infty(\mathbb{R}^{n-1}, C^{1,1}(\mathbb{R}))$, we may assume that $n = 1$, that $b \in C^{1,1}(\mathbb{R})$, and we need show that

$$\|(1 + \xi_+^2)\widehat{bf}\|_{L^2(\mathbb{R})} \lesssim \|b\|_{C^{1,1}} \|(1 + \xi_+^2)\hat{f}\|_{L^2(\mathbb{R})}.$$

If $\hat{f}$ is supported in $[0, \infty)$, the bound follows from the fact that

$$\|\langle D\rangle^2 (bf)\|_{L^2} \lesssim \|b\|_{C^{1,1}} \|\langle D\rangle^2 f\|_{L^2}.$$

Hence we may assume $\hat{f}$ is supported in $-(\infty, 0]$. Since

$$\|\chi_{(-\infty,2]}\widehat{bf}\|_{L^2} \le \|bf\|_{L^2} \le \|b\|_{L^\infty}\|f\|_{L^2} \le \|b\|_{C^{1,1}} \|(1 + \xi_+^2)\hat{f}\|_{L^2},$$

it suffices to then bound

$$\|\langle\xi\rangle^2 \widehat{bf}\|_{L^2([2,\infty))} \lesssim \|b\|_{C^{1,1}} \|f\|_{L^2}, \quad \operatorname{supp}(\hat{f}) \subset (-\infty, 0].$$

Given $h \in L^2(\mathbb{R})$ with $\hat{h}$ supported in $[2,\infty)$, using the functions ϕ_j and ψ_j from (A-2), we may write

$$\begin{aligned}\left|\int h\langle D\rangle^2(bf)\,dx\right| &= \left|\sum_{k\le j+2}\int (\psi_k(D)\langle D\rangle^2 h)(\psi_j(D)b)(\phi_{j+4}(D)f)\,dx\right| \\ &\le \sum_{k\le j+2} 4^{k-j}\int |2^{-2k}\psi_k(D)\langle D\rangle^2 h|\,|2^{2j}\psi_j(D)b|\,|\phi_{j+4}(D)f|\,dx \\ &\lesssim \|h\|_{L^2}\left(\sum_{j=0}^{\infty}\int |2^{2j}\psi_j(D)b|^2|\phi_{j+4}(D)f|^2\,dx\right)^{\frac12} \\ &\lesssim \|h\|_{L^2}\|b\|_{C^{1,1}}\|f\|_{L^2},\end{aligned}$$

where at the last step we use Theorem A.3. This completes the proof for ${}_3B_{jk}$.

We now establish (4-1) for the term ${}_4B(t) = T\,Q(t)T^*$. Recall that

$$Q(t) = \begin{cases} P(t)^{-1}R_1(t), & s_0 = 1,\\ R_1(t)P(t)^{-1}, & s_0 = 0,\end{cases}$$

where $R_1(t)$ is a convergent sum of terms of the form (2-3). We observe that if $M_{s_0} = M_{s_0}(t,x,\xi)$ denotes the weight for s_0, then

$$\|M_1 T g\|_{\ell^2 L^2} \approx \|M_0 T\langle D\rangle g\|_{\ell^2 L^2}.$$

Also, if $q(\xi) \in S^0(\mathbb{R}^n)$, then

$$\|M_{s_0} T q(D) T^* f\|_{\ell^2 L^2} \le C\|M_{s_0} f\|_{\ell^2 L^2},$$

since the operator $T_k\beta_k(D)q(D)\beta_j(D)T_j^*$ vanishes unless $|j-k|\le 1$, and, for $|j-k|\le 1$, is given by an integral kernel with bound

$$|K(x,\xi;y,\eta)| \le C_N(1+2^{k/2}|x-y|+2^{-k/2}|\xi-\eta|)^{-N}. \tag{4-11}$$

Since $T^*T = I$, it therefore suffices to show the bounds

$$\|M_0 T a T^* f\|_{\ell^2 L^2} \le C\|a\|_{C^{0,1}}\|M_0 f\|_{\ell^2 L^2}, \tag{4-12}$$

$$\|M_0 T[a,q(D)]T^* f\|_{\ell^2 L^2} \le C\|a\|_{C^{1,1}}\|M_0 f\|_{\ell^2 L^2}, \tag{4-13}$$

$$\|M_0 T\langle D\rangle P(t)^{-1} T^* f\|_{\ell^2 L^2} \le C\|M_0 f\|_{\ell^2 L^2}, \tag{4-14}$$

where in (4-13) the multiplier $q(\xi)$ belongs to $S^1_{\mathrm{cl}}(\mathbb{R}^n)$.

To establish (4-12), it suffices to prove

$$\|M_0 T_j\beta_j(D)a\beta_k(D)T_k^* f\|_{L^2} \le C\|a\|_{C^{0,1}}\|M_0 f\|_{L^2}, \quad |j-k|\le 1, \tag{4-15}$$

since the terms for $|j-k|\ge 2$ are handled by the arguments leading to (3-10), together with Lemma A.4 and the fact that $c \le M_0 \le \langle\xi\rangle$.

By taking adjoints, we may replace M_0 by M_0^{-1} in (4-15) and ignore the factor $\langle\xi\rangle$ in M_0, since $|j-k|\le 1$. Furthermore, we may replace a by the operator $(\phi_k(D)a)\phi_k(D)$ for a compactly supported

$\phi(\xi)$. As with the handling of the term ${}_3B_{kk}$, it suffices to prove that if $\|a\|_{C^{0,1}(\mathbb{R}^n)} \le 1$, then $(\phi_k(D)a)\phi_k(D)$ preserves the spaces with norms

$$\big\|\big(1+2^k \min(1, \operatorname{dist}^2(x,\Omega))\big)g(x)\big\|_{L^2}, \quad \|(1+2^{-k}\langle\omega,\xi\rangle_+^2)\hat{g}(\xi)\|_{L^2}. \tag{4-16}$$

Boundedness in the first norm is simple, since $\phi_k(D)$ is a convolution kernel that is rapidly decreasing on scale 2^{-k}. For the second norm, we can reduce to the one-dimensional case, and we need to prove that

$$\|(1+2^{-k}D_+^2)(\phi_k(D)a)(\phi_k(D)g)\|_{L^2(\mathbb{R})} \le C\|a\|_{C^{0,1}}\|(1+2^{-k}D_+^2)\hat{g}\|_{L^2(\mathbb{R})},$$

where D_+ is the operator with multiplier $\xi_+ = \max(\xi, 0)$.

Consider first the case that $\hat{g}$ is supported in $\xi \le 0$. Then, since $|\xi| \lesssim 2^k$ on the frequency support of $(\phi_k(D)a)(\phi_k(D)g)$, this follows from the bound

$$\|(1+D_+)(ag)\|_{L^2} \le C\|a\|_{C^{0,1}}\|(1+D_+)g\|_{L^2},$$

which holds by Theorem A.1 since ξ_+ is a classical first order multiplier. If $\hat{g}$ is supported in $\xi \ge 0$, it suffices to prove the bound

$$\|2^{-k}D^2(\phi_k(D)a)(\phi_k(D)g)\|_{L^2} \le C\|a\|_{C^{0,1}}(\|g\|_{L^2} + 2^{-k}\|D^2 g\|_{L^2}).$$

This holds by distributing derivatives and using the fact that $\|D^2\phi_k(D)a\|_{L^\infty} \lesssim 2^k\|a\|_{C^{0,1}}$, in addition to $\|D\phi_k(D)g\|_{L^2} \lesssim 2^k\|g\|_{L^2}$.

The estimate (4-13) is similarly reduced by Lemma A.4 to handling $|j-k| \le 1$. We then need to show that the commutator $[(\phi_k(D)a), \rho_k(D)q(D)]$ is bounded in the norms (4-16) with operator norm $\lesssim \|a\|_{C^{1,1}}$. Here $\rho_k(\xi)q(\xi)$ is an order 1 classical symbol dyadically localized to $|\xi| \approx 2^k$. Thus, the kernel $K(x,y)$ of the commutator has bounds

$$|K(x,y)| \le C_N\|a\|_{C^{0,1}}2^{kn}(1+2^k|x-y|)^{-N},$$

so boundedness in the first norm in (4-16) follows by the Schur test as the weight is slowly varying over distance 2^{-k}. For boundedness in the second norm, we assume $\langle\omega,\xi\rangle = \xi_1$, let $q_k(D) = \rho_k(D)q(D)$, and need to show that, if $\hat{g}$ vanishes for $|\xi| \ge 2^k$, then

$$\|(1+D_{1,+})[(\phi_k(D)a), q_k(D)]g\|_{L^2} \le C\|a\|_{C^{1,1}}\|(1+D_{1,+})g\|_{L^2}, \tag{4-17}$$

$$\|2^{-k}D_1^2[(\phi_k(D)a), q_k(D)]g\|_{L^2} \le C\|a\|_{C^{1,1}}(\|g\|_{L^2} + 2^{-k}\|D_1^2 g\|_{L^2}). \tag{4-18}$$

The estimate (4-17) follows from Corollary A.9, since we may replace the multiplier $1+\xi_{1,+}$ by its truncation to $|\xi| \lesssim 2^k$. The estimate (4-18) follows by distributing derivatives similar to those above, and using that $\|D_1 a\|_{C^{0,1}} + 2^{-k}\|D_1^2\phi_k(D)a\|_{C^{0,1}} \lesssim \|a\|_{C^{1,1}}$ together with Theorem A.1.

We now turn to the proof of (4-14). By Lemma A.10,

$$\langle D\rangle P(t)^{-1} = \langle D\rangle(p^\sharp(t,x,D)+c)^{-1}\sum_{n=0}^{\infty}\big(p^\flat(t,x,D)(p^\sharp(t,x,D)+c)^{-1}\big)^n,$$

where the sum converges as a map on $L^2(\mathbb{R}^n)$, uniformly over t. Since $(p^\sharp(t,x,D)+c)^{-1}$ is a pseudodifferential operator of class $S^{-1}_{1,1/2}$, it follows from (A-6) that

$$\big\|\big(p^\flat(t,x,D)(p^\sharp(t,x,D)+c)^{-1}\big)^2 g\big\|_{H^1} \le C\|g\|_{L^2}$$

uniformly in t. Thus, the sum of terms over $n\ge 2$ gives a bounded map from L^2 to H^1, and the bound (4-14) holds for these terms since $c\le M_0\le\langle\xi\rangle$.

It thus suffices to show that

$$\|M_0 T p^\flat(t,x,D)\langle D\rangle^{-1}T^* f\|_{\ell^2L^2}\le C\|M_0 f\|_{\ell^2L^2}, \tag{4-19}$$

$$\|M_0 T\langle D\rangle(p^\sharp(t,x,D)+c)^{-1}T^* f\|_{\ell^2L^2}\le C\|M_0 f\|_{\ell^2L^2}. \tag{4-20}$$

In proving (4-19), we take the symbol expansion (A-1) of p, and use that order 0 multipliers are bounded in the M_0 norm to replace $p^\flat(x,D)\langle D\rangle^{-1}$ by $a^\flat$, and thus need to show

$$\|M_0 T a^\flat T^* f\|_{\ell^2L^2}\le C_0\|a\|_{C^{0,1}}\|M_0 f\|_{\ell^2L^2}.$$

This is proven as for (4-12). Indeed, the off-diagonal terms of $a^\flat$ are the same as for multiplication by a, and if $|j-k|\le 1$, the bound holds for both a and $\phi_{\lfloor k/2\rfloor}(D)a$.

The bound for (4-20) is simpler. The operator $\langle D\rangle(p^\sharp(t,x,D)+c)^{-1}$ is a pseudodifferential operator of type $S^0_{1,1/2}$, so the off-diagonal part is a smoothing operator; in particular, it maps $L^2(\mathbb{R}^n)$ to $H^1(\mathbb{R}^n)$. And for $|j-k|\le 1$, the operator

$$T_j\beta_j(D)\langle D\rangle(p^\sharp(t,x,D)+c)^{-1}\beta_k(D)T_k^*$$

is an integral kernel operator with kernel satisfying (4-11). □

5. The space-time version: Proof of Theorem 2.2.

In this section we deduce the space-time wavefront estimate in Theorem 2.2 from the fixed-time wavefront estimate established in Theorem 4.1. We use the notation of Section 3, with $P=p(t,x,D)$ denoting a choice of $p_\pm(t,x,D)$, and $Q(t)$ constructed according to the choice of $s_0\in\{0,1\}$.

Lemma 5.1. *If $u\in C^0H^{s_0}$ satisfies $D_tu-P(t)u-Q(t)u=0$, then*

$$(t_0,x_0,p(t_0,x_0,\xi_0),\xi_0)\notin \mathrm{WF}_{s_0+1}(u)\Longrightarrow (x_0,\xi_0)\notin\mathrm{WF}_{s_0+1}(u(t_0)).$$

Proof. Let $\chi(t,x)$ and $\tilde\chi(t,x)$ denote cutoff functions, with $\tilde\chi=1$ on a neighborhood of the support of χ, and $\chi=1$ on a neighborhood of (t_0,x_0). Also, let $\Gamma(\xi)$ and $\widetilde\Gamma(\xi)$ denote conic cutoffs, equal to one on a neighborhood of ξ_0, with $\widetilde\Gamma=1$ on a neighborhood of the support of Γ.

Let $\phi\in C_c^\infty(\mathbb{R})$ equal 1 near 0. If the support of $\phi\in C_c^\infty(\mathbb{R})$ is suitably close to 0, and $\tilde\chi$ and $\widetilde\Gamma$ also have suitably small support, then $\gamma(t_0)\notin\mathrm{WF}_{s_0+1}(u)$ implies that

$$\widetilde\Gamma(D)\phi(1-p(t_0,x_0,D)D_t^{-1})(\tilde\chi u)\in H^{s_0+1}(\mathbb{R}^{1+n}).$$

On the other hand, (2-10) implies that $Q(t)u(t) \in C^0 H^{s_0}$, so Theorem B.4 implies

$$\big(1-\phi(1-p(t_0,x_0,D)D_t^{-1})\big)(\tilde{\chi}u) \in L^2 H^{s_0+1},$$

hence

$$\widetilde{\Gamma}(D)(\tilde{\chi}u) \in L^2 H^{s_0+1}. \tag{5-1}$$

We next show that

$$(D_t - P(t) - Q(t))\Gamma(D)(\chi u) \in L^1 H^{s_0+1}. \tag{5-2}$$

Together with (5-1) and Theorem B.5, this implies $\Gamma(D)(\chi u) \in C^0 H^{s_0+1}$, hence $(x_0,\xi_0) \notin \mathrm{WF}_{s_0+1}(u(t_0))$.

To establish (5-2), we write

$$\begin{aligned}(D_t - P(t) - Q(t))\Gamma(D)\chi u \\ &= [\Gamma(D)\chi, Q(t)]u - [P(t),\Gamma(D)\chi]\widetilde{\Gamma}(D)\tilde{\chi}u + \Gamma(D)\chi P(t)(1-\widetilde{\Gamma}(D)\tilde{\chi})u \\ &\qquad + P(t)\Gamma(D)\chi(1-\widetilde{\Gamma}(D)\tilde{\chi})u + \Gamma(D)(D_t\chi)u. \end{aligned} \tag{5-3}$$

The next to last term on the right belongs to L^1H^2, since $u \in C^0L^2$ and the cutoffs give a smoothing operator, and the last term belongs to $L^2H^{s_0+1}$ by (5-1), hence both are in $L^1H^{s_0+1}$.

Since $u \in C^0H^{s_0}$, the first term on the right in (5-3) belongs to $L^1H^{s_0+1}$, by (2-13) and the fact that χ is C_c^∞ and the components of Q are smooth symbols in the ξ variable.

For the second and third terms in (5-3), by the symbol expansion (A-1) we may substitute $p(t,x,D) = a(t,x)q(D)$, with $q(\xi)$ a symbol of order 1 and $a \in C^{0,1} \cap L^1C^{1,1}$. We note that, as a consequence of Lemma A.4,

$$\|\Gamma(D)\chi a(t,\cdot)q(D)(1-\widetilde{\Gamma}(D))u(t,\cdot)\|_{H^{s_0+1}} \le C\|\chi a(t,\cdot)\|_{C^{1,1}}\|u(t)\|_{H^{s_0}},$$

and, additionally, as a consequence of pseudolocality of $q(D)\tilde{\Gamma}(D)$, where $s_0+1 \le 2$,

$$\|\Gamma(D)\chi a(t,\cdot)q(D)\widetilde{\Gamma}(D)(1-\tilde{\chi})u(t,\cdot)\|_{H^{s_0+1}} \le C\|a(t,\cdot)\|_{C^{1,1}}\|u(t)\|_{H^{s_0}}.$$

Since $u \in C^0H^{s_0}$, this handles the third term on the right in (5-3).

Now consider the second term on the right in (5-3), and write

$$[aq(D),\Gamma(D)\chi] = a\Gamma(D)[q(D),\chi] + [a,\Gamma(D)]\chi q(D).$$

Consider the case $s_0 = 0$. By Corollary A.2, we have

$$\|[aq(D),\Gamma(D)\chi]v\|_{L^2H^1} \lesssim \|a\|_{L^\infty C^{0,1}}\|v\|_{L^2H^1},$$

which we apply to $v = \widetilde{\Gamma}(D)\tilde{\chi}u \in L^2H^1$.

In case $s_0 = 1$, we use that $v = \widetilde{\Gamma}(D)\tilde{\chi}u \in L^2H^2 \cap C^0H^1$ by (5-1) and since $u \in C^0H^1$. We again apply Corollary A.2 to obtain

$$\begin{aligned}\|D[a,\Gamma(D)]\chi q(D)v\|_{L^1H^1} &\le \|[(Da),\Gamma(D)]\chi q(D)v\|_{L^1H^1} + \|[a,\Gamma(D)]D\chi q(D)v\|_{L^1H^1} \\ &\lesssim \|Da\|_{L^1C^{0,1}}\|v\|_{L^\infty H^1} + \|a\|_{L^\infty C^{0,1}}\|v\|_{L^1H^2}.\end{aligned}$$

Similarly, we use that $\Gamma(D)[q(D),\chi]v \in L^2H^2 \cap C^0H^1$, and

$$\|aw\|_{L^1H^2} \le \|Da\|_{L^1C^{0,1}}\|w\|_{L^\infty H^1} + \|a\|_{L^\infty C^{0,1}}\|w\|_{L^1H^2},$$

a consequence of the Leibniz rule, to handle the remaining term. $\square$

We now observe that if the conditions of Theorem 2.2 hold and $s \in (-T, T)$, then, by Lemma 5.1 and Theorem 4.1, there is a function $\chi \in C_c^\infty(\mathbb{R}^{1+n})$ equal to 1 on a neighborhood of (s, x_s), and conic cutoff $\Gamma(\xi)$ equal to 1 on a neighborhood of ξ_s, so that

$$\Gamma(D)\chi u \in L^2H^{s_0+1},$$

where $\gamma(s) = (s, x_s, p(s, x_s, \xi_s), \xi_s)$. Since $p \approx |\xi|$, it follows that $\gamma(s) \notin \mathrm{WF}_{s_0+1}(u)$, which completes the proof of Theorem 2.2. $\square$

6. Piecewise regular coefficients

We work in this section with L of the form

$$L = D_t^2 - 2D_jb^j(t,x)D_t - D_ic^{ij}(t,x)D_j + d^0(t,x)D_t + d^j(t,x)D_j, \tag{6-1}$$

where the coefficients satisfy certain piecewise regularity conditions with respect to a decomposition of $(-T, T)\times\mathbb{R}^n$ into disjoint time slabs $-T = t_1 < t_2 < \cdots < t_{n-1} < t_n = T$. Given such a partition, we assume that the coefficients c^{ij} and b^j satisfy the conditions (1-3) and (1-4) separately on each time slab $(t_j, t_{j+1})\times\mathbb{R}^n$. In addition, we assume that c^{ij} and b^j are continuous on $[-T, T]\times\mathbb{R}^n$. This implies in particular that c^{ij} and b^j belong to $C^{0,1}([-T, T]\times\mathbb{R}^n)$, and that the map $t \to c^{ij}(t,\cdot)$, respectively $t \to b^j(t,\cdot)$, is continuous from $[-T, T]$ into $C^1(\mathbb{R}^n)$.

Similarly, we assume d^0 and d^j satisfy (1-6) separately on each time slab, hence on each slab they admit a continuous extension to $[t_j, t_{j+1}]\times\mathbb{R}^n$. We allow d_j to have jumps at t_j for $1 \le j \le n$. It is unimportant how d^j is defined at $t = t_j$, but for definiteness we assume it is right continuous.

We assume the coefficient d^0 belongs to $C^0([-T, T]\times\mathbb{R}^n)$, which with the above is equivalent to assuming $\partial_{t,x}d^0 \in L^1L^\infty((-T, T)\times\mathbb{R}^n)$. The continuity assumption on d^0 is needed for weak solutions of $Lu = 0$ to agree with solutions defined separately on each slab with matching Cauchy data at each t_j. At the end of this section, we indicate how to handle jumps in d^0.

For $s_0 \in \{0, 1, 2\}$ and Cauchy data of regularity $H^{s_0}\times H^{s_0-1}$ at some t_0, one obtains a solution to $Lu = 0$ of regularity $C^0H^{s_0}\cap C^1H^{s_0-1}$ by piecing together solutions on $[t_j, t_{j+1}]$, and imposing continuity of u and D_tu at t_j. Such a solution is easily verified to satisfy $\int uL^t\phi = 0$ for $\phi \in C_c^\infty((-T, T)\times\mathbb{R}^n)$, with L^t the formal transpose of L.

That this u is the unique weak solution of regularity $C^0H^{s_0}\cap C^1H^{s_0-1}$ follows immediately from uniqueness for the Cauchy problem on each time slab, by the assumed continuity condition in t of $(u(t), D_tu(t))$.

Since the first-order derivatives in x of c^{ij} and b^j satisfy the regularity conditions of d^0, one can convert between the standard form of L in the introduction and one of the form (6-1) and preserve the regularity assumptions. Since the first order derivatives in t of b^j satisfy the conditions on d^j for $1 \le j \le n$, one could also express a term of the form $D_t b^j D_j$ in the form (6-1). Indeed, up to addition of an $L^1 L^\infty$ function, the class (6-1) is closed under transpose.

If we factor the principal symbol of L as before, as

$$H(t,x,\tau,\xi) = (\tau - p_+(t,x,\xi))(\tau + p_-(t,x,\xi)),$$

then $p_\pm$ are C^1, and $\partial^2_{x,\xi} p_\pm$ belongs to $L^1 L^\infty$, hence the null bicharacteristics of L are well-defined C^1 curves.

Theorem 6.1. *Assume that the coefficients of L are as above. Suppose that $s_0 \in \{0,1\}$, that $Lu = 0$, and that $u \in C^0 H^{s_0} \cap C^1 H^{s_0-1}$.*

Then, if $\gamma(t)$ is a null bicharacteristic curve of L and $\gamma(t_0) \notin \mathrm{WF}_{s_0+1}(u)$ for some $t_0 \in (-T,T)$, then $\gamma \cap \mathrm{WF}_{s_0+1}(u) = \varnothing$.

Proof. For simplicity, we assume that the partition consists of $[-T,0] \cup [0,T]$. The general case follows easily. By openness of the wavefront set, we may then assume $t_0 \neq 0$, and without loss of generality take $t_0 < 0$.

We derive the result as a limiting case of Theorem 1.1, using uniformity of the wavefront set estimates over bounded sets of coefficients. Precisely, we use the fact that all of the bounds on wavefront sets involve only uniform control over appropriate norms of the coefficients. To define uniform cutoffs, we fix a smooth radial cutoff to the half-unit ball $\chi(t,x)$, supported in the unit ball, and let $\chi_{c,t_0,x_0}(t,x) = \chi(c^{-1}(t-t_0), c^{-1}(x-x_0))$. We also fix a conic cutoff Γ, rotationally symmetric about the ξ_1 axis and supported in the cone of angle $c\pi$,

$$\Gamma_c(\tau,\xi) = \chi(c^{-1}\xi_1^{-1}\tau, c^{-1}\xi_1^{-1}\xi', 0),$$

and define $\Gamma_{c,\tau_0,\xi_0}(\tau,\xi)$ by composing Γ_c with a rotation that centers it on the ray through (τ_0,ξ_0). The following result is then a consequence of the fact that the bounds and support of the cutoffs in the wavefront estimates in the proof of Theorem 1.1 depend only on bounds for the cited quantities in L.

Corollary 6.2. *Suppose that, for some $0 < c_0, C_0 < \infty$, the coefficients of L satisfy the bounds (1-2), (1-3), (1-4), and (1-6), where $\|\alpha\|_{L^1} \le C_0$.*

Suppose that $u \in C^0 H^{s_0} \cap C^1 H^{s_0-1}$ satisfies $Lu = 0$, and that

$$\sup_{t \in (-T,T)} (\|u(t)\|_{H^{s_0}} + \|D_t u(t)\|_{H^{s_0-1}}) \le C_0.$$

Let $\gamma(t) = (t, x_t, \tau_t, \xi_t)$ be a null bicharacteristic for L, and suppose that the following holds for some $0 < c_1, C_1 < \infty$ and some t_0:

$$\|\Gamma_{c_1,\tau_{t_0},\xi_{t_0}}(\boldsymbol{D})\chi_{c_1,t_0,x_{t_0}} u\|_{H^{s_0+1}} \le C_1.$$

Then, if $T' < T$, there are $0 < c_2, C_2 < \infty$, depending only on c_0, C_0, c_1, C_1, and T', so that

$$\|\Gamma_{c_2,\tau_t,\xi_t}(\boldsymbol{D})\chi_{c_2,t,x_t} u\|_{H^{s_0+1}} \le C_2$$

for all $|t| \le T'$.

We will consider a family of operators L_n of the form

$$L_n = D_t^2 - 2D_j b_n^j D_t - D_i c_n^{ij} D_j + d_n^0 D_t + d_n^j D_j,$$

which converge appropriately to L, and such that the coefficients of L_n satisfy (1-2), (1-3), (1-4), and (1-6), with constants c_0, C_0 uniform over n. Since the class L of Theorem 1.1 can be expressed in the form (6-1) with comparable c_0, C_0, the corollary applies to L_n.

To construct L_n, we fix an increasing function $h \in C^\infty(\mathbb{R})$ which vanishes for $s < -1$, equals 1 for $s > 1$, and so that $h(s)+h(-s) = 1$. For $c^{ij}(t, x)$ as above, we let c_-^{ij} denote its restriction to $t \in [-T, 0]$, which, using (1-7), we assume extended to a function on $[-T, T]$ satisfying (1-3) and (1-4) there. Similarly, let c_+^{ij} denote its restriction to $[0, T]$, appropriately extended to $[-T, T]$. Define

$$\begin{aligned} c_n^{ij}(t, x) &= h(-nt)c_-^{ij}(t, x) + h(nt)c_+^{ij}(t, x) \\ &= c_-^{ij}(t, x) + h(nt)(c_+^{ij}(t, x) - c_-^{ij}(t, x)). \end{aligned}$$

Since $c_+^{ij}(0, x) = c_-^{ij}(0, x)$, it is seen that the estimates (1-3) and (1-4) are satisfied by c_n^{ij} on $(-T, T)$ with uniform bounds for $\|\alpha_n\|_{L^1}$. Furthermore,

$$c_n^{ij}(t, x) = c^{ij}(t, x) \quad \text{if } |t| > \frac{1}{n}.$$

We apply this smoothing technique to the coefficients c^{ij} and b^j of L. Since d^0 is already globally regular, we set $d_n^0 = d^0$. We also define

$$d_n^j(t, x) = h(-nt)d_-^j(t, x) + h(nt)d_+^j(t, x), \quad 1 \le j \le n,$$

which satisfies (1-6) with uniform bounds on $\|\alpha_n\|_{L^1}$. We then define L_n to be the operator of form (6-1) with modified coefficients, and note that $L = L_n$ for $|t| > 1/n$.

If we factor the Hamiltonian of the principal part of L_n as

$$H_n(t, x, \tau, \xi) = (\tau - p_{n,+}(t, x, \xi))(\tau + p_{n,-}(t, x, \xi)),$$

then $p_{n,\pm} = p_\pm$ for $|t| > 1/n$, and $D_{x,\xi}\, p_{n,\pm}$ converges uniformly to $D_{x,\xi}\, p_\pm$ on compact sets. It follows that the null bicharacteristic of L_n through a given initial point converges uniformly on $[-T, T]$ to the null bicharacteristic of L through that point.

We henceforth assume n large so that $t_0 < -1/n$. Then the solution to $Lu = 0$ with given Cauchy data at t_0 satisfies $L_n u = 0$ for $-T < t < 1/n$, in particular for t near t_0. Thus, if we let u_n be the solution to $L_n u_n = 0$, with the same Cauchy data as u at t_0, then $u_n = u$ for $-T < t < -1/n$. In particular, for all n and c_1 small,

$$\|\Gamma_{c_1,\tau_{t_0},\xi_{t_0}}(\boldsymbol{D})\chi_{c_1,t_0,x_{t_0}} u_n\|_{H^{s_0+1}} \le C_1.$$

Thus, since the null bicharacteristic of L_n through $(t_0, x_0, \tau_0, \xi_0)$ converges uniformly to the null bicharacteristic of L through $(t_0, x_0, \tau_0, \xi_0)$, Corollary 6.2 shows that, for n large and some small $c_2 > 0$ with C_2 independent of n,

$$\|\Gamma_{c_2,\tau_t,\xi_t}(\boldsymbol{D})\chi_{c_2,t,x_t} u_n\|_{H^{s_0+1}} \le C_2.$$

We will prove that some subsequence u_{n_j} converges weakly as distributions to u, from which we obtain the desired result,

$$\|\Gamma_{c_2,\tau_t,\xi_t}(\boldsymbol{D})\chi_{c_2,t,x_t}u\|_{H^{s_0+1}} \le C_2.$$

To show the convergence, we observe that, by weak compactness, some subsequence $(u_{n_j}, D_t u_{n_j})$ converges weakly in $L^\infty H^{s_0} \times L^\infty H^{s_0-1}$ to $(v, D_t v) \in L^\infty H^{s_0} \times L^\infty H^{s_0-1}$. We next verify that v is in fact of regularity $C^0 H^{s_0} \cap C^1 H^{s_0-1}$ separately on $[-T, 0] \times \mathbb{R}^n$ and $[0, T] \times \mathbb{R}^n$. For $t < 0$, this is trivial, since $u_n = u$ for $t < -1/n$, hence $v = u$ for $t < 0$. For $t > 0$, if $s_0 = 1$, it follows from Theorem B.6, since $Lv = 0$ and $v \in H^1((-T, T) \times \mathbb{R}^n)$. If $s_0 = 0$, then Lemma B.1 yields that $L(\langle D\rangle^{-1}v) \in L^1 L^2$. Since $(\langle D\rangle^{-1}v,\, D_t\langle D\rangle^{-1}v) \in L^\infty H^1 \times L^\infty L^2$, the result again follows from Theorem B.6.

Thus v consists of regular solutions on $(-T, 0)$ and $(0, T)$ to $Lv = 0$, and it remains only to show that the Cauchy data match at $t = 0$, since $v = u$ for $t < 0$. To see that the data match, we note that, for $\psi(t, x) \in C_c^\infty((-T, T) \times \mathbb{R}^n)$,

$$0 = \int v(L^t\psi)\,dt\,dx.$$

Integration by parts separately on $t > 0$ and $t < 0$ leads to the condition

$$\begin{aligned}\int_{\mathbb{R}^n} (v(0^+, x) - v(0^-, x))(D_t\psi(0, x) - b^j(0, x)D_j\psi(0, x))\,dx \\ = \int_{\mathbb{R}^n} \big(D_t v(0^+, x) - D_t v(0^-, x) + d^0(0^+, x)v(0^+, x) - d^0(0^-, x)v(0^-, x)\big)\psi(0, x)dx.\end{aligned}$$

Since this vanishes for all ψ, we must have $v(0^+, x) = v(0^-, x)$, and if $d^0(0^+, x) = d^0(0^-, x)$, as we assume, then also $D_t v(0^+, x) = D_t v(0^-, x)$. □

We remark that if d^0 is piecewise regular with jumps at t_j, that is, of the same regularity as d^j, then the result still holds, but the solution u must be defined by piecing together $C^0 H^{s_0} \cap C^1 H^{s_0-1}$ solutions on $[t_j, t_{j+1}]$ with the following matching conditions on u at each t_j:

$$u(t_j^+) = u(t_j^-), \qquad D_t u(t_j^+) - D_t u(t_j^-) + d^0(t_j^+)u(t_j^+) - d^0(t_j^-)u(t_j^-) = 0. \tag{6-2}$$

The proof shows that the limiting solution v agrees with this solution u, hence the result of Corollary 6.2 holds for u satisfying (6-2). The one modification to the proof is to define d_n^0 similar to d_n^j, so that it meets the regularity conditions (1-6).

An example showing sharpness. We now show that the assumption of global H^{s_0} regularity on u cannot be lowered in Theorem 6.1, hence it is necessary for Theorem 1.1 to hold with bounds depending only on the appropriate norms of the coefficients. Precisely, we construct a piecewise smooth operator L and a corresponding null bicharacteristic γ, and for each $\sigma \le 2$, a solution $Lu = 0$ with u of H^σ regularity, such that u is microlocally smooth on $\gamma(t)$ for $t < -1$, but for all $\epsilon > 1$, we have $\gamma(t) \in \mathrm{WF}_{\sigma+\epsilon}(u)$ for $t \ge -1$.

Consider the following hyperbolic equation on $\mathbb{R}_t \times \mathbb{R}_x$:

$$\begin{cases}(\partial_t^2 + t\partial_x^2)u(t, x) = 0, & t \le -1,\\ (\partial_t^2 - \partial_x^2)u(t, x) = 0, & t \ge -1.\end{cases} \tag{6-3}$$

Let $A(s) = Ai(\omega s)$ be the solution to the Airy equation $A''(s) + sA(s) = 0$, where $\omega = e^{2\pi i/3}$. Then, for $s < 0$,

$$A(s) = e^{i\frac{2}{3}(-s)^{3/2}} a(-s), \quad a(s) \sim \sum_{k=0}^{\infty} a_k s^{-\frac{1}{4}-\frac{3}{2}k}.$$

Furthermore, $A(s) \neq 0$ for $s < 0$, and each $a_k \neq 0$.

If $\xi \geq 1$, we consider the solutions to (6-3)

$$u_\xi(t,x) = \begin{cases} e^{i\xi x} \dfrac{A(\xi^{2/3}t)}{A(-\xi^{2/3})}, & t \leq -1, \\ e^{i\xi x}(c_0(\xi)e^{-i\xi(t+1)} + c_1(\xi)e^{i\xi(t+1)}), & t \geq -1, \end{cases}$$

where the following matching conditions are met to yield $u \in C^{1,1}(\mathbb{R}^2)$:

$$c_0(\xi) + c_1(\xi) = 1, \quad -i\xi(c_0(\xi) - c_1(\xi)) = \xi^{2/3} \frac{A'(-\xi^{2/3})}{A(-\xi^{2/3})} = -i\xi\left(1 - i\xi^{-1/3}\frac{a'(\xi^{2/3})}{a(\xi^{2/3})}\right),$$

Then $c_0(\xi)$ and $c_1(\xi)$ are smooth on $\xi \geq 1$, and admit an asymptotic expansion

$$c_0(\xi) = 1 - c_1(\xi), \quad c_1(\xi) \sim \frac{i}{8}\xi^{-1}\left(1 + \sum_{k=1}^{\infty} d_k \xi^{-k}\right).$$

If $\xi \leq -1$, we set

$$u_\xi(t,x) = \overline{u_{-\xi}(t,x)}. \tag{6-4}$$

For $-1 \leq \xi \leq 1$, we take a combination of the solutions A and $\bar{A}$ in the definition of u_ξ so that $u_\xi(-1,x) = e^{i\xi x}$ and (6-4) holds. Then

$$u_\xi(t,x) = \begin{cases} e^{i\xi x + i\frac{2}{3}\xi(-t)^{3/2} - i\frac{2}{3}\xi} a(t,\xi), & t \leq -1, \\ e^{i\xi x}(c_0(\xi)e^{-i\xi(t+1)} + c_1(\xi)e^{i\xi(t+1)}), & t \geq -1, \end{cases}$$

where $a(t,\xi)$ and $c_0(\xi)$ are elliptic symbols in ξ of order 0, and $c_1(\xi)$ is elliptic of order -1.

Let $b(\xi) = \langle\xi\rangle^{-1/2-\sigma}\log(2+|\xi|)^{-1}$, and set

$$u(t,x) = \int_{-\infty}^{\infty} b(\xi) u_\xi(t,x)\, d\xi.$$

Then $u \in C^0 H^\sigma \cap C^1 H^{\sigma-1} \cap C^2 H^{\sigma-2}$. Consequently, $\mathrm{WF}_\sigma(u) = \varnothing$ if $\sigma \leq 2$.

For each $\sigma \in \mathbb{R}$ and $\epsilon > 0$, on the set $t < -1$, we see by stationary phase that

$$\mathrm{WF}_{\sigma+\epsilon} u = \left\{(t,x,\tau,\xi) : x = -\tfrac{2}{3}(-t)^{3/2} + \tfrac{2}{3},\ \tau + (-t)^{1/2}\xi = 0\right\}.$$

For $t > -1$ and $0 < \epsilon \leq 1$, since c_1 is of order -1, we have

$$\mathrm{WF}_{\sigma+\epsilon} u = \{(t,x,\tau,\xi) : x = 1 + t,\ \tau + \xi = 0\},$$

On the other hand, for $t > -1$ and $\epsilon > 1$, we have

$$\mathrm{WF}_{\sigma+\epsilon} u = \bigcup_{\pm} \{(t,x,\tau,\xi) : x = \pm(1+t),\ \tau \pm \xi = 0\}.$$

Thus, if $\epsilon > 1$, then $\mathrm{WF}_{\sigma+\epsilon}$ contains the null bicharacteristic $\gamma(t) = (t, -(1+t), 1, 1)$ for $t \geq -1$, but is microlocally smooth on its continuation for $t < -1$, $\gamma(t) = \bigl(t, \frac{2}{3}(-t)^{3/2} - \frac{2}{3}, (-t)^{1/2}, 1\bigr)$. $\square$

Appendix A: Paraproduct estimates

In this section we collect the paraproduct and commutator estimates used throughout this paper. By a standard multiplier of order m, we understand a function $q(\xi) \in C^\infty(\mathbb{R}^n)$ such that

$$|\partial_\xi^\alpha q(\xi)| \leq C_\alpha (1 + |\xi|)^{m-|\alpha|},$$

and denote the class of such multipliers by $S^m(\mathbb{R}^n)$. The best constants C_α form the seminorms of q. In the statements of this section, it is implicit that the constant C in any given operator bound for a multiplier depends on a finite number of the C_α. We say that $q \in S^m$ is a classical multiplier if, in addition,

$$q(r\xi) = r^m q(\xi), \quad r \geq 1, \ |\xi| \geq 1,$$

and denote this subspace by S^m_{cl}.

The homogeneous symbol $p(t, x, \xi)$ admits a convergent expansion on the set $|\xi| \geq 1$ of the form

$$p(t, x, \xi) = \sum_{l=1}^\infty a_l(t, x) q_l(\xi), \quad q_l(\xi) = |\xi| \omega_l(\xi/|\xi|) \in S^1_{\mathrm{cl}}(\mathbb{R}^n), \tag{A-1}$$

where ω_l are spherical harmonics, and $a_l(t, x)$ satisfies the regularity conditions in (1-3) and (1-4), with constants C_l that decrease rapidly in l. We may smoothly extend the $q_l(\xi)$ near 0 so that this expansion is valid for all ξ. The seminorms of q_l grow at most polynomially in l, so the bounds in prior sections on $R_1^\pm$, etc., are convergent.

The Coifman–Meyer commutator theorem [1978], which generalizes the Calderón commutator theorem [1965] for homogeneous multipliers, is the following.

Theorem A.1 (Coifman–Meyer commutator theorem). *Suppose that $a \in C^{0,1}(\mathbb{R}^n)$ and $q \in S^1(\mathbb{R}^n)$. Then*

$$\|[a, q(D)]f\|_{L^2} \leq C \|a\|_{C^{0,1}} \|f\|_{L^2}.$$

An immediate corollary, as seen by commuting or composing with D, is the following:

Corollary A.2. *If $q \in S^1(\mathbb{R}^n)$ and $a \in C^{1,1}(\mathbb{R}^n)$, then*

$$\|[a, q(D)]f\|_{H^s} \leq C \|a\|_{C^{1,1}} \|f\|_{H^s}, \quad -1 \leq s \leq 1.$$

If $q \in S^0(\mathbb{R}^n)$ and $a \in C^{0,1}(\mathbb{R}^n)$, respectively $a \in C^{1,1}(\mathbb{R}^n)$, then

$$\|[a, q(D)]f\|_{H^{s+1}} \leq C \|a\|_{C^{0,1}} \|f\|_{H^s}, \quad -1 \leq s \leq 0,$$
$$\|[a, q(D)]f\|_{H^{s+1}} \leq C \|a\|_{C^{1,1}} \|f\|_{H^s}, \quad -2 \leq s \leq 1.$$

A key ingredient in the proof of the commutator theorem is the following estimate, due to Carleson [1962] and Fefferman and Stein [1972]; for a proof, see [Stein 1993, II.2.4, IV.4.3].

