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LOCAL AND NONLOCAL BOUNDARY CONDITIONS FOR μ-TRANSMISSION AND FRACTIONAL ELLIPTIC PSEUDODIFFERENTIAL OPERATORS

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A classical pseudodifferential operator P on $\mathbb{R}^n$ satisfies the μ-transmission condition relative to a smooth open subset Ω when the symbol terms have a certain twisted parity on the normal to $\partial\Omega$. As shown recently by the author, this condition assures solvability of Dirichlet-type boundary problems for P in full scales of Sobolev spaces with a singularity $d^{\mu-k}$, $d(x) = \operatorname{dist}(x, \partial\Omega)$. Examples include fractional Laplacians $(-\Delta)^a$ and complex powers of strongly elliptic PDE.

We now introduce new boundary conditions, of Neumann type, or, more generally, nonlocal type. It is also shown how problems with data on $\mathbb{R}^n \setminus \Omega$ reduce to problems supported on $\overline{\Omega}$, and how the so-called "large" solutions arise. Moreover, the results are extended to general function spaces $F^s_{p,q}$ and $B^s_{p,q}$, including Hölder–Zygmund spaces $B^s_{\infty,\infty}$. This leads to optimal Hölder estimates, e.g., for Dirichlet solutions of $(-\Delta)^a u = f \in L_\infty(\Omega)$, $u \in d^a C^a(\overline{\Omega})$ when $0 < a < 1$, $a \neq \frac{1}{2}$.

Boundary value problems for elliptic pseudodifferential operators (ψdo's) P, on a smooth subset Ω of a Riemannian manifold Ω_1, have been studied under various hypotheses through the years. There is a well-known calculus initiated by Boutet de Monvel [Boutet de Monvel 1971; Rempel and Schulze 1982; Grubb 1984; 1990; 1996; 2009; Schrohe 2001] for integer-order ψdo's with the 0-transmission property (preserving C^∞ up to the boundary), including boundary value problems for elliptic differential operators and their inverses. There are theories treating more general operators with suitable factorizations of the principal symbol, initiated by Vishik and Eskin (see, e.g., [Eskin 1981; Shargorodsky 1994; Chkadua and Duduchava 2001]). Theories for operators without the transmission property have been developed by Schulze and coauthors, see, e.g., [Rempel and Schulze 1984; Harutyunyan and Schulze 2008], and theories where the boundary is considered as a singularity of the manifold have been developed in works of Melrose and coauthors, see, e.g., [Melrose 1993; Albin and Melrose 2009].

A category of ψdo's lying between the operators handled by the Boutet de Monvel calculus and the very general categories mentioned above consists of the ψdo's with a μ-transmission property, $\mu \in \mathbb{C}$, with respect to $\partial\Omega$. Only recently, a systematic study in H^s_p Sobolev spaces was given in [Grubb 2015a], departing from a result on such operators in C^∞-spaces by Hörmander [1985, Theorem 18.2.18] (in fact developed from the lecture notes [Hörmander 1965]). This category includes fractional Laplacians $(-\Delta)^a$ and complex powers of strongly elliptic differential operators, and also more generally polyhomogeneous ψdo's with symbol $p \sim \sum_{j\in\mathbb{N}_0} p_j$ having *even parity* (that is, $p_j(x, -\xi) = (-1)^j p_j(x, \xi)$ for $j \geq 0$)

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or a twisted parity involving a factor $e^{i\pi\varrho}$. The general μ-transmission operators have such a reflection property of the symbol at $\partial\Omega$ just in the normal direction; see (1-5) below. This allows regularity and solvability results not only for s in a finite interval, but for arbitrarily large s.

The fractional Laplacian and its generalizations, often formulated as singular integral operators, are currently of interest in probability theory, finance, mathematical physics and geometry.

The work [Grubb 2015a] showed the Fredholm solvability of homogeneous or nonhomogeneous Dirichlet-type problems in large scales of Sobolev spaces, for μ-transmission ψdo's. In the present paper, we introduce more general boundary conditions and find criteria for their solvability. There are the general nonlocal conditions $\gamma_0 Bu = \psi$, where B is a μ-transmission ψdo; in addition to this, local higher-order conditions such as a Neumann-type condition involving the normal derivative at $\partial\Omega$ are treated. The case of $N \times N$ systems of ψdo is briefly considered.

Moreover, we show by use of [Johnsen 1996] that the theory also works in the Besov–Triebel–Lizorkin spaces $B^s_{p,q}$ and $F^s_{p,q}$, with special attention to the spaces $B^s_{\infty,\infty}$, which coincide with Hölder spaces C^s for $s \in \mathbb{R}_+ \setminus \mathbb{N}$. In comparison with [Grubb 2015a], this allows for a sharpening of Hölder results for $(-\Delta)^a$ (and other a-transmission operators) as follows: Let $\overline{\Omega}$ be a compact subset of $\mathbb{R}^n$. For solutions $u \in e^+L_\infty(\Omega)$ of $r^+(-\Delta)^a u = f$,

$$f \in L_\infty(\Omega) \implies u \in e^+ d(x)^a C^a(\overline{\Omega}), \text{ when } a \in]0,1[,\ a \neq \tfrac{1}{2}, \tag{0-1}$$

which is optimal in the Hölder exponent. (For $a = \frac{1}{2}$, it holds with C^a replaced by $C^{a-\varepsilon}$. Also higher regularities are treated, and optimal Hölder estimates for nonhomogeneous Dirichlet and Neumann problems are likewise shown.) In a new work, Ros-Oton and Serra [2014a] have studied integral operators with homogeneous, positive, even kernel and obtained (0-1) with C^a replaced by $C^{a-\varepsilon}$; in the smooth case this is covered by the present theory. (We are concerned with linear operators; the nonlinear implications in [Ros-Oton and Serra 2014a] are not touched here.) Such operators were treated in cases without boundary by Caffarelli and Silvestre, see, e.g., [2009].

Furthermore, we show the equivalence of Dirichlet problems for u supported in $\overline{\Omega}$ with problems prescribing a value of u on the exterior $\mathbb{R}^n \setminus \Omega$, obtaining new results for the latter, which were treated recently by, for example, Felsinger, Kassmann and Voigt [Felsinger et al. 2014] and Abatangelo [2013].

For nonhomogeneous problems the solutions can be "large" at the boundary; cf. [Abatangelo 2013] and its references. We show how the solutions have a specific power singularity when the boundary data are nontrivial.

The case $a = \frac{1}{2}$ enters as a boundary integral operator in treatments of mixed boundary value problems for elliptic differential operators. The present results are applied to mixed problems in [Grubb 2015b].

Outline. In Section 1, we briefly recall the relevant definitions of operators and spaces. Section 2 presents the basic results on Dirichlet and Neumann problems for $(-\Delta)^a$, including situations with given exterior data, and derives conclusions in Hölder spaces. Section 3 explains the extension of the general results to Besov–Triebel–Lizorkin spaces, including $B^s_{\infty,\infty}$. Section 4 introduces new nonlocal boundary conditions $\gamma_0 Bu = \psi$, as well as local Neumann-type conditions; also $N \times N$ systems of ψdo are discussed. The

Appendix illustrates the theory by treating a particular constant-coefficient case, showing how the problems for $(1-\Delta)^a$ on $\mathbb{R}^n_+$ can be solved in full detail by explicit calculations.

1. Preliminaries

The notation of [Grubb 2015a] will be used. We shall give a brief account, and refer there for further details.

Consider a Riemannian n-dimensional C^∞ manifold Ω_1 (it can be $\mathbb{R}^n$) and an embedded smooth n-dimensional manifold $\overline{\Omega}$ with boundary $\partial\Omega$ and interior Ω. For $\Omega_1=\mathbb{R}^n$, Ω can be $\mathbb{R}^n_\pm=\{x\in\mathbb{R}^n \mid x_n \gtrless 0\}$; we will denote $(x_1,\dots,x_{n-1})$ by x'. In the general manifold case, $\overline{\Omega}$ is taken to be compact. For $\xi\in\mathbb{R}^n$, we let $(1+|\xi|^2)^{\frac12}=\langle\xi\rangle$, and denote by $[\xi]$ a positive C^∞-function equal to $|\xi|$ for $|\xi|\ge 1$ and $\ge\frac12$ for all ξ. Restriction from $\mathbb{R}^n$ to $\mathbb{R}^n_\pm$ (or from Ω_1 to Ω or $\complement\overline{\Omega}$, respectively) is denoted by $r^\pm$, extension by zero from $\mathbb{R}^n_\pm$ to $\mathbb{R}^n$ (or from Ω or $\complement\overline{\Omega}$, respectively, to Ω_1) is denoted by $e^\pm$.

A pseudodifferential operator (ψdo) P on $\mathbb{R}^n$ is defined from a symbol $p(x,\xi)$ on $\mathbb{R}^n\times\mathbb{R}^n$ by

$$Pu=p(x,D)u=\mathrm{OP}(p(x,\xi))u=(2\pi)^{-n}\int e^{ix\cdot\xi}p(x,\xi)\hat u\,d\xi=\mathscr{F}^{-1}_{\xi\to x}(p(x,\xi)\hat u(\xi)); \tag{1-1}$$

here, $\mathscr{F}$ is the Fourier transform $(\mathscr{F}u)(\xi)=\hat u(\xi)=\int_{\mathbb{R}^n}e^{-ix\cdot\xi}u(x)\,dx$. The symbol p is assumed to be such that for some $r\in\mathbb{R}$, $\partial_x^\beta\partial_\xi^\alpha p(x,\xi)$ is $O(\langle\xi\rangle^{r-|\alpha|})$ for all α,β (defining the symbol class $S^r_{1,0}(\mathbb{R}^n\times\mathbb{R}^n)$); the symbol then has order r. The definition of P is carried over to manifolds by use of local coordinates. We refer to textbooks such as [Hörmander 1985; Taylor 1981; Grubb 2009] for the rules of calculus; [Grubb 2009] moreover gives an account of the Boutet de Monvel calculus of pseudodifferential boundary problems, see also, e.g., [Grubb 1996; Schrohe 2001]. When P is a ψdo on $\mathbb{R}^n$ or Ω_1, $P_+=r^+Pe^+$ denotes its truncation to $\mathbb{R}^n_+$ or Ω, respectively.

Let $1<p<\infty$ (with $1/p'=1-1/p$), then we define for $s\in\mathbb{R}$ the spaces

$$\begin{aligned}
H^s_p(\mathbb{R}^n)&=\{u\in\mathscr{S}'(\mathbb{R}^n)\mid \mathscr{F}^{-1}(\langle\xi\rangle^s\hat u)\in L_p(\mathbb{R}^n)\},\\
\dot H^s_p(\overline{\mathbb{R}}^n_+)&=\{u\in H^s_p(\mathbb{R}^n)\mid \operatorname{supp}u\subset\overline{\mathbb{R}}^n_+\},\\
\overline{H}^s_p(\mathbb{R}^n_+)&=\{u\in\mathscr{D}'(\mathbb{R}^n_+)\mid u=r^+U \text{ for some } U\in H^s_p(\mathbb{R}^n)\};
\end{aligned} \tag{1-2}$$

here, $\operatorname{supp}u$ denotes the support of u. For a compact subset $\overline{\Omega}$ of Ω_1, the definition extends to define $\dot H^s_p(\overline{\Omega})$ and $\overline{H}^s_p(\Omega)$ by use of a finite system of local coordinates. We shall in the present paper moreover work in the Triebel–Lizorkin and Besov spaces $F^s_{p,q}$ and $B^s_{p,q}$, defined for $s\in\mathbb{R}$, $0<p,q\le\infty$ (we take $p<\infty$ in the F-case), and the derived spaces $\dot F^s_{p,q}$ and $\overline{F}^s_{p,q}$, etc. Here we refer to [Triebel 1995; Johnsen 1996] for basic definitions. ([Triebel 1995] writes $\widetilde{F}$ instead of $\dot F$, etc.; the present notation stems from Hörmander's works.) For a Hölder space C^t, $\dot C^t(\overline{\Omega})$ denotes the Hölder functions on Ω_1 supported in $\overline{\Omega}$. $B^s_{p,p}$ is also denoted by B^s_p when $p<\infty$, and $F^s_{p,p}=B^s_{p,p}$, $F^s_{p,2}=H^s_p$, $H^s_2=B^s_2$.

We shall use the conventions $\bigcup_{\varepsilon>0}H^{s+\varepsilon}_p=H^{s+0}_p$ and $\bigcap_{\varepsilon>0}H^{s-\varepsilon}_p=H^{s-0}_p$, applied in a similar way for the other scales of spaces.

The results hold in particular for $B^s_{\infty,\infty}$-spaces. These are interesting because $B^s_{\infty,\infty}(\mathbb{R}^n)$ equals the Hölder space $C^s(\mathbb{R}^n)$ when $s\in\mathbb{R}_+\setminus\mathbb{N}$. (There are similar statements for derived spaces over $\mathbb{R}^n_+$ and Ω.) The spaces $B^s_{\infty,\infty}(\mathbb{R}^n)$ can be identified with the Hölder–Zygmund spaces, often denoted $\mathscr{C}^s(\mathbb{R}^n)$ when

$s > 0$. There is a nice account of these spaces in Section 8.6 of [Hörmander 1997], where they are denoted by $C^s_*(\mathbb{R}^n)$ for all $s \in \mathbb{R}$; we shall use that label below, for simplicity of notation:

$$B^s_{\infty,\infty} = C^s_* \quad \text{for all } s \in \mathbb{R}. \tag{1-3}$$

For integer values of k one has, with $C^k_b(\mathbb{R}^n)$ denoting the space of functions with bounded continuous derivatives up to order k,

$$\begin{aligned} &C^k_b(\mathbb{R}^n) \subset C^{k-1,1}(\mathbb{R}^n) \subset C^k_*(\mathbb{R}^n) \subset C^{k-0}(\mathbb{R}^n) \text{ when } k \in \mathbb{N},\\ &C^0_b(\mathbb{R}^n) \subset L_\infty(\mathbb{R}^n) \subset C^0_*(\mathbb{R}^n), \end{aligned} \tag{1-4}$$

and similar statements for derived spaces.

A ψdo P is called classical (or polyhomogeneous) when the symbol p has an asymptotic expansion $p(x,\xi) \sim \sum_{j\in\mathbb{N}_0} p_j(x,\xi)$ with p_j homogeneous in ξ of degree $m-j$ for all j. Then P has order m. One can even allow m to be complex; then $p \in S^{\operatorname{Re} m}_{1,0}(\mathbb{R}^n \times \mathbb{R}^n)$, and the operator and symbol are still said to be of order m.

Here there is an additional definition: P satisfies *the μ-transmission condition* (in short, is of type μ) for some $\mu \in \mathbb{C}$ when, in local coordinates,

$$\partial_x^\beta \partial_\xi^\alpha p_j(x,-N) = e^{\pi i(m-2\mu-j-|\alpha|)} \partial_x^\beta \partial_\xi^\alpha p_j(x,N) \tag{1-5}$$

for all $x \in \partial\Omega$, all j, α, β, where N denotes the interior normal to $\partial\Omega$ at x. The implications of the μ-transmission property were a main subject of [Grubb 2015a].

A special role in the theory is played by the *order-reducing operators*. There is a simple definition of operators $\Xi^\mu_\pm$ on $\mathbb{R}^n$:

$$\Xi^\mu_\pm = \operatorname{OP}(([\xi'] \pm i\xi_n)^\mu)$$

(or with $[\xi']$ replaced by $\langle\xi'\rangle$); they preserve support in $\overline{\mathbb{R}}^n_\pm$, respectively. Here the function $([\xi'] \pm i\xi_n)^\mu$ does not satisfy all the estimates required for the class $S^{\operatorname{Re}\mu}(\mathbb{R}^n \times \mathbb{R}^n)$, but the operators are useful for some purposes. There is a more refined choice $\Lambda^\mu_\pm$ (with symbol $\lambda^\mu_\pm(\xi)$) that does satisfy all the estimates, and there is a definition $\Lambda^{(\mu)}_\pm$ in the manifold situation. These operators define homeomorphisms for all $s \in \mathbb{R}$ such as

$$\begin{aligned} \Lambda^{(\mu)}_+ &: \dot H^s_p(\overline\Omega) \xrightarrow{\sim} \dot H^{s-\operatorname{Re}\mu}_p(\overline\Omega),\\ \Lambda^{(\mu)}_{-,+} &: \overline H^s_p(\Omega) \xrightarrow{\sim} \overline H^{s-\operatorname{Re}\mu}_p(\Omega); \end{aligned} \tag{1-6}$$

here, $\Lambda^{(\mu)}_{-,+}$ is short for $r^+\Lambda^{(\mu)}_- e^+$, suitably extended to large negative s (see Remark 1.1 and Theorem 1.3 in [Grubb 2015a]).

