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SHARP CONSTANT FOR A k-PLANE TRANSFORM INEQUALITY

ALEXIS DROUOT

The k-plane transform $\mathcal{R}_k$ acting on test functions on $\mathbb{R}^d$ satisfies a dilation-invariant $L^p \to L^q$ inequality for some exponents p, q. We will make explicit some extremizers and the value of the best constant for any value of k and d, solving the endpoint case of a conjecture of Baernstein and Loss. This extends their own result for $k = 2$ and Christ's result for $k = d - 1$.

1. Introduction

Let us choose $d \geq 2$, $1 \leq k \leq d-1$ and denote by $\mathcal{G}_k$ the set of all k-planes in $\mathbb{R}^d$, meaning affine subspaces in $\mathbb{R}^d$ with dimension k. We define the k-plane transform of a continuous function with compact support $f : \mathbb{R}^d \to \mathbb{R}$ as

$$\mathcal{R}_k f(\Pi) = \int_{\Pi} f \, d\lambda_{\Pi},$$

where $\Pi \in \mathcal{G}_k$ and the measure λ_{Π} is the surface Lebesgue measure on Π. The operator $\mathcal{R}_k$ is known as the Radon transform for $k = d-1$ and as the X-ray transform for $k = 1$. It is known since the works of Oberlin and Stein [1982], Drury [1984] and Christ [1984] that $\mathcal{R}_k$ can be extended from $L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$ to $L^{d+1}(\mathcal{G}_k, \sigma_k)$ where σ_k is a measure defined as follows. Let us denote by $\mathcal{M}_k$ the submanifold of $\mathcal{G}_k$ of all k-planes containing 0. The Lebesgue measure on $\mathbb{R}^d$ induces a natural measure on $\mathcal{M}_k$: there exists a unique probability measure μ_k on $\mathcal{M}_k$ invariant in the sense that if Ω is an orthogonal map and P is a subset of $\mathcal{M}_k$, then $\mu_k(P) = \mu_k(\Omega P)$. The construction of this measure can be found in [Mattila 1995]. This induces a measure σ_k on $\mathcal{G}_k$ such that

$$\sigma_k(A) = \int_{\Pi \in \mathcal{M}_k} \lambda_{\Pi^\perp}(\{x \in \Pi^\perp : x + \Pi \in A\}) \, d\mu_k(\Pi), \tag{1-1}$$

where $\lambda_{\Pi^\perp}$ denotes the Lebesgue surface measure on the $(d-k)$-plane $\Pi^\perp$. Equation (1-1) defines a measure on $\mathcal{G}_k$ invariant under translations and rotations in the following sense: if Ω is an orthogonal map, P is a subset of $\mathcal{G}_k$, and $x \in \mathbb{R}^d$, then $\sigma_k(P) = \sigma_k(\Omega P + x)$.

The $L^{\frac{d+1}{k+1}}(\mathbb{R}^d)$ to $L^{d+1}(\mathcal{G}_k, \sigma_k)$-boundedness of $\mathcal{R}_k$ leads to the inequality

$$\|\mathcal{R}_k f\|_{L^{d+1}(\mathcal{G}_k, \sigma_k)} \leq A(k,d) \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{R}^d)} \tag{1-2}$$

for a certain constant $A(k,d)$, chosen to be optimal, that is,

$$A(k,d) = \sup\big\{\|\mathcal{R}_k f\|_{L^{d+1}(\mathcal{G}_k, \sigma_k)} : \|f\|_{L^{\frac{d+1}{k+1}}(\mathbb{R}^d)} = 1\big\}. \tag{1-3}$$

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Functions realizing the supremum in (1-3) are called extremizers of (1-2).

Here are some standard questions about this inequality:

(1) What is the best constant?

(2) What are the extremizers?

(3) Is any extremizing sequence relatively compact, modulo the group of symmetries?

(4) What can we say about functions satisfying $\|\mathcal{R}_k f\|_{d+1} \geq c\|f\|_{\frac{d+1}{k+1}}$?

Some of the answers are already known for some values of k. Baernstein and Loss [1997] solved the first question for the special case $k = 2$, and formulated a conjecture about the form of extremizers for a larger class of $L^p \to L^q$ inequalities. Christ solved their conjecture and answered all the above questions with the three papers [Christ 2011a; 2011b; 2011c] for the case $k = d - 1$.

By a quite different approach, we will give here a proof of Baernstein and Loss' conjecture *for any values of* k, d in the inequality (1-2). Note that this concerns only the endpoint case of their general conjecture. The value of the extremizers provides the explicit value of the best constant in (1-2). In a subsequent paper [Drouot 2013] we give a positive answer to the third question in the radial case, which is much easier than the general case.

Main result. Our main result is the following theorem:

Theorem 1.1. *The constant* $A(k, d)$ *in* (1-2) *is given by*

$$A(k,d) = \left[2^{k-d}\frac{|S^k|^d}{|S^d|^k}\right]^{\frac{1}{d+1}},$$

and some extremizers are given by

$$h(x) = \left[\frac{C}{1+|Lx|^2}\right]^{\frac{k+1}{2}}, \tag{1-4}$$

where L *is any invertible affine map on* $\mathbb{R}^d$, *and* C *is any positive constant.*

To find the best constant in the k-plane inequality (1-2) we will use the method of competing symmetries introduced in [Carlen and Loss 1990]. We will need the existence of an additional symmetry $\mathcal{S}$ of (1-2) that changes the level sets of functions — this could be seen as a problem but it actually gives very helpful information on the structure of the inequality. The choice of this symmetry is the generalization of a symmetry found in [Christ 2011c] in the special case of the Radon transform.

Nevertheless, the approach followed by Carlen and Loss led them to the values of *all* extremizers, using some additional work for the equality case in the rearrangement inequality. This does not work for us, and so we do not prove that the extremizers are unique modulo the invertible affine maps. However, we prove in Section 4 that if all extremizers are of the form $F \circ L$ with F radial and L an invertible map, then all extremizers are of the form (1-4). Using this result, Flock [2013] proved the following theorem:

Theorem 1.2. *All extremizers of* (1-2) *are of the form* (1-4).

