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PARABOLIC BOUNDARY HARNACK PRINCIPLES IN DOMAINS WITH THIN LIPSCHITZ COMPLEMENT

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We prove forward and backward parabolic boundary Harnack principles for nonnegative solutions of the heat equation in the complements of thin parabolic Lipschitz sets given as subgraphs

$$E = \{(x,t) : x_{n-1} \le f(x'',t),\ x_n = 0\} \subset \mathbb{R}^{n-1}\times\mathbb{R}$$

for parabolically Lipschitz functions f on $\mathbb{R}^{n-2}\times\mathbb{R}$.

We are motivated by applications to parabolic free boundary problems with thin (i.e., codimension-two) free boundaries. In particular, at the end of the paper we show how to prove the spatial $C^{1,\alpha}$-regularity of the free boundary in the parabolic Signorini problem.

1. Introduction

The purpose of this paper is to study forward and backward boundary Harnack principles for nonnegative solutions of the heat equation in certain domains in $\mathbb{R}^n\times\mathbb{R}$ which are, roughly speaking, complements of thin parabolically Lipschitz sets E. By the latter, we understand closed sets lying in the vertical hyperplane $\{x_n = 0\}$ which are locally given as subgraphs of parabolically Lipschitz functions (see Figure 1).

Such sets appear naturally in free boundary problems governed by parabolic equations, where the free boundary lies in a given hypersurface and thus has codimension two. Such free boundaries are also known as thin free boundaries. In particular, our study was motivated by the parabolic Signorini problem, recently studied in [Danielli et al. 2013].

The boundary Harnack principles that we prove in this paper provide important technical tools in problems with thin free boundaries. For instance, they open up the possibility of proving that the thin Lipschitz free boundaries have Hölder-continuous spatial normals, following the original idea in [Athanasopoulos and Caffarelli 1985]. In particular, we show that this argument can indeed be successfully carried out in the parabolic Signorini problem.

We have to point out that the elliptic counterparts of the results in this paper are very well known; see e.g. [Athanasopoulos and Caffarelli 1985; Caffarelli et al. 2008; Aikawa et al. 2003]. However, there are significant differences between the elliptic and parabolic boundary Harnack principles, mostly because of the time-lag in the parabolic Harnack inequality. This results in two types of boundary Harnack principles for parabolic equations: the forward one (also known as the Carleson estimate) and the backward one.

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Moreover, those results are known only for a much smaller class of domains than in the elliptic case. Thus, to put our results in a better perspective, we start with a discussion of the known results both in the elliptic and parabolic cases.

Elliptic boundary Harnack principle. The by-now classical boundary Harnack principle for harmonic functions [Kemper 1972a; Dahlberg 1977; Wu 1978] says that if D is a bounded Lipschitz domain in $\mathbb{R}^n$, $x_0 \in \partial D$, and u and v are positive harmonic functions on D vanishing on $B_r(x_0) \cap \partial D$ for a small $r > 0$, then there exist positive constants M and C, depending only on the dimension n and the Lipschitz constant of D, such that

$$\frac{u(x)}{v(x)} \le C \frac{u(y)}{v(y)} \quad \text{for } x, y \in B_{r/M}(x_0) \cap D.$$

Note that this result is scale-invariant, and hence, by a standard iterative argument, one then immediately obtains that the ratio u/v extends to $\overline{D} \cap B_{r/M}(x_0)$ as a Hölder-continuous function. Roughly speaking, this theorem says that two positive harmonic functions vanishing continuously on a certain part of the boundary will decay at the same rate near that part of the boundary.

This boundary Harnack principle depends heavily on the geometric structure of the domains. The scale-invariant boundary Harnack principle (among other classical theorems of real analysis) was extended in [Jerison and Kenig 1982] from Lipschitz domains to the so-called NTA (nontangentially accessible) domains. Moreover, if the Euclidean metric is replaced by the internal metric, then similar results hold for so-called uniform John domains [Aikawa et al. 2003; Aikawa 2005].

In particular, the boundary Harnack principle is known for domains of the type

$$D = B_1 \setminus E_f, \quad E_f = \{x \in \mathbb{R}^n : x_{n-1} \le f(x''), x_n = 0\},$$

where f is a Lipschitz function on $\mathbb{R}^{n-2}$ with $f(0) = 0$; it is used, for instance, in the thin obstacle problem [Athanasopoulos and Caffarelli 1985; Athanasopoulos et al. 2008; Caffarelli et al. 2008]. In fact, there is a relatively simple proof of the boundary Harnack principle for domains as above already indicated in [Athanasopoulos and Caffarelli 1985]: there exists a bi-Lipschitz transformation from D to a half-ball B_1^+, which is a Lipschitz domain. The harmonic functions in D transform to solutions of a uniformly elliptic equation in divergence form with bounded measurable coefficients in B_1^+, for which the boundary Harnack principle is known [Caffarelli et al. 1981].

Parabolic boundary Harnack principle. The parabolic version of the boundary Harnack principle is much more challenging than the elliptic one, mainly because of the time-lag issue in the parabolic Harnack inequality. The latter is called sometimes the forward Harnack inequality, to emphasize the way it works: for nonnegative caloric functions (solutions of the heat equation), if the earlier value is positive at some spatial point, after a necessary waiting time, one can expect that the value will become positive everywhere in a compact set containing that point. Under the condition that the caloric function vanishes on the lateral boundary of the domain, one may overcome the time-lag issue and get a backward-type Harnack principle (so, combining the two together, one gets an elliptic-type Harnack inequality).

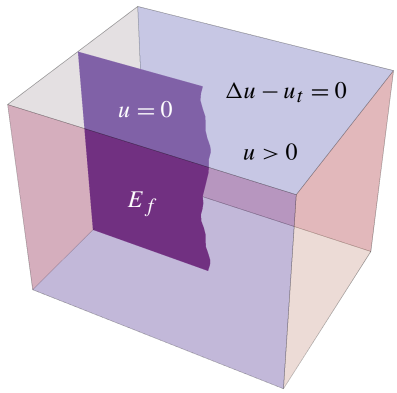

Figure 1. Domain with a thin Lipschitz complement.

The forward and backward boundary Harnack principle are known for parabolic Lipschitz domains, not necessarily cylindrical; see [Kemper 1972b; Fabes et al. 1984; Salsa 1981]. Moreover, they were shown more recently in [Hofmann et al. 2004] to hold for unbounded parabolically Reifenberg-flat domains. In this paper, we will generalize the parabolic boundary Harnack principle to the domains of the type (see Figure 1)

$$D = \Psi_1 \setminus E_f,$$

where

$$\Psi_1 = \{(x,t) : |x_i| < 1,\, i = 1, \dots, n-2,\, |x_{n-1}| < 4nL,\, |x_n| < 1,\, |t| < 1\},$$
$$E_f = \{(x,t) : x_{n-1} \le f(x'', t),\, x_n = 0\},$$

and $f(x'', t)$ is a parabolically Lipschitz function satisfying

$$|f(x'',t) - f(y'',s)| \le L(|x'' - y''|^2 + |t-s|)^{1/2}, \qquad f(0,0) = 0.$$

Note that D is not cylindrical (E_f is not time-invariant), and it does not fall into any category of domains on which the forward or backward Harnack principle is known. Inspired by the elliptic inner NTA domains (see e.g. [Athanasopoulos et al. 2008]), it seems natural to equip the domain D with the intrinsic geodesic distance $\rho_D((x,t),(y,s))$, where $\rho_D((x,t),(y,s))$ is defined as the infimum of the Euclidean length of rectifiable curves γ joining (x,t) and (y,s) in D, and consider the abstract completion D^* of D with respect to this inner metric ρ_D. We will not work directly with the inner metric in this paper since it seems easier to work with the Euclidean parabolic cylinders due to the time-lag issues and different scales in space and time variables. However, we do use the fact that the interior points of E_f (in the relative topology) correspond to two different boundary points in the completion D^*.

Even though we assume in this paper that E_f lies on the hyperplane $\{x_n = 0\}$ in $\mathbb{R}^n \times \mathbb{R}$, our proofs (except those on the doubling of the caloric measure and the backward boundary Harnack principle) are easily generalized to the case when E_f is a hypersurface which is Lipschitz in the space variable and independent of the time variable.

Structure of the paper. The paper is organized as follows.

In Section 2 we give basic definitions and introduce the notation used in this paper.

In Section 3 we consider the Perron–Wiener–Brelot (PWB) solution to the Dirichlet problem of the heat equation for D. We show that D is regular and has a Hölder-continuous barrier function at each parabolic boundary point.

In Section 4 we establish a forward boundary Harnack inequality for nonnegative caloric functions vanishing continuously on a part of the lateral boundary, following the lines of [Kemper 1972b].

In Section 5 we study the kernel functions for the heat operator. We show that each boundary point (y, s) in the interior of E_f (as a subset of the hyperplane $\{x_n = 0\}$) corresponds to two independent kernel functions. Hence, the parabolic Euclidean boundary for D is not homeomorphic to the parabolic Martin boundary.

In Section 6 we show the doubling property of the caloric measure with respect to D, which will imply a backward Harnack inequality for caloric functions vanishing on the whole lateral boundary.

Section 7 is dedicated to various forms of the boundary Harnack principle from Sections 4 and 6, including a version for solutions of the heat equation with a nonzero right-hand side. We conclude the section and the paper with an application to the parabolic Signorini problem.

2. Notation and preliminaries

2A. *Basic notation.*

$\mathbb{R}^n$	n-dimensional Euclidean space
$x' = (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1}$	for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$
$x'' = (x_1, \dots, x_{n-2}) \in \mathbb{R}^{n-2}$	for $x = (x_1, \dots, x_n) \in \mathbb{R}^n$

Sometimes it will be convenient to identify x', x'' with $(x', 0)$ and $(x'', 0, 0)$, respectively.

$x \cdot y = \sum_{i=1}^{n} x_i y_i$	the inner product for $x, y \in \mathbb{R}^n$
$\lvert x\rvert = (x \cdot x)^{1/2}$	the Euclidean norm of $x \in \mathbb{R}^n$
$\lVert (x, t)\rVert = (\lvert x\rvert^2 + \lvert t\rvert)^{1/2}$	the parabolic norm of $(x, t) \in \mathbb{R}^n \times \mathbb{R}$
$\bar{E}$, E°, ∂E	the closure, the interior, the boundary of E
$\partial_p E$	the parabolic boundary of E in $\mathbb{R}^n \times \mathbb{R}$
$B_r(x) := \{y \in \mathbb{R}^n : \lvert x - y\rvert < r\}$	open ball in $\mathbb{R}^n$
$B'_r(x')$, $B''_r(x'')$	(thin) open balls in $\mathbb{R}^{n-1}$, $\mathbb{R}^{n-2}$
$Q_r(x, t) := B_r(x) \times (t - r^2, t)$	lower parabolic cylinders in $\mathbb{R}^n \times \mathbb{R}$
$\operatorname{dist}_p(E, F) = \inf_{\substack{(x,t)\in E \\ (y,s)\in F}} \lVert (x - y, t - s)\rVert$	the parabolic distance between sets E, F

We will also need the notion of a *parabolic Harnack chain* in a domain $D \subset \mathbb{R}^n \times \mathbb{R}$. For two points

(z_1, h_1) and (z_2, h_2) in D with $h_2 - h_1 \geq \mu^2 |z_2 - z_1|^2$, $0 < \mu < 1$, we say that a sequence of parabolic cylinders $Q_{r_i}(x_i, t_i) \subset D$, $i = 1, \dots, N$, is a Harnack chain from (z_1, h_1) to (z_2, h_2) with constant μ if:

$$\begin{gathered}(z_1, h_1) \in Q_{r_1}(x_1, t_1), \quad (z_2, h_2) \in Q_{r_N}(x_N, t_N),\\ \mu r_i \leq \operatorname{dist}_p(Q_{r_i}(x_i, t_i), \partial_p D) \leq \frac{1}{\mu} r_i, \quad i = 1, \dots, N,\\ Q_{r_{i+1}}(x_{i+1}, t_{i+1}) \cap Q_{r_i}(x_i, t_i) \neq \varnothing, \quad i = 1, \dots, N-1,\\ t_{i+1} - t_i \geq \mu^2 r_i^2, \quad i = 1, \dots, N-1.\end{gathered}$$

The number N is called the length of the Harnack chain. By the parabolic Harnack inequality, if u is a nonnegative caloric function in D and there is a Harnack chain of length N and constant μ from (z_1, h_1) to (z_2, h_2), then

$$u(z_1, h_1) \leq C(\mu, n, N)\, u(z_2, h_2).$$

Further, for given $L \geq 1$ and $r > 0$ we also introduce the (elongated) parabolic boxes, specifically adjusted to our purposes:

$$\begin{aligned}\Psi''_r &= \{(x'', t) \in \mathbb{R}^{n-2} \times \mathbb{R} : |x_i| < r, i = 1, \dots, n-2, |t| < r^2\},\\ \Psi'_r &= \{(x', t) \in \mathbb{R}^{n-1} \times \mathbb{R} : (x'', t) \in \Psi''_r, |x_{n-1}| < 4nLr\},\\ \Psi_r &= \{(x, t) \in \mathbb{R}^n \times \mathbb{R} : (x', t) \in \Psi'_r, |x_n| < r\},\\ \Psi_r(y, s) &= (y, s) + \Psi_r.\end{aligned}$$

We also define the neighborhoods

$$\mathcal{N}_r(E) := \bigcup_{(y,s) \in E} \Psi_r(y, s) \quad \text{for any set } E \subset \mathbb{R}^n \times \mathbb{R}.$$

2B. ***Domains with thin Lipschitz complement.*** Let $f : \mathbb{R}^{n-2} \times \mathbb{R} \to \mathbb{R}$ be a parabolically Lipschitz function with a Lipschitz constant $L \geq 1$ in the sense that

$$|f(x'', t) - f(y'', s)| \leq L(|x'' - y''|^2 + |t - s|)^{1/2}, \quad (x'', t), (y'', s) \in \mathbb{R}^{n-2} \times \mathbb{R}$$

Then consider the following two sets:

$$\begin{aligned}G_f &= \{(x, t) : x_{n-1} = f(x'', t), x_n = 0\},\\ E_f &= \{(x, t) : x_{n-1} \leq f(x'', t), x_n = 0\}.\end{aligned}$$

We will call them the *thin Lipschitz graph* and *subgraph* respectively (with "thin" indicating their lower dimension). We are interested in the behavior of caloric functions in domains of the type $\Omega \setminus E_f$, where Ω is open in $\mathbb{R}^n \times \mathbb{R}$. We will say that $\Omega \setminus E_f$ is a domain with a *thin Lipschitz complement*.

We are interested mostly in local behavior of caloric functions near the points on G_f and therefore we concentrate our study on the case

$$D = D_f := \Psi_1 \setminus E_f$$

with a normalization condition

$$f(0,0)=0 \Longleftrightarrow (0,0)\in G_f.$$

We will state most of our results for D defined as above; however, the results will still hold if we replace Ψ_1 in the construction above with a rectangular box

$$\widetilde{\Psi}=\left(\prod_{i=1}^{n}(a_i,b_i)\right)\times(\alpha,\beta)$$

such that, for some constants $c_0, C_0>0$ depending on L and n, we have

$$\widetilde{\Psi}\subset\Psi_{C_0},\quad \Psi_{c_0}(y,s)\subset\widetilde{\Psi}\quad \text{for all } (y,s)\in G_f,\ s\in[\alpha+c_0^2,\beta-c_0^2],$$

and consider the complement

$$\widetilde{D}=\widetilde{D}_f:=\widetilde{\Psi}\setminus E_f.$$

Even more generally, one may take $\widetilde{\Psi}$ to be a cylindrical domain of the type $\widetilde{\Psi}=\mathbb{O}\times(\alpha,\beta)$ where $\mathbb{O}\subset\mathbb{R}^n$ has the property that $\mathbb{O}_\pm=\mathbb{O}\cap\{\pm x_n>0\}$ are Lipschitz domains. For instance, we can take $\mathbb{O}=B_1$. Again, most of the results that we state will be valid also in this case, with a possible change in constants that appear in estimates.

2C. ***Corkscrew points.*** Since we will be working in $D=\Psi_1\setminus E_f$ as above, it will be convenient to redefine the sets E_f and G_f as follows:

$$G_f=\{(x,t)\in\overline{\Psi_1}: x_{n-1}=f(x'',t),\ x_n=0\},$$
$$E_f=\{(x,t)\in\overline{\Psi_1}: x_{n-1}\le f(x'',t),\ x_n=0\},$$

so that they are subsets of $\overline{\Psi_1}$. It is easy to see from the definition of D that it is connected and that its parabolic boundary is given by

$$\partial_p D=\partial_p\Psi_1\cup E_f.$$

As we will see, the domain D has a parabolic NTA-like structure, with the catch that at points on E_f (and close to it) we need to define two pairs of future and past corkscrew points, pointing into D_+ and D_-, respectively, where

$$D_+=D\cap\{x_n>0\}=(\Psi_1)_+,\quad D_-=D\cap\{x_n<0\}=(\Psi_1)_-.$$

More specifically, fix $0<r<\frac{1}{4}$ and $(y,s)\in\mathcal{N}_r(E_f)\cap\partial_p D$, and define

$$\overline{A}_r^{\pm}(y,s)=(y'',y_{n-1}+r/2,\pm r/2,s+2r^2)\quad\text{if } s\in[-1,1-4r^2),$$
$$\underline{A}_r^{\pm}(y,s)=(y'',y_{n-1}+r/2,\pm r/2,s-2r^2)\quad\text{if } s\in(-1+4r^2,1].$$

Note that, by definition, we always have $\bar{A}_r^+(y,s)$, $\underline{A}_r^+(y,s) \in D_+$ and $\bar{A}_r^-(y,s)$, $\underline{A}_r^-(y,s) \in D_-$. We also have that

$$\bar{A}_r^\pm(y,s),\ \underline{A}_r^\pm(y,s) \in \Psi_{2r}(y,s),$$
$$\Psi_{r/2}(\bar{A}_r^\pm(y,s)) \cap \partial D = \Psi_{r/2}(\underline{A}_r^\pm(y,s)) \cap \partial D = \varnothing.$$

Moreover, the corkscrew points have the following property.

Lemma 2.1 (Harnack chain property I). *Let* $0 < r < \frac{1}{4}$, $(y,s) \in \partial_p D \cap \mathcal{N}_r(E_f)$ *and* $(x,t) \in D$ *be such that*

$$(x,t) \in \Psi_r(y,s) \quad \textit{and} \quad \Psi_{\gamma r}(x,t) \cap \partial_p D = \varnothing.$$

Then there exists a Harnack chain in D *with constant* μ *and length* N, *depending only on* γ, L, *and* n, *from* (x,t) *to either* $\bar{A}_r^+(y,s)$ *or* $\bar{A}_r^-(y,s)$, *provided* $s \le 1-4r^2$, *and from either* $\underline{A}_r^+(y,s)$ *or* $\underline{A}_r^-(y,s)$ *to* (x,t), *provided* $s \ge -1+4r^2$.

