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RESOLVENT ESTIMATES FOR THE MAGNETIC SCHRÖDINGER OPERATOR

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We prove optimal high-frequency resolvent estimates for self-adjoint operators of the form

$$G=-\Delta+ib(x)\cdot\nabla+i\nabla\cdot b(x)+V(x)$$

on $L^2(\mathbb{R}^n)$, $n\ge 3$, where $b(x)$ and $V(x)$ are large magnetic and electric potentials, respectively.

1. Introduction and statement of results

Let Δ be the (negative) Euclidean Laplacian on $\mathbb{R}^n$. It is well-known that the self-adjoint realization G_0 of the operator $-\Delta$ on $L^2(\mathbb{R}^n)$ has an absolutely continuous spectrum consisting of the interval $[0,+\infty)$ and satisfies the resolvent estimate

$$\left\|\langle x\rangle^{-s}\partial_x^{\alpha_1}(G_0-\lambda^2\pm i\varepsilon)^{-1}\partial_x^{\alpha_2}\langle x\rangle^{-s}\right\|_{L^2\to L^2}\le C\lambda^{|\alpha_1|+|\alpha_2|-1},\quad \lambda\ge 1, \tag{1-1}$$

for all multi-indices α_1 and α_2 such that $|\alpha_1|+|\alpha_2|\le 2$, where $s>\frac{1}{2}$, $0<\varepsilon\le 1$, and the constant $C>0$ does not depend on λ or ε. The same estimate still holds (see [Cardoso and Vodev 2002; Rodnianski and Tao 2011], for example) for λ large enough for perturbations of the form $-\Delta+V(x)$, where V is a real-valued function satisfying the conditions below. Note that (1-1) for $\alpha_1=\alpha_2=0$ together with the ellipticity of the operator G_0 imply that the estimate (1-1) holds for all multi-indices α_1 and α_2 such that $|\alpha_1|+|\alpha_2|\le 2$. This fact remains valid for more general elliptic perturbations of $-\Delta$.

The purpose of this work is to prove an analogue of (1-1) for perturbations by large magnetic and electric potentials, extending the recent results in [Cardoso et al. 2013; 2014a] to a larger class (most probably optimal) of magnetic potentials. More precisely, we study the high-frequency behavior of the resolvent of self-adjoint operators of the form

$$G=-\Delta+ib(x)\cdot\nabla+i\nabla\cdot b(x)+V(x)\quad\text{on } L^2(\mathbb{R}^n),\ n\ge 3,$$

where $b=(b_1,\dots,b_n)\in L^\infty(\mathbb{R}^n;\mathbb{R}^n)$ is a magnetic potential and $V\in L^\infty(\mathbb{R}^n;\mathbb{R})$ is an electric potential. Hereafter, the operator $\nabla\cdot b$ is defined by $(\nabla\cdot b)u=\nabla\cdot(bu)$. Introduce the polar coordinates $r=|x|$, $w=x/|x|\in\mathbb{S}^{n-1}$. We suppose that $b(x)=b^L(x)+b^S(x)$, $V(x)=V^L(x)+V^S(x)$ with long-range parts b^L and V^L belonging to $C^1([r_0,+\infty))$, $r_0\gg 1$ with respect to the radial variable r and satisfying the

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conditions

$$|V^L(rw)| \le C, \tag{1-2}$$

$$\partial_r V^L(rw) \le Cr^{-1-\delta}, \tag{1-3}$$

$$|\partial_r^k b^L(rw)| \le Cr^{-k-\delta}, \quad k=0,1, \tag{1-4}$$

for all $r \ge r_0$, $w \in \mathbb{S}^{n-1}$, with some constants $C, \delta > 0$. The short-range parts satisfy

$$|b^S(x)| + |V^S(x)| \le C\langle x\rangle^{-1-\delta}. \tag{1-5}$$

Note that in the case $b^L \equiv 0$, $V^L \equiv 0$ and b^S, V^S satisfying (1-5), the operator G has an absolutely continuous spectrum consisting of the interval $[0, +\infty)$ with no strictly positive eigenvalues (see [Koch and Tataru 2006]). It follows from our result below that in the more general case when the long-range parts are not identically zero the spectrum of the operator G has a similar structure in an interval of the form $[a, +\infty)$ with some constant $a > 0$. Our main result is the following:

Theorem 1.1. *Under the conditions* (1-2)–(1-5), *for every $s > \frac{1}{2}$ there exist constants $C, \lambda_0 > 0$ so that for $\lambda \ge \lambda_0$, $0 < \varepsilon \le 1$, $|\alpha_1|, |\alpha_2| \le 1$, we have the estimate*

$$\big\|\langle x\rangle^{-s}\partial_x^{\alpha_1}(G-\lambda^2 \pm i\varepsilon)^{-1}\partial_x^{\alpha_2}\langle x\rangle^{-s}\big\|_{L^2\to L^2} \le C\lambda^{|\alpha_1|+|\alpha_2|-1}. \tag{1-6}$$