The following special spaces, introduced by Hörmander, are particularly adapted to μ-transmission operators P:

$$\begin{aligned} H^{\mu(s)}_p(\overline{\mathbb{R}}^n_+) &= \Xi^{-\mu}_+ e^+ \overline H^{s-\operatorname{Re}\mu}_p(\mathbb{R}^n_+), \quad s > \operatorname{Re}\mu - 1/p',\\ H^{\mu(s)}_p(\overline\Omega) &= \Lambda^{(-\mu)}_+ e^+ \overline H^{s-\operatorname{Re}\mu}_p(\Omega), \quad s > \operatorname{Re}\mu - 1/p',\\ \mathscr{E}_\mu(\overline\Omega) &= e^+\{u(x) = d(x)^\mu v(x) \mid v \in C^\infty(\overline\Omega)\}; \end{aligned} \tag{1-7}$$

namely, r^+P (of order m) maps them into $\overline{H}_p^{s-\operatorname{Re} m}(\mathbb{R}^n_+)$, $\overline{H}_p^{s-\operatorname{Re} m}(\Omega)$ and $C^\infty(\overline{\Omega})$ respectively (see [Grubb 2015a] Sections 1.3, 2, 4), and they appear as domains of realizations of P in the elliptic case. In the third line, $\operatorname{Re}\mu > -1$ (for other μ, see [Grubb 2015a]) and $d(x)$ is a C^∞-function vanishing to order 1 at $\partial\Omega$ and positive on Ω, e.g., $d(x) = \operatorname{dist}(x, \partial\Omega)$ near $\partial\Omega$. One has that $H_p^{\mu(s)}(\overline{\Omega}) \supset \dot{H}_p^s(\overline{\Omega})$, and the distributions are locally in H_p^s on Ω, but at the boundary they in general have a singular behavior. More about that in the text below.

The order-reducing operators also operate in the Besov–Triebel–Lizorkin scales of spaces, satisfying the relevant versions of (1-6), and the definitions in (1-7) extend.

2. Three basic problems for the fractional Laplacian

As a useful introduction, we start out by giving a detailed presentation of boundary problems for the basic example of the fractional Laplacian.

Let $P_a = (-\Delta)^a$, $a > 0$, and let Ω be a bounded open subset of $\mathbb{R}^n$ with a C^∞-boundary $\partial\Omega = \Sigma$. P_a, acting as $u \mapsto \mathcal{F}^{-1}(|\xi|^{2a}\hat{u})$, is a pseudodifferential operator on $\mathbb{R}^n$ of order $2a$, and it is of type a and has factorization index a relative to Ω, as defined in [Grubb 2015a]. With terminology introduced by Hörmander in the notes [1965] and now exposed in [Grubb 2015a], we consider the following problems for P_a:

(1) *The homogeneous Dirichlet problem*:

$$\begin{cases} r^+P_a u = f & \text{on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}. \end{cases} \tag{2-1}$$

(2) *A nonhomogeneous Dirichlet problem* (with u less regular than in (2-1)):

$$\begin{cases} r^+P_a u = f & \text{on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, & \\ d(x)^{1-a}u = \varphi & \text{on } \Sigma. \end{cases} \tag{2-2}$$

(3) *A nonhomogeneous Neumann problem*:

$$\begin{cases} r^+P_a u = f & \text{on } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}, & \\ \partial_n(d(x)^{1-a}u) = \psi & \text{on } \Sigma. \end{cases} \tag{2-3}$$

It is shown in [Grubb 2015a] that (2-1) and (2-2) have good solvability properties in suitable Sobolev spaces and Hölder spaces, and we shall include (2-3) in the study below. In the following, we derive further properties of each of the three problems.

Remark 2.1. The theorems in Sections 2A and 2B below are also valid when $(-\Delta)^a$ is replaced by a general a-transmission ψdo P of order $2a$ and with factorization index a, except that bijectivity is replaced by the Fredholm property. They also hold when $\overline{\Omega}$ is a compact subset of a manifold Ω_1. The results in Section 2C extend to such operators when they are principally like $(-\Delta)^a$.

In the Appendix of this paper we have included a treatment of $(1-\Delta)^a$ on a half-space; it is a model case where one can obtain the solvability results directly by Fourier transformation.

2A. ***The homogeneous Dirichlet problem.*** From the point of view of functional analysis (as used for example in [Frank and Geisinger 2014]), it is natural to define the Dirichlet realization $P_{a,D}$ as the Friedrichs extension of the symmetric operator $P_{a,0}$ in $L_2(\Omega)$ acting like r^+P_a with domain $C_0^\infty(\Omega)$. There is an associated sesquilinear form

$$p_{a,0}(u,v) = (2\pi)^{-n}\int_{\mathbb{R}^n} |\xi|^{2a}\hat{u}(\xi)\overline{\hat{v}(\xi)}\,d\xi, \quad u, v \in C_0^\infty(\Omega). \tag{2-4}$$

Since $\big(\|u\|_{L_2}^2 + \int |\xi|^{2a}|\hat{u}|^2\,d\xi\big)^{\frac{1}{2}}$ is a norm equivalent with $\|u\|_{H_2^a}$, the completion of $C_0^\infty(\Omega)$ in this norm is $V = \dot{H}_2^a(\overline{\Omega})$, and $p_{a,0}$ extends to a continuous nonnegative symmetric sesquilinear form on V. A standard application of the Lax–Milgram lemma (e.g., as in [Grubb 2009, Chapter 12]) gives an operator $P_{a,D}$ that is selfadjoint nonnegative in $L_2(\Omega)$ and acts like $r^+P_a : \dot{H}_2^a(\overline{\Omega}) \to \overline{H}_2^{-a}(\Omega)$, with domain

$$D(P_{a,D}) = \big\{u \in \dot{H}_2^a(\overline{\Omega}) \mid r^+P_a u \in L_2(\Omega)\big\}. \tag{2-5}$$

The operator has compact resolvent, and the spectrum is a nondecreasing sequence of nonnegative eigenvalues going to infinity. As we shall document below, 0 is not an eigenvalue, so $P_{a,D}$ in fact has a positive lower bound and is invertible.

The results of [Grubb 2015a, Sections 4, 7] clarify the mapping properties and solvability properties further: For $1 < p < \infty$, r^+P_a maps continuously:

$$r^+P_a : H_p^{a(s)}(\overline{\Omega}) \to \overline{H}_p^{s-2a}(\Omega), \text{ when } s > a - 1/p'; \tag{2-6}$$

there is the regularity result

$$u \in \dot{H}_p^{a-1/p'+0}(\overline{\Omega}), \ \ r^+P_a u \in \overline{H}_p^{s-2a}(\Omega) \Longrightarrow u \in H_p^{a(s)}(\overline{\Omega}), \text{ when } s > a - 1/p', \tag{2-7}$$

and the mapping (2-6) is Fredholm. (It is even bijective, as seen below.) As an application of the results for $s = 2a$, $p = 2$, we have in particular that

$$D(P_{a,D}) = H_2^{a(2a)}(\overline{\Omega}) = \Lambda_+^{(-a)}e^+\overline{H}_2^a(\Omega); \tag{2-8}$$

see also Example 7.2 in [Grubb 2015a]. We recall from [Grubb 2015a, Theorem 5.4] that

$$H_p^{a(s)}(\overline{\Omega}) \begin{cases} = \dot{H}_p^s(\overline{\Omega}) & \text{when } a - 1/p' < s < a + 1/p, \\ \subset \dot{H}_p^{s-0}(\overline{\Omega}) & \text{when } s = a + 1/p, \\ \subset e^+d^a\overline{H}_p^{s-a}(\Omega) + \dot{H}_p^s(\overline{\Omega}) & \text{when } s > a + 1/p,\ s - a - 1/p \notin \mathbb{N}, \\ \subset e^+d^a\overline{H}_p^{s-a}(\Omega) + \dot{H}_p^{s-0}(\overline{\Omega}) & \text{when } s - a - 1/p \in \mathbb{N}. \end{cases} \tag{2-9}$$

In [Grubb 2015a, Section 7], we used Sobolev embedding theorems to draw conclusions for Hölder spaces. Slightly sharper (often optimal) results can be obtained if we use an extension of the results of [Grubb 2015a] to the general scales of Triebel–Lizorkin and Besov spaces $F_{p,q}^s$ and $B_{p,q}^s$. The extended theory will be presented in detail below in Sections 3–4; for the moment we shall borrow some results to give powerful statements for $(-\Delta)^a$, $0 < a < 1$. We recall that the notation $B_{\infty,\infty}^s$ is simplified to C_*^s,

and that C^s_* equals C^s (the ordinary Hölder space) for $s \in \mathbb{R}_+ \setminus \mathbb{N}$; see also (1-4). Moreover, as special cases of Definition 3.1 and Theorem 3.4 below for $p = q = \infty$,

$$\begin{aligned} C^{\mu(s)}_*(\overline{\Omega}) &= \Lambda_+^{(-\mu)} e^+ \overline{C}^{s-\operatorname{Re}\mu}_*(\Omega) && \text{for } s > \operatorname{Re}\mu - 1, \\ C^{\mu(s)}_*(\overline{\Omega}) &\subset \begin{cases} d(x)^\mu e^+ \overline{C}^{s-\operatorname{Re}\mu}_*(\Omega) + \dot{C}^s_*(\overline{\Omega}) & \text{when } s > \operatorname{Re}\mu,\ s - \operatorname{Re}\mu \notin \mathbb{N}, \\ d(x)^\mu e^+ \overline{C}^{s-\operatorname{Re}\mu}_*(\Omega) + \dot{C}^{s-0}_*(\overline{\Omega}) & \text{when } s > \operatorname{Re}\mu,\ s - \operatorname{Re}\mu \in \mathbb{N}. \end{cases} \end{aligned} \tag{2-10}$$

Note also that the distributions in $C^{\mu(s)}_*(\overline{\Omega})$ are locally in C^s_* on Ω, by the ellipticity of $\Lambda_+^{(-\mu)}$.

We focus in the following on the case $0 < a < 1$, assumed in the rest of this chapter. Here we find the following results, with conclusions formulated in ordinary Hölder spaces:

Theorem 2.2. *Let $s > a - 1$. If $u \in \dot{C}^{a-1+\varepsilon}_*(\overline{\Omega})$ for some $\varepsilon > 0$ (e.g., if $u \in e^+ L_\infty(\Omega)$), and $r^+ P u \in \overline{C}^{s-2a}_*(\Omega)$, then $u \in C^{a(s)}_*(\overline{\Omega})$. The mapping $r^+ P_a$ defines a bijection*

$$r^+ P_a : C^{a(s)}_*(\overline{\Omega}) \to \overline{C}^{s-2a}_*(\Omega). \tag{2-11}$$

In particular, for any $f \in L_\infty(\Omega)$, there exists a unique solution u of (2-1) *in $C^{a(2a)}_*$; it satisfies*

$$\begin{aligned} &u \in e^+ d(x)^a C^a(\overline{\Omega}) \cap C^{2a}(\Omega), \ \text{when } a \neq \tfrac{1}{2}, \\ &u \in \big(e^+ d(x)^{\frac{1}{2}} C^{\frac{1}{2}}(\overline{\Omega}) + \dot{C}^{1-0}(\overline{\Omega})\big) \cap C^{1-0}(\Omega) \subset e^+ d(x)^{\frac{1}{2}} C^{\frac{1}{2}-0}(\overline{\Omega}) \cap C^{1-0}(\Omega), \ \text{when } a = \tfrac{1}{2}. \end{aligned} \tag{2-12}$$

For $f \in C^t(\overline{\Omega})$, $t > 0$, the solution satisfies

$$u \in \begin{cases} e^+ d(x)^a C^{a+t}(\overline{\Omega}) \cap C^{2a+t}(\Omega) & \text{when } a + t \text{ and } 2a + t \notin \mathbb{N}, \\ \big(e^+ d(x)^a C^{a+t-0}(\overline{\Omega}) + \dot{C}^{2a+t-0}(\overline{\Omega})\big) \cap C^{2a+t}(\Omega) & \text{when } a + t \in \mathbb{N}, \\ \big(e^+ d(x)^a C^{a+t}(\overline{\Omega}) + \dot{C}^{2a+t-0}(\overline{\Omega})\big) \cap C^{2a+t-0}(\Omega) & \text{when } 2a + t \in \mathbb{N}. \end{cases} \tag{2-13}$$

Also, the mappings in (2-6) *are bijections for $s > a - 1/p'$.*

Proof. The first two statements are a special case of Theorem 3.2 below (see Example 3.3), except that we have replaced the Fredholm property with bijectivity. According to [Ros-Oton and Serra 2014b, Proposition 1.1] a weak solution (a solution in $\dot{H}^a_2(\overline{\Omega})$) of the problem (2-1) with $f \in L_\infty(\Omega)$ satisfies $\|u\|_{C^a} \leq C\|f\|_{L_\infty}$; in particular, it is unique. For $f \in \overline{H}_2^{-a}(\Omega)$, the Fredholm property of $r^+ P_a$ from $H^{a(a)}_2(\overline{\Omega}) = \dot{H}^a_2(\overline{\Omega})$ to $\overline{H}_2^{-a}(\Omega)$ is covered by [Grubb 2015a, Theorem 7.1] with $s = a$, $p = 2$. Moreover, the kernel $\mathcal{N}$ is in $\mathscr{E}_a(\overline{\Omega})$ by Theorem 3.5 below. If the kernel were nonzero, there would exist nontrivial null-solutions $u \in \mathscr{E}_a(\overline{\Omega})$, contradicting the uniqueness for $f \in L_\infty(\Omega)$ mentioned above. Thus $\mathcal{N} = 0$. Then the kernel of the Dirichlet realization $P_{a,D}$ in $L_2(\Omega)$ recalled above is likewise 0, and, since it is a selfadjoint operator with compact resolvent, it must be bijective. So the cokernel in $L_2(\Omega)$ is likewise 0. This shows the bijectivity of (2-6) in the case $s = 2a$, $p = 2$. In view of Theorem 3.5 below, this bijectivity carries over to all the other versions, including (2-6) for general $s > a - 1/p'$, and the mapping (2-11) in C^s_*-spaces for $s > a - 1$.

For (2-12) we use Theorem 3.4 (as recalled in (2-10)), noting that $\overline{C}^a_*(\Omega) = C^a(\overline{\Omega})$, that $\dot{C}^{2a}_*(\overline{\Omega}) = \dot{C}^{2a}(\overline{\Omega}) \subset d(x)^a C^a(\overline{\Omega})$ when $a \neq \frac{1}{2}$, and that $u \in C^{2a}(\Omega)$ by interior regularity when $a \neq \frac{1}{2}$, with slightly weaker statements when $a = \frac{1}{2}$. The rest of the statements follow similarly by use of (2-10) with $\mu = a$ and the various information on the relation between the C^s_*-spaces and standard Hölder spaces. □

Ros-Oton and Serra [2014b] showed, under weaker smoothness hypotheses on Ω, the inclusion $u \in d^a C^\alpha(\overline{\Omega})$ for any α with $0 < \alpha < \min\{a, 1-a\}$, and improved it in [Ros-Oton and Serra 2014a] to $\alpha = a - \varepsilon$. They observe that $\alpha > a$ cannot be obtained, so $\alpha = a$, which we obtain in (2-12), is optimal.

We also have as shown in [Grubb 2015a, Theorem 4.4] that for functions u supported in $\overline{\Omega}$ (see the first inclusion in (2-7)),

$$r^+ P_a u \in C^\infty(\overline{\Omega}) \iff u \in \mathscr{E}_a(\overline{\Omega}) \equiv \big\{u = e^+ d(x)^a v(x) \mid v \in C^\infty(\overline{\Omega})\big\}. \tag{2-14}$$

It is worth emphasizing that the functions in $\mathscr{E}_a$ have a nontrivially singular behavior at Σ when $a \notin \mathbb{N}_0$; $e^+C^\infty(\overline{\Omega})$ and $\mathscr{E}_a(\overline{\Omega})$ are very different spaces. The appearance of a factor d^{μ_0}, where μ_0 is the factorization index, was observed in C^∞-situations also in [Eskin 1981, p. 311] and in [Chkadua and Duduchava 2001, Theorem 2.1].

The solution operator is denoted by R; its form as a composition of pseudodifferential factors was given in [Grubb 2015a].

There is another point of view on the Dirichlet problem for P_a that we shall also discuss. In a number of papers (see, e.g., [Hoh and Jacob 1996; Felsinger et al. 2014] and their references), the Dirichlet problem for P_a (and other related operators) is formulated as

$$\begin{cases} P_a U = f & \text{in } \Omega, \\ U = g & \text{on } \complement\Omega. \end{cases} \tag{2-15}$$

Although the main aim is to determine U on Ω, the prescription of the values of U on $\complement\Omega$ is explained as necessitated by the nonlocality of P_a. As observed explicitly in [Hoh and Jacob 1996], the transmission property of [Boutet de Monvel 1971] is not satisfied; hence that theory of boundary problems for pseudodifferential operators is of no help. But now that we have the μ-transmission calculus, it is worth investigating what the methods can give.

The case $g = 0$ corresponds to the formulation (2-1). But also, in general, (2-15) can be reduced to (2-1) when the spaces are suitably chosen. For (2-15), let f be given in $\overline{H}_p^{s-2a}(\Omega)$ (with $s > a - 1/p'$), and let g be given in $\overline{H}_p^s(\complement\overline{\Omega})$; then we search for U in a Sobolev space over $\mathbb{R}^n$.

Let $G = \ell g$ be an extension of g to $H_p^s(\mathbb{R}^n)$. Then $u = U - G$ must satisfy

$$\begin{cases} r^+ P_a u = f - r^+ P_a G & \text{in } \Omega, \\ \operatorname{supp} u \subset \overline{\Omega}. \end{cases} \tag{2-16}$$

Here $P_a G \in H_{p,\mathrm{loc}}^{s-2a}(\mathbb{R}^n)$, so $f - r^+ P_a G \in \overline{H}_p^{s-2a}(\Omega)$.

According to our analysis of (2-1), there is a unique solution $u = R(f - r^+ P_a G) \in H_p^{a(s)}(\overline{\Omega})$ of (2-16). Then (2-15) has the solution $U = u + G \in H_p^{a(s)}(\overline{\Omega}) + H_p^s(\mathbb{R}^n)$. Moreover, there is at most one solution to (2-15) in this space, for if $U_1 = u_1 + G_1$ and $U_2 = u_2 + G_2$ are two solutions, then $v = u_1 - u_2 + G_1 - G_2$ is supported in $\overline{\Omega}$, hence lies in $H_p^{a(s)}(\overline{\Omega}) + \dot{H}_p^s(\overline{\Omega}) = H_p^{a(s)}(\overline{\Omega})$ and satisfies (2-1) with $f = 0$; hence it must be 0.