In particular, there exists a constant $C = C(\gamma, L, n) > 0$ *such that, for any nonnegative caloric function* u *in* D,

$$u(x,t) \le C \max\{u(\bar{A}_r^+(y,s)), u(\bar{A}_r^-(y,s))\} \quad \textit{if } s \le 1-4r^2,$$
$$u(x,t) \ge C^{-1} \min\{u(\underline{A}_r^+(y,s)), (\underline{A}_r^-(y,s))\} \quad \textit{if } s \ge -1+4r^2.$$

Proof. This is easily seen when $(y,s) \notin \mathcal{N}_r(G_f)$ (in this case the chain length N does not depend on L). When $(y,s) \in \mathcal{N}_r(G_f)$, one needs to use the parabolic Lipschitz continuity of f. □

Next, we want to define the corkscrew points when (y,s) is farther away from E_f. Namely, if $(y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$, we define a single pair of future and past corkscrew points by

$$\bar{A}_r(y,s) = (y(1-r), s+2r^2) \quad \text{if } s \in [-1, 1-4r^2),$$
$$\underline{A}_r(y,s) = (y(1-r), s-2r^2) \quad \text{if } s \in (-1+4r^2, 1].$$

Note that the points $\bar{A}_r(y,s)$ and $\underline{A}_r(y,s)$ will have properties similar to those of $\bar{A}_r^\pm(y,s)$ and $\underline{A}_r^\pm(y,s)$. That is,

$$\bar{A}_r(y,s),\ \underline{A}_r(y,s) \in \Psi_{2r}(y,s),$$
$$\Psi_{r/2}(\bar{A}_r(y,s)) \cap \partial D = \Psi_{r/2}(\underline{A}_r(y,s)) \cap \partial D = \varnothing,$$

and we have the following version of Lemma 2.1 above.

Lemma 2.2 (Harnack chain property II). *Let* $0 < r < \frac{1}{4}$, $(y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$ *and* $(x,t) \in D$ *be such that*

$$(x,t) \in \Psi_r(y,s) \quad \textit{and} \quad \Psi_{\gamma r}(x,t) \cap \partial_p D = \varnothing.$$

Then there exists a Harnack chain in D *with constant* μ *and length* N, *depending only on* γ, L, *and* n, *from* (x,t) *to* $\bar{A}_r(y,s)$, *provided* $s \le 1-4r^2$, *and from* $\underline{A}_r(y,s)$ *to* (x,t), *provided* $s \ge -1+4r^2$.

In particular, there exists a constant $C = C(\gamma, L, n) > 0$ such that, for any nonnegative caloric function u in D,

$$u(x,t) \le C\,u(\bar{A}_r(y,s)) \quad \text{if } s \le 1-4r^2,$$
$$u(x,t) \ge C^{-1}u(\underline{A}_r(y,s)) \quad \text{if } s \ge -1+4r^2. \qquad \square$$

To state our next lemma, we need to use a parabolic scaling operator on $\mathbb{R}^n \times \mathbb{R}$. For any $(y,s) \in \mathbb{R}^n \times \mathbb{R}$ and $r > 0$, we define

$$T^r_{(y,s)} : (x,t) \mapsto \left(\frac{x-y}{r}, \frac{t-s}{r^2}\right).$$

Lemma 2.3 (localization property). *For $0 < r < \frac{1}{4}$ and $(y,s) \in \partial_p D$, there exists a point $(\tilde{y},\tilde{s}) \in \partial_p D \cap \Psi_{2r}(y,s)$ and $\tilde{r} \in [r, 4r]$ such that*

$$\Psi_r(y,s) \cap D \subset \Psi_{\tilde{r}}(\tilde{y},\tilde{s}) \cap D \subset \Psi_{8r}(y,s) \cap D,$$

and the parabolic scaling $T^{\tilde{r}}_{(\tilde{y},\tilde{s})}(\Psi_{\tilde{r}}(\tilde{y},\tilde{s}) \cap D)$ is one of the following:

(1) *a rectangular box $\widetilde{\Psi}$ such that $\Psi_{c_0} \subset \widetilde{\Psi} \subset \Psi_{C_0}$ for some positive constants c_0 and C_0 depending on L and n, or*

(2) *a union of two rectangular boxes as in* (1) *with a common vertical side, or*

(3) *a domain $\widetilde{D}_{\tilde{f}} = \widetilde{\Psi} \setminus E_f$ with a thin Lipschitz complement, as defined at the end of Section 2B.*

Proof. Consider the following cases:

Case 1: $\Psi_r(y,s) \cap E_f = \varnothing$. In this case, we take $(\tilde{y},\tilde{s}) = (y,s)$ and $\rho = r$. Then $\Psi_r(y,s) \cap \Psi_1$ falls into category (1).

Case 2: $\Psi_r(y,s) \cap E_f \ne \varnothing$, but $\Psi_{2r}(y,s) \cap G_f = \varnothing$. In this case, we take $(\tilde{y},\tilde{s}) = (y,s)$ and $\rho = 2r$. Then $\Psi_{2r}(y,s) \cap D$ splits into the disjoint union of $\Psi_{2r}(y,s) \cap (\Psi_1)_\pm$, which falls into category (2).

Case 3: $\Psi_{2r}(y,s) \cap G_f \ne \varnothing$. In this case, choose $(\tilde{y},\tilde{s}) \in \Psi_{3r}(y,s) \cap G_f$ with the additional property that $-1 + r^2/4 \le \tilde{s} \le 1 - r^2/4$, and let $\rho = 4r$. Then $\Psi_\rho(\tilde{y},\tilde{s}) \cap D = (\Psi_\rho(\tilde{y},\tilde{s}) \setminus E_f) \cap \Psi_1$ falls into category (3). $\square$

3. Regularity of D for the heat equation

In this section we show that the domains D with thin Lipschitz complement E_f are regular for the heat equation by using the existence of an exterior thin cone at points on E_f and applying the Wiener-type criterion for the heat equation [Evans and Gariepy 1982]. Furthermore, we show the existence of Hölder-continuous local barriers at the points on E_f, which we will use in the next section to prove the Hölder continuity and regularity of the solutions up to the parabolic boundary.

3A. ***PWB solutions*** [Doob 1984; Lieberman 1996]. Given an open subset $\Omega \subset \mathbb{R}^n \times \mathbb{R}$, let $\partial\Omega$ be its Euclidean boundary. Define the parabolic boundary $\partial_p\Omega$ of Ω to be the set of all points $(x,t) \in \partial\Omega$ such that for any $\varepsilon > 0$ the lower parabolic cylinder $Q_\varepsilon(x,t)$ contains points not in Ω.

We say that a function $u : \Omega \to (-\infty, +\infty]$ is supercaloric if u is lower semicontinuous, finite on dense subsets of Ω, and satisfies the comparison principle in each parabolic cylinder $Q \Subset \Omega$: if $v \in C(\bar{Q})$ solves $\Delta v - \partial_t v = 0$ in Q and $v = u$ on $\partial_p Q$, then $v \le u$ in Q.

A subcaloric function is defined as the negative of a supercaloric function. A function is caloric if it is supercaloric and subcaloric.

Given any real-valued function g defined on $\partial_p \Omega$, we define the upper solution

$$\overline{H}_g = \inf\{u : u \text{ is supercaloric or identically } +\infty \text{ on each component of } \Omega,\\ \liminf_{(y,s)\to(x,t)} u(y,s) \ge g(x,t) \text{ for all } (x,t) \in \partial_p\Omega,\ u \text{ bounded below on } \Omega\},$$

and the lower solution

$$\underline{H}_g = \sup\{u : u \text{ is subcaloric or identically } -\infty \text{ on each component of } \Omega,\\ \limsup_{(y,s)\to(x,t)} u(y,s) \le g(x,t) \text{ for all } (x,t) \in \partial_p\Omega,\ u \text{ bounded above on } \Omega\}.$$

If $\overline{H}_g = \underline{H}_g$, then $H_g = \overline{H}_g = \underline{H}_g$ is the Perron–Wiener–Brelot (PWB) solution to the Dirichlet problem for g. It is shown in §1.VIII.4 and §1.XVIII.1 in [Doob 1984] that if g is a bounded continuous function, then the PWB solution H_g exists and is unique for any bounded domain Ω in $\mathbb{R}^n \times \mathbb{R}$.

Continuity of the PWB solution at points of $\partial_p\Omega$ is not automatically guaranteed. A point $(x,t) \in \partial_p\Omega$ is a regular boundary point if $\lim_{(y,s)\to(x,t)} H_g(y,s) = g(x,t)$ for every bounded continuous function g on $\partial_p D$. A necessary and sufficient condition for a parabolic boundary point to be regular is the existence of a local barrier for earlier time at that point (Theorem 3.26 in [Lieberman 1996]). By a local barrier at $(x,t) \in \partial_p\Omega$, we mean here a nonnegative continuous function w in $\overline{Q_r(x,t) \cap \Omega}$ for some $r > 0$ that has the following properties: (i) w is supercaloric in $Q_r(x,t) \cap \Omega$, and (ii) w vanishes only at (x,t).

3B. ***Regularity of D and barrier functions.*** For the domain D defined in the introduction, we have $\partial_p D = \partial_p \Psi_1 \cup E_f$. The regularity of $(x,t) \in \partial_p \Psi_1$ follows immediately from the exterior cone condition for the Lipschitz domain. For $(x,t) \in E_f$, instead of the full exterior cone we only know the existence of a flat exterior cone centered at (x,t) by the Lipschitz nature of the thin graph. This will still be enough for the regularity, by the Wiener-type criterion for the heat equation. We give the details below.

For $(x,t) \in E_f$, with f parabolically Lipschitz, there exist $c_1, c_2 > 1$, depending on n and L, such that the exterior of D contains a flat parabolic cone $\mathscr{C}(x,t)$, defined by

$$\mathscr{C}(x,t) = (x,t) + \mathscr{C},$$
$$\mathscr{C} = \{(y,s) \in \mathbb{R}^n \times \mathbb{R} : s \le 0,\ y_{n-1} \le -c_1|y''| - c_2\sqrt{-s},\ y_n = 0\}.$$

Then by the Wiener-type criterion for the heat equation [Evans and Gariepy 1982], the regularity of $(x,t) \in E_f$ will follow once we show that

$$\sum_{k=1}^{\infty} 2^{kn/2} \operatorname{cap}(\mathscr{A}(2^{-k}) \cap \mathscr{C}) = +\infty,$$

where

$$\mathscr{A}(c) = \{(y,s) : (4\pi c)^{-n/2} \le \Gamma(y,-s) \le (2\pi c)^{-n/2}\},$$

Γ is the heat kernel

$$\Gamma(y,s) = \begin{cases} (4\pi s)^{-n/2} e^{-|y|^2/4s} & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases}$$

and $\operatorname{cap}(K)$ is the thermal capacity of a compact set K, defined by

$$\operatorname{cap}(K) = \sup\{\mu(K) : \mu \text{ is a nonnegative Radon measure supported in } K, \text{ with } \mu * \Gamma \le 1 \text{ on } \mathbb{R}^n \times \mathbb{R}\}.$$

Since $\mathscr{C}$ is self-similar, it is enough to verify that

$$\operatorname{cap}(\mathscr{A}(1) \cap \mathscr{C}) > 0.$$

The latter is easy to see, since we can take as μ the restriction of H^n, the Hausdorff measure, to $\mathscr{A}(1) \cap \mathscr{C}$, and note that

$$(\mu * \Gamma)(x,t) = \int_{\mathscr{A}(1)\cap\mathscr{C}} \Gamma(x-y, t-s)\, dy'\, ds \le \int_{-1}^{0} \frac{1}{\sqrt{4\pi(t-s)^+}}\, ds \le \int_{-1}^{0} \frac{1}{\sqrt{4\pi(-s)}}\, ds < \infty$$

for any $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Since $H^n(\mathscr{A}(1) \cap \mathscr{C}) > 0$, we therefore conclude that $\operatorname{cap}(\mathscr{A}(1) \cap \mathscr{C}) > 0$. We have therefore established the following fact:

Proposition 3.1. *The domain $D = D_f$ is regular for the heat equation.* □

We next show that we can use the self-similarity of $\mathscr{C}$ to construct a Hölder-continuous barrier function at every $(x,t) \in E_f$.

Lemma 3.2. *There exists a nonnegative continuous function U on $\overline{\Psi_1}$ with the following properties*:

(i) $U > 0$ *in* $\overline{\Psi_1} \setminus \{(0,0)\}$ *and* $U(0,0) = 0$;

(ii) $\Delta U - \partial_t U = 0$ *in* $\Psi_1 \setminus \mathscr{C}$; *and*

(iii) $U(x,t) \le C(|x|^2 + |t|)^{\alpha/2}$ *for* $(x,t) \in \Psi_1$ *and some* $C > 0$ *and* $0 < \alpha < 1$ *depending only on n and L*.

Proof. Let U be a solution of the Dirichlet problem in $\Psi_1 \setminus \mathscr{C}$ with boundary values $U(x,t) = |x|^2 + |t|$ on $\partial_p(\Psi_1 \setminus \mathscr{C})$. Then U will be continuous on $\overline{\Psi_1}$ and will satisfy the following properties:

(i) $U > 0$ in $\overline{\Psi_1} \setminus \{(0,0)\}$ and $U(0,0) = 0$; and

(ii) $\Delta U - \partial_t U = 0$ in $\Psi_1 \setminus \mathscr{C}$.

In particular, there exists $c_0 > 0$ and $\lambda > 0$ such that

$$U \ge c_0 \text{ on } \partial_p \Psi_1 \quad \text{and} \quad U \le c_0/2 \text{ on } \Psi_\lambda.$$

We then can compare U with its own parabolic scaling. Indeed, let $M_U(r) = \sup_{\Psi_r} U$ for $0 < r < 1$. Then, by the comparison principle for the heat equation, we have

$$U(x,t) \le \frac{M_U(r)}{c_0} U(x/r, t/r^2) \quad \text{for } (x,t) \in \Psi_r.$$

(Carefully note that this inequality is satisfied on $\mathscr{C}$ by the homogeneity of the boundary data on $\mathscr{C}$.) Hence, we obtain that

$$M_U(\lambda r) \le \frac{M_U(r)}{2} \quad \text{for any } 0 < r < 1,$$

which implies the Hölder-continuity of U at the origin by the standard iteration. The proof is complete. □

4. Forward boundary Harnack inequalities

In this section, we show the boundary Hölder-regularity of the solutions to the Dirichlet problem and follow the lines of [Kemper 1972b] to show the forward boundary Harnack inequality (Carleson estimate).

We also need the notion of the caloric measure. Given a domain $\Omega \subset \mathbb{R}^n \times \mathbb{R}$ and $(x, t) \in \Omega$, the caloric measure on $\partial_p \Omega$ is denoted by $\omega_\Omega^{(x,t)}$. The following facts about caloric measures can be found in [Doob 1984]. For a Borel subset B of $\partial_p\Omega$, we have $\omega_\Omega^{(x,t)}(B) = H_{\chi_B}(x, t)$, which is the PWB solution to the Dirichlet problem

$$\Delta u - u_t = 0 \text{ in } \Omega; \quad u = \chi_B \text{ on } \partial_p \Omega,$$

where χ_B is the characteristic function of B. Given a bounded and continuous function g on $\partial_p\Omega$, the PWB solution to the Dirichlet problem

$$\Delta u - u_t = 0 \text{ in } \Omega; \quad u = g \text{ on } \partial_p \Omega$$

is given by $u(x, t) = \int_{\partial_p\Omega} g(y, s)\, d\omega_\Omega^{(x,t)}(y, s)$. For a regular domain Ω, one has the following useful property of caloric measures:

Proposition 4.1 [Doob 1984]. *If E is a fixed Borel subset of $\partial_p\Omega$, then the function $(x, t) \mapsto \omega_\Omega^{(x,t)}(E)$ extends to $(y, s) \in \partial_p\Omega$ continuously provided χ_E is continuous at (y, s).*

4A. ***Forward boundary Harnack principle.*** From now on, we will write the caloric measure with respect to $D = \Psi_1 \setminus E_f$ as $\omega^{(x,t)}$ for simplicity. Before we prove the forward boundary Harnack inequality, we first show the Hölder-continuity of the caloric functions up to the boundary, which follows from the estimates on the barrier function constructed in Section 3.

In what follows, for $0 < r < \frac{1}{4}$ and $(y, s) \in \partial_p D$, we will denote

$$\Delta_r(y, s) = \Psi_r(y, s) \cap \partial_p D,$$

and call it the *parabolic surface ball* at (y, s) of radius r.

Lemma 4.2. *Let $0 < r < \frac{1}{4}$ and $(y, s) \in \partial_p D$. Then there exist $C = C(n, L) > 0$ and $\alpha = \alpha(n, L) \in (0, 1)$ such that if u is positive and caloric in $\Psi_r(y, s) \cap D$ and u vanishes continuously on $\Delta_r(y, s)$, then*

$$u(x, t) \le C\left(\frac{|x - y|^2 + |t - s|}{r^2}\right)^{\alpha/2} M_u(r) \tag{4-1}$$

for all $(x, t) \in \Psi_r(y, s) \cap D$, where $M_u(r) = \sup_{\Psi_r(y,s)\cap D} u$.

Proof. Let U be the barrier function at $(0,0)$ in Lemma 3.2 and $c_0=\inf_{\partial_p\Psi_1}U>0$. We then use the parabolic scaling $T^r_{(y,s)}$ to construct a barrier function at (y,s). If $(y,s)\in\mathcal{N}_r(E_f)$, then there is an exterior cone $\mathscr{C}(y,s)$ at (y,s) with a universal opening, depending only on n and L, and

$$U^r_{(y,s)}:=U\circ T^r_{(y,s)}$$

will be a local barrier function at (y,s) and will satisfy

$$0\le U^r_{(y,s)}(x,t)\le C\left(\frac{|x-y|^2+|t-s|}{r^2}\right)^{\alpha/2}\quad\text{for }\ (x,t)\in\Psi_r(y,s).\tag{4-2}$$

This construction can be made also at $(y,s)\in\partial_pD\setminus\mathcal{N}_r(E_f)$, as these points also have the exterior cone property, and we may still use the same formula for $U^r_{(y,s)}$, but after a possible rotation of the coordinate axes in $\mathbb{R}^n$.

Then, by the maximum principle in $\Psi_r(y,s)\cap D$, we easily obtain that

$$u(x,t)\le\frac{M_u(r)}{c_0}U^r_{(y,s)}(x,t)\quad\text{for }(x,t)\in\Psi_r(y,s)\cap D.\tag{4-3}$$

Combining (4-2) and (4-3), we obtain (4-1). □

The main result in this section is the following forward boundary Harnack principle, also known as the Carleson estimate.

Theorem 4.3 (forward boundary Harnack principle or Carleson estimate). *Let $0<r<\frac14$, $(y,s)\in\partial_pD$ with $s\le 1-4r^2$, and u be a nonnegative caloric function in D, continuously vanishing on $\Delta_{3r}(y,s)$. Then there exists $C=C(n,L)>0$ such that, for $(x,t)\in\Psi_{r/2}(y,s)\cap D$,*

$$u(x,t)\le C\begin{cases}\max\{u(\bar A_r^+(y,s)),u(\bar A_r^-(y,s))\} & \textit{if } (y,s)\in\partial_pD\cap\mathcal{N}_r(E_f),\\ u(\bar A_r(y,s)) & \textit{if } (y,s)\in\partial_pD\setminus\mathcal{N}_r(E_f).\end{cases}\tag{4-4}$$

To prove the Carleson estimate above, we need the following two lemmas on the properties of the caloric measure in D, which correspond to Lemmas 1.1 and 1.2 in [Kemper 1972b], respectively.