Theorem A.3. *Suppose that $\phi,\,\psi\in\mathcal{S}(\mathbb{R}^n)$, and that $\psi(0)=0$. Let $\psi_j(D)=\psi(2^{-j}D)$, $\phi_j(D)=\phi(2^{-j}D)$. Then*

$$\Bigl(\sum_{j=0}^{\infty}\int_{\mathbb{R}^n}|\psi_j(D)a|^2|\phi_j(D)f|^2\,dx\Bigr)^{\frac12}\le C\|a\|_{\mathrm{BMO}}\|f\|_{L^2}.$$

Theorem A.3 yields smoothing estimates for the off-diagonal terms in paraproducts. To state these, form a Littlewood–Paley partition of unity $\{\psi_k(\xi)\}_{k=1}^{\infty}$ by taking $\psi_k(\xi)=\beta_k^2(\xi)$, with β_k as in (3-3). Then let

$$\phi_k(D)=\sum_{j=0}^{k-2}\psi_j(D),\qquad \rho_k(D)=\sum_{j=k-1}^{k+1}\psi_j(D). \tag{A-2}$$

If $g\in L^2(\mathbb{R}^n)$ and $a\in L^\infty(\mathbb{R}^n)$, we decompose $ag=R_ag+T_ag$, where

$$\begin{aligned}R_ag&=\sum_{|j-k|\ge2}\psi_j(D)(a\psi_k(D)g)\\&=\sum_{j=2}^{\infty}\psi_j(D)(a\phi_j(D)g)+\sum_{k=2}^{\infty}\phi_k(D)(a\psi_k(D)g)\end{aligned}$$

and

$$T_ag=\sum_{|j-k|\le1}\psi_j(D)(a\psi_k(D)g)=\sum_{j=0}^{\infty}\psi_j(D)((\phi_{j+4}(D)a)(\rho_j(D)g)).$$

With the exception of the last identity, in the above, a may be replaced by a general bounded linear operator on L^2.

Lemma A.4. *Suppose $a\in C^{1,1}(\mathbb{R}^n)$. If $-1\le\sigma\le1$, then*

$$\|\langle D\rangle^{1+\sigma}R_a(\langle D\rangle^{1-\sigma}g)\|_{L^2}\lesssim\|a\|_{C^{1,1}}\|g\|_{L^2}.$$

Suppose $a\in C^{0,1}(\mathbb{R}^n)$. If $0\le\sigma\le1$, then

$$\|\langle D\rangle^{\sigma}R_a(\langle D\rangle^{1-\sigma}g)\|_{L^2}\lesssim\|a\|_{C^{0,1}}\|g\|_{L^2}.$$

Proof. We prove the first estimate; the second follows by similar steps. By interpolation we may restrict attentions to $\sigma=\pm1$, and by considering adjoints we can assume that $\sigma=-1$. We may then replace $\langle D\rangle^2$ by D^2, which denotes an arbitrary second-order derivative. First consider

$$\sum_{k=2}^{\infty}\phi_k(D)(a\psi_k(D)D^2g)=\sum_{k=2}^{\infty}\phi_k(D)((\rho_k(D)a)\psi_k(D)D^2g).$$

We take the inner product with $h\in L^2$; by the Cauchy–Schwarz inequality and almost orthogonality over k of $\psi_k(D)g$, we can dominate the result by

$$\Bigl(\sum_{k=2}^{\infty}\int|2^{2k}\rho_k(D)a|^2\cdot|\phi_k(D)h|^2\,dx\Bigr)^{\frac12}\|g\|_{L^2}\lesssim\|D^2a\|_{\mathrm{BMO}}\|h\|_{L^2}\|g\|_{L^2},$$

where we use Theorem A.3 and write $2^{2j}\rho_j(D)a = \bar{\rho}_j(D)D^2a$.

Now consider the remaining term,

$$\sum_{j=2}^{\infty}\psi_j(D)(a\phi_j(D)D^2g) = \sum_{j=2}^{\infty}\psi_j(D)\big((\rho_j(D)a)\phi_j(D)D^2g\big).$$

By almost orthogonality over j we can dominate the L^2 norm of this sum by

$$\left(\sum_{j=2}^{\infty}\int |2^{2j}\rho_j(D)a|^2\cdot|2^{-2j}\phi_j(D)D^2g|^2\,dx\right)^{\frac{1}{2}} \lesssim \|D^2a\|_{\mathrm{BMO}}\|g\|_{L^2},$$

where we use Theorem A.3 and write $2^{-2j}\phi_j(D)D^2g = \bar{\phi}_j(D)g$. □

Corollary A.5. *For $a\in C^{0,1}(\mathbb{R}^n)$, define the operator $a^\flat$ by*

$$a^\flat g = \sum_{j=0}^{\infty}(a-(\phi_{\lfloor j/2\rfloor}(D)a))\psi_j(D)g.$$

Then

$$\|a^\flat g\|_{H^{s+1/2}} \le C\|a\|_{C^{0,1}}\|g\|_{H^s},\quad -1\le s\le \tfrac{1}{2}.$$

If $a\in C^{1,1}(\mathbb{R}^n)$, then

$$\|a^\flat g\|_{H^{s+1}} \le C\|a\|_{C^{1,1}}\|g\|_{H^s},\quad -2\le s\le 1.$$

Proof. We write

$$a^\flat g = R_a g + \sum_{j=0}^{\infty}\rho_j(D)(a-(\phi_{\lfloor j/2\rfloor}(D)a))\psi_j(D)g.$$

The desired bound for $R_a g$ follows from Lemma A.4, and for the second term it follows by orthogonality and the bound

$$\|a-\phi_{\lfloor j/2\rfloor}(D)a\|_{L^\infty} \le C\min(2^{-j/2}\|a\|_{C^{0,1}}, 2^{-j}\|a\|_{C^{1,1}}).$$ □

Corollary A.6. *Suppose $a\in C^{1,1}(\mathbb{R}^n)$ and $q\in S^1(\mathbb{R}^n)$. If $0\le\sigma\le 1$, then*

$$\|\langle D\rangle^\sigma R_{[a,q(D)]}(\langle D\rangle^{1-\sigma}g)\|_{L^2} \lesssim \|a\|_{C^{1,1}}\|g\|_{L^2}.$$

Proof. We note that $R_{[a,q(D)]} = [R_a, q(D)]$. The estimate then follows by Lemma A.4, since it yields that, for $0\le\sigma\le 1$,

$$\|q(D)\langle D\rangle^\sigma R_a(\langle D\rangle^{1-\sigma}g)\|_{L^2} + \|\langle D\rangle^\sigma R_a q(D)(\langle D\rangle^{1-\sigma}g)\|_{L^2} \lesssim \|a\|_{C^{1,1}}\|g\|_{L^2}.$$ □

We will need an extension of these results involving double commutators.

Lemma A.7. *Suppose that $a\in C^{1,1}(\mathbb{R}^n)$ and $b\in C^{0,1}(\mathbb{R}^n)$, and that $q_0, q_1\in S^1(\mathbb{R}^n)$ are Fourier multipliers on $\mathbb{R}^n$. Then we have*

$$\|[[a,q_0(D)],q_1(D)]g\|_{L^2} \le C\|a\|_{C^{1,1}}\|g\|_{L^2}, \tag{A-3}$$

$$\|[b,[a,q_0(D)]]q_1(D)g\|_{L^2} \le C\|a\|_{C^{1,1}}\|b\|_{C^{0,1}}\|g\|_{L^2}. \tag{A-4}$$

Proof. We start with the proof of (A-3). We decompose the multiplication operator a into $T_a + R_a$. By Corollary A.6,

$$\|[R_a, q_0(D)]q_1(D)g\|_{L^2} + \|q_1(D)[R_a, q_0(D)]g\|_{L^2} \le C\|a\|_{C^{1,1}}\|g\|_{L^2}.$$

For the term $[T_a, q_0(D)]$, since $\psi_k(D)$ and $\rho_j(D)$ commute with the $q(D)$ and have finite overlap of support, it suffices to prove that, uniformly over j,

$$\|\psi_j(D)[[a, q_0(D)], q_1(D)]\rho_j(D)g\|_{L^2} \le C\|g\|_{L^2}.$$

We may then replace $q_0(D)$ and $q_1(D)$ by their dyadic localization to $|\xi| \approx 2^j$, in which case they are represented by convolution kernels $K_{0,j}$ and $K_{1,j}$ for which

$$|K_j(x-y)| \le C_N 2^{j(n+1)}(1+2^j|x-y|)^{-N} \quad \text{for all } N.$$

After this substitution, we may ignore the factors $\psi_j(D)$ and $\rho_j(D)$. We next expand

$$a(x) - a(y) = a'(x)(x-y) + r(x,y)(x-y)^2, \quad \|r(x,y)\|_{L^\infty} \le C\|a\|_{C^{1,1}}. \tag{A-5}$$

The integral kernel $r(x,y)(x-y)^2 K_{0,j}(x-y)$ has operator norm $\lesssim 2^{-j}\|a\|_{C^{1,1}}$, whereas $K_{1,j}$ has operator norm $\lesssim 2^j$, hence this contribution to the double commutator is bounded on L^2. Letting $q_0'(D)$ denote the L^2-bounded operator with kernel $(x-y)K_{0,j}(x-y)$, the other term yields $[a', q_1(D)]q_0'(D)$, which is bounded on L^2 with norm $\|a\|_{C^{1,1}}$ by a similar argument, or using Theorem A.1.

To establish (A-4), we first use Corollary A.6 to see that

$$\|R_a q_0(D)g\|_{L^2} + \|q_0(D)R_a g\|_{L^2} \le C\|a\|_{C^{1,1}}\|g\|_{H^{-1}}.$$

We may thus replace a by T_a. In the j-th term for $[T_a, q_0(D)]$, we may replace $q_0(D)$ and $q_1(D)$ by their j-th dyadic localization as above. Expanding a as in (A-5), the second-order remainder term leads to a bounded operator. It thus suffices to show that

$$\left\| \left[b, \sum_j \psi_j(D)a'\rho_j(D)q_0'(D) \right] q_1(D)g \right\|_{L^2} \le C\|a'\|_{C^{0,1}}\|b\|_{C^{0,1}}\|g\|_{L^2},$$

where $q_0'(D)$ is a multiplier of order 0. We may write the commutator on the left-hand side as

$$a'[b, q_0'(D)]q_1(D) - [b, R_{a'}q_0'(D)]q_1(D).$$

The first term has the desired bound on L^2 by Corollary A.2, and the second term has the desired bound by Lemma A.4. $\square$

Remark A.8. The estimate (A-4) can be established with $\|a\|_{C^{0,1}}\|b\|_{C^{0,1}}\|g\|_{L^2}$ on the right-hand side. This is the second commutator estimate; see for example [Stein 1993]. The simpler estimate in (A-4) suffices for our purposes, however.

Corollary A.9. *Suppose that $q_1(\xi_1) \in S^1(\mathbb{R})$ and $q_0(\xi) \in S^1(\mathbb{R}^n)$. Then, uniformly over k,*

$$\|[[a, \rho_k(D)q_0(D)], \phi_k(D_1)q_1(D_1)]g\|_{L^2} \le C\|a\|_{C^{1,1}}\|g\|_{L^2}.$$

Proof. Let $q_{0,k} = \rho_k q_0$. As in the proof of (A-3), we write

$$[a, q_{0,k}(D)] = a'(x)q'_{0,k}(D) + r_k, \quad \|r_k g\|_{L^2} \le C2^{-k}\|a\|_{C^{1,1}}\|g\|_{L^2}.$$

The operator $\phi_k(D_1)q_1(D_1)$ has norm $\lesssim 2^k$, so $[r_k, \phi_k(D_1)q_1(D_1)]$ is suitably bounded. This leaves the term $[a'(x), \phi_k(D_1)q_1(D_1)]q'_{0,k}(D)$, which is bounded uniformly over k by Theorem A.1, since $a'(x)$ is a $C^{0,1}$ function of x_1, uniformly over $(x_2, \dots, x_n)$, with norm less than $\|a\|_{C^{1,1}}$. □

Lemma A.10. *Let $s_0 \in \{0, 1\}$ and $Q^{\pm}$ be constructed as in Section 2. Then, for $c > 0$ sufficiently large, the operator $2P(t_0) + Q^+(t_0) + Q^-(t_0) + c : H^{s_0} \to H^{s_0-1}$ has a bounded right inverse for each $t_0 \in (-T, T)$. Furthermore, with uniform bounds over $t \in (-T, T)$,*

$$(2P(t_0) + Q^+(t_0) + Q^-(t_0) + c)^{-1} : H^{s-1} \to H^s, \quad s_0 \le s \le s_0 + 1,$$

and the inverse is a continuous function of t_0 into the operator norm topology. Also, $P(t_0) + c : L^2 \to H^{-1}$ is invertible, and

$$(P(t_0) + c)^{-1} : H^{s-1} \to H^s, \quad 0 \le s \le 2,$$

with norm-continuity of the inverse over $t \in (-T, T)$.

Proof. Consider a fixed value of t_0, and let $p(x, \xi) = p(t_0, x, \xi)$. We use only $C^{0,1}$ bounds on the symbol $p(x, \xi)$, so all estimates on p in the following proof will be uniform over t and norm-continuous in t. We write $p(x, \xi) = p^\sharp(x, \xi) + p^\flat(x, \xi)$ with

$$p^\sharp(x, \xi) = \sum_{j=0}^{\infty} (\phi_{\lfloor j/2 \rfloor}(D)p)(x, \xi)\psi_j(\xi) \in S^1_{1,1/2},$$

where the frequency truncation is in the x variable. By Corollary A.5 and the symbol expansion (A-1),

$$\|p^\flat(x, D)g\|_{H^s} \le C\|g\|_{H^{s+1/2}}, \quad -\tfrac{1}{2} \le s \le 1. \tag{A-6}$$

Since $|p^\flat(x, \xi)| \le C(1 + |\xi|)^{1/2}$, it follows that $2p^\sharp(x, \xi) \ge c_1|\xi| - c_0$. For $c > c_0 + 1$, the symbol $(2p^\sharp(x, \xi) + c)^{-1}$ is a bounded family in $S^{-1}_{1,1/2}$, and $c^1/2(2p^\sharp(x, \xi) + c)^{-1}$ is a bounded family in $S^{-1/2}_{1,1/2}$. By the pseudodifferential calculus, the composition of the corresponding operator with $2p^\sharp(x, D) + c$ differs from the identity by a pseudodifferential operator that has seminorm bounds in $S^0_{1,1/2}$ of size $c^{-1/2}$. Hence, for c large, the operator $2p^\sharp(x, D) + c$ is left and right invertible on each given H^s, and in particular, for some c and all $|s| \le 2$,

$$\|(2p^\sharp(x, D) + c_{H^{s+1}} \le C_s\|g\|_{H^s},$$
$$\|(2p^\sharp(x, D) + c)^{-1}g\|_{H^{s+1/2}} \le C_s c^{-\frac{1}{2}}\|g\|_{H^s}.$$

Furthermore, the inverse is a pseudodifferential operator of class $S^{-1}_{1,1/2}$.

It follows that, if $s_0 - \frac{1}{2} \le s \le s_0$, where $s_0 \in \{0, 1\}$, then

$$\|(2p^\flat(x, D) + Q^+(t_0) + Q^-(t_0))(2p^\sharp(x, D) + c)^{-1}g\|_{H^s} \le C_s\|g\|_{H^{s-1/2}},$$
$$\|(2p^\flat(x, D) + Q^+(t_0) + Q^-(t_0))(2p^\sharp(x, D) + c)^{-1}g\|_{H^s} \le C_s c^{-\frac{1}{2}}\|g\|_{H^s}.$$

The operator $2P(t_0) + Q^-(t_0) + Q^+(t_0)$ as a map from $H^s \to H^{s-1}$ is invertible provided that

$$\sum_{n=0}^{\infty} (2p^\sharp(x, D) + c)^{-1}((2p^\flat(x, D) + Q^+(t_0) + Q^-(t_0))(2p^\sharp(x, D) + c)^{-1})^n$$

converges as a map $H^{s-1} \to H^s$, which by the above is true for $s_0 \le s \le s_0 + 1$. Continuity in t of the inverse follows by norm-continuity of $P(t)$ and $Q^\pm(t)$ as functions of t. The same proof works for $0 \le s \le 2$ with $Q_\pm(t_0)$ replaced by 0. $\square$

Appendix B: Energy estimates

In this section we establish the energy bounds and well-posedness results we use for L and its factors. Throughout this section, we assume L satisfies the conditions in the introduction, and make use of the equivalent form (6-1). Since we work with space-time mollification of L in this section, we assume that the coefficients of the operator L have been extended to $\mathbb{R}^{1+n}$, as in (1-7), with the same regularity conditions, and that the coefficients of L are constant for $|t| \ge T + 1$. The solution u is defined only on $(-T, T) \times \mathbb{R}^n$, however, and function space norms of u are with respect to that domain. Recall that $\boldsymbol{D} = (D_t, D_1, \dots, D_n) = (D_t, D)$.

The following result will be used when obtaining bounds for solutions of L^2 regularity.

Lemma B.1. *The commutator $[L, \langle D\rangle^{-1}] = \langle D\rangle^{-1}[L, \langle D\rangle]\langle D\rangle^{-1}$ admits an expansion of the form*

$$[L, \langle D\rangle^{-1}]u(t) = B_1(t)(\boldsymbol{D}u)(t) + B_2(t)(\boldsymbol{D}u)(t), \tag{B-1}$$

where

$$\|B_1(t)g\|_{L^2} \le C\|g\|_{H^{-1}}, \quad \|B_2(t)g\|_{H^1} \le C\alpha(t)\|g\|_{H^{-1}}.$$

Proof. We write $L = D_t^2 - 2D_j b^j D_t - D_i c^{ij} D_j + d^0 D_t + d^j D_j$, after absorbing derivatives of b^j and c^{ij} into d. We use the commutator bound for functions $c \in C^{0,1}(\mathbb{R}^n)$

$$\|[c, \langle D\rangle^{-1}]g\|_{H^1} \le C\|c\|_{C^{0,1}}\|g\|_{H^{-1}},$$

as seen by writing $[c, \langle D\rangle^{-1}] = \langle D\rangle^{-1}[c, \langle D\rangle]\langle D\rangle^{-1}$ and applying Theorem A.1.

The terms $D_i[c^{ij}, \langle D\rangle^{-1}]D_j$ and $D_j[b^j, \langle D\rangle^{-1}]D_t$ can thus be written as $B_1(t)\boldsymbol{D}$, and the terms $[d^j, \langle D\rangle^{-1}]D_j$ and $[d^0, \langle D\rangle^{-1}]D_t$ as $B_2(t)\boldsymbol{D}$. $\square$

Theorem B.2. *If $u \in H^1_{\mathrm{loc}}$ and $Lu \in L^2_{\mathrm{loc}}$, then $\mathrm{WF}_2(u) \subseteq \mathrm{char}(L)$. If $u \in C^0L^2 \cap C^1H^{-1}$ and $Lu \in L^1L^2$, then $\mathrm{WF}_1(u) \subseteq \mathrm{char}(L)$.*

Proof. The first result relies only on the Lipschitz nature of L. As in Lemma B.1, we write $L = \boldsymbol{D}A\boldsymbol{D}^T + d^0 D_t + d^j D_j$, where A is an $(n+1) \times (n+1)$ matrix function consisting of 1, b^j and c^{ij}. The terms $d^0 D_t u + d^j D_j u$ belong to L^2_{loc} by the assumed regularity of u, so we absorb them into F. We decompose multiplication by A into $A = A^\sharp + A^\flat$ as in Corollary A.5, but where the regularization $\phi_{\lfloor j/2\rfloor}(\boldsymbol{D})$ takes place over both the t and x variables.

Let $\Gamma(\tau,\xi)\chi(t,x)$ be supported away from the characteristic set of L, where $\Gamma \in S^0_{\mathrm{cl}}$ is a conic cutoff and $\chi \in C^\infty_c((-T,T)\times\mathbb{R}^n)$. Suppose $u \in H^1_{\mathrm{loc}}$, and write

$$\boldsymbol{D}A^\sharp\boldsymbol{D}^T\Gamma(\boldsymbol{D})(\chi u) = [\boldsymbol{D}A\boldsymbol{D}^T,\Gamma(\boldsymbol{D})\chi]u - \boldsymbol{D}A^\flat\boldsymbol{D}^T\Gamma(\boldsymbol{D})(\chi u).$$

The first term on the right belongs to L^2 by Corollary A.2, and the second to $H^{-1/2}$ by Corollary A.5. The operator $\boldsymbol{D}A^\sharp\boldsymbol{D}^T$ has symbol in $S^2_{1,1/2}$, and is elliptic away from the characteristic set of L, hence $\Gamma(\boldsymbol{D})\chi u \in H^{3/2}$. Corollary A.5 now yields that the second term on the right belongs to L^2, and we conclude $\Gamma(\boldsymbol{D})(\chi u) \in H^2$.

For the second result, we let $v = \langle D\rangle^{-1}u$. By Lemma B.1, $Lv \in L^2 + L^1H^1$. By Theorem B.6 below, there exists $w \in C^0H^2 \cap C^1H^1$ so that $L(v-w) \in L^2$. Since $v - w \in H^1_{\mathrm{loc}}$, we may apply the preceding result to see that $\mathrm{WF}_2(v-w) \subset \mathrm{char}(L)$. On the other hand $\langle D\rangle w \in H^1_{\mathrm{loc}}$, so $\mathrm{WF}_1(\langle D\rangle w) = \varnothing$. ☐

Under a strengthened regularity assumption we can obtain results for L^2 solutions.

Corollary B.3. *Suppose that* $L = \boldsymbol{D}A\boldsymbol{D}^T + d^0D_t + d^jD_j$, *where* A, d^0, *and* d^j *belong to* $C^{0,1}(\mathbb{R}^{1+n})$. *If* $u \in L^2_{\mathrm{loc}}$ *and* $Lu \in H^{-1}_{\mathrm{loc}}$, *then* $\mathrm{WF}_1(u) \subset \mathrm{char}(L)$.

Proof. Under the assumptions on d^0 and d^j, we have $\boldsymbol{D}A\boldsymbol{D}^Tu \in H^{-1}_{\mathrm{loc}}$. The proof then follows the same steps as for the first part of Theorem B.2. ☐

Theorem B.4. *Let* $s_0 \in \{0,1\}$. *Suppose that* $D_tu - p(t,x,D)u \in L^2H^{s_0}$ *and* $u \in L^2H^{s_0}$. *If* $\Gamma(\tau,\xi)\chi(t,x)$ *vanishes on a neighborhood of the characteristic set* $\tau = p(t,x,\xi)$, *where* $\chi \in C^\infty_c((-T,T)\times\mathbb{R}^n)$ *and* $\Gamma \in S^0_{\mathrm{cl}}$, *then* $\Gamma(\boldsymbol{D})(\chi u) \in L^2H^{s_0+1}$. *In particular,*

$$\mathrm{WF}_{s_0+1}(u) \subset \{\tau = p(t,x,\xi)\} \cup \{\xi = 0\}.$$

Proof. Consider first the case that $\Gamma(\tau,\xi)$ vanishes near the set $\xi = 0$. We write

$$\begin{aligned}(D_t - p^\sharp(t,x,D))\Gamma(\boldsymbol{D})(\chi u)\\ = \Gamma(\boldsymbol{D})\chi(D_t - p(t,x,D))u + \Gamma(\boldsymbol{D})(D_t\chi)u - [p,\Gamma(\boldsymbol{D})\chi]u + p^\flat(t,x,D)\Gamma(\boldsymbol{D})(\chi u),\end{aligned}$$

where the frequency regularization defining $p^\sharp$ takes place over both t and x variables. The first two terms on the right belong to $H^{s_0}(\mathbb{R}^{1+n})$, where, since Γ vanishes near $\xi = 0$, we have $\Gamma(\boldsymbol{D}) : L^2H^{s_0}(\mathbb{R}^{1+n}) \to H^{s_0}(\mathbb{R}^{1+n})$, and similarly the last term belongs to $H^{s_0-1/2}(\mathbb{R}^{1+n})$ by Corollary A.5. To see that the third term also belongs to H^{s_0}, we take the symbol expansion (A-1) to replace p by $a(t,x)q(D)$. The commutator of $q(D)$ and χ is bounded on H^{s_0}, so we check that

$$\|[a,\Gamma(\boldsymbol{D})]\langle D\rangle g\|_{H^{s_0}} \le C\|a\|_{C^{0,1}}\|g\|_{L^2H^{s_0}}.$$

This follows from Corollary A.2, since $\|\langle D\rangle g\|_{H^{s_0-1}} \le \|g\|_{L^2H^{s_0}}$ for $s_0 \le 1$.

The symbol $\tau - p^\sharp(t,x,\xi)$ has a microlocal parametrix of class $S^{-1}_{1,1/2}$ away from the set $\{\xi = 0\} \cup \{\tau = p(t,x,\xi)\}$, and the result follows as in Theorem B.2.

Suppose then that Γ and $\widetilde{\Gamma}$ are supported in a small cone about the τ axis, vanish near $\tau = 0$, with $\widetilde{\Gamma}\Gamma = \Gamma$. We write

$$(I - p(t,x,D)D_t^{-1}\widetilde{\Gamma}(\boldsymbol{D}))D_t\Gamma(\boldsymbol{D})\chi u = \Gamma(\boldsymbol{D})\chi(D_t - p(t,x,D))u + \Gamma(\boldsymbol{D})(D_t\chi)u - [p,\Gamma(\boldsymbol{D})\chi]u.$$

The right-hand side belongs to $L^2H^{s_0}$ by steps similar to those above.

The operator $p(t,x,D)D_t^{-1}\widetilde{\Gamma}(\boldsymbol{D})$ is of small norm on L^2H^s for $-1\le s\le 1$, as seen by the symbol expansion (A-1), since $|\xi|\ll\tau$ on the support of $\widetilde{\Gamma}$. We conclude that $D_t\Gamma(\boldsymbol{D})\chi u\in L^2H^{s_0}$, and hence that $\Gamma(\boldsymbol{D})\chi u\in L^2H^{s_0+1}$. □

Theorem B.5. *Let $s_0\in\{0,1\}$ and assume that $p(t,x,\xi)$ satisfies (1-9) and that $Q(t)$ satisfies (2-10). Let $E(t,t_0)$ be the wave group of Theorem 3.2. Suppose that, in the sense of distributions,*

$$D_tu-p(t,x,D)u-Q(t)u=F, \tag{B-2}$$

and $u\in L^2H^{s_0}$, $F\in L^1H^{s_0}$. Then $u\in C^0H^{s_0}$, and, for each $t_0\in(-T,T)$,

$$u(t)=E(t,t_0)u(t_0)+\int_{t_0}^{t}E(t,s)F(s)\,ds. \tag{B-3}$$

In particular, if $u\in L^2H^{s_0+1}$ and $F\in L^1H^{s_0+1}$, then $u\in C^0H^{s_0+1}$.

Proof. We start by proving uniqueness for the equation

$$D_tu-p(t,x,D)u=G,\quad \operatorname{supp}(u)\subset\{t>-T+\delta\},\quad \delta>0, \tag{B-4}$$

under the condition $u\in L^2$ and $G\in L^1L^2$, and $T<\infty$. This will show that, for such u,

$$u(t)=\int_{-T}^{t}E_0(t,s)G(s)\,ds, \tag{B-5}$$

where $E_0(t,t_0)$ is as in Theorem 3.2.

Suppose first that $u\in C^\infty((-T,T)\times\mathbb{R}^n)$ and is supported where $t>-T+\delta$ and $|x|\le R$, for some $\delta>0$ and $R<\infty$. Let u satisfy (B-4). We calculate

$$\partial_t\int|u(t,x)|^2\,dx=-2\operatorname{Im}\int\overline{u(t)}\,p(t,x,D)u(t)\,dx-2\operatorname{Im}\int\overline{u(t)}G(t)\,dx.$$

Since $p(t,x,\xi)$ is real, it follows by Theorem A.1 and the symbol expansion (A-1) that, uniformly over t,

$$\|p(t,x,D)^*u(t)-p(t,x,D)u(t)\|_{L^2}\le C\|u(t)\|_{L^2},$$

hence

$$\partial_t\|u(t)\|_{L^2}^2\le 2C\|u(t)\|_{L^2}^2+2\|G(t)\|_{L^2}\|u(t)\|_{L^2}.$$

By the Gronwall inequality,

$$\|u(t)\|_{L^2}\le e^{C(t+T)}\int_{-T}^{t}\|G(s)\|_{L^2}\,ds,$$

and, in particular,

$$\|u\|_{L^2((-T,T)\times\mathbb{R}^n)}\le C_T\|G\|_{L^1L^2((-T,T)\times\mathbb{R}^n)}. \tag{B-6}$$

Suppose now that $u\in L^2((-T,T)\times\mathbb{R}^n)$ satisfies (B-4). We choose $\chi\in C_c^\infty(\mathbb{R}^{1+n})$ supported in $t>0$ satisfying $\widehat{\chi}(0)=1$, and $\phi\in C_c^\infty(\mathbb{R}^{1+n})$ satisfying $\phi(0)=1$. Let J_ϵ denote the family of causal, compactly supported mollifiers

$$J_\epsilon u=\phi(\epsilon^{-1}(t,x))\widehat{\chi}(\epsilon\boldsymbol{D})u.$$

Note that J_ϵ is a uniformly bounded family of pseudodifferential operators of class $S^0_{1,0}$. By Theorem A.1, the following holds uniformly over $\epsilon > 0$:

$$\|[D_t - p(t,x,D), J_\epsilon]u\|_{L^2} \le C\|u\|_{L^2}.$$

Since $J_\epsilon \to I$ strongly on both H^1 and L^2, as well as L^1L^2, then by density of $H^1 \subset L^2$ it follows that

$$\lim_{\epsilon\to 0} \|[D_t - p(t,x,D), J_\epsilon]u\|_{L^2((-T,T)\times\mathbb{R}^n)} = 0.$$

It follows that (B-6) holds for general $u \in L^2$ and $G \in L^1L^2$ under the condition (B-4), yielding uniqueness of the solution, and thus the identity (B-5).

Now suppose that $u \in L^2H^{s_0}$ is supported in $t > -T+\delta$, and satisfies (B-2) with $F = 0$. Since $Q(t)$ and $E_0(t,s)$ are uniformly bounded on H^{s_0}, taking $G(t) = Q(t)u(t)$ in (B-5), we see that

$$\|u\|_{L^2((-T,-T+c),H^{s_0})} \le Cc^{\frac12}\|u\|_{L^2((-T,-T+c),H^{s_0})},$$

and, by a continuation argument, we must have $u \equiv 0$. Hence we have uniqueness for (B-2) for solutions supported in $t > -T+\delta$.

Thus, if $\psi(t) \in C^\infty(\mathbb{R})$ is supported in $t > -T$, and $u \in L^2H^{s_0}$ satisfies (B-2), then

$$\psi(t)u(t) = \int_{-T}^{t} E(t,s)((D_s\psi)(s)u(s) + \psi(s)F(s))\,ds, \tag{B-7}$$

since the right-hand side is a solution belonging to $C((-T,T),H^{s_0})$. Reversing time, we obtain the following bound for solutions to (B-2) without restrictions on the time-support of u:

$$\|u\|_{C((-T,T),H^{s_0})} \le C_T(\|u\|_{L^2H^{s_0}} + \|F\|_{L^1H^{s_0}}).$$

In particular, $u(t_0)$ is well defined in H^{s_0} for each $t_0 \in (-T,T)$. Now let ψ be an increasing function in $C^\infty(\mathbb{R})$, which vanishes for $t < t_0 - \epsilon$ and equals 1 for $t > t_0+\epsilon$. Letting $\epsilon \to 0$, the formula (B-7) shows that, for $t > t_0$,

$$u(t) = E(t,t_0)u(t_0) + \int_{t_0}^{t} E(t,s)F(s)\,ds, \tag{B-8}$$

and, by time reversal, this holds for all t, which establishes (B-3).

Finally, if $u \in L^2H^{s_0+1}$ and $F \in L^1H^{s_0+1}$, then (B-7) necessarily holds, and since $E(t,s)$ is a strongly continuous evolution group on H^{s_0+1}, the same steps as above show that $u \in C^0H^{s_0+1}$, and that (B-8) holds. □

Theorem B.6. *Given $t_0 \in (-T,T)$ and $u_0 \in L^2$, $u_1 \in H^{-1}$, $F \in L^1H^{-1}$, there exists a unique solution $u \in C^0L^2 \cap C^1H^{-1}$ to the Cauchy problem*

$$Lu = F, \quad u(t_0) = u_0, \quad D_tu(t_0) = u_1.$$

If $0 \le s \le 2$, and if $u_0 \in H^s$, $u_1 \in H^{s-1}$, $F \in L^1H^{s-1}$, then the solution satisfies $u \in C^0H^s \cap C^1H^{s-1}$. Also, if $u \in H^1((-T,T)\times\mathbb{R}^n)$ satisfies $Lu \in L^1L^2$, then $u \in C^0H^1 \cap C^1L^2$.

Proof. We start by proving the existence of such a solution to the Cauchy problem. Assume $0 \le s \le 2$ and $(u_0, u_1) \in H^s \times H^{s-1}$. We seek a solution u of the form

$$u(t) = \sum_{\pm} E_{0,\pm}(t, t_0) f_\pm + \int_{t_0}^{t} (E_{0,+} - E_{0,-})(t, s)(2P(s))^{-1} G(s)\, ds. \tag{B-9}$$

Here $E_{0,\pm}$ is the wave group (3-2) for $D_t \mp P^\pm$. We take $G \in L^1 H^{s-1}$ and set

$$f_\pm = (2P(t_0))^{-1}(P^\mp(t_0)u_0 \pm u_1) \in H^s,$$

that last inclusion holding by Lemma A.10. Recall that $P^+ + P^- = 2P$.

Applying L and using (2-2) and (2-16), the equation $Lu = F$ reduces to

$$G(t) + \int_{t_0}^{t} (R_1^+(t)E_{0,+}(t,s) - R_1^-(t)E_{0,-}(t,s))(2P(s))^{-1} G(s)\, ds = F(t) - \sum_{\pm} R_1^\pm(t) E_{0,\pm}(t, t_0) f_\pm.$$

By Theorem 3.2 and (2-5), the right-hand side belongs to $L^1 H^{s-1}$. Also, by Lemma A.10,

$$\|(R_1^+(t)E_{0,+}(t,s) - R_1^-(t)E_{0,-}(t,s))(2P(s))^{-1} G(s)\|_{H^{s-1}} \le \alpha(t)\|G(s)\|_{H^{s-1}},$$

so that the Volterra equation for G is uniquely solvable on $L^1 H^{s-1}$, with solution given by a convergent expansion. Note that if $\|F(t)\|_{H^{s-1}} \le C\alpha(t)$, the same holds for G. Then $u \in C^0 H^s \cap C^1 H^{s-1}$ follows by (B-9) and Theorem 3.2.

We now consider uniqueness. Suppose first that $u \in C^1((-T, T), C^2(\mathbb{R}^n))$ satisfies $Lu = F \in L^1 C^0$, and assume that u is supported in $|x| \le R$, some $R < \infty$. It follows from $Lu = F$ that $D_t^2 u \in L^1 C^0$. Using integration by parts, we calculate

$$\begin{aligned}\partial_t \int \Big(|D_t u(t,x)|^2 + \sum_{i,j=1}^{n} c^{ij}(t,x) D_i u(t,x) \overline{D_j u(t,x)} + |u(t,x)|^2 \Big)\, dx \\ = 2i \operatorname{Im} \int F(t,x) \overline{D_t u(t,x)}\, dx + \int (B(t,x)(u, \boldsymbol{D}u)(t,x)) \cdot \overline{(u, \boldsymbol{D}u)(t,x)}\, dx,\end{aligned}$$

where $B(t,x)$ is an $(n+2) \times (n+2)$ matrix whose coefficients consist of first-order space-time derivatives of the coefficients b^j, c^{ij}, as well as d^0 and d^j. Hence

$$\|B^{ij}(t)\|_{C^0} \le C, \qquad \|\boldsymbol{D}B^{ij}(t)\|_{L^\infty} \le C\alpha(t).$$

By the positive definite condition on c^{ij} and the Gronwall inequality, we conclude

$$\|(u, \boldsymbol{D}u)(t)\|_{L^2} \le C e^{\int_{t_0}^t \alpha} \|(u, \boldsymbol{D}u)(t_0)\|_{L^2} + \int_{t_0}^{t} e^{\int_s^t \alpha} \|F(s)\|_{L^2}\, ds. \tag{B-10}$$

By mollification and truncation with respect to the x variable, the bound (B-10) holds under the assumption $u \in C^0 H^1 \cap C^1 L^2$ and $F \in L^1 L^2$.

Suppose that $u \in C^0 L^2 \cap C^1 H^{-1}$ satisfies $Lu = F \in L^1 H^{-1}$. By Lemma B.1, $v = \langle D\rangle^{-1} u$ satisfies

$$Lv = B_0(t)(\boldsymbol{D}v)(t) + \langle D\rangle^{-1} F(t), \qquad \|B_0(t)g\|_{L^2} \le C\alpha(t)\|g\|_{L^2}.$$

The Gronwall inequality and (B-10) applied to v then imply

$$\|(u, \boldsymbol{D}u)(t)\|_{H^{-1}} \le C e^{\int_{t_0}^t \alpha} \|(u, \boldsymbol{D}u)(t_0)\|_{H^{-1}} + \int_{t_0}^t e^{\int_s^t \alpha} \|F(s)\|_{H^{-1}}\, ds, \tag{B-11}$$

from which the desired uniqueness follows.

To complete the proof of Theorem B.6, suppose now that $u \in H^1((-T, T) \times \mathbb{R}^n)$ satisfies $Lu = 0$. We want to show that

$$u \in C^0 H^1 \cap C^1 L^2, \tag{B-12}$$

since, by the above, the inhomogeneous problem admits a solution of this regularity.

We consider $\psi(t)u$ as in the proof of Theorem B.5, where $\psi = 0$ near either $\pm T$, and easily verify that $L(\psi(t)u(t)) \in L^2((-T, T) \times \mathbb{R}^n)$. By the above, there exists a solution of regularity (B-12) with inhomogeneity $L(\psi(t)u)$, which also vanishes for t near the chosen $\pm T$. Hence it suffices to prove that if $u \in H^1((-T, T) \times \mathbb{R}^n)$ satisfies $Lu = 0$ with $u = 0$ for t near either $\pm T$, then $u \equiv 0$. For this we note that (B-10) implies

$$\|u\|_{H^1((-T,T)\times\mathbb{R}^n)} \le C\|F\|_{L^2((-T,T)\times\mathbb{R}^n)},$$

if u satisfies (B-12) and vanishes near either $\pm T$. The same inequality holds if $u \in H^1$ and $F \in L^2$, as seen by using the space-time mollifiers J_ϵ from Theorem B.5 and noting that $[L, J_\epsilon]$ maps H^1 to L^2, uniformly in ϵ. □

Remark B.7. Theorem B.6 together with finite propagation velocity shows that Theorem 1.1 holds for solutions on an open set; that is, it holds for solutions to $Lu = 0$ with $u \in C^0((-T, T), H^{s_0}_{\mathrm{loc}}(\Omega))$, $D_t u \in C^0((-T, T), H^{s_0-1}_{\mathrm{loc}}(\Omega))$, for an open set $\Omega \subset \mathbb{R}^n$, as long as γ remains above Ω. To see this, assume that $|D_\xi p_\pm| \le C$. If $\chi \in C_c^\infty(\Omega)$ equals 1 on a ball of radius $2r$ about the spatial projection of $\gamma(t_0)$, then, by finite propagation velocity, u agrees on the ball of radius r and $|t - t_0| \le C^{-1}r$ with the solution $\tilde{u}$ on $(-T, T) \times \mathbb{R}^n$ to $L\tilde{u} = 0$, where $\tilde{u}$ has Cauchy data $(\chi u(t_0), \chi D_t u(t_0))$ at $t = t_0$. By Theorem B.6, $\tilde{u} \in C^0 H^{s_0} \cap C^1 H^{s_0-1}$, and Theorem 1.1 implies that $\gamma(t) \notin \mathrm{WF}_{s_0+1}(u)$ for $|t - t_0| \le C^{-1}r$. The argument may then be repeated starting at $t_0 \pm C^{-1}r$. □

Remark B.8. Under increased regularity of the coefficients, solutions $u \in L^2$ to $Lu = 0$ satisfy (1-1). Suppose $L = D_t^2 - 2D_j b^j D_t - D_i c^{ij} D_j + d^0 D_t + d^j D_j$, where b^j and c^{ij} satisfy (1-3) and (1-4) as before, but we make the stronger assumption that $d^0, d^j \in C^{0,1}((-T, T) \times \mathbb{R}^n)$. Suppose that $Lu \in L^2 H^{-1}$. By Corollary B.3, we then have $\boldsymbol{D}u \in L^2_{\mathrm{loc}} H^{-1}$. Hence $L(\psi(t)u) \in L^2 H^{-1}$ for $\psi \in C_c^\infty((-T, T))$.