This reduction allows a study of higher regularity of the solutions. The treatment in [Felsinger et al. 2014] seems primarily directed towards the regularity involved in variational formulations ($p = 2$, $s = a$)

where Vishik and Eskin's results would be applicable; moreover, [Felsinger et al. 2014] allows a less smooth boundary.

We have shown:

Theorem 2.3. *Let $s > a - 1/p'$, and let $f \in \overline{H}_p^{s-2a}(\Omega)$ and $g \in \overline{H}_p^s(\complement\overline{\Omega})$ be given. Then the problem* (2-15) *has the unique solution $U = u + G \in H_p^{a(s)}(\overline{\Omega}) + H_p^s(\mathbb{R}^n)$, where $G \in H_p^s(\mathbb{R}^n)$ is an extension of g and*

$$u = R(f - r^+ P_a G) \in H_p^{a(s)}(\overline{\Omega}); \tag{2-17}$$

here, R is the solution operator for (2-1).

Observe in particular that the solution is independent of the choice of an extension operator $\ell : g \mapsto G$.

There is an immediate corollary for solutions in Hölder spaces (as in [Grubb 2015a, Section 7]):

Corollary 2.4. *Let $p > n/a$. For $f \in L_p(\Omega)$, $g \in C^{2a+0}(\complement\Omega) \cap \overline{H}_p^{2a}(\complement\overline{\Omega})$, the solution of* (2-15) *according to* Theorem 2.3 *satisfies*

$$U \in e^+ d^a C^{a-n/p}(\overline{\Omega}) + C^{2a+0}(\mathbb{R}^n) \cap H_p^{2a}(\mathbb{R}^n), \tag{2-18}$$

if $2a - n/p \neq 1$. If $2a - n/p$ equals 1, we need to add the space $\dot{C}^{1-0}(\overline{\Omega})$.

Proof. The intersection with $\overline{H}_p^{2a}(\complement\overline{\Omega})$ serves as a bound at ∞. We extend g to a function $G \in C^{2a+0}(\mathbb{R}^n)$; then $G \in C^{2a+0}(\mathbb{R}^n) \cap H_p^{2a}(\mathbb{R}^n)$ (since $C^{t+0} \subset H_p^t$ over bounded sets). Theorem 2.3 now gives the existence of a solution $U = u + G$, where $u \in H_p^{a(2a)}(\overline{\Omega})$. By [Grubb 2015a, Corollary 5.5] (see (2-9) above), this is contained in $d^a C^{a-n/p}(\overline{\Omega})$ when $2a - n/p \neq 1$ ($a - 1/p$ and $a - n/p$ are already noninteger). If $2a - p/n = 1$, then we have to add the space $\dot{C}^{1-0}(\overline{\Omega})$, due to the embedding $\dot{H}_p^{1+n/p}(\overline{\Omega}) \subset \dot{C}^{1-0}(\overline{\Omega})$. □

Results for problems with $f \in L_\infty(\Omega)$ or Hölder spaces were obtained in [Grubb 2015a] by letting $p \to \infty$; here we shall obtain sharper results by applying the general method to the C_*^s-scale. Repeating the proof of Theorem 2.3 in this scale, we find:

Theorem 2.5. *Let $s > a - 1$, and let $f \in \overline{C}_*^{s-2a}(\Omega)$ and $g \in \overline{C}_*^s(\complement\overline{\Omega})$ be given. Then the problem* (2-15) *has the unique solution $U = u + G \in C_*^{a(s)}(\overline{\Omega}) + C_*^s(\mathbb{R}^n)$, where $G \in C_*^s(\mathbb{R}^n)$ is an extension of g and*

$$u = R(f - r^+ P_a G) \in C_*^{a(s)}(\overline{\Omega}); \tag{2-19}$$

here, R is the solution operator for (2-1).

Let us spell this out in more detail for $s = 2a$ and $s = 2a + t$ in terms of ordinary Hölder spaces. In Corollary 2.6(1), we take g to be compactly supported in $\complement\Omega$; in (2) and (3), a very general term supported away from $\overline{\Omega}$ is added (it can in particular lie in C_*^{2a+t}). Recall from (1-4) that $L_\infty \subset C_*^0$.

Corollary 2.6. (1) *For $f \in L_\infty(\Omega)$, $g \in C_{\text{comp}}^{2a}(\complement\Omega)$, the solution of* (2-15) *according to* Theorem 2.5 *satisfies*

$$U \in e^+ d^a C^a(\overline{\Omega}) \cap C^{2a}(\Omega) + C_{\text{comp}}^{2a}(\mathbb{R}^n), \tag{2-20}$$

with C^{2a} replaced by C^{1-0} if $a = \frac{1}{2}$, and the same for C_{comp}^{2a}.

(2) *Let X be any of the function spaces $F^\sigma_{p,q}(\mathbb{R}^n)$ or $B^\sigma_{p,q}(\mathbb{R}^n)$, and denote by $X_{\rm ext}$ the subset of elements with support disjoint from $\overline{\Omega}$. For $f \in L_\infty(\Omega)$, $g \in C^{2a}_{\rm comp}(\complement\Omega) + X_{\rm ext}$, there exists a solution U of* (2-15) *satisfying*

$$U \in e^+ d^a C^a(\overline{\Omega}) \cap C^{2a}(\Omega) + C^{2a}_{\rm comp}(\mathbb{R}^n) + X_{\rm ext}, \tag{2-21}$$

with C^{2a} replaced by C^{1-0} if $a = \frac{1}{2}$, and the same for $C^{2a}_{\rm comp}$.

(3) *For $f \in C^t(\overline{\Omega})$, $g \in C^{2a+t}_{\rm comp}(\complement\Omega) + X_{\rm ext}$, $t > 0$, the solution according to* (2) *satisfies*

$$U \in e^+ d^a C^{a+t}(\overline{\Omega}) \cap C^{2a+t}(\Omega) + C^{2a+t}_{\rm comp}(\mathbb{R}^n) + X_{\rm ext}, \tag{2-22}$$

with C^{a+t}, C^{2a+t} and $C^{2a+t}_{\rm comp}$ replaced by C^{a+t-0} C^{2a+t-0} and $C^{2a+t-0}_{\rm comp}$, respectively, when the exponents hit an integer.

Proof. (1) That $g \in C^{2a}_{\rm comp}(\complement\Omega)$ means that g is in C^{2a} over the closed set $\complement\Omega$ and vanishes outside a large ball; it extends to a function $G \in C^{2a}_{\rm comp}(\mathbb{R}^n)$. Since $C^{2a}_{\rm comp}(\mathbb{R}^n) \subset C^{2a}_{{\rm comp},*}(\mathbb{R}^n)$, the construction in Theorem 2.5 gives a solution $U = u + G$, where u is as in (2-12).

(2) The function spaces are as described, for example, in [Johnsen 1996], with $\sigma \in \mathbb{R}$, $0 < p, q \le \infty$ ($p < \infty$ in the F-case), and ψdo's are well-defined in these spaces. We write $g = g_1 + g_2$, where $g_1 \in C^{2a}_{\rm comp}(\complement\Omega)$ and $g_2 \in X_{\rm ext}$. The problem (2-15) with g replaced by g_1 has a solution $u_1 + G_1$ as in (1). For the problem (2-15) with f replaced by 0 and g replaced by g_2 we take $G_2 = g_2$. Then $P_a G_2$ is C^∞ on a neighborhood of $\overline{\Omega}$ (by the pseudolocal property of pseudodifferential operators, see, e.g., [Grubb 2009, p. 177]), so the reduced problem has a solution $u_2 \in \mathscr{E}_a(\overline{\Omega})$, and the given problem then has the solution $u_2 + g_2$.

The sum of the solutions $u_1 + G_1 + u_2 + g_2$ solves (2-15) and lies in the asserted space.

(3) This is shown in a similar way, using (2-13). □

Remark 2.7. Note that according to this corollary, the effect on the solution over $\overline{\Omega}$ of an exterior contribution to g supported at a distance from $\overline{\Omega}$ is only a term in $\mathscr{E}_a(\overline{\Omega})$.

2B. ***A nonhomogeneous Dirichlet problem.*** For the nonhomogeneous Dirichlet problem (2-2), the crucial observation that leads to its solvability is that we can identify $\mathscr{E}_{a-1}(\overline{\Omega})/\mathscr{E}_a(\overline{\Omega})$ with $C^\infty(\Sigma)$ by use of the mapping

$$\gamma_{a-1,0} : u \mapsto \Gamma(a)(d(x)^{1-a}u)|_\Sigma \equiv \Gamma(a)\gamma_0(d^{1-a}u). \tag{2-23}$$

(The gamma-function is included for consistency in calculations of Fourier transformations and Taylor expansions.) Namely, using normal and tangential coordinates $x = y' + y_n\vec{n}(y')$ on a tubular neighborhood $U_\delta = \{y' + y_n\vec{n}(y') \mid y' \in \Sigma, |y_n| < \delta\}$ of Σ (where $\vec{n}(y')$ denotes the interior normal at y'), we have for $v \in C^\infty(\overline{\Omega})$ that

$$v(x) = v(y' + y_n\vec{n}) = v_0(y') + y_n w(x) \quad \text{on } U_\delta \cap \overline{\Omega},$$

where $v_0 \in C^\infty(\Sigma)$ is the restriction of v to Σ (also denoted $\gamma_0 v$), and w is C^∞ on $U_\delta \cap \overline{\Omega}$. Now, when $u \in \mathscr{E}_{a-1}(\overline{\Omega})$ is written as $u = e^+\Gamma(a)^{-1}d(x)^{a-1}v$ with $v \in C^\infty(\overline{\Omega})$ and $d(x)$ taken as y_n on U_δ, then

$$u(x) = \Gamma(a)^{-1}d(x)^{a-1}v_0(y') + \Gamma(a)^{-1}d(x)^a w(x) \quad \text{on } U_\delta \cap \overline{\Omega}, \tag{2-24}$$

where $\Gamma(a)^{-1}d(x)^a w$ is a function in $\mathscr{E}_a(\overline{\Omega})$. Here, v_0 is determined uniquely from v and hence $\gamma_{a-1,0}u$ is determined uniquely from u, and the null-space of the mapping $u \mapsto \gamma_{a-1,0}u$ is $\mathscr{E}_a(\overline{\Omega})$. See also Section 5 of [Grubb 2015a]; there it is moreover shown that the mapping

$$\gamma_{a-1,0} : \mathscr{E}_{a-1}(\overline{\Omega}) \to C^\infty(\Sigma), \quad \text{with null-space } \mathscr{E}_a(\overline{\Omega}),$$

extends to a continuous surjective mapping

$$\gamma_{a-1,0} : H_p^{(a-1)(s)}(\overline{\Omega}) \to B_p^{s-a+1/p'}(\Sigma), \quad \text{with null-space } H_p^{a(s)}(\overline{\Omega}), \quad \text{for } s > a - 1/p'. \tag{2-25}$$

Now since we have the bijectivity of r^+P_a in (2-6), we can simply adjoin the mapping (2-25) and conclude the bijectivity of

$$\begin{pmatrix} r^+P_a \\ \gamma_{a-1,0} \end{pmatrix} : H_p^{(a-1)(s)}(\overline{\Omega}) \xrightarrow{\sim} \overline{H}_p^{s-2a}(\Omega) \times B_p^{s-a+1/p'}(\Sigma). \tag{2-26}$$

This gives the unique solvability of the problem (2-2) in these spaces. There is an inverse

$$\begin{pmatrix} R & K \end{pmatrix} = \begin{pmatrix} r^+P_a \\ \gamma_{a-1,0} \end{pmatrix}^{-1},$$

where R is the inverse of (2-6) as introduced above and K is a mapping going from Σ to $\overline{\Omega}$. (Further details in [Grubb 2015a, Section 6].)

In C_*^s-spaces, we likewise have an extension of the mapping $\gamma_{a-1,0}$:

$$\gamma_{a-1,0} : C_*^{(a-1)(s)}(\overline{\Omega}) \to C_*^{s-a+1}(\Sigma), \quad \text{with null-space } C_*^{a(s)}(\overline{\Omega}), \text{ for } s > a - 1. \tag{2-27}$$

Then the result is as follows (as a special case of Theorem 3.2 below), with conclusions in Hölder spaces:

Theorem 2.8. *Let $s > a - 1$. The mapping $\{r^+P_a, \gamma_{a-1,0}\}$ defines a bijection*

$$\{r^+P_a, \gamma_{a-1,0}\} : C_*^{(a-1)(s)}(\overline{\Omega}) \to \overline{C}_*^{s-2a}(\Omega) \times C_*^{s-a+1}(\Sigma). \tag{2-28}$$

In particular, for any $f \in L_\infty(\Omega)$, $\varphi \in C^{a+1}(\Sigma)$, there exists a unique solution u of (2-2) *in $C_*^{(a-1)(2a)}(\overline{\Omega})$; it satisfies*

$$u \in \begin{cases} e^+d(x)^{a-1}C^{a+1}(\overline{\Omega}) + \dot{C}^{2a}(\overline{\Omega}) & \text{when } a \neq \frac{1}{2}, \\ e^+d(x)^{-\frac{1}{2}}C^{\frac{3}{2}}(\overline{\Omega}) + \dot{C}^{1-0}(\overline{\Omega}) & \text{when } a = \frac{1}{2}. \end{cases} \tag{2-29}$$

For $f \in C^t(\overline{\Omega})$, $\varphi \in C^{a+1+t}(\Sigma)$, $t > 0$, the solution satisfies

$$u \in \begin{cases} e^+d(x)^{a-1}C^{a+1+t}(\overline{\Omega}) + \dot{C}^{2a+t}(\overline{\Omega}) & \text{when } a + t \text{ and } 2a + t \notin \mathbb{N}, \\ e^+d(x)^{a-1}C^{a+1+t-0}(\overline{\Omega}) + \dot{C}^{2a+t-0}(\overline{\Omega}) & \text{when } a + t \in \mathbb{N}, \\ e^+d(x)^{a-1}C^{a+1+t}(\overline{\Omega}) + \dot{C}^{2a+t-0}(\overline{\Omega}) & \text{when } 2a + t \in \mathbb{N}. \end{cases} \tag{2-30}$$

Proof. The bijectivity holds in view of the bijectivity in Theorem 2.2, and (2-27). The implications (2-29) and (2-30) follow from (2-10) with $\mu = a-1$, together with the embedding properties recalled in Section 1. Note that since $a+1 > 2a$, there is no need to mention an intersection with $C^{2a(+t)}(\Omega)$. □

This gives a sharpening of Theorem 7.4 in [Grubb 2015a]. We moreover recall that as shown in [Grubb 2015a, Theorem 7.1], for functions $u \in H_p^{(a-1)(s)}(\overline{\Omega})$ for some s, p with $s > a - 1/p'$,

$$f \in C^\infty(\overline{\Omega}),\ \varphi \in C^\infty(\Sigma) \iff u \in \mathscr{E}_{a-1}(\overline{\Omega}). \tag{2-31}$$

Also for the nonhomogeneous Dirichlet problem, there exist formulations where the support condition on u is replaced by a prescription of its value on $\complement\Omega$. Abatangelo [2013] considers problems of the type

$$\begin{cases} r^+P_aU = f & \text{on } \Omega, \\ U = g & \text{on } \complement\Omega, \\ \gamma_{a-1,0}U = \varphi & \text{on } \Sigma. \end{cases} \tag{2-32}$$

(The boundary condition in [Abatangelo 2013] takes the form of the third line when Ω is a ball, but is described in a more general way for other domains.)

For (2-32), let f, g, φ be given, with

$$\{f, g, \varphi\} \in \overline{H}_p^{s-2a}(\Omega) \times \overline{H}_p^s\big(\complement\overline{\Omega}\big) \times B_p^{s-a+1/p'}(\Sigma), \quad \text{with } s > a - 1/p'. \tag{2-33}$$

Then we search for a solution U in a Sobolev space over $\mathbb{R}^n$ that allows definition of $\gamma_{a-1,0}U$.

We want to take as G an extension of g to $H_p^s(\mathbb{R}^n)$. If $s > n/p$, such that $H_p^s(\mathbb{R}^n) \subset C^0(\mathbb{R}^n)$, we have that $\gamma_{a-1,0} : G \mapsto \Gamma(a)\gamma_0(d(x)^{1-a}G)$ is well-defined and gives 0 for $G \in H_p^s(\mathbb{R}^n)$ (since $a < 1$). If $s < 1/p$, we can take G as the extension by 0 on Ω $\big($since $\overline{H}_p^s\big(\complement\overline{\Omega}\big)$ is identified with $\dot{H}_p^s\big(\complement\Omega\big)$ when $-1/p' < s < 1/p\big)$. If $1/p \le s \le n/p$, we can also use the extension by 0 and note that the boundary value from Ω is zero, but G is only in $H_p^{1/p-0}(\mathbb{R}^n)$. Now $U_1 = U - G$ must satisfy

$$\begin{cases} r^+P_aU_1 = f - r^+P_aG & \text{in } \Omega, \\ \operatorname{supp} U_1 \subset \overline{\Omega}, \\ \gamma_{a-1,0}U_1 = \varphi. \end{cases} \tag{2-34}$$

We continue the analysis for $s \notin [1/p, n/p]$; when $s > 0$, this can be achieved by taking p sufficiently large.