Lemma 4.4. *For $0<r<\frac14$, $(y,s)\in\partial_pD$ with $s\le1-4r^2$, and $\gamma\in(0,1)$, there exists $C=C(\gamma,L)>0$ such that*

$$\omega^{(x,t)}(\Delta_r(y,s))\ge C\quad \textit{for } (x,t)\in\Psi_{\gamma r}(y,s)\cap D.$$

Proof. Suppose first that $(y,s)\in\mathcal{N}_r(E_f)$. Consider the caloric function

$$v(x,t):=\omega^{(x,t)}_{\Psi_r(y,s)\setminus\mathscr{C}(y,s)}(\mathscr{C}(y,s)),$$

where $\mathscr{C}(y,s)$ is the flat exterior cone defined in Section 3. The domain $\Psi_r(y,s)\setminus\mathscr{C}(y,s)$ is regular; hence, by Proposition 4.1, $v(x,t)$ is continuous on $\overline{\Psi_{\gamma r}(y,s)}$. We next claim that there exists $C=C(\gamma,n,L)>0$ such that

$$v(x,t)\ge C\quad\text{in }\Psi_{\gamma r}(y,s).$$

Indeed, consider the normalized version of v,

$$v_0(x,t) := \omega^{(x,t)}_{\Psi_1\setminus\mathscr{C}}(\mathscr{C}),$$

which is related to v through the identity $v = v_0 \circ T^r_{(y,s)}$. Then, from the continuity of v_0 in $\overline{\Psi_\gamma}$, the equality $v_0 = 1$ on $\mathscr{C}$, and the strong maximum principle we obtain that $v_0 \geq C = C(\gamma, n, L) > 0$ on $\overline{\Psi_\gamma}$. Using the parabolic scaling, we obtain the claimed inequality for v. Moreover, applying the comparison principle to $v(x,t)$ and $\omega^{(x,t)}(\Delta_r(y,s))$ in $D \cap \Psi_r(y,s)$, we have

$$\omega^{(x,t)}(\Delta_r(y,s)) \geq v(x,t) \geq C \quad \text{for } (x,t) \in D \cap \Psi_{\gamma r}(y,s).$$

In the case when $(y,s) \in \partial_p D \setminus \mathscr{N}_r(E_f)$, we may modify the proof by changing the flat cone $\mathscr{C}(y,s)$ with the full cone contained in the complement of D, or directly applying Lemma 1.1 in [Kemper 1972b]. □

Lemma 4.5. *For $0 < r < \frac{1}{4}$, $(y,s) \in \partial_p D$ with $s \leq 1 - 4r^2$, there exists a constant $C = C(n, L) > 0$ such that, for any $r' \in (0, r)$ and $(x,t) \in D \setminus \Psi_r(y,s)$, we have*

$$\omega^{(x,t)}(\Delta_{r'}(y,s)) \leq C \begin{cases} \omega^{\bar{A}_r(y,s)}(\Delta_{r'}(y,s)) & \text{if } (y,s) \notin \mathscr{N}_r(E_f), \\ \max\{\omega^{\bar{A}^+_r(y,s)}(\Delta_{r'}(y,s)), \omega^{\bar{A}^-_r(y,s)}(\Delta_{r'}(y,s))\} & \text{if } (y,s) \in \mathscr{N}_r(E_f). \end{cases} \tag{4-5}$$

Proof. For notational simplicity, we define

$$\Delta' := \Delta_{r'}(y,s), \quad \Delta := \Delta_r(y,s), \quad \Psi^k := \Psi_{2^{k-1}r'}(y,s),$$
$$\bar{A}^\pm_k := \bar{A}^\pm_{2^{k-1}r'}(y,s) \quad \text{if } \Psi^k \cap E_f \neq \varnothing,$$
$$\bar{A}_k := \bar{A}_{2^{k-1}r'}(y,s) \quad \text{if } \Psi^k \cap E_f = \varnothing \text{ for } k = 0, 1, \dots, \ell \text{ with } 2^{\ell-1}r' < 3r/4 < 2^\ell r'.$$

We want to clarify here that for $(y,s) \notin E_f$ and small r' and k, it may happen that Ψ^k does not intersect E_f. To be more specific, let ℓ_0 be the smallest nonnegative integer such that $\Psi^{\ell_0} \cap E_f \neq \varnothing$. Then we define $\bar{A}_k$ for $0 \leq k \leq \min\{\ell_0 - 1, \ell\}$ and the pair $\bar{A}^\pm_k$ for $\ell_0 \leq k \leq \ell$.

To prove the lemma, we want to show that there exists a universal constant C, in particular independent of k, such that, for $(x,t) \in D \setminus \Psi^k$,

$$\omega^{(x,t)}(\Delta') \leq C \begin{cases} \omega^{\bar{A}_k}(\Delta') & \text{if } 1 \leq k \leq \min\{\ell_0 - 1, \ell\}, \\ \max\{\omega^{\bar{A}^+_k}(\Delta'), \omega^{\bar{A}^-_k}(\Delta')\} & \text{if } \ell_0 \leq k \leq \ell. \end{cases} \tag{S_k}$$

Once this is established, (4-5) will follow from (S_l) and the Harnack inequality.

The proof of (S_k) is going to be by induction in k. We start with the observation that, by the Harnack inequality, there is $C_1 > 0$, independent of k and r', such that

$$\begin{aligned} \omega^{\bar{A}_k}(\Delta') &\leq C_1 \omega^{\bar{A}_{k+1}}(\Delta') && \text{for } 0 \leq k \leq \min\{\ell_0 - 2, \ell - 1\}, \\ \omega^{\bar{A}_{\ell_0 - 1}}(\Delta') &\leq C_1 \max\{\omega^{\bar{A}^+_{\ell_0}}(\Delta'), \omega^{\bar{A}^-_{\ell_0}}(\Delta')\} && \text{if } \ell_0 \leq \ell, \\ \omega^{\bar{A}^\pm_k}(\Delta') &\leq C_1 \omega^{\bar{A}^\pm_{k+1}}(\Delta') && \text{for } \ell_0 \leq k \leq \ell - 1. \end{aligned} \tag{4-6}$$

Proof of (S_1): Without loss of generality, assume $(y,s) \in \partial_p D \cap \bar{D}_+$.

Case 1: Suppose first that $\Psi^1 \cap E_f = \varnothing$, i.e., $\ell_0 > 1$. In this case, $\bar{A}_0 = \bar{A}_{r'/2}(y,s) \in \Psi_{(3/4)r'}(y,s)$, and by Lemma 4.4 there exists a universal $C_0 > 0$ such that $\omega^{\bar{A}_0}(\Delta') \geq C_0$. By (4-6) we have $\omega^{\bar{A}_0}(\Delta') \leq C_1 \omega^{\bar{A}_1}(\Delta')$. Letting $C_2 = C_1/C_0$, we then have

$$\omega^{(x,t)}(\Delta') \leq 1 \leq C_2 \omega^{\bar{A}_1}(\Delta'). \tag{4-7}$$

Case 2: Suppose now that $\Psi^1 \cap E_f \neq \varnothing$, but $\Psi^0 \cap E_f = \varnothing$, i.e., $\ell_0 = 1$. In this case, we start as in Case 1, and finish by applying the second inequality in (4-6), which yields

$$\omega^{(x,t)}(\Delta') \leq 1 \leq C_2 \max\{\omega^{\bar{A}_1^+}(\Delta'), \omega^{\bar{A}_1^-}(\Delta')\}. \tag{4-8}$$

Case 3: Finally, assume that $\Psi^0 \cap E_f \neq \varnothing$, i.e., $\ell_0 = 0$. Without loss of generality, assume also that $(y,s) \in \partial_p D \cap \bar{D}_+$. In this case, $\bar{A}_0^+ \in \Psi_{(3/4)r'}(y,s)$, and therefore $\omega^{\bar{A}_0^+}(\Delta') \geq C_0$. Besides, by (4-6), we have that $\omega^{\bar{A}_0^+}(\Delta') \leq C_1 \omega^{\bar{A}_1^+}(\Delta')$, which yields

$$\omega^{(x,t)}(\Delta') \leq 1 \leq C_2 \omega^{\bar{A}_1^+}(\Delta'). \tag{4-9}$$

This proves (S_1) with the constant $C = C_2$.

We now turn to the proof of the induction step.

Proof of $(S_k) \implies (S_{k+1})$: More precisely, we will show that if (S_k) holds with some universal constant C (to be specified) then (S_{k+1}) also holds with the same constant.

By the maximum principle, we need to verify (S_{k+1}) for $(x,t) \in \partial_p(D \setminus \Psi^{k+1})$. Since $\omega^{(x,t)}(\Delta')$ vanishes on $(\partial_p D) \setminus \Psi^{k+1}$, we may assume that $(x,t) \in (\partial \Psi^{k+1}) \cap D$. We will need to consider three cases, as in the proof of (S_1):

1. $\Psi^{k+1} \cap E_f = \varnothing$, i.e., $\ell_0 > k+1$;
2. $\Psi^{k+1} \cap E_f \neq \varnothing$, but $\Psi^k \cap E_f = \varnothing$, i.e., $\ell_0 = k+1$;
3. $\Psi^k \cap E_f \neq \varnothing$, i.e., $\ell_0 \leq k$.

Since the proof is similar in all three cases, we will treat only Case 2 in detail.

Case 2: Suppose that $\Psi^{k+1} \cap E_f \neq \varnothing$ but $\Psi^k \cap E_f = \varnothing$. We consider two subcases, depending on whether $(x,t) \in \partial \Psi^{k+1}$ is close to $\partial_p D$ or not.

Case 2a: First, assume that $(x,t) \in \mathcal{N}_{\mu 2^k r'}(\partial_p D)$ for some small positive $\mu = \mu(L,n) < \frac{1}{2}$ (to be specified). Take $(z,h) \in \Psi_{\mu 2^k r'}(x,t) \cap \partial_p D$, and observe that $\omega^{(x,t)}(\Delta')$ is caloric in $\Psi_{2^{k-1}r'}(z,h) \cap D$ and vanishes continuously on $\Delta_{2^{k-1}r'}(z,h)$ (by Proposition 4.1). Besides, by the induction assumption that (S_k) holds, we have

$$\omega^{(x,t)}(\Delta') \leq C \omega^{\bar{A}_k}(\Delta') \quad \text{for } (x,t) \in \Psi_{2^{k-1}r'}(z,h) \cap D \subset D \setminus \Psi^k.$$

Hence, by Lemma 4.2, if $\mu = \mu(n,L) > 0$ is small enough, we obtain that

$$\omega^{(x,t)}(\Delta') \leq \frac{1}{C_1} C \omega^{\bar{A}_k}(\Delta') \quad \text{for } (x,t) \in \Psi_{\mu 2^k r'}(z,h).$$

Here C_1 is the constant in (4-6). This, combined with (4-6), gives

$$\omega^{(x,t)}(\Delta') \le \frac{C}{C_1}\omega^{\bar{A}_k}(\Delta') \le \frac{C}{C_1}\cdot C_1 \max\{\omega^{\bar{A}^+_{k+1}}(\Delta'), \omega^{\bar{A}^-_{k+1}}(\Delta')\} = C\max\{\omega^{\bar{A}^+_{k+1}}(\Delta'), \omega^{\bar{A}^-_{k+1}}(\Delta')\}.$$

This proves (S_{k+1}) for $(x,t) \in \mathcal{N}_{\mu 2^k r'}(\partial_p D) \cap \partial\Psi^{k+1}$.

Case 2b: Assume now that $\Psi_{\mu 2^k r'}(x,t) \cap \partial_p D = \varnothing$. In this case, it is easy to see that we can construct a parabolic Harnack chain in D of universal length from (x,t) to either $\bar{A}^+_{k+1}$ or $\bar{A}^-_{k+1}$, which implies that, for some universal constant $C_3 > 0$,

$$\omega^{(x,t)}(\Delta') \le C_3 \max\{\omega^{\bar{A}^+_{k+1}}(\Delta'), \omega^{\bar{A}^-_{k+1}}(\Delta')\}.$$

Thus, combining Cases 2a and 2b, we obtain that (S_{k+1}) holds provided $C = \max\{C_2, C_3\}$. This completes the proof of our induction step in Case 2. As we mentioned earlier, Cases 1 and 3 are obtained by a small modification from the respective cases in the proof of (S_1). This completes the proof of the lemma. □

Now we prove the Carleson estimate. With Lemma 4.4 and Lemma 4.5 at hand, we use ideas similar to those in [Salsa 1981].

Proof of Theorem 4.3. We start with the remark that if $(y,s) \notin \mathcal{N}_{r/4}(E_f)$ then we can restrict u to D_+ or D_- and obtain the second estimate in (4-4) from the known result for parabolic Lipschitz domains. We thus consider only the case $(y,s) \in \mathcal{N}_{r/4}(E_f)$. Besides, replacing (y,s) with $(y',s') \in \Psi_{r/4}(y,s) \cap E_f$, we may further assume that $(y,s) \in E_f$, but then we will need to change the assumption that u vanishes on $\Delta_{2r}(y,s)$ and prove the estimate (4-4) for $(x,t) \in \Psi_r(y,s) \cap D$.

With these assumptions in mind, let $0 < r < \frac{1}{4}$ and $R = 8r$. Let $\widetilde{D}_R(y,s) := \Psi_{\tilde{R}}(\tilde{y},\tilde{s}) \cap D$ be given by the localization property Lemma 2.3. Note that we will be either in Case (2) or (3) of that lemma; moreover, we can choose $(\tilde{y},\tilde{s}) = (y,s)$.

For notational brevity, let

$$\omega_R^{(x,t)} := \omega^{(x,t)}_{\widetilde{D}_R(y,s)}$$

be the caloric measure with respect to $\widetilde{D}_R(y,s)$. We will also omit the center (y,s) from the notations $\widetilde{D}_R(y,s)$, $\Psi_\rho(y,s)$ and $\Delta_\rho(y,s)$.

Since u is caloric in $\widetilde{D}_R$ and continuously vanishes up to Δ_{2r}, we have

$$u(x,t) = \int_{(\partial_p \widetilde{D}_R)\setminus\Delta_{2r}} u(z,h)\, d\omega_R^{(x,t)}(z,h), \quad (x,t) \in \widetilde{D}_R. \tag{4-10}$$

Note that for $(x,t) \in \Psi_r \cap D$, we have $(x,t) \notin \Psi_{r/2}(z,h)$ for any $(z,h) \in (\partial_p \widetilde{D}_R) \setminus \Delta_{2r}$. Hence, applying Lemma 4.5[1] to $\omega_R^{(x,t)}$ in $\widetilde{D}_R$, we will have that, for $(x,t) \in \Psi_r \cap D$ and sufficiently small r',

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h)) \le C \max\left\{\omega_R^{\bar{A}^+_{r/2,R}(z,h)}(\Delta_{r'}(z,h)), \omega_R^{\bar{A}^-_{r/2,R}(z,h)}(\Delta_{r'}(z,h))\right\}$$

[1]We have to scale the domain $\widetilde{D}_R$ with $T^{\tilde{R}}_{(\tilde{y},\tilde{s})}$ first and apply Lemma 4.5 to $r/2\tilde{R} < \frac{1}{8}$ if we are in case (3) of the localization property Lemma 2.3; in the case (2) we apply the known results for parabolic Lipschitz domains.

for $(z,h)\in\mathcal{N}_{r/2}(E_f)\cap(\partial_p\widetilde{D}_R)\setminus\Delta_{2r}$, and

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h))\le C\,\omega_R^{\bar{A}_{r/2,R}(z,h)}(\Delta_{r'}(z,h))$$

for $(z,h)\in\partial_p\widetilde{D}_R\setminus(\mathcal{N}_{r/2}(E_f)\cup\Delta_{2r})$, where $C=C(L,n)$ and by $\bar{A}^{\pm}_{r/2,R}$ and $\bar{A}_{r/2,R}$ we denote the corkscrew points with respect to the domain $\widetilde{D}_R$. To proceed, we note that, for $(z,h)\in\partial_p\widetilde{D}_R$ with $h>s+r^2$, by the maximum principle we have

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h))=0$$

for any $(x,t)\in\Psi_r\cap D$, provided r' is small enough. For $(z,h)\in(\partial_p\widetilde{D}_R)\setminus\Delta_{2r}$ with $h\le s+r^2$, we note that with the help of Lemmas 2.1 and 2.2 we can construct a Harnack chain of controllable length in D from $\bar{A}^{\pm}_{r/2,R}(z,h)$ or $\bar{A}_{r/2,R}(z,h)$ to $\bar{A}^{+}_{r}(y,s)$ or $\bar{A}^{-}_{r}(y,s)$ (corkscrew points with respect to the original D). This implies that, for $(x,t)\in\Psi_r\cap D$ and $(z,h)\in\partial_p\widetilde{D}_R\setminus\Delta_{2r}$,

$$\omega_R^{(x,t)}(\Delta_{r'}(z,h))\le C\max\{\omega_R^{\bar{A}^{+}_{r}(y,s)}(\Delta_{r'}(z,h)),\omega_R^{\bar{A}^{-}_{r}(y,s)}(\Delta_{r'}(z,h))\}. \tag{4-11}$$

We now want to apply Besicovitch's theorem on the differentiation of Radon measures. However, since $\partial_p\widetilde{D}_R$ locally is not topologically equivalent to a Euclidean space, we make the following symmetrization argument. For $x\in\mathbb{R}^n$, let $\hat{x}$ be its mirror image with respect to the hyperplane $\{x_n=0\}$. We then can write

$$\begin{aligned}u(x,t)+u(\hat{x},t)&=\int_{\partial_p\widetilde{D}_R\setminus\Delta_{2r}}[u(z,h)+u(\hat{z},h)]\,d\omega_R^{(x,t)}(z,h)\\&=\frac{1}{2}\int_{\partial_p\widetilde{D}_R\setminus\Delta_{2r}}[u(z,h)+u(\hat{z},h)]\big(d\omega_R^{(x,t)}(z,h)+d\omega_R^{(\hat{x},t)}(z,h)\big)\\&=\int_{\partial_p((\widetilde{D}_R)_+)\setminus\Delta_{2r}}[u(z,h)+u(\hat{z},h)]\chi\big(d\omega_R^{(x,t)}(z,h)+d\omega_R^{(\hat{x},t)}(z,h)\big),\end{aligned}$$

where $\chi=\frac{1}{2}$ on $\partial_p((\widetilde{D}_R)_+)\cap\{x_n=0\}$ and $\chi=1$ on the remaining part of $\partial_p((\widetilde{D}_R)_+)$ and the measures $d\omega_R^{(x,t)}$ and $d\omega_R^{(\hat{x},t)}$ are extended as zero on the thin space outside E_f, i.e., on $\partial_p((\widetilde{D}_R)_+)\setminus\partial_p\widetilde{D}_R$. We then use the estimate (4-11) for (x,t) and $(\hat{x},t)$ in $\Psi_r\cap D$. Note that in this situation we can apply Besicovitch's theorem on differentiation, since we can locally project $\partial_p((\widetilde{D}_R)_+)$ to hyperplanes, similarly to [Hunt and Wheeden 1970]. This will yield

$$\frac{d\omega_R^{(x,t)}(z,h)+d\omega_R^{(\hat{x},t)}(z,h)}{d\omega_R^{\bar{A}^{+}_{r}(y,s)}(z,h)+d\omega_R^{\bar{A}^{-}_{r}(y,s)}(z,h)}\le C\,\frac{d\omega_R^{(x,t)}(z,h)}{d\omega_R^{\bar{A}_{r}(y,s)}(z,h)}\le C \tag{4-12}$$

for $(z,h)\in\partial_p((\widetilde{D}_R)_+)\setminus\Delta_{2r}$ and $(x,t)\in\Psi_r\cap D$. Hence, we obtain

$$\begin{aligned}u(x,t)+u(\hat{x},t)&\le C\int_{\partial_p((\widetilde{D}_R)_+)\setminus\Delta_{2r}}[u(z,h)+u(\hat{z},h)]\big(d\omega_R^{\bar{A}^{+}_{r}(y,s)}(z,h)+d\omega_R^{\bar{A}^{-}_{r}(y,s)}(z,h)\big)\\&\le C\big(u(\bar{A}^{+}_{r}(y,s))+u(\bar{A}^{-}_{r}(y,s))\big)\\&\le C\max\{u(\bar{A}^{+}_{r}(y,s)),u(\bar{A}^{-}_{r}(y,s))\},\quad (x,t)\in\Psi_r\cap D.\end{aligned}$$

This completes the proof of the theorem. □

The following theorem is a useful consequence of Theorem 4.3; with that in hand, its proof is similar to that of Theorem 1.1 in [Fabes et al. 1986]. Hence, we only state the theorem here without giving a proof.