Suppose now that $u \in L^2((-T, T) \times \mathbb{R}^n)$ satisfies $Lu = 0$. By Theorem B.6, there is a solution $v \in C^0 L^2 \cap C^1 H^{-1}$ satisfying $Lv = L(\psi(t)u)$. Hence, to prove $u \in C^0 L^2 \cap C^1 H^{-1}$, it suffices to prove uniqueness of L^2 solutions to $Lu = F \in L^2 H^{-1}$ when u is supported in $|t| \le T - \delta$.

Fix $\chi \in C_c^\infty(\mathbb{R}^{1+n})$ with $\widehat{\chi}(0) = 1$. Then the commutator $[L, \widehat{\chi}(\epsilon \boldsymbol{D})]$ maps L^2 to $L^2 H^{-1}$, uniformly in ϵ. This follows from Corollary A.2 by the $C^{0,1}$ regularity of the coefficients, and the form of L; in particular the D_t^2 term commutes with $\widehat{\chi}(\epsilon \boldsymbol{D})$. By a density argument,

$$\lim_{\epsilon \to 0} \|[L, \widehat{\chi}(\epsilon \boldsymbol{D})]u\|_{L^2 H^{-1}} = 0, \qquad u \in L^2.$$

It follows from (B-11) that

$$\|u\|_{L^2((-T,T)\times\mathbb{R}^n)} \le C\|F\|_{L^1H^{-1}},$$

from which the uniqueness of solutions follows. □

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Received 18 Feb 2014. Accepted 11 Apr 2014.

HART F. SMITH: hart@math.washington.edu
Department of Mathematics, University of Washington, Box 354350, Seattle, WA 98195-4350, United States

ANALYSIS AND PDE
Vol. 7, No. 5, 2014
dx.doi.org/10.2140/apde.2014.7.1179

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WELL-POSEDNESS OF LAGRANGIAN FLOWS AND CONTINUITY EQUATIONS IN METRIC MEASURE SPACES

LUIGI AMBROSIO AND DARIO TREVISAN

We establish, in a rather general setting, an analogue of DiPerna–Lions theory on well-posedness of flows of ODEs associated to Sobolev vector fields. Key results are a well-posedness result for the continuity equation associated to suitably defined Sobolev vector fields, via a commutator estimate, and an abstract superposition principle in (possibly extended) metric measure spaces, via an embedding into $\mathbb{R}^\infty$.

When specialized to the setting of Euclidean or infinite-dimensional (e.g., Gaussian) spaces, large parts of previously known results are recovered at once. Moreover, the class of $\mathsf{RCD}(\mathsf{K}, \infty)$ metric measure spaces, introduced by Ambrosio, Gigli and Savaré [*Duke Math. J.* **163**:7 (2014) 1405–1490] and the object of extensive recent research, fits into our framework. Therefore we provide, for the first time, well-posedness results for ODEs under low regularity assumptions on the velocity and in a nonsmooth context.

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1. Introduction

DiPerna–Lions theory, initiated in the seminal paper [DiPerna and Lions 1989], provides existence, stability and uniqueness results for ODEs associated to large classes of nonsmooth vector fields, most notably that of Sobolev vector fields. In more recent times Ambrosio [2004] extended the theory to include BV vector fields and, at the same time, he introduced a more probabilistic axiomatization based on the duality between flows and continuity equation, while the approach of [DiPerna and Lions 1989] relied on characteristics and the transport equation. In more recent years the theory developed in many different directions, including larger classes of vector fields, quantitative convergence estimates, mild regularity properties of the flow, and non-Euclidean spaces, including infinite-dimensional ones. We refer

MSC2010: primary 49J52; secondary 35K90.
Keywords: continuity equation, flows, DiPerna–Lions theory, Γ-calculus.

to [Ambrosio 2008; Ambrosio and Crippa 2008] for more exhaustive, but still incomplete, description of the developments on this topic.

The aim of this paper is to extend the theory of well-posedness for the continuity equation and the theory of flows to metric measure spaces $(X, d, \mathfrak{m})$. Roughly speaking, and obviously under additional structural assumptions, we prove that if $\{\boldsymbol{b}_t\}_{t\in(0,T)}$ is a time-dependent family of Sobolev vector fields then there is a unique flow associated to $\boldsymbol{b}_t$, namely a family of absolutely continuous maps $\{\mathbb{X}(\cdot, x)\}_{x\in X}$ from $[0, T]$ to X satisfying:

(i) $\mathbb{X}(\cdot, x)$ solves the possibly nonautonomous ODE associated to $\boldsymbol{b}_t$ for $\mathfrak{m}$-a.e. $x \in X$;

(ii) the push-forward measures $\mathbb{X}(t, \cdot)_\#\mathfrak{m}$ are absolutely continuous w.r.t. $\mathfrak{m}$ and have uniformly bounded densities.

Of course the notions of "Sobolev vector field" and even "vector field", as well as the notion of solution to the ODE have to be properly understood in this nonsmooth context, where not even local coordinates are available. As far as we know, these are the first well-posedness results for ODEs under low regularity assumptions and in a nonsmooth context.

One motivation for writing this paper has been the theory of "Riemannian" metric measure spaces developed by the first author in collaboration with N. Gigli and G. Savaré, leading to a series of papers [Ambrosio et al. 2014a; 2014b; 2014c] and to further developments in [Gigli 2012; 2013]. In this perspective, it is important to develop new calculus tools in metric measure spaces. For instance, in the proof of the splitting theorem in [Gigli 2013] a key role is played by the flow associated to the gradient of a c-concave harmonic function, whose flow lines provide the fibers of the product decomposition; therefore a natural question is under which regularity assumption on the potential V the gradient flow associated to V has a unique solution, where uniqueness is not understood pointwise, but in the sense of the DiPerna–Lions theory (see Theorem 8.3 and Theorem 9.7 for a partial answer to this question). We also point out the recent paper [Gigli and Bang-Xian 2014], where continuity equations in metric measure spaces are introduced and studied in connection with absolutely continuous curves with respect to the Wasserstein distance W_2, thus relying mainly on a "Lagrangian" point of view.

The paper is basically organized in three parts. In the "Eulerian" part, which has independent interest, we study the well-posedness of continuity equations. In the "Lagrangian" part we define the notion of solution to the ODE and relate well-posedness of the continuity equation to existence and uniqueness of the flow (in the same spirit as [Ambrosio 2004; 2008], where the context was Euclidean). Eventually, in the third part we see how a large class of previous results can be seen as particular cases of ours. On the technical side, these are the main ingredients: for the first part, a new intrinsic way to write down the so-called commutator estimate, obtained with Γ-calculus tools (this point of view is new even for such "nice" spaces as Euclidean spaces and Riemannian manifolds); for the second part, a more general version of the so-called superposition principle (see, for instance, [Ambrosio et al. 2005, Theorem 8.2.1], in the setting of Euclidean spaces), that allows us to lift, not canonically in general, nonnegative solutions of the continuity equation to measures on paths.

We pass now to a more detailed description of the three parts.

Part 1. This part consists of five sections, from Section 2 to Section 6. Section 2 is devoted to the description of our abstract setup, which is the typical one of Γ-calculus and of the theory of Dirichlet forms; for the moment the distance is absent and we are given only a topology τ on X and a reference measure $\mathfrak{m}$ on X, which is required to be Borel, nonnegative and σ-finite. On $L^2(\mathfrak{m})$ we are given a symmetric, densely defined and strongly local Dirichlet form $(\mathscr{E}, D(\mathscr{E}))$ whose semigroup P is assumed to be Markovian. We write $\mathbb{V} := D(\mathscr{E})$ and assume that a *carré du champ* $\Gamma : \mathbb{V} \times \mathbb{V} \to L^1(\mathfrak{m})$ is defined, and that we are given a "nice" algebra $\mathcal{A}$ which is dense in $\mathbb{V}$ and which plays the role of the C_c^∞ functions in the theory of distributions.

Using $\mathcal{A}$, we can define in Section 3 "vector fields" as derivations, in the same spirit as [Weaver 2000] (see also [Ambrosio and Kirchheim 2000] for parallel developments in the theory of currents); a derivation $\boldsymbol{b}$ is a linear map from $\mathcal{A}$ to the space of real-valued Borel functions on X, satisfying the Leibniz rule $\boldsymbol{b}(fg) = f\boldsymbol{b}(g) + g\boldsymbol{b}(f)$, and a pointwise $\mathfrak{m}$-a.e. bound in terms of Γ. We will use the more intuitive notation (since differentials of functions are covectors) $f \mapsto df(\boldsymbol{b})$ for the action of a derivation $\boldsymbol{b}$ on f. An important example is provided by *gradient* derivations $\boldsymbol{b}_V$ induced by $V \in \mathbb{V}$, acting as follows:

$$df(\boldsymbol{b}_V) := \Gamma(V, f).$$

Although we will not need more than this, we would like to mention the forthcoming [Gigli 2014], which provides equivalent axiomatizations, in which the Leibniz rule is not an axiom anymore, and it is shown that gradient derivations generate, in a suitable sense, all derivations. Besides the basic example of gradient derivations, the carré du champ also provides, by duality, a natural pointwise norm on derivations; such duality can be used to define, via integration by parts, a notion of divergence $\operatorname{div} \boldsymbol{b}$ for a derivation (the divergence depends only on $\mathfrak{m}$, not on Γ). In Section 4 we prove existence of solutions to the weak formulation of the continuity equation $du_t/dt + \operatorname{div}(u_t\boldsymbol{b}_t) = 0$ induced by a family $(\boldsymbol{b}_t)$ of derivations,

$$\frac{d}{dt}\int f u_t \, d\mathfrak{m} = \int df(\boldsymbol{b}_t)u_t \, d\mathfrak{m} \quad \forall f \in \mathcal{A}.$$

The strategy of the proof is classical: first we add a viscosity term and get a $\mathbb{V}$-valued solution by Hilbert space techniques, then we take a vanishing viscosity limit. Together with existence, we recover also higher (or lower, since our measure $\mathfrak{m}$ might be not finite and therefore there might be no inclusion between L^p spaces) integrability estimates on u, depending on the initial condition $\bar{u}$. Also, under a suitable assumption (4-3) on $\mathcal{A}$, we prove that the L^1 norm is independent of time. Section 5 is devoted to the proof of uniqueness of solutions to the continuity equation. The classical proof in [DiPerna and Lions 1989] is based on a smoothing scheme that, in our context, is played by the semigroup P (an approach already proved to be successful in [Ambrosio and Figalli 2009; Trevisan 2013] in Wiener spaces). For fixed t, one has to estimate carefully the so-called commutator

$$\mathscr{C}^\alpha(\boldsymbol{b}_t, u_t) := \operatorname{div}((\mathsf{P}_\alpha u_t)\boldsymbol{b}_t) - \mathsf{P}_\alpha(\operatorname{div}(u_t\boldsymbol{b}_t))$$

as $\alpha \to 0$. The main new idea here is to imitate Bakry and Émery's Γ-calculus (see, e.g., the recent monograph [Bakry et al. 2014]), interpolating and writing, at least formally,

$$\begin{aligned}\mathscr{C}^\alpha(\boldsymbol{b}_t, u_t) &= \int_0^\alpha \frac{d}{ds}\mathsf{P}_{\alpha-s}\big(\operatorname{div}(\mathsf{P}_s(u_t)\boldsymbol{b}_t)\big)\,ds \\ &= \int_0^\alpha \big[-\Delta\mathsf{P}_{\alpha-s}\big(\operatorname{div}(\mathsf{P}_s(u_t)\boldsymbol{b}_t)\big) + \mathsf{P}_{\alpha-s}\big(\operatorname{div}(\Delta\mathsf{P}_s(u_t)\boldsymbol{b}_t)\big)\big]\,ds.\end{aligned} \tag{1-1}$$

It turns out that an estimate of the commutator involves only the symmetric part of the derivative (this, in the Euclidean case, was already observed in [Capuzzo Dolcetta and Perthame 1996] for regularizations induced by even convolution kernels). This structure can be recovered in our context: inspired by the definition of the Hessian in [Bakry 1997] we define the symmetric part $D^{\mathrm{sym}}\boldsymbol{c}$ of the gradient of a deformation $\boldsymbol{c}$ by

$$\int D^{\mathrm{sym}}\boldsymbol{c}(f,g)\,d\mathfrak{m} := -\frac{1}{2}\int \big[df(\boldsymbol{c})\Delta g + dg(\boldsymbol{c})\Delta f - (\operatorname{div}\boldsymbol{c})\Gamma(f,g)\big]\,d\mathfrak{m}. \tag{1-2}$$

Using this definition in (1-1) (assuming here for simplicity $\operatorname{div}\boldsymbol{b}_t = 0$), we establish the identity

$$\int f\,\mathscr{C}^\alpha(\boldsymbol{b}_t,u_t)\,d\mathfrak{m} = 2\int_0^\alpha\int D^{\mathrm{sym}}\boldsymbol{b}_t(\mathsf{P}_{\alpha-s}f, \mathsf{P}_s u_t)\,d\mathfrak{m}\,ds \quad \forall f\in\mathcal{A}. \tag{1-3}$$

Then, we assume the validity of the estimates (see Definition 5.2 for a more general setup with different powers)

$$\left|\int D^{\mathrm{sym}}\boldsymbol{b}_t(f,g)\,d\mathfrak{m}\right| \le c\left(\int\Gamma(f)^2\,d\mathfrak{m}\right)^{\frac14}\left(\int\Gamma(g)^2\,d\mathfrak{m}\right)^{\frac14}, \tag{1-4}$$

which, in a smooth context, amount to an L^2 control on the symmetric part of derivative. Luckily, the control (1-4) on $D^{\mathrm{sym}}\boldsymbol{b}_t$ can be combined with (1-1) and (1-3) to obtain strong convergence to 0 of the commutator as $\alpha\to 0$ and therefore well-posedness of the continuity equation. This procedure works assuming some regularizing properties of the semigroup P, especially the validity of

$$\left(\int\Gamma(\mathsf{P}_t f)^2\,d\mathfrak{m}\right)^{\frac14} \le \frac{c}{\sqrt{t}}\left(\int |f|^4\,d\mathfrak{m}\right)^{\frac14} \quad \text{for every } f\in L^2\cap L^4(\mathfrak{m}),\ t\in(0,1),$$

for some constant $c\ge 0$ (see Theorem 5.4). In particular, these hold assuming an abstract curvature lower bound on the underlying space, as discussed in Section 6, where we crucially exploit the recent results in [Savaré 2014; Ambrosio et al. 2013] to show that our structural assumptions on P and on $\mathcal{A}$ are fulfilled in the presence of lower bounds on the curvature. Furthermore, gradient derivations associated to sufficiently regular functions satisfy (1-4).

Finally, we remark that, as in [DiPerna and Lions 1989], analogous well-posedness results could be obtained for weak solutions to the inhomogeneous transport equation

$$\frac{d}{dt}u_t + du_t(\boldsymbol{b}_t) = c_t u_t + w_t,$$

under suitable assumptions on c_t and w_t. We confined our discussion to the case of the homogeneous continuity equation (corresponding to $c_t = -\operatorname{div} \boldsymbol{b}_t$ and $w_t = 0$) for the sake of simplicity and for the relevance of this PDE in connection with the theory of flows.

Part 2. This part consists of two sections. In Section 7 we show how solutions u to the continuity equation $du_t/dt + \operatorname{div}(u_t \boldsymbol{b}_t) = 0$ can be lifted to measures $\boldsymbol{\eta}$ in $C([0, T]; X)$. Namely, we would like that $(e_t)_\# \boldsymbol{\eta} = u_t \mathfrak{m}$ for all $t \in (0, T)$ and that $\boldsymbol{\eta}$ is concentrated on solutions η to the ODE $\dot{\eta} = \boldsymbol{b}_t(\eta)$. This statement is well-known in Euclidean spaces (or even Hilbert spaces) [Ambrosio et al. 2005, Theorem 8.2.1]; in terms of currents, it could be seen as a particular case of Smirnov's [1993] decomposition of 1-currents as superposition of rectifiable currents. Here, we realized that the most appropriate setup for the validity of this principle is $\mathbb{R}^\infty$; see Theorem 7.1, where only the Polish structure of $\mathbb{R}^\infty$ matters and neither distance nor reference measure come into play.

In order to extend this principle from $\mathbb{R}^\infty$ to our abstract setup we assume the existence of a sequence $(g_k) \subset \{f \in \mathcal{A} : \|\Gamma(f)\|_\infty \le 1\}$ satisfying:

$$\operatorname{span}(g_k) \text{ is dense in } \mathbb{V} \text{ and any function } g_k \text{ is } \tau\text{-continuous}; \tag{1-5}$$

$$\exists \lim_{n\to\infty} g_k(x_n) \text{ in } \mathbb{R} \text{ for all } k \quad \Longrightarrow \quad \exists \lim_{n\to\infty} x_n \text{ in } X. \tag{1-6}$$

This way, the embedding $J : X \to \mathbb{R}^\infty$ mapping x to $(g_k(x))$ provides an homeomorphism of X with $J(X)$ and we can first read the solution to the continuity equation in $\mathbb{R}^\infty$ (setting $\nu_t := J_\#(u_t \mathfrak{m})$, with an appropriate choice of the velocity in $\mathbb{R}^\infty$) and then pull back the lifting obtained in $\mathscr{P}(C([0, T]; \mathbb{R}^\infty))$ to obtain $\boldsymbol{\eta} \in \mathscr{P}(C([0, T]; X))$; see Theorem 7.6. It turns out that $\boldsymbol{\eta}$ is concentrated on curves η satisfying

$$\frac{d}{dt}(f \circ \eta) = df(\boldsymbol{b}_t) \circ \eta \quad \text{in the sense of distributions in } (0, T), \text{ for all } f \in \mathcal{A}, \tag{1-7}$$

which is the natural notion of solution to the ODE $\dot{\eta} = \boldsymbol{b}_t(\eta)$ in our context (again, consistent with the fact that a vector can be identified with a derivation). We show, in addition, that this property implies absolute continuity of $\boldsymbol{\eta}$-almost every curve η with respect to the possibly extended distance $d(x, y) := \sup_k |g_k(x) - g_k(y)|$, with metric derivative $|\dot{\eta}|$ estimated from above by $|\boldsymbol{b}_t| \circ \eta$. Notice also that, in our setup, the distance appears only now. Also, we remark that a similar change of variables appears in the recent paper [Kolesnikov and Röckner 2014], but not in a Lagrangian perspective: it is used therein to prove well-posedness of the continuity equation when the reference measure is log-concave (see Section 9E).

Section 8 is devoted to the proof of Theorem 8.3, which links well-posedness of the continuity equation in the class of nonnegative functions $L^1_t(L^1_x \cap L^\infty_x)$ with initial data $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$ to the existence and uniqueness of the flow $\mathbb{X}$ according to (i), (ii) above, where (i) is now understood as in (1-7). The proof of Theorem 8.3 is based on two facts: first, the possibility to lift solutions u to probabilities $\boldsymbol{\eta}$, discussed in the previous section; second, the fact that the restriction of $\boldsymbol{\eta}$ to any Borel set still induces a solution to the continuity equation with the same velocity field. Therefore we can "localize" $\boldsymbol{\eta}$ to show that, whenever some branching of trajectories occurs, then there is nonuniqueness at the level of the continuity equation.

Let us comment that, in this abstract setting, it seems more profitable to the authors to deal uniquely with continuity equations, instead of transport equations as in [DiPerna and Lions 1989], since the latter require in their very definition a choice of "coordinates", while the former arise naturally as the description of evolution of underlying measures.

Part 3. This part consists of Section 9 only, where we specialize the general theory to settings where continuity equations and associated flows have already been considered, and to $\mathsf{RCD}(\mathsf{K},\infty)$-metric measure spaces. Since the transfer mechanism of well-posedness from the PDE to the ODE levels is quite general, we mainly focus on the continuity equation. Moreover, in these particular settings (except for $\mathsf{RCD}(\mathsf{K},\infty)$ spaces), the proof of existence for solutions turns out to be a much easier task than in the general framework, due to explicit and componentwise approximations by smooth vector fields. Therefore, we limit ourselves to compare uniqueness results.

In Section 9A, we show how the classical DiPerna–Lions theory [1989] fits into our setting; in short, we recover almost all the well-posedness results in [DiPerna and Lions 1989], with the notable exception of $W^{1,1}_{\mathrm{loc}}$-regular vector fields. In Section 9B we also describe how our techniques provide intrinsic proofs, i.e., without reducing to local coordinates, of analogous results for weighted Riemannian manifolds.

In Section 9C and Section 9D, we deal with (infinite-dimensional) Gaussian frameworks, comparing our results to those established respectively in [Ambrosio and Figalli 2009; Da Prato et al. 2014]; large parts of these can be obtained as consequences of our general theory, which turns out to be more flexible, e.g., we can allow for vector fields that do not necessarily take values in the Cameron–Martin space (see at the end of Section 9D), which is not admissible in their work. In Section 9E we consider the even more general setting of log-concave measures and make a comparison with some of the results contained in [Kolesnikov and Röckner 2014]. The strength of our approach is immediately revealed; for example, we are not limited as they are to uniformly log-concave measures.

We conclude in Section 9F by describing how the theory specializes to the setting of $\mathsf{RCD}(\mathsf{K},\infty)$-metric measure spaces, that is one of our original motivations for this work. We show that Lagrangian flows do exist in many cases (Theorem 9.7) and provide instances of so-called *test plans*. In the case of gradient derivations, we also show that the trajectories satisfy a global energy dissipation identity (Theorem 9.6).

2. Notation and abstract setup

Let (X,τ) be a Polish topological space, endowed with a σ-finite Borel measure $\mathfrak{m}$ with full support (i.e., $\operatorname{supp}\mathfrak{m}=X$) and

$$\begin{gathered}\text{a strongly local, densely defined and symmetric Dirichlet form } \mathcal{E} \text{ on } L^2(X,\mathcal{B}(X),\mathfrak{m})\\ \text{enjoying a carré du champ } \Gamma: D(\mathcal{E})\times D(\mathcal{E})\to L^1(X,\mathcal{B}(X),\mathfrak{m}) \text{ and}\\ \text{generating a Markov semigroup } (\mathsf{P}_t)_{t\geq 0} \text{ on } L^2(X,\mathcal{B}(X),\mathfrak{m}).\end{gathered}\tag{2-1}$$

The precise meaning of (2-1) is recalled below in this section.

To keep notation simple, we write $L^p(\mathfrak{m})$ instead of $L^p(X,\mathcal{B}(X),\mathfrak{m})$ and denote $L^p(\mathfrak{m})$ norms by $\|\cdot\|_p$. We also write $L^0(\mathfrak{m})$ for the space of $\mathfrak{m}$-a.e. equivalence classes of Borel functions $f: X\mapsto[-\infty,+\infty]$ that take finite values $\mathfrak{m}$-a.e. in X.

Since (X,τ) is Polish and $\mathfrak{m}$ is σ-finite, the spaces $L^p(\mathfrak{m})$ are separable for $p\in[1,\infty)$. We shall also use the duality relations

$$(L^p(\mathfrak{m})+L^q(\mathfrak{m}))^* = L^{p'}\cap L^{q'}(\mathfrak{m}),\quad \text{p}, q\in[1,\infty)$$

and the notation $\|\cdot\|_{L^p+L^q}$, $\|\cdot\|_{L^{p'}\cap L^{q'}}$. In addition, we will use that the spaces $L^p(\mathfrak{m})$, $1\le p\le\infty$ (and $p=0$), are complete lattices with respect to the order relation induced by the inequality $\mathfrak{m}$-a.e. in X. This follows at once from the general fact that, for any family of Borel functions $f_i : X\to[-\infty,+\infty]$, there exists $f : X\to[-\infty,+\infty]$ Borel such that $f\ge f_i$ $\mathfrak{m}$-a.e. in X for all $i\in I$, and $f\le g$ $\mathfrak{m}$-a.e. in X for any function g with the same property. Existence of f can be achieved, for instance, by considering the maximization of

$$J\mapsto\int \tan^{-1}\bigl(\sup_{i\in J} f_i\bigr)\,\vartheta\,d\mathfrak{m}$$

among the finite subfamilies J of I, with ϑ a positive function in $L^1(\mathfrak{m})$ (notice that the pointwise supremum could lead to a function which is not $\mathfrak{m}$-measurable).

2A. *Dirichlet form and carré du champ.* A symmetric Dirichlet form $\mathscr{E}$ is a $L^2(\mathfrak{m})$-lower semicontinuous quadratic form satisfying the Markov property

$$\mathscr{E}(\eta\circ f)\le\mathscr{E}(f)\quad\text{for every normal contraction } \eta:\mathbb{R}\to\mathbb{R}, \tag{2-2}$$

i.e., a 1-Lipschitz map satisfying $\eta(0)=0$. We refer to [Bouleau and Hirsch 1991; Fukushima et al. 2011] for equivalent formulations of (2-2). Recall that

$$\mathbb{V}:=\mathrm{D}(\mathscr{E})\subset L^2(\mathfrak{m})\quad\text{endowed with } \|f\|_{\mathbb{V}}^2:=\|f\|_2^2+\mathscr{E}(f)$$

is a Hilbert space. Furthermore, $\mathbb{V}$ is separable because $L^2(\mathfrak{m})$ is separable (see [Ambrosio et al. 2014c, Lemma 4.9] for the simple proof).

We still denote by $\mathscr{E}(\cdot,\cdot):\mathbb{V}\times\mathbb{V}\to\mathbb{R}$ the associated continuous and symmetric bilinear form

$$\mathscr{E}(f,g):=\tfrac14(\mathscr{E}(f+g)-\mathscr{E}(f-g)).$$

We will assume strong locality of $\mathscr{E}$, namely:

$$\text{for all } f,g\in\mathbb{V},\quad \mathscr{E}(f,g)=0,\quad\text{if } (f+a)g=0\ \ \mathfrak{m}\text{-a.e. in } X,\ \text{ for some } a\in\mathbb{R}.$$

It is possible to prove (see [Bouleau and Hirsch 1991, Proposition 2.3.2], for instance) that $\mathbb{V}\cap L^\infty(\mathfrak{m})$ is an algebra with respect to pointwise multiplication, so that for every $f\in\mathbb{V}\cap L^\infty(\mathfrak{m})$ the linear form on $\mathbb{V}\cap L^\infty(\mathfrak{m})$,

$$\Gamma[f;\varphi]:=2\mathscr{E}(f,f\varphi)-\mathscr{E}(f^2,\varphi),\quad \varphi\in\mathbb{V}\cap L^\infty(\mathfrak{m}), \tag{2-3}$$

is well-defined and, for every normal contraction $\eta:\mathbb{R}\to\mathbb{R}$, it satisfies [Bouleau and Hirsch 1991, Proposition 2.3.3]

$$0\le\Gamma[\eta\circ f;\varphi]\le\Gamma[f;\varphi]\le\|\varphi\|_\infty\,\mathscr{E}(f)\quad\text{for all } f,\ \varphi\in\mathbb{V}\cap L^\infty(\mathfrak{m}),\ \varphi\ge0. \tag{2-4}$$

The inequality (2-4) shows that for every nonnegative $\varphi \in \mathbb{V} \cap L^\infty(\mathfrak{m})$ the function $f \mapsto \Gamma[f;\varphi]$ is a quadratic form in $\mathbb{V} \cap L^\infty(\mathfrak{m})$ that satisfies the Markov property and can be extended by continuity to $\mathbb{V}$.

We assume that for all $f \in \mathbb{V}$ the linear form $\varphi \mapsto \Gamma[f;\varphi]$ can be represented by an absolutely continuous measure w.r.t. $\mathfrak{m}$ with density $\Gamma(f) \in L^1_+(\mathfrak{m})$, the so-called carré du champ. Since $\mathscr{E}$ is strongly local, [Bouleau and Hirsch 1991, Theorem I.6.1.1] yields the representation formula

$$\mathscr{E}(f,f) = \int_X \Gamma(f)\,d\mathfrak{m} \quad \text{for all } f \in \mathbb{V}. \tag{2-5}$$

It is not difficult to check that Γ as defined by (2-5) (see [Bouleau and Hirsch 1991, Definition I.4.1.2], for example) is a quadratic continuous map defined in $\mathbb{V}$ with values in $L^1_+(\mathfrak{m})$, and that $\Gamma[f-g;\varphi] \geq 0$ for all $\varphi \in \mathbb{V} \cap L^\infty(\mathfrak{m})$ yields

$$|\Gamma(f,g)| \leq \sqrt{\Gamma(f)}\sqrt{\Gamma(g)} \quad \mathfrak{m}\text{-a.e. in } X. \tag{2-6}$$

We use the Γ notation also for the symmetric, bilinear and continuous map

$$\Gamma(f,g) := \tfrac{1}{4}(\Gamma(f+g) - \Gamma(f-g)) \in L^1(\mathfrak{m}), \qquad f, g \in \mathbb{V},$$

which, thanks to (2-5), represents the bilinear form $\mathscr{E}$ by the formula

$$\mathscr{E}(f,g) = \frac{1}{2}\int_X \Gamma(f,g)\,d\mathfrak{m} \quad \text{for all } f,\ g \in \mathbb{V}.$$

Because of the Markov property and locality, $\Gamma(\cdot\,,\cdot)$ satisfies the chain rule [Bouleau and Hirsch 1991, Corollary I.7.1.2]

$$\Gamma(\eta(f),g) = \eta'(f)\Gamma(f,g) \quad \text{for all } f,\ g \in \mathbb{V} \text{ and } \eta : \mathbb{R} \to \mathbb{R} \text{ Lipschitz with } \eta(0) = 0, \tag{2-7}$$

and the Leibniz rule

$$\Gamma(fg,h) = f\Gamma(g,h) + g\Gamma(f,h) \quad \text{for all } f,\ g,\ h \in \mathbb{V} \cap L^\infty(\mathfrak{m}).$$

Notice that, by [Bouleau and Hirsch 1991, Theorem I.7.1.1], (2-7) is well-defined since, for every Borel set $N \subset \mathbb{R}$ (in particular the set where η is not differentiable),

$$\mathscr{L}^1(N) = 0 \quad \Longrightarrow \quad \Gamma(f) = 0 \ \ \mathfrak{m}\text{-a.e. on } f^{-1}(N). \tag{2-8}$$

For $p \in [1,\infty]$, we introduce the spaces

$$\mathbb{V}_p := \left\{ u \in \mathbb{V} \cap L^p(\mathfrak{m}) : \int (\Gamma(u))^{p/2}\,d\mathfrak{m} < \infty \right\}, \qquad p \in [1,\infty), \tag{2-9}$$

with the obvious extension to $p = \infty$. As in [Bouleau and Hirsch 1991, §I.6.2], one can endow each $\mathbb{V}_p$ with the norm

$$\|f\|_{\mathbb{V}_p} = \|f\|_{\mathbb{V}} + \|f\|_p + \|\Gamma(u)^{1/2}\|_p, \tag{2-10}$$

thus obtaining a Banach space, akin to the intersection of classical Sobolev spaces $W^{1,2} \cap W^{1,p}$. Notice that $\mathbb{V}_2 = \mathbb{V}$, with an equivalent norm. The Banach space structure plays a major role only starting from Section 5, but the notation $f \in \mathbb{V}_p$ is conveniently used throughout.

2B. ***Laplace operator and Markov semigroup.*** The Dirichlet form $\mathscr{E}$ induces a densely defined, negative and selfadjoint operator $\Delta : D(\Delta) \subset \mathbb{V} \to L^2(\mathfrak{m})$, defined by the integration by parts formula $\mathscr{E}(f, g) = -\int_X g \, \Delta f \, d\mathfrak{m}$ for all $g \in \mathbb{V}$. The operator Δ is of "diffusion" type, since it satisfies the following chain rule for every $\eta \in C^2(\mathbb{R})$ with $\eta(0) = 0$ and bounded first and second derivatives [Bouleau and Hirsch 1991, Corollary I.6.1.4]: whenever $f \in D(\Delta)$ with $\Gamma(f) \in L^2(\mathfrak{m})$, then $\eta(f) \in D(\Delta)$ and

$$\Delta\eta(f) = \eta'(f)\Delta f + \eta''(f)\Gamma(f). \tag{2-11}$$

The "heat flow" P_t associated to $\mathscr{E}$ is well-defined starting from any initial condition $f \in L^2(\mathfrak{m})$. Recall that in this framework the heat flow $(\mathsf{P}_t)_{t\geq 0}$ is an analytic Markov semigroup and that $f_t = \mathsf{P}_t f$ can be characterized as the unique C^1 map $f : (0, \infty) \to L^2(\mathfrak{m})$ with values in $D(\Delta)$ satisfying

$$\begin{cases} \dfrac{d}{dt} f_t = \Delta f_t & \text{for } t \in (0, \infty), \\ \lim_{t\downarrow 0} f_t = f & \text{in } L^2(\mathfrak{m}). \end{cases}$$

Because of this, Δ can equivalently be characterized in terms of the strong convergence $(\mathsf{P}_t f - f)/t \to \Delta f$ in $L^2(\mathfrak{m})$ as $t \downarrow 0$.

Furthermore, we have the regularization estimates (in the more general context of gradient flows of convex functionals, see [Ambrosio et al. 2005, Theorem 4.0.4(ii)], for instance)

$$\mathscr{E}(\mathsf{P}_t u, \mathsf{P}_t u) \leq \inf_{v\in\mathbb{V}} \left\{ \mathscr{E}(v, v) + \frac{\|v-u\|_2^2}{2t} \right\} < \infty \quad \forall t > 0,\ u \in L^2(\mathfrak{m}), \tag{2-12}$$

$$\|\Delta\mathsf{P}_t u\|_2^2 \leq \inf_{v\in D(\Delta)} \left\{ \|\Delta v\|_2^2 + \frac{\|v-u\|_2^2}{t^2} \right\} < \infty \quad \forall t > 0,\ u \in L^2(\mathfrak{m}). \tag{2-13}$$

For $p \in (1, \infty)$, we shall also need an L^p version of (2-13), namely

$$\|\Delta\mathsf{P}_t f\|_p \leq \frac{c_p^\Delta}{t}\|f\|_p \quad \text{for every } f \in L^p \cap L^2(\mathfrak{m}) \text{ and every } t \in (0, 1). \tag{2-14}$$

This can be obtained as a consequence of the fact that P is analytic [Stein 1970, Theorem III.1]; these are actually equivalent [Yosida 1980, §X.10].

As an easy corollary of (2-14), we obtain the following estimate.

Corollary 2.1. *Let $p \in (1, \infty)$ and let c_p^Δ be the constant in (2-14). Then*

$$\|\mathsf{P}_t f - \mathsf{P}_{t-t'} f\|_p \leq \min\left\{ c_p^\Delta \log\left(1 + \frac{t'}{t-t'}\right), 2 \right\} \|f\|_p \quad \forall f \in L^p \cap L^2(\mathfrak{m}),$$

for every $t,\ t' \in (0, 1)$ with $t' \leq t$.

Proof. The estimate with the constant 2 follows from L^p contractivity. For the other one, we apply (2-14) as follows:

$$\|\mathsf{P}_t f - \mathsf{P}_{t-t'} f\|_p \leq \int_0^{t'} \|\Delta\mathsf{P}_{t-t'+r} f\|_p \, dr \leq \int_0^{t'} \frac{c_p^\Delta}{t-t'+r} \, dr \|f\|_p = c_p^\Delta \log\left(1 + \frac{t'}{t-t'}\right)\|f\|_p. \qquad \square$$

One useful consequence of the Markov property is the L^p contraction of $(\mathsf{P}_t)_{t\geq 0}$ from $L^p \cap L^2$ to $L^p \cap L^2$. Because of the density of $L^p \cap L^2$ in L^p when $p \in [1, \infty)$, this allows us to uniquely extend P_t to a strongly continuous semigroup of linear contractions in $L^p(\mathfrak{m})$, $p \in [1, \infty)$, for which we retain the same notation. Furthermore, $(\mathsf{P}_t)_{t\geq 0}$ is sub-Markovian (see [Bouleau and Hirsch 1991, Proposition I.3.2.1]), since it preserves one-sided essential bounds: $f \leq C$ (resp. $f \geq C$) $\mathfrak{m}$-a.e. in X for some $C \geq 0$ (resp. $C \leq 0$) implies $\mathsf{P}_t f \leq C$ (resp. $\mathsf{P}_t f \geq C$) $\mathfrak{m}$-a.e. in X, for all $t \geq 0$.

Finally, it is easy to check, using L^1-contractivity of P, that the dual semigroup $\mathsf{P}_t^\infty : L^\infty(\mathfrak{m}) \to L^\infty(\mathfrak{m})$ given by

$$\int g \mathsf{P}_t^\infty f \, d\mathfrak{m} = \int f \mathsf{P}_t g \, d\mathfrak{m}, \qquad f \in L^\infty(\mathfrak{m}),\ g \in L^1(\mathfrak{m}), \tag{2-15}$$

is well-defined. It is a contraction semigroup in $L^\infty(\mathfrak{m})$, sequentially w^*-continuous, and it coincides with P on $L^2 \cap L^\infty(\mathfrak{m})$.

2C. *The algebra* $\mathcal{A}$. Throughout the paper we assume that an algebra $\mathcal{A} \subset \mathbb{V}$ is prescribed, with

$$\mathcal{A} \subset \bigcap_{p \in [1,\infty]} L^p(\mathfrak{m}), \qquad \mathcal{A} \text{ dense in } \mathbb{V}, \tag{2-16}$$

and

$$\Phi(f_1, \dots, f_n) \in \mathcal{A} \quad \text{whenever } \Phi \in C^1(\mathbb{R}^n),\ f_1, \dots, f_n \in \mathcal{A}. \tag{2-17}$$

Additional conditions on $\mathcal{A}$ will be considered in specific sections of the paper. A particular role is played by the condition $\mathcal{A} \subset \mathbb{V}_p$, for $p \in [2, \infty]$. By interpolation, if such an inclusion holds, then it holds for every q between 2 and p. Concerning the inclusion $\mathcal{A} \subset \mathbb{V}_p$ for $p \in [1, 2]$, we prove:

Lemma 2.2. *Let $\mathscr{A} \subset \mathbb{V}$ be dense in $\mathbb{V}$ and satisfy (2-17). Then, there exists $\mathcal{A} \subset \mathscr{A}$, such that (2-16) and (2-17) hold, and*

$$\mathcal{A} \textit{ is contained in and dense in } \mathbb{V}_p, \textit{for every } p \in [1, 2]. \tag{2-18}$$

In particular, without any loss of generality, we assume throughout that (2-18) holds.

Proof. We define

$$\mathcal{A} = \{\Phi(f) : f \in \mathscr{A},\ \Phi \in \mathscr{F}\} \subset \mathscr{A},$$

where $\mathscr{F}$ consists of all functions $\Phi : \mathbb{R} \to \mathbb{R}$ bounded and Lipschitz, continuously differentiable and null at the origin, with $\Phi'(x)/x$ bounded in $\mathbb{R}$. By the chain rule and Hölder's inequality, it follows that $\mathcal{A} \subset L^p(\mathfrak{m})$ for all $p \in [1, \infty]$ and that (2-17) holds. We address the density of $\mathcal{A}$ in $\mathbb{V}_p$ for $p \in [1, 2]$.

We consider Lipschitz functions $\phi_n : \mathbb{R} \mapsto [0, 1]$ such that $\phi_n(z) = 0$ for $|z| \leq 1/(2n)$ and for $|z| \geq 2n$, while $\phi_n(z) = 1$ for $|z| \in [1/n, n]$, and we set $\Phi_n(z) = \int_0^z \phi_n(t)dt$. Notice that $\Phi_n \equiv 0$ on $[-1/(2n), 1/(2n)]$, that Φ_n belongs to $\mathscr{F}$, and that $\Phi_n'(z) = \phi_n(z) \to 1$ as $n \to \infty$. It is easily seen, by the chain rule, that $\Phi_n(f) \to f$ in $\mathbb{V}_p$ as $n \to \infty$ for all $f \in \mathbb{V}_p$, therefore density is achieved if we show that all functions $\Phi_n(f)$ belong to the closure of $\mathcal{A}$. Since by assumption there exist $f_k \in \mathscr{A}$ convergent to f in $\mathbb{V}$ (and $\mathfrak{m}$-a.e. on X), it will be sufficient to show that, for every $n \geq 1$, $\Phi_n(f_k)$ converge to $\Phi_n(f)$ in $\mathbb{V}_p$ as $k \to \infty$.

We claim that, as $k \to \infty$, $\phi_n(f_k)$ converge to $\phi_n(f)$ in $L^q(\mathfrak{m})$ for every $q \in [1, \infty)$. To prove the claim, it suffices to consider separately the sets $\{|f| \geq 1/(3n)\}$ and $\{|f| < 1/(3n)\}$. On the first set, which has finite $\mathfrak{m}$-measure, we can use $\mathfrak{m}$-a.e. and dominated convergence to achieve the thesis, taking into account the boundedness of ϕ_n (since n is fixed); on the second set, we have

$$|\phi_n(f_k) - \phi_n(f)| = \chi_{\{|f_k| \geq 1/(2n)\}} |\phi_n(f_k) - \phi_n(f)| \leq \chi_{\{|f_k - f| \geq 1/(6n)\}} \min\{2, \mathrm{Lip}(\phi_n)|f_k - f|\}$$

and we can use Hölder's inequality for $q < 2$ and uniform boundedness for $q \geq 2$.