Since $P_aG \in H_{p,\mathrm{loc}}^{s-2a}(\mathbb{R}^n)$, $f - r^+P_aG \in \overline{H}_p^{s-2a}(\Omega)$. In this way, we have reduced the problem to the form (2-3), where we have the solution operator $\begin{pmatrix} R & K \end{pmatrix}$, see (2-26) and the following. This implies that (2-32) has the solution

$$U = R(f - r^+P_aG) + K\varphi + G \in H_p^{a(s)}(\overline{\Omega}) + H_p^{(a-1)(s)}(\overline{\Omega}) + H_p^s(\mathbb{R}^n). \tag{2-35}$$

It is unique, since zero data give a zero solution (as we know from (2-15) in the case $\varphi = 0$). Recall that $H_p^{a(s)}(\overline{\Omega}) \subset H_p^{(a-1)(s)}(\overline{\Omega})$.

This shows the first part of the following theorem:

Theorem 2.9. (1) *Let $s > a - 1/p'$ (if $s > 0$ assume moreover that $s \notin [1/p, n/p]$), and let f, g, φ be given as in* (2-33). *Let $G \in H_p^s(\mathbb{R}^n)$ be an extension of g (by zero if $s < 1/p$).*

The problem (2-32) *has the unique solution* (2-35) *in $H_p^{(a-1)(s)}(\overline{\Omega}) + H_p^s(\mathbb{R}^n)$.*

(2) *Let $s > a - 1$, $s \neq 0$, and let f, g, φ be given, with*

$$\{f, g, \varphi\} \in \overline{C}_*^{s-2a}(\Omega) \times \overline{C}_*^s\big(\complement\overline{\Omega}\big) \times C_*^{s-a+1}(\Sigma). \tag{2-36}$$

Let $G \in C_*^s(\mathbb{R}^n)$ be an extension of g (by zero if $s < 0$).

The problem (2-32) *has the unique solution*

$$U = R(f - r^+ P_a G) + K\varphi + G \in C_*^{(a-1)(s)}(\overline{\Omega}) + C_*^s(\mathbb{R}^n). \tag{2-37}$$

Proof. (1) was shown above, and (2) is shown in an analogous way:

For $s > 0$, the extension G has boundary value $\gamma_{a-1,0}G = \Gamma(a)\gamma_0(d^{1-a}G) = 0$ since G is continuous and $1 - a > 0$, and for $s < 0$ the boundary value from Ω is 0, since G is extended by zero (using the identification of $\overline{C}_*^s\big(\complement\overline{\Omega}\big)$ with $\dot{C}_*^s\big(\complement\Omega\big)$ when $-1 < s < 0$). We then apply Theorem 2.8 to $u = U - G$. $\square$

This reduction allows a study of higher regularity of the solutions. The treatment in [Abatangelo 2013] seems primarily directed towards solutions for not very smooth data. The boundary of Ω is only assumed $C^{1,1}$ there.

Remark 2.10. When $s > a + n/p$, we note that since $H_p^{a(s)}(\overline{\Omega}) \subset e^+ d(x)^a C^0(\overline{\Omega}) \subset C^0(\mathbb{R}^n)$ (see (2-9) or [Grubb 2015a, Corollary 5.5]), the solution (2-35) is the sum of a continuous function and a term $K\varphi \in H_p^{(a-1)(s)}(\overline{\Omega})$ that stems solely from the boundary value φ. To further describe $K\varphi$, consider a localized situation, where Ω is replaced by $\mathbb{R}^n_+$, $d(x)$ is replaced by x_n, and P_a is carried over to a similar operator P (of type and factorization index a). As shown in the proof of [Grubb 2015a, Theorem 6.5], the solution $K\varphi$ (in a parametrix sense) of

$$r^+ Pu = 0 \text{ in } \mathbb{R}^n_+, \quad \gamma_{a-1,0}u = \varphi \text{ at } x_n = 0,$$

is of the form $K\varphi = z + w$, where

$$z = K_{a-1,0}\varphi = \Xi_+^{1-a} e^+ K_0 \varphi = e^+ c_{a-1} x_n^{a-1} K_0 \varphi, \qquad w = -Rr^+ Pz \in H^{a(s)}(\overline{\mathbb{R}}^n_+) \subset C^0(\mathbb{R}^n);$$

here K_0 is the standard Poisson operator sending $\varphi \in B_p^{s-a+1/p'}(\mathbb{R}^{n-1})$ into

$$K_0\varphi = \mathcal{F}^{-1}_{\xi\to x}(\hat{\varphi}(\xi')([\xi'] + i\xi_n)^{-1}) = \mathcal{F}^{-1}_{\xi'\to x'}(\hat{\varphi}(\xi')e^{-[\xi']x_n}) \in \overline{H}_p^{s-a+1}(\mathbb{R}^n_+),$$

with $\gamma_0 K_0 \varphi = \varphi$ (see also Corollary 5.3 and the proof of Theorem 5.4 in [Grubb 2015a]). Then

$$z = e^+ c_{a-1} x_n^{a-1} K_0 \varphi \in e^+ x_n^{a-1} \overline{H}_p^{s-a+1}(\mathbb{R}^n_+) \subset e^+ x_n^{a-1} C^{s-a+1-n/p}(\overline{\mathbb{R}}^n_+),$$

with $K_0\varphi \neq 0$ at $\{x_n = 0\}$ when $\varphi \neq 0$. For higher s, the factor $K_0\varphi$ lies in higher-order Sobolev and Hölder spaces, but is always nontrivial at $\{x_n = 0\}$ when $\varphi \neq 0$.

When this is carried back to the manifold situation, we have that U is the sum of a term in $C^0(\mathbb{R}^n)$ and a term $e^+d(x)^{a-1}v$, $v\in \overline{H}_p^{s-a+1}(\Omega)$, where v is nonzero at $\partial\Omega$ when $\varphi\neq 0$. Since $a<1$, this term blows up at the boundary.

Hence the solutions are "large" at the boundary in this precise sense, consisting of a continuous function plus a term containing the factor $d(x)^{a-1}$ nontrivially. See also (2-31).

It is a theme of [Abatangelo 2013] that there exist "large" solutions of the nonhomogeneous Dirichlet problem; we here see that this is *not an exception but a rule* of the setup, provided naturally by the part of the solution mapping going from Σ to $\overline{\Omega}$.

Theorem 2.9(1) gives the following result in Hölder spaces when $f\in L_p(\Omega)=\overline{H}_p^0(\Omega)$.

Corollary 2.11. *Let $p>n/a$. For $f\in L_p(\Omega)$, $g\in C^{2a+0}\big(\complement\Omega\big)\cap \overline{H}_p^{2a}\big(\complement\overline{\Omega}\big)$ and $\varphi\in C^{a+1/p'+0}(\Sigma)$, the solution U of* (2-32) *according to* Theorem 2.8 *satisfies*

$$U\in e^+d^{a-1}C^{a+1-n/p}(\overline{\Omega})+\dot{C}^{2a-n/p}(\overline{\Omega})+C^{2a+0}(\mathbb{R}^n)\cap H_p^{2a}(\mathbb{R}^n),\tag{2-38}$$

with $C^{2a-n/p}$ replaced by C^{1-0} if $2a-n/p=1$.

Proof. Note that $2a>n/p$. We extend g as in Corollary 2.4 to a function $G\in C^{2a+0}(\mathbb{R}^n)\cap H_p^{2a}(\mathbb{R}^n)$, and note that $\varphi\in C^{a+1/p'+0}(\Sigma)\subset B_p^{a+1/p'}(\Sigma)$. Theorem 2.9(1) shows that there is a (unique) solution $U=u+K\varphi+G$ with

$$u+K\varphi\in H_p^{(a-1)(2a)}(\overline{\Omega})\subset e^+d^{a-1}C^{a+1-n/p}(\overline{\Omega})+\dot{C}^{2a-n/p}(\overline{\Omega})$$

(one may consult [Grubb 2015a, (7.15)]), with the mentioned modification if $2a-n/p$ is integer. □

For $f\in L_\infty(\Omega)$ or $C^t(\overline{\Omega})$, we get the sharpest results by applying the statement for C_*^s-spaces:

Corollary 2.12. *(1) For $f\in L_\infty(\Omega)$, $g\in C_{\rm comp}^{2a}\big(\complement\Omega\big)$ and $\varphi\in C^{a+1}(\Sigma)$, the solution of* (2-32) *satisfies*

$$U\in e^+d^{a-1}C^{a+1}(\overline{\Omega})+C_{\rm comp}^{2a}(\mathbb{R}^n),\tag{2-39}$$

with $C_{\rm comp}^{2a}$ replaced by $C_{\rm comp}^{1-0}$ if $a=\frac12$.

(2) Let X be any of the function spaces $F_{p,q}^\sigma(\mathbb{R}^n)$ or $B_{p,q}^\sigma(\mathbb{R}^n)$, and denote by $X_{\rm ext}$ the subset of elements with support disjoint from $\overline{\Omega}$. For $f\in L_\infty(\Omega)$, $g\in C_{\rm comp}^{2a}\big(\complement\Omega\big)+X_{\rm ext}$ and $\varphi\in C^{a+1}(\Sigma)$, there exists a solution of (2-32) *satisfying*

$$U\in e^+d^{a-1}C^{a+1}(\overline{\Omega})+C_{\rm comp}^{2a}(\mathbb{R}^n)+X_{\rm ext},\tag{2-40}$$

with $C_{\rm comp}^{2a}$ replaced by $C_{\rm comp}^{1-0}$ if $a=\frac12$.

(3) For $f\in C^t(\overline{\Omega})$, $g\in C_{\rm comp}^{2a+t}\big(\complement\Omega\big)+X_{\rm ext}$ and $\varphi\in C^{a+1+t}(\Sigma)$, the solution according to (2) *satisfies*

$$U\in e^+d^{a-1}C^{a+1+t}(\overline{\Omega})+C_{\rm comp}^{2a+t}(\mathbb{R}^n)+X_{\rm ext},$$

with C^{a+1+t} and $C_{\rm comp}^{2a+t}$ replaced by $C^{a+1+t-0}$ and $C_{\rm comp}^{2a+t-0}$, respectively, when the exponents hit an integer.

Proof. We apply Theorem 2.9(2) in essentially the same way as in Corollary 2.6; details can be omitted. □

2C. ***A nonhomogeneous Neumann problem.*** The Neumann boundary value defined in connection with $(-\Delta)^a$ is

$$\gamma_{a-1,1}u = \Gamma(a+1)\gamma_0\big(\partial_n(d(x)^{1-a}u)\big); \tag{2-41}$$

it is proportional to the second coefficient in the Taylor expansion of $d^{1-a}u$ in the normal variable at the boundary (like $\gamma_0 w$ when w is as in (2-24)).

We here have, by use of Theorem 4.3 below:

Theorem 2.13. *The mapping $\{r^+P_a, \gamma_{a-1,1}\}$ defines a Fredholm operator*

$$\{r^+P_a, \gamma_{a-1,1}\} : H_p^{(a-1)(s)}(\overline{\Omega}) \to \overline{H}_p^{s-2a}(\overline{\Omega}) \times B_p^{s-a-1/p}(\Sigma), \tag{2-42}$$

for $s > a + 1/p$.

Proof. The continuity of the mapping (2-42) follows from [Grubb 2015a, Theorem 5.1] with $\mu = a-1$, $M = 2$. The Fredholm property follows from Theorem 4.3 below in a special case (see (3-2)) by piecing together a parametrix from the parametrix construction in local coordinates given there. We use that the parametrix exists since P_a in local coordinates has principal symbol $|\xi|^{2a}$. □

There is a similar version in C_*^s-spaces, with consequences for Hölder estimates:

Theorem 2.14. *Let $s > a$. The mapping $\{r^+P_a, \gamma_{a-1,1}\}$ defines a Fredholm operator*

$$\{r^+P_a, \gamma_{a-1,1}\} : C_*^{(a-1)(s)}(\overline{\Omega}) \to \overline{C}_*^{s-2a}(\Omega) \times C_*^{s-a}(\Sigma). \tag{2-43}$$

In particular, for $\{f, \psi\} \in L_\infty(\Omega) \times C^a(\Sigma)$ subject to a certain finite set of linear constraints there exists a solution u of (2-3) *in $C_*^{(a-1)(2a)}(\overline{\Omega})$; it is unique modulo a finite dimensional linear subspace $\mathcal{N} \subset \mathscr{E}_{a-1}(\overline{\Omega})$ and satisfies*

$$u \in \begin{cases} e^+d(x)^{a-1}C^{a+1}(\overline{\Omega}) + \dot{C}^{2a}(\overline{\Omega}) & \text{when } a \neq \frac{1}{2}, \\ e^+d(x)^{-\frac{1}{2}}C^{\frac{3}{2}}(\overline{\Omega}) + \dot{C}^{1-0}(\overline{\Omega}) & \text{when } a = \frac{1}{2}. \end{cases} \tag{2-44}$$

For $f \in C^t(\overline{\Omega})$, $\psi \in C^{a+t}(\Sigma)$, $t > 0$, the solution satisfies

$$u \in \begin{cases} e^+d(x)^{a-1}C^{a+1+t}(\overline{\Omega}) + \dot{C}^{2a+t}(\overline{\Omega}) & \text{when } a+t \text{ and } 2a+t \notin \mathbb{N}, \\ e^+d(x)^{a-1}C^{a+1+t-0}(\overline{\Omega}) + \dot{C}^{2a+t-0}(\overline{\Omega}) & \text{when } a+t \in \mathbb{N}, \\ e^+d(x)^{a-1}C^{a+1+t}(\overline{\Omega}) + \dot{C}^{2a+t-0}(\overline{\Omega}) & \text{when } 2a+t \in \mathbb{N}. \end{cases} \tag{2-45}$$

Proof. The first statement is the analogue of Theorem 2.13, now derived from Theorem 4.3, for $p = q = \infty$. In the next, detailed statements we formulate the Fredholm property explicitly, using also Theorem 3.5 on the smoothness of the kernel. Here the inclusions (2-44) and (2-45) follow from the description (2-10) of $C_*^{(a-1)(s)}(\overline{\Omega})$ as in the proof of Theorem 2.8. □

Also in the Neumann case, one can formulate versions of the theorems with u prescribed on $\mathbb{R}^n \setminus \Omega$, and show their equivalence with the set-up for u supported in $\overline{\Omega}$; we think this is sufficiently exemplified by the treatment of the Dirichlet condition above that we can leave details to the interested reader.

3. Boundary problems in general spaces

One of the conclusions in [Grubb 2015a] of the study of the ψdo P of order $m \in \mathbb{C}$, with factorization index and type $\mu_0 \in \mathbb{C}$, was that it could be linked, by the help of the special order-reducing operators $\Lambda_\pm^{(\mu)}$, to an operator

$$Q = \Lambda_-^{(\mu_0 - m)} P \Lambda_+^{(-\mu_0)} \tag{3-1}$$

of order 0 and with factorization index and type 0, which could be treated by the help of the calculus of Boutet de Monvel on H_p^s-spaces, as accounted for in [Grubb 1990]. Results for P and its boundary value problems could then be deduced from those for Q in the case of a homogeneous boundary condition. With a natural definition of boundary operators $\gamma_{\mu,k}$, nonhomogeneous boundary conditions could also be treated. In particular, we found the structure of parametrices of r^+P, with homogeneous or nonhomogeneous Dirichlet-type conditions, as compositions of operators belonging to the Boutet de Monvel calculus with the special order-reducing operators; see Theorems 4.4, 6.1 and 6.5 of [Grubb 2015a].

The results of [Grubb 1990] have been extended to the much more general families of spaces $F_{p,q}^s$ (Triebel–Lizorkin spaces) and $B_{p,q}^s$ (Besov spaces) by Johnsen [1996]. He shows that elliptic systems on a compact manifold with a smooth boundary, belonging to the Boutet de Monvel calculus, have Fredholm solvability also in these more general spaces, with C^∞ kernels and range complements (cokernels) independent of s, p, q. Here $0 < p, q \le \infty$ is allowed for the $B_{p,q}^s$-spaces, and the same goes for the $F_{p,q}^s$-spaces, except that p is taken $< \infty$ (to avoid long explanations of exceptional cases). The parameter s is taken $> s_0$, for a suitable s_0 depending on p and the order and class of the involved operators. We refer to [Johnsen 1996] (or to Triebel's books) for detailed descriptions of the spaces, recalling just that for $1 < p < \infty$,

$$\begin{aligned} &F_{2,2}^s = B_{2,2}^s = H_2^s, && L_2\text{-Sobolev spaces,} \\ &F_{p,2}^s = H_p^s, && \text{Bessel-potential spaces,} \\ &B_{p,p}^s = B_p^s, && \text{Besov spaces.} \end{aligned} \tag{3-2}$$

Here the Bessel-potential spaces H_p^s are also called W_p^s (or $W^{s,p}$) for $s \in \mathbb{N}_0$, and the Besov spaces B_p^s are also called W_p^s (or $W^{s,p}$) for $s \in \mathbb{R}_+ \setminus \mathbb{N}$, under the common name Sobolev–Slobodetskii spaces. Recall moreover that $F_{p,p}^s = B_{p,p}^s$ for $0 < p < \infty$ (also denoted B_p^s).

We return to the general situation of $\overline{\Omega}$ smoothly embedded in a Riemannian manifold Ω_1, with $\overline{\mathbb{R}}_+^n \subset \mathbb{R}^n$ used in localizations. Hörmander's notation $\dot{F}$, $\overline{F}$ and $\dot{B}$, $\overline{B}$ will be used for the general scales, in the same way as for H_p^s; see (1-2) and the following.