Theorem 4.6. *For $0 < r < \frac{1}{4}$, $(y, s) \in \partial_p D$ with $s \le 1 - 4r^2$, let u be caloric in D and continuously vanishing on $\partial_p D \setminus \Delta_{r/2}(y, s)$. Then there exists $C = C(n, L)$ such that, for $(x, t) \in D \setminus \Psi_r(y, s)$, we have*

$$u(x,t) \le C \begin{cases} \max\{u(\bar{A}_r^+(y,s)), u(\bar{A}_r^-(y,s))\} & \text{if } (y,s) \in \mathcal{N}_r(E_f), \\ u(\bar{A}_r(y,s)) & \text{if } (y,s) \notin \mathcal{N}_r(E_f). \end{cases} \tag{4-13}$$

Moreover, applying Lemma 4.4 and the maximum principle, for $(x, t) \in D \setminus \Psi_r(y, s)$, we have

$$u(x,t) \le C\omega^{(x,t)}(\Delta_{2r}(y,s)) \times \begin{cases} \max\{u(\bar{A}_r^+(y,s)), u(\bar{A}_r^-(y,s))\} & \text{if } (y,s) \in \mathcal{N}_r(E_f), \\ u(\bar{A}_r(y,s)) & \text{if } (y,s) \notin \mathcal{N}_r(E_f). \end{cases}$$

5. Kernel functions

Before proceeding to the backward boundary Harnack principle, we need the notion of kernel functions associated to the heat operator and the domain D. In [Fabes et al. 1986], the backward Harnack principle is a consequence of the global comparison principle (Theorem 6.4) by a simple time-shifting argument. In our case, since D is not cylindrical, this simple argument does not work. So we will first prove some properties of the kernel functions which can be used to show the doubling property of the caloric measures, as in [Wu 1979]. Then, using arguments as in [Fabes et al. 1986], we obtain the backward Harnack principle.

5A. *Existence of kernel functions.* Let $(X, T) \in D$ be fixed. Given $(y, s) \in \partial_p D$ with $s < T$, a function $K(x, t; y, s)$ defined in D is called a kernel function at (y, s) for the heat equation with respect to (X, T) if:

(i) $K(\cdot, \cdot; y, s) \ge 0$ in D,

(ii) $(\Delta - \partial_t)K(\cdot, \cdot; y, s) = 0$ in D,

(iii) $\lim_{\substack{(x,t)\to(z,h)\\(x,t)\in D}} K(x, t; y, s) = 0$ for $(z, h) \in \partial_p D \setminus \{(y, s)\}$, and

(iv) $K(X, T; y, s) = 1$.

If $s \ge T$, $K(x, t; y, s)$ will be taken identically equal to zero. We note that, by the maximum principle, $K(x, t; y, s) = 0$ when $t < s$.

The existence of the kernel functions (for the heat operator on domain D) follows directly from Theorem 4.3. Let $(y, s) \in \partial_p D$ with $s < T - \delta^2$ for some $\delta > 0$, and consider

$$v_n(x,t) = \frac{\omega^{(x,t)}(\Delta_{1/n}(y,s))}{\omega^{(X,T)}(\Delta_{1/n}(y,s))}, \quad (x,t) \in D,\ \frac{1}{n} < \delta. \tag{5-1}$$

We clearly have $v_n(x, t) \ge 0$, $(\Delta - \partial_t)v_n(x, t) = 0$ in D and $v_n(X, T) = 1$. Given $\varepsilon \in \left(0, \frac{1}{4}\right)$ small, by Theorem 4.6 and the Harnack inequality, $\{v_n\}$ is uniformly bounded on $\overline{D \setminus \Psi_\varepsilon(y, s)}$ if $n \ge 2/\varepsilon$. Moreover, by the up-to-the-boundary regularity (see Proposition 4.1 and Lemma 4.2), the family $\{v_n\}$ is uniformly

Hölder in $\overline{D \setminus \Psi_\varepsilon(y,s)}$. Hence, up to a subsequence, $\{v_n\}$ converges uniformly on $\overline{D \setminus \Psi_\varepsilon(y,s)}$ to some nonnegative caloric function v satisfying $v(X,T)=1$. Since ε can be taken arbitrarily small, v vanishes on $\partial_p D \setminus \{(y,s)\}$. Therefore, $v(x,t)$ is a kernel function at (y,s).

Convention 5.1. From now on, to avoid cumbersome details we will make a time extension of the domain D for $1 \le t < 2$ by looking at

$$\widetilde{D} = \widetilde{\Psi} \setminus E_f, \quad \widetilde{\Psi} = (-1,1)^n \times (-1,2),$$

as in Section 2B. We then fix (X,T) with $T = \frac{3}{2}$ and $X \in \{x_n = 0\}$, $X_{n-1} > 3nL$, and normalize all kernels $K(\cdot,\cdot;\cdot,\cdot)$ at this point (X,T). In this way, we will be able to state the results in this section for our original domain D. Alternatively, we could fix $(X,T) \in D$, and then state the results in the part of the domain $D \cap \{(x,t) : -1 < t < T - \delta^2\}$ with some $\delta > 0$, with the additional dependence of constants on δ.

5B. ***Nonuniqueness of kernel functions at $E_f \setminus G_f$.*** The idea is this: if we consider the completion D^* of the domain D with respect to the inner metric ρ_D and let $\partial^* D = D^* \setminus D$, then it is clear that each Euclidean boundary point $(y,s) \in G_f$ and $(y,s) \in \partial_p \Psi_1$ will correspond to only one $(y,s)^* \in \partial^* D$, and each $(y,s) \in E_f \setminus G_f$ will correspond to exactly two points $(y,s)^*_+$, $(y,s)^*_- \in \partial^* D$. It is not hard to imagine that the kernel functions corresponding to $(y,s)^*_+$ and $(y,s)^*_-$ are linearly independent, and they are the two linearly independent kernel functions at (y,s). In this section we will make this idea precise by considering the two-sided caloric measures ϑ_+ and ϑ_-. We will study the properties of ϑ_+ and ϑ_- and their relationship with the caloric measure ω_D.

First we introduce some more notation. Given $(y,s) \in \partial_p D \setminus G_f$, let

$$r_0 = \sup\big\{r \in \big(0, \tfrac{1}{4}\big) : \Delta_{2r}(y,s) \cap G_f = \varnothing\big\}. \tag{5-2}$$

Note that r_0 is a constant depending on (y,s), and is such that, for any $0 < r < r_0$, $\Psi_{2r}(y,s) \cap D$ is either separated by E_f into two disjoint sets Ψ^+_{2r} and Ψ^-_{2r}, or $\Psi_{2r}(y,s) \cap D \subset D_+$ (or D_-). We define, for $0 < r < r_0$, the shifting operators F^+_r and F^-_r:

$$F^+_r(x,t) = (x'', x_{n-1} + 4nLr, x_n + r, t + 4r^2), \tag{5-3}$$

$$F^-_r(x,t) = (x'', x_{n-1} + 4nLr, x_n - r, t + 4r^2). \tag{5-4}$$

For any $0 < r < r_0$, define

$$D^+_r = D \setminus (E^+_{r,1} \cup E^+_{r,2} \cup E^+_{r,3} \cup E^+_{r,4}), \tag{5-5}$$

where

$$\begin{aligned}
E^+_{r,1} &:= \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : x_{n-1} \le f(x'',t), -r \le x_n \le 0\},\\
E^+_{r,2} &:= \{(x,t) : 1 - r \le x_n \le 1\},\\
E^+_{r,3} &:= \{(x,t) : 4nL(1-r) \le x_{n-1} \le 4nL\},\\
E^+_{r,4} &:= \{(x,t) : 1 - 4r^2 \le t \le 1\}.
\end{aligned}$$

It is easy to see that $D_r^+ \subset D$ and $F_r^+(D_r^+) \subset D$. Similarly, we can define $D_r^- \subset D$ satisfying $F_r^-(D_r^-) \subset D$. Notice that $D_r^+ \nearrow D$, $D_r^- \nearrow D$ as $r \searrow 0$. Moreover, it is clear that, for each $r \in \left(0, \frac{1}{4}\right)$,

$$\mathcal{N}_{1/4}(E_f) \cap \partial_p D \subset (\partial_p D_r^+ \cup \partial_p D_r^-) \cap \partial_p D, \tag{5-6}$$

$$E_f \subset \partial_p D_r^+ \cap \partial_p D_r^-. \tag{5-7}$$

Let ω_r^+ and ω_r^- denote the caloric measures with respect to D_r^+ and D_r^-, respectively. Given $(x, t) \in D$ and $r > 0$ small enough such that $(x, t) \in D_r^+ \cap D_r^-$, $\omega_r^{\pm^{(x,t)}}$ are Radon measures on $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$ (recall $D_\pm = D \cap \{x_n \gtrless 0\}$. Moreover, if K is a relatively compact Borel subset of $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$, then, by the comparison principle, $\omega_r^{\pm^{(x,t)}}(K) \le \omega_{r'}^{\pm^{(x,t)}}(K)$ for $0 < r' < r$. Hence, there exist Radon measures $\vartheta_\pm^{(x,t)}$ on $\partial_p(D_r^\pm) \cap \partial_p(D_\pm)$ such that

$$\omega_r^{\pm^{(x,t)}}\big|_{\partial_p(D_r^\pm)\cap\partial_p(D_\pm)} \overset{*}{\rightharpoonup} \vartheta_\pm^{(x,t)}, \qquad r \to 0.$$

For $(y, s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$ and $0 < r < r_0$, denote

$$\Delta_r^\pm(y, s) := \Delta_r(y, s) \cap \partial_p D_\pm \quad \text{if } \Delta_r(y, s) \cap \partial_p(D_\pm) \neq \varnothing.$$

Note that if $\Delta_r(y, s) \subset E_f$ then $\Delta_r^\pm(y, s) = \Delta_r(y, s)$. It is easy to see that $(x, t) \mapsto \vartheta_\pm^{(x,t)}(\Delta_r^\pm(y, s))$ are caloric in D.

To simplify the notation we will write Δ_r, $\Delta_r^\pm$ instead of $\Delta_r(y, s)$, $\Delta_r^\pm(y, s)$. If $\Delta_r(y, s) \cap \partial_p(D_+)$ (or $\Delta_r(y, s) \cap \partial_p(D_-)$) is empty, we set $\vartheta_+^{(x,t)}(\Delta_r^+(y, s)) = 0$ (or $\vartheta_-^{(x,t)}(\Delta_r^-(y, s)) = 0$).

We also note that, with Convention 5.1 in mind, the future corkscrew points $\bar{A}_r^\pm(y, s)$ or $\bar{A}_r(y, s)$, $0 < r < r_0$, are defined for all $s \in [-1, 1]$.

Proposition 5.2. *Given $(y, s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$, for $0 < r < r_0$, we have*:

(i) $$\sup_{(x,t)\in\partial_p D_{r'}^+\cap D} \vartheta_+^{(x,t)}(\Delta_r^+) \to 0 \quad \textit{and} \quad \sup_{(x,t)\in\partial_p D_{r'}^-\cap D} \vartheta_-^{(x,t)}(\Delta_r^-) \to 0 \quad \textit{as } r' \to 0.$$

(ii) $\vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-) = \omega^{(x,t)}(\Delta_r)$ *for* $(x, t) \in D$.

(iii) *There exists a constant $C = C(n, L)$ such that, for any $0 < r' < r$,*

$$\vartheta_+^{(x,t)}(\Delta_{r'}^+) \le C\vartheta_+^{\bar{A}_r^+(y,s)}(\Delta_{r'}^+)\vartheta_+^{(x,t)}(\Delta_{2r}^+) \quad \textit{for } (x, t) \in D \setminus \Psi_r^+(y, s),$$

$$\vartheta_-^{(x,t)}(\Delta_{r'}^-) \le C\vartheta_-^{\bar{A}_r^-(y,s)}(\Delta_{r'}^-)\vartheta_-^{(x,t)}(\Delta_{2r}^-) \quad \textit{for } (x, t) \in D \setminus \Psi_r^-(y, s).$$

(iv) *For (X, T) as defined above and $(y, s) \in E_f \setminus G_f$, there exists a positive constant $C = C(n, L, r_0)$ such that*

$$C^{-1}\vartheta_+^{(X,T)}(\Delta_r^+) \le \vartheta_-^{(X,T)}(\Delta_r^-) \le C\vartheta_+^{(X,T)}(\Delta_r^+).$$

Proof of (i). We assume that $\Delta_r^\pm \neq \varnothing$. If either of them is empty, the conclusion obviously holds.

For $0 < r < r_0$, we have

$$\partial_p D_r^+ \cap D = \{(x, t) \in D : x_{n-1} = 4nL(1 - r) \text{ or } x_n = 1 - r\}$$
$$\cup \{(x, t) \in D : x_{n-1} \le f(x'', t), x_n = -r \text{ or } x_{n-1} = f(x'', t), -r \le x_n < 0\}.$$

Given $(y,s) \in (\mathcal{N}_{1/4}(E_f) \cap \partial_p D) \setminus G_f$, let $0 < r'' < r' < r_0$; then $\omega_{r''}^{+(x,t)}(\Delta_r^+(y,s))$ is caloric in $D_{r''}^+$, and from the way r_0 is chosen, vanishes continuously on $\Delta_{r_0}(z,h)$ for each $(z,h) \in \partial_p D_{r''}^+ \cap D$. Notice that

$$\partial_p D_{r'}^+ \cap D \subset \bigcup_{(z,h)\in\partial_p D_{r''}^+ \cap D} \Psi_{r_0}(z,h),$$

hence, applying Lemma 4.2 in each $\Psi_{r_0}(z,h) \cap D_{r''}^+$, we obtain constants $C = C(n, L)$ and $\gamma = \gamma(n, L)$, $\gamma \in (0, 1)$, such that

$$\omega_{r''}^{+(x,t)}(\Delta_r^+) \le C\left(\frac{|x-z| + |t-h|^{1/2}}{r_0}\right)^{\gamma} \le C\left(\frac{r'}{r_0}\right)^{\gamma} \quad \text{for all } (x,t) \in \partial_p D_{r'}^+ \cap D. \tag{5-8}$$

The constants C and γ above do not depend on $(z,h) \in \partial_p D_{r''}^+ \cap D$, r or r'' because of the existence of the exterior flat parabolic cones centered at each (z,h) with an uniform opening depending only on n and L.

Let $r'' \to 0$ in (5-8), we then get

$$\vartheta_+^{(x,t)}(\Delta_r^+) \le C\left(\frac{r'}{r_0}\right)^{\gamma} \quad \text{uniformly for } (x,t) \in \partial_p D_{r'}^+ \cap D.$$

Therefore

$$\lim_{r'\to 0} \sup_{(x,t)\in\partial_p D_{r'}^+ \cap D} \vartheta_+^{(x,t)}(\Delta_r^+) = 0,$$

which finishes the proof.

Proof of (ii): Let χ_{Δ_r} be the characteristic function of Δ_r on $\partial_p D$. Let g_n be a sequence of nonnegative continuous functions on $\partial_p D$ such that $g_n \nearrow \chi_{\Delta_r}$. Let u_n be the solution to the heat equation in D with boundary values g_n. Then, by the maximum principle, $u_n(x,t) \nearrow \omega^{(x,t)}(\Delta_r)$ for $(x,t) \in D$.

Now we estimate $\vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-)$. Let $u_{n,r'}^+(x,t)$ be the solution to the heat equation in $D_{r'}^+$ with boundary value equal to g_n on $\partial_p D_{r'}^+ \cap \partial_p D$ and equal to $\vartheta_+^{(x,t)}(\Delta_r^+)$ otherwise. Since $\vartheta_+^{(x,t)}(\Delta_r^+) = \lim_{r''\to 0} \omega_{r''}^{+(x,t)}(\Delta_r^+)$ takes the boundary value $\chi_{\Delta_r^+}$ on $\partial_p D_{r'}^+ \cap \partial_p D$, then, by the maximum principle, we have $u_{n,r'}^+(x,t) \le \vartheta_+^{(x,t)}(\Delta_r^+)$ for $(x,t) \in D_{r'}^+$. Similarly, $u_{n,r'}^-(x,t) \le \vartheta_-^{(x,t)}(\Delta_r^-)$ for $(x,t) \in D_{r'}^-$. Therefore, for $(x,t) \in D_{r'}^+ \cap D_{r'}^-$ and $0 < r' < r$ sufficiently small, we have

$$u_{n,r'}^+(x,t) + u_{n,r'}^-(x,t) \le \vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-). \tag{5-9}$$

Let $r' \searrow 0$; then $D_{r'}^+ \cap D_{r'}^- \nearrow D$. By the comparison principle, there is a nonnegative function $\tilde{u}_n$ in Ψ_1, caloric in D, such that

$$u_{n,r'}^+(x,t) + u_{n,r'}^-(x,t) \nearrow \tilde{u}_n(x,t) \quad \text{as } r' \searrow 0,\ (x,t) \in D. \tag{5-10}$$

By (i) just shown above and (5-9),

$$\sup_{\partial_p D_{r'}^+ \cap D} u_{n,r'}^+(x,t) + \sup_{\partial_p D_{r'}^- \cap D} u_{n,r'}^-(x,t) \le \sup_{\partial_p D_{r'}^+ \cap D} \vartheta_+^{(x,t)}(\Delta_r^+) + \sup_{\partial_p D_{r'}^- \cap D} \vartheta_-^{(x,t)}(\Delta_r^-) \to 0 \quad \text{as } r' \to 0,$$

hence it is not hard to see that $\tilde{u}_n$ takes the boundary value g_n continuously on $\partial_p D$. Hence, by the maximum principle, $\tilde{u}_n = u_n$ in D. This, combined with (5-9) and (5-10), gives

$$u_n(x,t) \le \vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-). \tag{5-11}$$

Letting $n \to \infty$ in (5-11), we obtain

$$\omega^{(x,t)}(\Delta_r) \le \vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^+).$$

By taking the approximation $g_n \searrow \chi_{\Delta_r}$, $0 \le g_n \le 2$ and $\operatorname{supp} g_n \subset \mathcal{N}_{2r}(E_f) \cap \partial_p D$, we obtain the reverse inequality, and hence the equality.