For the rest of the paper, let us note the following:

- Let A and B be positive functions. We will say that $A \lesssim B$ when there exists a universal constant C, which depends only on the dimension d and on the integer k, such that $A \leq CB$. $A \gtrsim B$ means $B \lesssim A$, and $A \sim B$ will be used when $A \lesssim B$ and $B \lesssim A$.
- A radial function will be considered throughout the paper either as a function on $\mathbb{R}^d$ or as a function of the Euclidean norm, depending on the context.
- $|E|$ denotes the Lebesgue measure of a set E, except in the case of a sphere.
- $d(0, \Pi)$ denotes the Euclidean distance between 0 and a k-plane Π, that is,
$$d(0, \Pi) = \inf_{y \in \Pi} |y|.$$
- $|S^{m-1}|$ denotes the Lebesgue surface measure of the Euclidean sphere of $\mathbb{R}^m$.
- e_d is the vector $(0, \ldots, 0, 1)$.
- For a vector x in $\mathbb{R}^d$, we will write $x = (x', x'')$ with $x' \in \mathbb{R}^{d-1}$ and $x'' \in \mathbb{R}$.
- $\|f\|_p$ denotes the L^p-norm of f with respect to a contextual measure.
- $\mathbb{R}^+$ is the set $(0, \infty)$.

2. Preliminaries

In this section we introduce some standard notions which will be useful for what follows. We will talk about the theory of radial, nonincreasing rearrangements of a function and about the special form of the k-plane transform for radial functions.

Let us consider a measure μ on $\mathbb{R}^d$ and a measurable subset E of $\mathbb{R}^d$. E^* denotes the unique closed ball centered at the origin such that $\mu(E^*) = \mu(E)$. Now for a measurable function f from $\mathbb{R}^d$ to $[0, \infty]$, and $t \geq 0$, let us denote
$$E_f(t) = \{x \in \mathbb{R}^d : |f(x)| \geq t\}.$$

Then we have the following proposition:

Proposition 2.1. *Let f be a measurable function from $\mathbb{R}^d$ to $\mathbb{R} \cup \{\pm\infty\}$. There exists a unique function f^* from $\mathbb{R}^d$ to $[0, \infty]$ such that*
$$E_{|f|}(t)^* = E_{f^*}(t). \tag{2-1}$$
Moreover, f^* is radial, and nonincreasing as a function of the norm. Furthermore, for all nonnegative functions $g, h \in L^p$ with $1 \leq p \leq \infty$, we have:

(i) $\|g\|_p = \|g^*\|_p$,

(ii) $\|g^* - h^*\|_p \leq \|g - h\|_p$,

(iii) *if $g \leq h$, then $g^* \leq h^*$,*

(iv) *for all $\lambda \geq 0$, $\lambda g^* = (\lambda g)^*$.*

Points (i) to (iv) show that the nonlinear operator $f \mapsto f^*$ is actually a properly contractive operator (see Section 3). The map f^* is called the symmetric rearrangement of f (with respect to the measure μ).

We are now applying this theory to the k-plane transform. Christ [1984] proved that the k-plane transform satisfies the rearrangement inequality

$$\|\mathcal{R}g\|_q \leq \|\mathcal{R}(g^*)\|_q. \tag{2-2}$$

That way, we can look for extremizers in the class of radial, nonincreasing functions. It obviously makes the study much easier, passing from functions on $\mathbb{R}^d$ to nonincreasing functions on $[0, \infty)$.

The geometric origin of the k-plane transform leads us to introduce the operator $\mathcal{T}$ defined on continuous, compactly supported functions on $\mathbb{R}^+$ as

$$\mathcal{T} f(r) = \int_0^\infty f\big(\sqrt{s^2 + r^2}\big) s^{k-1}\, ds.$$

Then we have the following:

Lemma 2.2. *For all radial, continuous, compactly supported functions f on $\mathbb{R}^d$ and $\Pi \in \mathcal{G}$ such that $d(0, \Pi) = r$, we have*

$$\mathcal{R} f(\Pi) = |S^{k-1}| \cdot \mathcal{T} f(r). \tag{2-3}$$

For a proof, see, for instance, [Baernstein and Loss 1997]. The equation (2-3) shows that $\mathcal{T}$ is *almost* the k-plane transform. $\mathcal{T}$ acts on some Lebesgue spaces that we need to explicitly define, using the correspondence (2-3). Its domain is of course the space $L^p(\mathbb{R}^+, r^{d-1}\, dr)$. On the other hand, we have

$$\|\mathcal{R} f\|_q^q = \int_{\mathcal{G}} |\mathcal{R} f(\Pi)|^q \, d\sigma(\Pi) = |S^{k-1}|^q |S^{d-k-1}| \int_{r=0}^\infty |\mathcal{T} f(r)|^q r^{d-k-1}\, dr,$$

where the last line is obtained thanks to the formula (1.1) in [Baernstein and Loss 1997]. This shows that $\mathcal{T}$ maps $L^p(\mathbb{R}^+, r^{d-1} dr)$ to $L^q(\mathbb{R}^+, r^{d-k-1} dr)$.

3. Best constant and value of extremizers for the k-plane inequality

Here we want to prove the following:

Theorem 3.1. *An extremizer for the inequality* (1-2) *is given by*

$$f(x) = \left[\frac{1}{1 + |x|^2}\right]^{\frac{k+1}{2}}. \tag{3-1}$$

As a matter of fact, since any invertible affine map is a symmetry of the inequality (1-2), this theorem is equivalent to Theorem 1.1.

Let us explain the process of the proof before the details. Our purpose here is to introduce two operators $V, \mathcal{S}$ acting on L^p, such that V and $\mathcal{S}$ preserve the L^p-norm *and*

$$\|\mathcal{R} f\|_q = \|\mathcal{R}\mathcal{S} f\|_q, \quad \|\mathcal{R} f\|_q \leq \|\mathcal{R} V f\|_q. \tag{3-2}$$

This means that V and $\mathcal{S}$ globally increase the functional $f \mapsto \|\mathcal{R}f\|_q/\|f\|_p$. Now using additional properties of $\mathcal{S}$ and V, we will apply a theorem from [Carlen and Loss 1990] to show that for any choice of $f \in L^p$ with norm 1, the sequence $(V\mathcal{S})^n f$ converges to an explicit function h that does not depend on f. Using (3-2), h must be an extremizer, and h is explicitly known.

In practice, the operator V will be the symmetric rearrangement $f \mapsto f^*$, and $\mathcal{S}$ will be a symmetry of the inequality. The operator $\mathcal{S}$ is special in a certain sense: it does not preserve the class of radial functions. Thus, if we were able to construct an extremizer such that $\mathcal{S}h = h$ and $Vh = h$, the explicit value of h could be determined. A way to construct such an extremizer is described in the next section. But we can already note that an extremizer satisfying this condition must satisfy $(V\mathcal{S})^n h = h$ for all n; this way, considering the sequence $(V\mathcal{S})^n f$ is probably a good idea.