To show convergence of $\Phi_n(f_k)$ to $\Phi_n(f)$ in $\mathbb{V}_p$ as $k \to \infty$, we use the following straightforward identity, valid for any $h_1,\ h_2 \in \mathbb{V}$ and $\Phi \in \mathcal{F}$:

$$\Gamma(\Phi(h_1) - \Phi(h_2)) = (\Phi'(h_1) - \Phi'(h_2))^2 \Gamma(h_1, h_2) + \Phi'(h_1)^2 \Gamma(h_1, h_1 - h_2) + \Phi'(h_2)^2 \Gamma(h_2, h_2 - h_1).$$

Adding and subtracting $\Phi'(h_2)^2 \Gamma(h_1, h_1 - h_2)$, and taking $\Phi = \Phi_n$, since $0 \leq \phi_n \leq 1$ we obtain the inequality

$$\begin{aligned}\Gamma(\Phi_n(h_1) - \Phi_n(h_2))^{1/2} \leq |\phi_n(h_1) - \phi_n(h_2)| \Gamma(h_1)^{1/4} \Gamma(h_2)^{1/4} + \phi_n(h_2) \Gamma(h_1 - h_2)^{1/2} \\ + 2|\phi_n(h_1) - \phi_n(h_2))|^{1/2} \Gamma(h_1)^{1/4} \Gamma(h_1 - h_2)^{1/4}. \quad (2\text{-}19)\end{aligned}$$

Finally, we let $h_1 = f$ and $h_2 = f_k$ in this inequality and use the convergence of $\phi_n(f_k)$ to $\phi_n(f)$ in every $L^q(\mathfrak{m})$ space, $q \in [1, \infty)$, as well as Hölder's inequality, to deduce that the right-hand side above converges to 0 in $L^p(\mathfrak{m})$ as $k \to \infty$. □

We also deduce density in $L^p \cap L^q$-spaces, thanks to the following lemma.

Lemma 2.3. *There exists a countable set $\mathcal{D} \subset \mathcal{A}$ that is dense in $L^p \cap L^q(\mathfrak{m})$ for all $1 \leq p \leq q < \infty$ and w^*-dense in $L^\infty(\mathfrak{m})$.*

Proof. Since $\mathbb{V}$ is dense in $L^2(\mathfrak{m})$ and we assume that $\mathcal{A}$ is dense in $\mathbb{V}$, we obtain that $\mathcal{A}$ is dense in $L^2(\mathfrak{m})$.

We consider first the case $p = q \in [2, \infty]$. Let $h \in L^{p'}(\mathfrak{m})$. Assuming $\int h\varphi\, d\mathfrak{m} = 0$ for all $\varphi \in \mathcal{A}$, to prove density in the w^* topology (and then in the strong topology if $p < \infty$) we have to prove that $h = 0$. Let $\delta > 0$, set $f_\delta = \operatorname{sign} h \chi_{\{|h| > \delta\}}$ (set equal to 0 wherever $h = 0$) and find an equibounded sequence $(\varphi_n) \subset \mathcal{A}$ convergent in $L^2(\mathfrak{m})$ to f_δ. Since (φ_n) are uniformly bounded in $L^\infty(\mathfrak{m})$, we obtain strong convergence to f_δ in L^p for $p \in [2, \infty)$ and w^*-convergence for $p = \infty$. It follows that $\int_{\{|h| > \delta\}} |h|\, d\mathfrak{m} = 0$ and we can let $\delta \downarrow 0$ to get $h = 0$.

To cover the cases $p = q \in [1, 2)$, by interpolation we need only to consider $p = 1$. Given $f \in L^1(\mathfrak{m})$ nonnegative, we can find $\varphi_n \in \mathcal{A}$ convergent to $\sqrt{f}$ in $L^2(\mathfrak{m})$. It follows that the functions φ_n^2 belong to $\mathcal{A}$ and converge to f in $L^1(\mathfrak{m})$. In order to remove the sign assumption, we split $f \in L^1(\mathfrak{m})$ into its positive and negative parts.

Finally, in the case $p < q$ we can use the density of bounded functions to reduce ourselves to the case of approximation of a bounded function $f \in L^p \cap L^q(\mathfrak{m})$ by functions in $\mathcal{A}$. Since f can be approximated by equibounded functions $f_n \in \mathcal{A}$ in L^p norm, we need only to use the fact that $f_n \to f$ also in L^q norm.

Finally, a simple inspection of the proof shows that we can achieve the same density result with a countable subset of $\mathcal{A}$, since $\mathbb{V}$ is separable. □

Remark 2.4. Under the additional condition

$$\mathcal{A} \text{ is invariant under the action of } \mathsf{P}_t, \tag{2-20}$$

our basic assumption that $\mathcal{A}$ is dense in $\mathbb{V}$ can be weakened to the assumption that $\mathcal{A}$ is dense in $L^2(\mathfrak{m})$; indeed, standard semigroup theory shows that an invariant subspace is dense in $\mathbb{V}$ if and only if it is dense in $L^2(\mathfrak{m})$; see, for instance, [Ambrosio et al. 2014c, Lemma 4.9], but also Lemma 5.6 below.

3. Derivations

Since $\mathcal{A}$ might be regarded as an abstract space of test functions, we introduce derivations as linear operators acting on it, satisfying a Leibniz rule and a pointwise $\mathfrak{m}$-a.e. upper bound in terms of Γ (even though for some results an integral bound would be sufficient).

Definition 3.1 (derivation). A derivation is a linear operator $\boldsymbol{b} : \mathcal{A} \to L^0(\mathfrak{m})$, $f \mapsto df(\boldsymbol{b})$, satisfying

$$d(fg)(\boldsymbol{b}) = f dg(\boldsymbol{b}) + g df(\boldsymbol{b}) \quad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f,\ g \in \mathcal{A}, \text{ and}$$
$$|df(\boldsymbol{b})| \le g\sqrt{\Gamma(f)} \quad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f \in \mathcal{A}, \text{ for some } g \in L^0(\mathfrak{m}).$$

The smallest function g with this property will be denoted by $|\boldsymbol{b}|$. For $p,\ q \in [1,\infty]$, we say that a derivation $\boldsymbol{b}$ is in $L^p + L^q$ if $|\boldsymbol{b}| \in L^p(\mathfrak{m}) + L^q(\mathfrak{m})$.

Existence of the smallest function g can easily be achieved using the fact that $L^0(\mathfrak{m})$ is a complete lattice, that is by considering the supremum among all functions $f \in \mathcal{A}$ of the expression $|df(\boldsymbol{b})|\Gamma(f)^{-1/2}$ (set equal to 0 on the set $\{\Gamma(f) = 0\}$).

N. Gigli pointed out to us that linearity and the $\mathfrak{m}$-a.e. upper bound are sufficient to entail "locality" and thus the Leibniz and chain rules, with a proof contained in the work in preparation [Gigli 2014], akin to that of [Ambrosio and Kirchheim 2000, Theorem 3.5]. Since our work focuses on the continuity equation and related Lagrangian flows, but not on the fine structure of the space of derivations, for the sake of simplicity we have chosen to retain this slightly redundant definition and deduce only the validity of the chain rule.

Proposition 3.2 (chain rule for derivations). *Let $\boldsymbol{b}$ be a derivation and let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be a smooth function with $\Phi(0) = 0$. Then, for any $\boldsymbol{f} = (f_1, \dots, f_n) \in \mathcal{A}^n$, there holds*

$$d\Phi(\boldsymbol{f})(\boldsymbol{b}) = \sum_{i=1}^{n} \partial_i \Phi(\boldsymbol{f}) df_i(\boldsymbol{b}) \quad \mathfrak{m}\text{-a.e. in } X. \tag{3-1}$$

Proof. Since $\Phi(\boldsymbol{f}) \in \mathcal{A}$, $\boldsymbol{b}(\Phi(\boldsymbol{f}))$ is well-defined. Arguing by induction and linearity, the Leibniz rule entails that (3-1) holds when Φ is a polynomial in n variables with $\Phi(0) = 0$. Since $\boldsymbol{f}$ is bounded, the thesis follows by approximating Φ by a sequence (p_k) of polynomials converging to Φ, together with their derivatives, uniformly on compact sets. □

Remark 3.3 (derivations $u\boldsymbol{b}$). Let $\boldsymbol{b}$ be a derivation in L^q for some $q \in [1,\infty]$ and let $u \in L^r(\mathfrak{m})$, with $q^{-1} + r^{-1} \le 1$. Then, $f \mapsto u df(\boldsymbol{b})$ defines a derivation $u\boldsymbol{b}$ in $L^{s'}$, where $(s')^{-1} = q^{-1} + r^{-1}$, i.e., $q^{-1} + r^{-1} + s^{-1} = 1$. By linearity, similar remarks apply when $\boldsymbol{b}$ is a derivation in $L^p + L^q$.

Example 3.4 (gradient derivations). The main example is provided by derivations $\boldsymbol{b}_g$ induced by $g \in \mathbb{V}$, of the form

$$f \in \mathcal{A} \mapsto df(\boldsymbol{b}_g) := \Gamma(f, g) \in L^1(\mathfrak{m}). \tag{3-2}$$

These derivations belong to L^2, because (2-6) yields $|\boldsymbol{b}_g| \le \sqrt{\Gamma(g)}$. Since $\mathcal{A}$ is dense in $\mathbb{V}$, it is not difficult to show that equality holds.

By linearity, the L^∞-module generated by this class of examples (i.e., finite sums $\sum_i \chi_i \boldsymbol{b}_{g_i}$ with $\chi_i \in L^\infty(\mathfrak{m})$ and $g_i \in \mathbb{V}$) still consists of derivations in L^2.

Definition 3.5 (divergence). Let $p,\ q \in [1, \infty]$, assume that $\mathcal{A} \subset \mathbb{V}_{p'} \cap \mathbb{V}_{q'}$ and let $\boldsymbol{b}$ be a derivation in $L^p + L^q$. The distributional divergence $\operatorname{div} \boldsymbol{b}$ is the linear operator on $\mathcal{A}$ defined by

$$\mathcal{A} \ni f \mapsto -\int df(\boldsymbol{b})\, d\mathfrak{m}.$$

We say that $\operatorname{div} \boldsymbol{b} \in L^p(\mathfrak{m}) + L^q(\mathfrak{m})$ if the distribution $\operatorname{div} \boldsymbol{b}$ is induced by $g \in L^p(\mathfrak{m}) + L^q(\mathfrak{m})$, that is

$$\int df(\boldsymbol{b})\, d\mathfrak{m} = -\int fg\, d\mathfrak{m} \quad \text{for all } f \in \mathcal{A}.$$

Analogously, we say that $\operatorname{div} \boldsymbol{b}^- \in L^p(\mathfrak{m})$ if there exists a nonnegative $g \in L^p(\mathfrak{m})$ such that

$$\int df(\boldsymbol{b})\, d\mathfrak{m} \le \int fg\, d\mathfrak{m}, \quad \text{for all } f \in \mathcal{A},\ f \ge 0.$$

Notice that we impose the additional condition $\mathcal{A} \subset \mathbb{V}_{p'} \cap \mathbb{V}_{q'}$, to ensure integrability of $df(\boldsymbol{b})$.

As we did for $|\boldsymbol{b}|$, we define $\operatorname{div} \boldsymbol{b}^-$ as the smallest nonnegative function g in $L^p(\mathfrak{m})$ for which the inequality above holds. Existence of the minimal g follows by a simple convexity argument, because the class of admissible functions g is convex and closed in $L^p(\mathfrak{m})$ (if $p = \infty$, one has to consider the w^* topology).

Example 3.6 (divergence of gradients). The distributional divergence of the "gradient" derivation $\boldsymbol{b}_g$ induced by $g \in \mathbb{V}$ as in (3-2) coincides with the Laplacian Δg, still understood in distributional terms.

Although the definitions given above are sufficient for many purposes, the following extensions will be technically useful in Section 4C, and in Section 5 for the case $q \in [1, \infty)$.

Remark 3.7 (derivations in $L^2 + L^\infty$ extend to $\mathbb{V}$). When a derivation $\boldsymbol{b}$ belongs to $L^2 + L^\infty$, we can use the density of $\mathcal{A}$ in $\mathbb{V}$ to extend $\boldsymbol{b}$ uniquely to a continuous derivation, still denoted by $\boldsymbol{b}$, defined on $\mathbb{V}$, with values in the space $L^1(\mathfrak{m}) + L^2(\mathfrak{m})$. For all $u \in \mathbb{V}$, it still satisfies

$$|du(\boldsymbol{b})| \le |\boldsymbol{b}| \sqrt{\Gamma(u)} \quad \mathfrak{m}\text{-a.e. in } X.$$

A similar remark holds for derivations belonging to $L^q + L^\infty$, for some $q \in [1, \infty)$, if $\mathcal{A}$ is assumed to be dense in $\mathbb{V}_r$, for some $r \in [1, \infty)$ with $q^{-1} + r^{-1} \le 1$. The extension is then a continuous linear operator $\boldsymbol{b}$ mapping $\mathbb{V}_r$ into $L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m})$, where $q^{-1} + r^{-1} + s^{-1} = 1$.

By a similar density argument as above, any derivation $\boldsymbol{b}$ could be extended uniquely to a derivation defined on $\mathbb{V}$, with values in the space $L^0(\mathfrak{m})$. However, such an extension is not useful when dealing with integral functionals defined initially on $\mathcal{A}$, e.g., divergence or weak solutions to the continuity equation, because these are not continuous with respect to the topology of $L^0(\mathfrak{m})$. Therefore, we avoid in what follows considering such an extension, except for the case in the remark above.

We conclude this section noticing that if $\boldsymbol{b}$ is a derivation in L^2+L^∞, with $\operatorname{div}\boldsymbol{b} \in L^2(\mathfrak{m})+L^\infty(\mathfrak{m})$, the following integration by parts formula can be proved by approximation with functions in $\mathcal{A}$:

$$\int du(\boldsymbol{b}) f\, d\mathfrak{m} = -\int df(\boldsymbol{b}) u\, d\mathfrak{m} + \int uf \operatorname{div}\boldsymbol{b}\, d\mathfrak{m} \quad \forall f \in \mathcal{A},\ \forall u \in \mathbb{V}. \tag{3-3}$$

4. Existence of solutions to the continuity equation

Let $I=(0,T)$ with $T\in(0,\infty)$. In this section we prove existence of weak solutions to the continuity equation

$$\frac{d}{dt}u_t + \operatorname{div}(u_t\boldsymbol{b}_t) = w_t u_t \quad \text{in } I\times X, \tag{4-1}$$

under suitable growth assumptions on $\boldsymbol{b}_t$ and its divergence.

Remark 4.1. Starting from this section, we always assume that $\mathcal{A}$ is contained in $\mathbb{V}_\infty$, i.e., $\Gamma(f)\in L^\infty(\mathfrak{m})$ for every $f\in\mathcal{A}$. We are motivated by the examples and by the clarity that we gain in the exposition, although some variants of our results could be slightly reformulated and proved without this assumption.

Before we address the definition of (4-1), let us remark that a Borel family of derivations $\boldsymbol{b}=(\boldsymbol{b}_t)_{t\in I}$ is by definition a map $t\mapsto \boldsymbol{b}_t$, taking values in the space of derivations on X, such that $t\mapsto df(\boldsymbol{b}_t)$ is Borel in I for all $f\in\mathcal{A}$ and there exists a Borel function $g: I\times X\mapsto[0,\infty)$ satisfying

$$|\boldsymbol{b}_t| \le g(t,\cdot) \quad \mathfrak{m}\text{-a.e. in } X, \text{ for a.e. } t\in I.$$

As in the autonomous case we denote by $|\boldsymbol{b}|$ the smallest function g (in the $\mathscr{L}^1\otimes\mathfrak{m}$-a.e. sense) with this property. We say that Borel family of derivations $(\boldsymbol{b}_t)_{t\in I}$ belongs to $L^r_t(L^p_x+L^q_x)$ if $|\boldsymbol{b}|\in L^r_t(L^p_x+L^q_x)$.

Definition 4.2 (weak solutions to the continuity equation with initial condition $\bar u$). Let $p, q\in[1,\infty]$, $\bar u\in L^p\cap L^q(\mathfrak{m})$, let $(\boldsymbol{b}_t)_{t\in I}$ be a Borel family of derivations in $L^1_t(L^{p'}_x+L^{q'}_x)$ and let $w\in L^1_t(L^{p'}_x+L^{q'}_x)$. We say that $u\in L^\infty_t(L^p_x\cap L^q_x)$ solves (4-1) with the initial condition $u_0=\bar u$ in the weak sense if

$$\int_0^T\int[-\psi'\varphi - \psi\, d\varphi(\boldsymbol{b}_t) - w_t]u_t\, d\mathfrak{m}dt = \psi(0)\int \varphi\bar u\, d\mathfrak{m}, \tag{4-2}$$

for all $\varphi\in\mathcal{A}$ and all $\psi\in C^1([0,T])$ with $\psi(T)=0$.

As usual with weak formulations of PDEs, the definition above has many advantages, the main one being to provide a meaning to (4-1) without any regularity assumption on u. Notice that, without the assumption $\mathcal{A}\subset\mathbb{V}_\infty$, one could define weak solutions $u\in L^\infty_t(L^2_x)$ to the equation associated to $\boldsymbol{b}$ in $L^1_t(L^\infty_x)$.

In order to prove the mass-conservation property of solutions to the continuity equation we assume the existence of $(f_n) \subset \mathcal{A}$ satisfying

$$0 \le f_n \le 1, \quad f_n \uparrow 1 \ \ \mathfrak{m}\text{-a.e. in } X, \quad \sqrt{\Gamma(f_n)} \rightharpoonup 0 \ \text{ weakly-}* \text{ in } L^\infty(\mathfrak{m}). \tag{4-3}$$

The following theorem is our main result about existence; we address the case $w = 0$ only, the general case following from a Duhamel's principle that we do not pursue here.

Theorem 4.3 (existence of weak solutions in $L^\infty_t(L^1_x \cap L^2_x)$). *Assume that $\mathcal{A} \subset \mathbb{V}_\infty$, let $\bar{u} \in L^1 \cap L^r(\mathfrak{m})$ for some $r \in [2, \infty]$ and let $\boldsymbol{b} = (\boldsymbol{b}_t)_{t \in I}$ be a Borel family of derivations with $|\boldsymbol{b}| \in L^1_t(L^2_x + L^\infty_x)$, $\operatorname{div} \boldsymbol{b} \in L^1_t(L^2_x + L^\infty_x)$, and $\operatorname{div} \boldsymbol{b}^- \in L^1_t(L^\infty_x)$. Then, there exists a weakly continuous in $[0, T)$ (in duality with $\mathcal{A}$) solution $u \in L^\infty_t(L^1_x \cap L^r_x)$ of (4-1) according to Definition 4.2 with $u_0 = \bar{u}$ and $w_t = 0$. Furthermore, if $\bar{u} \ge 0$, we can build a solution u in such a way that $u_t \ge 0$ for all $t \in I$. Finally, if (4-3) holds, then*

$$\int u_t \, d\mathfrak{m} = \int \bar{u} \, d\mathfrak{m} \quad \forall t \in [0, T). \tag{4-4}$$

To prove existence of a solution u to (4-1) with $w_t = 0$, we rely on a suitable approximation of the equation. Following a classical strategy, we approximate the original equation by adding a diffusion term, i.e., we solve, still in the weak sense of duality with test functions $\psi(t)\varphi(x)$,

$$\partial_t u_t + \operatorname{div}(u_t \boldsymbol{b}_t) = \sigma \Delta u_t, \tag{4-5}$$

where $\sigma > 0$. By Hilbert space techniques, we show existence of a solution with some extra regularity, namely $u \in L^2(I; \mathbb{V})$. We use this extra regularity to derive a priori estimates and then we take weak limits as $\sigma \downarrow 0$.

Let us remark that such a technique forces the introduction of stronger assumptions than those known to prove existence in particular classes of spaces (e.g., Euclidean or Gaussian), where ad hoc methods are available; here, we trade some strength in the result in favor of generality.

4A. *Auxiliary Hilbert spaces.* In all of what follows, we consider the Gelfand triple

$$\mathbb{V} \subset L^2(\mathfrak{m}) = (L^2(\mathfrak{m}))^* \subset \mathbb{V}',$$

that is we regard $\mathbb{V}$ as a dense subspace in $\mathbb{V}'$ (proper if $\mathbb{V} \ne L^2(\mathfrak{m})$) by means of

$$\phi \mapsto (\phi^* : f \mapsto \textstyle\int f \phi \, d\mathfrak{m}).$$

Notice that this is different from the identification $\mathbb{V} \sim \mathbb{V}'$ provided by the Riesz–Fischer theorem applied to the Hilbert space $\mathbb{V}$ (which has been applied to $L^2(\mathfrak{m})$ instead).

Given a vector space F, we introduce a space of F-valued test functions on I, namely

$$\Phi_F := \operatorname{span}\{\psi \cdot \phi : \psi \in C^1([0, T]), \ \psi(T) = 0, \ \phi \in F\}.$$

We notice that, for every $\varphi \in \Phi_F$, the function $t \mapsto \varphi_t$ is Lipschitz and continuously differentiable from I to F, and there exists $\varphi_0 = \lim_{t \downarrow 0} \varphi_t$ in F (while $\lim_{t \uparrow T} \varphi_t = 0$ in F by construction).

Assuming that F is a separable Hilbert space, starting from Φ_F one can consider completions with respect to different norms. The classical space

$$L^2(I;F), \quad \langle \varphi, \tilde\varphi\rangle_{L^2(F)} = \int_I \langle \varphi_t, \tilde\varphi_t\rangle_F \, dt,$$

is indeed the closure of Φ_F with respect to the norm induced by the scalar product above. Similarly, the space $H^1(I;F)$ is obtained by completing Φ_F with respect to the norm

$$\langle \varphi, \tilde\varphi\rangle_{H^1(F)} = \int_I \langle \varphi_t, \tilde\varphi_t\rangle_F + \left\langle \frac{d}{dt}\varphi_t, \frac{d}{dt}\tilde\varphi_t \right\rangle_F dt.$$

Arguing by mollification as in the case $F=\mathbb{R}^n$, it is not difficult to prove that $H^1(I;F) = W^{1,2}(I;F)$, where the latter space is defined as the subspace of functions $\varphi \in L^2(I;F)$ such that there exists $g \in L^2(I;F)$ which represents the distributional derivative of φ, i.e.,

$$\int_I \left\langle \varphi_t, \frac{d}{dt}\tilde\varphi_t \right\rangle_F dt = -\int_0^T \langle g_t, \tilde\varphi_t\rangle_F \, dt \quad \text{for every } \tilde\varphi \in \Phi_F \text{ with } \tilde\varphi_0 = 0.$$

4B. *Existence under additional ellipticity.* We address now the existence of some $u \in L^2(I;\mathbb{V})$ that solves the following weak formulation of (4-5) with the initial condition $u_0 = \bar u$:

$$\int_0^T \int [-\partial_t\varphi_t - d\varphi_t(\boldsymbol{b}_t)]u_t + \sigma\Gamma(\varphi_t, u_t)\, d\mathfrak{m}\, dt = \int \varphi_0 \bar u \, d\mathfrak{m} \quad \forall \varphi \in \Phi_{\mathcal{A}}. \tag{4-6}$$

We still assume that $\mathcal{A} \subset \mathbb{V}_\infty$, and that $\sigma \in (0, 1/2]$, $|\boldsymbol{b}| \in L^\infty_t(L^2_x + L^\infty_x)$, $\operatorname{div} \boldsymbol{b}^- \in L^\infty_t(L^\infty_x)$, $\bar u \in L^2(\mathfrak{m})$. Notice that the assumptions on $|\boldsymbol{b}|$ and $\operatorname{div}\boldsymbol{b}^-$ are stronger than those in Theorem 4.3, but only with respect to integrability in time.

We obtain, together with existence, the a priori estimate

$$\|e^{-\lambda t}u\|_{L^2(I;\mathbb{V})} \le \frac{\|\bar u\|_2}{\sigma} \quad \text{with } \lambda := \tfrac{1}{2}|\operatorname{div}\boldsymbol{b}^-|_\infty + \sigma. \tag{4-7}$$

To this aim, we change variables setting $h_t = e^{-\lambda t}u_t$ and we pass to the equivalent weak formulation

$$\int_0^T \int [-\partial_t\varphi_t + \lambda - d\varphi_t(\boldsymbol{b}_t)]h_t + \sigma\Gamma(\varphi_t, h_t)\, d\mathfrak{m}\, dt = \int \varphi_0 \bar u \, d\mathfrak{m}, \quad \forall \varphi \in \Phi_{\mathcal{A}}. \tag{4-8}$$

From now on we shall use the notation $\tilde{\mathfrak{m}}$ for the product measure $\mathscr{L}^1 \otimes \mathfrak{m}$ in $I \times X$. Existence of h is a consequence of J. L. Lions' extension of the Lax–Milgram theorem, whose statement is recalled below [Showalter 1997, Theorem III.2.1, Corollary III.2.3] applied with $H = L^2(I;\mathbb{V})$, $V = \Phi_{\mathcal{A}}$ endowed with the norm

$$\|\varphi\|_V^2 = \|\varphi\|^2_{L^2(I;\mathbb{V})} + \|\varphi_0\|_2^2, \tag{4-9}$$

$$B(\varphi, h) = \int [-\partial_t\varphi + \lambda\varphi - d\varphi(\boldsymbol{b})]h + \sigma\Gamma(\varphi, h)\, d\tilde{\mathfrak{m}}, \quad \ell(\varphi) = \int \varphi_0 \bar u \, d\mathfrak{m}.$$

Theorem 4.4 (Lions). *Let V, H be respectively a normed and a Hilbert space, with V continuously embedded in H, with $\|v\|_H \le \|v\|_V$ for all $v \in V$, and let $B : V \times H \to \mathbb{R}$ be bilinear, with $B(v,\cdot)$*

continuous for all $v \in V$. If B is coercive, namely there exists $c > 0$ satisfying $B(v, v) \geq c\|v\|_V^2$ for all $v \in V$, then for all $\ell \in V'$ there exists $h \in H$ such that $B(\,\cdot\,, h) = \ell$ and

$$\|h\|_H \leq \frac{\|\ell\|_{V'}}{c}. \tag{4-10}$$

Let us start by proving continuity; to this end, let $\varphi \in V$. The linear functional $h \mapsto B(\varphi, h)$ is $L^2(I; \mathbb{V})$-continuous for all $\phi \in V$, since we can estimate $|B(\varphi, h)|$ from above with

$$\|h\|_{L^2(I;\mathbb{V})}\big[\|\partial_t\varphi\|_{L^2_t(L^2_x)} + \lambda\|\varphi\|_{L^2_t(L^2_x)} + \|\boldsymbol{b}\|_{L^2_t(L^2_x+L^\infty_x)}\|\sqrt{\Gamma(\varphi)}\|_{L^\infty_t(L^2_x\cap L^\infty_x)} + \sigma\|\sqrt{\Gamma(\varphi)}\|_{L^2_t(L^2_x)}\big].$$

The functional ℓ satisfies $\|\ell\|_{V'} \leq \|\bar{u}\|_2$, immediately from the definition of $\|\cdot\|_V$ in (4-9).

To conclude the verification of the assumptions of Theorem 4.4, we show coercivity (here the change of variables we did and the choice of λ play a role):

$$\begin{aligned}\int [\lambda\varphi - d\varphi(\boldsymbol{b})]\varphi \, d\widetilde{\mathfrak{m}} &= \lambda\|\varphi\|^2_{L^2_t(L^2_x)} - \frac{1}{2}\int d\varphi^2(\boldsymbol{b})\, d\widetilde{\mathfrak{m}} \\ &\geq \lambda\|\varphi\|^2_{L^2_t(L^2_x)} - \frac{1}{2}\int \varphi^2 \operatorname{div}\boldsymbol{b}^- \, d\widetilde{\mathfrak{m}} \\ &\geq \big(\lambda - \tfrac{1}{2}\|\operatorname{div}\boldsymbol{b}^-\|_\infty\big)\|\varphi\|^2_{L^2_t(L^2_x)} = \sigma\|\varphi\|^2_{L^2_t(L^2_x)}.\end{aligned} \tag{4-11}$$

Since $\varphi \in V = \Phi_{\mathcal{A}}$, we obtain $\partial_t\varphi_t^2 = 2\varphi_t\partial_t\varphi_t$ and $\int -2\varphi_t\partial\varphi\, d\widetilde{\mathfrak{m}} = \int \varphi_0^2\, d\mathfrak{m}$. Hence, inequality (4-11) entails that

$$\int [-\partial_t\varphi + \lambda\varphi - d\varphi(\boldsymbol{b})]\varphi + \sigma\Gamma(\varphi)\, d\widetilde{\mathfrak{m}} \geq \frac{1}{2}\int \varphi_0^2\, d\mathfrak{m} + \sigma\|\varphi\|^2_{L^2_t(L^2_x)} + \sigma\|\sqrt{\Gamma(\varphi)}\|^2_{L^2_t(L^2_x)},$$

Since $\sigma \leq \frac{1}{2}$, it follows from these two inequalities that

$$B(\varphi, \varphi) \geq \sigma\|\varphi\|_V^2. \tag{4-12}$$

Finally, (4-7) follows at once from (4-10) and (4-12), taking into account that $\|\ell\|_{V'} \leq \|\bar{u}\|_2$.

4C. *A priori estimates.* In this section we still consider weak solutions to

$$\int_0^T\int -[\partial_t\varphi_t + d\varphi_t(\boldsymbol{b}_t)]u_t + \sigma\Gamma(\varphi_t, u_t)\, d\mathfrak{m}dt = \int \varphi_0\bar{u}\, d\mathfrak{m} \quad \forall\varphi \in \Phi_{\mathcal{A}}, \tag{4-13}$$

obtained in the previous section. In order to state pointwise in time L^r estimates in space, we use the following remark.

Remark 4.5 (equivalent formulation). Assuming $\mathcal{A} \subset \mathbb{V}_\infty$, $u \in L^2(I; \mathbb{V})$ and $|\boldsymbol{b}| \in L^1_t(L^2_x + L^\infty_x)$, an equivalent formulation of (4-13), in terms of absolute continuity and pointwise derivatives w.r.t. time, is the following: we are requiring that, for every $f \in \mathcal{A}$, $t \mapsto \int f u_t\, d\mathfrak{m}$ is absolutely continuous in I and that its a.e. derivative in I is $\int (df(\boldsymbol{b}_t)u_t + \sigma\Gamma(f, u_t))\, d\mathfrak{m}$. In addition, the Cauchy initial condition is encoded by

$$\lim_{t\downarrow 0}\int f u_t\, d\mathfrak{m} = \int f\bar{u}\, d\mathfrak{m} \quad \text{for every } f \in \mathcal{A} \tag{4-14}$$

(notice also that $\bar{u}$ is uniquely determined by (4-14), thanks to the density of $\mathcal{A}$ in $L^2(\mathfrak{m})$).

Indeed, it is clear that the definition above implies the formula for the distributional derivative, because for absolutely continuous functions the two concepts coincide; the converse can be obtained using the set $\mathscr{D}$ of Lemma 2.3 to redefine u_t in a negligible set of times in order to get a weakly continuous representative in the duality with $\mathcal{A}$; see [Ambrosio et al. 2005, Lemma 8.1.2] for details.

We prove, by a suitable approximation, the following result:

Theorem 4.6. *Assume that $\mathcal{A} \subset \mathbb{V}_\infty$, $|\boldsymbol{b}| \in L^\infty_t(L^2_x + L^\infty_x)$, $\operatorname{div} \boldsymbol{b} \in L^\infty_t(L^2_x + L^\infty_x)$, $\operatorname{div} \boldsymbol{b}^- \in L^\infty_t(L^\infty_x)$, and that the initial condition $\bar{u}$ belongs to $L^p \cap L^q(\mathfrak{m})$, with $1 \le p \le 2 \le q \le \infty$. Then there exists a weakly continuous (in duality with $\mathcal{A}$) solution*

$$u \in L^\infty_t(L^p_x \cap L^q_x) \cap L^2(I; \mathbb{V})$$

to (4-13) satisfying

$$\sup_{(0,T)} \|u_t^\pm\|_r \le \|\bar{u}^\pm\|_r \exp\Big(\Big(1 - \frac{1}{r}\Big)\|\operatorname{div} \boldsymbol{b}^-\|_{L^1_t(L^\infty_x)}\Big), \tag{4-15}$$

for every $r \in [p, q]$. In particular, if $\bar{u} \ge 0$, then $u_t \ge 0$ for all $t \in (0, T)$.

At this stage, it is technically useful to introduce another formulation of the continuity equation, suitable for $\mathbb{V}$-valued solutions u, with the derivation acting on u.

Remark 4.7 (transport weak formulation). Using (3-3) we obtain an equivalent weak formulation of (4-13), namely

$$\int_0^T \int -u_t \partial_t \varphi_t + du_t(\boldsymbol{b}_t)\varphi_t + u_t\varphi_t \operatorname{div} \boldsymbol{b}_t + \sigma\Gamma(\varphi_t, u_t)\, d\mathfrak{m}dt = \int \varphi_0 \bar{u}\, d\mathfrak{m}, \quad \forall \varphi \in \Phi_{\mathcal{A}}. \tag{4-16}$$

Remark 4.8 (basic formal identity). Before we address the proof of the a priori estimates, let us remark that these, and uniqueness as well, strongly rely on the formal identity

$$\frac{d}{dt}\int \beta(u_t)\, d\mathfrak{m} = -\int [\beta'(u_t)u_t - \beta(u_t)] \operatorname{div} \boldsymbol{b}_t\, d\mathfrak{m}, \tag{4-17}$$

which comes from chain rule in (4-2) and the formal identity $\int \operatorname{div}(\beta(u_t)\boldsymbol{b}_t) = 0$. To establish existence, however, this computation is made rigorous by approximating the PDE (by vanishing viscosity, in Theorem 4.6, or by other approximations), while to obtain uniqueness in Section 5B we approximate u. In both cases technical assumptions on $\boldsymbol{b}$ will be needed.

A natural choice in (4-17) is a convex "entropy" function $\beta : \mathbb{R} \to \mathbb{R}$ with $\beta(0) = 0$. In order to give a meaning to the identity (4-17) also when β is not C^1 ($z \mapsto z^+$ will be a typical choice of β) we define

$$\mathscr{L}_\beta(z) := \begin{cases} z\beta'_+(z) - \beta(z) & \text{if } z \ge 0, \\ z\beta'_-(z) - \beta(z) & \text{if } z \le 0, \end{cases} \tag{4-18}$$

where we write $\beta'_\pm(z) := \lim_{y \to z^\pm} \beta'(y)$. Notice that the convexity of β and the condition $\beta(0) = 0$ give that $\mathscr{L}_\beta$ is nonnegative; for instance, if $z \ge 0$, there holds

$$\beta(0) = 0 \ge \beta(z) - z\beta'_-(z) \ge \beta(z) - z\beta'_+(z).$$

The argument for $z \le 0$ follows from $\mathscr{L}_{\tilde\beta}(-z) = \mathscr{L}_\beta(z)$, where $\tilde\beta(z) = \beta(-z)$.

In order to approximate β with functions with linear growth in $\mathbb{R}$, we will consider the approximations

$$\beta_n(z) := \begin{cases} \beta(-n)+\beta'_-(-n)(z+n) & \text{if } z<-n, \\ \beta(z) & \text{if } -n\le z\le n, \\ \beta(n)+\beta'_+(n)(z-n) & \text{if } z>n, \end{cases} \tag{4-19}$$

that satisfy $\mathscr{L}_{\beta_n}(z)=\mathscr{L}_\beta(-n\vee z\wedge n)$, so that $\mathscr{L}_{\beta_n}\uparrow\mathscr{L}_\beta$ as $n\to\infty$. On the other hand, in order to pass from smooth to nonsmooth β, we will also need the following property, whose proof is elementary and motivates our precise definition of $\mathscr{L}_\beta$ in (4-18):

$$\limsup_{i\to\infty}\mathscr{L}_{\beta_i}\le\mathscr{L}_\beta \quad \text{whenever } \beta_i \text{ are convex, } \beta_i\to\beta \text{ uniformly on compact sets.} \tag{4-20}$$

Proof of Theorem 4.6. By Remark 4.5 we can assume with no loss of generality that $t\mapsto u_t$ is weakly continuous in $[0,T)$, in the duality with $\mathcal{A}$.

We assume first that a weak solution u satisfies the strong continuity property

$$\lim_{t\downarrow 0}u_t=\bar u \quad \text{in } L^2(\mathfrak{m}). \tag{4-21}$$

We shall remove this assumption at the end of the proof.

We claim that, for any convex function $\beta:\mathbb{R}\to[0,\infty)$ where $\beta(0)=0$ and $\beta'(z)/z$ is bounded on $\mathbb{R}$, the inequality

$$\frac{d}{dt}\int\beta(u_t)\,d\mathfrak{m}\le\int\mathscr{L}_\beta(u_t)\operatorname{div}\boldsymbol{b}_t^-\,d\mathfrak{m} \tag{4-22}$$

holds in the sense of distributions in $(0,T)$. The assumption on the behavior of β near to the origin is needed to ensure that both $\beta(u)$ and $\mathscr{L}_\beta(u)$ belong to $L^2_t(L^1_x)$, since at present we only know that $u\in L^2_t(L^2_x)$. By approximation, taking (4-19) and (4-20) into account, we can assume with no loss of generality that $\beta\in C^1$ with bounded derivative.

In the proof of (4-22), motivated by the necessity to get strong differentiability w.r.t. time, we shall use the regularization $u_t^s:=\mathsf{P}_s u_t$ and the following elementary remark [Showalter 1997, Prop. III.1.1].

Remark 4.9. Let X be a Banach space and let $f,\ g\in L^1((0,T);X)$ satisfy $\partial_t f=g$ in the weak sense, namely

$$-\int_0^T\psi'(t)\int\phi(f)\,d\mathfrak{m}dt=\int_0^T\psi(t)\int\phi(g)\,d\mathfrak{m}dt,$$

for every $\psi\in C^1_c(0,T)$, $\phi\in\mathscr{D}\subset X^*$, dense w.r.t. the $\sigma(X^*,X)$ topology. Then, f admits a unique absolutely continuous representative from I to X and this representative is strongly differentiable a.e. in I, with derivative equal to g.

Notice that X may not have the Radon–Nikodym property so that it might be the case that not all absolutely continuous maps with values in X are strongly differentiable a.e. in their domain. Indeed, we are going to apply it with $X=L^1(\mathfrak{m})+L^2(\mathfrak{m})$, so that $X^*=L^2\cap L^\infty(\mathfrak{m})$, and $\mathscr{D}=\mathcal{A}$.

It is immediate to check, replacing φ in (4-16) by $\mathsf{P}_s\varphi$ and using (3-3), that for any $s>0$ the function $t\mapsto u_t^s$ solves

$$\frac{d}{dt}u_t^s+\operatorname{div}(\boldsymbol{b}_t u_t^s)=\sigma\Delta u_t^s+\mathscr{C}_t^s$$

in the weak sense of duality with $\mathcal{A}$, where $\mathscr{C}_t^s$ is the commutator between semigroup and divergence, namely

$$\mathscr{C}_t^s := \operatorname{div}(\boldsymbol{b}_t u_t^s) - \mathsf{P}_s(\operatorname{div}(\boldsymbol{b}_t u_t)).$$

Therefore, using (3-3) once more and expanding

$$\mathscr{C}_t^s = u_t^s \operatorname{div} \boldsymbol{b}_t + du_t^s(\boldsymbol{b}_t) - \mathsf{P}_s(u_t \operatorname{div} \boldsymbol{b}_t) - \mathsf{P}_s(du_t(\boldsymbol{b}_t))$$

we may use the assumption $\operatorname{div} \boldsymbol{b} \in L_t^\infty(L_x^2 + L_x^\infty)$ and the continuity of derivations to obtain that $\mathscr{C}_t^s \to 0$ strongly in $L_t^2(L_x^1 + L_x^2)$ as $s \downarrow 0$. Similarly, expanding $\operatorname{div}(\boldsymbol{b}_t u_t^s) = u_t^s \operatorname{div} \boldsymbol{b}_t + du_t^s(\boldsymbol{b}_t)$ and using the regularization estimate (2-13) to estimate the Laplacian term in the derivative of u_t, we obtain $\frac{d}{dt} u_t^s \in L_t^2(L_x^1 + L_x^2)$ in the weak sense of duality with $\mathcal{A}$; therefore by Remark 4.9, $t \mapsto u_t^s$ is strongly $(L^1 + L^2)$-differentiable a.e. in $(0, T)$ and absolutely continuous.