Proof of (iii): We only show it for ϑ_+, and assume additionally that $\Delta_{r'}^\pm \ne \varnothing$.

First, for $0 < r'' < r' < r_0$, by Lemma 1.1 in [Kemper 1972b], there exists $C = C(n) \ge 0$ such that

$$\omega_{\Psi_{2r'}(y,s)\cap D_+}^{\bar{A}_{r'}^+(y,s)}(\Delta_{r'}^+) \ge C.$$

Applying the comparison principle in $\Psi_{2r'}(y,s) \cap D_+$, we have

$$\vartheta_+^{\bar{A}_{r'}^+(y,s)}(\Delta_{r'}^+) \ge C. \tag{5-12}$$

Next, for $0 < r'' < r' < r_0$, applying the same induction arguments as in Lemma 4.5, we have

$$\omega_{r''}^{+(x,t)}(\Delta_{r'}^+) \le C\omega_{r''}^{+\bar{A}_r^+(y,s)}(\Delta_{r'}^+) \quad \text{for } (x,t) \in D_{r''}^+ \setminus (\Psi_r(y,s))_+, \tag{5-13}$$

where $C = C(n, L)$ is independent of r' and r''. The reason that C is uniform in r'' is as follows. By the maximum principle, it is enough to show (5-13) for $(x,t) \in \partial(\Psi_r(y,s))_+ \cap D_{r''}^+$, which is contained in D_+. Hence, the same iteration procedure as in Lemma 4.5, but only on the D_+ side, gives (5-13), and the proof is uniform in r''. Therefore, letting $r'' \to 0$ in (5-13), we obtain

$$\vartheta_+^{(x,t)}(\Delta_{r'}^+) \le C\vartheta_+^{\bar{A}_r^+(y,s)}(\Delta_{r'}^+).$$

Applying Lemma 4.4 and the maximum principle, we deduce (iii).

Proof of (iv): Applying (iii), (ii), the Harnack inequality and Lemma 4.4, we have that, for given $(y,s) \in E_f \setminus G_f$ and $0 < r < r_0$,

$$\vartheta_-^{(X,T)}(\Delta_r^-) \le C\vartheta_-^{\bar{A}_{r_0}^-(y,s)}(\Delta_r^-) \le C\omega^{\bar{A}_{r_0}^-(y,s)}(\Delta_r) \le C\omega^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r) \le C\vartheta_+^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r^+) \le C\vartheta_+^{(X,T)}(\Delta_r^+)$$

for $C = C(n, L, r_0)$. The second-last inequality holds because

$$\vartheta_+^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r^+) \ge \vartheta_-^{\bar{A}_{2r_0}^+(y,s)}(\Delta_r^-), \tag{5-14}$$

which follows from the x_n-symmetry of D and the comparison principle. Equation (5-14), together with (ii) just shown above, yields the result. □

Now we use ϑ_+ and ϑ_- to construct two linearly independent kernel functions at $(y,s) \in E_f \setminus G_f$.

Theorem 5.3. *For $(y,s) \in E_f \setminus G_f$, there exist at least two linearly independent kernel functions at (y,s).*

Proof. Given $(y,s)\in E_f\setminus G_f$, let r_0 be as in (5-2). For $m>1/r_0$, we consider the sequence

$$v_m^+(x,t)=\frac{\vartheta_+^{(x,t)}(\Delta_{1/m}^+(y,s))}{\vartheta_+^{(X,T)}(\Delta_{1/m}^+(y,s))},\quad (x,t)\in D. \tag{5-15}$$

By Proposition 5.2(iii) and the same arguments as in Section 5A, we have, up to a subsequence, that $v_m(x,t)$ converges to a kernel function at (y,s) normalized at (X,T). We denote it by $K^+(x,t;y,s)$.

If we consider instead

$$v_m^-(x,t)=\frac{\vartheta_-^{(x,t)}(\Delta_{1/m}^-(y,s))}{\vartheta_-^{(X,T)}(\Delta_{1/m}^-(y,s))},\quad (x,t)\in D, \tag{5-16}$$

we will obtain another kernel function at (y,s), which we will denote $K^-(x,t;y,s)$.

We now show that, for fixed (y,s), $K^+(\cdot,\cdot;y,s)$ and $K^-(\cdot,\cdot;y,s)$ are linearly independent. In fact, by Proposition 5.2(i), (5-15) and (5-16), we have $K^+(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from D_- and $K^-(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from D_+. If $K^+(\cdot,\cdot;y,s)=K^-(\cdot,\cdot;y,s)$, then we also have $K^+(x,t;y,s)\to 0$ as $(x,t)\to(y,s)$ from D_+, which will mean that $K^+(x,t;y,s)$ is a caloric function continuously vanishing on the whole of $\partial_p D$. By the maximum principle, K^+ will vanish in the entire domain D, which contradicts the normalization condition $K^+(X,T;y,s)=1$. Moreover, since $K^+(X,T;y,s)=K^-(X,T;y,s)=1$, it is impossible that $K^+(\cdot,\cdot;y,s)=\lambda K^-(\cdot,\cdot;y,s)$ for a constant $\lambda\neq 1$. Hence K^+ and K^- are linearly independent. □

Remark 5.4. The nonuniqueness of the kernel functions at (y,s) shows that the parabolic Martin boundary of D is not homeomorphic to the Euclidean parabolic boundary $\partial_p D$.

Next we show that K^+ and K^- in fact span the space of all the kernel functions at (y,s). We use an argument similar to the one in [Kemper 1972b].

Lemma 5.5. *Let $(y,s)\in E_f\setminus G_f$. There exists a positive constant $C=C(n,L,r_0)$ such that, if u is a kernel function at (y,s) in D, we have either*

$$u\ge CK^+ \tag{5-17}$$

or

$$u\ge CK^-. \tag{5-18}$$

Here K^+, K^- are the kernel functions at (y,s) constructed from (5-15) *and* (5-16).

Proof. For $0<r<r_0$, we consider $u_r^\pm:D_r^\pm\to\mathbb{R}$, where $u_r^\pm(x,t)=u(F_r^\pm(x,t))$. The functions $u_r^\pm$ are caloric in $D_r^\pm$ and continuous up to the boundary. Then, for $(x,t)\in D_r^\pm$,

$$\begin{aligned}u_r^\pm(x,t)&=\int_{\partial_p D_r^\pm}u_r^\pm(z,h)\,d\omega_r^{\pm^{(x,t)}}(z,h)\\&\ge\int_{\Delta_r^\pm(y,s)}u_r^\pm(z,h)\,d\omega_r^{\pm^{(x,t)}}(z,h)\\&\ge\inf_{(z,h)\in\Delta_r^\pm(y,s)}u_r^\pm(z,h)\omega_r^{\pm^{(x,t)}}(\Delta_r^\pm(y,s)).\end{aligned}$$

Note that the parabolic distance between $F_r^{\pm}(\Delta_r^{\pm}(y,s))$ and $\partial_p D$ is equivalent to r, and the time lag between it and $\bar{A}_r^{\pm}(y,s)$ is equivalent to r^2; hence, by the Harnack inequality, there exists $C = C(n, L)$ such that

$$\inf_{(z,h)\in\Delta_r^{\pm}(y,s)} u_r^{\pm}(z,h) \geq Cu(\bar{A}_r^{\pm}(y,s)).$$

Hence,

$$u_r^{\pm}(x,t) \geq Cu(\bar{A}_r^{\pm}(y,s))\omega_r^{\pm^{(x,t)}}(\Delta_r^{\pm}(y,s)) \quad \text{for } (x,t)\in D_r^{\pm}. \tag{5-19}$$

On the other hand, u is a kernel function at (y,s), and u vanishes on $\partial_p D \setminus \Delta_{r/4}(y,s)$ for any $0 < r < 1$. Applying Theorem 4.6, we obtain

$$u(x,t) \leq C\max\{u(\bar{A}_{r/2}^{+}(y,s)), u(\bar{A}_{r/2}^{-}(y,s))\}\omega^{(x,t)}(\Delta_r(y,s)) \quad \text{for } (x,t)\in D\setminus\Psi_{r/2}(y,s). \tag{5-20}$$

Case 1: $u(\bar{A}_{r/2}^{+}(y,s)) \geq u(\bar{A}_{r/2}^{-}(y,s))$ in (5-20).

By Proposition 5.2(ii) and the Harnack inequality,

$$u(x,t) \leq Cu(\bar{A}_r^{+}(y,s))(\vartheta_+^{(x,t)}(\Delta_r^+) + \vartheta_-^{(x,t)}(\Delta_r^-)), \quad (x,t)\in D\setminus\Psi_{r/2}(y,s).$$

In particular,

$$1 = u(X,T) \leq Cu(\bar{A}_r^{+}(y,s))(\vartheta_+^{(X,T)}(\Delta_r^+) + \vartheta_-^{(X,T)}(\Delta_r^-)). \tag{5-21}$$

Now (5-19) for u_r^+, (5-21) and Proposition 5.2(iv) yield the existence of $C_1 = C_1(n, L, r_0)$ such that, for any $0 < r < r_0$,

$$u_r^+(x,t) \geq C\frac{\omega_r^{+^{(x,t)}}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+) + \vartheta_-^{(X,T)}(\Delta_r^-)} \geq C_1\frac{\omega_r^{+^{(x,t)}}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)}, \quad (x,t)\in D_r^+. \tag{5-22}$$

Since, by the maximum principle in D_r^+,

$$\omega_r^{+^{(x,t)}}(\Delta_r^+) \geq \vartheta_+^{(x,t)}(\Delta_r^+) - \sup_{(z,h)\in\partial_p D_r^+\cap D}\vartheta_+^{(z,h)}(\Delta_r^+), \tag{5-23}$$

then (5-22) can be written as

$$u_r^+(x,t) \geq C_1\left(\frac{\vartheta_+^{(x,t)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} - \sup_{(z,h)\in\partial_p D_r^+\cap D}\frac{\vartheta_+^{(z,h)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)}\right), \quad (x,t)\in D_r^+. \tag{5-24}$$

By Proposition 5.2(iii) and the Harnack inequality, there exists $C_2 = C_2(n, L, r_0)$ such that, for $(z,h)\in \partial_p D_r^+\cap D$,

$$\frac{\vartheta_+^{(z,h)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} \leq C\frac{\vartheta_+^{\bar{A}_{r_0}^+}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)}\cdot\vartheta_+^{(z,h)}(\Delta_{r_0}^+) \leq C_2\vartheta_+^{(z,h)}(\Delta_{r_0}^+). \tag{5-25}$$

Hence, (5-24) and (5-25) imply

$$u_r^+(x,t) \geq C_1\left(\frac{\vartheta_+^{(x,t)}(\Delta_r^+)}{\vartheta_+^{(X,T)}(\Delta_r^+)} - C_2\sup_{(z,h)\in\partial_p D_r^+\cap D}\vartheta_+^{(z,h)}(\Delta_{r_0}^+)\right), \quad (x,t)\in D_r^+.$$

Case 2: $u(\bar{A}^+_{r/2}(y,s)) \le u(\bar{A}^-_{r/2}(y,s))$ in (5-20). Similarly,

$$u_r^-(x,t) \ge C_1\left(\frac{\vartheta_-^{(x,t)}(\Delta_r^-)}{\vartheta_-^{(X,T)}(\Delta_r^-)} - C_2 \sup_{(z,h)\in\partial_p D_r^-\cap D} \vartheta_-^{(z,h)}(\Delta_{r_0}^-)\right), \quad (x,t)\in D_r^-.$$

Note that as $r \searrow 0$, $D_r^\pm \nearrow D$ and $u_r^\pm \to u$. Let $r_j \to 0$ be such that either Case 1 applies for all r_j, or Case 2 applies. Hence, over a subsequence, it follows by Proposition 5.2(i) and (5-15) that either

$$u(x,t) \ge C_1 \lim_{r_j\to 0}\left(\frac{\vartheta_+^{(x,t)}(\Delta_{r_j}^+)}{\vartheta_+^{(X,T)}(\Delta_{r_j}^+)} - C_2 \sup_{(z,h)\in\partial_p D_{r_j}^+\cap D} \vartheta_+^{(z,h)}(\Delta_{r_0}^+)\right) = C_1 K^+(x,t) \quad \text{for all } (x,t)\in D,$$

or

$$u(x,t) \ge C_1 K^-(x,t) \quad \text{for all } (x,t)\in D. \qquad \square$$

The next theorem says that $K^+(\,\cdot\,,\cdot\,;y,s)$ and $K^-(\,\cdot\,,\cdot\,;y,s)$ span the space of kernel functions at (y,s).

Theorem 5.6. *If u is a kernel function at $(y,s)\in E_f\setminus G_f$ normalized at (X,T), then there exists a constant $\lambda\in[0,1]$, which may depend on (y,s), such that $u(\,\cdot\,,\cdot\,) = \lambda K^+(\,\cdot\,,\cdot\,;y,s) + (1-\lambda)K^-(\,\cdot\,,\cdot\,;y,s)$ in D, where K^+ and K^- are kernel functions obtained from* (5-15) *and* (5-16).

Proof. By Lemma 5.5, if u is a kernel function at (y,s), then either (i) $u \ge CK^+$ or (ii) $u \ge CK^-$ with $C = C(r_0,n,L)$.

If (i) holds, let

$$\lambda = \sup\{C : u(x,t) \ge CK^+(x,t) \text{ for all } (x,t)\in D\};$$

then we must have $\lambda \le 1$, because $u(X,T) = K^+(X,T) = 1$. If $\lambda = 1$, then $u(x,t) = K^+(x,t)$ for all $(x,t)\in D$, by the strong maximum principle, and we are done. If $\lambda < 1$, consider

$$u_1(x,t) := \frac{u(x,t) - \lambda K^+(x,t)}{1-\lambda},$$

which is another kernel function at (y,s) satisfying either (i) or (ii). If (i) holds for u_1 for some $C > 0$, then $u(x,t) \ge (C(1-\lambda)+\lambda)K^+(x,t)$, with $C(1-\lambda)+\lambda > \lambda$, which contradicts the definition of λ as a supremum. Hence (ii) must be true for u_1. Let

$$\tilde\lambda = \sup\{C : u_1(x,t) \ge CK^-(x,t)\ \forall (x,t)\in D\}.$$

The same reason as above gives $\tilde\lambda \le 1$. We claim $\tilde\lambda = 1$.

Proof of the claim: If not, then $\tilde\lambda < 1$. We get that

$$u_2(x,t) := \frac{u_1(x,t) - \tilde\lambda K^-(x,t)}{1-\tilde\lambda}$$

is again a kernel function at (y,s). If u_2 satisfies (i) for some $C > 0$, then

$$u_1(x,t) \ge u_1(x,t) - \tilde\lambda K^-(x,t) \ge C(1-\tilde\lambda)K^+(x,t),$$

which implies

$$u(x,t) \geq (\lambda + C(1-\tilde{\lambda}))K^+(x,t),$$

again a contradiction to the definition of λ. Hence, u_2 has to satisfy (ii) for some $C > 0$, and then we have

$$u_2(x,t) \geq (C(1-\tilde{\lambda}) + \tilde{\lambda})K^-(x,t),$$

but this contradicts the definition of $\tilde{\lambda}$. This completes the proof of the claim.

The fact that $\tilde{\lambda} = 1$ implies that $u_1(x,t) = K^-(x,t)$ in D, by the strong maximum principle. Hence, if (i) applies to u, we have $u(x,t) = \lambda K^+(x,t) + (1-\lambda)K^-(x,t)$ with $\lambda \in (0,1]$. If (ii) applies to u, we get the equality with $\lambda \in [0,1)$. □

5C. ***Radon–Nikodym derivative as a kernel function.*** We first show that the kernel function at $(y,s) \in G_f$ or $(y,s) \in \partial_p D \setminus E_f$ is unique. The proof for the uniqueness is similar to Lemma 1.6 and Theorem 1.7 in [Kemper 1972b]. More precisely, we will need the direction-shift operator F_r^0:

$$F_r^0(x,t) = (x'', x_{n-1} + 4nLr, x_n, t + 8r^2), \quad 0 < r < \tfrac{1}{4}, \tag{5-26}$$
$$D_r^0 = \{(x,t) \in D : F_r^0(x,t) \in D\}.$$

Let ω_r^0 denote the caloric measure for D_r^0. Note that D_r^0 is also a cylindrical domain with a thin Lipschitz complement.

Theorem 5.7. *For all $(y,s) \in \partial_p D$, the limit of* (5-1) *exists. If we denote the limit by $K_0(\cdot,\cdot;y,s)$, i.e.,*

$$K_0(x,t;y,s) = \lim_{n\to\infty} \frac{\omega^{(x,t)}(\Delta_{1/n}(y,s))}{\omega^{(X,T)}(\Delta_{1/n}(y,s))},$$

then:

(i) *For $(y,s) \in G_f$ or $(y,s) \in \partial_p D \setminus E_f$, K_0 is the unique kernel function at (y,s).*

(ii) *If $(y,s) \in E_f \setminus G_f$, then K_0 is a kernel function at (y,s), and*

$$K_0(x,t;y,s) = \tfrac{1}{2}K^+(x,t;y,s) + \tfrac{1}{2}K^-(x,t;y,s), \tag{5-27}$$

where K^+ and K^- are kernel functions at (y,s) given by the limits of (5-15) *and* (5-16), *respectively.*

Proof. For $(y,s) \in G_f$ and r small enough, we denote $\bar{\bar{A}}_r(y,s) = (y'', y_{n-1} + 4nrL, 0, s + 4r^2)$, which is on $\{x_n = 0\}$ and has a time-lag $2r^2$ above $\bar{A}_r^{\pm}$. Then, by the Harnack inequality,

$$\omega^{\bar{A}_r^{\pm}(y,s)}(\Delta_{r'}(y,s)) \leq C(n,L)\omega^{\bar{\bar{A}}_r(y,s)}(\Delta_{r'}(y,s)) \qquad \text{for all } 0 < r' < r.$$

Then one can proceed as in Lemma 1.6 of [ibid.] by using F_r^0, D_r^0, ω^0 to show that any kernel function u (at (y,s)) satisfies $u \geq CK_0$ for some $C > 0$. Then the uniqueness follows from Theorem 1.7 and Remark 1.8 of [ibid.].

For $(y,s) \in \partial_p D \setminus E_f$, for r sufficiently small one has either $\Psi_r(y,s) \cap D \subset D_+$ or $\Psi_r(y,s) \cap D \subset D_-$. In either case, one can proceed as in Lemma 1.6, Theorem 1.7 and Remark 1.8 of [ibid.].

For $(y, s) \in E_f \setminus G_f$, by Theorem 5.6, $K_0(x, t; y, s) = \lambda K^+(x, t; y, s) + (1-\lambda)K^-(x, t; y, s)$ for some $\lambda \in [0, 1]$. By Proposition 5.2(ii), the symmetry of the domain about x_{n-1} and the definitions of $K^\pm$, one has $\lambda = \frac{1}{2}$. □

Remark 5.8. From Theorem 5.7, we can conclude that the Radon–Nikodym derivative $d\omega^{(x,t)}/d\omega^{(X,T)}$ exists at every $(y, s) \in \partial_p D$ and it is the kernel function $K_0(x, t; y, s)$ with respect to (X, T).