Competing operators. As we said, we are following the approach introduced in [Carlen and Loss 1990]. We might also refer to the book [Bianchini et al. 2011]. First, we sum up the general results stated Chapter II, §3.4 of this book: let $\mathcal{B}$ be a Banach space of real valued functions, with norm $\|\cdot\|$. Let $\mathcal{B}^+$ be the cone of nonnegative functions, and assume that $\mathcal{B}^+$ is closed. Let us introduce some definitions:

Definition 3.2. An operator A on $\mathcal{B}$ is called properly contractive provided that:

(i) A is norm-preserving on $\mathcal{B}^+$, i.e., $\|Af\| = \|f\|$ for all $f \in \mathcal{B}^+$.

(ii) A is contractive on $\mathcal{B}^+$, i.e., for all $f, g \in \mathcal{B}^+$, $\|Af - Ag\| \le \|f - g\|$.

(iii) A is order-preserving on $\mathcal{B}^+$, i.e., for all $f, g \in \mathcal{B}^+$, $f \le g \implies Af \le Ag$.

(iv) A is homogeneous of degree one on $\mathcal{B}^+$, i.e., for all $f \in \mathcal{B}^+, \lambda \ge 0$, $A(\lambda f) = \lambda Af$.

Note that we do not need A to be linear. Some examples of such operators are for instance the radial nonincreasing rearrangement $f \mapsto f^*$ or any linear isometry on $\mathcal{B}$.

Definition 3.3. Given a pair of properly contractive operators $\mathcal{S}$ and V, it is said that $\mathcal{S}$ competes with V if, for $f \in \mathcal{B}^+$,

$$f \in R(V) \cap \mathcal{S}R(V) \implies \mathcal{S}f = f.$$

Here R denotes the range.

Theorem 3.4. *Suppose that $\mathcal{S}$ and V are both properly contractive, that $V^2 = V$ and that $\mathcal{S}$ competes with V. Suppose further that there is a dense set $\widetilde{\mathcal{B}} \subset \mathcal{B}^+$ and sets K_N satisfying $\bigcup_N K_N = \widetilde{\mathcal{B}}$ and for all integers N, $\mathcal{S}K_N \subset K_N$, $VK_N \subset K_N$, and VK_N is relatively compact in $\mathcal{B}$. Finally, suppose that there exists a function $h \in \mathcal{B}^+$ with $\mathcal{S}h = Vh = h$ and such that, for all $f \in \mathcal{B}^+$,*

$$\|Vf - h\| = \|f - h\| \implies Vf = f. \tag{3-3}$$

Then, for any $f \in \mathcal{B}^+$,

$$Tf \equiv \lim_{n\to\infty} (V\mathcal{S})^n f$$

exists. Moreover, $\mathcal{S}T = T$ and $VT = T$.

An additional symmetry. Now we come back to the work of Christ. Using a correspondence between a convolution operator that he studied in [Christ 2011a; 2011b; 2012], and the Radon transform, he proved in [Christ 2011c] the existence of an additional symmetry for the Radon transform inequality. It is defined as

$$\mathcal{I} f(u,s) = \frac{1}{|s|^d} f\Big(\frac{u}{s}, \frac{1}{s}\Big).$$

It then satisfies $\|\mathcal{I} f\|_{\frac{d+1}{d}} = \|f\|_{\frac{d+1}{d}}$ and $\|\mathcal{R}_{d-1}\mathcal{I} f\|_{d+1} = \|\mathcal{R}_{d-1} f\|_{d+1}$. Fortunately, it happens that this symmetry, slightly modified, also works for the $L^p \to L^q$ inequality related to the k-plane transform.

Lemma 3.5. *Let $\mathcal{S}$ be the operator defined on L^p as*

$$\mathcal{S} f(u,s) = \frac{1}{|s|^{k+1}} f\Big(\frac{u}{s}, \frac{1}{s}\Big),$$

where $(u,s) \in \mathbb{R}^{d-1} \times (\mathbb{R} - \{0\})$. Then $\mathcal{S}$ is an isometry of L^p and satisfies the identity

$$\|\mathcal{R}\mathcal{S} f\|_q = \|\mathcal{R} f\|_q \tag{3-4}$$

for any nonnegative function f.

Proof. Let us check first that $\mathcal{S}$ is an isometry of L^p. Let us call

$$\Phi(x) = \left(\frac{x'}{x''}, \frac{1}{x''}\right)$$

for $x = (x', x'') \in \mathbb{R}^{d-1} \times (\mathbb{R} - \{0\})$. Then its Jacobian determinant is

$$J\Phi(x) = \frac{1}{|x''|^{d+1}},$$

which shows that $\|\mathcal{S} f\|_p = \|f\|_p$. Then we just have to prove (3-4). The proof is just calculation. Denote the unique k-plane containing the linearly independent points $x_0, \dots, x_k \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$ by $\Pi(x_0, \dots, x_k)$ and let $\widetilde{\mathcal{R}} f$ be

$$\widetilde{\mathcal{R}} f(x_0, \dots, x_k) = \int_{\mathbb{R}^k} f\big(x_0 + \lambda_1(x_1 - x_0) + \dots + \lambda_k(x_k - x_0)\big)\, d\lambda_1 \dots d\lambda_k.$$

Thus we have the correspondence

$$V(x_0, \dots, x_k) \cdot \widetilde{\mathcal{R}} f(x_0, \dots, x_k) = \mathcal{R} f(\Pi(x_0, \dots, x_k)), \tag{3-5}$$

where $V(x_0, \dots, x_k)$ is the volume of the k-simplex $(x_0, \dots, x_k)$.

Lemma 3.6. *For all $f \in C_0^\infty$, for all $x_0, \dots, x_k \in \mathbb{R}^d \times \dots \times \mathbb{R}^d$, linearly independent and such that $\Phi(x_0), \dots, \Phi(x_k)$ exist and are linearly independent,*

$$(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \dots, x_k) = \frac{(\widetilde{\mathcal{R}} f)(\Phi(x_0), \dots, \Phi(x_k))}{|x_0'' \cdots x_k''|}.$$

Proof. Let us call $\alpha = x_0'' + \lambda_1(x_1'' - x_0'') + \cdots + \lambda_k(x_k'' - x_0'')$ and $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$. Thus

$$(\widetilde{\mathscr{R}}\mathscr{S} f)(x_0, \ldots, x_k) = \int_{\mathbb{R}^k} \frac{1}{|\alpha|^{k+1}} f\left(\frac{x_0' + \lambda_1(x_1' - x_0') + \cdots + \lambda_k(x_k' - x_0') + e_d}{\alpha}\right) d\lambda. \tag{3-6}$$

Let us make the change of variables

$$\lambda_1' = \alpha^{-1}\lambda_1, \quad \ldots, \quad \lambda_{k-1}' = \alpha^{-1}\lambda_{k-1}, \quad \lambda_k' = \alpha^{-1}. \tag{3-7}$$