In the present paper, we shall in particular be interested in the case of the scale of spaces $B_{\infty,\infty}^s = C_*^s$ (see the text around (1-3)), which gives a shortcut to sharp results on solvability in Hölder spaces.

Since we are mostly interested in results for large p, we shall assume $p \ge 1$, which simplifies the quotations from [Johnsen 1996]; namely, the condition $s > \max\{1/p-1, n/p-n\}$ simplifies to $s > 1/p-1$, since $1/p - 1 \ge n/p - n$ when $p \ge 1$. (In situations where $p < 1$ would be needed, e.g., in bootstrap regularity arguments, one can supply the presentation here with the appropriate results from [Johnsen 1996].) The usual notation $1/p' = 1 - 1/p$ is understood as 0 or 1 when $p = 1$ or ∞, respectively. We assume $p \le \infty$ in B-cases, $p < \infty$ in F-cases, and take $0 < q \le \infty$.

The scales $F^s_{p,q}$ and $B^s_{p,q}$ have analogous roles in definitions over $\overline{\Omega}$, but the trace mappings on them are slightly different: when $s > 1/p$,

$$\gamma_0 : \overline{F}^s_{p,q}(\Omega) \to B^{s-1/p}_{p,p}(\partial\Omega) \quad \text{and} \quad \gamma_0 : \overline{B}^s_{p,q}(\Omega) \to B^{s-1/p}_{p,q}(\partial\Omega), \tag{3-3}$$

continuously and surjectively. (One could also write $F^s_{p,p}$ instead of $B^s_{p,p}$; in [Johnsen 1996], both cases occur.)

To reduce repetitive formulations, we shall introduce the common notation

$$X^s_{p,q} \text{ stands for either } F^s_{p,q} \text{ or } B^s_{p,q}, \text{ as necessary,} \tag{3-4}$$

with the same choice in each place if the notation appears several times in the same calculation. Formulas involving boundary operators will be given explicitly in the two different cases resulting from (3-3).

In addition to the mapping and Fredholm properties established for Boutet de Monvel systems in [Johnsen 1996], we need the following generalizations of (1-6) (as in [Grubb 2015a, (1.11)–(1.20)]):

$$\begin{aligned}
\Xi^\mu_+ \text{ and } \Lambda^\mu_+ &: \dot{X}^s_{p,q}(\overline{\mathbb{R}}^n_+) \xrightarrow{\sim} \dot{X}^{s-\operatorname{Re}\mu}_{p,q}(\overline{\mathbb{R}}^n_+), \text{ with inverses } \Xi^{-\mu}_+ \text{ and } \Lambda^{-\mu}_+,\\
\Xi^\mu_{-,+} \text{ and } \Lambda^\mu_{-,+} &: \overline{X}^s_{p,q}(\mathbb{R}^n_+) \xrightarrow{\sim} \overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\mathbb{R}^n_+), \text{ with inverses } \Xi^{-\mu}_{-,+} \text{ and } \Lambda^{-\mu}_{-,+},\\
\Lambda^{(\mu)}_+ &: \dot{X}^s_{p,q}(\overline{\Omega}) \xrightarrow{\sim} \dot{X}^{s-\operatorname{Re}\mu}_{p,q}(\overline{\Omega}),\\
\Lambda^{(\mu)}_{-,+} &: \overline{X}^s_{p,q}(\Omega) \xrightarrow{\sim} \overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\Omega),
\end{aligned} \tag{3-5}$$

valid for all $s \in \mathbb{R}$. The cases with integer μ are covered by [Johnsen 1996] as a direct extension of the presentation in [Grubb 1990]; the cases of more general μ likewise extend, since the support-preserving properties extend.

We can then define (analogously to the definitions and observations in [Grubb 2015a, Sections 1.2, 1.3]):

Definition 3.1. Let $s > \operatorname{Re}\mu - 1/p'$.

(1) A distribution u on $\mathbb{R}^n$ is in $X^{\mu(s)}_{p,q}(\overline{\mathbb{R}}^n_+)$ if and only if $\Xi^\mu_+ u \in \dot{X}^{-1/p'+0}_{p,q}(\overline{\mathbb{R}}^n_+)$ and $r^+\Xi^\mu_+ u \in \overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\mathbb{R}^n_+)$. In fact, $r^+\Xi^\mu_+$ maps $X^{\mu(s)}_{p,q}(\overline{\mathbb{R}}^n_+)$ bijectively onto $\overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\mathbb{R}^n_+)$, with inverse $\Xi^{-\mu}_+ e^+$, and

$$X^{\mu(s)}_{p,q}(\overline{\mathbb{R}}^n_+) = \Xi^{-\mu}_+ e^+ \overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\mathbb{R}^n_+), \tag{3-6}$$

with the inherited norm. Here $\Lambda^{-\mu}_+$ can equivalently be used.

(2) A distribution u on Ω_1 is in $X^{\mu(s)}_{p,q}(\overline{\Omega})$ if and only if $\Lambda^{(\mu)}_+ u \in \dot{X}^{-1/p'+0}_{p,q}(\overline{\Omega})$ and $r^+\Lambda^{(\mu)}_+ u \in \overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\Omega)$. In fact, $r^+\Lambda^{(\mu)}_+$ maps $X^{\mu(s)}_{p,q}(\overline{\Omega})$ bijectively onto $\overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\Omega)$, with inverse $\Lambda^{(-\mu)}_+ e^+$, and

$$X^{\mu(s)}_{p,q}(\overline{\Omega}) = \Lambda^{(-\mu)}_+ e^+ \overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\Omega), \tag{3-7}$$

with the inherited norm.

The distributions in $X^{\mu(s)}_{p,q}(\overline{\mathbb{R}}^n_+)$ and $X^{\mu(s)}_{p,q}(\overline{\Omega})$ are locally in $X^s_{p,q}$ over $\mathbb{R}^n_+$ and Ω, respectively, by interior regularity.

By use of the mapping properties of the standard trace operators γ_j described in [Johnsen 1996], and use of (3-5) above, the trace operators $\varrho_{\mu,M}$ introduced in [Grubb 2015a, Section 5] extend to the general spaces

$$\varrho_{\mu,M} = \{\gamma_{\mu,0}, \gamma_{\mu,1}, \dots, \gamma_{\mu,M-1}\} : \begin{cases} F^{\mu(s)}_{p,q}(\overline{\Omega}) \to \prod_{0\le j<M} B^{s-\operatorname{Re}\mu-j-1/p}_{p,p}(\partial\Omega), \\ B^{\mu(s)}_{p,q}(\overline{\Omega}) \to \prod_{0\le j<M} B^{s-\operatorname{Re}\mu-j-1/p}_{p,q}(\partial\Omega), \end{cases} \tag{3-8}$$

for $s > \operatorname{Re}\mu + M - 1/p'$; they are surjective with kernels $F^{(\mu+M)(s)}_{p,q}(\overline{\Omega})$ and $B^{(\mu+M)(s)}_{p,q}(\overline{\Omega})$.

We can now formulate some important results from [Grubb 2015a] in these scales of spaces. Recall that when P is of type μ, it is also of type μ' for $\mu - \mu' \in \mathbb{Z}$.

Theorem 3.2. (1) *Let the ψdo P on Ω_1 be of order $m \in \mathbb{C}$ and of type $\mu \in \mathbb{C}$ relative to the boundary of the smooth compact subset $\overline{\Omega} \subset \Omega_1$. Then when $s > \operatorname{Re}\mu - 1/p'$, r^+P maps $X^{\mu(s)}_{p,q}(\overline{\Omega})$ continuously into $\overline{X}^{s-\operatorname{Re} m}_{p,q}(\Omega)$.*

(2) *Assume in addition that P is elliptic and has factorization index μ_0, where $\mu - \mu_0 \in \mathbb{Z}$. Let $s > \operatorname{Re}\mu_0 - 1/p'$. If $u \in \dot{X}^{\sigma}_{p,q}(\overline{\Omega})$ for some $\sigma > \operatorname{Re}\mu_0 - 1/p'$ and $r^+Pu \in \overline{X}^{s-\operatorname{Re} m}_{p,q}(\Omega)$, then $u \in X^{\mu_0(s)}_{p,q}(\overline{\Omega})$. The mapping r^+P defines a Fredholm operator*

$$r^+P : X^{\mu_0(s)}_{p,q}(\overline{\Omega}) \to \overline{X}^{s-\operatorname{Re} m}_{p,q}(\Omega). \tag{3-9}$$

Moreover, $\{r^+P, \gamma_{\mu_0-1,0}\}$ defines a Fredholm operator

$$\{r^+P, \gamma_{\mu_0-1,0}\} : \begin{cases} F^{(\mu_0-1)(s)}_{p,q}(\overline{\Omega}) \to \overline{F}^{s-\operatorname{Re} m}_{p,q}(\Omega) \times B^{s-\operatorname{Re}\mu_0+1-1/p}_{p,p}(\partial\Omega), \\ B^{(\mu_0-1)(s)}_{p,q}(\overline{\Omega}) \to \overline{B}^{s-\operatorname{Re} m}_{p,q}(\Omega) \times B^{s-\operatorname{Re}\mu_0+1-1/p}_{p,q}(\partial\Omega). \end{cases} \tag{3-10}$$

(3) *Let P be as in* (2)*, and let $\mu' = \mu_0 - M$ for a positive integer M. Then when $s > \operatorname{Re}\mu_0 - 1/p'$, $\{r^+P, \varrho_{\mu',M}\}$ defines a Fredholm operator*

$$\{r^+P, \varrho_{\mu',M}\} : \begin{cases} F^{\mu'(s)}_{p,q}(\overline{\Omega}) \to \overline{F}^{s-\operatorname{Re} m}_{p,q}(\Omega) \times \prod_{0\le j<M} B^{s-\operatorname{Re}\mu'-j-1/p}_{p,p}(\partial\Omega), \\ B^{\mu'(s)}_{p,q}(\overline{\Omega}) \to \overline{B}^{s-\operatorname{Re} m}_{p,q}(\Omega) \times \prod_{0\le j<M} B^{s-\operatorname{Re}\mu'-j-1/p}_{p,q}(\partial\Omega). \end{cases} \tag{3-11}$$

Proof. (1) This is the extension of [Grubb 2015a, Theorem 4.2] to the general spaces. We recall that the proof consist of a reduction of the study of r^+P to the consideration of Q_+ (with Q as in (3-1) for $\mu = \mu_0$) of type 0; this works well in the present spaces.

(2)–(3). Here, (3-9) is obtained by a generalization of [Grubb 2015a, Theorem 4.4] and its proof to the current spaces. Now (3-11) is obtained as in [Grubb 2015a, Theorem 6.1] by adjoining the mapping (3-8) (with $\mu = \mu'$) to r^+P. Here (3-10) is the special case $M = 1$, as in [Grubb 2015a, Corollary 6.2] □

The parametrices R and $\begin{pmatrix} R & K \end{pmatrix}$ described by formulas in [Grubb 2015a, Theorems 4.4, 6.5] also work in the present spaces.

Example 3.3. As an example, we have for the choice $X = B$, $p = q = \infty$, i.e., $X^s_{p,q} = B^s_{\infty,\infty} = C^s_*$, that Theorem 3.2(2) shows the following:

Let P be elliptic of order m and of type μ_0, with factorization index μ_0, and let $s > \operatorname{Re}\mu_0 - 1$. If $u \in \dot{C}^{\sigma}_*(\overline{\Omega})$ for some $\sigma > \operatorname{Re}\mu_0 - 1$ and $r^+Pu \in \overline{C}^{s-\operatorname{Re} m}_*(\Omega)$, then $u \in C^{\mu_0(s)}_*(\overline{\Omega})$. The mapping r^+P

defines a Fredholm operator

$$r^+P : C_*^{\mu_0(s)}(\overline{\Omega}) \to \overline{C}_*^{s-\operatorname{Re} m}(\Omega). \tag{3-12}$$

Moreover, $\{r^+P, \gamma_{\mu_0-1,0}\}$ defines a Fredholm operator

$$\{r^+P, \gamma_{\mu_0-1,0}\} : C_*^{(\mu_0-1)(s)}(\overline{\Omega}) \to \overline{C}_*^{s-\operatorname{Re} m}(\Omega) \times C_*^{s-\operatorname{Re}\mu_0+1}(\partial\Omega). \tag{3-13}$$

For $\operatorname{Re}\mu > -1/p'$, the spaces $X_{p,q}^{\mu(s)}(\overline{\mathbb{R}}^n_+)$ and $X_{p,q}^{\mu(s)}(\overline{\Omega})$ are further described by the following generalization of [Grubb 2015a, Theorem 5.4]:

Theorem 3.4. *One has for* $\operatorname{Re}\mu > -1$, $s > \operatorname{Re}\mu - 1/p'$, *with* $M \in \mathbb{N}$:

$$\begin{aligned} &X_{p,q}^{\mu(s)}(\overline{\mathbb{R}}^n_+) \begin{cases} = \dot{X}_{p,q}^{s}(\overline{\mathbb{R}}^n_+) & \text{if } s - \operatorname{Re}\mu \in]-1/p', 1/p[, \\ \subset \dot{X}_{p,q}^{s-0}(\overline{\mathbb{R}}^n_+) & \text{if } s - \operatorname{Re}\mu = 1/p, \end{cases} \\ &X_{p,q}^{\mu(s)}(\overline{\mathbb{R}}^n_+) \subset e^+ x_n^\mu \overline{X}_{p,q}^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+) + \begin{cases} \dot{X}_{p,q}^{s}(\overline{\mathbb{R}}^n_+) & \text{if } s - \operatorname{Re}\mu \in M +]-1/p', 1/p[, \\ \dot{X}_{p,q}^{s-0}(\overline{\mathbb{R}}^n_+) & \text{if } s - \operatorname{Re}\mu = M + 1/p. \end{cases} \end{aligned} \tag{3-14}$$

The inclusions (3-14) *also hold in the manifold situation, with* $\mathbb{R}^n_+$ *replaced by* Ω *and* x_n *replaced by* $d(x)$.

Proof. The first statement in (3-14) follows since $e^+\overline{X}_{p,q}^{t}(\mathbb{R}^n_+) = \dot{X}_{p,q}^{t}(\overline{\mathbb{R}}^n_+)$ for $-1/p' < t < 1/p$; see [Johnsen 1996, (2.51)–(2.52)].

For the second statement we use the representation of u as in [Grubb 2015a, (5.13)–(5.14)], in the same way as in the proof of Theorem 5.4 there. The crucial fact is that the Poisson operator K_0 maps $\gamma_{\mu,0}u \in B_{p,p}^{s-\operatorname{Re}\mu-1/p}(\mathbb{R}^{n-1})$ and $B_{p,q}^{s-\operatorname{Re}\mu-1/p}(\mathbb{R}^{n-1})$ into $\overline{F}_{p,q}^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+)$ and $\overline{B}_{p,q}^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+)$, respectively (by [Johnsen 1996]), defining a term

$$v_0 = e^+K_{\mu,0}\gamma_{\mu,0}u = c_\mu e^+ x_n^\mu K_0\gamma_{\mu,0}u \in e^+x_n^\mu \overline{X}_{p,q}^{s-\operatorname{Re}\mu}(\mathbb{R}^n_+),$$

with similar descriptions of terms $e^+K_{\mu,j}\gamma_{\mu,j}u$ for j up to $M-1$, such that u by subtraction of these terms gives a term in $\dot{X}_{p,q}^{s}(\overline{\mathbb{R}}^n_+)$ (with s replaced by $s-0$ if $s - \operatorname{Re}\mu - 1/p$ hits an integer). □

Moreover, it is important to observe the following invariance property of kernels and cokernels (typical in elliptic theory):

Theorem 3.5. *For the Fredholm operators considered in* Theorem 3.2, *the kernel is a finite-dimensional subspace* $\mathcal{N}$ *of* $\mathscr{E}_\mu(\overline{\Omega})$, *independent of the choice of* s, p, q *and* F *or* B.

There is a finite-dimensional range complement $\mathcal{M} \subset C^\infty(\overline{\Omega})$ *for* (3-9), *and* $\mathcal{M}_1 \subset C^\infty(\overline{\Omega}) \times C^\infty(\partial\Omega)^M$ *for* (3-10)–(3-11), *that is independent of the choice of* s, p, q, F, B.

Proof. This follows from the similar statement for operators in the Boutet de Monvel calculus in [Johnsen 1996, Section 5.1] when we apply the mappings $\Lambda_\pm^{(\mu)}$, etc., in the reduction of the homogeneous Dirichlet problem to a problem in the Boutet de Monvel calculus. □

4. More general boundary conditions

In Theorem 3.2, we obtain the Fredholm solvability of Dirichlet-type problems defined by operators

$$\{r^+P,\gamma_{\mu-1,0}\}:\begin{cases}F^{(\mu-1)(s)}_{p,q}(\overline{\Omega})\to \overline{F}^{s-\operatorname{Re}m}_{p,q}(\Omega)\times B^{s-\operatorname{Re}\mu+1/p'}_{p,p}(\partial\Omega),\\ B^{(\mu-1)(s)}_{p,q}(\overline{\Omega})\to \overline{B}^{s-\operatorname{Re}m}_{p,q}(\Omega)\times B^{s-\operatorname{Re}\mu+1/p'}_{p,q}(\partial\Omega),\end{cases}\tag{4-1}$$

for $s>\operatorname{Re}\mu-1/p'$, where P is elliptic of order m, is of type μ, and has factorization index μ (called μ_0 there). In Theorem 6.5 of [Grubb 2015a] we constructed a parametrix in local coordinates, which in the Besov–Triebel–Lizorkin scales maps as

$$\begin{pmatrix}R_D & K_D\end{pmatrix}:\begin{cases}\overline{F}^{s-\operatorname{Re}m}_{p,q}(\mathbb{R}^n_+)\times B^{s-\operatorname{Re}\mu+1/p'}_{p,p}(\mathbb{R}^{n-1})\to F^{(\mu-1)(s)}_{p,q}(\overline{\mathbb{R}}^n_+),\\ \overline{B}^{s-\operatorname{Re}m}_{p,q}(\mathbb{R}^n_+)\times B^{s-\operatorname{Re}\mu+1/p'}_{p,q}(\mathbb{R}^{n-1})\to B^{(\mu-1)(s)}_{p,q}(\overline{\mathbb{R}}^n_+),\end{cases}\tag{4-2}$$

where $R_D=\Lambda_+^{-\mu}e^+\widetilde{Q_+}\Lambda_{-,+}^{\mu-m}$ and $K_D=\Xi_+^{1-\mu}e^+K'$ or $\Lambda_+^{1-\mu}e^+K''$. Here $\widetilde{Q_+}$ is a parametrix of Q_+ (where Q is recalled in (3-1)), and K' and K'' are Poisson operators in the Boutet de Monvel calculus of order 0.