Since β is convex, we can start from the inequality

$$\int \beta(u_t^s)\, d\mathfrak{m} - \int \beta(u_{t'}^s)\, d\mathfrak{m} \le \int \beta'(u_t^s)(u_t^s - u_{t'}^s)\, d\mathfrak{m}$$

and use the uniform boundedness of $\beta'(z)$ and of $\beta'(z)/z$ to obtain that $\beta'(u_t^s) \in L_t^2(L_x^2 \cap L_x^\infty)$; hence

$$\int \beta(u_t^s)\, d\mathfrak{m} - \int \beta(u_{t'}^s)\, d\mathfrak{m} \le g(t) \left| \int_{t'}^t \left\| \frac{d}{dr} u_r^s \right\|_{L^1+L^2} dr \right|$$

with $g(t) = \|\beta'(u_t^s)\|_{L^2 \cap L^\infty} \in L^2(0, T)$. Since (again by the convexity of β) $t \mapsto \int \beta(u_t^s)\, d\mathfrak{m}$ is lower semicontinuous, a straightforward application of a calculus lemma [Ambrosio et al. 2014b, Lemma 2.9] entails that $t \mapsto \int \beta(u_t^s)\ d\mathfrak{m}$ is absolutely continuous in $(0, T)$ and that

$$\frac{d}{dt} \int \beta(u_t^s)\, d\mathfrak{m} = \int \beta'(u_t^s) \left[-\operatorname{div}(\boldsymbol{b}_t u_t^s) + \sigma \Delta u_t^s + \mathscr{C}_t^s \right] d\mathfrak{m} \quad \text{for a.e. } t \in (0, T). \tag{4-23}$$

Since $\beta(u_t^s) \in \mathbb{V}$ we get $\int \beta'(u_t^s) \Delta u_t^s\, d\mathfrak{m} = -\int \beta''(u_t^s) \Gamma(u_t^s)\, d\mathfrak{m} \le 0$, hence we may disregard this term. Using the chain rule twice and $\operatorname{div} \boldsymbol{b} \in L_t^2(L_x^2 + L_x^\infty)$, $\mathscr{L}_\beta(u) \in L_t^2(L_x^1)$ gives

$$\begin{aligned}
\frac{d}{dt} \int \beta(u_t^s)\, d\mathfrak{m} &\le -\int \beta'(u_t^s) u_t^s \operatorname{div} \boldsymbol{b}_t + d\beta(u_t^s)(\boldsymbol{b}_t)\, d\mathfrak{m} + \int \beta'(u_t^s) \mathscr{C}_t^s\, d\mathfrak{m} \\
&= -\int (\beta'(u_t^s) u_t^s - \beta(u_t^s)) \operatorname{div} \boldsymbol{b}_t\, d\mathfrak{m} + \int \beta'(u_t^s) \mathscr{C}_t^s\, d\mathfrak{m} \\
&\le \int (\beta'(u_t^s) u_t^s - \beta(u_t^s)) \operatorname{div} \boldsymbol{b}_t^-\, d\mathfrak{m} + \int \beta'(u_t^s) \mathscr{C}_t^s\, d\mathfrak{m}.
\end{aligned}$$

Eventually, since $\beta'(u_t^s)$ are bounded in $L_t^2(L_x^2 \cap L_x^\infty)$, uniformly w.r.t. s, we let $s \downarrow 0$ to obtain (4-22) (convergence of the first term in the right-hand side follows from dominated convergence and convergence in $L^2(\mathfrak{m})$ of $u_t^s \to u_t$).

We now prove (4-15). Let $r \in [p, q]$, let $\beta(z) = (z^+)^r$ and notice that $\mathscr{L}_\beta(z) = (r-1)\beta(z)$. We cannot apply (4-22) directly to β, because $\beta'(z)/z$ is unbounded near 0. If $r < 2$, we let

$$\beta_n(z) := \begin{cases} \dfrac{(z^+)^2}{2\epsilon^{2-r}} & \text{if } z \le \epsilon, \\ (z^+)^r - \dfrac{\epsilon^r}{2} & \text{if } z \ge \epsilon, \end{cases}$$

where $\varepsilon = 1/n$, so that β_n are convex, $\beta_n'(z)/z$ is bounded, $\mathscr{L}_{\beta_n} \le \beta_n$, and $\beta_n \to \beta$ as $n \to \infty$.

If $r \ge 2$, we use the approximations β_n in (4-19), that satisfy $\mathscr{L}_{\beta_n}(z) = \mathscr{L}_\beta(z \wedge n)$, so that we still have $\mathscr{L}_{\beta_n} \le (r-1)\beta_n$, and $\beta_n'(z)/z$ is bounded.

Now, in both cases it is sufficient to apply Gronwall's lemma to the differential inequality (4-22) with $\beta = \beta_n$ (here we use the assumption (4-21) to ensure that the value at 0 is the expected one) and then let $n \to \infty$ to conclude with Fatou's lemma.

The correspondent inequalities for $\beta(z) = (z^-)^r$ are settled similarly.

Finally, the assumption (4-21) can be removed by considering the solutions u_t^ε relative to the same initial condition and to the derivations

$$\boldsymbol{b}_t^\varepsilon := \begin{cases} \boldsymbol{b}_t & \text{if } t \in [\epsilon, T), \\ 0 & \text{if } t \in (0, \epsilon). \end{cases}$$

Since u_t^ε coincides with $\mathsf{P}_{\sigma t}\bar{u}$ for $t \in (0, \epsilon)$, (4-21) is fulfilled. Then, we can take weak limits in $L_t^\infty(L_x^p \cap L_x^q) \cap L^2(I; \mathbb{V})$ as $\varepsilon \downarrow 0$ to obtain a function u satisfying the desired properties. $\square$

4D. ***Vanishing viscosity and proof of Theorem 4.3.*** Let $\boldsymbol{b} = (\boldsymbol{b}_t)_{t\in I}$ and $\bar{u} \in L^1 \cap L^r(\mathfrak{m})$ $(r \ge 2)$ satisfy the assumptions of Theorem 4.3. Let $\delta > 0$, let ρ be a mollifying kernel in $C_c^1(0,1)$ and set $\boldsymbol{b}_t^\delta := \int_0^1 \boldsymbol{b}_{t+s\delta}\rho(s)\,ds$ (where $\boldsymbol{b}_t = 0$ for $t > T$), i.e., we let

$$\varphi \mapsto d\varphi(\boldsymbol{b}_t^\delta) = \int_0^1 d\varphi(\boldsymbol{b}_{t+s\delta})\rho(s)\,ds.$$

Since $|\boldsymbol{b}| \in L_t^1(L_x^2)$, $\operatorname{div}\boldsymbol{b} \in L_t^1(L_x^2 + L_x^\infty)$, it follows that $|\boldsymbol{b}^\delta| \in L_t^\infty(L_x^2)$, $\operatorname{div}\boldsymbol{b}^\delta \in L_t^\infty(L_x^2 + L_x^\infty)$ and the assumption $\operatorname{div}\boldsymbol{b}^- \in L_t^1(L_x^1 \cap L_x^\infty)$ entails $(\operatorname{div}\boldsymbol{b}^\delta)^- \in L_t^\infty(L_x^1 \cap L_x^\infty)$. Moreover, as $\delta \downarrow 0$, $d\varphi(\boldsymbol{b}^\delta)$ converges to $d\varphi(\boldsymbol{b})$ in $L_t^1(L_x^2 + L_x^\infty)$, for every $\varphi \in \mathcal{A}$ and $\|(\operatorname{div}\boldsymbol{b}^\delta)^-\|_{L_t^1(L_x^\infty)}$ converges to $\|(\operatorname{div}\boldsymbol{b})^-\|_{L_t^1(L_x^\infty)}$.

For fixed $\delta > 0$, consider a sequence $u^n = u^{\delta,n}$ of solutions to (4-13) with $\boldsymbol{b}^\delta$ in place of $\boldsymbol{b}$, $\sigma = 1/n$, $n \ge 2$, as provided by Theorem 4.6 with $p = 1$ and $q = r$, and notice that (4-7) gives

$$\frac{1}{n}\|e^{-(1+\lambda)t}u^n\|_{L^2(I;\mathbb{V})} \le \|\bar{u}\|_2,$$

so that $v^n := u^n/n$ is bounded in $L^2(I;\mathbb{V})$. We would like to pass to the limit as $n \to \infty$ in

$$\int_0^T \int -\big[\partial_t\varphi_t + d\varphi_t(\boldsymbol{b}_t^\delta)\big]u_t^n + \Gamma(\varphi_t, v_t^n)\,d\mathfrak{m}\,dt = \int \varphi_0\bar{u}\,d\mathfrak{m} \quad \forall \varphi \in \Phi_{\mathcal{A}}. \tag{4-24}$$

Inequality (4-15) entails that $(u^n)_n$ is bounded in $L_t^\infty(L_x^1 \cap L_x^r)$, so v^n weakly converges to 0 in $L^2(I;\mathbb{V})$. In addition, there exists a subsequence $n(k)$ such that $(u^{n(k)})_k$ converges, in duality with $L_t^1(L_x^2 + L_x^\infty)$, to some $u := u^\delta \in L_t^\infty(L_x^1 \cap L_x^r)$. This gives that u^δ is a weak solution to the continuity equation with $\boldsymbol{b}^\delta$ in place of $\boldsymbol{b}$.

We then let $\delta \downarrow 0$ and again extract a subsequence $\delta(k)$ such that $(u^{\delta(k)})_k$ converges, in duality with

$L^1_t(L^{r'}_x + L^\infty_x)$, to some $u := L^\infty_t(L^1_x \cap L^r_x)$ and is a weak solution to the continuity equation, thus concluding the proof of Theorem 4.3, except for conservation of mass.

Finally, we prove conservation of mass for any weak solution to the continuity equation, assuming the existence of $f_n \in \mathcal{A}$ as in (4-3). The proof is based on the simple observation that our assumptions on $\boldsymbol{b}$ and u imply $\boldsymbol{c} := u\boldsymbol{b} \in L^1_t(L^1_x)$, and therefore

$$\lim_{n\to\infty} \int_0^T \int |df_n(\boldsymbol{c}_t)|\, d\mathfrak{m}\, dt = 0.$$

Since

$$\lim_{n\to\infty} \int u_t f_n \, d\mathfrak{m} = \int u_t \, d\mathfrak{m} \quad \forall t \in [0,T), \qquad \text{and} \qquad \frac{d}{dt}\int u_t f_n\, d\mathfrak{m} = \int df_n(\boldsymbol{c}_t)\, d\mathfrak{m},$$

we conclude that $\int u_t \, d\mathfrak{m} = \int \bar{u}\, d\mathfrak{m}$ for all $t \in [0,T)$.

5. Uniqueness of solutions to the continuity equation

In this section, we provide conditions that ensure uniqueness, in certain classes, for the continuity equation; these involve further regularity of $\boldsymbol{b}$, expressed in terms of bounds on its divergence and its deformation (introduced below), density assumptions of $\mathcal{A}$ in $\mathbb{V}_p$ and the validity of inequalities which correspond, in the smooth setting, to integral bounds on the gradient of the kernel of P.

Definition 5.1 (L^p-Γ inequality). Let $p \in [1,\infty]$. We say that the L^p-Γ inequality holds if there exists $c_p > 0$ satisfying

$$\|\sqrt{\Gamma(\mathsf{P}_t f)}\|_p \le \frac{c_p}{\sqrt{t}} \|f\|_p \quad \text{for every } f \in L^2 \cap L^p(\mathfrak{m}),\ t \in (0,1).$$

Although the L^p-Γ inequality is expressed for $t \in (0,1)$, from its validity and L^p contractivity of P, we easily deduce that

$$\|\sqrt{\Gamma(\mathsf{P}_t f)}\|_p \le c_p (t\wedge 1)^{-1/2} \|f\|_p \quad \text{for every } f \in L^2 \cap L^p(\mathfrak{m}),\ t \in (0,\infty). \tag{5-1}$$

Notice also that, thanks to (2-12), the L^2-Γ inequality always holds, with $c_2 = 1/\sqrt{2}$. By Marcinkiewicz interpolation, we obtain that if the L^p-Γ inequality holds then, for every q between 2 and p, the L^q-Γ inequality holds as well.

Definition 5.2 (derivations with deformations of type (r,s)). Let $q \in [1,\infty]$, let $\boldsymbol{b}$ be a derivation in $L^q + L^\infty$ with $\operatorname{div} \boldsymbol{b} \in L^q(\mathfrak{m}) + L^\infty(\mathfrak{m})$, let $r,\ s \in [1,\infty]$ with $q^{-1} + r^{-1} + s^{-1} = 1$ and assume that $\mathcal{A}$ is dense both in $\mathbb{V}_r$ and in $\mathbb{V}_s$. We say that the deformation of $\boldsymbol{b}$ is of type (r,s) if there exists $c \ge 0$ satisfying

$$\left| \int D^{\mathrm{sym}}\boldsymbol{b}(f,g)\, d\mathfrak{m} \right| \le c \|\sqrt{\Gamma(f)}\|_r \|\sqrt{\Gamma(g)}\|_s, \tag{5-2}$$

for all $f \in \mathbb{V}_r$ with $\Delta f \in L^r \cap L^2(\mathfrak{m})$ and all $g \in \mathbb{V}_s$ with $\Delta g \in L^s \cap L^2(\mathfrak{m})$, where

$$\int D^{\mathrm{sym}}\boldsymbol{b}(f,g)\, d\mathfrak{m} := -\frac{1}{2}\int [df(\boldsymbol{b})\Delta g + dg(\boldsymbol{b})\Delta f - (\operatorname{div}\boldsymbol{b})\Gamma(f,g)]\, d\mathfrak{m}. \tag{5-3}$$

We let $\|D^{\mathrm{sym}}\boldsymbol{b}\|_{r,s}$ be the smallest constant c in (5-2).

The density assumption of $\mathcal{A}$ in $\mathbb{V}_r$ and $\mathbb{V}_s$ is necessary to extend the derivation $\boldsymbol{b}$ to all of $\mathbb{V}_r$ and $\mathbb{V}_s$, by Remark 3.7. Notice that the expression $\int D^{\mathrm{sym}}\boldsymbol{b}(f,g)\,d\mathfrak{m}$ is symmetric with respect to f, g, so the role of r and s above can be interchanged.

Remark 5.3 (deformation in the smooth case). Let $(X, \langle\cdot,\cdot\rangle)$ be a compact Riemannian manifold, let $\mathfrak{m}$ be its associated Riemannian volume and let $\Gamma(f,g)=\langle\nabla f,\nabla g\rangle$. Let $df(\boldsymbol{b})=\langle b,\nabla f\rangle$ for some smooth vector field b and let Db be the covariant derivative of b. The expression

$$\langle\nabla g,\nabla\langle b,\nabla f\rangle\rangle+\langle\nabla f,\nabla\langle b,\nabla g\rangle\rangle-\langle b,\nabla\langle\nabla f,\nabla g\rangle\rangle=\langle Db\nabla g,\nabla f\rangle+\langle Db\nabla f,\nabla g\rangle$$

gives exactly twice the symmetric part of the tensor Db, namely $2\langle D^{\mathrm{sym}}bf,g\rangle$. Integrating over X and then integrating by parts, we obtain twice the expression in (5-3), so that the derivation $\boldsymbol{b}$ associated to a smooth field b is of type (r,s) if $|D^{\mathrm{sym}}b|\in L^q(\mathfrak{m})$, where $q\in[1,\infty]$ satisfies $q^{-1}+r^{-1}+s^{-1}=1$.

Theorem 5.4 (uniqueness of solutions). *Let $1<s\le r<\infty$, $q\in(1,\infty]$ satisfy $q^{-1}+r^{-1}+s^{-1}=1$. Assume the existence of $(f_n)\subset\mathcal{A}$ that satisfy (4-3) and that, for $p\in\{r,s\}$, $\mathcal{A}$ is dense in $\mathbb{V}_p$ and the L^p-Γ inequality holds. Let $\boldsymbol{b}=(\boldsymbol{b}_t)_{t\in(0,T)}$ be a Borel family of derivations, with*

$$|\boldsymbol{b}|\in L^1_t(L^q_x+L^\infty_x),\ \operatorname{div}\boldsymbol{b}\in L^1_t(L^q_x+L^\infty_x),\ \operatorname{div}\boldsymbol{b}^-\in L^1_t(L^\infty_x)\ \text{and}\ \|D^{\mathrm{sym}}\boldsymbol{b}_t\|_{r,s}\in L^1(0,T).$$

Then, for every initial condition $\bar u\in L^r\cap L^2(\mathfrak{m})$, there exists at most one weak solution u in $(0,T)\times X$ to the continuity equation $du_t/dt+\operatorname{div}(u_t\boldsymbol{b}_t)=0$ in the class

$$\{u\in L^\infty_t(L^r_x\cap L^2_x):t\mapsto u_t\ \text{is weakly continuous in}\ [0,T)\ \text{and}\ u_0=\bar u\}.$$

The proof of this result is given in Section 5B and relies upon the strong convergence to 0 as $\alpha\downarrow 0$ of the commutator between divergence and action of the semigroup

$$\mathscr{C}^\alpha(u_t,\boldsymbol{b}_t):=\operatorname{div}((\mathsf{P}_\alpha u_t)\boldsymbol{b}_t)-\mathsf{P}_\alpha(\operatorname{div}(u_t\boldsymbol{b}_t)),\tag{5-4}$$

proved in Lemma 5.8 in the next section. We end this section with some comments on the density assumption on $\mathcal{A}$.

Remark 5.5 (on the density of $\mathcal{A}$ in $\mathbb{V}_p$). The assumption that $\mathcal{A}\subset\mathbb{V}_p$ is dense for $p\in\{r,s\}$ is fundamental to show that the semigroup approximation $t\mapsto\mathsf{P}_\alpha u_t$ is a solution to another continuity equation, (5-14) below. This follows by the extension of the derivation on $\mathbb{V}_p$ provided by Remark 3.7. One could argue that the invariance condition (2-20) is sufficient to define $\boldsymbol{b}(\mathsf{P}_\alpha f)$, whenever $f\in\mathcal{A}$; indeed Theorem 5.4 holds, assuming (2-20) in place of the density of $\mathcal{A}$ in $\mathbb{V}_p$, and the same proof goes through, with minor modifications (e.g., in Definition 5.2 above we require $f,g\in\mathcal{A}$). In view of Remark 2.4, one could also wonder whether (2-20) and the L^p-Γ inequality are sufficient to entail density in $\mathbb{V}_p$; the next lemma provides a partial affirmative answer (see Proposition 6.5 for an application of the lemma, assuming curvature lower bounds).

Lemma 5.6. *Let $p\in[2,\infty)$, assume that (2-20) and the L^p-Γ inequality hold and that*

$$\limsup_{t\downarrow 0}\|\sqrt{\Gamma(\mathsf{P}_t f)}\|_p\le\|\sqrt{\Gamma(f)}\|_p\quad\text{for every } f\in\mathbb{V}_p.\tag{5-5}$$

Then, $\mathcal{A}$ is dense in $\mathbb{V}_p$.

Proof. Let $f \in \mathbb{V}_p$. Notice first that, since $\mathsf{P}_t f$ converges to f in $\mathbb{V}$ as $t \downarrow 0$, Fatou's lemma gives

$$\|\sqrt{\Gamma(f)}\|_p \le \liminf_{t\downarrow 0} \|\sqrt{\Gamma(\mathsf{P}_t f)}\|_p$$

which combined with (5-5) gives convergence of $\Gamma(\mathsf{P}_t f)^{1/2}$ to $\Gamma(f)^{1/2}$ in $L^p(\mathfrak{m})$.

To prove density, we let $f \in \mathbb{V}_p$ and consider the functions $\Phi_n : \mathbb{R} \to \mathbb{R}$, with derivative ϕ_n, introduced in Lemma 2.2; since, by the chain rule, the $\Phi_n(f)$ converge to f in $\mathbb{V}_p$, it is sufficient to approximate each $\Phi_n(f)$ in $\mathbb{V}_p$ with elements of $\mathcal{A}$.

We first show that $\lim_{t\downarrow 0} \Phi_n(\mathsf{P}_t f) = \Phi_n(f)$ in $\mathbb{V}_p$. Since convergence in $\mathbb{V}$ and in $L^p(\mathfrak{m})$ is obvious, we prove $\Gamma(\Phi_n(\mathsf{P}_t f) - \Phi_n(f))^{1/2} \to 0$ in $L^p(\mathfrak{m})$. We let $h_1 = \mathsf{P}_t f$ and $h_2 = f$ in (2-19) to get

$$\Gamma(\Phi_n(\mathsf{P}_t f) - \Phi_n(f))^{1/2} \le |\phi_n(\mathsf{P}_t f) - \phi_n(f)|\Gamma(f)^{1/4}\Gamma(\mathsf{P}_t f)^{1/4} + \phi_n(f)\Gamma(\mathsf{P}_t f - f)^{1/2} \\ + 2|\phi_n(\mathsf{P}_t f) - \phi_n(f)|^{1/2}\Gamma(f)^{1/4}(\Gamma(f)^{1/4} + \Gamma(\mathsf{P}_t f)^{1/4}). \quad (5\text{-}6)$$

To handle the integral of the p-power of the term $\phi_n(f)\Gamma(\mathsf{P}_t f - f)^{1/2}$ we notice that, since $\Gamma(\mathsf{P}_t f)^{1/2}$ converges to $\Gamma(f)^{1/2}$ in $L^p(\mathfrak{m})$, they converge also in $L^p(\mathfrak{m}')$ with $\mathfrak{m}' = \phi_n(f)^p\mathfrak{m}$. Because $\mathfrak{m}'$ is finite we obtain that $\Gamma(\mathsf{P}_t f)^{p/2}$ are equiintegrable with respect to $\mathfrak{m}'$, and then the Lebesgue–Vitali convergence ensures convergence to 0. The first term can be handled similarly, by adding and subtracting $|\phi_n(\mathsf{P}_t f) - \phi_n(f)|^p\Gamma(f)^{p/4}\Gamma(f)^{p/4}$ and using dominated convergence, since $0 \le \phi_n \le 1$; the integral of the p-th power of the last term can be estimated with dominated convergence for $\int |\phi_n(\mathsf{P}_t f) - \phi_n(f)|^{p/2}\Gamma(f)^{p/2}\,d\mathfrak{m}$ and with the same argument as we used for the first term for $\int |\phi_n(\mathsf{P}_t f) - \phi_n(f)|^{p/2}\Gamma(f)^{p/4}\Gamma(\mathsf{P}_t f)^{p/4}\,d\mathfrak{m}$.

We proceed then to approximate $\Phi_n(\mathsf{P}_t f)$ in $\mathbb{V}_p$ by elements of $\mathcal{A}$, at fixed $n \ge 1$ and $t > 0$. Let $(f_k) \subset \mathcal{A}$ converge to f in $L^2 \cap L^p(\mathfrak{m})$. We show that $\Phi_n(\mathsf{P}_t f_k)$ converge to $\Phi_n(\mathsf{P}_t f)$ in $\mathbb{V}_p$. Notice that $\Phi_n(\mathsf{P}_t f_k)$ belong to $\mathcal{A}$, because of (2-20) and (2-17). Since convergence in $L^2 \cap L^p(\mathfrak{m})$ holds, convergence in $\mathbb{V}_p$ follows again by (2-19) with $h_1 = \mathsf{P}_t f_k$ and $h_2 = \mathsf{P}_t f$, because

$$\Gamma(\Phi_n(\mathsf{P}_t f_k) - \Phi_n(\mathsf{P}_t f))^{1/2} \le |\phi_n(\mathsf{P}_t f_k) - \phi_n(\mathsf{P}_t f)|\Gamma(\mathsf{P}_t f)^{1/4}\Gamma(\mathsf{P}_t f_k)^{1/4} + \phi_n(\mathsf{P}_t f)\Gamma(\mathsf{P}_t f_k - \mathsf{P}_t f)^{1/2} \\ + 2|\phi_n(\mathsf{P}_t f_k) - \phi_n(\mathsf{P}_t f)|^{1/2}\Gamma(\mathsf{P}_t f)^{1/4}(\Gamma(\mathsf{P}_t f_k)^{1/4} + \Gamma(\mathsf{P}_t f)^{1/4}).$$

By the L^2-Γ inequality and the L^p-Γ inequality, $\Gamma(\mathsf{P}_t f_k)^{1/2}$ converges to $\Gamma(\mathsf{P}_t f)^{1/2}$ in $L^2 \cap L^p(\mathfrak{m})$ as $k \to \infty$ and we can argue as we did in connection with (5-6) to obtain that $\Gamma(\Phi_n(\mathsf{P}_t f_k) - \Phi_n(\mathsf{P}_t f))^{1/2} \to 0$ in $L^p(\mathfrak{m})$. □

Actually, the proof above entails the following result. Let $p \in [1, \infty)$, assume that the L^p-Γ inequality holds, and let $\mathcal{A} \subset \mathbb{V}$ satisfy (2-17), (2-20), (5-5), and be dense in $L^2 \cap L^p(\mathfrak{m})$. Then $\mathcal{A}$ is dense in $\mathbb{V}_p$. Finally, notice that this gives another proof of Remark 2.4.

5A. ***The commutator lemma.*** We first collect some easy consequences of the L^r-Γ inequality, for some $r \in (1, \infty)$, which allow for an approximation of the derivation $\boldsymbol{b}$, via the action of P_α, as expressed in the next proposition. We denote by B^α the linear operator thus obtained, to stress the fact that it is not a derivation.

Proposition 5.7. *Let $r,\ s \in (1,\infty)$, $q \in (1,\infty]$ satisfy $q^{-1}+r^{-1}+s^{-1}=1$. Let $\boldsymbol{b}$ be a derivation in L^q+L^∞ and assume that $\mathcal{A} \subset \mathbb{V}_r$ is dense and that the L^r-Γ inequality holds.*

(1) *For every $\alpha \in (0,\infty)$, the map*

$$\mathcal{A} \ni f \mapsto d(\mathsf{P}_\alpha f)(\boldsymbol{b})$$

extends uniquely to $\mathsf{B}^\alpha \in \mathcal{L}(L^r \cap L^2(\mathfrak{m}), L^{s'}(\mathfrak{m})+L^2(\mathfrak{m}))$, with

$$\|\mathsf{B}^\alpha\| \le \max\{c_r, c_2\}(\alpha \wedge 1)^{-1/2}\|\boldsymbol{b}\|_{L^q+L^\infty}. \tag{5-7}$$

(2) *For all $f \in L^r \cap L^2(\mathfrak{m})$ the map $\alpha \mapsto \mathsf{B}^\alpha(f)$ is continuous from $(0,\infty)$ to $L^{s'}(\mathfrak{m})+L^2(\mathfrak{m})$ and, if $\Delta f \in L^r \cap L^2(\mathfrak{m})$, it is $C^1((0,\infty); L^{s'}(\mathfrak{m})+L^2(\mathfrak{m}))$, with*

$$\frac{d}{d\alpha}\mathsf{B}^\alpha(f) = \mathsf{B}^\alpha(\Delta f).$$

(3) *Assume that $u \in L^r \cap L^2(\mathfrak{m})$, $\operatorname{div}\boldsymbol{b} \in L^q(\mathfrak{m})+L^\infty(\mathfrak{m})$. Then,*

$$\operatorname{div}(\beta(\mathsf{P}_\alpha u)\boldsymbol{b}) = \beta(\mathsf{P}_\alpha u)\operatorname{div}\boldsymbol{b} + \beta'(\mathsf{P}_\alpha u)\mathsf{B}^\alpha(u) \in L^{s'}(\mathfrak{m})+L^2(\mathfrak{m}) \tag{5-8}$$

for all $\alpha>0$ and all $\beta \in C^1(\mathbb{R})\cap \operatorname{Lip}(\mathbb{R})$ with $\beta(0)=0$. In particular (5-8) with $\beta(z)=z$ gives

$$\operatorname{div}((\mathsf{P}_\alpha u)\boldsymbol{b}) = (\mathsf{P}_\alpha u)\operatorname{div}\boldsymbol{b} + \mathsf{B}^\alpha(u) \in L^{s'}(\mathfrak{m})+L^2(\mathfrak{m}). \tag{5-9}$$

(4) *Assume $u \in L^r \cap L^2(\mathfrak{m})$ and $\operatorname{div}\boldsymbol{b} \in L^q(\mathfrak{m})+L^\infty(\mathfrak{m})$. Then $\mathscr{C}^\alpha(\mathsf{P}_\delta u, \boldsymbol{b}) \in L^{s'}(\mathfrak{m})+L^2(\mathfrak{m})$ for every $\delta>0$ and*

$$\lim_{\alpha\downarrow 0}\|\mathscr{C}^\alpha(\mathsf{P}_\delta u, \boldsymbol{b})\|_{L^{s'}+L^2} = 0. \tag{5-10}$$

Proof. (1) By Remark 3.7, if $\boldsymbol{c}$ is a derivation in L^q, then we can extend it to a linear operator on $\mathbb{V}_r$, thus $d(\mathsf{P}_\alpha f)(\boldsymbol{c})$ is well-defined. Since the L^r-Γ inequality holds for every $f \in \mathcal{A}$, we get

$$\|d(\mathsf{P}_\alpha f)(\boldsymbol{c})\|_{s'} \le \|\boldsymbol{c}\|_q \|\sqrt{\Gamma(\mathsf{P}_\alpha f)}\|_r \le c_r(\alpha\wedge 1)^{-1/2}\|\boldsymbol{c}\|_q\|f\|_r.$$

Analogously, if $\boldsymbol{c}$ is a derivation in L^∞, $d(\mathsf{P}_\alpha f)(\boldsymbol{c})$ is well-defined and there holds

$$\|d(\mathsf{P}_\alpha f)(\boldsymbol{c})\|_2 \le \|\boldsymbol{c}\|_\infty \|\sqrt{\Gamma(\mathsf{P}_\alpha f)}\|_2 \le c_2(\alpha\wedge 1)^{-1/2}\|\boldsymbol{c}\|_\infty\|f\|_2.$$

This gives $\|\mathsf{B}^\alpha(f)\|_{L^{s'}+L^2} \le \max\{c_r,c_2\}(\alpha\wedge 1)^{-1/2}\|\boldsymbol{b}\|_{L^q+L^\infty}\|f\|_{L^r\cap L^2}$ on $\mathcal{A}$. By density of $\mathcal{A}$ in $L^r\cap L^2(\mathfrak{m})$, this provides the existence of B^s and the estimate on its norm.

(2) The semigroup law and the uniqueness of the extension give

$$\mathsf{B}^{\alpha+\sigma}(f) = \mathsf{B}^\alpha(\mathsf{P}_\sigma f) \quad \text{for every } f \in L^r\cap L^2(\mathfrak{m}),\ \alpha,\ \sigma \in (0,\infty).$$

Then, continuity follows easily, combining identity with (5-7) and the strong continuity of P_s:

$$\|\mathsf{B}^{\alpha+\sigma}(f)-\mathsf{B}^\alpha(f)\|_{L^{s'}+L^2} \le \max\{c_r,c_2\}(\alpha\wedge 1)^{-1/2}\|\boldsymbol{b}\|_{L^q+L^\infty}\|\mathsf{P}_\sigma f - f\|_{L^r\cap L^2}.$$

A similar argument shows differentiability if $\Delta f \in L^r\cap L^2(\mathfrak{m})$.

(3) We obtain (5-9) by (3-3). By the chain rule, the identity (5-8) follows.

(4) To prove that $\mathscr{C}^\alpha(\mathsf{P}_\delta u, \boldsymbol{b}) \in L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m})$, it is sufficient to apply (5-9) twice, to get

$$-\mathscr{C}^\alpha(\mathsf{P}_\delta u, \boldsymbol{b}) = \mathsf{P}_\alpha[(\mathsf{P}_\delta u)\operatorname{div}\boldsymbol{b}] + \mathsf{P}_\alpha(\mathsf{B}^\delta(u)) - (\mathsf{P}_{\alpha+\delta}u)\operatorname{div}\boldsymbol{b} - \mathsf{B}^{\alpha+\delta}(u) \in L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m}).$$

By strong continuity of $\alpha \mapsto \mathsf{P}_\alpha$ at $\alpha = 0$ and continuity of $\alpha \mapsto \mathsf{B}^\alpha(u)$ in $(0, \infty)$, the same expression shows that $\mathscr{C}^\alpha(\mathsf{P}_\delta u, \boldsymbol{b}) \to 0$ in $L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m})$ as $\alpha \downarrow 0$. □

We are now in a position to state and prove the following crucial lemma.

Lemma 5.8 (commutator estimate). *Let r, $s \in (1, \infty)$, $q \in (1, \infty]$ satisfy $q^{-1} + r^{-1} + s^{-1} = 1$. Let $\boldsymbol{b}$ be a derivation in $L^q + L^\infty$ with $\operatorname{div}\boldsymbol{b} \in L^q(\mathfrak{m}) + L^\infty(\mathfrak{m})$ and deformation of type (r, s). Assume that $\mathcal{A}$ is dense in $\mathbb{V}_p$ and that the L^p-Γ inequality holds for $p \in \{r, s\}$. Then*

$$\|\mathscr{C}^\alpha(u, \boldsymbol{b})\|_{L^{s'}+L^2} \le c\|u\|_{L^r\cap L^2}\big[\|D^{\mathrm{sym}}\boldsymbol{b}\|_{r,s} + \|\operatorname{div}\boldsymbol{b}\|_{L^q+L^\infty}\big] \tag{5-11}$$

for all $u \in L^r \cap L^2(\mathfrak{m})$ and all $\alpha \in (0, 1)$, where c is a constant depending only on the constants c_r and c_s in (5-1) and the constants c_r^Δ and c_s^Δ in (2-14). Moreover, $\mathscr{C}^\alpha(u, \boldsymbol{b}) \to 0$ in $L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m})$ as $\alpha \downarrow 0$.

Proof. For brevity, we introduce the notation $g^\alpha := \mathsf{P}_\alpha g$. By duality and density, inequality (5-11) is equivalent to the validity of

$$\int df^\alpha(\boldsymbol{b})u\,d\mathfrak{m} - \int df(\boldsymbol{b})u^\alpha\,d\mathfrak{m} \le c\big[\|D^{\mathrm{sym}}\boldsymbol{b}\|_{r,s} + \|\operatorname{div} b\|_{L^q+L^\infty}\big]\|u\|_{L^r\cap L^2}\|f\|_{L^s\cap L^2}, \tag{5-12}$$

for every f of the form $f = \mathsf{P}_\varepsilon\varphi$, for some $\varphi \in \mathcal{A}$, $\varepsilon > 0$. Since both sides are continuous in u with respect to $L^r \cap L^2(\mathfrak{m})$ convergence, it is also enough to establish it in a dense set; we therefore let $u = \mathsf{P}_\delta v$ for some $v \in \mathcal{A}$, $\delta > 0$.

We also notice that, by Proposition 5.7, we know that for such a choice of u, $\mathscr{C}^\alpha(u, \boldsymbol{b}) \to 0$ in $L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m})$ as $\alpha \downarrow 0$. Thus, once (5-11) is obtained, the same convergence as $\alpha \downarrow 0$ holds for every $u \in L^r \cap L^2(\mathfrak{m})$, from a standard density argument.

Then, we have to estimate

$$\int df^\alpha(\boldsymbol{b})u\,d\mathfrak{m} - \int df(\boldsymbol{b})u^\alpha\,d\mathfrak{m} = F(\alpha) - F(0),$$

where we let $F(\sigma) = \int df^\sigma(\boldsymbol{b})u^{\alpha-\sigma}\,d\mathfrak{m}$, for $\sigma \in [0, \alpha]$. Our assumption on $f = \mathsf{P}_\varepsilon\varphi$ entails, via Proposition 5.7, that the map $\sigma \mapsto df^\sigma(\boldsymbol{b}) = \mathsf{B}^\varepsilon(\varphi^\sigma)$ is $C^1([0, \alpha], L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m}))$, with

$$\frac{d}{d\sigma}[df^\sigma(\boldsymbol{b})] = \mathsf{B}^\varepsilon(\Delta\varphi^\sigma).$$

On the other hand, (2-14) entails that $\Delta u = \Delta\mathsf{P}_\delta v \in L^r \cap L^2(\mathfrak{m})$ and so $\sigma \mapsto u^\sigma$ in $C^1([0, \alpha], L^r \cap L^2(\mathfrak{m}))$. Thus, we are in a position to apply the Leibniz rule to obtain

$$F(\alpha) - F(0) = \int_0^\alpha \left(\int \mathsf{B}^\varepsilon(\Delta\varphi^\sigma)u^{\alpha-\sigma} - df^\sigma(\boldsymbol{b})\Delta u^{\alpha-\sigma}\,d\mathfrak{m}\right)d\sigma.$$

By applying (5-9) with $\Delta\varphi^\sigma$ in place of u, we integrate by parts to obtain

$$\int \mathsf{B}^\varepsilon(\Delta\varphi^\sigma)u^{\alpha-\sigma}\,d\mathfrak{m} = -\int \Delta f^\sigma\,du^{\alpha-\sigma}(\boldsymbol{b}) + (\operatorname{div}\boldsymbol{b})(\Delta f^\sigma)u^{\alpha-\sigma}\,d\mathfrak{m}.$$

We now estimate separately the terms

$$I := -\int \Delta f^\sigma \, du^{\alpha-\sigma}(\boldsymbol{b}) + df^\sigma(\boldsymbol{b})\Delta u^{\alpha-\sigma} \, d\mathfrak{m}, \quad II := -\int (\operatorname{div} \boldsymbol{b})(\Delta f^\sigma) u^{\alpha-\sigma} \, d\mathfrak{m},$$

at fixed $\sigma \in (0, \alpha)$ and then integrate over σ.

To handle the first term, we add and subtract $\int (\operatorname{div} \boldsymbol{b})\Gamma(f^\sigma, u^{\alpha-\sigma}) \, d\mathfrak{m}$, and thus recognize twice the deformation of $\boldsymbol{b}$, applied to f^σ and $u^{\alpha-\sigma}$, which are admissible functions in the sense of Definition 5.2, because of (5-1) and (2-14):

$$I = 2\int D^{\mathrm{sym}}\boldsymbol{b}(f^\sigma, u^{\alpha-\sigma}) \, d\mathfrak{m} - \int (\operatorname{div} \boldsymbol{b})\Gamma(f^\sigma, u^{\alpha-\sigma}) \, d\mathfrak{m}.$$

We use the assumption on $D^{\mathrm{sym}}\boldsymbol{b}$, $\operatorname{div} \boldsymbol{b}$ and L^r-Γ and L^s-Γ as well as L^2-Γ inequalities to obtain that

$$|I| \le \left[2\|D^{\mathrm{sym}}\boldsymbol{b}\|_{r,s} + \|\operatorname{div} \boldsymbol{b}\|_{L^q+L^\infty}\right] \frac{c}{\sqrt{\alpha(\alpha-\sigma)}} \|f\|_{L^s\cap L^2} \|u\|_{L^r\cap L^2},$$

with $c = c_r + c_s + c_2$. To handle integration over $\sigma \in (0, \alpha)$, we use

$$\int_0^\alpha \frac{d\sigma}{\sqrt{\sigma(\alpha-\sigma)}} = \pi.$$

To estimate the second term, we add and subtract

$$\int (\operatorname{div} \boldsymbol{b})(\Delta f^\sigma) u^\alpha \, d\mathfrak{m} = \frac{d}{d\sigma} \int (\operatorname{div} \boldsymbol{b}) f^\sigma u^\alpha \, d\mathfrak{m},$$

obtaining

$$II = \int (\operatorname{div} \boldsymbol{b})(\Delta f^\sigma)(u^\alpha - u^{\alpha-\sigma}) \, d\mathfrak{m} - \frac{d}{d\sigma} \int (\operatorname{div} \boldsymbol{b}) f^\sigma u^s \, d\mathfrak{m}.$$

We then estimate the first part of II by means of (2-14) and Corollary 2.1, to get

$$\frac{c^\Delta}{\sigma} \min\left\{2, c^\Delta \log\left(1 + \frac{\sigma}{\alpha-\sigma}\right)\right\} \|f\|_{L^s\cap L^2} \|u\|_{L^r\cap L^2},$$

with $c^\Delta = c_s^\Delta + c_r^\Delta + c_2^\Delta$.

The remaining part of II is estimated once we integrate over $\sigma \in (0, \alpha)$, as

$$-\int_0^\alpha \frac{d}{d\sigma} \int (\operatorname{div} \boldsymbol{b}) f^\sigma u^\alpha \, d\mathfrak{m} d\sigma = \int \operatorname{div} \boldsymbol{b}(f - f^\alpha) u^\alpha \le 2\|\operatorname{div} \boldsymbol{b}\|_{L^q+L^\infty} \|f\|_{L^s\cap L^2} \|u\|_{L^r\cap L^2}.$$

To conclude, we notice that

$$\int_0^\alpha \min\left\{\frac{2}{\sigma}, \frac{c^\Delta}{\sigma} \log\left(1 + \frac{\sigma}{\alpha-\sigma}\right)\right\} d\sigma \le \max\{2, c^\Delta\} \int_0^\alpha \min\left\{\frac{1}{\sigma}, \frac{1}{\alpha-\sigma}\right\} d\sigma = 2\log 2 \max\{2, c^\Delta\};$$

thus the proof of (5-12) is complete. □

Remark 5.9 (time-dependent commutator estimate). By integrating the commutator estimate with respect to time, we can achieve a similar estimate for time-dependent derivations $\boldsymbol{b}$ of type (r, s) satisfying

$$|\boldsymbol{b}| \in L^1_t(L^q_x + L^\infty_x), \quad \operatorname{div} \boldsymbol{b} \in L^1_t(L^q_x + L^\infty_x) \quad \text{and} \quad \|D^{\mathrm{sym}}\boldsymbol{b}_t\|_{r,s} \in L^1(I),$$

still assuming the validity of the L^p-Γ inequalities for $p \in \{r, s\}$:

$$\int_I \|\mathscr{C}^\alpha(u_t, \boldsymbol{b}_t)\|_{L^{s'}+L^2}\, dt \le c\|u\|_{L^\infty_t(L^r_x\cap L^2_x)} \int_I \|D^{\mathrm{sym}}\boldsymbol{b}_t\|_{r,s} + \|\mathrm{div}\, \boldsymbol{b}_t\|_{L^q+L^\infty}\, dt$$

for all $u \in L^\infty_t(L^r_x \cap L^2_x)$ and $\alpha \in (0, \infty)$. Moreover, dominated convergence gives

$$\lim_{\alpha\downarrow 0} \int_I \|\mathscr{C}^\alpha(u_t, \boldsymbol{b}_t)\|_{L^{s'}+L^2}\, dt = 0. \tag{5-13}$$

5B. *Proof of Theorem 5.4.* The proof of Theorem 5.4 is similar to that of Theorem 4.6, but it crucially exploits Lemma 5.8 to show that the error terms are negligible.