Then

$$d\lambda' = \frac{|x_k'' - x_0''|}{|\alpha|^{k+1}}\, d\lambda. \tag{3-8}$$

A proof of this formula is given in the Appendix. The equation (3-6) becomes

$$(\widetilde{\mathscr{R}}\mathscr{S} f)(x_0, \ldots, x_k) = \int_{\mathbb{R}^k} f\left(y_k + \lambda_k'(x_0' + e_d - x_0'' y_k) + \sum_{i=1}^{k-1} \lambda_i'(x_i' - x_0' - (x_i'' - x_0'')y_k)\right) \frac{d\lambda'}{|x_k'' - x_0''|},$$

where

$$y_i = \frac{x_i' - x_0'}{x_i'' - x_0''}.$$

This formula is somehow important: it shows that we are still integrating f over a k-plane. Which one? When we computed $\widetilde{\mathscr{R}}\mathscr{S} f(x_0, \ldots, x_k)$, we were interested only in the values of f on $\Phi(\Pi(x_0, \ldots, x_k))$. That way it is simple to guess that $\widetilde{\mathscr{R}}\mathscr{S} f(x_0, \ldots, x_k)$ is closely related to $\Pi(\Phi(x_0), \ldots, \Phi(x_k))$. And indeed, we just have to check that any of the points x_j can be written as

$$x_j = y_k + \lambda_k'(x_0' + e_d - y_k) + \sum_{i=1}^{k-1} \lambda_i'(x_i' - x_0' - (x_i'' - x_0'')y_k) \tag{3-9}$$

for a suitable choice of λ'. Indeed, taking $\lambda = e_j$ and λ' given by (3-7) for this choice of λ, we get the equality (3-9). Let us now make the other change of variables

$$\lambda_1' = \frac{\mu_1}{x_1'' - x_0''}, \quad \ldots, \quad \lambda_{k-1}' = \frac{\mu_{k-1}}{x_{k-1}'' - x_0''}, \quad \lambda_k' = \frac{\mu_k}{x_0''}.$$

We finally get

$$(\widetilde{\mathscr{R}}\mathscr{S} f)(x_0, \ldots, x_k) = \int_{\mathbb{R}^k} f\left(y_k' + \mu_k(\Phi(x_0) - y_k') + \sum_{i=1}^{k-1} \mu_i(y_i' - y_k')\right) \frac{d\mu}{|x_0''| \prod_{i=1}^{k-1} |x_i'' - x_0''|}.$$

Let us come back to Equation (3-5), the correspondence between $\mathscr{R}$ and $\widetilde{\mathscr{R}}$. We want to find a relation between $(\widetilde{\mathscr{R}}\mathscr{S} f)(x_0, \ldots, x_k)$ and $(\widetilde{\mathscr{R}} f)(\Phi(x_0), \ldots, \Phi(x_k))$. The above algebra tells us that this is equivalent to finding a relation between the two volumes

$$V(\Phi(x_0), y_1, \ldots, y_k) \quad \text{and} \quad V(\Phi(x_0), \Phi(x_1), \ldots, \Phi(x_k)).$$

Lemma 3.7. *$V(\Phi(x_0), y_1, \dots, y_k)$ and $V(\Phi(x_0), \Phi(x_1), \dots, \Phi(x_k))$ are related through*

$$\frac{V(\Phi(x_0), \Phi(x_1), \dots, \Phi(x_k))}{V(\Phi(x_0), y_1, \dots, y_k)} = \prod_{i=1}^{k} \left| \frac{x_0''}{x_i''} - 1 \right|.$$

Proof. A direct calculation shows

$$\frac{x_i''}{x_i'' - x_0''}[\Phi(x_i) - \Phi(x_0)] = \frac{x_0'' x_i' + x_0'' e_d - x_i'' x_0' - x_i'' e_d}{x_0''(x_i'' - x_0'')}$$

and on the other hand, by definition of y_i and $\Phi(x_0)$,

$$y_i - \Phi(x_0) = \frac{x_0'' x_i' + x_0'' e_d - x_i'' x_0' - x_i'' e_d}{x_0''(x_i'' - x_0'')},$$

which proves the equality

$$\Phi(x_i) - \Phi(x_0) = \left(1 - \frac{x_0''}{x_i''}\right)[y_i - \Phi(x_0)].$$

Thus, using that

$$V(\Phi(x_0), \Phi(x_1), \dots, \Phi(x_k)) = V(0, \Phi(x_1) - \Phi(x_0), \dots, \Phi(x_k) - \Phi(x_0)),$$

Lemma 3.7 is proved. ☐

Let us go back to the proof of Lemma 3.6. Using the correspondence described in (3-5) and the previous lemma, we finally get the equality

$$(\widetilde{\mathcal{R}}\mathcal{S} f)(x_0, \dots, x_k) = \frac{(\widetilde{\mathcal{R}} f)(\Phi(x_0), \dots, \Phi(x_k))}{|x_0'' \cdots x_k''|}.$$ ☐

At last, let us return to the proof of Lemma 3.5. Since the set of bad points $x_0, \dots, x_k$ (we mean points which do not satisfy the natural assumptions of Lemma 3.6) has null Lebesgue measure in $(\mathbb{R}^d)^{k+1}$, we do not consider them. Let us use Drury's formula [1984]:

$$\|\mathcal{R} f\|_q^q = \int_{(\mathbb{R}^d)^{k+1}} dx_0 \dots dx_k \, f(x_0) \cdots f(x_k) \cdot \widetilde{\mathcal{R}} f(x_0, \dots, x_k)^{d-k}. \tag{3-10}$$

Now all that remains to be done is an easy change of variable $z_i = \Phi(x_i)$. Indeed,

$$\begin{aligned}
\|\mathcal{R}\mathcal{S} f\|_q^q
&= \int_{(\mathbb{R}^d)^{k+1}} dx_0 \dots dx_k \frac{1}{|x_0''|^{k+1}} f(\Phi(x_0)) \cdots \frac{1}{|x_k''|^{k+1}} f(\Phi(x_k)) \cdot (\widetilde{\mathcal{R}}\mathcal{S} f(x_0, \dots, x_k))^{d-k} \\
&= \int_{(\mathbb{R}^d)^{k+1}} dx_0 \dots dx_k \frac{1}{|x_0''|^{d+1}} f(\Phi(x_0)) \cdots \frac{1}{|x_k''|^{d+1}} f(\Phi(x_k)) \cdot (\widetilde{\mathcal{R}} f(\Phi(x_0), \dots, \Phi(x_k)))^{d-k} \\
&= \int_{(\mathbb{R}^d)^{k+1}} dz_0 \dots dz_k \, f(z_0) \cdots f(z_k) \cdot \widetilde{\mathcal{R}} f(z_0, \dots, z_k)^{d-k} = \|\mathcal{R} f\|_q^q.
\end{aligned}$$

This completes the proof. ☐

It is a good time to prove a claim we made earlier: affine maps are symmetries.