4A. ***Boundary operators of type γ_0B.*** We shall now describe a general way to let other boundary operators enter in lieu of $\gamma_{\mu-1,0}$. The point is to reduce the problem to a problem in the Boutet de Monvel calculus (with ψdo's of type 0 and integer order). We can assume that the family of auxiliary operators $\Lambda_\pm^{(\varrho)}$ is chosen such that $(\Lambda_\pm^{(\varrho)})^{-1}=\Lambda_\pm^{(-\varrho)}$.

Theorem 4.1. *Let P be elliptic of order $m\in\mathbb{C}$ on Ω_1, having type μ and factorization index μ with respect to the smooth compact subset $\overline{\Omega}$. Let B be a ψdo of order $m_0+\mu$ and of type μ, with m_0 integer. Consider the mapping*

$$\{r^+P,\gamma_0r^+B\}:\begin{cases}F^{(\mu-1)(s)}_{p,q}(\overline{\Omega})\to \overline{F}^{s-\operatorname{Re}m}_{p,q}(\Omega)\times B^{s-m_0-\operatorname{Re}\mu+1/p'}_{p,p}(\partial\Omega),\\ B^{(\mu-1)(s)}_{p,q}(\overline{\Omega})\to \overline{B}^{s-\operatorname{Re}m}_{p,q}(\Omega)\times B^{s-m_0-\operatorname{Re}\mu+1/p'}_{p,q}(\partial\Omega),\end{cases}\tag{4-3}$$

for $s>\operatorname{Re}\mu+\max\{m_0,0\}-1/p'$.

(1) *For $u\in X^{(\mu-1)(s)}_{p,q}(\overline{\mathbb{R}}^n_+)$, the problem*

$$r^+Pu=f\ \text{on}\ \Omega,\quad \gamma_0r^+Bu=\psi\ \text{on}\ \partial\Omega,\tag{4-4}$$

can be reduced to an equivalent problem

$$P'_+w=g\ \text{on}\ \Omega,\quad \gamma_0B'_+w=\psi\ \text{on}\ \partial\Omega,\tag{4-5}$$

where $w=r^+\Lambda_+^{(\mu-1)}u\in\overline{X}^{s-\operatorname{Re}\mu+1}_{p,q}(\Omega)$, $g=\Lambda_{-,+}^{(\mu-m)}f\in\overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\Omega)$, and where

$$P'=\Lambda_-^{(\mu-m)}P\Lambda_+^{(1-\mu)},\quad B'=B\Lambda_+^{(1-\mu)},\tag{4-6}$$

are ψdo's of order 1 and m_0+1, respectively, and type 0.

(2) *The problem* (4-4) *is Fredholm solvable for* $s > \operatorname{Re}\mu + \max\{m_0, 0\} - 1/p'$ *if and only if the problem* (4-5) *is Fredholm solvable, as a mapping*

$$\{P'_+, \gamma_0 B'_+\} : \begin{cases} \overline{F}^{t+1}_{p,q}(\Omega) \to \overline{F}^{t}_{p,q}(\Omega) \times B^{t-m_0+1/p'}_{p,p}(\partial\Omega), \\ \overline{B}^{t+1}_{p,q}(\Omega) \to \overline{B}^{t}_{p,q}(\Omega) \times B^{t-m_0+1/p'}_{p,q}(\partial\Omega), \end{cases} \tag{4-7}$$

for $t > \max\{m_0, 0\} - 1/p'$.

(3) *The operator in* (4-7) *belongs to the Boutet de Monvel calculus; therefore Fredholm solvability holds if and only if (in addition to the invertibility of the interior symbol) the boundary symbol operator is bijective at each* $(x', \xi') \in T^*(\partial\Omega) \setminus 0$. *This can also be formulated as the unique solvability of the model problem for* (4-4) *at each* $x' \in \partial\Omega$, $\xi' \neq 0$.

(4) In the transition between (4-4) *and* (4-5), $\begin{pmatrix} R'_B & K'_B \end{pmatrix}$ *is a parametrix for* (4-5) *if and only if*

$$\begin{pmatrix} R_B & K_B \end{pmatrix} = \begin{pmatrix} \Lambda_+^{(1-\mu)} e^+ R'_B \Lambda_{-,+}^{(\mu-m)} & \Lambda_+^{(1-\mu)} e^+ K'_B \end{pmatrix} \tag{4-8}$$

is a parametrix for (4-4).

Proof. The mapping (4-3) is well-defined, since $r^+B : X^{(\mu-1)(s)}_{p,q}(\overline{\Omega}) \to \overline{X}^{s-m_0-\operatorname{Re}\mu}_{p,q}(\Omega)$ by Theorem 3.2(1), and γ_0 acts as in (3-3).

(1) Let us go through the transition between (4-4) and (4-5), as already laid out in the formulation of the theorem.

We have from Definition 3.1 that $u \in X^{(\mu-1)(s)}_{p,q}(\overline{\Omega})$ if and only if $w = r^+\Lambda_+^{(\mu-1)}u \in \overline{X}^{s-\operatorname{Re}\mu+1}_{p,q}(\Omega)$; here $u = \Lambda_+^{(1-\mu)}e^+w$. Moreover, since $\Lambda_{-,+}^{(\varrho)} : \overline{X}^{t}_{p,q}(\Omega) \xrightarrow{\sim} \overline{X}^{t-\operatorname{Re}\varrho}_{p,q}(\Omega)$ for all ϱ and t, $f \in \overline{X}^{s-\operatorname{Re}m}_{p,q}(\Omega)$ if and only if $g = \Lambda_{-,+}^{(\mu-m)} f \in \overline{X}^{s-\operatorname{Re}\mu}_{p,q}(\Omega)$. Hence the first equation in (4-4) carries over to

$$\Lambda_{-,+}^{(\mu-m)} r^+ P \Lambda_+^{(1-\mu)} e^+ w = g.$$

Here $\Lambda_{-,+}^{(\mu-m)} r^+ P \Lambda_+^{(1-\mu)} e^+ w$ can be simplified to $r^+\Lambda_-^{(\mu-m)} P \Lambda_+^{(1-\mu)} e^+ w = P'_+ w$, as accounted for in the proof of Theorem 4.4 in [Grubb 2015a] in a similar situation. The boundary condition in (4-4) carries over to that in (4-5) since $B'_+ w = r^+ B \Lambda_+^{(1-\mu)} e^+ w = r^+ B u$.

The order and type of the operators is clear from the definitions.

(2) Since the transition takes place by use of bijections, the Fredholm property carries over between the two situations.

(3) The model problem is the problem defined from the principal symbols of the involved operators at a boundary point x', in a local coordinate system where Ω is replaced by $\mathbb{R}^n_+$ and the operator is applied only in the x_n-direction for fixed $\xi' \neq 0$. The hereby-defined operator on $\mathbb{R}_+$ is called the boundary symbol operator in the Boutet de Monvel calculus. The first statement in (3) is just a reference to facts from the Boutet de Monvel calculus. The second statement follows immediately when the transition is applied on the principal symbol level.

(4) Finally, when $w = R'_B g + K'_B \psi$, we have

$$u = \Lambda_+^{(1-\mu)} e^+ w = \Lambda_+^{(1-\mu)} e^+ (R'_B g + K'_B \psi) = \Lambda_+^{(1-\mu)} e^+ R'_B \Lambda_{-,+}^{(\mu-m)} f + \Lambda_+^{(1-\mu)} e^+ K'_B \psi,$$

showing the last statement. □

The search for a parametrix here requires the analysis of model problems in Sobolev-type spaces over $\mathbb{R}_+$. It can be an advantage to reduce this question to the boundary, where it suffices to investigate the ellipticity of a ψdo (i.e., invertibility of its principal symbol), as in classical treatments of differential and pseudodifferential problems.

Theorem 4.2. *Consider the problem* (4-3)–(4-4) *in* Theorem 4.1, *and its transformed version* (4-5).

(1) *The nonhomogeneous Dirichlet system for* P', $\{P'_+, \gamma_0\}$, *is elliptic, and has a parametrix for* $s > 1/p$:

$$\begin{pmatrix} R'_D & K'_D \end{pmatrix} : \begin{cases} \overline{F}^{s-1}_{p,q}(\Omega) \times B^{s-1/p}_{p,p}(\partial\Omega) \to \overline{F}^{s}_{p,q}(\Omega), \\ \overline{B}^{s-1}_{p,q}(\Omega) \times B^{s-1/p}_{p,q}(\partial\Omega) \to \overline{B}^{s}_{p,q}(\Omega). \end{cases} \tag{4-9}$$

(2) *Define*

$$S'_B = \gamma_0 B'_+ K'_D, \tag{4-10}$$

a ψ*do on* $\partial\Omega$ *of order* m_0. *Then* (4-3) *defines a Fredholm operator if and only if* S'_B *is elliptic. When this is so, if* $\widetilde{S}'_B$ *denotes a parametrix, then* $\{r^+P, \gamma_0 r^+ B\}$ *has the parametrix* $\begin{pmatrix} R_B & K_B \end{pmatrix}$, *where*

$$R_B = \Lambda_+^{(1-\mu)} (I - K'_D \widetilde{S}'_B \gamma_0 B'_+) R'_D \Lambda_{-,+}^{(\mu-m)}, \quad K_B = \Lambda_+^{(1-\mu)} K'_D \widetilde{S}'_B. \tag{4-11}$$

Proof. We begin by discussing the solvability of the type 0 problem (4-5) with $B' = I$. Set $Q_1 = \Lambda_-^{(\mu-m)} P \Lambda_+^{(1-\mu)} \Lambda_+^{(-1)}$; it is very similar to the operator $Q = \Lambda_-^{(\mu-m)} P \Lambda_+^{(-\mu)}$ used in [Grubb 2015a, Theorems 4.2 and 4.4] being of order 0, type 0 and having factorization index 0. Then we can write

$$P' = Q_1 \Lambda_+^{(1)}, \quad P'_+ = r^+ Q_1 \Lambda_+^{(1)} e^+ = r^+ Q_1 e^+ r^+ \Lambda_+^{(1)} e^+ = Q_{1,+} \Lambda_{+,+}^{(1)}, \tag{4-12}$$

where we used that $r^- \Lambda_+^{(1)} e^+$ is 0 on $\overline{X}^s_{p,q}(\Omega)$ for $s > 1/p$.

The operator $\Lambda_+^{(1)}$ defines an elliptic (bijective) system for $s > 1/p$,

$$\{\Lambda_{+,+}^{(1)}, \gamma_0\} : \begin{cases} \overline{F}^{s}_{p,q}(\Omega) \xrightarrow{\sim} \overline{F}^{s-1}_{p,q}(\Omega) \times B^{s-1/p}_{p,p}(\partial\Omega), \\ \overline{B}^{s}_{p,q}(\Omega) \xrightarrow{\sim} \overline{B}^{s-1}_{p,q}(\Omega) \times B^{s-1/p}_{p,q}(\partial\Omega). \end{cases} \tag{4-13}$$

This is shown in [Grubb 1990, Theorem 5.1] for $q = 2$ in the F-case, and extends to the Besov–Triebel–Lizorkin spaces by the results of [Johnsen 1996]. Composition with the operator $Q_{1,+}$ preserves this ellipticity, so $\{P'_+, \gamma_0\}$ forms an elliptic system with regards to the mapping property

$$\{P'_+, \gamma_0\} : \begin{cases} \overline{F}^{s}_{p,q}(\Omega) \to \overline{F}^{s-1}_{p,q}(\Omega) \times B^{s-1/p}_{p,p}(\partial\Omega), \\ \overline{B}^{s}_{p,q}(\Omega) \to \overline{B}^{s-1}_{p,q}(\Omega) \times B^{s-1/p}_{p,q}(\partial\Omega), \end{cases} \tag{4-14}$$

for $s > 1/p$. Hence there is a parametrix

$$\begin{pmatrix} R'_D & K'_D \end{pmatrix}$$

of this Dirichlet problem, continuous in the opposite direction of (4-14). This shows (1).

Next, we can discuss the general problem (4-5) by the help of this special problem; such a discussion is standard within the Boutet de Monvel calculus. Define S'_B by (4-10), it is a ψdo on $\partial\Omega$ of order m_0 by the rules of calculus. If it is elliptic, it has a parametrix, which we denote $\widetilde{S}'_B$.

On the principal symbol level, the discussion takes place for exact operators; here we denote principal symbols of the involved operators P', B', K'_D, etc., by p', b', k'_D, etc. To solve the model problem (at a point (x', ξ') with $\xi' \neq 0$), with $g \in L_2(\mathbb{R}_+)$, $\psi \in \mathbb{C}$,

$$p'_+(x', \xi', D_n)w(x_n) = g(x_n) \text{ on } \mathbb{R}_+, \quad \gamma_0 b'_+(x', \xi', D_n)w(x_n) = \psi \text{ at } x_n = 0, \tag{4-15}$$

let $z = w - r'_D g$; then z should satisfy

$$p'_+ z = 0, \quad \gamma_0 b'_+ z = \psi - \gamma_0 b'_+ r'_D g \equiv \zeta. \tag{4-16}$$

Assuming that z satisfies the first equation, set

$$\gamma_0 z = \varphi; \text{ then } z = k'_D \varphi,$$

as the solution of the semihomogeneous Dirichlet problem for p'_+. To adapt z to the second part of (4.16), we require that $\gamma_0 b'_+ z = \zeta$; here

$$\gamma_0 b'_+ z = \gamma_0 b'_+ k'_D \varphi = s'_B \varphi,$$

when we define s'_B by (4-10) on the principal symbol level; it is just a complex number depending on (x', ξ'). The equation

$$s'_B \varphi = \zeta \tag{4-17}$$

is uniquely solvable precisely when $s'_B \neq 0$. In that case, (4-17) is solved uniquely by $\varphi = (s'_B)^{-1}\zeta$.

With this choice of φ, $z = k'_D \varphi$ is the unique solution of (4-16), and $w = r'_D g + z$ is the unique solution of (4-15). The formula in complete detail is

$$w = r'_D g + k'_D (s'_B)^{-1} \zeta = (I - k'_D (s'_B)^{-1} \gamma_0 b'_+) r'_D g + k'_D (s'_B)^{-1} \psi. \tag{4-18}$$

Expressed for the full operators, this shows that the problem (4-5) is elliptic precisely when the ψdo S'_B is so.

For the full operators, a similar construction can be carried out in a parametrix sense, but it is perhaps simpler to test directly by compositions that the operator

$$\begin{pmatrix} R'_B & K'_B \end{pmatrix} = \begin{pmatrix} (I - K'_D \widetilde{S}'_B \gamma_0 B'_+) R'_D & K'_D \widetilde{S}'_B \end{pmatrix}, \tag{4-19}$$

defined in analogy with (4-18), is a parametrix for $\{P'_+, \gamma_0 B'_+\}$: since $R'_D P'_+ + K'_D\gamma_0 = I + \mathcal{R}$ and $\widetilde{S}'_B\gamma_0 B'_+ K_D = \widetilde{S}'_B S'_B = I + \mathcal{S}$, with operators $\mathcal{R}$ and $\mathcal{S}$ of order $-\infty$, we have

$$\begin{aligned}\begin{pmatrix} R'_B & K'_B\end{pmatrix}\begin{pmatrix} P'_+ \\ \gamma_0 B'_+\end{pmatrix} &= (I - K'_D\widetilde{S}'_B\gamma_0 B'_+)R'_D P'_+ + K'_D\widetilde{S}'_B\gamma_0 B'_+\\ &= (I - K'_D\widetilde{S}'_B\gamma_0 B'_+)(1 + \mathcal{R} - K'_D\gamma_0) + K'_D\widetilde{S}'_B\gamma_0 B'_+\\ &= I - K'_D\widetilde{S}'_B\gamma_0 B'_+ - K'_D\gamma_0 + K'_D\widetilde{S}'_B\gamma_0 B'_+ K'_D\gamma_0 + K'_D\widetilde{S}'_B\gamma_0 B'_+ + \mathcal{R}_1\\ &= I + \mathcal{R}_2,\end{aligned} \tag{4-20}$$

with operators $\mathcal{R}_1$ and $\mathcal{R}_2$ of order $-\infty$. The composition in the opposite order is similarly checked.

All this takes place in the Boutet de Monvel calculus. For our original problem we now find the parametrix as in (4-11), by the transition described in Theorem 4.1. □

The order assumption on B was made for the sake of arriving at operators to which the Boutet de Monvel calculus applies. We think that m_0 could be allowed to be noninteger, with some more effort, drawing on results from [Grubb and Hörmander 1990].