This kind of resolvent estimates plays an important role in proving uniform local energy decay, dispersive, smoothing and Strichartz estimates for solutions to the corresponding wave and Schrödinger equations (see [Cardoso et al. 2013; 2014b; Erdoğan et al. 2009], for example). In particular, it follows from the above theorem that the smoothing and Strichartz estimates for solutions to the corresponding Schrödinger equation proved in [Erdoğan et al. 2009] hold true without the continuity condition on the magnetic potential.

Theorem 1.1 is proved in [Cardoso et al. 2013] assuming additionally that $b^S(x)$ is continuous with respect to the radial variable r uniformly in w. In the case $b^L \equiv 0$, $V^L \equiv 0$ and b^S, V^S satisfying (1-5), the estimate (1-6) is proved in [Erdoğan et al. 2009] under the extra assumption that $b(x)$ is continuous in x. In fact, no continuity of the magnetic potential is needed in order to have (1-6), as shown in [Cardoso et al. 2014a]. Instead, it was supposed in [Cardoso et al. 2014a] that $\operatorname{div} b^L$ and $\operatorname{div} b^S$ exist as functions in L^∞. This assumption allows us to conclude that the perturbation (which is a first-order differential operator) sends the Sobolev space H^1 into L^2, a fact used in an essential way in [Cardoso et al. 2014a]. Thus, our goal in the present paper is to remove this technical condition on the magnetic potential. To this end, we propose a new approach inspired by the *global* Carleman estimates proved recently in [Datchev 2014] in a different context. In what follows we will describe the main points of our proof.

There are two main difficulties in proving the above theorem. The first one is that, under our assumptions, the commutator of the gradient and the magnetic potential is not an L^∞ function. Consequently, the perturbation does not send the Sobolev space H^1 into L^2. Instead, it is bounded from H^1 into H^{-1}. Secondly, the magnetic potential is large, and therefore it is hard to apply perturbation arguments similar to those used in [Cardoso et al. 2013]. Thus, to prove Theorem 1.1 we first observe that (1-6) is equivalent to a semiclassical a priori estimate on weighted Sobolev spaces (see (2-10) below). Furthermore, we derive this a priori estimate from a semiclassical Carleman estimate on weighted Sobolev spaces (see (2-7) below) with a suitably chosen phase function independent of the semiclassical parameter. To get this Carleman estimate we first prove a semiclassical Carleman estimate on weighted Sobolev spaces for the long-range part of the operator (see Theorem 2.1 below) and we then apply a perturbation argument. Note that the estimate (2-1) is valid for any phase function $\varphi(r) \in C^2(\mathbb{R})$ whose first derivative $\varphi'(r)$ is of compact support and nonnegative. The main feature of our Carleman estimate is that it is uniform with respect to the phase function φ (that is, the constant C_1 does not depend on φ), and the weight in the right-hand side is smaller than the usual one (that is, $(\langle x\rangle^{-2s}+\varphi'(|x|))^{-1/2}$ instead of $\langle x\rangle^{s}$). Thus, we can make this weight small on an arbitrary compact set by choosing the phase function properly. Moreover, in the right-hand side we have the better semiclassical Sobolev H^{-1} norm instead of the L^2 one, which is crucial for the application we make here. Note also that Carleman estimates similar to (2-1) and (2-7) have recently been proved in [Datchev 2014] for operators of the form $-h^2\Delta + V(x,h)$, where V is a real-valued long-range potential which is C^1 with respect to the radial variable r. There are, however, several important differences between the Carleman estimates in [Datchev 2014] and ours. First, the phase function in [Datchev 2014] is of the form $\varphi = \varphi_1(r)/h$, where φ_1 does not depend on h and must satisfy some conditions. Thus, the Carleman estimates in [Datchev 2014] lead to the conclusion that the resolvent in that case is bounded by $e^{C/h}$, $C > 0$ being a constant. Secondly, in [Datchev 2014] the Carleman estimates are not uniform with respect to the phase function and the norm in the right-hand side is L^2 (and not H^{-1}). Finally, the operator in [Datchev 2014] does not contain a magnetic potential.

To prove Theorem 2.1 we make use of methods originating from [Cardoso and Vodev 2002]. Note that in [Cardoso and Vodev 2002] the high-frequency behavior of the resolvent of operators of the form $-\Delta_g + V$ is studied, where V is a real-valued scalar potential and Δ_g is the negative Laplace–Beltrami operator on unbounded Riemannian manifolds, such as, for example, asymptotically Euclidean and hyperbolic ones. Similar techniques have been also used in [Rodnianski and Tao 2011], where actually all ranges of frequencies are covered. In these two papers, however, no perturbations by magnetic potentials are studied.