Lemma 3.8. *Let $f \in L^p$ and L be an invertible affine map. Then*

$$\frac{\|\mathcal{R}(f \circ L)\|_q}{\|f \circ L\|_p} = \frac{\|\mathcal{R} f\|_q}{\|f\|_p}.$$

Proof. The proof is a direct consequence of the correspondence formula (3-5) and of Drury's formula (3-10). Indeed, let L be an invertible affine map; then

$$\widetilde{\mathcal{R}}(f \circ L)(x_0, \dots, x_k) = \widetilde{\mathcal{R}} f(Lx_0, \dots, Lx_k),$$

and with the change of variable $z_i = Lx_i$ in Drury's formula we get

$$\|\mathcal{R}(f \circ L)\|_q = |\det(L)|^{-\frac{1}{p}} \|\mathcal{R} f\|_q,$$

which ends the proof. □

Our goal is now to apply the general Theorem 3.4 about competing symmetries. The operator $\mathcal{S}$ and the rearrangement operator $V : f \mapsto f^*$ increase the L^q-norm of the k-plane transform, and preserve the norm of L^p-functions.

Proposition 3.9. *The operators V and $\mathcal{S}$ satisfy the assumptions of Theorem 3.4, with the Banach space $\mathcal{B} = L^p$.*

Proof. $\mathcal{S}$ and V are both properly contractive operators. Let us check that $\mathcal{S}$ competes with V: choose $f, g \in L^p$, radial, nonincreasing, such that $f = \mathcal{S} g$. Then

$$f(u,s) = \frac{1}{|s|^{k+1}} g\Big(\frac{u}{s}, \frac{1}{s}\Big), \tag{3-11}$$

and, specializing to $s = 1$, we get $f(u,1) = g(u,1)$. Since both f and g are radial, $f(x) = g(x)$ for all $|x| \geq 1$. Let us choose $s < 1$. Specializing (3-11) to $u = 0$, we get

$$f(0,s) = \frac{1}{|s|^{k+1}} g\Big(0, \frac{1}{s}\Big) = \frac{1}{|s|^{k+1}} f\Big(0, \frac{1}{s}\Big).$$

But

$$f\Big(0, \frac{1}{s}\Big) = |s|^{k+1} g(0,s),$$

which shows that $f(0,s) = g(0,s)$. Now again, since both f and g are radial, $f = g$ and $f = \mathcal{S} f$.

We now have to check that $\mathcal{S}$ and V satisfy the assumptions of Theorem 3.4. We follow the arguments of Carlen in [Bianchini et al. 2011]. Let us define

$$h(x) = \left[\frac{1}{1+|x|^2}\right]^{\frac{k+1}{2}}.$$

Then $\mathcal{S} h = h$, $Vh = h$, and so with

$$K_N = \{f \in L^p : 0 \leq f \leq Nh\}$$

it is straightforward to check that $VK_N \subset K_N$ and $\mathcal{S} K_N \subset K_N$. Moreover VK_N is a compact subset of L^p. Indeed, let us consider a sequence $f_n \in VK_N$. Then f_n is radial, nonincreasing, and since h lies in L^∞ the sequence f_n is bounded in L^∞. Thus, by Helly's principle, f_n admits a subsequence that converges almost everywhere. But since $0 \le f_n \le Nh$, the dominated convergence theorem shows that this subsequence also converges in L^p, which implies that VK_N is relatively compact. At last, $\widetilde{L}^p = \bigcup_N K_N$ is a dense subset of nonnegative elements of L^p (since nonnegative, continuous, compactly supported functions are dense in L^p).

The hardest part is to prove the assumption (3-3). Fortunately, since h is strictly nonincreasing, it has already been done in [Carlen and Loss 1990]. □

We now close this subsection with the final key lemma for the explicit value of extremizers:

Lemma 3.10. *Let $h \in L^p$ such that $Vh = \mathcal{S}h = h$. Then there exists a constant C such that*

$$h(x) = C\left[\frac{1}{1+|x|^2}\right]^{\frac{k+1}{2}}.$$

Proof. Since h satisfies $\mathcal{S}h = Vh = h$, then h is equal to its own rearrangement and so is defined on (at least) $\mathbb{R}^d - \{0\}$. Moreover, $\mathcal{S}h$ must be radial. This leads to

$$\mathcal{S}h\big(u, \sqrt{1+|u|^2}\big) = \left[\frac{1}{1+|u|^2}\right]^{\frac{k+1}{2}} h\left(\frac{u}{\sqrt{1+|u|^2}}, \frac{1}{\sqrt{1+|u|^2}}\right) = \left[\frac{1}{1+|u|^2}\right]^{\frac{k+1}{2}} h(e_d),$$

using that h is radial. But, since $h = \mathcal{S}h$ is also radial,

$$\mathcal{S}h\big(u, \sqrt{1+|u|^2}\big) = \mathcal{S}h\big(0, \sqrt{1+2|u|^2}\big) = h\big(0, \sqrt{1+2|u|^2}\big).$$

Thus, we get the equality

$$h(x) = h(0, |x|) = \left[\frac{2}{1+|x|^2}\right]^{\frac{k+1}{2}} h(e_d) \tag{3-12}$$

for all $x \in \mathbb{R}^d$ such that $|x| \ge 1$. For $|x| < 1$, the equality $\mathcal{S}h = h$ shows that (3-12) is also right, which proves the lemma. □

Proof of the main theorem. Now we have all the material that we need to prove Theorem 3.1. Let $f_0 \ge 0$ be any function with L^p-norm equal to 1. Let us define the limit

$$h_0 = Tf_0 = \lim_{n\to\infty} (V\mathcal{S})^n f_0.$$

Using that $\mathcal{R}$ is bounded from $L^p \to L^q$, and equations (2-2), (3-4),

$$\|\mathcal{R}h_0\|_q = \lim_{n\to\infty} \|\mathcal{R}(V\mathcal{S})^n f_0\|_q \ge \|\mathcal{R}f_0\|_q. \tag{3-13}$$

Moreover, by Theorem 3.4, $Vh_0 = \mathcal{S}h_0 = h_0$, so h satisfies the assumptions of Lemma 3.10. We then get

$$h_0(x) = h_0(e_d)\left[\frac{2}{1+|x|^2}\right]^{\frac{k+1}{2}}.$$

Because of normalization and positivity of f_0, $h_0(e_d)$ can take only one value. It then follows from (3-13) that h_0 maximizes the norms of $\mathcal{R} f_0$, and thus it is an extremizer.