The following corollary is an easy consequence of Theorems 5.6 and 5.7.

Corollary 5.9. *For fixed $(x, t) \in D$, the function $(y, s) \mapsto K_0(x, t; y, s)$ is continuous on $\partial_p D$, where K_0 is given by the limit of* (5-1).

Proof. Given $(y, s) \in \partial_p D$, let $(y_m, s_m) \in \partial_p D$ with $(y_m, s_m) \to (y, s)$ as $m \to \infty$.

If $(y, s) \in G_f$ or $\partial_p D \setminus E_f$, continuity follows from the uniqueness of the kernel function.

If $(y, s) \in E_f \setminus G_f$, by Theorem 5.7(ii), for each m we have

$$K_0(x, t; y_m, s_m) = \tfrac{1}{2}K^+(x, t; y_m, s_m) + \tfrac{1}{2}K^-(x, t; y_m, s_m). \tag{5-28}$$

Given $\varepsilon > 0$, $K^+(\cdot\,,\cdot\,; y_m, s_m)$ is uniformly bounded and equicontinuous on $D \setminus \Psi_\varepsilon(y, s)$ for m large enough. Hence, by a similar argument as in Section 5A, up to a subsequence, $K^+(\cdot\,,\cdot\,; y_m, s_m) \to v^+(\cdot\,,\cdot\,; y, s)$ uniformly on compact subsets, where $v^+(\cdot\,,\cdot\,; y, s)$ is some kernel function at (y, s). Moreover, by Theorem 5.6, we have

$$v^+(\cdot\,,\cdot\,; y, s) = \lambda K^+(\cdot\,,\cdot\,; y, s) + (1-\lambda)K^-(\cdot\,,\cdot\,; y, s) \quad \text{for some } \lambda \in [0, 1]. \tag{5-29}$$

By Proposition 5.2(i),

$$\sup_{(x,t)\in\partial_p D_r^+ \cap D} K^+(x, t; y_m, s_m) \to 0 \quad \text{as } r \to 0,$$

which is uniform in m from the proof of the proposition. Hence, after $m \to \infty$, v^+ satisfies

$$\sup_{(x,t)\in\partial_p D_r^+ \cap D} v^+(x, t) \to 0 \quad \text{as } r \to 0,$$

which, combined with

$$K^-(x, t; y, s) \not\to 0 \quad \text{as } (x, t) \to (y, s), \text{ for } (x, t) \in D_-,$$

gives $\lambda = 1$ in (5-29).

Similarly, up to a subsequence, $K^-(x, t; y_m, s_m) \to K^-(x, t; y, s)$.

Thus, along a subsequence, $K(\cdot\,,\cdot\,; y_m, s_m) \to K_0(\cdot\,,\cdot\,; y, s)$ by (5-27). Since this holds for all the convergent subsequences, then $K_0(x, t; y, s)$ is continuous on $\partial_p D$ for fixed (x, t). □

By using Corollary 5.9, Remark 5.8 and Theorem 4.6, we can prove some uniform behavior of K_0 on $\partial_p D$, as in Lemmas 2.2 and 2.3 of [Kemper 1972b]. We state the results in the following two lemmas and omit the proof of the first.

Lemma 5.10. *Let $(y,s)\in\partial_p D$. Then, for $0<r<\frac{1}{4}$,*

$$\sup_{(y',s')\in\partial_p D\setminus\Delta_r(y,s)} K_0(x,t;y',s')\to 0 \quad \text{as } (x,t)\to(y,s) \text{ in } D.$$

The following lemma says that if D' is a domain obtained by a perturbation of a portion of $\partial_p D$ where $\omega^{(x,t)}$ vanishes, then the caloric measure $\omega_{D'}$ is equivalent to ω_D on the common boundary of D' and D. We recall here that ω_r^0 is the caloric measure with respect to the domain D_r^0 defined in (5-26), and $\omega_r^\pm$ is the caloric measure with respect to $D_r^\pm$ defined in (5-5).

Lemma 5.11. (i) *Let $0<r<\frac{1}{4}$ and $(y,s)\in G_f\cup(\partial_p D\setminus E_f)$ with $s>-1+4r^2$. Then there exist $\rho_0=\rho_0(n,L)>0$ and $C=C(n,L)>0$ such that, for $0<\rho<\rho_0$, we have*

$$\omega_\rho^{0^{(X',T')}}(\Delta_r(y,s))\geq C\omega^{(X',T')}(\Delta_r(y,s)),\quad (X',T')\in\Psi_{1/4}(X,T), \tag{5-30}$$

provided also $r<|y_n|$ for $(y,s)\in\partial_p D\setminus E_f$.

(ii) *Let $(y,s)\in(\mathcal{N}_r(E_f)\cap\partial_p D)\setminus G_f$. Then there exists $\delta_0=\delta_0(n,L)>0$, such that, for $0<r'<\delta_0$, we have*

$$\omega_{r'}^{+^{(X',T')}}(\Delta_r^+(y,s))+\omega_{r'}^{-^{(X',T')}}(\Delta_r^-(y,s))\geq\tfrac{1}{2}\omega^{(X',T')}(\Delta_r(y,s)) \tag{5-31}$$

for $(X',T')\in\Psi_{1/4}(X,T)$ and $0<r<r_0$, where r_0 is the constant defined in (5-2).

Proof. To show (5-31) we first argue similarly as in [Kemper 1972b] to show there exists $\delta_0=\delta_0(n,L)>0$ such that, for any $0<r'<\delta_0$,

$$\omega_{r'}^{\pm^{(X',T')}}(\Delta_r^\pm(y,s))\geq\tfrac{1}{2}\vartheta_\pm^{(X',T')}(\Delta_r^\pm(y,s)) \tag{5-32}$$

for each $\Delta_r^\pm(y,s)$ with $0<r<r_0$. Then using Proposition 5.2(ii) we get the conclusion. □

6. Backward boundary Harnack principle

In this section, we follow the lines of [Fabes et al. 1984] to build up a backward Harnack inequality for nonnegative caloric functions in D. To prove this kind of inequality, we have to ask that these functions vanish on the *lateral boundary*

$$S:=\partial_p D\cap\{s>-1\},$$

or at least a portion of it. This will allow to control the time-lag issue in the parabolic Harnack inequality.

Some of the proofs in this section follow the lines of the corresponding proofs in [ibid.]. For that reason, we will omit the parts that don't require modifications or additional arguments.

For (x,t) and $(y,s)\in D$, denote by $G(x,t;y,s)$ the Green's function for the heat equation in the domain D. Since D is a regular domain, the Green's function can be written in the form

$$G(x,t;y,s)=\Gamma(x,t;y,s)-V(x,t;y,s),$$

where $\Gamma(\cdot,\cdot;y,s)$ is the fundamental solution of the heat equation with pole at (y,s), and $V(\cdot,\cdot;y,s)$ is a caloric function in D that equals $\Gamma(\cdot,\cdot;y,s)$ on $\partial_p D$. We note that, by the maximum principle, we have $G(x,t;y,s)=0$ whenever $(x,t)\in D$ with $t\leq s$.

In this section, similarly to Section 5, we will work under Convention 5.1. In particular, in Green's function we will allow the pole (y, s) to be in $\widetilde{D}$ with $s \ge 1$. But in that case we simply have $G(x, t; y, s) = 0$ for all $(x, t) \in D$.

Lemma 6.1. *Let $0 < r < \frac{1}{4}$ and $(y, s) \in S$ with $s \ge -1 + 8r^2$. Then there exists a constant $C = C(n, L) > 0$ such that, for $(x, t) \in D \cap \{t \ge s + 4r^2\}$, we have*

$$C^{-1} r^n \max\{G(x, t; \bar{A}_r^{\pm}(y, s))\} \le \omega^{(x,t)}(\Delta_r(y, s)) \le C r^n \max\{G(x, t; \underline{A}_r^{\pm}(y, s))\} \quad \text{if } (y, s) \in \mathcal{N}_r(E_f), \tag{6-1}$$

and

$$C^{-1} r^n G(x, t; \bar{A}_r(y, s)) \le \omega^{(x,t)}(\Delta_r(y, s)) \le C r^n G(x, t; \underline{A}_r(y, s)) \quad \text{if } (y, s) \notin \mathcal{N}_r(E_f). \tag{6-2}$$

Proof. The proof uses Lemma 4.4 and Theorem 4.3, and is similar to that of Lemma 1 in [ibid.]. □

Theorem 6.2 (interior backward Harnack inequality). *Let u be a positive caloric function in D vanishing continuously on S. Then, for any compact $K \Subset D$, there exists a constant $C = C(n, L, \operatorname{dist}_p(K, \partial_p D))$ such that*

$$\max_K u \le C \min_K u.$$

Proof. The proof is similar to that of Theorem 1 in [ibid.], and uses Theorem 4.3 and the Harnack inequality. □

Theorem 6.3 (local comparison theorem). *Let $0 < r < \frac{1}{4}$, $(y, s) \in S$ with $s \ge -1 + 18r^2$, and u, v be two positive caloric functions in $\Psi_{3r}(y, s) \cap D$ vanishing continuously on $\Delta_{3r}(y, s)$. Then there exists $C = C(n, L) > 0$ such that, for $(x, t) \in \Psi_{r/8}(y, s) \cap D$, we have*

$$\frac{u(x, t)}{v(x, t)} \le C \frac{\max\{u(\bar{A}_r^{+}(y, s)), u(\bar{A}_r^{-}(y, s))\}}{\min\{v(\underline{A}_r^{+}(y, s)), v(\underline{A}_r^{-}(y, s))\}} \quad \text{if } (y, s) \in \mathcal{N}_r(E_f), \tag{6-3}$$

and

$$\frac{u(x, t)}{v(x, t)} \le C \frac{u(\bar{A}_r(y, s))}{v(\underline{A}_r(y, s))} \quad \text{if } (y, s) \notin \mathcal{N}(E_f). \tag{6-4}$$

Proof. The proof is similar to that of Theorem 3 in [ibid.]. First, note that if $\Psi_{r/8}(y, s) \cap E_f = \varnothing$, we can consider the restrictions of u and v to D_+ or D_- (which are Lipschitz cylinders) and apply the arguments from [ibid.] directly there. Thus, we may assume that $\Psi_{r/8}(y, s) \cap E_f \neq \varnothing$. If we now argue as in the proof of the localization property (Lemma 2.3) by replacing (y, s) and r with $(\tilde{y}, \tilde{s}) \in \Psi_{(3/8)r}(y, s) \cap E_f$, we may further assume that $(y, s) \in E_f$, and that $\Psi_r(y, s) \cap D$ falls either into category (2) or (3) in the localization property. For definiteness, we will assume category (3). To account for the possible change in (y, s), we then change the hypothesis to assume that $u = 0$ on $\Delta_{2r}(y, s)$, and prove (6-3) for $(x, t) \in \Psi_{r/2}(y, s) \cap D$.

With this simplification in mind, we proceed as in the proof of Theorem 3 in [ibid.]. By using Lemma 6.1 and Theorem 4.6, we first show

$$\omega_r^{(x,t)}(\alpha_r) \le C \omega_r^{(x,t)}(\beta_r), \quad (x, t) \in \Psi_{r/2}(y, s) \cap D, \tag{6-5}$$

where $\alpha_r = \partial_p(\Psi_r(y,s)\cap D)\setminus S$, $\beta_r = \partial_p(\Psi_r(y,s)\cap D)\setminus \mathcal{N}_{\mu r}(S)$ with a small fixed $\mu\in(0,1)$, and where ω_r denotes the caloric measure with respect to $\Psi_r(y,s)\cap D$. Then by Theorem 4.3, the Harnack inequality and the maximum principle, we obtain

$$u(x,t)\le C\max\{u(\bar{A}_r^+(y,s)),u(\bar{A}_r^-(y,s))\}\omega_r^{(x,t)}(\alpha_r),$$
$$v(x,t)\ge C\min\{v(\underline{A}_r^+(y,s)),v(\underline{A}_r^-(y,s))\}\omega_r^{(x,t)}(\beta_r),$$

which, combined with (6-5), completes the proof. □

Theorem 6.4 (global comparison theorem). *Let u, v be two positive caloric functions in D, vanishing continuously on S, and let (x_0,t_0) be a fixed point in D. If $\delta>0$, then there exists $C=C(n,L,\delta)>0$ such that*

$$\frac{u(x,t)}{v(x,t)}\le C\frac{u(x_0,t_0)}{v(x_0,t_0)}\quad \text{for all } (x,t)\in D\cap\{t>-1+\delta^2\}. \tag{6-6}$$

Proof. This is an easy consequence of Theorems 6.2 and 6.3. □

Now we show the doubling properties of the caloric measure at the lateral boundary points by using the properties of the kernel functions we showed in Section 5. The idea of the proof is similar to that of Lemma 2.2 in [Wu 1979], but with a more careful inspection of the different types of boundary points.

To proceed, we will need to define the time-invariant corkscrew points at (y,s) on the lateral boundary, in addition to future and past corkscrew points. Namely, for $(y,s)\in S$, we let

$$\begin{aligned} A_r(y,s)&=(y(1-r),s) &&\text{if } \Psi_r(y,s)\cap E_f=\varnothing,\\ A_r^{\pm}(y,s)&=(y'',y_{n-1}+r/2,\pm r/2,s) &&\text{if } \Psi_r(y,s)\cap E_f\neq\varnothing.\end{aligned}$$

Theorem 6.5 (doubling at the lateral boundary points). *For $0<r<\frac14$ and $(y,s)\in S$ with $s\ge -1+8r^2$, there exist $\varepsilon_0=\varepsilon_0(n,L)>0$ small and $C=C(n,L)>0$ such that, for any $r<\varepsilon_0$, we have:*

(i) *If $(y,s)\in E_f$ and $\Psi_{2r}(y,s)\cap G_f\neq\varnothing$, then*

$$C^{-1}r^nG(X,T;A_r^{\pm}(y,s))\le \omega^{(X,T)}(\Delta_r(y,s))\le Cr^nG(X,T;A_r^{\pm}(y,s)). \tag{6-7}$$

(ii) *If $(y,s)\in \mathcal{N}_r(E_f)\cap\partial_pD$ and $\Psi_{2r}(y,s)\cap G_f=\varnothing$, then*

$$C^{-1}r^nG(X,T;A_r^{+}(y,s))\le \vartheta_+^{(X,T)}(\Delta_r^+(y,s))\le Cr^nG(X,T;A_r^{+}(y,s)), \tag{6-8}$$

$$C^{-1}r^nG(X,T;A_r^{-}(y,s))\le \vartheta_-^{(X,T)}(\Delta_r^-(y,s))\le Cr^nG(X,T;A_r^{-}(y,s)). \tag{6-9}$$

(iii) *If $(y,s)\in\partial_pD\setminus\mathcal{N}_r(E_f)$, then*

$$C^{-1}r^nG(X,T;A_r(y,s))\le \omega^{(X,T)}(\Delta_r(y,s))\le Cr^nG(X,T;A_r(y,s)). \tag{6-10}$$

Moreover, there is a constant $C=C(n,L)>0$ such that:

- *For $(y,s)\in S\cap\{s\ge -1+8r^2\}$,*

$$\omega^{(X,T)}(\Delta_{2r}(y,s))\le C\omega^{(X,T)}(\Delta_r(y,s))u(x,t). \tag{6-11}$$

- *For* $(y,s)\in\mathcal{N}_r(E_f)\cap S\cap\{s\geq -1+8r^2\}$,

$$\vartheta_+^{(X,T)}(\Delta_{2r}^+(y,s))\leq C\vartheta_+^{(X,T)}(\Delta_r^+(y,s)),$$
$$\vartheta_-^{(X,T)}(\Delta_{2r}^-(y,s))\leq C\vartheta_-^{(X,T)}(\Delta_r^-(y,s)). \tag{6-12}$$

Proof. We start by showing the estimates from above in (6-7) and (6-8).

Case 1: $(y,s)\in E_f$ and $\Psi_{2r}(y,s)\cap G_f\neq\varnothing$. By Lemma 2.3, there is $(\tilde{y},\tilde{s})\in G_f$ such that

$$\Psi_r(y,s)\cap D\subset\Psi_{4r}(\tilde{y},\tilde{s})\cap D\subset\Psi_{8r}(y,s)\cap D.$$

It is not hard to check, by (5-26), that $F_r^0(\Delta_{4r}(\tilde{y},\tilde{s}))\subset D$. Moreover, the parabolic distance between $F_r^0(\Delta_{4r}(\tilde{y},\tilde{s}))$ and $\partial_p D$, and the t-coordinate distance from $F_r^0(\Delta_{4r}(\tilde{y},\tilde{s}))$ down to $A_r^\pm$, are greater than cr for some universal c which only depends on n and L. Therefore, by the estimate of Green's function as in [Wu 1979], we have

$$G(x,t;A_r^\pm(y,s))\geq C(n,L)r^{-n},\quad (x,t)\in F_r^0(\Delta_{4r}(\tilde{y},\tilde{s})).$$

Applying the maximum principle to $F_r^0(D_r^0)$, we have

$$G(x,t;A_r^\pm(y,s))\geq C(n,L)r^{-n}\omega_r^{0^{F_r^{0^{-1}}(x,t)}}(\Delta_{4r}(\tilde{y},\tilde{s})).$$

In particular,

$$G(X,T;A_r^\pm(y,s))\geq C(n,L)r^{-n}\omega_r^{0^{F_r^{0^{-1}}(X,T)}}(\Delta_{4r}(\tilde{y},\tilde{s})).$$

Let $(X_r,T_r):=F_r^{0^{-1}}(X,T)$ and take $(X',T')\in D$ with $T'=T-\frac14$, $X'=X$, so that $T'>\frac14+T_r$. Then we obtain, by the Harnack inequality, that

$$G(X,T;A_r^\pm(y,s))\geq C(n,L)r^{-n}\omega_r^{0^{(X',T')}}(\Delta_{4r}(\tilde{y},\tilde{s})). \tag{6-13}$$

By Lemma 5.11(i), for $0<r<\min\{\frac14,\rho_0\}$, there exists $C=C(n,L)$, independent of r, such that

$$\omega_r^{0^{(X',T')}}(\Delta_{4r}(\tilde{y},\tilde{s}))\geq C\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s})). \tag{6-14}$$

By Theorem 5.7, for each $(\tilde{y},\tilde{s})\in G_f$,

$$K_0(X',T';\tilde{y},\tilde{s})=\lim_{r\to 0}\frac{\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s}))}{\omega^{(X,T)}(\Delta_{4r}(\tilde{y},\tilde{s}))}>0,$$

and by Corollary 5.9, for (X',T') fixed, $K_0(X',T';\cdot,\cdot)$ is continuous on $\partial_p D$. Therefore, in the compact set G_f, there exists $c>0$, only depending on n, L, such that $K_0(X',T';\tilde{y},\tilde{s})\geq c>0$ for any $(\tilde{y},\tilde{s})\in G_f$. Hence, by the Radon–Nikodym theorem for $0<r<\min\{\frac14,\rho_0\}$, we have

$$\omega^{(X',T')}(\Delta_{4r}(\tilde{y},\tilde{s}))\geq\frac{c}{2}\omega^{(X,T)}(\Delta_{4r}(\tilde{y},\tilde{s}))\geq\frac{c}{2}\omega^{(X,T)}(\Delta_r(y,s)). \tag{6-15}$$

Combining (6-13), (6-14) and (6-15), we obtain the estimate from above in (6-7) for Case 1.