2. Proof of Theorem 1.1

Set $h = \lambda^{-1}$, $P(h) = h^2 G$, $\tilde b(x,h) = hb(x)$, $\tilde b^L(x,h) = h\chi(|x|)b^L(x)$, $\tilde b^S(x,h) = \tilde b(x,h) - \tilde b^L(x,h)$, $\widetilde V(x,h) = h^2 V(x)$, $\widetilde V^L(x,h) = h^2\chi(|x|)V^L(x)$, $\widetilde V^S(x,h) = \widetilde V(x,h) - \widetilde V^L(x,h)$, where $\chi \in C^\infty(\mathbb{R})$, $\chi(r) = 0$ for $r \le r_0 + 1$, $\chi(r) = 1$ for $r \ge r_0 + 2$. Throughout this paper, $H^1(\mathbb{R}^n)$ will denote the Sobolev space equipped with the semiclassical norm

$$\|u\|_{H^1}^2 = \sum_{0\le|\alpha|\le 1} \|\mathcal{D}_x^\alpha u\|_{L^2}^2 ,$$

where $\mathcal{D}_x = ih\partial_x$. Furthermore, H^{-1} will denote the dual space of H^1 with respect to the scalar product $\langle\,\cdot\,,\cdot\,\rangle_{L^2}$ with the norm

$$\|v\|_{H^{-1}} = \sup_{0\ne u\in H^1} \frac{|\langle u, v\rangle_{L^2}|}{\|u\|_{H^1}} .$$

Let $\rho \in C^\infty(\mathbb{R})$ be a function independent of h such that $0 \le \rho \le 1$ and $\rho(\sigma) = 1$ for $\sigma \le 0$, $\rho(\sigma) = 0$ for $\sigma \ge 1$. Define the function $\varphi(r) \in C^\infty(\mathbb{R})$ as follows: $\varphi(0) = 0$ and

$$\varphi'(r) = \tau\rho(r - A),$$

where $\tau, A \ge 1$ are parameters independent of h to be fixed later on. Introduce the operator

$$P^L(h) = -h^2\Delta + ih\tilde{b}^L(x,h)\cdot\nabla + ih\nabla\cdot\tilde{b}^L(x,h) + \widetilde{V}^L(x,h)$$

and set

$$P^L_\varphi(h) = e^\varphi P^L(h) e^{-\varphi},$$
$$P_\varphi(h) = e^\varphi P(h) e^{-\varphi} = P^L_\varphi(h) + ih\tilde{b}^S(x,h)\cdot\nabla + ih\nabla\cdot\tilde{b}^S(x,h) - 2ih\tilde{b}^S(x,h)\cdot\nabla\varphi + \widetilde{V}^S(x,h),$$
$$\mu(x) = \sqrt{\langle x\rangle^{-2s} + \varphi'(|x|)}.$$

In this section we will show that Theorem 1.1 follows from:

Theorem 2.1. *Suppose* (1-2), (1-3), (1-4) *hold and let* $\frac{1}{2} < s < \frac{1}{2}(1+\delta)$. *Then, for all functions* $f \in H^1(\mathbb{R}^n)$ *such that* $\langle x\rangle^s(P^L_\varphi(h) - 1 \pm i\varepsilon)f \in H^{-1}(\mathbb{R}^n)$, *we have the a priori estimate*

$$\|\langle x\rangle^{-s} f\|_{H^1} \le \frac{C_1}{h}\|\mu^{-1}(P^L_\varphi(h) - 1 \pm i\varepsilon)f\|_{H^{-1}} + C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|f\|_{L^2} \tag{2-1}$$

for $0 < \varepsilon \le 1$, $0 < h \le h_0(\tau, A) \ll 1$, *with a constant* $C_1 > 0$ *independent of* $f, \varepsilon, h, \tau, A$, *and a constant* $C_2 > 0$ *independent of* f, ε, h.