The treatment can be extended to problems with vector-valued boundary conditions $\gamma_0 r^+ B$, where we also involve $\varrho_{\mu,M}$ for $M > 1$; see (3-8).

4B. ***The Neumann boundary operator $\gamma_{\mu_0-1,1}$.*** For ease of comparison to [Grubb 2015a], we denote the μ used above by μ_0 here.

The boundary conditions with B of noninteger order $m_0 + \mu_0$ are generally nonlocal, since B is so. But there do exist local boundary conditions too. For example, the Dirichlet-type operator $\gamma_{\mu_0-1,0}$ is local; see (2-23). So are the systems (see (3-8)) $\varrho_{\mu_0-M,M} = \{\gamma_{\mu_0-M,0}, \dots, \gamma_{\mu_0-M,M-1}\}$, which also define Fredholm operators together with r^+P; see Theorem 3.2(3). Note that $\{r^+P, \varrho_{\mu_0-M,M}\}$ operates on a larger space $X^{(\mu_0-M)(s)}_{p,q}(\overline{\Omega})$ than $X^{(\mu_0-1)(s)}_{p,q}(\overline{\Omega})$ when $M > 1$.

What we shall show now is that one can impose a *higher-order local* boundary condition defined on $X^{(\mu_0-1)(s)}_{p,q}(\overline{\Omega})$ *itself*, leading to a meaningful boundary value problem with Fredholm solvability under a reasonable ellipticity condition.

Here we treat the Neumann-type condition $\gamma_{\mu_0-1,1}u = \psi$, recalling from [Grubb 2015a, (5.3)ff.] that

$$\gamma_{\mu_0-1,1}u = \Gamma(\mu_0+1)\gamma_0\big(\partial_n(d(x)^{1-\mu_0}u)\big). \tag{4-21}$$

By application of (3-8) with $M = 2$, $\mu = \mu_0 - 1$,

$$\gamma_{\mu_0-1,1} = \gamma_{\mu,M-1} : \begin{cases} F^{(\mu_0-1)(s)}_{p,q}(\overline{\Omega}) \to B^{s-\operatorname{Re}\mu_0-1/p}_{p,p}(\partial\Omega),\\ B^{(\mu_0-1)(s)}_{p,q}(\overline{\Omega}) \to B^{s-\operatorname{Re}\mu_0-1/p}_{p,q}(\partial\Omega),\end{cases} \tag{4-22}$$

is well-defined for $s > \operatorname{Re}\mu + M - 1/p' = \operatorname{Re}\mu_0 + 1/p$.

The discussion of ellipticity takes place in local coordinates, so let us now assume that we are in a localized situation where P is given on $\mathbb{R}^n$, globally estimated, elliptic of order m and of type μ_0 and with factorization index μ_0 relative to the subset $\mathbb{R}^n_+$, as in [Grubb 2015a, Theorem 6.5].

For $\mathbb{R}^n_+$, we can express $\gamma_{\mu_0-1,1}$ in terms of auxiliary operators by

$$\gamma_{\mu_0-1,1}u = \gamma_0\partial_n\Xi_+^{\mu_0-1}u - (\mu_0-1)[D']\gamma_0\Xi_+^{\mu_0-1}u; \tag{4-23}$$

see the calculations after Corollary 5.3 in [Grubb 2015a]. (In the manifold situation there is a certain freedom in choosing $d(x)$ and ∂_n, so we are tacitly assuming that a choice has been made that carries over to $d(x) = x_n$, $\partial_n = \partial/\partial x_n$ in the localization.)

There is an obstacle to applying the results of Section 4A to this, namely, that $\Xi_+^{\mu_0-1}$ is not truly a ψdo! This is a difficult fact that has been observed throughout the development of the theory. However, in connection with boundary conditions, operators like Ξ_+^{μ} work to some extent like the truly pseudodifferential operators Λ_+^{μ}. It is for this reason that we gave two versions of the operator K_D in (4-2) and the following, stemming from [Grubb 2015a, Theorem 6.5], in which Lemma 6.6 there was used.

Theorem 4.3. *Let P be given on $\mathbb{R}^n$, globally estimated, elliptic of order m and of type μ_0 and with factorization index μ_0 relative to the subset $\mathbb{R}^n_+$, and let $\begin{pmatrix} R_D & K_D \end{pmatrix}$ be a parametrix of the nonhomogeneous Dirichlet problem, as recalled in* (4-2) *and the following, with $K_D = \Xi_+^{1-\mu_0}e^+K'$ for a certain Poisson operator K' of order* 0.

Consider the Neumann-type problem

$$r^+Pu = f, \quad \gamma_{\mu_0-1,1}u = \psi, \tag{4-24}$$

where

$$\{r^+P, \gamma_{\mu_0-1,1}\} : \begin{cases} F_{p,q}^{(\mu_0-1)(s)}(\overline{\mathbb{R}}^n_+) \to \overline{F}_{p,q}^{s-\operatorname{Re} m}(\mathbb{R}^n_+) \times B_{p,p}^{s-\operatorname{Re}\mu_0-1/p}(\mathbb{R}^{n-1}), \\ B_{p,q}^{(\mu_0-1)(s)}(\overline{\mathbb{R}}^n_+) \to \overline{B}_{p,q}^{s-\operatorname{Re} m}(\mathbb{R}^n_+) \times B_{p,q}^{s-\operatorname{Re}\mu_0-1/p}(\mathbb{R}^{n-1}), \end{cases} \tag{4-25}$$

for $s > \mu_0 + 1/p$.

(1) *The operator*

$$S_N = \gamma_{\mu_0-1,1}K_D \tag{4-26}$$

equals $(\gamma_0\partial_n - (\mu_0-1)[D']\gamma_0)K'$ and is a ψdo on $\mathbb{R}^{n-1}$ of order 1.

(2) *If S_N is elliptic, then, with a parametrix of S_N denoted $\widetilde{S}_N$, there is the parametrix for $\{r^+P, \gamma_{\mu_0-1,1}\}$*

$$\begin{pmatrix} R_N & K_N \end{pmatrix} = \begin{pmatrix} (I - K_D\widetilde{S}_N\gamma_{\mu_0-1,1})R_D & K_D\widetilde{S}_N \end{pmatrix}. \tag{4-27}$$

(3) *Ellipticity holds in particular when the principal symbol of P equals $c(x)|\xi|^{2\mu_0}$, with $\operatorname{Re}\mu_0 > 0$, $c(x) \neq 0$.*

Proof. (1) By the formulas for $\gamma_{\mu_0-1,1}$ and K_D,

$$S_N = \gamma_{\mu_0-1,1}K_D = (\gamma_0\partial_n - (\mu_0-1)[D']\gamma_0)\Xi_+^{\mu_0-1}\Xi_+^{1-\mu_0}K' = (\gamma_0\partial_n - (\mu_0-1)[D']\gamma_0)K',$$

and it follows from the rules of the Boutet de Monvel calculus that this is a ψdo on $\mathbb{R}^{n-1}$ of order 1.

(2) In the elliptic case, one checks that (4-27) is a parametrix by calculations as in Theorem 4.2.

(3) In this case, the model problem for $\{r^+P, \gamma_{\mu_0-1,1}\}$ can be reduced to that for $\{r^+(1-\Delta)^{\mu_0}, \gamma_{\mu_0-1,1}\}$. For the latter, we have shown unique solvability in Theorem A.2 and Remark A.3 in the appendix. $\square$

Remark 4.4. The operator S_N is in fact the *Dirichlet-to-Neumann* operator for P, sending the Dirichlet data over into the Neumann data for solutions of $r^+Pu = 0$ in an approximate sense (modulo operators of order $-\infty$). From the calculations in the Appendix we see that its principal symbol equals $-\mu_0|\xi'|$ when P is principally equal to $(-\Delta)^{\mu_0}$, with $\operatorname{Re}\mu_0 > 0$.

4C. ***Systems, further perspectives.*** The factorization property used above will not in general hold for systems ($N \times N$-matrices) in a convenient way with smooth dependence on ξ', even if every element of the matrix has a factorization. But with the μ-transmission property we can establish an extremely useful connection to systems in the Boutet de Monvel calculus:

Proposition 4.5. *Let N be an integer ≥ 1, and let P be an $N \times N$-system, $P = (P_{jk})_{j,k=1,\dots,N}$, of classical ψdo's P_{jk} of order $m \in \mathbb{C}$ on Ω_1 and of type $\mu \in \mathbb{C}$ relative to Ω. Let $\mu_0 \in \mu + \mathbb{Z}$. Then the operator*

$$Q = \Lambda_-^{(\mu_0 - m)} P \Lambda_+^{(-\mu_0)} \tag{4-28}$$

is of order and type 0, *and hence belongs to the Boutet de Monvel calculus.*

Proof. The factors $\Lambda_-^{(\mu_0-m)}$ and $\Lambda_+^{(-\mu_0)}$ should be understood as diagonal matrices with $\Lambda_-^{(\mu_0-m)}$ and $\Lambda_+^{(-\mu_0)}$, respectively, in the diagonal. When they are composed with P, they act on each entry by defining an operator of order and type 0 by the symbol composition rules. □

This will allow for a general application of the Boutet de Monvel theory in the discussion of boundary value problems. Leaving the most general case for future works, we shall in the present paper just draw conclusions for systems where the operator (4-28) defines a system Q_+ that is in itself elliptic. Let us give a name to such cases, where the present considerations will apply without further efforts:

Definition 4.6. Let N be an integer ≥ 1, and let P be an elliptic $N \times N$-system, $P = (P_{jk})_{j,k=1,\dots,N}$, of classical ψdo's P_{jk} of order $m \in \mathbb{C}$ on Ω_1 and of type $\mu \in \mathbb{C}$ relative to Ω. Let $\mu_0 \in \mu + \mathbb{Z}$. Then P is said to be μ_0-*reducible* when the operator Q, defined in (4-28) of order and type 0, has the property that Q_+ *is elliptic in the Boutet de Monvel calculus* (without auxiliary boundary operators).

The condition in the definition means that in local coordinates at the boundary, the model operator $q_0(x', 0, \xi', D_n)_+$ is bijective in $L_2(\mathbb{R}_+)^N$. It holds for $N = 1$ for the operators with factorization index μ_0, as accounted for in the proof of [Grubb 2015a, Theorem 4.4]. Another important case is where the operator P (a scalar or a system) is strongly elliptic, as observed in [Eskin 1981, Example 17.1].

Lemma 4.7. *Let $N \geq 1$, and let P be of order $m \in \mathbb{R}_+$ on Ω_1 and of type $\mu_0 = m/2$ relative to Ω. If P is strongly elliptic, i.e., satisfies in local coordinates (with $c > 0$),*

$$\operatorname{Re}(p_0(x,\xi)v, v) \geq c|\xi|^m|v|^2 \text{ for all } \xi \in \mathbb{R}^n,\ v \in \mathbb{C}^N,$$

then P is μ_0-reducible.

Proof. Here Q equals $\Lambda_-^{(-m/2)} P \Lambda_+^{(-m/2)}$. This is strongly elliptic of order 0, because the principal symbols of $\Lambda_-^{(-m/2)}$ and $\Lambda_+^{(-m/2)}$ are conjugates and homogeneous elliptic of order $-m/2$:

$$\operatorname{Re}(q_0(x,\xi)v, v) = \operatorname{Re}\big(p_0(x,\xi)\lambda_{+,0}^{-m/2}(\xi)v,\ \lambda_{+,0}^{-m/2}(\xi)v\big) \geq c|\xi|^m|\lambda_{+,0}^{-m/2}(\xi)v|^2 \geq c'|v|^2,$$

for all $\xi \in \mathbb{R}^n$, $v \in \mathbb{C}^N$, in local coordinates. Thus for each $x' \in \partial\Omega$, $\xi' \neq 0$, the model operator $q_0(x', 0, \xi', D_n)$ on $\mathbb{R}$ satisfies

$$\operatorname{Re}(q_0 u, u) \geq C\|u\|^2_{L_2(\mathbb{R})^N} \quad \text{for } u \in L_2(\mathbb{R})^N,$$

as seen by Fourier transformation in ξ_n. In particular, the restriction of r^+q_0 to $C_0^\infty(\mathbb{R}_+)^N$ satisfies the above inequality, and the inequality extends to its closure, $r^+q_0e^+$, defined on $L_2(\mathbb{R}_+)^N$, which is therefore injective. Similar considerations hold for the adjoint, so indeed, $q_0(x', 0, \xi', D_n)_+$ is bijective in $L_2(\mathbb{R}_+)^N$. □

Theorem 4.8. *Let P be an elliptic $N \times N$ system, $P = (P_{jk})_{j,k=1,\dots,N}$, of classical ψdo's P_{jk} of order $m \in \mathbb{C}$ on Ω_1 and of type $\mu_0 \in \mathbb{C}$ relative to Ω.*

Define Q by (4-28) *and assume that P is μ_0-reducible. Then we have*:

(1) *Let $s > \operatorname{Re}\mu_0 - 1/p'$. If $u \in \dot{X}^\sigma_{p,q}(\overline{\Omega})^N$ for some $\sigma > \operatorname{Re}\mu_0 - 1/p'$ and $r^+Pu \in \overline{X}^{s-\operatorname{Re}m}_{p,q}(\Omega)^N$, then $u \in X^{\mu_0(s)}_{p,q}(\overline{\Omega})^N$. The mapping*

$$r^+P : X^{\mu_0(s)}_{p,q}(\overline{\Omega})^N \to \overline{X}^{s-\operatorname{Re}m}_{p,q}(\Omega)^N \tag{4-29}$$

is Fredholm, and has the parametrix

$$R = \Lambda_+^{(-\mu_0)} e^+ \widetilde{Q_+} \Lambda_{-,+}^{(\mu_0-m)} : \overline{X}^{s-\operatorname{Re}m}_{p,q}(\Omega)^N \to X^{\mu_0(s)}_{p,q}(\overline{\Omega})^N, \tag{4-30}$$

where $\widetilde{Q_+}$ is a parametrix of Q_+. It has the structure $\widetilde{Q}_+ + G$ with G a singular Green operator of order and class 0.

(2) *In particular, if $r^+Pu \in C^\infty(\overline{\Omega})^N$, then $u \in \mathscr{E}_{\mu_0}(\overline{\Omega})^N$, and the mapping*

$$r^+P : \mathscr{E}_{\mu_0}(\overline{\Omega})^N \to C^\infty(\overline{\Omega})^N \tag{4-31}$$

is Fredholm.

(3) *Moreover, let $\mu = \mu_0 - M$ for a positive integer M. Then when $s > \operatorname{Re}\mu_0 - 1/p'$, $\{r^+P, \varrho_{\mu,M}\}$ defines a Fredholm operator*

$$\{r^+P, \varrho_{\mu,M}\} : \begin{cases} F^{\mu(s)}_{p,q}(\overline{\Omega})^N \to \overline{F}^{s-\operatorname{Re}m}_{p,q}(\Omega)^N \times \prod_{0\leq j<M} B^{s-\operatorname{Re}\mu-j-1/p}_{p,p}(\partial\Omega)^N, \\ B^{\mu(s)}_{p,q}(\overline{\Omega})^N \to \overline{B}^{s-\operatorname{Re}m}_{p,q}(\Omega)^N \times \prod_{0\leq j<M} B^{s-\operatorname{Re}\mu-j-1/p}_{p,q}(\partial\Omega)^N. \end{cases} \tag{4-32}$$

Proof. The proof goes as in [Grubb 2015a, Theorems 4.4 and 6.1]:

(1) We replace the equation

$$r^+Pu = f \in \overline{X}^{s-\operatorname{Re}m}_{p,q}(\Omega)^N, \tag{4-33}$$

by composition on the left with $\Lambda_{-,+}^{(\mu_0-m)}$, by the equivalent problem

$$\Lambda_{-,+}^{(\mu_0-m)} r^+Pu = g, \quad \text{where } g = \Lambda_{-,+}^{(\mu_0-m)} f \in \overline{X}^{s-\operatorname{Re}\mu_0}_{p,q}(\Omega)^N, \tag{4-34}$$

using the homeomorphism properties of $\Lambda_{-,+}^{(\mu_0-m)}$, applied to vectors. Here $f=\Lambda_{-,+}^{(m-\mu_0)}g$. Moreover (see Remark 1.1 in [Grubb 2015a]),

$$\Lambda_{-,+}^{(\mu_0-m)}r^+Pu=r^+\Lambda_-^{(\mu_0-m)}Pu.$$

Next, we set $v=r^+\Lambda_+^{(\mu_0)}u$; then $u=\Lambda_+^{(-\mu_0)}e^+v$, and equation (4-33) becomes

$$Q_+v=g,\quad \text{with } g \text{ given in } \overline{X}_{p,q}^{s-\operatorname{Re}\mu_0}(\Omega), \tag{4-35}$$

where Q is defined by (4-28).

The properties of P imply that Q is elliptic of order 0 and type 0, and hence belongs to the Boutet de Monvel calculus. The rest of the argument takes place within that calculus. By our assumption, $Q_+=r^+Qe^+$ defines an elliptic boundary problem (without auxiliary trace or Poisson operators) there, and Q_+ is continuous in $\overline{X}_{p,q}^t(\Omega)$ for $t>-1/p'$. By the ellipticity, Q_+ has a parametrix $\widetilde{Q_+}$, continuous in the opposite direction, and with the mentioned structure. Since $v\in\dot{X}_{p,q}^{-1/p'+0}(\overline{\Omega})$ by hypothesis, solutions of $Q_+v=g$ with $g\in\overline{X}_{p,q}^t(\Omega)$ for some $t>-1/p'$ are in $\overline{X}_{p,q}^t(\Omega)$. Moreover,

$$Q_+:\overline{X}_{p,q}^t(\Omega)\to\overline{X}_{p,q}^t(\Omega)\quad\text{is Fredholm for all } t>-1/p'.$$

When carried back to the original functions, this shows (1).