Case 2: $(y,s) \in \mathcal{N}_r(E_f) \cap \partial_p D$ and $\Psi_{2r}(y,s) \cap G_f = \varnothing$.

In this case, $\Psi_{2r}(y,s) \cap D$ splits into the disjoint union of $\Psi_{2r}(y,s) \cap D_\pm$. We use F_r^+ and F_r^-, defined in (5-3) and (5-4), and apply the same arguments as in Case 1 in D_r^+ and D_r^-. Then

$$\omega_r^{\pm(X,T)}(\Delta_r^\pm(y,s)) \le Cr^n G(X,T;A_r^\pm(y,s)).$$

Taking $0 < r < \delta_0$, where $\delta_0 = \delta_0(n,L)$ is the constant in Lemma 5.11(ii), we have

$$\vartheta_\pm^{(X,T)}(\Delta_r(y,s)) \le 2\omega_r^{\pm(X,T)}(\Delta_r(y,s)) \le Cr^n G(X,T;A_r^\pm(y,s)).$$

Case 3: $(y,s) \in \partial_p D \setminus \mathcal{N}_r(E_f)$. We argue similarly to Cases 1 and 2.

Taking $\varepsilon_0 = \min\{\rho_0, \delta_0, \frac{1}{4}\}$, we complete the proof of the estimates from above in (6-7)–(6-10).

The proof of the estimate from below in (6-7)–(6-10) is the same as in [Wu 1979]. For (6-7) it is a consequence of Lemma 4.4 and the maximum principle. (6-8) and (6-9) follow from (5-12) and the maximum principle. The doubling properties of caloric measure $\omega^{(x,t)}$ and $\theta_\pm^{(x,t)}$ are easy consequences of (6-7)–(6-10) and Proposition 5.2(ii) for $0 < r < \varepsilon_0/2$. For $r > \varepsilon_0/2$ we use Lemma 4.4 and (5-12). □

Theorem 6.5 implies the following backward Harnack principle.

Theorem 6.6 (backward boundary Harnack principle). *Let u be a positive caloric function in D vanishing continuously on S, and let $\delta > 0$. Then there exists a positive constant $C = C(n,L,\delta)$ such that, for $(y,s) \in \partial_p D \cap \{s > -1+\delta^2\}$ and for $0 < r < r(n,L,\delta)$ sufficiently small, we have*

$$\left.\begin{aligned} C^{-1}u(\underline{A}_r^+(y,s)) \le u(\bar{A}_r^+(y,s)) \le Cu(\underline{A}_r^+(y,s)) \\ C^{-1}u(\underline{A}_r^-(y,s)) \le u(\bar{A}_r^-(y,s)) \le Cu(\underline{A}_r^-(y,s)) \end{aligned}\right\} \quad \text{if } (y,s) \in \mathcal{N}_r(E_f)$$

and

$$C^{-1}u(\underline{A}_r(y,s)) \le u(\bar{A}_r(y,s)) \le Cu(\underline{A}_r(y,s)) \quad \text{if } (y,s) \notin \mathcal{N}_r(E_f). \tag{6-16}$$

Proof. Once we have Theorem 6.5, which is an analogue of Lemma 2.2 in [Wu 1979], we can proceed as in Theorem 4 in [Fabes et al. 1984] to show the backward Harnack principle. □

Remark 6.7. From (6-7), and using the same proof as in Theorem 6.6, we can conclude that, for any positive caloric function u vanishing continuously on S and $(y,s) \in G_f$, there exists $C = C(n,L,\delta) > 0$ such that

$$C^{-1}u(\bar{A}_r^-(y,s)) \le u(\bar{A}_r^+(y,s)) \le Cu(\bar{A}_r^-(y,s)),$$
$$C^{-1}u(\underline{A}_r^-(y,s)) \le u(\underline{A}_r^+(y,s)) \le Cu(\underline{A}_r^-(y,s)).$$

7. Various versions of boundary Harnack

In the applications, it is very useful to have a local version of the backward Harnack for solutions vanishing only on a portion of the lateral boundary S. For the parabolically Lipschitz domains this was proved in [Athanasopoulos et al. 1996] as a consequence of the (global) backward Harnack principle.

To state the results, we use the following corkscrew points associated with $(y, s) \in G_f$: for $0 < r < \frac{1}{4}$, let

$$\bar{A}_r(y, s) = (y'', y_{n-1} + 4nLr, 0, s + 2r^2),$$
$$\underline{A}_r(y, s) = (y'', y_{n-1} + 4nLr, 0, s - 2r^2),$$
$$A_r(y, s) = (y'', y_{n-1} + 4nLr, 0, s).$$

When $(y, s) = (0, 0)$, we simply write $\bar{A}_r$, $\underline{A}_r$ and A_r, in addition to Ψ_r, Δ_r, $\bar{A}_r^{\pm}$, $\underline{A}_r^{\pm}$.

Theorem 7.1. *Let u be a nonnegative caloric function in D, vanishing continuously on E_f. Let $m = u(\underline{A}_{3/4})$, $M = \sup_D u$. Then there exists a constant $C = C(n, L, M/m)$ such that, for any $0 < r < \frac{1}{4}$, we have*

$$u(\bar{A}_r) \le C u(\underline{A}_r). \tag{7-1}$$

Proof. Using Theorems 6.6 and 6.5 and following the lines of Theorem 13.7 in [Caffarelli and Salsa 2005], we have

$$u(\bar{A}_{2r}^{\pm}) \le C u(\underline{A}_{2r}^{\pm}), \quad 0 < r < \tfrac{1}{4},$$

for $C = C(n, L, M/m)$. Then (7-1) follows from Theorem 6.6 and the observation that there is a Harnack chain with constant $\mu = \mu(n, L)$ and length $N = N(n, L)$ joining $\bar{A}_r$ to $\bar{A}_{2r}^{\pm}$ and $\underline{A}_{2r}^{\pm}$ to $\underline{A}_r$. □

Theorem 7.1 implies the boundary Hölder-regularity of the quotient of two negative caloric functions vanishing on E_f. The proof of the following corollary is the same as for Corollary 13.8 in [Caffarelli and Salsa 2005], and is therefore omitted.

Theorem 7.2. *Let u_1, u_2 be nonnegative caloric functions in D continuously vanishing on E_f. Let $M_i = \sup_D u_i$ and $m_i = u_i(\underline{A}_{3/4})$ with $i = 1, 2$. Then we have*

$$C^{-1}\frac{u_1(A_{1/4})}{u_2(A_{1/4})} \le \frac{u_1(x,t)}{u_2(x,t)} \le C\frac{u_1(A_{1/4})}{u_2(A_{1/4})} \quad \text{for } (x,t) \cap \Psi_{1/8} \cap D, \tag{7-2}$$

where $C = C(n, L, M_1/m_1, M_2/m_2)$. Moreover, if u_1 and u_2 are symmetric in x_n, then u_1/u_2 extends to a function in $C^\alpha(\Psi_{1/8})$ for some $0 < \alpha < 1$, where the exponent α and the C^α-norm depend only on $n, L, M_1/m_1, M_2/m_2$. □

Remark 7.3. The symmetry condition in the latter part of the theorem is important to guarantee the continuous extension of u_1/u_2 to the Euclidean closure $\overline{\Psi_{1/8} \setminus E_f} = \overline{\Psi_{1/8}}$, since the limits at $E_f \setminus G_f$ as we approach from different sides may be different. Without the symmetry condition, one may still prove that u_1/u_2 extends to a C^α function on the completion $(\Psi_{1/8} \setminus E_f)^*$ with respect to the inner metric.

For a more general application, we need to have a boundary Harnack inequality for u satisfying a nonhomogeneous equation with bounded right-hand side, but additionally with a nondegeneracy condition. The method we use here is similar to the one used in the elliptic case [Caffarelli et al. 2008].

Theorem 7.4. *Let u be a nonnegative function in D, continuously vanishing on E_f, and satisfying*

$$|\Delta u - \partial_t u| \le C_0 \quad \text{in } D, \tag{7-3}$$
$$u(x,t) \ge c_0 \operatorname{dist}_p((x,t), E_f)^\gamma \quad \text{in } D, \tag{7-4}$$

where $0<\gamma<2$, $c_0>0$, $C_0\ge 0$. *Then there exists* $C=C(n,L,\gamma,C_0,c_0)>0$ *such that, for* $0<r<\frac{1}{4}$, *we have*

$$u(x,t)\le Cu(\bar{A}_r),\quad (x,t)\in\Psi_r. \tag{7-5}$$

Moreover, if $M=\sup_D u$, *then there exists a constant* $C=C(n,L,\gamma,C_0,c_0,M)$ *such that, for any* $0<r<\frac{1}{4}$, *we have*

$$u(\bar{A}_r)\le Cu(\underline{A}_r). \tag{7-6}$$

Proof. Let u^* be a solution to the heat equation in $\Psi_{2r}\cap D$ that is equal to u on $\partial_p(\Psi_{2r}\cap D)$. Then, by the Carleson estimate, we have $u^*(x,t)\le C(n,L)u^*(\bar{A}_r)$ for $(x,t)\in\Psi_r$.

On the other hand, we have

$$u^*(x,t)+C(|x|^2-t-8r^2)\le u(x,t)\quad \text{on } \partial_p(\Psi_{2r}\cap D),$$
$$(\Delta-\partial_t)(u^*(x,t)+C(|x|^2-t-8r^2))\ge C(2n-1)\ge(\Delta-\partial_t)u(x,t)\quad \text{in } \Psi_{2r}\cap D$$

for $C\ge C_0/(2n-1)$. Hence, by the comparison principle, we have $u^*-u\le Cr^2$ in $\Psi_{2r}\cap D$ for $C=C(C_0,n)$. Similarly, $u-u^*\le Cr^2$, and hence $|u-u^*|\le Cr^2$ in $\Psi_{2r}\cap D$. Consequently,

$$u(x,t)\le C(n,L)(u(\bar{A}_r)+C(C_0,n)r^2),\quad (x,t)\in\Psi_r. \tag{7-7}$$

Next, note that, by the nondegeneracy condition (7-4),

$$u(\bar{A}_r)\ge c_0r^\gamma\ge c_0r^2,\quad r\in(0,1). \tag{7-8}$$

Thus, combining (7-7) and (7-8), we obtain (7-5).

The proof of (7-6) follows in a similar manner from Theorem 7.1 for u^*. □

Remark 7.5. In fact, the nondegeneracy condition (7-4) is necessary. An easy counterexample is $u(x,t)=x_{n-1}^2x_n^2$ in Ψ_1 and $E_f=\{(x,t):x_{n-1}\le 0, x_n=0\}\cap\Psi_1$. Then $u(\bar{A}_r)=0$ for $r\in(0,1)$, but obviously u does not vanish in $\Psi_r\cap D$.

We next state a generalization of the local comparison theorem.

Theorem 7.6. *Let* u_1, u_2 *be nonnegative functions in* D, *continuously vanishing on* E_f, *and satisfying*

$$|\Delta u_i-\partial_t u_i|\le C_0\quad \text{in } D,$$
$$u_i(x,t)\ge c_0\,\mathrm{dist}_p((x,t),E_f)^\gamma\quad \text{in } D$$

for $i=1,2$, *where* $0<\gamma<2$, $c_0>0$, $C_0\ge 0$. *Let* $M=\max\{\sup_D u_1,\sup_D u_2\}$. *Then there exists a constant* $C=C(n,L,\gamma,C_0,c_0,M)>0$ *such that*

$$C^{-1}\frac{u_1(A_{1/4})}{u_2(A_{1/4})}\le\frac{u_1(x,t)}{u_2(x,t)}\le C\frac{u_1(A_{1/4})}{u_2(A_{1/4})},\quad (x,t)\in\Psi_{1/8}\cap D. \tag{7-9}$$

Moreover, if u_1 *and* u_2 *are symmetric in* x_n, *then* u_1/u_2 *extends to a function in* $C^\alpha(\overline{\Psi_{1/8}})$ *for some* $0<\alpha<1$, *with* α *and the* C^α*-norm depending only on* $n, L, \gamma, C_0, c_0, M$.

To prove this theorem, we will also need the following two lemmas, which are essentially Lemmas 11.5 and 11.8 in [Danielli et al. 2013]. The proofs are therefore omitted.

Lemma 7.7. *Let Λ be a subset of $\mathbb{R}^{n-1}\times(-\infty,0]$, and $h(x,t)$ a continuous function in Ψ_1. Then, for any $\delta_0>0$, there exists $\varepsilon_0>0$ depending only on δ_0 and n such that, if*:

(i) $h\ge 0$ *on* $\Psi_1\cap\Lambda$,

(ii) $(\Delta-\partial_t)h\le\varepsilon_0$ *in* $\Psi_1\setminus\Lambda$,

(iii) $h\ge-\varepsilon_0$ *in* Ψ_1,

(iv) $h\ge\delta_0$ *in* $\Psi_1\cap\{|x_n|\ge\beta_n\}$, $\beta_n=1/(32\sqrt{n-1})$,

then $h\ge 0$ *in* $\Psi_{1/2}$. □

Lemma 7.8. *For any $\delta_0>0$, there exists $\varepsilon_0>0$ and $c_0>0$, depending only on δ_0 and n, such that, if h is a continuous function on $\Psi_1\cap\{0\le x_n\le\beta_n\}$, $\beta_n=1/(32\sqrt{n-1})$, satisfying*:

(i) $(\Delta-\partial_t)h\le\varepsilon_0$ *in* $\Psi_1\cap\{0<x_n<\beta_n\}$,

(ii) $h\ge 0$ *in* $\Psi_1\cap\{0<x_n<\beta_n\}$,

(iii) $h\ge\delta_0$ *on* $\Psi_1\cap\{x_n=\beta_n\}$,

then

$$h(x,t)\ge c_0x_n\quad in\ \Psi_{1/2}\cap\{0<x_n<\beta_n\}.$$ □

Proof of Theorem 7.6. We first note that, arguing as in the proof of Theorem 7.4 and using Theorem 7.1, we will have that

$$u_i(x,t)\le Cu_i(A_{1/4}),\quad (x,t)\in\Psi_{1/8},\tag{7-10}$$

for $C=C(n,L,\gamma,C_0,c_0,M)$. Next, dividing u_i by $u_i(A_{1/4})$, we can assume $u_i(A_{1/4})=1$. Then consider the rescalings

$$u_{i\rho}(x,t)=\frac{u_i(\rho x,\rho^2t)}{\rho^\gamma},\quad \rho\in(0,1),\ i=1,2.$$

It is immediate to verify that, for $(x,t)\in\Psi_{1/(8\rho)}\cap D$, the functions $u_{i\rho}$ satisfy

$$|(\Delta-\partial_t)u_{i\rho}(x,t)|\le C_0\rho^{2-\gamma},\tag{7-11}$$

$$u_{i\rho}(x,t)\ge c_0\operatorname{dist}_p((x,t),E_{f_\rho})^\gamma,\tag{7-12}$$

$$u_{i\rho}(x,t)\le\frac{C}{\rho^\gamma},\quad \text{where } C \text{ is the constant in (7-10)},\tag{7-13}$$

where $f_\rho(x'',t)=(1/\rho)f(\rho x'',\rho^2t)$ is the scaling of f. By (7-12), there exists $c_n>0$ such that

$$u_{i\rho}(x,t)\ge c_0c_n,\quad (x,t)\in\Psi_{1/(8\rho)}\cap\{|x_n|\ge\beta_n\}.\tag{7-14}$$

Consider now the difference

$$h=u_{2\rho}-su_{1\rho}$$

for a small positive s, specified below. By (7-11), (7-14) and (7-13), one can choose a positive $\rho = \rho(n, L, \gamma, C_0, c_0, M) < \frac{1}{16}$ and $s = s(\rho, n, c_0, C) > 0$ such that

$$\begin{aligned} &h(x,t) \geq c_0 c_n - s \cdot \frac{C}{\rho^\gamma} \geq \frac{c_0 c_n}{2}, && (x,t) \in \Psi_{1/(8\rho)} \cap \{|x_n| \geq \beta_n\}, \\ &h(x,t) \geq -s \cdot \frac{C}{\rho^\gamma} \geq -\varepsilon_0, && (x,t) \in \Psi_{1/(8\rho)}, \\ &|(\Delta - \partial_t) h(x,t)| \leq C_0 \rho^{2-\gamma} \leq \varepsilon_0, && (x,t) \in \Psi_{1/(8\rho)} \cap D, \end{aligned}$$

where $\varepsilon_0 = \varepsilon_0(c_0, c_n, n)$ is the constant in Lemma 11.5 of [Danielli et al. 2013]. Thus, by that result, $h > 0$ in $\Psi_{1/2} \cap D$, which implies

$$\frac{u_1(x,t)}{u_2(x,t)} \leq \frac{1}{s}, \quad (x,t) \in \Psi_{\rho/2} \cap D. \tag{7-15}$$

By moving the origin to any $(z, h) \in \Psi_{1/8} \cap E_f$, we will therefore obtain the bound

$$\frac{u_1(x,t)}{u_2(x,t)} \leq C(n, L, \gamma, C_0, c_0, M) \tag{7-16}$$

for any $(x,t) \in \Psi_{1/8} \cap \mathcal{N}_{\rho/2}(E_f) \cap D$. On the other hand, for $(x,t) \in \Psi_{1/8} \setminus \mathcal{N}_{\rho/2}(E_f)$, the estimate (7-16) will follow from (7-4) and (7-10). Hence, (7-16) holds for any $(x,t) \in \Psi_{1/8} \cap D$, which gives the bound from above in (7-9). Changing the roles of u_1 and u_2, we get the bound from below.

The proof of C^α-regularity follows by iteration from (7-9), similarly to the proof of Corollary 13.8 in [Caffarelli and Salsa 2005]; however, we need to make sure that at every step the nondegeneracy condition is satisfied. We will only verify the Hölder-continuity of u_1/u_2 at the origin, the rest being standard.