Let us first see that (2-1) implies the estimate

$$\|\langle x\rangle^{-s} f\|_{H^1} \le \frac{2C_1}{h}\|\langle x\rangle^s(P_\varphi(h) - 1 \pm i\varepsilon)f\|_{H^{-1}} + 2C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|f\|_{L^2}. \tag{2-2}$$

Using that $\mu(x) \ge \tau^{1/2}$ for $|x| \le A$ and $\mu(x) \ge \langle x\rangle^{-s}$ for $|x| \ge A+1$ together with the condition (1-4), we get (for $0 < s - \frac{1}{2} \ll 1$)

$$\langle x\rangle^s\mu(x)^{-1}\big(|\tilde{b}^S(x,h)| + |\widetilde{V}^S(x,h)|\big) \le Ch(\tau^{-1/2} + A^{2s-1-\delta}), \tag{2-3}$$
$$\langle x\rangle^s\mu(x)^{-1}|\tilde{b}^S(x,h)|\,|\nabla\varphi| \le O_{\tau,A}(h). \tag{2-4}$$

By (2-3) and (2-4),

$$\|\mu^{-1}(P_\varphi(h) - P^L_\varphi(h))\langle x\rangle^s\|_{H^1\to H^{-1}} \le Ch(\tau^{-1/2} + A^{2s-1-\delta} + O(h)). \tag{2-5}$$

By (2-1) and (2-5),

$$\begin{aligned}
&\|\langle x\rangle^{-s} f\|_{H^1}\\
&\le \frac{C_1}{h}\|\mu^{-1}(P_\varphi(h) - 1 \pm i\varepsilon)f\|_{H^{-1}} + \frac{C_1}{h}\|\mu^{-1}(P_\varphi(h) - P^L_\varphi(h))f\|_{H^{-1}} + C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|f\|_{L^2}\\
&\le \frac{C_1}{h}\|\langle x\rangle^s(P_\varphi(h) - 1 \pm i\varepsilon)f\|_{H^{-1}} + C(\tau^{-1/2} + A^{2s-1-\delta} + O(h))\|\langle x\rangle^{-s} f\|_{H^1} + C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|f\|_{L^2}.
\end{aligned} \tag{2-6}$$

Taking now τ^{-1}, A^{-1} and h small enough, we can absorb the second term in the right-hand side of (2-6) to obtain (2-2).

Applying (2-2) with $f = e^{\varphi} g$ we obtain the Carleman estimate

$$\|\langle x\rangle^{-s} e^{\varphi} g\|_{H^1} \le \frac{2C_1}{h}\|\langle x\rangle^{s} e^{\varphi}(P(h)-1\pm i\varepsilon)g\|_{H^{-1}} + 2C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|e^{\varphi} g\|_{L^2}. \tag{2-7}$$

Since the function φ does not depend on h, the function e^{φ} is bounded by positive constants both from below and from above. Thus, we deduce from (2-7) the a priori estimate

$$\|\langle x\rangle^{-s} g\|_{H^1} \le \frac{\widetilde{C}_1}{h}\|\langle x\rangle^{s}(P(h)-1\pm i\varepsilon)g\|_{H^{-1}} + \widetilde{C}_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|g\|_{L^2} \tag{2-8}$$

with constants $\widetilde{C}_1, \widetilde{C}_2 > 0$ independent of h, ε and g. On the other hand, since the operator $P(h)$ is symmetric on $L^2(\mathbb{R}^n)$, we have

$$\varepsilon\|g\|^2_{L^2} = \mp \operatorname{Im}\langle (P(h)-1\pm i\varepsilon)g, g\rangle_{L^2} \le \gamma^{-1}h^{-1}\|\langle x\rangle^{s}(P(h)-1\pm i\varepsilon)g\|^2_{H^{-1}} + \gamma h\|\langle x\rangle^{-s} g\|^2_{H^1} \tag{2-9}$$

for every $\gamma > 0$. Taking γ small enough, independent of h, we deduce from (2-8) and (2-9) the a priori estimate

$$\|\langle x\rangle^{-s} g\|_{H^1} \le \frac{C}{h}\|\langle x\rangle^{s}(P(h)-1\pm i\varepsilon)g\|_{H^{-1}} \tag{2-10}$$

with a constant $C > 0$ independent of h, ε and g. It is easy to see now that (2-10) implies the resolvent estimate (1-6) for $0 < s - \frac{1}{2} \ll 1$. On the other hand, we clearly have that, if (1-6) holds for some $s_0 > \frac{1}{2}$, it holds for all $s \ge s_0$. Hence (1-6) holds for all $s > \frac{1}{2}$.

3. Proof of Theorem 2.1

We will first prove the following:

Proposition 3.1. *Under the conditions of Theorem 2.1 we have the estimate*

$$\|\langle x\rangle^{-s} f\|_{H^1} \le \frac{C_1}{h}\|\mu^{-1}(P^L_{\varphi}(h)-1\pm i\varepsilon)f\|_{L^2} + C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|f\|_{H^1} \tag{3-1}$$

for every $0 < \varepsilon \le 1$, $0 < h \le h_0(\tau, A) \ll 1$, *with a constant* $C_1 > 0$ *independent of* $f, \varepsilon, h, \tau, A$, *and a constant* $C_2 > 0$ *independent of* f, ε, h.