Value of the best constant. Here we compute the value of the best constant. We use the correspondence (2-3) described in the previous section, and only think about $\mathcal{T}$ and its related measurable spaces instead of $\mathcal{R}$. Let h be the radial extremizer

$$h(r) = \left[\frac{1}{1+r^2}\right]^{\frac{k+1}{2}}.$$

A family of integrals will be useful to compute its L^p-norm and the L^q-norm of $\mathcal{T}h$. These integrals are defined as

$$\int_0^\infty \frac{t^m}{(1+t^2)^{\frac{n}{2}}}\,dt.$$

A calculation shows that

$$\int_0^\infty \frac{t^m}{(1+t^2)^{\frac{n}{2}}}\,dt = \frac{\Gamma\big(\frac{1}{2}(m+1)\big)\Gamma\big(\frac{1}{2}(n-m-1)\big)}{\Gamma\big(\frac{1}{2}n\big)},$$

where Γ is the standard Euler gamma function. Then

$$\|h\|_p^p = \int_0^\infty \frac{r^{d-1}\,dr}{(1+r^2)^{\frac{d+1}{2}}} = \frac{\Gamma\big(\frac{1}{2}d\big)\Gamma\big(\frac{1}{2}\big)}{2\Gamma\big(\frac{1}{2}(d+1)\big)}.$$

Moreover,

$$\mathcal{T}h(r) = \frac{1}{\sqrt{1+r^2}}\int_0^\infty \frac{u^{k-1}\,du}{(1+|u|^2)^{\frac{k+1}{2}}},$$

and this leads to

$$\|\mathcal{T}h\|_q^q = \left(\frac{\Gamma(k-1)\Gamma(k+1)}{\Gamma(2k)}\right)^{d+1}\frac{\Gamma(d-k-1)\Gamma(d+1)}{\Gamma(2d-k)}.$$

The use of the fundamental relation

$$\tfrac{1}{2}|S^{n-1}|\Gamma\big(\tfrac{1}{2}n\big) = \pi^{\frac{n}{2}}$$

leads to

$$A(k,d) = \frac{\|\mathcal{R}h\|_q}{\|h\|_p} = \pi^{\frac{d-k}{2(d+1)}}\cdot\Gamma\big(\tfrac{1}{2}(d+1)\big)^{\frac{k}{d+1}}\cdot\Gamma\big(\tfrac{1}{2}(k+1)\big)^{-\frac{d}{d+1}} = \left[2^{k-d}\frac{|S^k|^d}{|S^d|^k}\right]^{\frac{1}{d+1}}.$$

4. The question of uniqueness

We shall discuss here the question of the uniqueness of extremizers of (1-2). For the sake of simplicity, we will assume $d \geq 3$. This is not a restricting assumption: indeed, for the case $d = 2$, the only k-plane transform is the Radon transform, and this has been thoroughly studied in [Christ 2011c].

The uniqueness problem for the Radon transform was solved in the same reference. The main tool for the proof is the following:

Theorem 4.1. *Let $k = d - 1$, and let f be a nonnegative extremizer. Then there exist a radial, nonincreasing, nonnegative extremizer F and an invertible affine map L such that $f = F \circ L$.*

Then it turned out that the work was almost all done. Christ characterized all the extremizers using the uniqueness Theorem 4.1 two times, in a certain sense. His approach is very interesting because the question of uniqueness is curiously intertwined with the question of existence. Here we want to develop a different approach, for an arbitrary $1 \le k \le d-1$, *assuming* that a result similar to Theorem 4.1 is true. More accurately, we want to prove the following:

Theorem 4.2. *Let $1 \le k \le d-1$. Assume that any extremizer for the k-plane transform inequality* (1-2) *can be written $F \circ L$ with F a radial, nonincreasing extremizer and L an affine map. Then any nonincreasing radial extremizer is of the form*

$$x \mapsto \left(\frac{1}{a+b|x|}\right)^{\frac{k+1}{2}}. \tag{4-1}$$

As we mentioned in the introduction, the *ad hoc* assumption in this theorem was proved to be true by Flock [2013], inducing the complete characterization of extremizers.

One of the main tools here will be the use of the symmetry $\mathcal{S}$ combined with the fact that an extremizer is a radial function composed with an affine map. Thus we will use again the competing symmetry theory. From now we will assume that k is such that any extremizer for (1-2) can be written $f \circ L$ with f radial and L an affine map. Our main lemma follows; it shows that radial extremizers enjoy additional symmetries.

Lemma 4.3. *Let f be a radial, nonincreasing extremizer for* (1-2). *Then there exists a real number $\mu > 0$ such that*

$$(V\mathcal{S})^2 f(r) = \mu^{\frac{d}{p}} f(\mu r).$$

Proof. Since f is a radial, nonincreasing extremizer, then f is not the (almost everywhere) null function: there exists $\lambda_0 > 0$ such that $f(\lambda_0 e_d) \neq 0$. Because of dilation-invariance, we can assume $\lambda_0 = 1$.

$\mathcal{S} f$ is also an extremizer. It follows that there exist $F : \mathbb{R}^+ \to \mathbb{R}$, nonincreasing, a linear invertible map L and a vector $x_0 \in \mathbb{R}^d$ such that

$$\mathcal{S} f(x) = F(|x_0 + Lx|). \tag{4-2}$$

Computing $\mathcal{S} f\big(u, \sqrt{|u|^2+1}\big)$, we get

$$f(e_d)\left[\frac{1}{1+|u|^2}\right]^{\frac{k+1}{2}} = F\big(|x_0 + Lu + \sqrt{1+|u|^2}Le_d|\big) \tag{4-3}$$

for all $u \in \mathbb{R}^{d-1} \times \{0\}$. Let $C = f(e_d) \neq 0$, and $I \subset \mathbb{R}^+$ the interval made of points that can be written $|x_0 + Lu + \sqrt{1+|u|^2}Le_d|$ for some $u \in \mathbb{R}^{d-1} \times \{0\}$. We claim that the map F is strictly decreasing on I. Indeed, let us assume that there exists $0 < \alpha < \beta$ such that F is constant on $[\alpha, \beta]$. Pick $u \in \mathbb{R}^{d-1} \times \{0\}$ such that $|x_0 + Lu + \sqrt{1+|u|^2}Le_d| \in (\alpha, \beta)$. For t close to 1, $|x_0 + Ltu + \sqrt{1+t^2|u|^2}Le_d| \in (\alpha, \beta)$, and thus for t close to 1 the map

$$t \mapsto F\big(|x_0 + Lu + \sqrt{1+|u|^2}Le_d|\big)$$

is constant. Because of (4-3), this is a contradiction.