For $k \in \mathbb{N}$ and $\lambda > 0$ to be specified below, let

$$l_k = \inf_{\Psi_{\lambda^k} \cap D} \frac{u_1}{u_2}, \quad L_k = \sup_{\Psi_{\lambda^k} \cap D} \frac{u_1}{u_2}.$$

Then we know that $1/C \leq l_k \leq L_k \leq C$ for $\lambda \leq \frac{1}{8}$. Let also

$$\mu_k = \frac{u_1(\underline{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \in [l_k, L_k].$$

Then there are two possibilities:

$$\text{either} \quad L_k - \mu_k \geq \tfrac{1}{2}(L_k - l_k) \quad \text{or} \quad \mu_k - l_k \geq \tfrac{1}{2}(L_k - l_k).$$

For definiteness, assume that we are in the latter case, the former case being treated similarly. Then consider the two functions

$$v_1(x,t) = \frac{u_1(\lambda^k x, \lambda^{2k} t) - l_k u_2(\lambda^k x, \lambda^{2k} t)}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})}, \quad v_2(x,t) = \frac{u_2(\lambda^k x, \lambda^{2k} t)}{u_2(\underline{A}_{\lambda^k/4})}.$$

In $\Psi_1 \setminus E_{f_{\lambda^k}}$, we will have

$$|(\Delta - \partial_t) v_1(x,t)| \le \frac{\lambda^{2k}(1+l_k)C_0}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})},$$
$$|(\Delta - \partial_t) v_2(x,t)| \le \frac{\lambda^{2k} C_0}{u_2(\underline{A}_{\lambda^k/4})}.$$

To proceed, fix a small $\eta_0 > 0$, to be specified below. From the nondegeneracy of u_2, we immediately have

$$|(\Delta - \partial_t) v_2(x,t)| \le C\lambda^{(2-\gamma)k} < \eta_0$$

if we take λ small enough. For v_1, we have a dichotomy:

$$\text{either} \quad |(\Delta - \partial_t) v_1(x,t)| \le \eta_0 \quad \text{or} \quad \mu_k - l_k \le C\lambda^{(2-\gamma)k}.$$

In the latter case, we obtain

$$L_k - l_k \le 2(\mu_k - l_k) \le C\lambda^{(2-\gamma)k}. \tag{7-17}$$

In the former case, we notice that both functions $v = v_1, v_2$ satisfy

$$v \ge 0, \quad v(\underline{A}_{1/4}) = 1 \quad \text{and} \quad |(\Delta - \partial_t) v(x,t)| \le \eta_0 \quad \text{in } \Psi_1 \setminus E_{f_{\lambda^k}},$$

and that v vanishes continuously on $\Psi_1 \cap E_{f_{\lambda^k}}$. We next establish a nondegeneracy property for such v. Indeed, first note that, by the parabolic Harnack inequality (see Theorems 6.17 and 6.18 in [Lieberman 1996]), for small enough η_0, we will have that

$$v \ge c_n \quad \text{on } \Psi_{1/8} \cap \{|x_n| \ge \beta_n/8\}.$$

Then, by invoking Lemma 7.8, we will obtain that

$$v(x,t) \ge c_n |x_n| \quad \text{in } \Psi_{1/16} \setminus E_{f_{\lambda^k}}. \tag{7-18}$$

We further claim that

$$v(x,t) \ge c \operatorname{dist}_p((x,t), E_{f_{\lambda^k}}) \quad \text{in } \Psi_{1/32} \setminus E_{f_{\lambda^k}}. \tag{7-19}$$

To this end, for $(x,t) \in \Psi_{1/32} \setminus E_{f_{\lambda^k}}$, let $d = \sup\{r : \Psi_r(x,t) \cap E_{f_{\lambda^k}} = \varnothing\}$, and consider the box $\Psi_d(x,t)$. Without loss of generality, assume $x_n \ge 0$. Then let $(x_*, t_*) = (x', x_n + d, t - d^2) \in \partial_p \Psi_d(x,t)$. From (7-18), we have that

$$v(x_*, t_*) \ge c_n(x_n + d) \ge c_n d,$$

and, applying the parabolic Harnack inequality, we obtain

$$v(x,t) \ge c_n v(x_*, t_*) - C_n \eta_0 d^2 \ge c_n d$$

provided η_0 is sufficiently small. Hence, (7-19) follows.

Having the nondegeneracy, we also have the bound from above for the functions v_1 and v_2. Indeed, by Theorem 7.4 for v_1 and v_2, we have

$$\sup_{\Psi_1} v_1 \le C v_1(\bar{A}_{1/4}) = C\frac{u_1(\bar{A}_{\lambda^k/4}) - l_k u_2(\bar{A}_{\lambda^k/4})}{u_1(\underline{A}_{\lambda^k/4}) - l_k u_2(\underline{A}_{\lambda^k/4})} \le C\frac{u_2(\bar{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})}\frac{L_k - l_k}{\mu_k - l_k} \le C \tag{7-20}$$

and

$$\sup_{\Psi_1} v_2 \le C v_2(\bar{A}_{1/4}) = C\frac{u_2(\bar{A}_{\lambda^k/4})}{u_2(\underline{A}_{\lambda^k/4})} \le C, \tag{7-21}$$

where we have also invoked the second part of Theorem 7.4 for u_2.

We have thus verified all conditions necessary for applying the estimate (7-9) to the functions v_1 and v_2. Particularly, the inequality from below, applied in $\Psi_{8\lambda} \setminus E_{f_{\lambda^k}}$, will give

$$\inf_{\Psi_\lambda \setminus E_{f_{\lambda^k}}} \frac{v_1}{v_2} \ge c\frac{v_1(A_{2\lambda})}{v_2(A_{2\lambda})} \ge c\lambda$$

for a small $c > 0$, or equivalently

$$l_{k+1} - l_k \ge c\lambda(\mu_k - l_k) \ge \frac{c\lambda}{2}(L_k - l_k).$$

Hence, we will have

$$L_{k+1} - l_{k+1} \le L_k - l_k - (l_{k+1} - l_k) \le \left(1 - \frac{c\lambda}{2}\right)(L_k - l_k). \tag{7-22}$$

Summarizing, (7-17) and (7-22) give a dichotomy: for any $k \in \mathbb{N}$,

$$\text{either} \quad L_k - l_k \le C\lambda^{(2-\gamma)k} \quad \text{or} \quad L_{k+1} - l_{k+1} \le (1 - c\lambda/2)(L_k - l_k).$$

This clearly implies that

$$L_k - l_k \le C\beta^k \quad \text{for some } \beta \in (0, 1),$$

for any $k \in \mathbb{N}$, which is nothing but the Hölder-continuity of u_1/u_2 at the origin. □

We next want to prove a variant of Theorem 7.6, but with the Ψ_r replaced with their lower halves

$$\Theta_r = \Psi_r \cap \{t \le 0\}.$$

Theorem 7.9. *Let u_1, u_2 be nonnegative functions in $\Theta_1 \setminus E_f$, continuously vanishing on $\Theta_1 \cap E_f$, and satisfying*

$$|\Delta u_i - \partial_t u_i| \le C_0 \quad \text{in } \Theta_1 \setminus E_f,$$
$$u_i(x, t) \ge c_0 \operatorname{dist}_p((x, t), E_f) \quad \text{in } \Theta_1 \setminus E_f$$

for $i = 1, 2$, for some $c_0 > 0$, $C_0 \ge 0$. Let also $M = \max\{\sup_D u_1, \sup_D u_2\}$. If u_1 and u_2 are symmetric in x_n, then u_1/u_2 extends to a function in $C^\alpha(\bar{\Theta}_{1/8})$ for some $0 < \alpha < 1$, with α and C^α-norm depending only on $n, L, \gamma, C_0, c_0, M$.

The idea is that the functions u_i can be extended to Ψ_δ, for some $\delta>0$, while still keeping the same inequalities, including the nondegeneracy condition.

Lemma 7.10. *Let u be a nonnegative continuous function on Θ_1 such that*

$$\begin{aligned} u&=0 && \text{in } \Theta_1\cap E_f,\\ |(\Delta-\partial_t)u|&\le C_0 && \text{in } \Theta_1\setminus E_f,\\ u(x,t)&\ge c_0\operatorname{dist}_p((x,t),E_f) && \text{in } \Theta_1\setminus E_f,\end{aligned}$$

for some $C_0\ge 0$, $c_0>0$. Then there exist positive δ and $\tilde c_0$, depending only on n, L, c_0 and C_0, and a nonnegative extension $\tilde u$ of u to Ψ_δ, such that

$$\begin{aligned} \tilde u&=0 && \text{in } \Psi_\delta\cap E_f,\\ |(\Delta-\partial_t)\tilde u|&\le C_0 && \text{in } \Psi_\delta\setminus E_f,\\ \tilde u(x,t)&\ge \tilde c_0\operatorname{dist}_p((x,t),E_f) && \text{in } \Psi_\delta\setminus E_f.\end{aligned}$$

Moreover, we will also have that $\sup_{\Psi_\delta}\tilde u\le\sup_{\Theta_1}u$.

Proof. We first continuously extend the function u from the parabolic boundary $\partial_p\Theta_{1/2}$ to $\partial_p\Psi_{1/2}$ by keeping it nonnegative and bounded above by the same constant. Further, put $u=0$ on $E_f\cap(\Psi_{1/2}\setminus\Theta_{1/2})$. Then extend u to $\Psi_{1/2}$ by solving the Dirichlet problem for the heat equation in $(\Psi_{1/2}\setminus\Theta_{1/2})\setminus E_f$, with already defined boundary values. We still denote the extended function by u.

Then it is easy to see that u is nonnegative in $\Psi_{1/2}$, $\sup_{\Psi_{1/2}}u\le\sup_{\Theta_1}u$, u vanishes on $\Psi_{1/2}\cap E_f$ and $|(\Delta-\partial_t)u|\le C_0$ in $\Psi_{1/2}\setminus E_f$. Note that we still have the nondegeneracy property $u(x,t)\ge c_0\operatorname{dist}_p((x,t),E_f)$ for in $\Theta_{1/2}\setminus E_f$, so it remains to prove the nondegeneracy for $t\ge 0$. We will be able to do it in a small box Ψ_δ as a consequence of Lemma 7.8.

For $0<\delta<\frac12$, consider the rescalings

$$u_\delta(x,t)=\frac{u(\delta x,\delta^2 t)}{\delta},\quad (x,t)\in\Psi_{1/(2\delta)}.$$

Then we have

$$\begin{aligned} |(\Delta-\partial_t)u_\delta|&\le C_0\delta && \text{in } \Psi_1\setminus E_{f_\delta},\\ u_\delta(x,t)&\ge c_0|x_n| && \text{in } \Theta_1,\end{aligned}$$

where $f_\delta(x'',t)=(1/\delta)f(\delta x'',\delta^2 t)$ is the rescaling of f. Then, by using the parabolic Harnack inequality (see Theorems 6.17 and 6.18 in [Lieberman 1996]) in $\Theta_1^\pm$, we obtain that

$$u_\delta(x,t)\ge c_nc_0-C_nC_0\delta>c_1\quad \text{on } \{|x_n|=\beta_n/2\}\cap\Psi_{1/2}.$$

Further, choosing δ small and applying Lemma 7.8, we deduce that

$$u_\delta(x,t)\ge c_2|x_n|\quad \text{in } \Psi_{1/4}.$$

Then, repeating the arguments based on the parabolic Harnack inequality, as for the inequality (7-19), we obtain

$$u(x,t) \geq C\,\mathrm{dist}_p((x,t), E_{f_\delta}) \quad \text{in } \Psi_{1/8}.$$

Scaling back, this gives

$$u(x,t) \geq C\,\mathrm{dist}_p((x,t), E_f) \quad \text{in } \Psi_{\delta/8}.$$ □

Proof of Theorem 7.9. Extend the functions u_i as in Lemma 7.10 and apply Theorem 7.6. If we repeat this at every $(y,s) \in \Theta_{1/8} \cap G_f$, we will obtain the Hölder-regularity of u_1/u_2 in $\mathcal{N}_{\delta/8}(\Theta_{1/8} \cap G_f) \cap \{t \leq 0\}$. For the remaining part of $\Theta_{1/8}$, we argue as in the proof of localization property Lemma 2.3, cases (1) and (2), and use the corresponding results for parabolically Lipschitz domains. □

7A. ***Parabolic Signorini problem.*** In this subsection, we discuss an application of the boundary Harnack principle to the parabolic Signorini problem. The idea of such applications goes back to [Athanasopoulos and Caffarelli 1985]. The particular result that we will discuss here can be found also in [Danielli et al. 2013], with the same proof based on our Theorem 7.9.

In what follows, we will use $H^{\ell,\ell/2}$, $\ell > 0$, to denote the parabolic Hölder classes, as defined for instance in [Ladyženskaja et al. 1968].

For a given function $\varphi \in H^{\ell,\ell/2}(Q_1')$, $\ell \geq 2$, known as the *thin obstacle*, we say that a function v solves the *parabolic Signorini problem* if $v \in W_2^{2,1}(Q_1^+) \cap H^{1+\alpha,(1+\alpha)/2}(\overline{Q_1^+})$, $\alpha > 0$, and

$$(\Delta - \partial_t)v = 0 \quad \text{in } Q_1^+, \tag{7-23}$$

$$v \geq \varphi, \quad -\partial_{x_n} v \geq 0, \quad (v-\varphi)\partial_{x_n} v = 0 \quad \text{on } Q_1'. \tag{7-24}$$

This kind of problem appears in many applications, such as thermics (boundary heat control), biochemistry (semipermeable membranes and osmosis), and elastostatics (the original Signorini problem). We refer to the book [Duvaut and Lions 1976] for the derivation of such models as well as for some basic existence and uniqueness results.

The regularity that we impose on the solutions of (7-23)–(7-24) is also well known in the literature; see, e.g., [Athanasopoulous 1982; Ural′tseva 1985; Arkhipova and Uraltseva 1996]. It was proved recently in [Danielli et al. 2013] that one can actually take $\alpha = \frac{1}{2}$ in the regularity assumptions on v, which is the optimal regularity, as can be seen from the explicit example

$$v(x,t) = \mathrm{Re}(x_{n-1} + i x_n)^{3/2},$$

which solves the Signorini problem with $\varphi = 0$. One of the main objects of study in the Signorini problem is the *free boundary*

$$G(v) = \partial_{Q_1'}(\{v > \varphi\} \cap Q_1'),$$

where $\partial_{Q_1'}$ is the boundary in the relative topology of Q_1'.

As the initial step in the study, we make the following reduction. We observe that the difference

$$u(x,t) = v(x,t) - \varphi(x',t)$$

will satisfy

$$(\Delta - \partial_t)u = g \quad \text{in } Q_1^+, \tag{7-25}$$

$$u \geq 0, \quad -\partial_{x_n} u \geq 0, \quad u\partial_{x_n} u = 0 \quad \text{on } Q_1', \tag{7-26}$$

where $g = -(\Delta_{x'} - \partial_t)\varphi \in H^{\ell-2,(\ell-2)/2}$. That is, one can make the thin obstacle equal to 0 at the expense of getting a nonzero right-hand side in the equation for u. For our purposes, this simple reduction will be sufficient, however, to take the full advantage of the regularity of φ. When $\ell > 2$, one may need to subtract an additional polynomial from u to guarantee the decay rate

$$|g(x,t)| \leq M(|x|^2 + |t|)^{(\ell-2)/2}$$

near the origin; see Proposition 4.4 in [Danielli et al. 2013]. With the reduction above, the free boundary $G(v)$ becomes

$$G(u) = \partial_{Q_1'}(\{u > 0\} \cap Q_1').$$

Further, it will be convenient to consider the even extension of u in the x_{n-1} variable to the entire Q_1, i.e., by putting $u(x', x_n, t) = u(x', -x_n, t)$. Then, such an extended function will satisfy

$$(\Delta - \partial_t)u = g \quad \text{in } Q_1 \setminus \Lambda(u),$$

where g has also been extended by even symmetry in x_n, and where

$$\Lambda(u) = \{u = 0\} \cap Q_1',$$

the so-called *coincidence set*.

As shown in [ibid.], a successful study of the properties of the free boundary near $(x_0, t_0) \in G(u) \cap Q_{1/2}'$ can be made by considering the rescalings

$$u_r(x,t) = u_r^{(x_0,t_0)}(x,t) = \frac{u(x_0 + rx, t_0 + r^2 t)}{H_u^{(x_0,t_0)}(r)^{1/2}}$$

for $r > 0$ and then studying the limits of u_r as $r = r_j \to 0+$ (so-called blowups). Here

$$H_u^{(x_0,t_0)}(r) := \frac{1}{r^2} \int_{t_0 - r^2}^{t_0} \int_{\mathbb{R}^n} u(x,t)^2 \psi^2(x) \Gamma(x_0 - x, t_0 - t)\, dx\, dt,$$

where $\psi(x) = \psi(|x|)$ is a cutoff function that equals 1 on $B_{3/4}$. Then a point $(x_0, t_0) \in G(u) \cap B_{1/2}$ is called regular if u_r converges in the appropriate sense to

$$u_0(x,t) = c_n \operatorname{Re}(x_{n-1} + i x_n)^{3/2}$$

as $r = r_j \to 0+$, after a possible rotation of coordinate axes in $\mathbb{R}^{n-1}$. See [ibid.] for more details. Let $\mathcal{R}(u)$ be the set of regular points of u.

Proposition 7.11 [Danielli et al. 2013]. *Let u be a solution of the parabolic Signorini problem* (7-25)–(7-26) *in Q_1^+ with $g \in H^{1,1/2}(Q_1^+)$. Then the regular set $\mathcal{R}(u)$ is a relatively open subset of $G(u)$. Moreover, if $(0,0) \in \mathcal{R}(u)$, then there exists $\rho = \rho_u > 0$ and a parabolically Lipschitz function f such that*

$$G(u) \cap Q'_\rho = \mathcal{R}(u) \cap Q'_\rho = G_f \cap Q'_\rho$$
$$\Lambda(u) \cap Q'_\rho = E_f \cap Q'_\rho.$$

Furthermore, for any $0 < \eta < 1$, we can find $\rho > 0$ such that

$$\partial_e u \geq 0 \quad \textit{in } Q_\rho$$

for any unit direction $e \in \mathbb{R}^{n-1}$ such that $e \cdot e_{n-1} > \eta$, and moreover

$$\partial_e u(x,t) \geq c \operatorname{dist}_p((x,t), E_f) \quad \textit{in } Q_\rho$$

for some $c > 0$. □

We next show that an application of Theorem 7.9 implies the following result.

Theorem 7.12. *Let u be as in Proposition 7.11 and $(0,0) \in \mathcal{R}(u)$. Then there exists $\delta < \rho$ such that $\nabla'' f \in H^{\alpha,\alpha/2}(Q'_\delta)$ for some $\alpha > 0$, i.e., $\mathcal{R}(u)$ has Hölder-continuous spatial normals in Q'_δ.*

Proof. We will work in parabolic boxes $\Theta_\delta = \Psi_\delta \cap \{t \leq 0\}$ instead of cylinders Q_δ. For a small $\varepsilon > 0$, let $e = (\cos\varepsilon)e_{n-1} + (\sin\varepsilon)e_j$ for some $j = 1, \dots, n-2$, and consider the two functions

$$u_1 = \partial_e u \quad \text{and} \quad u_2 = \partial_{e_{n-1}} u.$$

Then, by Proposition 7.11, the conditions of Theorem 7.9 are satisfied (after a rescaling), provided $\cos\varepsilon > \eta$. Thus, if we fix such $\varepsilon > 0$, we will have that for some $\delta > 0$ and $0 < \alpha < 1$,

$$\frac{\partial_e u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta).$$

This gives that

$$\frac{\partial_{e_j} u}{\partial_{e_{n-1}} u} \in H^{\alpha,\alpha/2}(\Theta_\delta), \qquad j = 1, \dots, n-2.$$

Hence the level surfaces $\{u = \sigma\} \cap \Theta'_\delta$ are given as graphs

$$x_{n-1} = f_\sigma(x'', t), \qquad x'' \in \Theta''_\delta,$$

with estimate on $\|\nabla'' f_\sigma\|_{H^{\alpha,\alpha/2}(\Theta''_\delta)}$ that is uniform in $\sigma > 0$. Consequently, this implies that

$$\nabla'' f \in H^{\alpha,\alpha/2}(\Theta''_\delta),$$

and completes the proof of the theorem. □

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