Proof. We pass to the polar coordinates $(r, w) \in \mathbb{R}^+ \times \mathbb{S}^{n-1}$, $r = |x|$, $w = x/|x|$, and recall that $L^2(\mathbb{R}^n) \cong L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, r^{n-1}\,dr\,dw)$. Denote by X the Hilbert space $L^2(\mathbb{R}^+ \times \mathbb{S}^{n-1}, dr\,dw)$. We also denote by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ the norm and the scalar product on $L^2(\mathbb{S}^{n-1})$. We will make use of the identity

$$r^{(n-1)/2}\Delta r^{-(n-1)/2} = \partial_r^2 + \frac{\widetilde{\Delta}_w}{r^2}, \tag{3-2}$$

where $\widetilde{\Delta}_w = \Delta_w - \frac{1}{4}(n-1)(n-3)$ and Δ_w denotes the negative Laplace–Beltrami operator on $\mathbb{S}^{n-1}$. Observe also that

$$r^{(n-1)/2}\partial_{x_j} r^{-(n-1)/2} = w_j\partial_r + r^{-1}q_j(w, \partial_w), \tag{3-3}$$

where $w_j = x_j/|x|$ and q_j is a first-order differential operator on $\mathbb{S}^{n-1}$, independent of r, antisymmetric on $L^2(\mathbb{S}^{n-1})$. It is easy to see that the operators $Q_j(w,\mathcal{D}_w) = ihq_j(w,\partial_w)$ and $\Lambda_w = -h^2\widetilde{\Delta}_w \geq 0$ satisfy the estimate

$$\|Q_j(w,\mathcal{D}_w)v\| \leq C\|\Lambda_w^{1/2}v\| + Ch\|v\| \quad \text{for all } v \in H^1(\mathbb{S}^{n-1}), \tag{3-4}$$

with a constant $C > 0$ independent of h and v. Set $u = r^{(n-1)/2}f$,

$$\mathcal{P}^{\pm}(h) = r^{(n-1)/2}(P^L(h) - 1 \pm i\varepsilon)r^{-(n-1)/2},$$

$$\mathcal{P}^{\pm}_{\varphi}(h) = r^{(n-1)/2}(P^L_{\varphi}(h) - 1 \pm i\varepsilon)r^{-(n-1)/2} = e^{\varphi}\mathcal{P}^{\pm}(h)e^{-\varphi}.$$

Using (3-2) and (3-3) we can write the operator $\mathcal{P}^{\pm}(h)$ in the coordinates (r, w) as follows:

$$\mathcal{P}^{\pm}(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - 1 \pm i\varepsilon + \widetilde{V}^L + \sum_{j=1}^{n} w_j\big(\tilde{b}_j^L(rw,h)\mathcal{D}_r + \mathcal{D}_r\tilde{b}_j^L(rw,h)\big)$$
$$+ r^{-1}\sum_{j=1}^{n}(\tilde{b}_j^L(rw,h)Q_j(w,\mathcal{D}_w) + Q_j(w,\mathcal{D}_w)\tilde{b}_j^L(rw,h)),$$

where we have put $\mathcal{D}_r = ih\partial_r$. Since the function φ depends only on the variable r, this implies

$$\mathcal{P}^{\pm}_{\varphi}(h) = \mathcal{D}_r^2 + \frac{\Lambda_w}{r^2} - 1 \pm i\varepsilon + \widetilde{V}^L + W - 2ih\varphi'\mathcal{D}_r + \sum_{j=1}^{n} w_j(\tilde{b}_j^L(rw,h)\mathcal{D}_r + \mathcal{D}_r\tilde{b}_j^L(rw,h))$$
$$+ r^{-1}\sum_{j=1}^{n}\big(\tilde{b}_j^L(rw,h)Q_j(w,\mathcal{D}_w) + Q_j(w,\mathcal{D}_w)\tilde{b}_j^L(rw,h)\big),$$

where

$$W = -h^2\varphi'(r)^2 - h^2\varphi''(r) - 2ih\varphi'\sum_{j=1}^{n} w_j\tilde{b}_j^L.$$