The function F is then injective on I. Formula (4-3) shows that $|x_0 + Lu + \sqrt{1+|u|^2}Le_d|$ must be a function of $|u|^2$ only. To conclude the proof, we require the following lemma:

Lemma 4.4. *Let L be an invertible linear map such that $|x_0 + Lu + \sqrt{1+|u|^2}Le_d|$ depends only on $|u|$. Then $L(\mathbb{R}^{d-1} \times \{0\}) \subset (\operatorname{span}(Le_d))^\perp$, and $L|_{\mathbb{R}^{d-1}\times\{0\}}$ preserves the norm, modulo a multiplicative constant. Moreover, there exists $s_0 \in \mathbb{R}^d$ such that $x_0 = s_0 Le_d$.*

Proof. Let us choose $u = r\theta \in S^{d-2} \times \{0\}$. Then

$$\big|x_0 + rL\theta + \sqrt{1+r^2}Le_d\big|^2 = r^2|L\theta|^2 + \big|\sqrt{1+r^2}Le_d + x_0\big|^2 + 2r\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle$$

depends only on r, and so does $r^2|L\theta|^2 + 2r\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle$. As a consequence, $|L\theta|$ is a constant and $\langle L\theta, \sqrt{2}Le_d + x_0\rangle$ is a constant. Here we must assume $d \geq 3$, so the sphere S^{d-2} contains an infinity of points.

The condition that $|L\theta|$ is constant holds only if $L|_{\mathbb{R}^{d-1}\times\{0\}}$ preserves the norm, modulo a multiplicative constant. Thus the quantity

$$\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle$$

must depend only on r. Specializing at θ and $-\theta$, for all r, $\langle L\theta, \sqrt{1+r^2}Le_d + x_0\rangle = 0$. But since L is invertible, the space spanned by $L\theta$ has dimension $d-1$. Thus the space spanned by the vectors $\sqrt{1+r^2}Le_d + x_0$ for $r \geq 0$ has dimension 1, which proves that there exists s_0 such that $s_0 Le_d = x_0$. □

Composing with an isometry, we can assume that $L(\mathbb{R}^{d-1} \times \{0\}) \subset \mathbb{R}^{d-1} \times \{0\}$. Moreover, $|Lu|$ depends only on $|u|$, which implies that L restricted to $\mathbb{R}^{d-1} \times \{0\}$ must be a multiple of an isometry. We then deduce that there exist $a > 0, b > 0, s_0$ such that $|L(u + se_d) + x_0|^2 = a^2|u|^2 + b^2(s+s_0)^2$, *for all* $(u, s) \in \mathbb{R}^{d-1} \times \mathbb{R}$. Thus we get the fundamental relation between f and F:

$$\mathcal{S}f(u + se_d) = F\big(\sqrt{a^2|u|^2 + b^2(s+s_0)^2}\big).$$

Now, changing F to $G = F(\sqrt{ab}\,\cdot)$, G remains nonincreasing, and we get

$$\mathcal{S}f(u + se_d) = F\big(\sqrt{a^2|u|^2 + b^2(s+s_0)^2}\big) = G\left(\sqrt{\frac{a}{b}|u|^2 + \frac{b}{a}(s+s_0)^2}\right),$$

reducing the number of unknown parameters in our system. Thus, we have accomplished the first step in our identification program: we know how the operator $\mathcal{S}$ acts on radial extremizers. Now we have to understand how V acts on functions g whose form is

$$g : u + se_d \mapsto G\left(\sqrt{c|u|^2 + \frac{1}{c}(s+s_0)^2}\right).$$

First, we can assume that $s_0 = 0$: indeed, $g(\cdot - s_0 e_d)^* = g^*$. Moreover, G is decreasing and so the level sets of g are ellipsoids $c|u|^2 + c^{-1}s^2 \leq R^2$. The corresponding rearranged sets are balls of radius R', with R' satisfying the relation

$$R'^d = \frac{R^{d-1}}{c^{\frac{d-1}{2}}} c^{\frac{1}{2}} R = \frac{R^d}{c^{\frac{d-2}{2}}}.$$

Thus

$$Vg(se_d) = G(c^{\frac{d-2}{2d}} s) = \frac{1}{(c^{\frac{d-1}{d}} s)^{k+1}} f\left(\frac{e_d}{c^{\frac{d-1}{d}} s}\right),$$

coming back to the relation defining G, and using that f is radial. And then

$$Vg(se_d) = \frac{1}{(c^{\frac{d-1}{d}} s)^{k+1}} f\left(\frac{e_d}{c^{\frac{d-1}{d}} s}\right).$$

This characterizes the action of the operator $V\mathcal{S}$ on radial extremizers. More simply, calling $\lambda = c^{\frac{d-1}{d}}$, we have

$$V\mathcal{S} f(x) = \frac{1}{\lambda^{k+1}|x|^{k+1}} f\left(\frac{e_d}{\lambda|x|}\right).$$

Let us use again the competing symmetry theory: to construct an explicit extremizer of (1-2) we used iterations of $V\mathcal{S}$, applied to *any* function. Let us choose f_0 a radial extremizer. Then $V\mathcal{S} f_0$ is still a radial extremizer, and we know that there exists λ such that

$$V\mathcal{S} f_0(r) = \left(\frac{1}{\lambda r}\right)^{k+1} f_0\left(\frac{1}{\lambda r}\right).$$

Let us do that again: there exists λ' such that

$$(V\mathcal{S})^2 f_0(r) = \left(\frac{1}{\lambda' r}\right)^{k+1} (V\mathcal{S} f_0)\left(\frac{1}{\lambda' r}\right) = \left(\frac{1}{\lambda' r}\frac{\lambda' r}{\lambda}\right)^{k+1} f_0\left(\frac{\lambda' r}{\lambda}\right) = \frac{1}{\lambda^{k+1}} f_0\left(\frac{\lambda' r}{\lambda}\right).$$

Since the operator $V\mathcal{S}$ preserves the norm, we must have $\lambda\lambda'^d = 1$. Using the parameter μ such that $\lambda' = \mu\lambda$, we conclude the proof of Lemma 4.3. □

That proves that the operator $V\mathcal{S}$ acts on radial, nonincreasing extremizers as a dilation. Now let us consider $f_n = (V\mathcal{S})^{2n} f_0$. For each n, there exists μ_n such that

$$(V\mathcal{S})^{2n} f_0(r) = (\mu_n)^{\frac{d}{p}} f_0(\mu_n r).$$

But the sequence f_n converges in L^p to the extremizer h described in Theorem 3.1. Thus it converges weakly to a nonzero function, which is possible if and only if μ_n converges to a nonzero value. That ends the proof of Theorem 4.2: every nonnegative radial extremizer can be written

$$x \mapsto \left[\frac{1}{a + b|x|^2}\right]^{\frac{k+1}{2}}$$

with $a, b > 0$.