Let $(f_n) \subset \mathcal{A}$ be a sequence given by (4-3). Starting from $|z|^{1+r/s}$, we define β as in (4-19), namely

$$\beta(z) := \begin{cases} 1 + \dfrac{r+s}{s}(z-1) & \text{if } z > 1, \\ |z|^{1+r/s} & \text{if } |z| \le 1, \\ 1 - \dfrac{r+s}{s}(z+1) & \text{if } z < -1, \end{cases}$$

so that $\mathscr{L}_\beta \le (r/s)\beta$ and β has linear growth at infinity.

By the linearity of the equation we can assume $\bar{u} = 0$ and the goal is to prove that $u = 0$. We first extend the time interval $I = (0, T)$ to $(-1, T)$, setting $\boldsymbol{b}_t = 0$ for $t \in (-1, 0)$ and given the weakly continuous (in duality with $\mathcal{A}$) solution in $[0, T)$, with $u \in L^\infty(L^r_x \cap L^2_x)$, we extend it to a weakly continuous solution in $(-1, T)$, setting $u_t = 0$ for $t \in (-1, 0)$.

For every $\alpha > 0$, let $u^\alpha_t = \mathsf{P}_\alpha u_t \in L^\infty(L^r_x \cap L^2_x)$. As in the proof of Theorem 4.6, replacing φ in (4-16) by $\mathsf{P}_s\varphi$ (recall Remark 5.5), we can check that $t \mapsto u^\alpha_t$ is a weakly continuous solution to the continuity equation

$$\partial_t u^\alpha_t + \mathrm{div}(u^\alpha_t \boldsymbol{b}_t) = \mathscr{C}^\alpha(u_t, \boldsymbol{b}_t). \tag{5-14}$$

By (5-9) in Proposition 5.7 and (5-11) in Lemma 5.8, this equation entails that

$$\frac{d}{dt} u^\alpha_t = \mathscr{C}^\alpha(u_t, \boldsymbol{b}_t) - \mathrm{div}(u^\alpha_t \boldsymbol{b}_t) \in L^1_t(L^{s'}_x + L^2_x),$$

for a.e. $t \in (-1, T)$. Since $t \mapsto \int f_n \beta(u^\alpha_t)\, d\mathfrak{m}$ is lower semicontinuous (because β is convex and $t \mapsto u_t$ is weakly continuous) and since $|\beta'(z)| \sim |z|^{r/s}$ near the origin and $r \ge s$ imply that $\beta'(u^\alpha_t)$ is uniformly bounded in $L^s \cap L^2(\mathfrak{m})$, we can argue as in the proof of (4-23) to obtain that $t \mapsto \int f_n \beta(u^\alpha_t)\, d\mathfrak{m}$ is absolutely continuous and

$$\frac{d}{dt}\int f_n \beta(u^\alpha_t)\, d\mathfrak{m} = \int f_n \beta'(u^\alpha_t) \frac{d}{dt} u^\alpha_t\, d\mathfrak{m} = \int f_n \beta'(u^\alpha_t) \mathscr{C}^\alpha(u^\alpha_t, \boldsymbol{b}_t) - f_n \beta'(u^\alpha_t)\, \mathrm{div}(u^\alpha_t \boldsymbol{b}_t)\, d\mathfrak{m},$$

for a.e. $t \in I$. Now, setting $\Psi_n(t, \alpha) := \int f_n \beta(u^\alpha_t)\, d\mathfrak{m}$, identities (5-8) and (5-9) in Proposition 5.7 give

$$\frac{d}{dt}\Psi_n(t, \alpha) = \int f_n \beta'(u^\alpha_t) \mathscr{C}^\alpha(u^\alpha_t, \boldsymbol{b}_t)\, d\mathfrak{m} - \int f_n\, \mathrm{div}(\beta(u^\alpha_t)\boldsymbol{b}_t) + f_n \mathscr{L}_\beta(u^\alpha_t)\, \mathrm{div}\, \boldsymbol{b}_t\, d\mathfrak{m},$$

for a.e. $t \in I$. Hence, denoting $L_t := (r/s)\|\mathrm{div}\, \boldsymbol{b}_t^-\|_\infty \in L^1(-1,T)$, we can use the inequality $\mathscr{L}_\beta \le (r/s)\beta$ to get

$$\frac{d}{dt}\Psi_n(t,\alpha) \le L_t \Psi_n(t,\alpha) + \int f_n \beta'(u_t^\alpha)\mathscr{C}^\alpha(u_t^\alpha, \boldsymbol{b}_t)\, d\mathfrak{m} + \int \beta(u_t^\alpha) df_n(\boldsymbol{b}_t)\, d\mathfrak{m}.$$

Now we let $\alpha \downarrow 0$ and use the strong convergence of commutators in $L^{s'}(\mathfrak{m}) + L^2(\mathfrak{m})$ and the boundedness of $\beta'(u_t^\alpha)$ in $L^s \cap L^2(\mathfrak{m})$ to obtain that $t \mapsto \int_X f_n \beta(u_t)$ is absolutely continuous, and that

$$\frac{d}{dt}\int_X f_n\beta(u_t)\, d\mathfrak{m} \le L_t \int_X f_n\beta(u_t)\, d\mathfrak{m} + \int \beta(u_t) df_n(\boldsymbol{b}_t)\, d\mathfrak{m}.$$

By integration, taking into account that $\int_X f_n\beta(u_t)\, d\mathfrak{m} \equiv 0$ on $(-1,0)$, we get

$$\log\left(\frac{1}{\delta}\int_X f_n\beta(u_t)\, d\mathfrak{m} + 1\right) \le \|L\|_1 + \int_0^T\!\!\int \beta(u_s) df_n(\boldsymbol{b}_s)\, d\mathfrak{m}\, ds \quad \text{for all } t \in (-1,T) \text{ and all } \delta > 0.$$

Eventually we use (4-3) and the monotone convergence theorem to obtain

$$\log\left(\frac{1}{\delta}\int_X \beta(u_t)\, d\mathfrak{m} + 1\right) \le \|L\|_1 \quad \text{for all } t \in (-1,T) \text{ and } \delta > 0.$$

Letting $\delta \downarrow 0$ gives $u = 0$.

6. Curvature assumptions and their implications

In this section we add to the basic setting (2-1) a suitable curvature condition, and see the implication of this assumption on the structural conditions of density of $\mathcal{A}$ in the spaces $\mathbb{V}_p$ and the existence of $f_n \in \mathcal{A}$ in (4-3) made in the previous sections.

In the sequel K denotes a generic but fixed real number, and I_K denotes the real function

$$\mathrm{I}_K(t) := \int_0^t \mathrm{e}^{Kr}\, dr = \begin{cases} \dfrac{1}{K}(\mathrm{e}^{Kt} - 1) & \text{if } K \ne 0, \\ t & \text{if } K = 0. \end{cases}$$

Definition 6.1 (Bakry–Émery conditions). We say that $\mathsf{BE}_2(\mathsf{K},\infty)$ holds if

$$\Gamma(\mathsf{P}_t f) \le \mathrm{e}^{-2Kt}\, \mathsf{P}_t(\Gamma(f)) \quad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f \in \mathbb{V},\ t \ge 0. \tag{6-1}$$

We say that $\mathsf{BE}_1(\mathsf{K},\infty)$ holds if

$$\sqrt{\Gamma(\mathsf{P}_t f)} \le \mathrm{e}^{-Kt}\, \mathsf{P}_t\big(\sqrt{\Gamma(f)}\big) \quad \mathfrak{m}\text{-a.e. in } X, \text{ for every } f \in \mathbb{V},\ t \ge 0. \tag{6-2}$$

We stated both the curvature conditions for the sake of completeness only, but we remark that $\mathsf{BE}_2(\mathsf{K},\infty)$ is sufficient for many of the results we are interested in this section. Obviously, $\mathsf{BE}_1(\mathsf{K},\infty)$ implies $\mathsf{BE}_2(\mathsf{K},\infty)$; the converse, first proved by Bakry [1985], has been recently extended to a nonsmooth setting by Savaré [2014, Corollary 3.5] under the assumption that $\mathscr{E}$ is *quasiregular*. The quasiregularity property has many equivalent characterizations; a transparent one is, for instance, in terms of the existence of a sequence of compact sets $F_k \subset X$ such that

$$\bigcup_k \{f \in \mathbb{V} : f = 0 \ \mathfrak{m}\text{-a.e. in } X \setminus F_k\}$$

is dense in $\mathbb{V}$.

The validity of the following inequality is actually equivalent to $\mathsf{BE}_2(\mathsf{K}, \infty)$; see, for instance, [Ambrosio et al. 2014a, Corollary 2.3] for a proof.

Proposition 6.2 (reverse Poincaré inequalities). *If* $\mathsf{BE}_2(\mathsf{K}, \infty)$ *holds, then*

$$2\mathrm{I}_{2K}(t)\Gamma(\mathsf{P}_t f) \le \mathsf{P}_t f^2 - (\mathsf{P}_t f)^2 \quad \mathfrak{m}\text{-a.e. in } X, \tag{6-3}$$

for all $t > 0$, $f \in L^2(\mathfrak{m})$.

Corollary 6.3 (L^p-Γ inequalities). *If* $\mathsf{BE}_2(\mathsf{K}, \infty)$ *holds, then* L^p-Γ *inequalities hold for* $p \in [2, \infty]$.

Proof. The validity of L^p-Γ inequalities for $p \in [2, \infty]$ is obtained by integrating (6-3),

$$(2\mathrm{I}_{2K}(t))^{p/2} \int \Gamma(\mathsf{P}_t f)^{p/2}\, d\mathfrak{m} \le \int (\mathsf{P}_t f^2)^{p/2}\, d\mathfrak{m} \le \int f^p\, d\mathfrak{m}$$

and using $2\mathrm{I}_{2K}(t)^{-1} = O(t^{-1})$ as $t \downarrow 0$. □

Another consequence of $\mathsf{BE}_2(\mathsf{K}, \infty)$ is the following higher integrability of $\Gamma(f)$, recently proved in [Ambrosio et al. 2013, Theorem 3.1] assuming higher integrability of f and Δf.

Theorem 6.4 (gradient interpolation). *Assume that* $\mathsf{BE}_2(\mathsf{K}, \infty)$ *holds and let* $\lambda \ge K^-$, $p \in \{2, \infty\}$. *For all* $f \in L^2 \cap L^\infty(\mathfrak{m})$ *with* $\Delta f \in L^p(\mathfrak{m})$, *it holds that* $\Gamma(f) \in L^p(\mathfrak{m})$ *with*

$$\|\Gamma(f)\|_p \le c\|f\|_\infty \|\Delta f + \lambda f\|_p \tag{6-4}$$

for a universal (*independent of* f, λ, K, X, $\mathfrak{m}$) *constant* c.

Finally, we will need two more consequences of the $\mathsf{BE}_2(\mathsf{K}, \infty)$ condition, proved under the quasiregularity assumption in [Savaré 2014]. The first one, first proved in [Savaré 2014, Lemma 3.2] and then slightly improved in [Ambrosio et al. 2013, Theorem 5.5], is the implication

$$f \in \mathbb{V},\ \Delta f \in L^4(\mathfrak{m}) \quad \Longrightarrow \quad \Gamma(f) \in \mathbb{V}. \tag{6-5}$$

In particular, this implication provides L^4-integrability of $\sqrt{\Gamma(f)}$, consistently with the integrability of the Laplacian. Secondly, and particularly useful for the quantitative estimate, we have

$$\Gamma(\Gamma(f)) \le 4\gamma_{2,K}[f]\,\Gamma(f) \quad \mathfrak{m}\text{-a.e. in } X, \text{ whenever } f \in \mathbb{V},\ \Delta f \in L^4(\mathfrak{m}), \tag{6-6}$$

first proved in [Savaré 2014, Theorem 3.4] and then slightly improved in [Ambrosio et al. 2013, Corollary 5.7]. The function $\gamma_{2,K}[f]$ in (6-6) is nonnegative, it satisfies the L^1 estimate

$$\int \gamma_{2,K}[f]\, d\mathfrak{m} \le \int_X ((\Delta f)^2 - K\Gamma(f))\, d\mathfrak{m} \tag{6-7}$$

and it can be represented as the density w.r.t. $\mathfrak{m}$ of the nonnegative (and possibly singular w.r.t. $\mathfrak{m}$) measure defined by

$$\mathbb{V} \ni \varphi \mapsto \int_X -\tfrac{1}{2}\Gamma(\Gamma(f), \varphi) + \Delta f\, \Gamma(f, \varphi) + ((\Delta f)^2 - K\Gamma(f))\varphi \, d\mathfrak{m}. \tag{6-8}$$

The nonnegativity of this measure is one of the equivalent formulations of $\mathsf{BE}_2(\mathsf{K}, \infty)$; see [Savaré 2014, §3] for a more detailed discussion.

6A. *Choice of the algebra* $\mathcal{A}$. We first prove that the following "minimal" choice for the algebra $\mathcal{A}$ provides (2-16), (2-17) and optimal density conditions.

Proposition 6.5. *Under assumption* $\mathsf{BE}_2(\mathsf{K}, \infty)$, *the algebra*

$$\mathcal{A}_1 := \Big\{ f \in \bigcap_{1 \le p \le \infty} L^p(\mathfrak{m}) : f \in \mathbb{V},\ \sqrt{\Gamma(f)} \in \bigcap_{1 \le p \le \infty} L^p(\mathfrak{m}) \Big\} \tag{6-9}$$

satisfies (2-16), (2-17) *and is dense in every space* $\mathbb{V}_p$, *for* $p \in [1, \infty)$.

Proof. Since (2-17) is obviously satisfied by the chain rule, we need only to show density of $\mathcal{A}_1$. First, we consider the algebra $\mathscr{A} = \mathbb{V}_2 \cap \mathbb{V}_\infty$, which satisfies the invariance condition (2-20) because of (6-1). Moreover, for $p \in [2, \infty)$, the validity of the L^p-Γ inequality entails that $\mathscr{A}$ is dense in $L^2 \cap L^p$, and taking the $L^{p/2}$ norm in (6-1) gives that (5-5) holds. By Lemma 5.6 (in particular, the remark below its proof) we conclude that $\mathscr{A}$ is dense in $\mathbb{V}_p$, for every $p \in [2, \infty)$.

To establish density of $\mathcal{A}_1$ in $\mathbb{V}_p$ for $p \in [1, \infty)$ it is sufficient to notice that the "refining" procedure in Lemma 2.2 applied to $\mathscr{A}$ preserves all the densities in $\mathbb{V}_p$ for $p \in [2, \infty)$, and provides an algebra contained in $\mathcal{A}_1$. □

Retaining the density condition and the algebra property, one can also consider classes smaller than $\mathcal{A}_1$, including, for instance, bounds in $L^p(\mathfrak{m})$ for the Laplacian.

6B. *Conservation of mass.* In this section we prove that the curvature condition, together with the conservativity condition $\mathsf{P}^\infty_t 1 = 1$ for all $t > 0$ (recall that $\mathsf{P}^\infty_t : L^\infty(\mathfrak{m}) \to L^\infty(\mathfrak{m})$ is the dual semigroup in (2-15)), imply the existence of a sequence $(f_n) \subset \mathcal{A}_1$ as in (4-3). Notice that the conservativity is loosely related to a mass conservation property, for the continuity equation with derivation induced by the logarithmic derivative of the density; therefore, even though sufficient conditions adapted to the prescribed derivation $\boldsymbol{b}$ could be considered as well, it is natural to consider the conservativity of P in connection with (4-3).

Proposition 6.6. *If* $\mathsf{BE}_2(\mathsf{K}, \infty)$ *holds and* P *is conservative, then there exist* $(f_n) \subset \mathcal{A}_1$ *satisfying* (4-3).

Proof. Let $(g_n) \subset L^1 \cap L^\infty(\mathfrak{m})$ be a nondecreasing sequence of functions (whose existence is ensured by the σ-finiteness assumption on $\mathfrak{m}$) with

$$0 \le g_n \le 1 \quad \text{for every } n \ge 1, \qquad \text{and} \qquad \lim_{n\to\infty} g_n = 1 \quad \mathfrak{m}\text{-a.e. in } X.$$

These conditions imply in particular that $g_n \to 1$ weakly* in $L^\infty(\mathfrak{m})$.

Let $h_n = \int_0^1 \mathsf{P}_s g_n ds = \int_0^1 \mathsf{P}^\infty_s g_n ds$ and define $f_n := \mathsf{P}_1 h_n = \mathsf{P}^\infty_1 h_n$. By linearity and continuity of P^∞ we obtain that $f_n \to \mathsf{P}^\infty_1 1 = 1$ weakly* in $L^\infty(\mathfrak{m})$. In addition, expanding the squares, it is easily seen that

$$\lim_{n\to\infty} \int (1-f_n)^2 v \, d\mathfrak{m} = 0 \quad \forall v \in L^1(\mathfrak{m}).$$

Hence, by a diagonal argument we can assume (possibly extracting a subsequence) that $f_n \to 1$ $\mathfrak{m}$-a.e. in X.

Since $h_n \le 1$, the reverse Poincaré inequality (6-3) entails

$$\Gamma(f_n) \le \frac{\mathsf{P}_1 h_n^2 - (f_n)^2}{2\mathrm{I}_{2K}(1)} \le \frac{1-(f_n)^2}{2\mathrm{I}_{2K}(1)} \quad \mathfrak{m}\text{-a.e. in } X.$$

Taking the square roots of both sides and using the a.e. convergence of f_n we obtain, thanks to dominated convergence, that $\sqrt{\Gamma(f_n)}$ weakly* converge to 0 in $L^\infty(\mathfrak{m})$.

Finally, we discuss the regularity of f_n. Since

$$\Delta f_n = \int_1^2 \Delta \mathsf{P}_s g_n \, ds = \mathsf{P}_2 g_n - \mathsf{P}_1 g_n \in L^\infty(\mathfrak{m}),$$

we can use Theorem 6.4 to obtain $\sqrt{\Gamma(f_n)} \in L^\infty(\mathfrak{m})$. In order to obtain integrability of the gradient for powers between 1 and 2 we can replace f_n by $k_n := \Phi_1(f_n)/\Phi_1(1)$, with $\Phi_1 : \mathbb{R} \to \mathbb{R}$ as introduced in Lemma 2.2. □

6C. *Derivations associated to gradients and their deformation.* In this section, we study in more detail the class of "gradient" derivations $\boldsymbol{b}_V$ in (3-2). More generally, we analyze the regularity of the derivation $f \mapsto \omega\Gamma(f, V)$ associated to sufficiently regular V and ω in $\mathbb{V}$.

For $p \in (1, \infty]$, let us denote

$$D_{L^p}(\Delta) := \{f \in \mathbb{V} \cap L^p(\mathfrak{m}) : \Delta f \in L^p(\mathfrak{m})\}. \tag{6-10}$$

Thanks to the implication (6-5), $D_{L^4}(\Delta) \subset \mathbb{V}_4$ and the Hessian

$$(f, g) \mapsto H[V](f, g) := \tfrac{1}{2}\big[\Gamma(f, \Gamma(V, g)) + \Gamma(g, \Gamma(V, f)) - \Gamma(V, \Gamma(f, g))\big] \in L^1(\mathfrak{m}) \tag{6-11}$$

is well-defined on $D_{L^4}(\Delta) \times D_{L^4}(\Delta)$. Notice that the expression $H[V](f, g)$ is symmetric in (f, g), that $(V, f, g) \mapsto H[V](f, g)$ is multilinear, and that

$$H[V](f, g_1 g_2) = H[V](f, g_1) g_2 + g_1 H[V](f, g_2).$$

By [Savaré 2014, Theorem 3.4], we have the estimate

$$|H[V](f, g)| \le \sqrt{\gamma_{2,K}[V]}\sqrt{\Gamma(f)}\sqrt{\Gamma(g)} \quad \mathfrak{m}\text{-a.e. in } X, \tag{6-12}$$

for every $f, g \in D_{L^4}(\Delta)$.

Theorem 6.7. *If* $\mathsf{BE}_2(\mathsf{K}, \infty)$ *holds and* $\mathcal{E}$ *is quasiregular, then for all* $V \in D(\Delta)$, $\omega \in \mathbb{V} \cap L^\infty(\mathfrak{m})$ *with* $\sqrt{\Gamma(\omega)} \in L^\infty(\mathfrak{m})$, *and* $c \in \mathbb{R}$, *the derivation* $\boldsymbol{b} = (\omega + c)\boldsymbol{b}_V$ *has deformation of type* $(4, 4)$ *according to*

Definition 5.2 with $q = 2$, and it satisfies

$$\|D^{\mathrm{sym}}\boldsymbol{b}\|_{4,4} \le \|\omega + c\|_\infty \|(\Delta V)^2 - K\Gamma(V)\|_1 + \|\sqrt{\Gamma(\omega)}\|_\infty \|\sqrt{\Gamma(V)}\|_2. \tag{6-13}$$

Proof. Assume first that $V \in D_{L^4}(\Delta)$. Let $f, g \in D_{L^4}(\Delta)$. After integrating by parts the Laplacians of f and g, the very definition of $D^{\mathrm{sym}}\boldsymbol{b}$ gives

$$\int D^{\mathrm{sym}}\boldsymbol{b}(f, g)\, d\mathfrak{m} = \int (\omega + c) H[V](f, g) + \tfrac{1}{2}[\Gamma(\omega, f)\Gamma(V, g) + \Gamma(\omega, g)\Gamma(V, f)]\, d\mathfrak{m}. \tag{6-14}$$

By Hölder inequality, we can use (6-12) to estimate $\left|\int D^{\mathrm{sym}}\boldsymbol{b}(f, g)\, d\mathfrak{m}\right|$ from above with

$$\left[\|\omega\|_\infty \|\sqrt{\gamma_{2,K}[V]}\|_2 + \|\sqrt{\Gamma(\omega)}\|_\infty \|\sqrt{\Gamma(V)}\|_2\right] \|\sqrt{\Gamma(f)}\|_4 \|\sqrt{\Gamma(g)}\|_4.$$

Thus, by definition of $\|D^{\mathrm{sym}}\boldsymbol{b}\|_{4,4}$, (6-13) follows, taking also (6-7) into account. To pass to the general case $V \in D(\Delta)$, it is sufficient to approximate V with $V_n \in D_{L^4}(\Delta)$ in such a way that $V_n \to V$ in $\mathbb{V}$ and $\Delta V_n \to \Delta V$ in $L^2(\mathfrak{m})$ and notice that $\int D^{\mathrm{sym}}\boldsymbol{b}_n(f, g)\, d\mathfrak{m}$ converge to $\int D^{\mathrm{sym}}\boldsymbol{b}(f, g)\, d\mathfrak{m}$ directly from (5-3). The existence of such an approximating sequence is obtained arguing as in [Ambrosio et al. 2013, Lemma 4.2], i.e., given $f \in D(\Delta)$, we let $h = f - \Delta f \in L^2(\mathfrak{m})$,

$$h_n := \max\{\min\{h, n\}, -n\} \in L^2 \cap L^\infty(\mathfrak{m})$$

and define f_n as the unique (weak) solution to $f_n - \Delta f_n = h_n$. The maximum principle for Δ (or equivalently the fact that the resolvent operator $R_1 = (I - \Delta)^{-1}$ is Markov) gives $f_n \in L^2 \cap L^\infty(\mathfrak{m})$, thus $\Delta f_n \in L^2 \cap L^\infty(\mathfrak{m})$ and by L^2-continuity of R_1, as $n \to \infty$, both h_n and f_n converge, respectively towards h and f. By difference, also Δf_n converge towards Δf in $L^2(\mathfrak{m})$ and this also easily gives convergence of f_n to f in $\mathbb{V}$. □

We end this section with a technical result that will be useful when dealing with probability measures on vector spaces, in particular in Section 9E.

Proposition 6.8. *Assume that $\mathfrak{m}(X) = 1$, $\mathsf{BE}_2(\mathsf{K}, \infty)$ holds, and $\mathcal{E}$ is quasiregular. Let $(V_i)_{i \ge 1} \subset D_{L^4}(\Delta)$ generate an algebra dense in $\mathbb{V}$ and satisfy $\Gamma(V_i, V_j) = \delta_{i,j}$ $\mathfrak{m}$-a.e. in X. Then,*

(a) $\Gamma(f) = \sum_{i \ge 1} \Gamma(V_i, f)^2$ *$\mathfrak{m}$-a.e. in X, for every $f \in \mathbb{V}$;*

(b) $H[V_i] = 0$, *for every $i \ge 1$.*

Moreover, for every $q \in [1, \infty]$ and $b = (b^i) \in L^q(X; \ell^2))$, the associated derivation $\boldsymbol{b}$ given by

$$f \mapsto df(\boldsymbol{b}) = \sum_i b^i \Gamma(V_i, f)$$

satisfies $|\boldsymbol{b}|^2 \le \sum_i |b^i|^2$ and therefore belongs to L^q. In addition, if $r, s \in [4, \infty)$ satisfy $q^{-1} + r^{-1} + s^{-1} = 1$, $\operatorname{div} \boldsymbol{b} \in L^q(\mathfrak{m})$, and $b_i \in \mathbb{V}$ for every $i \ge 1$, then

$$\|D^{\mathrm{sym}}\boldsymbol{b}\|_{r,s} \le \frac{1}{2} \left\| \left(\sum_{i,j} |\Gamma(V_j, b_i) + \Gamma(V_i, b_j)|^2 \right)^{\frac{1}{2}} \right\|_q. \tag{6-15}$$

Proof. When $f = \psi(V_1, \dots, V_n)$ belongs to the algebra generated by (V_i), the first identity is immediate from $\Gamma(V_i, V_j) = \delta_{i,j}$. The general case of (a) follows by density.

From the definition (6-11) of Hessian, $H[V_i](V_j, V_k) = 0$ for every $i, j, k \geq 1$. For fixed $i, j \geq 1$, the derivation $g \mapsto H[V_i](V_j, g)$ belongs to $L^2(\mathfrak{m})$ in virtue of (6-12); thus it can be extended by density of $\mathcal{A}$ to all of $\mathbb{V}$. By the chain rule, the extended derivation is identically zero on the algebra generated by (V_i); thus by density it is the null derivation. In particular, for $g \in \mathcal{A}$, $H[V_i](V_j, g) = 0$, for every $j \geq 1$. Keeping $g \in \mathcal{A}$ fixed, we argue similarly and obtain that $H[V_i](f, g) = 0$ $\mathfrak{m}$-a.e. in X, for every $f, g \in \mathcal{A}$, thus proving (b).

If only a finite number of b^i's is different from 0, and they belong to $\mathbb{V}$, the claimed estimate (6-15) follows immediately by linearity, (6-14) and (b) above. The general case follows by "cylindrical" approximation, where the assumption $r, s \geq 4$ plays a role. Indeed, given $f \in \mathbb{V}_r \cap D_{L^r}(\Delta)$ and $g \in \mathbb{V}_s \cap D_{L^s}(\Delta)$ we have $f, g \in D_{L^4}(\Delta)$, thus $\Gamma(f, g) \in \mathbb{V}$ and we can integrate by parts the last term in (5-3), obtaining

$$\int D^{\mathrm{sym}}\boldsymbol{b}(f, g)\, d\mathfrak{m} = -\frac{1}{2}\int df(\boldsymbol{b})\Delta g + dg(\boldsymbol{b})\Delta f + d(\Gamma(f, g))(\boldsymbol{b})\, d\mathfrak{m}. \tag{6-16}$$

Let $N \geq 1$ and let $\boldsymbol{b}_N$ be the derivation associated to the sequence $(b^1, \dots, b^N, 0, 0, \dots)$. Given $h \in \mathbb{V}$,

$$|d(h)\boldsymbol{b}_N - d(h)\boldsymbol{b}| \leq \Gamma(h)^{\frac{1}{2}}\biggl(\sum_{i>N}|b^i|^2\biggr)^{\frac{1}{2}} \qquad \mathfrak{m}\text{-a.e. in } X.$$

By this estimate with $h = f$, $h = g$ and $h = \Gamma(f, g)$, Hölder's inequality and dominated convergence we conclude that the sequence $\int D^{\mathrm{sym}}\boldsymbol{b}_N(f, g)\, d\mathfrak{m}$ converges towards $\int D^{\mathrm{sym}}\boldsymbol{b}(f, g)\, d\mathfrak{m}$ as $N \to \infty$, entailing (6-15). □

Notice that the assumption $r, s \in [4, \infty)$ is used only to obtain $\Gamma(f, g) \in \mathbb{V}$ and thus (6-16). The same argument indeed shows that, for $r, s \in [1, \infty)$ and $q \in (1, \infty]$ with $q^{-1} + r^{-1} + s^{-1} = 1$, if $\mathcal{A}$ is dense in the space $\mathbb{V}_p \cap D_{L^p}(\Delta)$, endowed with the norm $\|f\| = \|f\|_{\mathbb{V}_p} + \|\Delta f\|_{L^2 \cap L^p}$, for $p \in \{r, s\}$ and it satisfies $\Gamma(f, g) \in \mathcal{A}$ for $f, g \in \mathcal{A}$, then the last statement in Proposition 6.8 holds, regardless of the condition $r, s \in [4, \infty)$.

7. The superposition principle in $\mathbb{R}^\infty$ and in metric measure spaces

In this section we write $\mathbb{R}^\infty$ for $\mathbb{R}^\mathbb{N}$ endowed with the product topology and we shall denote by $\pi^n := (p_1, \dots, p_n) : \mathbb{R}^\infty \to \mathbb{R}^n$ the canonical projections from $\mathbb{R}^\infty$ to $\mathbb{R}^n$. On the space $\mathbb{R}^\infty$ we consider the complete and separable distance

$$d_\infty(x, y) := \sum_{n=1}^\infty 2^{-n}\min\{1, |p_n(x) - p_n(y)|\}.$$

Accordingly, we consider the space $C([0, T]; \mathbb{R}^\infty)$ endowed with the distance

$$\delta(\eta, \tilde{\eta}) := \sum_{n=1}^\infty 2^{-n}\max_{t\in[0,T]}\min\{1, |p_n(\eta(t)) - p_n(\tilde{\eta}(t))|\},$$

which makes $C([0,T];\mathbb{R}^\infty)$ complete and separable as well. We shall also consider the subspace $\mathrm{AC}_w([0,T];\mathbb{R}^\infty)$ of $C([0,T];\mathbb{R}^\infty)$ consisting of all η such that $p_i\circ\eta\in\mathrm{AC}([0,T])$ for all $i\geq 1$. Notice that for this class of curves the derivative $\eta'\in\mathbb{R}^\infty$ can still be defined a.e. in $(0,T)$, arguing componentwise. We use the notation AC_w to avoid the confusion with the space of absolutely continuous maps from $[0,T]$ to $(\mathbb{R}^\infty, d_\infty)$.

It is immediate to check that for any choice of convex superlinear and lower semicontinuous functions $\Psi_n:[0,\infty)\to[0,\infty]$, and for lower semicontinuous functions $\Phi_n:[0,\infty)\to[0,\infty]$ with $\Phi_n(v)\to\infty$ as $v\to\infty$, the functional $\mathscr{A}:C([0,T];\mathbb{R}^\infty)\to[0,\infty]$ defined by

$$\mathscr{A}(\eta):=\begin{cases}\sum_{n=1}^{\infty}\left[\Phi_n(p_n\circ\eta(0))+\int_0^T\Psi_n(|(p_n\circ\eta)'|)\,dt\right] & \text{if } \eta\in\mathrm{AC}_w([0,T];\mathbb{R}^\infty),\\ \infty & \text{if } \eta\in C([0,T];\mathbb{R}^\infty)\setminus\mathrm{AC}_w([0,T];\mathbb{R}^\infty),\end{cases}$$

is coercive in $C([0,T];\mathbb{R}^\infty)$, that is to say all sublevels $\{\mathscr{A}\leq M\}$ are compact in $C([0,T];\mathbb{R}^\infty)$.

We call a smooth cylindrical function any $f:\mathbb{R}^\infty\to\mathbb{R}$ representable in the form

$$f(x)=\psi(\pi_n(x))=\psi(p_1(x),\dots,p_n(x)),\qquad x\in\mathbb{R}^\infty,$$

with $\psi:\mathbb{R}^n\to\mathbb{R}$ bounded and continuously differentiable with bounded derivative. When we want to emphasize n, we say that f is n-cylindrical. Given ψ smooth cylindrical, we define $\nabla f:\mathbb{R}^\infty\to c_0$ (where c_0 is the space of sequences (x_n) null for n large enough) by

$$\nabla f(x):=\left(\frac{\partial\psi}{\partial z_1}(\pi_n(x)),\dots,\frac{\partial\psi}{\partial z_n}(\pi_n(x)),0,0,\dots\right). \tag{7-1}$$

We fix a Borel vector field $\boldsymbol{c}:(0,T)\times\mathbb{R}^\infty\to\mathbb{R}^\infty$ and a weakly continuous (in duality with smooth cylindrical functions) family of Borel probability measures $\{\nu_t\}_{t\in(0,T)}$ in $\mathbb{R}^\infty$ satisfying

$$\int_0^T\int|p_i(\boldsymbol{c}_t)|d\nu_t dt<\infty\quad\forall i\geq 1, \tag{7-2}$$

and, in the sense of distributions,

$$\frac{d}{dt}\int f\,d\nu_t=\int(\boldsymbol{c}_t,\nabla f)d\nu_t\quad\text{in } (0,T),\text{ for all } f \text{ smooth cylindrical.} \tag{7-3}$$

Theorem 7.1 (superposition principle in $\mathbb{R}^\infty$). *Under assumptions (7-2) and (7-3), there exists a Borel probability measure $\boldsymbol{\lambda}$ in $C([0,T];\mathbb{R}^\infty)$ satisfying $(e_t)_\#\boldsymbol{\lambda}=\nu_t$ for all $t\in(0,T)$, concentrated on $\gamma\in\mathrm{AC}_w([0,T];\mathbb{R}^\infty)$ which are solutions to the ODE $\dot\gamma=\boldsymbol{c}_t(\gamma)$ a.e. in $(0,T)$.*

Proof. The statement is known in finite-dimensional spaces; for example, see [Ambrosio et al. 2005, Theorem 8.2.1] for the case when $\int\int|\boldsymbol{c}_t|^r d\nu_t dt<\infty$ for some $r>1$, and [Ambrosio and Crippa 2008, Theorem 12] for the case $r=1$. For $i\geq 1$ we choose convex, superlinear, lower semicontinuous functions $\Psi_i:[0,\infty)\to[0,\infty]$ with

$$\int_0^T\int\Psi_i(|p_i(\boldsymbol{c}_t)|)\,d\nu_t dt\leq 2^{-i} \tag{7-4}$$

and coercive $\Phi_i : [0, \infty) \to [0, \infty)$ satisfying

$$\int \Phi_i(p_i(x))\, d\nu_0(x) \le 2^{-i}, \tag{7-5}$$

and define $\mathscr{A}$ accordingly.

Defining $\nu_t^n := (\pi^n)_\# \nu_t$ and $\boldsymbol{c}_{t,i}^n$, $1 \le i \le n$, as the density of $(\pi^n)_\#(p_i(\boldsymbol{c}_t)\nu_t)$ w.r.t. to ν_t^n, it is immediate to check with Jensen's inequality that

$$\int_0^T\!\!\int \Psi_i(|\boldsymbol{c}_{t,i}^n|)\, d\nu_t^n dt \le \int_0^T\!\!\int \Psi_i(|p_i(\boldsymbol{c}_t)|)\, d\nu_t dt, \quad i \ge 1, \tag{7-6}$$

and that ν_t^n solves the continuity equation in $\mathbb{R}^n$ relative to the vector field $\boldsymbol{c}^n = (\boldsymbol{c}_i^n, \dots, \boldsymbol{c}_n^n)$. Therefore the finite-dimensional statement provides probability measures $\boldsymbol{\lambda}_n$ in $C([0, T]; \mathbb{R}^n)$, concentrated on absolutely continuous a.e. solutions to the ODE $\dot\gamma = \boldsymbol{c}_t^n(\gamma)$ and satisfying $(e_t)_\# \boldsymbol{\lambda}_n = \nu_t^n$ for all $t \in [0, T]$.

In order to pass to the limit as $n \to \infty$, it is convenient to view $\boldsymbol{\lambda}_n$ as probability measures in $C([0, T]; \mathbb{R}^\infty)$ concentrated on curves γ such that $p_i(\gamma)$ is null for $i > n$, and ν^n as probability measures in $\mathbb{R}^\infty$ concentrated on $\{x \in \mathbb{R}^\infty : p_i(x) = 0,\ \forall i > n\} \subset c_0$. Accordingly, if we set $\boldsymbol{c}_{t,i}^n \equiv 0$ for $i > n$, we retain the property that $\boldsymbol{\lambda}_n$ is concentrated on absolutely continuous solutions to the ODE $\dot\gamma = \boldsymbol{c}_t^n(\gamma)$ and satisfies $(e_t)_\# \boldsymbol{\lambda}_n = \nu_t^n$ for all $t \in [0, T]$.

Using (7-6) and our choice of Ψ_i and Φ_i we immediately obtain

$$\int \mathscr{A}(\gamma)\, d\boldsymbol{\lambda}_n(\gamma) \le 2;$$

hence the sequence $(\boldsymbol{\lambda}_n)$ is tight in $\mathscr{P}(C([0, T]; \mathbb{R}^\infty))$.

We claim that any limit point $\boldsymbol{\lambda}$ fulfills the properties stated in the lemma. Just for notational simplicity, we assume in the sequel that the whole family $(\boldsymbol{\lambda}_n)$ weakly converges to $\boldsymbol{\lambda}$. The lower semicontinuity of $\mathscr{A}$ gives $\int \mathscr{A}\, d\boldsymbol{\lambda} < \infty$; hence $\boldsymbol{\lambda}$ is concentrated on $\mathrm{AC}_w([0, T]; \mathbb{R}^\infty)$. Furthermore, since

$$\gamma \mapsto \pi_k \circ \gamma(t), \quad t \in [0, T],$$

are continuous maps from $C([0, T]; \mathbb{R}^\infty)$ to $\mathbb{R}^k$, by passing to the limit as $n \to \infty$ in the identity $(\pi_k)_\sharp (e_t)_\sharp \boldsymbol{\lambda}_n = (\pi_k)_\sharp \nu_t^n$ it follows that $(\pi_k)_\sharp (e_t)_\sharp \boldsymbol{\lambda} = (\pi_k)_\sharp \nu_t$ for all k. We can now use the fact that cylindrical functions generate the Borel σ-algebra of $\mathbb{R}^\infty$ to obtain that $(e_t)_\sharp \boldsymbol{\lambda} = \nu_t$.

It remains to prove that $\boldsymbol{\lambda}$ is concentrated on solutions to the ODE $\dot\gamma = \boldsymbol{c}_t(\gamma)$. To this aim, it suffices to show that

$$\int \left| p_i \circ \gamma(t) - p_i \circ \gamma(0) - \int_0^t p_i \circ \boldsymbol{c}_s(\gamma(s))\, ds \right| d\boldsymbol{\lambda}(\gamma) = 0, \tag{7-7}$$

for any $t \in [0, T]$ and $i \ge 1$. The technical difficulty is that this test function, due to the lack of regularity of $\boldsymbol{c}$, is not continuous in $C([0, T]; \mathbb{R}^\infty)$. To this aim, we prove first that

$$\int \left| p_i \circ \gamma(t) - p_i \circ \gamma(0) - \int_0^t \boldsymbol{d}_s(\gamma(s))\, ds \right| d\boldsymbol{\lambda}(\gamma) \le \int_{(0,T)\times\mathbb{R}^\infty} |p_i \circ \boldsymbol{c} - \boldsymbol{d}|\, d\nu_t dt, \tag{7-8}$$

for any bounded Borel function $\boldsymbol{d}$ where $\boldsymbol{d}(t, \cdot)$ is k-cylindrical for all $t \in (0, T)$, with k independent of t. It is clear that the space $\{\boldsymbol{d} \in L^1(\nu_t\, dt) : \boldsymbol{d}(t, \cdot) \text{ is cylindrical for all } t \in (0, T)\}$ is dense in $L^1(\nu_t\, dt)$; by a further approximation, also the space

$$\bigcup_{k=1}^{\infty}\{\boldsymbol{d}\in L^1(\nu_t\,dt):\boldsymbol{d}(t,\cdot)\text{ is }k\text{-cylindrical for all }t\in(0,T)\}$$

is dense. Hence, choosing a sequence $(\boldsymbol{d}^m)$ of functions admissible for (7-8) converging to $p_i\circ\boldsymbol{c}$ in $L^1(\nu_t\,dt)$ and noticing that

$$\int_{(0,T)\times\mathbb{R}^\infty}|p_i\circ\boldsymbol{c}_s(\gamma(s))-\boldsymbol{d}^m_s(\gamma(s))|\,ds\,d\boldsymbol{\lambda}(\gamma)=\int_{(0,T)\times\mathbb{R}^\infty}|p_i\circ\boldsymbol{c}-\boldsymbol{d}^m|\,d\nu_t dt\to 0,$$

we can take the limit in (7-8) with $\boldsymbol{d}=\boldsymbol{d}^m$ to obtain (7-7).