Set

$$\Phi_s(r) = \|\langle r\rangle^{-s}u(r,\cdot)\|^2 + \|\langle r\rangle^{-s}\mathcal{D}_r u(r,\cdot)\|^2 + \|\langle r\rangle^{-s}r^{-1}\Lambda_w^{1/2}u(r,\cdot)\|^2,$$
$$\Psi_s = \|\langle r\rangle^{-s}u\|^2_{L^2(X)} + \|\langle r\rangle^{-s}\mathcal{D}_r u\|^2_{L^2(X)} + \|\langle r\rangle^{-s}r^{-1}\Lambda_w^{1/2}u\|^2_{L^2(X)} = \int_0^{\infty}\Phi_s(r)\,dr,$$
$$M^{\pm}(r) = \|\mathcal{P}^{\pm}_{\varphi}(h)u(r,\cdot)\|^2,$$
$$\mathcal{M}^{\pm} = \int_0^{\infty}\mu^{-2}M^{\pm}(r)\,dr,$$
$$N(r) = \|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2,$$
$$\mathcal{N} = \int_0^{\infty}N(r)\,dr,$$
$$E(r) = -\langle(r^{-2}\Lambda_w - 1 + \widetilde{V}^L)u(r,\cdot), u(r,\cdot)\rangle + \|\mathcal{D}_r u(r,\cdot)\|^2$$
$$- 2r^{-1}\sum_{j=1}^{n}\operatorname{Re}\langle\tilde{b}_j^L(rw,h)Q_j(w,\mathcal{D}_w)u(r,\cdot), u(r,\cdot)\rangle.$$

To prove (3-1) we will make use of the method of [Cardoso and Vodev 2002; Rodnianski and Tao 2011] (used there in the case when the magnetic potential is identically zero), which is based on the observation that the first derivative of the function $E(r)$ has a nice lower bound. The situation is more complex in the presence of a nontrivial magnetic potential, but we will show in what follows that the method still works. To be more precise, observe first that, in view of (1-1), (1-3) and (3-4), we have

$$E(r) \geq -\|r^{-1}\Lambda_w^{1/2}u(r,\cdot)\|^2 + \tfrac{1}{2}\|u(r,\cdot)\|^2 + \|\mathcal{D}_r u(r,\cdot)\|^2 - O(h)\Phi_{(1+\delta)/2}(r), \tag{3-5}$$

provided h is taken small enough. Furthermore, using that $\operatorname{Im}\langle \tilde{b}_j^L \mathcal{D}_r u, \mathcal{D}_r u\rangle = 0$ and $Q_j^* = Q_j$, it is easy to check that $E(r)$ satisfies the identity — see also [Cardoso et al. 2013; 2014a], where the same identity is used in an essential way —

$$\begin{aligned}
E'(r) :=& \frac{dE(r)}{dr}\\
=& \frac{2}{r}\langle r^{-2}\Lambda_w u(r,\cdot), u(r,\cdot)\rangle - \left\langle \frac{\partial \widetilde{V}^L}{\partial r} u(r,\cdot), u(r,\cdot)\right\rangle\\
&- 2\sum_{j=1}^n \operatorname{Re}\left\langle \frac{\partial(\tilde{b}_j^L(rw,h)/r)}{\partial r} Q_j(w,\mathcal{D}_w)u(r,\cdot), u(r,\cdot)\right\rangle\\
&- 2\sum_{j=1}^n \operatorname{Re}\left\langle w_j \frac{\partial \tilde{b}_j^L(rw,h)}{\partial r} u(r,\cdot), \mathcal{D}_r u(r,\cdot)\right\rangle + 2h^{-1}\operatorname{Im}\langle \mathcal{P}_\varphi^{\pm}(h)u(r,\cdot), \mathcal{D}_r u(r,\cdot)\rangle\\
&\mp 2\varepsilon h^{-1}\operatorname{Re}\langle u(r,\cdot), \mathcal{D}_r u(r,\cdot)\rangle + 4\langle \varphi' \mathcal{D}_r u(r,\cdot), \mathcal{D}_r u(r,\cdot)\rangle\\
&- 2h^{-1}\operatorname{Im}\langle W u(r,\cdot), \mathcal{D}_r u(r,\cdot)\rangle.
\end{aligned} \tag{3-6}$$

In view of (1-2), (1-3), (3-4) and (3-6), we obtain the inequality

$$\begin{aligned}
E'(r) \geq& \frac{2}{r}\|r^{-1}\Lambda_w^{1/2}u(r,\cdot)\|^2 + 4\varphi'\|\mathcal{D}_r u(r,\cdot)\|^2 - 2h^{-1}\|\mathcal{P}_\varphi^{\pm}(h)u(r,\cdot)\|\|\mathcal{D}_r(r,\cdot)\|\\
&- O(h)\Phi_{(1+\delta)/2}(r) - O(\varepsilon h^{-1})N(r).
\end{aligned} \tag{3-7}$$