Appendix

Here we prove the jacobian formula (3-8). Define

$$\Psi(\lambda_1,\dots\lambda_k)=(\alpha^{-1}\lambda_1,\dots,\alpha^{-1}\lambda_{k-1},\lambda_k).$$

We want to compute $J\psi(\lambda)=|\det(\nabla\psi)(\lambda_1,\dots,\lambda_k)|$. Note first that

$$\frac{\partial\alpha^{-1}}{\partial\lambda_i}=\alpha^{-2}(x_i''-x_0'').$$

Thus

$$J\psi(\lambda)=\begin{vmatrix}-\alpha^{-2}(x_1''-x_0'')\lambda_1+\alpha^{-1} & \dots & -\alpha^{-2}(x_1''-x_0'')\lambda_{k-1} & -\alpha^{-2}(x_1''-x_0'')\\ \vdots & \ddots & \vdots & \vdots\\ -\alpha^{-2}(x_{k-1}''-x_0'')\lambda_1 & \dots & -\alpha^{-2}(x_{k-1}''-x_0'')\lambda_{k-1}+\alpha^{-1} & -\alpha^{-2}(x_{k-1}''-x_0'')\\ -\alpha^{-2}(x_k''-x_0'')\lambda_1 & \dots & -\alpha^{-2}(x_k''-x_0'')\lambda_{k-1} & -\alpha^{-2}(x_k''-x_0'')\end{vmatrix}$$

$$=|\alpha|^{-k-1}|x_k''-x_0''|\begin{vmatrix}y_1\lambda_1+1 & \dots & y_1\lambda_{k-1} & y_1\\ \vdots & \ddots & \vdots & \vdots\\ y_{k-1}\lambda_1 & \dots & y_{k-1}\lambda_{k-1}+1 & y_{k-1}\\ \lambda_1 & \dots & \lambda_{k-1} & 1\end{vmatrix},$$

where $y_i=-\alpha^{-1}(x_i''-x_0'')$. We claim that the determinant appearing in the last line is always equal to 1. Indeed, consider the polynomial

$$P(z)=\det\begin{pmatrix}y_1\lambda_1+1 & \dots & y_1\lambda_{k-1} & y_1\\ \vdots & \ddots & \vdots & \vdots\\ y_{k-1}\lambda_1 & \dots & y_{k-1}\lambda_{k-1}+1 & y_{k-1}\\ \lambda_1 & \dots & \lambda_{k-1} & z\end{pmatrix}.$$

It is of degree 1 in z. Moreover, we have

$$P'(1)=\det\begin{pmatrix}y_1\lambda_1+1 & \dots & y_1\lambda_{k-1}\\ \vdots & \ddots & \vdots\\ y_{k-1}\lambda_1 & \dots & y_{k-1}\lambda_{k-1}+1\end{pmatrix}=1+\langle y,\lambda\rangle, \tag{A-1}$$

$$P(2)=2+\langle y,\lambda\rangle. \tag{A-2}$$

Here $\langle y,\lambda\rangle=\sum_{i=1}^{k-1}\lambda_i y_i$. The formulas (A-1), (A-2) both come from the following lemma:

Lemma A.1. *If $u,v\in\mathbb{R}^p$, then*

$$\det(\mathbb{1}+u^tv)=1+\langle u,v\rangle.$$

Proof. The matrix u^tv is of rank one. As a consequence, its only eigenvalue is its trace $\langle u,v\rangle$. The characteristic polynomial of $-u^tv$ is then

$$\det(z\mathbb{1}+u^tv)=z^{p-1}(z+\langle u,v\rangle).$$

Evaluating this at $z=1$ proves the lemma. ☐

Applying this lemma to $u=\lambda$, $v=y$ leads to (A-1), and $u=(\lambda,1)$, $v=(y,1)$ leads to (A-2). Thus

$$P(z)=(1+\langle\lambda,y\rangle)z-\langle\lambda,y\rangle.$$

Evaluate this at $z=1$ to get the asserted claim, and then

$$J\psi(\lambda)=|\alpha|^{-k-1}|x_k''-x_0''|.$$

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References

[Baernstein and Loss 1997] A. Baernstein, II and M. Loss, "Some conjectures about L^p norms of k-plane transforms", *Rend. Sem. Mat. Fis. Milano* **67** (1997), 9–26. MR 2001h:44005 Zbl 1016.44002

[Bianchini et al. 2011] S. Bianchini, E. A. Carlen, A. Mielke, and C. Villani, *Nonlinear PDE's and applications* (Cetraro, Italy, 2008), Lecture Notes in Mathematics **2028**, Springer, Heidelberg, 2011. MR 2012h:49002

[Carlen and Loss 1990] E. A. Carlen and M. Loss, "Extremals of functionals with competing symmetries", *J. Funct. Anal.* **88**:2 (1990), 437–456. MR 91f:42021 Zbl 0705.46016

[Christ 1984] M. Christ, "Estimates for the k-plane transform", *Indiana Univ. Math. J.* **33**:6 (1984), 891–910. MR 86k:44004 Zbl 0597.44003

[Christ 2011a] M. Christ, "Quasiextremals for a Radon-like transform", preprint, 2011. arXiv 1106.0722

[Christ 2011b] M. Christ, "On extremals for a Radon-like transform", preprint, 2011. arXiv 1106.0728

[Christ 2011c] M. Christ, "Extremizers for a Radon transform inequality", preprint, 2011. arXiv 1106.0719

[Christ and Xue 2012] M. Christ and Q. Xue, "Smoothness of extremizers of a convolution inequality", *J. Math. Pures Appl.* (9) **97**:2 (2012), 120–141. MR 2875294 Zbl 1234.42010

[Drouot 2013] A. Drouot, "Precompactness of radial extremizing sequences for a k-plane transform inequality", preprint, 2013. arXiv 1205.3251

[Drury 1984] S. W. Drury, "Generalizations of Riesz potentials and L^p estimates for certain k-plane transforms", *Illinois J. Math.* **28**:3 (1984), 495–512. MR 85h:44004 Zbl 0552.43005

[Flock 2013] T. Flock, "Uniqueness of extremizers for an endpoint inequality of the k-plane transform", preprint, 2013. arXiv 1307.6551

[Mattila 1995] P. Mattila, *Geometry of sets and measures in Euclidean spaces: Fractals and rectifiability*, Cambridge Studies in Advanced Mathematics **44**, Cambridge University Press, 1995. MR 96h:28006 Zbl 0819.28004

[Oberlin and Stein 1982] D. M. Oberlin and E. M. Stein, "Mapping properties of the Radon transform", *Indiana Univ. Math. J.* **31**:5 (1982), 641–650. MR 84a:44002 Zbl 0548.44003

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ALEXIS DROUOT: alexis.drouot@gmail.com
Department of Mathematics, UC Berkeley, Berkeley, 94704, United States

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