It remains to show (7-8). We first prove

$$\limsup_{n\to\infty}\int_{(0,T)\times\mathbb{R}^\infty}|p_i\circ\boldsymbol{c}^n-\boldsymbol{d}|\,d\nu^n_s\,ds\le\int_{(0,T)\times\mathbb{R}^\infty}|p_i\circ\boldsymbol{c}-\boldsymbol{d}|\,d\nu_t dt \tag{7-9}$$

for all bounded Borel functions $\boldsymbol{d}$ with $\boldsymbol{d}(t,\cdot)$ k-cylindrical for all $t\in(0,T)$, with k independent of t. The proof is elementary, because for $n\ge k$ and $t\in(0,T)$ we have

$$(p_i\circ\boldsymbol{c}^n_t-\boldsymbol{d}_t)\nu^n_t=(\pi_n)_\#((p_i\circ\boldsymbol{c}_t-\boldsymbol{d}_t)\nu_t).$$

Now we can prove (7-8), with a limiting argument based on the fact that (7-7) holds for $\boldsymbol{c}^n$ and $\boldsymbol{\lambda}_n$:

$$\begin{aligned}\int\left|p_i\circ\gamma(t)-p_i\circ\gamma(0)-\int_0^t\boldsymbol{d}_s(\gamma(s))\,ds\right|d\boldsymbol{\lambda}_n(\gamma)&=\int\left|\int_0^t\big(p_i\circ\boldsymbol{c}^n_s(\gamma(s))-\boldsymbol{d}_s(\gamma(s))\big)\,ds\right|d\boldsymbol{\lambda}_n(\gamma)\\&\le\iint_0^t|p_i\circ\boldsymbol{c}^n_s-\boldsymbol{d}_s|(\gamma(s))\,ds\,d\boldsymbol{\lambda}_n(\gamma)\\&\le\int_{(0,T)\times\mathbb{R}^\infty}|p_i\circ\boldsymbol{c}^n-\boldsymbol{d}|\,d\nu^n_s ds.\end{aligned}$$

Since $\boldsymbol{d}_s$ is cylindrical for all s and uniformly bounded w.r.t. s, the map

$$\gamma\mapsto\left|p_i\circ\gamma(t)-p_i\circ\gamma(0)-\int_0^t\boldsymbol{d}_s(\gamma(s))\,ds\right|$$

belongs to $C\big(C([0,T];\mathbb{R}^\infty)\big)$ and is nonnegative. Hence, taking the limit in the chain of inequalities above and using (7-9), we obtain (7-8). □

We next consider the case of a (possibly extended) metric measure space $(X,\tau,\mathfrak{m},d)$. Starting from the basic setup of Section 2, we have only a topology τ and the measure $\mathfrak{m}$. We assume the existence of a countable set $\mathcal{A}^*\subset\{f\in\mathcal{A}:\ \|\Gamma(f)\|_\infty\le 1\}$ satisfying:

$$\mathbb{R}\mathcal{A}^*\text{ is dense in }\mathbb{V}\text{ and any function in }\mathcal{A}^*\text{ has a }\tau\text{-continuous representative;} \tag{7-10}$$

$$\exists\lim_{n\to\infty}f(x_n)\text{ in }\mathbb{R}\text{ for all }f\in\mathcal{A}^*\quad\Longrightarrow\quad\exists\lim_{n\to\infty}x_n\text{ in }X. \tag{7-11}$$

Since $\operatorname{supp}\mathfrak{m}=X$, the τ-continuous representative of an $\mathfrak{m}$-measurable function is unique if it exists, and for this reason we do not use a distinguished notation for the continuous representative of functions in

$\mathcal{A}^*$ in (7-12) and in the sequel. Notice that (7-11) implies that the family $\mathcal{A}^*$ separates the points of X and that (7-10) and (7-11) can easily be fulfilled in many cases when an a priori distance d is given, considering the distance functions from a countable and dense set of points; see Section 9F for details.

Remark 7.2 (extended distance induced by $\mathcal{A}^*$). Following [Biroli and Mosco 1995] (see also [Sturm 1995; Stollmann 2010]) we build $d_{\mathcal{A}^*} : X \times X \to [0, \infty]$ as

$$d_{\mathcal{A}^*}(x, y) = \sup\{|f(x) - f(y)| : f \in \mathcal{A}^*\}, \quad x, y \in X. \tag{7-12}$$

A priori, $d_{\mathcal{A}^*}$ is an extended distance in the sense of [Ambrosio et al. 2014b], since it may take the value ∞; nevertheless, by definition, all functions in $\mathcal{A}^*$ are 1-Lipschitz w.r.t. $d_{\mathcal{A}^*}$ and $d_{\mathcal{A}^*}$ is the smallest extended distance with this property. In particular the derivative $d(f \circ \eta)/dt$, which occurs in the next definition, makes sense a.e. in $(0, T)$ when $f \in \mathcal{A}^*$ and $\eta \in \mathrm{AC}([0, T]; (X, d_{\mathcal{A}^*}))$ because $f \circ \eta$ belongs to $\mathrm{AC}([0, T])$. However, we will not use the topology induced by $d_{\mathcal{A}^*}$, which could be much finer than the topology τ and, in the next definition, we will require only continuity of $\eta : [0, T] \to X$ (with the topology τ in the target space X) and $W^{1,1}(0, T)$ regularity of $f \circ \eta$, for $f \in \mathcal{A}$. A posteriori, in Lemma 7.4 we are going to recover some absolute continuity for η, with respect to $d_{\mathcal{A}^*}$. In any case, whenever $f \in \mathcal{A}$ has a continuous representative (as it happens when $f \in \mathcal{A}^*$), the continuity of $f \circ \eta$ in conjunction with Sobolev regularity gives $f \circ \eta \in \mathrm{AC}([0, T])$.

Definition 7.3 (ODE induced by a family $(\boldsymbol{b}_t)$ of derivations). Let $\boldsymbol{\eta} \in \mathscr{P}(C([0, T]; X))$ and let $(\boldsymbol{b}_t)_{t\in(0,T)}$ be a Borel family of derivations. We say that $\boldsymbol{\eta}$ is concentrated on solutions to the ODE $\dot\eta = \boldsymbol{b}_t(\eta)$ if

$$f \circ \eta \in W^{1,1}(0, T) \quad \text{and} \quad \frac{d}{dt}(f \circ \eta)(t) = df(\boldsymbol{b}_t)(\eta(t)), \quad \text{for a.e. } t \in (0, T),$$

for $\boldsymbol{\eta}$-a.e. $\eta \in C([0, T]; X)$, for all $f \in \mathcal{A}$.

Notice that the property of being concentrated on solutions to the ODE implicitly depends on the choice of Borel representatives of the maps f and $(t, x) \mapsto df(\boldsymbol{b}_t)(x)$, $f \in \mathcal{A}$. As such, it should be handled with care. We will see, however, that in the class of regular flows of Definition 8.1 this sensitivity to the choice of Borel representatives disappears; see Remark 8.2.

The following simple lemma shows that time marginals of measures $\boldsymbol{\eta}$ concentrated on solutions to the ODE $\dot\eta = \boldsymbol{b}_t(\eta)$ provide weakly continuous solutions to the continuity equation.

Given a derivation $\boldsymbol{b}$, we introduce the quantity

$$|\boldsymbol{b}|_* = \sup\{|df(\boldsymbol{b})| : f \in \mathcal{A}^*\}. \tag{7-13}$$

Notice that $|\boldsymbol{b}|_*$ is well-defined up to $\mathfrak{m}$-a.e. equivalence and that one has $|\boldsymbol{b}|_* \le |\boldsymbol{b}|$ $\mathfrak{m}$-a.e. in X. Also in view of (7-14) below, it is natural to investigate the validity of the equality $|\boldsymbol{b}| = |\boldsymbol{b}|_*$ $\mathfrak{m}$-a.e. in X. We are able to prove this in the setting of RCD spaces; see Lemma 9.2 below.

Lemma 7.4. *Let $\boldsymbol{\eta} \in \mathscr{P}(C([0, T]; X))$ be concentrated on solutions η to the ODE $\dot\eta = \boldsymbol{b}_t(\eta)$, where $|\boldsymbol{b}| \in L^1_t(L^p_x)$ for some $p \in [1, \infty]$ and $\mu_t := (e_t)_\#\boldsymbol{\eta} \in \mathscr{P}(X)$ are representable as $u_t\mathfrak{m}$ with $u \in L^\infty_t(L^{p'}_x)$. Then, the following two properties hold*:

(a) *The family* $(u_t)_{t\in(0,T)}$ *is a weakly continuous solution to the continuity equation.*

(b) $\boldsymbol{\eta}$ *is concentrated on* $\mathrm{AC}([0,T];(X,d_{\mathcal{A}^*}))$, *with*

$$|\dot\eta|(t)=|\boldsymbol{b}_t|_*(\eta(t))\quad \textit{for a.e. } t\in(0,T),\ \textit{for } \boldsymbol{\eta}\textit{-a.e. } \eta. \tag{7-14}$$

Remark 7.5. Arguing as in the last part of [Ambrosio et al. 2005, Theorem 8.3.1] one can prove that $u\in L^\infty_t(L^\infty_x)$ implies that $(\mu_t)_t$ is an absolutely continuous curve in the Wasserstein space W_p naturally associated to $d_{\mathcal{A}^*}$ (see [Gigli and Bang-Xian 2014] for a more systematic investigation of this connection in metric measure spaces).

Proof. We integrate w.r.t. $\boldsymbol{\eta}$ the weak formulation

$$\int_0^t -\psi'(t)\, f\circ\eta(t)\,dt=\int_0^T \psi(t)\,df(\boldsymbol{b}_t)(\eta(t))\,dt$$

with $f\in\mathcal{A}$, $\psi\in C^1_c(0,T)$ to recover the weak formulation of the continuity equation for (u_t).

Given $f\in\mathcal{A}^*$, for $\boldsymbol{\eta}$-a.e. η the map $t\mapsto f\circ\eta(t)$ is absolutely continuous, with

$$f\circ\eta(t)-f\circ\eta(s)=\int_s^t df(\boldsymbol{b}_r)(\eta(r))\,dr\quad \text{for all } s,\ t\in[0,T].$$

In particular one has $df(\boldsymbol{b}_t)(\eta(t)=(f\circ\eta)'(t)$ a.e. in $(0,T)$, for $\boldsymbol{\eta}$-a.e. η.

By Fubini's theorem and the fact that the marginals of $\boldsymbol{\eta}$ are absolutely continuous w.r.t. $\mathfrak{m}$ we obtain that, for $\boldsymbol{\eta}$-a.e. η, one has

$$\sup_{f\in\mathcal{A}^*}|(f\circ\eta)'(t)|=\sup_{f\in\mathcal{A}^*}|df(\boldsymbol{b}_t)(\eta(t))|=|\boldsymbol{b}_t|_*(\eta(t))\quad \text{for a.e. } t\in(0,T),$$

and therefore

$$d_{\mathcal{A}^*}(\eta(t),\eta(s))=\sup_{f\in\mathcal{A}^*}|(f\circ\eta)(t)-(f\circ\eta)(s)|\le\int_s^t|\boldsymbol{b}_t|_*(\eta(r))\,dr\quad \text{for all } s,\ t\in[0,T],$$

proving that $\eta\in\mathrm{AC}([0,T];(X,d_{\mathcal{A}^*}))$, with $|\dot\eta|(t)\le|\boldsymbol{b}_t|_*(\eta(t))$, for a.e. $t\in(0,T)$. The converse inequality follows from the fact that every $f\in\mathcal{A}^*$ is 1-Lipschitz with respect to $d_{\mathcal{A}^*}$; thus for $\boldsymbol{\eta}$-a.e. η one has

$$|\boldsymbol{b}_t|_*(\eta(t))=\sup_{f\in\mathcal{A}^*}|(f\circ\eta)'(t)|\le|\dot\eta|(t)\quad \text{for a.e. } t\in(0,T).\qquad\square$$

Even though, as we explained in Remark 7.2, the (extended) distance is hidden in the choice of the family $\mathcal{A}^*$, we call the next result "superposition in metric measure spaces", because in most cases $\mathcal{A}^*$ consists precisely of distance functions from a countable dense set (see also the recent papers [Bate 2013; Schioppa 2014] for related results on the existence of suitable measures in the space of curves, and derivations).

Theorem 7.6 (superposition principle in metric measure spaces). *Assume* (7-10), (7-11). *Let* $\boldsymbol{b}=(\boldsymbol{b}_t)_{t\in(0,T)}$ *be a Borel family of derivations and let* $\mu_t=u_t\mathfrak{m}\in\mathscr{P}(X)$, $0\le t\le T$, *be a weakly continuous solution to the continuity equation*

$$\partial_t\mu_t+\mathrm{div}(\boldsymbol{b}_t\mu_t)=0 \tag{7-15}$$

with

$$u \in L^\infty_t(L^p_x), \qquad \int_0^T \int |\boldsymbol{b}_t|^r \, d\mu_t dt < \infty, \qquad \frac{1}{r} + \frac{1}{p} \le \frac{1}{2}. \tag{7-16}$$

Then there exists $\boldsymbol{\eta} \in \mathcal{P}(C([0,T];X))$ *satisfying*

(a) $\boldsymbol{\eta}$ *is concentrated on solutions* η *to the ODE* $\dot\eta = \boldsymbol{b}_t(\eta)$, *according to Definition 7.3*;

(b) $\mu_t = (e_t)_\# \boldsymbol{\eta}$ *for any* $t \in [0,T]$.

Proof. We enumerate by f_i, $i \ge 1$, the elements of $\mathcal{A}^*$ and define a continuous and injective map $J : X \to \mathbb{R}^\infty$ by

$$J(x) := (f_1(x), f_2(x), f_3(x), \ldots). \tag{7-17}$$

A simple consequence of (7-11), besides the injectivity we already observed, is that $J(X)$ is a closed subset of $\mathbb{R}^\infty$ and that J^{-1} is continuous from $J(X)$ to X.

Defining $\nu_t \in \mathcal{P}(\mathbb{R}^\infty)$ by $\nu_t := J_\# \mu_t$, $\boldsymbol{c} : (0,T) \times \mathbb{R}^\infty \to \mathbb{R}^\infty$ by

$$\boldsymbol{c}^i_t := \begin{cases} (df_i(\boldsymbol{b}_t)) \circ J^{-1} & \text{on } J(X), \\ 0 & \text{otherwise,} \end{cases}$$

and noticing that

$$|\boldsymbol{c}^i_t| \circ J \le |\boldsymbol{b}_t| \quad \mathfrak{m}\text{-a.e. in } X, \tag{7-18}$$

the chain rule (see Proposition 3.2)

$$d\phi(\boldsymbol{b}_t)(x) = \sum_{i=1}^n \frac{\partial \psi}{\partial z_i}(f_1(x), \ldots, f_n(x)) \boldsymbol{c}^i_t(x)$$

for $\phi(x) = \psi(f_1(x), \ldots, f_n(x))$ shows that the assumptions of Theorem 7.1 are satisfied by ν_t with velocity $\boldsymbol{c}$, because (7-18) and $\mu_t \ll \mathfrak{m}$ give $|\boldsymbol{c}^i_t| \le |\boldsymbol{b}_t| \circ J^{-1}$ ν_t-a.e. in $\mathbb{R}^\infty$.

As a consequence we can apply Theorem 7.1 to obtain $\boldsymbol{\lambda} \in \mathcal{P}(C([0,T];\mathbb{R}^\infty))$ concentrated on solutions $\gamma \in \mathrm{AC}([0,T];\mathbb{R}^\infty)$ to the ODE $\dot\gamma = \boldsymbol{c}_t(\gamma)$ such that $(e_t)_\# \boldsymbol{\lambda} = \nu_t$ for all $t \in [0,T]$. Since all measures ν_t are concentrated on $J(X)$,

$$\gamma(t) \in J(X) \qquad \text{for } \boldsymbol{\lambda}\text{-a.e. } \gamma, \text{ for all } t \in [0,T] \cap \mathbb{Q}.$$

Then, the closedness of $J(X)$ and the continuity of γ give $\gamma([0,T]) \subset J(X)$ for $\boldsymbol{\lambda}$-a.e. γ. For this reason, it makes sense to define

$$\boldsymbol{\eta} := \Theta_\# \boldsymbol{\lambda},$$

where $\Theta : C([0,T]; J(X)) \to C([0,T];X)$ is the map $\gamma \mapsto \Theta(\gamma) := J^{-1} \circ \gamma$. Since $(J^{-1})_\# \nu_t = \mu_t$, we obtain immediately that $(e_t)_\# \boldsymbol{\eta} = \mu_t$.

Let $i \ge 1$ be fixed. Since $f_i \circ \Theta(\gamma) = p_i \circ \gamma$, taking the definition of $\boldsymbol{c}_i$ into account we obtain that $f_i \circ \eta$ is absolutely continuous in $[0,T]$ and that

$$(f_i \circ \eta)'(t) = df_i(\boldsymbol{b}_t)(\eta(t)) \qquad \text{a.e. in } (0,T), \text{ for } \boldsymbol{\eta}\text{-a.e. } \eta. \tag{7-19}$$

We will complete the proof by showing that (7-19) extends from $\mathcal{A}^*$ to all of $\mathcal{A}$. By the chain rule we observe, first of all, that (7-19) extends from f_i to smooth truncations of f_i. Therefore, by the density of $\mathcal{A}^*$ in $\mathbb{V}$, for any $f \in \mathcal{A}$ we can find g_n satisfying:

(a) $g_n \to f$ in $\mathbb{V}$ and $\|g_n\|_\infty \le \|f\|_\infty + 1$;

(b) $g_n \circ \eta \in \mathrm{AC}([0, T])$ and $(g_n \circ \eta)'(t) = dg_n(\boldsymbol{b}_t)(\eta(t))$ a.e. in $(0, T)$, for $\boldsymbol{\eta}$-a.e. η.

Since

$$\iint_0^T |(f - g_n)(\eta(t))| \, dt \, d\boldsymbol{\eta}(\eta) = \int_0^T \int |f - g_n| u_t \, d\mathfrak{m} \, dt \to 0 \tag{7-20}$$

we can assume, possibly refining the sequence (g_n), that $g_n \circ \eta \to f \circ \eta$ in $L^1(0, T)$, for $\boldsymbol{\eta}$-a.e. η.

In order to achieve Sobolev regularity of $f \circ \eta$ it remains to show convergence of the derivatives of $g_n \circ \eta$, namely $dg_n(\boldsymbol{b}_t)(\eta(t))$, to $df(\boldsymbol{b}_t)(\eta(t))$. Arguing as in (7-20), we get

$$\iint_0^T |df(\boldsymbol{b}_t)(\eta(t)) - dg_n(\boldsymbol{b}_t)(\eta(t))| \, dt \, d\boldsymbol{\eta}(\eta) = \int_0^T \int |d(f - g_n)(\boldsymbol{b}_t)| u_t \, d\mathfrak{m} \, dt \to 0,$$

because of (7-16) and the convergence $\Gamma(f - g_n) \to 0$ in $L^1(\mathfrak{m})$. Therefore, possibly refining (g_n) once more, $dg_n(\boldsymbol{b})(\eta) \to df(\boldsymbol{b})(\eta)$ in $L^1(0, T)$, for $\boldsymbol{\eta}$-a.e. η. □

8. Regular Lagrangian flows

In this section we consider a Borel family of derivations $\boldsymbol{b} = (\boldsymbol{b}_t)_{t \in (0,T)}$ satisfying

$$\boldsymbol{b} \in L^1_t(L^1_x + L^\infty_x). \tag{8-1}$$

Under the assumption that the continuity equation has uniqueness of solutions in the class

$$\mathscr{L}_+ := \{u \in L^\infty_t(L^1_x \cap L^\infty_x) : t \mapsto u_t \text{ is weakly continuous in } [0, T],\ u \ge 0\} \tag{8-2}$$

for any initial datum $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$, and existence of solutions in the class

$$\bigl\{u \in \mathscr{L}_+ : \ \|u_t\|_\infty \le C(\boldsymbol{b}) \|u_0\|_\infty \ \forall t \in [0, T]\bigr\}, \tag{8-3}$$

for any nonnegative initial datum $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$, we prove existence and uniqueness of the regular flow $\mathbb{X}$ associated to $\boldsymbol{b}$. Here, the need for a class as large as possible where uniqueness holds is hidden in the proof of Theorem 8.4, where solutions are built by taking the time marginals of suitable probability measures on curves and uniqueness leads to a nonbranching result. The concept of regular flow, adapted from [Ambrosio 2004], is the following.

Definition 8.1 (regular flows). We say that $\mathbb{X} : [0, T] \times X \to X$ is a regular flow (relative to $\boldsymbol{b}$) if:

(i) $\mathbb{X}(0, x) = x$ and $\mathbb{X}(\cdot, x) \in C([0, T]; X)$ for all $x \in X$;

(ii) for all $f \in \mathcal{A}$, $f(\mathbb{X}(\cdot, x)) \in W^{1,1}(0, T)$ and $df(\mathbb{X}(t, x))/dt = df(\boldsymbol{b}_t)(\mathbb{X}(t, x))$ for a.e. $t \in (0, T)$, for $\mathfrak{m}$-a.e. $x \in X$;

(iii) there exists a constant $C = C(\mathbb{X})$ satisfying $\mathbb{X}(t, \cdot)_\# \mathfrak{m} \le C \mathfrak{m}$ for all $t \in [0, T]$.

Remark 8.2 (invariance under modifications of $\boldsymbol{b}$ and f). Assume that $\boldsymbol{b}$ and $\tilde{\boldsymbol{b}}$ satisfy:

$$\text{for all } f \in \mathcal{A}, \quad df(\boldsymbol{b}) = df(\tilde{\boldsymbol{b}}) \quad \mathscr{L}^1 \otimes \mathfrak{m}\text{-a.e. in } (0,T)\times X. \tag{8-4}$$

Then $\mathbb{X}$ is a regular flow relative to $\boldsymbol{b}$ if and only if $\mathbb{X}$ is a regular flow relative to $\tilde{\boldsymbol{b}}$. Indeed, let us fix $f \in \mathcal{A}$ and notice that for all $t \in (0,T)$ such that $\mathfrak{m}(\{df(\boldsymbol{b}_t) \neq df(\tilde{\boldsymbol{b}}_t)\}) = 0$, condition (iii) of Definition 8.1 gives

$$df(\boldsymbol{b}_t)(\mathbb{X}(t,x)) = df(\tilde{\boldsymbol{b}}_t)(\mathbb{X}(t,x)) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X.$$

Thanks to (8-4) and Fubini's theorem, the condition $\mathfrak{m}(\{df(\boldsymbol{b}_t) \neq df(\tilde{\boldsymbol{b}}_t)\}) = 0$ is satisfied for a.e. $t \in (0,T)$. Hence, we may apply Fubini's theorem once more to get

$$df(\boldsymbol{b}_t)(\mathbb{X}(t,x)) = df(\tilde{\boldsymbol{b}}_t)(\mathbb{X}(t,x)) \quad \text{a.e. in } (0,T), \text{ for } \mathfrak{m}\text{-a.e. } x \in X.$$

With a similar argument, one can show that if we modify not only $df(\boldsymbol{b})$ but also f in an $\mathfrak{m}$-negligible set, to obtain a Borel representative $\tilde{f}$, then $f(\mathbb{X}(\cdot,x)) \in W^{1,1}(0,T)$ and $df(\mathbb{X}(t,x))/dt = \boldsymbol{b}_t(\mathbb{X}(t,x))$ for a.e. $t \in (0,T)$ if and only if $\tilde{f}(\mathbb{X}(\cdot,x)) \in W^{1,1}(0,T)$ and $d\tilde{f}(\mathbb{X}(t,x))/dt = \boldsymbol{b}_t(\mathbb{X}(t,x))$ for a.e. $t \in (0,T)$, because Fubini's theorem gives $\tilde{f}(\mathbb{X}(t,x)) = f(\mathbb{X}(t,x))$ for a.e. $t \in (0,T)$, for $\mathfrak{m}$-a.e. $x \in X$. For this reason the choice of a Borel representative of $f \in \mathcal{A}$ is not really important. Whenever this is possible, the natural choice of course is given by the continuous representative.

The main result of the section is the following existence and uniqueness result. We stress that uniqueness is understood in the pathwise sense, namely $\mathbb{X}(\cdot,x) = \mathbb{Y}(\cdot,x)$ in $[0,T]$ for $\mathfrak{m}$-a.e. $x \in X$, whenever $\mathbb{X}$ and $\mathbb{Y}$ are regular Lagrangian flows relative to $\boldsymbol{b}$.

Theorem 8.3 (existence and uniqueness of the regular Lagrangian flow). *Assume (8-1) and that the continuity equation induced by $\boldsymbol{b}$ has uniqueness of solutions in $\mathscr{L}_+$ for all initial data $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$, as well as existence of solutions in the class (8-3) for all nonnegative initial data $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$. Then there exists a unique regular Lagrangian flow relative to $\boldsymbol{b}$.*

Proof. Let $B \in \mathscr{B}(X)$ with positive and finite $\mathfrak{m}$-measure and let us build first a "generalized" flow starting from B. To this aim, we take $\bar{u} = \chi_B/\mathfrak{m}(B)$ as initial datum and we apply first the assumption on existence of a solution $u \in \mathscr{L}_+$ starting from $\bar{u}$, with $u_t \le C(\boldsymbol{b})/\mathfrak{m}(B)$, and then the superposition principle stated in Theorem 7.6 to obtain $\boldsymbol{\eta} \in \mathscr{P}(C([0,T];X))$ whose time marginals are $u_t\mathfrak{m}$, concentrated on solutions to the ODE $\dot{\eta} = \boldsymbol{b}_t(\eta)$. Then, Theorem 8.4 below (which uses the uniqueness part of our assumptions relative to the continuity equation) provides a representation

$$\boldsymbol{\eta} = \frac{1}{\mathfrak{m}(B)} \int_B \delta_{\eta_x}\, d\mathfrak{m}(x),$$

with $\eta_x \in C([0,T];X)$, such that $\eta_x(0) = x$ and $\dot{\eta}_x = \boldsymbol{b}_t(\eta)$. Setting $\mathbb{X}(\cdot,x) = \eta_x(\cdot)$ for $x \in B$, it follows that $\mathbb{X} : B \times [0,T]$ is a regular flow, relative to $\boldsymbol{b}$, with the only difference that (i) and (ii) in Definition 8.1 have to be understood for $\mathfrak{m}$-a.e. $x \in B$, and

$$\mathbb{X}(t,\cdot)_\#(\bar{u}\mathfrak{m}) = (e_t)_\#\boldsymbol{\eta} = u_t\mathfrak{m} \le \frac{C(\boldsymbol{b})}{\mathfrak{m}(B)}\mathfrak{m}. \tag{8-5}$$

Next we prove consistency of these "local" flows $\mathbb{X}_B$. If $B_1 \subset B_2$ with $\mathfrak{m}(B_1) > 0$ and $\mathfrak{m}(B_2) < \infty$, we can consider the measure

$$\boldsymbol{\eta} := \frac{1}{2\mathfrak{m}(B_1)} \int_{B_1} \bigl(\delta_{X_{B_1}(\cdot,x)} + \delta_{X_{B_2}(\cdot,x)}\bigr)\, d\mathfrak{m}(x) \in \mathscr{P}(C([0,T];X))$$

to obtain from Theorem 8.4 that $\mathbb{X}_{B_1}(\cdot,x) = \mathbb{X}_{B_2}(\cdot,x)$ for $\mathfrak{m}$-a.e. $x \in B_1$.

Having gained consistency, we can build a regular Lagrangian flow by considering a nondecreasing sequence of Borel sets B_n with positive and finite $\mathfrak{m}$-measure whose union $\mathfrak{m}$-almost covers X and the corresponding local flows $\mathbb{X}_n : B_n \times [0,T] \to X$. Notice that we needed a quantitative upper bound on $\mathbb{X}_n(t,\cdot)_\#(\chi_{B_n}\mathfrak{m})$ precisely in order to be able to pass to the limit in condition (iii) of Definition 8.1, since (8-5) gives $\mathbb{X}(t,\cdot)_\#(\chi_B\mathfrak{m}) \le C(\boldsymbol{b})\mathfrak{m}$.

This completes the existence part. The uniqueness part can be proved using Theorem 8.4 once more and the same argument used to show consistency of the "local" flows. □

Theorem 8.4 (no splitting criterion). *Assume (8-1) and that the continuity equation induced by $\boldsymbol{b}$ has at most one solution in $\mathscr{L}_+$ for all $\bar{u} \in L^1 \cap L^\infty(\mathfrak{m})$. Let $\boldsymbol{\eta} \in \mathscr{P}(C([0,T];X))$ satisfy:*

(i) *$\boldsymbol{\eta}$ is concentrated on solutions η to the ODE $\dot{\eta} = \boldsymbol{b}_t(\eta)$;*

(ii) *there exists $L_0 \in [0,\infty)$ satisfying*

$$(e_t)_\#\boldsymbol{\eta} \le L_0\mathfrak{m} \quad \forall\, t \in [0,T]. \tag{8-6}$$

Then, the conditional measures $\boldsymbol{\eta}_x \in \mathscr{P}(C([0,T];X))$ induced by the map e_0 are Dirac masses for $(e_0)_\#\boldsymbol{\eta}$-a.e. x; equivalently, there exist $\eta_x \in C([0,T];X)$ solving the ODE $\dot{\eta}_x = \boldsymbol{b}_t(\eta_x)$, $\eta_x(0) = x$, and satisfying $\boldsymbol{\eta} = \int \delta_{\eta_x}\, d(e_0)_\#\boldsymbol{\eta}(x)$.

Proof. Using the uniqueness assumption at the level of the continuity equation, as well as the implication provided by Lemma 7.4, the decomposition procedure of [Ambrosio and Crippa 2008, Theorem 18] (that slightly improves the original argument of [Ambrosio 2004, Theorem 5.4], where comparison principle for the continuity equation was assumed) gives the result. □

9. Examples

In this section, on one hand we illustrate relevant classes of metric measure spaces for which our abstract theory applies. On the other hand we try to compare our results on the well-posedness of the continuity equation with the ones obtained in other papers, for particular classes of spaces. Several variants of the existence and uniqueness results are possible, varying the regularity and the growth conditions imposed on $\boldsymbol{b}$ and on the density u_t; we focus mainly on the issue of uniqueness, since existence in particular classes of spaces (e.g., the Euclidean ones) can be often be obtained by ad hoc methods (such as convolution of the components of the vector field, which preserve bounds on divergence) not available in general spaces. Also, we will not discuss the existence/uniqueness of the flow, which follow automatically from well-posedness at the PDE level using the transfer mechanisms presented in Section 8. We list the examples that follow, to some extent, in order of chronology and level of complexity.

9A. ***Euclidean spaces: DiPerna–Lions theory.*** The theory of well-posedness for flows and for transport and continuity equations was initiated by DiPerna and Lions [1989] and it (quite obviously) fits into our abstract setting. More explicitly, in the basic setup (2-1) we let $X = \mathbb{R}^n$, $\mathfrak{m} = \mathscr{L}^n$ (the Lebesgue measure) and

$$\mathscr{E}(f) = \int |\nabla f|^2(x)\, d\mathscr{L}^n(x) \quad \text{for } f \in W^{1,2}(\mathbb{R}^n),$$

so that Δ is the usual Laplacian and $(\mathsf{P}_t)_t$ is the heat semigroup, that corresponds (up to a factor 2 in the time scale) to the transition semigroup of the Brownian motion, which is conservative. The algebra $\mathcal{A}$ of Section 2C can be chosen to be the space of Lipschitz functions with compact support.

Given a Borel vector field $b = \sum_{i=1}^n b^i e_i$, with $b \in (L^1 + L^\infty)^n$, its associated derivation $\boldsymbol{b}$ is

$$\mathcal{A} \ni f \mapsto df(\boldsymbol{b}) = \sum_{i=1}^n b_i \frac{\partial f}{\partial x^i}.$$

Obviously, $\operatorname{div} \boldsymbol{b}$ is the usual distributional divergence and $D^{\mathrm{sym}}\boldsymbol{b}$ is the symmetric part of the distributional derivative of $\boldsymbol{b}$. Then, the uniqueness Theorem 5.4 above corresponds to [DiPerna and Lions 1989, Corollary II.1], as long as $q \in (1, \infty]$.

On the other hand, in Euclidean spaces the strong local convergence of commutators depends on local regularity assumptions on $\boldsymbol{b}$ (and the use of convolutions with compact support), while our setting is intrinsically global. In order to adapt our methods to this case, one could "localize the Dirichlet form" by considering $X = B_r(0)$ and the form

$$\mathscr{E}_r(f) = \int_{B_r} |\nabla f|^2 d\mathscr{L}^n \quad \text{for } f \in H^1(B_r).$$

Thus Δ would be the Laplacian with Neumann boundary conditions and $(\mathsf{P}_t)_t$ would be the semigroup correspondent to the Brownian motion reflected at the boundary $\partial B_r(0)$, which is still conservative. Since the ball is convex, it can be proved that $\mathsf{BE}_2(0, \infty)$ still holds; see, for instance, [Ambrosio et al. 2014c, Theorem 6.20].

A second major difference is that uniqueness assuming the regularity $b \in (W^{1,1})^n$ (or even $b \in (\mathrm{BV})^n$, the case considered in [Ambrosio 2004]) is not covered. Indeed, the BV case seems difficult to reach in the abstract setting, due to the present lack of a covariant derivative (but see [Gigli 2014]).

9B. ***Weighted Riemannian manifolds.*** Our arguments extend the classical DiPerna–Lions theory to the setting of weighted Riemannian manifolds. Of course, in order to prove strong convergence of commutators and the fact that solutions are renormalized one can always argue by local charts, but computations become more cumbersome compared to the Euclidean case, and here the advantages of our intrinsic approach become more manifest.

Let $(M, \boldsymbol{g})$ be a smooth Riemannian manifold and let μ be its associated Riemannian volume measure. Assume that the Ricci curvature tensor $\mathrm{Ric}_{\boldsymbol{g}}$ is pointwise bounded from below (in the sense of quadratic forms) by some constant $K \in \mathbb{R}$. More generally, one can add a "weight" $V : M \to \mathbb{R}$ to the measure: consider a smooth nonnegative function and assume that the Bakry–Émery curvature tensor is bounded from below by $K \in \mathbb{R}$, i.e.,

$$\mathrm{Ric}_{\boldsymbol{g}} + \mathrm{Hess}(V) \geq K.$$

The form (on smooth compactly supported functions)

$$f \mapsto \mathscr{E}_V(f) = \int_M \boldsymbol{g}(\nabla f, \nabla f) e^{-V} \, d\mu,$$

is closable and we are in the setup (2-1). Once more, the algebra $\mathcal{A}$ of Section 2C can be chosen to be the space of Lipschitz functions with compact support.

When $V = 0$, Bochner's formula entails that $\mathsf{BE}_2(\mathsf{K}, \infty)$ holds and it is a classical result due to S.-T. Yau that the heat semigroup is conservative. In the case of weighted measures, analogous results can be found in [Bakry 1994, Proposition 6.2] for the curvature bound and in [Grigor′yan 1999, Theorem 9.1] for the conservativity of P, relying on a correspondent volume comparison theorem; see [Wei and Wylie 2009, Theorem 1.2], for example.

Given a Borel vector field b, i.e., a Borel section of the tangent bundle of M, its associated derivation $\boldsymbol{b}$ acts on smooth functions by

$$f \mapsto df(\boldsymbol{b}) = \boldsymbol{g}(b, \nabla f).$$

The divergence can be given in terms of the μ-distributional divergence of b by

$$\operatorname{div} \boldsymbol{b} = \operatorname{div} b - \boldsymbol{g}(\boldsymbol{b}, \nabla V),$$

while the deformation is the symmetric part of the distributional covariant derivative; see Remark 5.3.

9C. *Abstract Wiener spaces.* Let $(X, \gamma, \mathscr{H})$ be an abstract Wiener space, i.e., X is a separable Banach space, γ is a centered nondegenerate Gaussian measure on X with covariance operator $Q : X^* \mapsto X$ and $\mathscr{H} \subset X$ is its associated Cameron–Martin space, which is naturally endowed with a Hilbertian norm. Moreover, $QX^* \subset \mathscr{H}$.

We define the set of smooth cylindrical functions $\mathscr{F}\mathscr{C}_b^\infty(X)$ as the set of all functions $f(x)$ representable as $\varphi(x_1^*(x), \dots, x_n^*(x))$, with $\varphi : \mathbb{R}^n \to \mathbb{R}$ smooth and bounded, $x_i^* \in X^*$ for $i \in \{1, \dots, n\}$, for some integer $n \geq 1$.

We introduce a notion of "gradient" on functions $f \in \mathscr{F}\mathscr{C}_b^\infty(X)$ letting $\nabla_{\mathscr{H}} f = Q df$, where df is the Fréchet differential of f. With these definitions, for $f = \varphi(x_1^*, \dots, x_n^*)$, and any orthonormal basis (h_i) of $\mathscr{H}$, we have

$$\nabla_{\mathscr{H}} f(x) = \sum_{j=1}^n \frac{\partial \varphi}{\partial z_j} Q x_j^* = \sum_i \frac{\partial f}{\partial h_i}(x) h_i, \quad \text{where } \frac{\partial f}{\partial h_i}(x) = \lim_{\varepsilon \to 0} \frac{f(x + \varepsilon h_i) - f(x)}{\varepsilon}.$$

It is well-known [Bouleau and Hirsch 1991] that Sobolev–Malliavin calculus on $(X, \gamma, \mathscr{H})$ fits into the setting (2-1), considering the closure of the quadratic form

$$\mathscr{E}(f) = \int_X |\nabla_{\mathscr{H}} f|_{\mathscr{H}}^2 \, d\gamma \quad \text{for every } f \in \mathscr{F}\mathscr{C}_b^\infty(X).$$

The domain $\mathbb{V}$ coincides with the space $W^{1,2}(X,\gamma)$. The semigroup P is the Ornstein–Uhlenbeck semigroup, given by Mehler's formula

$$\mathsf{P}_t f(x) = \int_X f(e^{-t}x + \sqrt{1-e^{-2t}}\,y)\,d\gamma(y) \quad \text{for } \gamma\text{-a.e. } x \in X.$$

From this, it is easy to show that $\mathsf{BE}_2(1,\infty)$ holds (indeed, on cylindrical functions, $\nabla_{\mathscr{H}}\mathsf{P}_t f = e^{-t}\mathsf{P}_t\nabla_{\mathscr{H}} f$, understanding the action of the semigroup componentwise); it is a classical result that $\mathscr{E}$ is *quasiregular*, e.g., [Bogachev 1998, Theorem 5.9.9]. We let $\mathscr{A} = \mathscr{F}\mathscr{C}_b^\infty(X)$, which is well-known to be dense in every L^p-space and satisfy (2-20) by Mehler's formula above: in particular we obtain density in $\mathbb{V}_p$ spaces by Lemma 5.6 and Lemma 2.2.

Given an $\mathscr{H}$-valued field $b = \sum_i b^i h_i$, we introduce the derivation $f \mapsto \boldsymbol{b}(f) = \sum_i b^i\,\partial f/\partial h_i$ and we briefly compare our well-posedness results for the continuity equation with those contained in [Ambrosio and Figalli 2009]. Combining Proposition 6.8 and the subsequent remark, we obtain that our notion of deformation for $\boldsymbol{b}$ is comparable to that of $(\nabla b)^{\mathrm{sym}}$ introduced in [Ambrosio and Figalli 2009, Definition 2.6]. Precisely, it can be proved that if $b \in LD^q(\gamma;\mathscr{H})$ for some $q > 1$, then the deformation of $\boldsymbol{b}$ is of type (r,s), for any r, s, with $q^{-1}+r^{-1}+s^{-1} = 1$. It is then easy to realize that Theorem 5.4 entails the uniqueness part of [Ambrosio and Figalli 2009, Theorem 3.1], with the exception, as we observed in connection to the Euclidean theory, of the case $b \in W^{1,1}(X,\gamma;\mathscr{H})$ (the case $b \in \mathrm{BV}(X,\gamma;\mathscr{H})$ has been recently settled in [Trevisan 2013]).

9D. ***Gaussian Hilbert spaces.*** We let $X = H$ be a separable Hilbert space, with norm $|\cdot|$, in the setting introduced in the previous section, namely γ is a Gaussian centered and nondegenerate measure in H. By identifying $H = H^*$ via the Riesz isomorphism induced by the norm, the covariance operator $Q : H \to H$ is a symmetric positive trace class operator, thus compact. In this setting the Cameron–Martin space is $\mathscr{H} = Q^{1/2}H$, with the norm $|h|_{\mathscr{H}} = |Q^{-1/2}h|$.