Since $\Phi_{(1+\delta)/2}(r) \leq \Phi_s(r)$ for $\frac{1}{2} < s \leq \frac{1}{2}(1+\delta)$, we obtain from (3-7)

$$\begin{aligned}
E'(r) \geq& \frac{2}{r}\|r^{-1}\Lambda_w^{1/2}u(r,\cdot)\|^2 + 4\varphi'\|\mathcal{D}_r u(r,\cdot)\|^2 - \gamma^{-1}h^{-2}\mu^{-2}M^{\pm}(r)\\
&- \gamma\mu^2\|\mathcal{D}_r(r,\cdot)\|^2 - O(h)\Phi_s(r) - O(\varepsilon h^{-1})N(r)\\
\geq& \frac{2}{r}\|r^{-1}\Lambda_w^{1/2}u(r,\cdot)\|^2 - \gamma^{-1}h^{-2}\mu^{-2}M^{\pm}(r) - O(h+\gamma)\Phi_s(r) - O(\varepsilon h^{-1})N(r)
\end{aligned} \tag{3-8}$$

for every $0 < \gamma \ll 1$. By (3-5) and (3-8),

$$\langle r\rangle^{-2s}(E(r) + rE'(r)) \geq \Phi_s(r) - \gamma^{-1}h^{-2}\mu^{-2}M^{\pm}(r) - O(h+\gamma)\Phi_s(r) - O(\varepsilon h^{-1})N(r). \tag{3-9}$$

Integrating (3-8) from t ($t > 0$) to $+\infty$ we get

$$E(t) = -\int_t^\infty E'(r)\,dr \leq O(\gamma^{-1}h^{-2})\mathcal{M}^{\pm} + O(\varepsilon h^{-1})\mathcal{N} + O(h+\gamma)\Psi_s. \tag{3-10}$$

Let $\psi > 0$ be a function independent of h and such that $\int_0^\infty \psi(r)\,dr < \infty$. Multiplying (3-10) by $\psi(t)$ and integrating from 0 to $+\infty$, we get

$$\int_0^\infty \psi(r)E(r)\,dr \le O(\gamma^{-1}h^{-2})\mathcal{M}^\pm + O(\varepsilon h^{-1})\mathcal{N} + O(h+\gamma)\Psi_s. \tag{3-11}$$

Observe now that we have the identity

$$\int_0^\infty \langle r\rangle^{-2s}(E(r)+rE'(r))\,dr = \int_0^\infty \psi(r)E(r)\,dr, \tag{3-12}$$

where $\psi(r) = 2sr\langle r\rangle^{-2s-1}$. Combining (3-9), (3-11) and (3-12) and taking γ and h small enough, we conclude

$$\Psi_s \le O(h^{-2})\mathcal{M}^\pm + O(\varepsilon h^{-1})\mathcal{N}. \tag{3-13}$$

Clearly, (3-13) implies (3-1). □

We will now show that (2-1) follows from (3-1) and the following:

Lemma 3.2. *Let $\ell \in \mathbb{R}$. Then we have the estimate*

$$\|\mu^{-\ell}(P_\varphi^L(h)-i)^{-1}\mu^\ell\|_{H^{-1}\to H^1} \le C \tag{3-14}$$

for $0 < h \le h_0(\tau, A) \ll 1$, with a constant $C > 0$ independent of h, τ and A.

We are going to use (3-1) with $f = (P_\varphi^L(h)-i)^{-1}g$. In view of the identity

$$1 = (1-i\mp i\varepsilon)(P_\varphi^L(h)-i)^{-1} + (P_\varphi^L(h)-i)^{-1}(P_\varphi^L(h)-1\pm i\varepsilon)$$

and Lemma 3.2, we have

$$\begin{aligned}
\|\langle x\rangle^{-s}g\|_{H^1} &\le 2\|\langle x\rangle^{-s}(P_\varphi^L(h)-i)^{-1}g\|_{H^1} + \|\langle x\rangle^{-s}(P_\varphi^L(h)-i)^{-1}(P_\varphi^L(h)-1\pm i\varepsilon)g\|_{H^1}A\\
&\le \frac{2C_1}{h}\|\mu^{-1}(P_\varphi^L(h)-i)^{-1}(P_\varphi^L(h)-1\pm i\varepsilon)g\|_{L^2}\\
&\quad + 2C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|(P_\varphi^L(h)-i)^{-1}g\|_{H^1} + C_3\|(P_\varphi^L(h)-i)^{-1}(P_\varphi^L(h)-1\pm i\varepsilon)g\|_{H^1}\\
&\le \frac{2C_1}{h}\|\mu^{-1}(P_\varphi^L(h)-i)^{-1}\mu\|_{H^{-1}\to L^2}\|\mu^{-1}(P_\varphi^L(h)-1\pm i\varepsilon)g\|_{H^{-1}}\\
&\qquad + 2C_2\left(\frac{\varepsilon}{h}\right)^{1/2}\|(P_\varphi^L(h)-i)^{-1}\|_{L^2\to H^1}\|g\|_{L^2}\\
&\qquad\qquad + C_3\|(P_\varphi^L(h)-i)^{-1}\|_{H^{-1}\to H^1}\|(P_\varphi^L(h)-1\pm i\varepsilon)g\|_{H^{-1}}\\
&\le \frac{C_1'}{h}\|\mu^{-1}(P_\varphi^L(h)-1\pm i\varepsilon)g\|_{H^{-1}} + C_2'\left(\frac{\varepsilon}{h}\right)^{1/2}\|g\|_{L^2} + C_3'\|(P_\varphi^L(h)-1\pm i\varepsilon)g\|_{H^{-1}}
\end{aligned} \tag{3-15}$$