(2) This follows by letting $s\to\infty$, using that $\bigcap_s X_{p,q}^{\mu(s)}(\overline{\Omega})^N=\mathscr{E}_\mu(\overline{\Omega})^N$.

(3) We use that the mapping $\varrho_{\mu,M}$ in (3-8) extends immediately to vector-valued functions

$$\varrho_{\mu,M}:\begin{cases}F_{p,q}^{\mu(s)}(\overline{\Omega})^N\to\prod_{0\le j<M}B_{p,p}^{s-\operatorname{Re}\mu-j-1/p}(\partial\Omega)^N,\\ B_{p,q}^{\mu(s)}(\overline{\Omega})^N\to\prod_{0\le j<M}B_{p,q}^{s-\operatorname{Re}\mu-j-1/p}(\partial\Omega)^N,\end{cases} \tag{4-36}$$

when $s>\operatorname{Re}\mu_0-1/p'$, surjective with null-space $X_{p,q}^{\mu_0(s)}(\overline{\Omega})^N$ (recall $\mu=\mu_0-M$). When we adjoin this mapping to (4-29), we obtain (4-32). □

One of the things we obtain here is that results from [Eskin 1981] (extended to L_p in [Shargorodsky 1994; Chkadua and Duduchava 2001]), on solvability for s in an interval of length 1 around $\operatorname{Re}\mu_0$, are lifted to regularity and Fredholm properties for all larger s, with exact information on the domain, also in general scales of function spaces. Moreover, our theorem is obtained via a systematic variable-coefficient calculus, whereas the results in [Eskin 1981] are derived from constant-coefficient considerations by ad hoc perturbation methods in L_2-Sobolev spaces.

Also the results on other boundary conditions in the present paper extend to suitable systems. One can moreover extend the results to operators in vector bundles (since they can be locally expressed by matrices of operators).

The Boutet de Monvel theory is not an easy theory (as the elaborate presentations [Boutet de Monvel 1971; Rempel and Schulze 1982; Grubb 1984; 1990; 1996; 2009; Schrohe 2001] in the literature show), but one could have feared that a theory for the more general μ-transmission operators and their boundary problems would be a step up in difficulty. Fortunately, as we have seen, many of the issues can be dealt with by reductions using the special operators $\Lambda_\pm^{(\mu)}$ to cases where the type 0 theory applies.

There is currently also an interest in problems with less smooth symbols. For this connection, we mention that there do exist pseudodifferential theories for such problems, also with boundary conditions; see [Abels 2005; Grubb 2014] and their references. One finds that a lack of smoothness in the x-variable narrows down the interval of parameters s where one has good solvability properties, and compositions are delicate. It is also possible to work under limitations on the number of standard estimates in ξ.

Appendix: Calculations in an explicit example

Pseudodifferential methods are a refinement of the application of the Fourier transform, making it useful even for variable-coefficient partial differential operators, and, for example, allowing generalizations to operators of noninteger order. But to explain some basic mechanisms, it may be useful to consider a simple "constant-coefficient" case, where explicit elementary calculations can be made, not requiring intricate composition rules. This is the case for $(1-\Delta)^a$ $(a>0)$ on $\mathbb{R}^n_+$, where everything can be worked out by hand in exact detail (in the spirit of the elementary [Grubb 2009, Chapter 9]). We here restrict the attention to H^s_p-spaces.

The symbol of $(1-\Delta)^a$ factors as

$$(\langle\xi'\rangle^2+\xi_n^2)^a=(\langle\xi'\rangle-i\xi_n)^a(\langle\xi'\rangle+i\xi_n)^a. \tag{A-1}$$

Now we shall use the definitions of simple order-reducing operators $\Xi^t_\pm$ and Poisson operators K_j from [Grubb 2015a], with $\langle\xi'\rangle$ instead of $[\xi']$, because they fit particularly well with the factors in (A-1). We shall often abbreviate $\langle\xi'\rangle$ to σ.

The *homogeneous Dirichlet problem*

$$r^+(1-\Delta)^a u=f,\quad \text{with } f \text{ given in } \overline{H}^{s-2a}_p(\mathbb{R}^n_+), \tag{A-2}$$

$s>a-1/p'$, has a unique solution u in $\dot H^{a-1/p'+0}_p(\overline{\mathbb{R}}^n_+)$ determined as follows:

With $\Xi^t_\pm=\mathrm{OP}\big((\langle\xi'\rangle+i\xi_n)^t\big)$, we have that $(1-\Delta)^a=\Xi^a_-\Xi^a_+$ on $\mathbb{R}^n$. Let $v=r^+\Xi^a_+u$; it is in $\overline{H}^{-1/p'+0}_p(\mathbb{R}^n_+)=\dot H^{-1/p'+0}_p(\overline{\mathbb{R}}^n_+)$, and $u=\Xi^{-a}_+e^+v$. Then (A-2) becomes

$$r^+\Xi^a_-e^+v=f. \tag{A-3}$$

Here $r^+\Xi^a_-e^+=\Xi^a_{-,+}$ is known to map $\overline{H}^t_p(\mathbb{R}^n_+)$ homeomorphically onto $\overline{H}^{t-a}_p(\mathbb{R}^n_+)$ for all $t\in\mathbb{R}$, with inverse $\Xi^{-a}_{-,+}$ (see, e.g., [Grubb 2015a, Section 1].) In particular, with f given in $\overline{H}^{s-2a}_p(\mathbb{R}^n_+)$, (A-3) has the unique solution $v=\Xi^{-a}_{-,+}f\in\overline{H}^{s-a}_p(\mathbb{R}^n_+)$. Then (A-2) has the unique solution

$$u=\Xi^{-a}_+e^+\Xi^{-a}_{-,+}f\equiv R_Df, \tag{A-4}$$

and it belongs to $H^{a(s)}_p(\overline{\mathbb{R}}^n_+)$ by the definition of that space. Thus the solution operator for (A-2) is $R_D=\Xi^{-a}_+e^+\Xi^{-a}_{-,+}$. (This is a simple variant of the proof of [Grubb 2015a, Theorem 4.4].)

Next, we go to the larger space $H^{(a-1)(s)}_p(\overline{\mathbb{R}}^n_+)$, still assuming $s>a-1/p'$, where we study the *nonhomogeneous Dirichlet problem*. By [Grubb 2015a, Theorem 5.1] with $\mu=a-1$ and $M=1$, we

have a mapping $\gamma_{a-1,0}$, acting as

$$\gamma_{a-1,0} : u \mapsto \Gamma(a)\gamma_0(x_n^{1-a}u),$$

also equal to $\gamma_0\Xi_+^{a-1}u$, and sending $H_p^{(a-1)(s)}(\overline{\mathbb{R}}^n_+)$ onto $B_p^{s-a+1-1/p}(\mathbb{R}^{n-1})$ with kernel $H_p^{a(s)}(\overline{\mathbb{R}}^n_+)$. Together with $(1-\Delta)^a$, it therefore defines a homeomorphism for $s > a - 1/p'$,

$$\{r^+(1-\Delta)^a, \gamma_{a-1,0}\} : H_p^{(a-1)(s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_p^{s-2a}(\mathbb{R}^n_+) \times B_p^{s-a+1-1/p}(\mathbb{R}^{n-1}). \tag{A-5}$$

It represents the problem

$$r^+(1-\Delta)^a u = f, \quad \gamma_{a-1,0}u = \varphi, \tag{A-6}$$

which we regard as the nonhomogeneous Dirichlet problem for $(1-\Delta)^a$. The solution operator in the case $\varphi = 0$ is clearly R_D defined above, since the kernel of $\gamma_{a-1,0}$ is $H_p^{a(s)}(\overline{\mathbb{R}}^n_+)$.

Also, the solution operator for the problem (A-6) with $f = 0$ can be found explicitly:

On the boundary symbol level we consider the problem (recall $\sigma = \langle\xi'\rangle$)

$$(\sigma - \partial_n)^a(\sigma + \partial_n)^a u(x_n) = 0 \quad \text{on } \mathbb{R}_+. \tag{A-7}$$

Since $\mathrm{OP}_n((\sigma - i\xi_n)^\mu)$ preserves support in $\overline{\mathbb{R}}_-$ for all μ, u must equivalently satisfy

$$(\sigma + \partial_n)^a u(x_n) = 0 \quad \text{on } \mathbb{R}_+. \tag{A-8}$$

This has the distribution solution

$$u(x_n) = \mathscr{F}^{-1}_{\xi_n\to x_n}(\sigma + i\xi_n)^{-a} = \Gamma(a)^{-1}x_n^{a-1}e^+r^+e^{-\sigma x_n} \tag{A-9}$$

(see, e.g., [Hörmander 1983, Example 7.1.17] or [Grubb 2015a, (2.5)]), and the derivatives $\partial_n^k u$ are likewise solutions, since

$$(\sigma + i\xi_n)^a(i\xi_n)^k(\sigma + i\xi_n)^{-a} = (i\xi_n)^k = \mathscr{F}_{x_n\to\xi_n}\delta_0^{(k)},$$

where $\delta_0^{(k)}$ is supported in $\{0\}$. The undifferentiated function matches our problem. Set

$$\tilde{k}_{a-1,0}(x_n, \xi') = \Gamma(a)^{-1}x_n^{a-1}e^+r^+e^{-\sigma x_n} = \mathscr{F}^{-1}_{\xi_n\to x_n}(\sigma + i\xi_n)^{-a}; \tag{A-10}$$

then, since $\gamma_{a-1,0}\tilde{k}_{a-1,0} = 1$, the mapping $\mathbb{C} \ni \varphi \mapsto \varphi \cdot r^+\tilde{k}_{a-1,0}$ solves the problem

$$(\sigma + \partial_n)^a u(x_n) = 0 \text{ on } \mathbb{R}_+, \quad \gamma_{a-1,0}u = \varphi. \tag{A-11}$$

Using the Fourier transform in ξ' also, we find that (A-6) with $f = 0$ has the solution

$$u(x) = K_{a-1,0}\varphi \equiv \mathscr{F}^{-1}_{\xi'\to x'}\big(\tilde{k}_{a-1,0}(x_n, \xi')\hat{\varphi}(\xi')\big). \tag{A-12}$$

It can be denoted $\mathrm{OPK}(\tilde{k}_{a-1,0})\varphi$, by a generalization of the notation from the Boutet de Monvel calculus. We moreover define $k_{a-1,0}(\xi) = \mathscr{F}_{x_n\to\xi_n}\tilde{k}_{a-1,0}(x_n, \xi') = (\sigma + i\xi_n)^{-a}$; $\tilde{k}_{a-1,0}$ and $k_{a-1,0}$ are the symbol-kernel and symbol of $K_{a-1,0}$, respectively.

Note that

$$k_{a-1,0}(\xi',\xi_n) = (\langle\xi'\rangle + i\xi_n)^{-a} = (\langle\xi'\rangle + i\xi_n)^{1-a}(\langle\xi'\rangle + i\xi_n)^{-1}, \text{ hence}$$
$$K_{a-1,0} = \Xi_+^{1-a} K_0, \tag{A-13}$$

where $K_0 = \mathrm{OPK}\big((\langle\xi'\rangle + i\xi_n)^{-1}\big)$ is *the Poisson operator for the Dirichlet problem for* $1-\Delta$,

$$K_0\varphi = \mathscr{F}^{-1}_{\xi\to x}((\langle\xi'\rangle + i\xi_n)^{-1}\hat{\varphi}(\xi'))$$

(see, e.g., [Grubb 2009, Chapter 9]). K_0 is usually considered as mapping into a space over $\mathbb{R}^n_+$, and it is well-known that $K_0 : B_p^{t-1/p}(\mathbb{R}^{n-1}) \to \overline{H}^t_p(\mathbb{R}^n_+)$ for all $t\in\mathbb{R}$. However, the above formula shows that it in fact maps into distributions on $\mathbb{R}^n$ supported in $\overline{\mathbb{R}}^n_+$, so we can, with a slight abuse of notation, identify K_0 with e^+K_0, mapping into $e^+\overline{H}^t_p(\mathbb{R}^n_+)$, and conclude that

$$K_{a-1,0} : B_p^{s-a+1-1/p}(\mathbb{R}^{n-1}) \to H_p^{(a-1)(s)}(\overline{\mathbb{R}}^n_+) \quad \text{for all } s\in\mathbb{R}. \tag{A-14}$$

We have shown:

Theorem A.1. *Let $a>0$. The nonhomogeneous Dirichlet problem* (A-6) *for $(1-\Delta)^a$ on $\mathbb{R}^n_+$ is uniquely solvable, in that the operator* (A-5) *for $s > a-1/p'$ has inverse*

$$\begin{pmatrix} r^+(1-\Delta)^a \\ \gamma_{a-1,0}\end{pmatrix}^{-1} = \begin{pmatrix} R_D & K_{a-1,0}\end{pmatrix}, \tag{A-15}$$

where R_D and $K_{a-1,0}$ are defined in (A-4) *and* (A-12).

Third, we consider the boundary problem

$$r^+(1-\Delta)^a u = f, \quad \gamma_{a-1,1}u = \psi, \tag{A-16}$$

which we shall view as a *nonhomogeneous Neumann problem* for $(1-\Delta)^a$. We here assume $s > (a-1)+2-1/p' = a+1/p$, to use the construction in [Grubb 2015a, Theorem 5.1] with $\mu = a-1$, $M=2$. Recall from [Grubb 2015a, (5.3)ff.], that $\gamma_{a-1,1}$ acts as

$$\gamma_{a-1,1} : u \mapsto \Gamma(a+1)\gamma_0(\partial_n(x_n^{1-a}u)). \tag{A-17}$$

Moreover, we can infer from the text after Corollary 5.3 in [Grubb 2015a] (with $[\xi']$ replaced by $\langle\xi'\rangle$) that

$$\gamma_{a-1,1}u = \gamma_0\partial_n\,\Xi_+^{a-1}u - (a-1)\,\langle D'\rangle\,\gamma_{a-1,0}u$$

for $u\in H_p^{(a-1)(s)}(\overline{\mathbb{R}}^n_+)$ with $s > a+1/p$. Then, for a null solution z written in the form $z = K_{a-1,0}\varphi = \Xi_+^{1-a}K_0\varphi$ (recall (A-13)), we have, since $\gamma_0\partial_n K_0 = -\langle D'\rangle$,

$$\gamma_{a-1,1}z = \gamma_0\partial_n\,\Xi_+^{a-1}z - (a-1)\,\langle D'\rangle\,\gamma_{a-1,0}z = \gamma_0\partial_n K_0\varphi - (a-1)\langle D'\rangle\varphi = -a\,\langle D'\rangle\varphi.$$

Hence in order for z to solve (A-16) with $f=0$, φ must satisfy

$$\psi = -a\,\langle D'\rangle\varphi.$$

Since $a \neq 0$, the coefficient $-a\langle D'\rangle$ is an elliptic invertible ψdo, so (A-16) with $f = 0$ is uniquely solvable with solution

$$z = K_N\psi, \text{ where } K_N = -K_{a-1,0}a^{-1}\langle D'\rangle^{-1} = -\Xi_+^{1-a}K_0a^{-1}\langle D'\rangle^{-1}. \tag{A-18}$$

To solve (A-16) with a given $f \neq 0$, and $\psi = 0$, we let $v = R_D f$ and reduce to the problem for $z = u - v$:

$$r^+(1-\Delta)^a(u-v) = 0, \quad \gamma_{a-1,1}(u-v) = -\gamma_{a-1,1}R_D f.$$

This has the unique solution

$$u - v = -K_N\gamma_{a-1,1}R_D f; \quad \text{and hence } u = R_D f - K_N\gamma_{a-1,1}R_D f.$$

Altogether, we find:

Theorem A.2. *The Neumann problem* (A-16) *for* $(1-\Delta)^a$ *on* $\mathbb{R}^n_+$ *is uniquely solvable, in that the operator*

$$\{r^+(1-\Delta)^a, \gamma_{a-1,1}\} : H_p^{(a-1)(s)}(\overline{\mathbb{R}}^n_+) \to \overline{H}_p^{s-2a}(\mathbb{R}^n_+) \times B_p^{s-a-1/p}(\mathbb{R}^{n-1}), \tag{A-19}$$

for $s > a + 1/p$ *is a homeomorphism, with inverse*

$$\begin{pmatrix} R_N & K_N \end{pmatrix} = \begin{pmatrix} (I - K_N\gamma_{a-1,1})R_D & K_N \end{pmatrix}, \tag{A-20}$$

with R_D *and* K_N *described in* (A-4) *and* (A-18).

Note that there is here a *Dirichlet-to-Neumann operator* P_{DN} sending the Dirichlet-type data over into Neumann-type data for solutions of $r^+(1-\Delta)^a u = 0$:

$$P_{DN} = -a\langle D'\rangle. \tag{A-21}$$

Remark A.3. We have here assumed a real in order to relate to the fractional powers of the Laplacian, but all the above goes through in the same way if a is replaced by a complex μ with $\operatorname{Re}\mu > 0$; then in Sobolev exponents and inequalities for s, a should be replaced by $\operatorname{Re}\mu$.

One can also let higher order boundary operators $\gamma_{a-1,j}$ enter in a similar way, defining single boundary conditions.

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