We let $(e_i) \subset H$ be an orthonormal basis of H consisting of eigenvectors of Q, with eigenvalues (λ_i), i.e., $Qe_i = \lambda_i e_i$ for every $i \ge 1$: in this setting, we define the class of smooth cylindrical functions $\mathscr{F}\mathscr{C}_b^\infty(H)$ as those functions $f : X \to \mathbb{R}$ of the form $f(x) = \varphi(\langle e_i, x\rangle, \ldots \langle e_n, x\rangle)$, with $\varphi : \mathbb{R}^n \to \mathbb{R}$ smooth and bounded. Given $f \in \mathscr{F}\mathscr{C}_b^\infty(H)$, from its Fréchet derivative $df : H \mapsto H^*$ we introduce $\nabla f : H \mapsto H$ via $H = H^*$, in coordinates:

$$\nabla f(x) = \sum_i \partial_i f(x) e_i, \quad \text{where } \partial_i f(x) = \lim_{\varepsilon\to 0}\frac{f(x+\varepsilon e_i) - f(x)}{\varepsilon}.$$

To recover the abstract setting of the previous section, notice that the family $h_i = \lambda_i^{1/2} e_i$ is an orthonormal basis of $\mathscr{H}$ and that $\partial/\partial h_i = \lambda_i^{-1/2}\partial_i$; thus $Q\nabla f = \nabla_{\mathscr{H}} f$. For $\alpha \in \mathbb{R}$, we introduce the form

$$\mathscr{E}^\alpha(f) = \int_X \left|Q^{(1-\alpha)/2}\nabla f\right|^2 d\gamma, \qquad f \in \mathscr{F}\mathscr{C}_b^\infty(H),$$

which is closable: its domain is the space $W^{1,2}_\alpha(H,\gamma)$; see [Da Prato 2004, Chapters 1 and 2] for more details. Evidently, we recover (2-1), with $\Gamma(f) = \sum_i \lambda_i^{1-\alpha}\left|\partial_i f\right|^2$. Notice that the associated distance is the one induced by the norm $|Q^{(\alpha-1)/2}x|$, which is extended if and only if $\alpha < 1$.

The associated semigroup can be still be seen as the transition semigroup of an infinite-dimensional stochastic differential equation, and its infinitesimal generator Δ_α is given by

$$\Delta_\alpha f(x) = \mathrm{Tr}[Q^{1-\alpha} D^2 f(x)] - \langle x, Q^{-\alpha}\nabla f(x)\rangle, \qquad f \in \mathscr{F}\mathscr{C}_b^\infty(H).$$

It can be shown that $\mathsf{BE}_2(1,\infty)$ holds [Da Prato 2004, Proposition 2.60]. We let $\mathcal{A} = \mathscr{F}\mathscr{C}_b^\infty(H)$, which is dense in every $L^p(\mathfrak{m})$ space and satisfies (2-20), thus obtaining density results in $\mathbb{V}_p$, $p \in [1,\infty)$, by Lemma 5.6 and Lemma 2.2.

For $\alpha = 0$, we recover the abstract Wiener space setting discussed above, while for $\alpha = 1$ we obtain the setting of [Da Prato et al. 2014]. We show that our results encompass those in [Da Prato et al. 2014] and analogues hold for any $\alpha \in \mathbb{R}$.

Given $b : H \mapsto H$, $b = \sum_i b_i e_i$ Borel, we consider the map

$$\mathscr{F}\mathscr{C}_b^\infty(H) \ni f \mapsto df(\boldsymbol{b}) := \langle b, \nabla f\rangle_H = \sum_i b_i \partial_i f.$$

If $|Q^{(\alpha-1)/2} b| \in L^q(H,\gamma)$ for some $q \in [1,\infty]$, then $\boldsymbol{b}$ is a well-defined derivation, with $|\boldsymbol{b}| \le |Q^{(\alpha-1)/2} b|$.

The Cameron–Martin theorem entails an integration by parts formula [Da Prato 2004, Theorem 1.4 and Lemma 1.5] that reads in our notation as

$$\operatorname{div} \boldsymbol{e}_i(x) = -\frac{\langle e_i, x\rangle}{\lambda_i}, \quad \text{where } df(\boldsymbol{e}_i) = \partial_i f.$$

On smooth "cylindrical" fields $b = \sum_i^n b_i e_i$, this gives

$$\operatorname{div} \boldsymbol{b}(x) = \sum_i \partial_i b_i(x) - \frac{\langle e_i, x\rangle}{\lambda_i} b_i,$$

where the series reduces to a finite sum. Notice that the expression does not depend on α but only on γ, in agreement with the notion of divergence as dual to derivation.

Notice also that the boundedness of the Gaussian Riesz transform [Bogachev 1998, Proposition 5.88] entails that if $\boldsymbol{b} \in W^{1,p}(H,\gamma,\mathscr{H})$, then $\operatorname{div} \boldsymbol{b} \in L^p(H,\gamma)$. These are only sufficient conditions and their assumptions would force us to limit the discussion to $\mathscr{H}$-valued fields, as in [Da Prato et al. 2014, Section 5]. Our results hold even for some classes of fields not taking values in $\mathscr{H}$; see at the end of this section.

Arguing on smooth cylindrical functions,

$$\int D^{\mathrm{sym}}\boldsymbol{b}(f,g)\,d\gamma = \int \sum_{i,j} \tfrac{1}{2}\left[\left(\frac{\lambda_i}{\lambda_j}\right)^{\frac{1-\alpha}{2}} \partial_i b_j + \left(\frac{\lambda_j}{\lambda_i}\right)^{\frac{1-\alpha}{2}} \partial_j b_i\right]\big(\lambda_i^{(1-\alpha)/2}\partial_i f\big)\big(\lambda_j^{(1-\alpha)/2}\partial_j f\big)\,d\gamma, \tag{9-1}$$

thus our bound on $D^{\mathrm{sym}}\boldsymbol{b}$ is implied by an L^q bound of the Hilbert–Schmidt norm of the expression in square brackets above (a fact that could also be seen as a consequence of Proposition 6.8 and the subsequent remark, setting $V_i(x) = \lambda_i^{(\alpha-1)/2}\langle e_i, x\rangle$, for $i \ge 1$).

Comparing our setting with that in [Da Prato et al. 2014], it is clear that Theorem 2.3 therein is a consequence of Theorem 5.4.

We end this section by considering a field b taking values outside $\mathcal{H}$, to which our theory applies (although well-posedness was already shown in [Mayer-Wolf and Zakai 2005]). Assume that each eigenvalue of Q admits a two-dimensional eigenspace, thus, slightly changing the notation, we write $(e_i, \tilde{e}_i)$ for an orthonormal basis of H consisting of eigenvectors of Q. We let

$$b = \sum_{i=1}^{\infty} \lambda_i^{1/2} [(\operatorname{div} \tilde{\boldsymbol{e}}_i) e_i - (\operatorname{div} \boldsymbol{e}_i) \tilde{e}_i]; \quad \text{thus} \int |Q^{(\alpha-1)/2} b|^2 \, d\gamma = \sum_{i=1}^{\infty} \lambda_i^{\alpha}.$$

The series above converges if $\alpha = 1$, and it does not if $\alpha = 0$. Since $(\operatorname{div} \boldsymbol{e}_i, \operatorname{div} \tilde{\boldsymbol{e}}_i)_i$ are independent, Kolmogorov's 0–1 law entails that b is well-defined as an H-valued map, but $b(x) \notin \mathcal{H}$ for γ-a.e. $x \in H$. The derivation $\boldsymbol{b}$ is therefore well-defined if $\alpha = 1$, and $|\boldsymbol{b}| \in L^2(\mathfrak{m})$. From its structure and (9-1), both its divergence and its deformation are seen to be identically 0, thus our results apply.

9E. *Log-concave measures.* Let $(H, |\cdot|)$ be a separable Hilbert space and let γ be a *log-concave* probability measure on H, i.e., for all open sets $B,\ C \subset H$,

$$\log \gamma((1-t)B + tC) \geq (1-t) \log \gamma(B) + t \log \gamma(C) \quad \text{for every } t \in [0, 1].$$

Assume also that γ is nondegenerate, namely that it is not concentrated on a proper closed subspace of H. Consider the quadratic form

$$\mathscr{E}_\gamma(f) = \int |\nabla f|^2 \, d\gamma, \quad \text{defined for } f \in C^1_b(H),$$

where $C^1_b(H)$ denotes the space of continuously Fréchet differentiable functions which are bounded with bounded differential.

It is shown in [Ambrosio et al. 2009] that the $\mathscr{E}_\gamma$ is closable, extending previous results obtained under more restrictive assumptions on γ. Actually, in [Ambrosio et al. 2009] the so-called EVI property for the associated semigroup P is proved, and since in [Ambrosio et al. 2014c] this is proved to be one of the equivalent characterizations of RCD, it follows that $(H, |\cdot|, \gamma)$ is an $\mathsf{RCD}(0, \infty)$ space; thus the results in Section 9F below apply and we already obtain an abstract well-posedness result under no additional assumption on γ. Recall that in that abstract setting $\mathcal{A}$ can be taken as the space of Lipschitz functions with bounded support.

Let $(e_i)_{i\geq 1} \subset H$ be an orthonormal basis. For every $f \in \mathbb{V}$, there exist $f_n \in C^1_b(H)$ such that $f_n \to f$ in $L^2(\gamma)$ and

$$\lim_{n,m\to\infty} \mathscr{E}_\gamma(f_n - f_m) \to 0;$$

thus an H-valued "gradient" $\nabla f = \sum_i \partial_i f e_i$ is γ-a.e. defined in H.

Let $b : H \mapsto H$, $b = \sum_i b_i e_i$; we associate the derivation $f \mapsto df(\boldsymbol{b}) = \sum_i b_i f_i$, thus $|\boldsymbol{b}| \leq |b|$. For $v \in H$, we write $\boldsymbol{v}$ for the constant derivation corresponding to the constant vector field equal to v, and $\boldsymbol{e}_i$ for the derivation corresponding to e_i.

Let us remark that, in this very general setting setting, bounds on the divergence of a given field $\boldsymbol{b}$ seem to be difficult to obtain, even for constant vector fields; this is due to the fact that presently it is not known whether every log-concave measure γ admits at least one nonzero direction v such that $\operatorname{div} \boldsymbol{v} \in L^1(\mathfrak{m})$

[Bogachev 2010, §4.3]. On the other hand, our abstract arguments do not require any absolute continuity of γ with respect to a Gaussian or other product measures and, combining our abstract well-posedness results with Theorem 6.7, we are able to provide nontrivial derivations that admit a well-posed flow, e.g., gradient derivations of functions in $D_{L^\infty}(\Delta)$, such as those of the form $\int_1^2 \mathsf{P}_t f\,dt$, for $f \in L^\infty(\mathfrak{m})$.

To state an explicit sufficient condition to bound the deformation of $\boldsymbol{b}$, we assume that $\operatorname{div} \boldsymbol{e}_i \ll \mathfrak{m}$ and, denoting by β_i the density, we require that $\beta_i \in \mathbb{V}$ for $i \geq 1$, or equivalently that the function $x \mapsto V_i(x) = \langle e_i, x\rangle$ satisfies $\Delta V_i \in \mathbb{V}$, thus, provided that $r, s \geq 4$, Proposition 6.8 gives $\|D^{\mathrm{sym}}\boldsymbol{b}\|_{r,s} < \infty$ if $[\partial_i b_j + \partial_j b_i]_{i,j} \in L^q(\gamma; \ell^2(\mathbb{N}\otimes\mathbb{N}))$.

We conclude by comparing our results in this setting with [Kolesnikov and Röckner 2014, Theorem 7.6], where uniqueness for the continuity equation is obtained in the case of log-concave measures formally given by $\gamma = e^{-V} d\mathscr{L}^\infty$, for convex Hamiltonians V of specific form. In particular, the assumptions on β_i imposed therein are stronger than ours. Their assumptions on the field b in [Kolesnikov and Röckner 2014] entail that $|\boldsymbol{b}| \in L^{a_1}(\gamma)$, for some $a_1 > 1$ and that $[\partial_i b_j + \partial_j b_i]_{i,j} \in L^{a_2}(\gamma; \ell^2(\mathbb{N}\otimes\mathbb{N}))$, for some $a_2 > 4$. Moreover, to deduce uniqueness, $\operatorname{div} \boldsymbol{b} \in L^q(\gamma)$ for some $q > 1$ is also assumed. Therefore, if $a_1 \geq 2$ and $q \geq 2$, we are in a position to recover, via Proposition 6.8, such a uniqueness result as a special case of Theorem 5.4.

9F. $\mathsf{RCD}(\mathsf{K},\infty)$ *metric measure spaces.* Recall that the class $\mathsf{CD}(\mathsf{K},\infty)$, introduced and deeply studied in [Lott and Villani 2009; Sturm 2006a; 2006b] consists of complete metric measure spaces such that the Shannon relative entropy w.r.t. $\mathfrak{m}$ is K-convex along Wasserstein geodesics; see [Villani 2009] for a full account of the theory and its geometric and functional implications. The class of $\mathsf{RCD}(\mathsf{K},\infty)$ metric measure spaces was first introduced in [Ambrosio et al. 2014c], from a metric perspective, as a smaller class than $\mathsf{CD}(\mathsf{K},\infty)$, with the additional requirement that the so-called Cheeger energy is quadratic; with this axiom, Finsler geometries are ruled out and stronger structural (and stability) properties can be established. Subsequently, connections with the theory of Dirichlet forms gave rise to a series of works, [Ambrosio et al. 2013; 2014a; Savaré 2014]. For a brief introduction to the setting and its notation, we refer to Sections 4.1 and 4.2 in [Savaré 2014], and in particular to Theorem 4.1 therein, which collects nontrivial equivalences among different conditions.

We will use the notation $W^{1,2}(X, d, \mathfrak{m})$ for the Sobolev space, Ch for the Cheeger energy arising from the relaxation in $L^2(X, \mathfrak{m})$ of the local Lipschitz constant

$$|\mathrm{D} f|(x) := \limsup_{y\to x} \frac{|f(y) - f(x)|}{d(y, x)} \tag{9-2}$$

of $L^2(\mathfrak{m})$ and Lipschitz maps $f : X \to \mathbb{R}$.

To introduce $\mathsf{RCD}(\mathsf{K},\infty)$ spaces we restrict the discussion to metric measure spaces $(X, d, \mathfrak{m})$ satisfying the following three conditions:

(i) (X, d) is a complete and separable length space;

(ii) $\mathfrak{m}$ is a nonnegative Borel measure with $\operatorname{supp}(\mathfrak{m}) = X$, satisfying

$$\mathfrak{m}(B_r(x)) \leq c\,\mathrm{e}^{Ar^2} \quad \text{for suitable constants } c \geq 0,\ A \geq 0; \tag{9-3}$$

(iii) $(X, \mathsf{d}, \mathfrak{m})$ is infinitesimally Hilbertian according to the terminology introduced in [Gigli 2012], i.e., the Cheeger energy Ch is a quadratic form.

As explained in [Ambrosio et al. 2014a; 2014c], the quadratic form Ch canonically induces a strongly regular, strongly local Dirichlet form $\mathscr{E}$ in (X, τ) (where τ is the topology induced by the distance d), as well as a carré du champ $\Gamma : D(\mathscr{E}) \times D(\mathscr{E}) \to L^1(\mathfrak{m})$. Thus, we recover the basic setting of (2-1) and we can identify $W^{1,2}(X, d, \mathfrak{m})$ with $\mathbb{V}$. In addition, P is conservative because of (9-3) and the definition of Ch provides the approximation property

$$\exists\, f_n \in \mathrm{Lip}(X) \cap L^2(\mathfrak{m}) \text{ with } f_n \to f \text{ in } L^2(\mathfrak{m}) \text{ and } |\mathrm{D} f_n| \to \sqrt{\Gamma(f)} \text{ in } L^2(\mathfrak{m})i, \tag{9-4}$$

for all $f \in \mathbb{V}$.

The above discussions justify the following definition of $\mathsf{RCD}(\mathsf{K}, \infty)$. It is not the original one given in [Ambrosio et al. 2014c] we mentioned at the beginning of this section, but it is more appropriate for our purposes; the equivalence of the two definitions is given in [Ambrosio et al. 2014a].

Definition 9.1 ($\mathsf{RCD}(\mathsf{K}, \infty)$ metric measure spaces). We say that $(X, d, \mathfrak{m})$, satisfying (a), (b), (c) above, is an $\mathsf{RCD}(\mathsf{K}, \infty)$ space if:

(i) the Dirichlet form associated to the Cheeger energy of $(X, d, \mathfrak{m})$ satisfies $\mathsf{BE}_2(\mathsf{K}, \infty)$ according to Definition 6.1;

(ii) any $f \in W^{1,2}(X, d, \mathfrak{m}) \cap L^\infty(\mathfrak{m})$ with $\left\|\Gamma(f)\right\|_\infty \le 1$ has a 1-Lipschitz representative.

From [Ambrosio et al. 2014c, Lemma 6.7] we obtain that $\mathscr{E}$ is *quasiregular*. We let $\mathscr{A}$ be the class of Lipschitz functions with bounded support. It is easily seen that $\mathscr{A}$ is dense in $\mathbb{V}$.

Lemma 9.2. *There exists a countable set $\mathscr{A}^* \subset \mathscr{A}$ with $\|\Gamma(f)\|_\infty \le 1$ for every $f \in \mathscr{A}^*$, such that (7-10) and (7-11) are satisfied, the distance $d_{\mathscr{A}^*}$ defined by (7-12) in Remark 7.2 coincides with d, and for any derivation $\boldsymbol{b}$ one has*

$$|\boldsymbol{b}| = |\boldsymbol{b}|_* \quad \mathfrak{m}\text{-a.e. in } X, \qquad \text{where } |\boldsymbol{b}|_* \text{ is defined in (7-13).} \tag{9-5}$$

Proof. Since both (X, d) and $\mathbb{V}$ are separable, it is not difficult to exhibit a countable family $\mathscr{A}^* \subset \mathscr{A}$ such that (7-10) and (7-11) are satisfied: let $(x_h) \subset X$ be dense, and set $f_{h,k} := (d(x_h, \cdot) - k)^- \in \mathscr{A}$ for $h,\ k \in \mathbb{N}$, then define

$$\mathscr{B} := \bigcup_{h,\,k=0}^{\infty} \{f_{h,k}\} \cup \bigcup_{h=0}^{\infty} \{g_h\},$$

with $(g_h) \subset \mathscr{A}$ dense in $\mathbb{V}$. Then, defining $\mathscr{A}^* = \{f \in \mathscr{B} : \|\Gamma(f)\|_\infty \le 1\} \subset \mathscr{A}$, since $\mathbb{R}\mathscr{A}^* = \mathscr{B}$ we obtain (7-10), while (7-11) follows easily from the fact that all functions $f_{h,k}$ belong to $\mathscr{A}^*$. To show that the distances coincide, notice that $d_{\mathscr{A}^*} \le d$ is obvious, while $d \le d_{\mathscr{A}^*}$ follows from taking $f = f_{h,k}$ in (7-12), with x_h arbitrarily close to x and k larger than $d(x, y)$.

We show that, up to a further enlargement of $\mathscr{A}^*$, (9-5) holds (notice that $d = d_{\mathscr{A}^*}$ and (7-11) holds automatically for the enlargement, and we need only to retain (7-10)). The problem reduces to arguing that for $f \in \mathscr{A}^*$ one can improve the inequality $|df(\boldsymbol{b})| \le |\boldsymbol{b}|_*$ to $|df(\boldsymbol{b})| \le \Gamma(f)^{1/2} |\boldsymbol{b}|_*$ $\mathfrak{m}$-a.e. in X. This is based on a localization procedure akin to [Ambrosio et al. 2014a, Proposition 3.11], which leads to the key inequality (9-8) below; the thesis follows then by a density argument, using the curvature assumption.

For any $\varepsilon > 0$, let $S_\varepsilon \in C^1(\mathbb{R})$ be a 1-Lipschitz truncation function, given by $S_\varepsilon(r) = \varepsilon S_1(r/\varepsilon)$, where $S_1(r)$ is a 1-Lipschitz function with

$$\begin{cases} S_1(r) = 1 & \text{for } r \le 1, \\ S_1(r) = 0 & \text{for } r \ge 3. \end{cases} \tag{9-6}$$

We notice that $S_\varepsilon(r) = \varepsilon$ if $r \le \varepsilon$.

Fix $f \in \mathbb{V} \cap C_b(X)$ and assume that $\zeta : X \to [0, \infty)$ is a bounded upper-semicontinuous function, such that $\Gamma(f)^{1/2} \le \zeta$ $\mathfrak{m}$-a.e. in X. For any $h \ge 1$, $\varepsilon > 0$, $M > 0$, such that $M \ge \sup_{B(x_h, 3\varepsilon)} \zeta$, we introduce the following "localization" of f at $x_h \in X$ (as above, $(x_h)_{h \ge 1} \subset X$ is dense in X):

$$T_{h,\varepsilon,M}(f)(\,\cdot\,) := \frac{f(\,\cdot\,) - f(x_h)}{M} \wedge [S_\varepsilon \circ d(\,\cdot\,, x_h)] \vee [-S_\varepsilon \circ d(\,\cdot\,, x_h)].$$

The following properties are easy to check:

(a) $T_{h,\varepsilon,M}(f) \in \mathbb{V}$ is supported in $B(x_h, 3\varepsilon)$, and $T_{h,\varepsilon,M}(f)(x_h) = 0$.

(b) $\Gamma(T_{h,\varepsilon,M}(f))^{1/2} \le (\Gamma(f)^{1/2}/M) \le 1$ on $B(x_h, 3\varepsilon)$ and $\Gamma(T_{h,\varepsilon,M}(f)) = 0$ outside $B(x_h, 3\varepsilon)$; thus $T_{h,\varepsilon,M}(f)$ is 1-Lipschitz by condition (b) in Definition 9.1.

(c) Combining (a) and (b), we have $|T_{h,\varepsilon,M}(f)(x)| \le d(x_h, x)$ in X, thus $|T_{h,\varepsilon,M}(f)(x)| < \varepsilon$ in $B(x_h, \varepsilon)$, so that $T_{h,\varepsilon,M}(f) = (f - f(x_h))/M$ in $B(x_h, \varepsilon)$.

From (a) and (b) we obtain $T_{h,\varepsilon,M} f \in \mathcal{A}$, which together with (c) leads to the identity

$$df(\boldsymbol{b}) = M dT_{h,\varepsilon,M}(f)(\boldsymbol{b}), \quad \mathfrak{m}\text{-a.e. in } B(x_h, \varepsilon). \tag{9-7}$$

Indeed, for every $g \in \mathcal{A}$ and $a \in \mathbb{R}$, it holds that $dg(\boldsymbol{b}) = 0$ $\mathfrak{m}$-a.e. in the set $\{g = a\}$ as a consequence of (2-8), with $N = \{a\}$, and the upper bound $|dg(\boldsymbol{b})| \le |\boldsymbol{b}|\Gamma(g)^{1/2}$. In the situation that we are considering, take $g = f - MT_{h,\varepsilon,M} f$ and $a = f(x_h)$.

Let us assume that $T_{h,\varepsilon,M}(f) \in \mathcal{A}^*$, for every $h \ge 1$ and rational numbers ε, $M > 0$ such that $M \ge \sup_{B(x_h, 3\varepsilon)} \zeta$. We claim that

$$|df(\boldsymbol{b})| \le \zeta |\boldsymbol{b}|_* \quad \mathfrak{m}\text{-a.e. in } X. \tag{9-8}$$

Indeed, from (9-7), we deduce

$$|df(\boldsymbol{b})|(x) = M|dT_{h,\varepsilon,M}(f)(\boldsymbol{b})| \le M|\boldsymbol{b}|_*(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in B(x_h, 3\varepsilon).$$

We pass to the infimum upon M (which is rational and greater than $\sup_{B(x_h, 3\varepsilon)} \zeta$) and $h \ge 1$, then we let $\varepsilon \downarrow 0$, to obtain

$$|df(\boldsymbol{b})|(x) \le \limsup_{\varepsilon \downarrow 0} \inf_{h : d(x_h, x) < \varepsilon} \sup_{B(x_h, 3\varepsilon)} \zeta |\boldsymbol{b}|_*(x) \le \zeta(x) |\boldsymbol{b}|_*(x) \quad \text{for } \mathfrak{m}\text{-a.e. } x \in X,$$

where the second inequality holds by upper-semicontinuity of ζ.

Thanks to the curvature assumption, it is not difficult to show that the class of functions $f \in \mathbb{V} \cap C_b(X)$ that admit functions ζ as above is not empty: by [Ambrosio et al. 2014a, Theorem 3.17] the operator P_t maps $L^2 \cap L^\infty(\mathfrak{m})$ into $C_b(X)$ for every $t > 0$. In addition, if $f \in \mathbb{V}$, it holds

$$\Gamma(\mathsf{P}_t f) \le e^{-2Kt} \mathsf{P}_t(\Gamma(f)) \quad \mathfrak{m}\text{-a.e. in } X.$$

Thus, if $\Gamma(f) \in L^\infty(\mathfrak{m})$ we may let ζ^2 be the continuous version of the expression in the right-hand side above that we denote by $e^{-2Kt}\tilde{\mathsf{P}}_t(\Gamma(f))$; see also [Ambrosio et al. 2014a, Proposition 3.2].

We are in a position to prove that, up to enlarging $\mathcal{A}^*$, (9-5) holds. More precisely, we let $(f_n)_{n \ge 1} \subseteq \mathcal{A}$ be any countable family, with $\Gamma(f_n) \le 1$ $\mathfrak{m}$-a.e. in X, for $n \ge 1$, and such that the dilations $(\lambda f_n)_{\lambda \in \mathbb{R}, n \ge 1}$ provide a dense set in $\mathbb{V}$. We enlarge $\mathcal{A}^*$ with the union of all functions

$$T_{\varepsilon,h,M}(\mathsf{P}_t f_n),$$

for n, $h \ge 1$ and rational numbers t, ε, $M > 0$ such that $M^2 \ge \sup_{B(x_h,3\varepsilon)} e^{-2Kt}\tilde{\mathsf{P}}_t(\Gamma(f))$.

For every $n \ge 1$ and rational $t > 0$, (9-8) with $\mathsf{P}_t f_n$ in place of f and $\zeta = e^{-Kt}[\tilde{\mathsf{P}}_t(\Gamma(f))]^{1/2}$ gives

$$|d\mathsf{P}_t f_n(\boldsymbol{b})| \le e^{-Kt}[\tilde{\mathsf{P}}_t(\Gamma(f_n))]^{1/2} |\boldsymbol{b}|_* \quad \mathfrak{m}\text{-a.e. in } X.$$

We let $t \downarrow 0$ to deduce that

$$|df_n(\boldsymbol{b})| \le \Gamma(f_n)^{1/2} |\boldsymbol{b}|_* \quad \mathfrak{m}\text{-a.e. in } X.$$

By homogeneity, a similar inequality holds for λf_n in place of f_n, for every $\lambda \in \mathbb{R}$. To conclude, let $g \in \mathcal{A}$ and let $(g_k)_k \subseteq (\lambda f_n)_{\lambda \in \mathbb{R}, n \ge 1}$ converge towards g in $\mathbb{V}$. Then

$$|dg(\boldsymbol{b})| \le \liminf_{k \to \infty} \Gamma(g_k)^{1/2} |\boldsymbol{b}|_* + \Gamma(g_k - g)^{1/2} |\boldsymbol{b}| = \Gamma(g)^{1/2} |\boldsymbol{b}|_* \quad \mathfrak{m}\text{-a.e. in } X,$$

and we deduce that $|\boldsymbol{b}| \le |\boldsymbol{b}|_*$ $\mathfrak{m}$-a.e. in X. □

We discuss now the fine regularity properties of functions in $\mathbb{V}$, recalling some results developed in [Ambrosio et al. 2014b]. We start with the notion of 2-plan.

Definition 9.3 (2-plans). We say that a positive finite measure $\boldsymbol{\eta}$ in $\mathscr{P}(C([0, T]; X))$ is a 2-plan if $\boldsymbol{\eta}$ is concentrated on $\mathrm{AC}([0, T]; (X, d))$ and the following two properties hold:

(i) $\displaystyle\iint_0^T |\dot{\eta}|^2(t)\, dt\, d\boldsymbol{\eta}(\eta) < \infty$;

(ii) there exists $C \in [0, \infty)$ satisfying $(e_t)_\# \boldsymbol{\eta} \le C\mathfrak{m}$ for all $t \in [0, T]$.

Accordingly, we say that $V : X \to \mathbb{R}$ is $W^{1,2}$ *along* 2*-almost every curve* if, for all $s \le t$ and all 2-plans $\boldsymbol{\eta}$, the family of inequalities

$$\int |V(\eta(s)) - V(\eta(t))|\, d\boldsymbol{\eta}(\eta) \le \iint_s^t g(\eta(r))|\dot{\eta}(r)|\, dr\, d\boldsymbol{\eta}(\eta) \quad \text{for all } s,\ t \in [0, T) \text{ with } s \le t, \tag{9-9}$$

holds for some $g \in L^2(\mathfrak{m})$. Since Lipschitz functions with bounded support are dense in $\mathbb{V}$, a density argument [Ambrosio et al. 2014b, Theorem 5.14] based on (9-4) provides the following result.

Proposition 9.4. *Any $V \in \mathbb{V}$ is $W^{1,2}$ along 2-almost every curve. In addition,* (9-9) *holds with $g = \sqrt{\Gamma(V)}$.*

Actually, a much finer result could be established [Ambrosio et al. 2014b, §5], namely the existence of a representative $\tilde V$ of V in the $L^2(\mathfrak{m})$ equivalence class, with the property that $\tilde V \circ \eta$ is absolutely continuous for $\boldsymbol{\pi}$-a.e. η for any 2-plan $\boldsymbol{\pi}$, with $|(\tilde V \circ \eta)'| \le \sqrt{\Gamma(V)}|\dot\eta|$ a.e. in $(0, T)$. However, we shall not need this fact in the sequel. Here we notice only that since $\chi_B\boldsymbol{\eta}$ is a 2-plan for any Borel set $B \subset C([0, T]; X)$, it follows from (9-9) with $g = \sqrt{\Gamma(V)}$ that

$$|V(\eta(s)) - V(\eta(t))| \le \iint_s^t \sqrt{\Gamma(V)}(\eta(r))|\dot\eta(r)|\,dr \quad \text{for } \boldsymbol{\eta}\text{-a.e. } \eta \tag{9-10}$$

for all $s,\ t \in [0, T)$ with $s \le t$.

Now, we would like to relate these known facts to solutions to the ODE $\dot\eta = \boldsymbol{b}_t(\eta)$. The first connection between 2-plans and probability measures concentrated on solutions to the ODE is provided by the following proposition.

Proposition 9.5. *Let $\boldsymbol{b} = (\boldsymbol{b}_t)$ be a Borel family of derivations with $|\boldsymbol{b}| \in L^1_t(L^2)$ and let $u \in L^\infty_t(L^\infty_x)$. Let $\boldsymbol{\eta}$ be concentrated on solutions to the ODE $\dot\eta = \boldsymbol{b}_t(\eta)$, with $(e_t)_\#\boldsymbol{\eta} = u_t\mathfrak{m}$ for all $t \in (0, T)$. Then $\boldsymbol{\eta}$ is a 2-plan.*

Proof. The fact that $\boldsymbol{\eta}$ has bounded marginals follows from the assumption $u \in L^\infty_t(L^\infty_x)$. By Lemma 7.4 and the identification $d = d_{\mathcal{A}^*}$, $\boldsymbol{\eta}$ is concentrated on $\mathrm{AC}([0, T]; (X, d))$, with $|\dot\eta|(t) = |\boldsymbol{b}_t|(\eta(t))$, $\mathscr{L}^1$-a.e. in $(0, T)$ for $\boldsymbol{\eta}$-a.e. η. Thus,

$$\iint_0^T |\dot\eta|^2(t)\,dt\,d\boldsymbol{\eta}(\eta) = \int_0^T\!\!\int |\boldsymbol{b}_t|^2 u_t\,d\mathfrak{m}\,dt < \infty. \qquad \square$$

We now focus on the case of a "gradient" and time-independent derivation $\boldsymbol{b}_V$ associated to $V \in \mathbb{V}$. Recall that in this case $|\boldsymbol{b}_V|^2 = \Gamma(V)$ $\mathfrak{m}$-a.e. in X.

Theorem 9.6. *Let $V \in D(\Delta)$ with $\Delta V^- \in L^\infty(\mathfrak{m})$. Then, there exist weakly continuous solutions (in $[0, T)$, in duality with $\mathcal{A}$) $u \in L^\infty_t(L^1_x \cap L^\infty_x)$ to the continuity equation, for any initial condition $\bar u \in L^1 \cap L^\infty(\mathfrak{m})$. In addition, if $\boldsymbol{\eta}$ is given by Theorem 7.6 (namely $\boldsymbol{\eta}$ is concentrated on solutions to the ODE $\dot\eta = \boldsymbol{b}_V(\eta)$ and $(e_t)_\#\boldsymbol{\eta} = u_t$, for all $t \in (0, T)$), then:*

(a) *$\boldsymbol{\eta}$ is concentrated on curves η satisfying $|\dot\eta|(t) = \Gamma(V)^{1/2}(\eta(t))$, for a.e. $t \in (0, T)$;*

(b) *for all $s,\ t \in [0, T)$ with $s \le t$,*

$$V \circ \eta(t) - V \circ \eta(s) = \int_s^t \Gamma(V)(\eta(r))\,dr \quad \textit{for } \boldsymbol{\eta}\textit{-a.e. } \eta.$$

Proof. The proof of the first statement follows immediately by Theorem 4.3 with $r = \infty$. Since

$$\int_s^t\!\!\int \Gamma(V, f)u_r\,d\mathfrak{m}\,dr = \int f u_t - \int f u_s \quad \text{for all } s,\ t \in [0, T) \text{ with } s \le t$$

for all $f \in \mathcal{A}$, we can use the density of $\mathcal{A}$ in $\mathbb{V}$ and a simple limiting procedure to obtain

$$\int_s^t\!\!\int \Gamma(V)u_r\,d\mathfrak{m}\,dr = \int V u_t - \int V u_s \quad \text{for all } s,\ t \in [0, T) \text{ with } s \le t. \tag{9-11}$$

If $\boldsymbol{\eta}$ is as in the statement of the theorem, since $\boldsymbol{\eta}$ is a 2-plan we can combine Proposition 9.4 and the inequality $|\dot\eta| \le |\boldsymbol{b}_V|(\eta)$ stated in Lemma 7.4 to get

$$\int V(\eta(t)) - V(\eta(s))\, d\boldsymbol{\eta}(\eta) \le \iint_s^t \Gamma(V)^{1/2}(\eta(r))|\dot\eta|(r)\, dr\, d\boldsymbol{\eta}(\eta) \le \iint_s^t \Gamma(V)(\eta(r))\, d\boldsymbol{\eta}(\eta),$$

for all $s,\ t \in [0, T)$ with $s \le t$. Since $(e_r)_\# \boldsymbol{\eta} = u_r \mathfrak{m}$ for all $r \in [0, T)$, it follows that

$$\int V u_t - \int V u_s = \int V(\eta(t)) - V(\eta(s))\, d\boldsymbol{\eta}(\eta) \le \int_s^t \Gamma(V) u_r\, d\mathfrak{m}\, dr. \tag{9-12}$$

Combining (9-11) and (9-12) it follows that all the intermediate inequalities we integrated w.r.t. $\boldsymbol{\eta}$ are actually identities, so that, for $\boldsymbol{\eta}$-a.e. η, $|\dot\eta| = \sqrt{\Gamma(V)} \circ \eta$ a.e. in $(0, T)$, and equality holds in (9-10). $\square$

In particular, one could prove that $\boldsymbol{\eta}$ is a 2-plan representing the 2-weak gradient of V, according to [Gigli 2012, Definition 3.7], where a weaker asymptotic energy dissipation inequality was required at $t = 0$. Our global energy dissipation is stronger, but it requires additional bounds on the Laplacian.

We can also prove uniqueness for the continuity equation, considering just for simplicity still the autonomous version.

Theorem 9.7. *Let $V \in D(\Delta)$ with $\Delta V^- \in L^\infty(\mathfrak{m})$. Then the continuity equation induced by $\boldsymbol{b}_V$ has existence and uniqueness in $L^\infty_t(L^1_x \cap L^\infty_x)$ for any initial condition $\bar u \in L^1 \cap L^\infty(\mathfrak{m})$.*

Proof. We already discussed existence in Theorem 9.6. For uniqueness, we want to apply Theorem 5.4 with $q = 2$ and $r = s = 4$ (which provides uniqueness in the larger class $L^2 \cap L^4(\mathfrak{m})$). In order to do this we need only to know that (4-3) holds (this follows by conservativity of P and $\mathsf{BE}_2(\mathsf{K}, \infty)$), that L^4-Γ inequalities hold in $\mathsf{RCD}(\mathsf{K}, \infty)$ spaces (this follows by $\mathsf{BE}_2(\mathsf{K}, \infty)$, thanks to Corollary 6.3) and that the deformation of $\boldsymbol{b}_V$ is of type $(4, 4)$ (this follows by Theorem 6.7). $\square$

Acknowledgments

Ambrosio acknowledges the support of the ERC ADG GeMeThNES. Trevisan has been partially supported by PRIN10-11 grant from MIUR for the project Calculus of Variations. Both authors are members of the GNAMPA group of the Istituto Nazionale di Alta Matematica (INdAM).

The authors thank G. Savaré for pointing out to them, after reading a preliminary version of this work, the possibility to dispense from the full curvature assumption, an observation that led to the present organization of the paper. They also thank F. Ricci and G. Da Prato for useful discussions.

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Received 19 Feb 2014. Accepted 12 Jul 2014.

LUIGI AMBROSIO: luigi.ambrosio@sns.it
Classe di Scienze, Scuola Normale Superiore, I-56126 Pisa, Italy

DARIO TREVISAN: dario.trevisan@sns.it
Classe di Scienze, Scuola Normale Superiore, I-56126 Pisa, Italy

ANALYSIS AND PDE
Vol. 7, No. 5, 2014
dx.doi.org/10.2140/apde.2014.7.1235

ERRATUM TO POISSON STATISTICS FOR EIGENVALUES OF CONTINUUM RANDOM SCHRÖDINGER OPERATORS

JEAN-MICHEL COMBES, FRANÇOIS GERMINET AND ABEL KLEIN

Volume **3**:1 (2010), 49–80

The following results in the paper in question are not proved, due to a mistake in the derivation of inequality (5-8): Theorem 2.1, Theorem 2.2, Lemma 5.1.

All references are to [Combes et al. 2010]. The inequality stated in (5-8),

$$\mathbb{E}\big\{\big(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\big)\big(\operatorname{tr} P_{\boldsymbol{\omega}}^{(\Lambda)}(I)-1\big)\big\} \le Q_1 Q_2 \rho_+ |I| \sum_{j\in\widetilde{\Lambda}} \mathbb{E}_{\boldsymbol{\omega}_j^\perp}\big\{\Phi_{b,\tau}^{(\Lambda)}(\boldsymbol{\omega}_j^\perp)\big\},$$

is not correct. This inequality was derived by taking the expectation of both sides of (5-6) and estimating $\mathbb{E}\big\{\operatorname{tr}\big\{\sqrt{u_j}\, P_{\boldsymbol{\omega}}^{(\Lambda)}(I)\sqrt{u_j}\, S_j^{(\Lambda)}\big\}\big\}$ by the spectral-averaging estimate stated in (3-5). Unfortunately, this argument implicitly assumes $\Phi_{b,\tau}^{(\Lambda)}(\boldsymbol{\omega}_j^\perp) \ge 0$ (see (5-7)), which cannot be guaranteed. This error invalidates the proof of Lemma 5.1. As a consequence, for continuum Anderson Hamiltonians, a Minami estimate (Theorem 2.2) and Poisson statistics for eigenvalues (Theorem 2.1) remain conjectures.

The only results affected by the error in Lemma 5.1 are Theorems 2.1 and 2.2. Everything else in Theorem 2.3 is correct. (Note that the proof of that theorem works with a weaker form of the Minami estimate.) The Wegner estimates of Section 4, including Wegner estimates with constants that go to zero as either the energy approaches the bottom of the spectrum or the disorder goes to infinity, are correct. The new approach to Minami's estimate for the discrete Anderson model in Section 3 is correct.

Acknowledgement

The authors would like to thank Sasha Sodin for calling their attention to the error in (5-8).

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Klein was supported in part by NSF Grant DMS-0457474.
MSC2010: primary 82B44; secondary 47B80, 60H25.
Keywords: Minami estimate.

Received 17 Sep 2013. Accepted 1 Mar 2014.

JEAN-MICHEL COMBES: combes@cpt.univ-mrs.fr
Département de Mathématiques, Université de Toulon, 83130 La Garde, France

and

Centre de Physique Théorique, CNRS Luminy, 13288 Marseille, France

FRANÇOIS GERMINET: germinet@math.u-cergy.fr
Département of Mathématiques, Université de Cergy-Pontoise, 95000 Cergy-Pontoise, France

ABEL KLEIN: aklein@uci.edu
Department of Mathematics, University of California, Irvine, Irvine, CA 92697, United States

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