with a constant $C_1' > 0$ independent of ε, h, τ, A and g, and constants $C_2', C_3' > 0$ independent of ε, h and g. Since the function μ is bounded on $\mathbb{R}^n$, there exists $0 < h_0(\varphi) \ll 1$ such that for $0 < h \le h_0$ the last term in the right-hand side of (3-15) can be bounded by the first one. Thus we get (2-1) from (3-15).

4. Proof of Lemma 3.2

It is easy to see that the estimate (3-14) holds with $\ell = 0$ and $P_\varphi^L(h)$ replaced by $-h^2\Delta$. Indeed, in this case the $L^2 \to L^2$ bound is trivial, while the $H^{-1} \to H^1$ bound follows from the fact that $\|f\|_{H^s} \sim \|(1-h^2\Delta)^{s/2} f\|_{L^2}$, $s = -1, 1$. We will use this to show that (3-14) with $\ell = 0$ still holds for first-order perturbations of the form $-h^2\Delta + Q(h)$, where

$$Q(h) = \sum_{|\alpha|=1} q_\alpha^{(1)}(x,h)\mathscr{D}_x^\alpha + \sum_{|\alpha|=1} \mathscr{D}_x^\alpha q_\alpha^{(2)}(x,h) + q_0(x,h)$$

with coefficients satisfying

$$|q_\alpha^{(1)}(x,h)| + |q_\alpha^{(2)}(x,h)| + |q_0(x,h)| \le Ch \quad \text{for all } x \in \mathbb{R}^n. \tag{4-1}$$

Clearly, (4-1) implies

$$\|Q(h)\|_{H^1 \to H^{-1}} \le Ch. \tag{4-2}$$

By (4-2) and the resolvent identity

$$(-h^2\Delta + Q(h) - i)^{-1} = (-h^2\Delta - i)^{-1} + (-h^2\Delta - i)^{-1} Q(h)(-h^2\Delta + Q(h) - i)^{-1},$$

we get

$$\begin{aligned}
&\|(-h^2\Delta + Q(h) - i)^{-1}\|_{H^{-1}\to H^1} \\
&\le \|(-h^2\Delta - i)^{-1}\|_{H^{-1}\to H^1} + \|(-h^2\Delta - i)^{-1}\|_{H^{-1}\to H^1} \|Q(h)\|_{H^1\to H^{-1}} \|(-h^2\Delta + Q(h) - i)^{-1}\|_{H^{-1}\to H^1} \\
&\le C + O(h)\|(-h^2\Delta + Q(h) - i)^{-1}\|_{H^{-1}\to H^1}.
\end{aligned} \tag{4-3}$$

Now, taking h small enough (depending on the coefficients of $Q(h)$) we can absorb the last term in the right-hand side of (4-3) and obtain the desired estimate with a constant $C > 0$ independent of $q_\alpha^{(1)}$, $q_\alpha^{(2)}$, q_0 and h.

Thus, to prove (3-14) it suffices to show that the operator $\mu^{-\ell} P_\varphi^L(h)\mu^\ell$ equals $-h^2\Delta$ plus a first-order differential operator with coefficients satisfying (4-1). To do so, observe first that $\mu^{-\ell} P_\varphi^L(h)\mu^\ell = P_\psi^L(h)$, where $\psi = \varphi - \ell \log \mu$. Furthermore, we have

$$P_\psi^L(h) = -h^2\Delta + (i\tilde{b}^L - h\nabla\psi)\cdot h\nabla + h\nabla\cdot(i\tilde{b}^L - h\nabla\psi) - h^2|\nabla\psi|^2 - 2ih\tilde{b}^L\cdot\nabla\psi + \widetilde{V}^L.$$

It is easy to see that $|\psi'(r)|$ is bounded on $\mathbb{R}$, and hence $|\nabla\psi(|x|)|$ is bounded on $\mathbb{R}^n$. This together with the assumptions on $\tilde{b}^L$ and $\widetilde{V}^L$ imply the desired properties of the coefficients of the operator $P_\psi^L(h)$. □

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