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MIRCEA PETRACHE AND TRISTAN RIVIÈRE

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GLOBAL GAUGES AND GLOBAL EXTENSIONS IN OPTIMAL SPACES

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We consider the problem of extending functions $\phi : \mathbb{S}^n \to \mathbb{S}^n$ to functions $u : B^{n+1} \to \mathbb{S}^n$ for $n = 2, 3$. We assume ϕ belongs to the critical space $W^{1,n}$ and we construct a $W^{1,(n+1,\infty)}$-controlled extension u. The Lorentz–Sobolev space $W^{1,(n+1,\infty)}$ is optimal for such controlled extension. Then we use these results to construct global controlled gauges for L^4-connections over trivial SU(2)-bundles in 4 dimensions. This result is a global version of the local Sobolev control of connections obtained by K. Uhlenbeck.

1. Introduction

The use of Hodge decomposition is by now one of the classical tools in the study of elliptic systems and is related to important breakthroughs such as the famous "div–curl"-type theorems [Coifman et al. 1993]. More recently, in [Rivière 2007], such use allowed the solution of S. Hildebrandt's [1982] conjecture. At the same time, it has helped establish important links to apparently unrelated fields of geometry, such as the study of conformally invariant geometric problems in 2 dimensions [Hélein 1996] and the study of Yang–Mills bundles and gauge theory [Uhlenbeck 1982b], with the introduction of controlled Coulomb gauges.

The study of 2-dimensional problems using controlled gauges has already given its fruits, and in connection to the discovery of H. Wente's inequality (which gave the basis for introducing the Lorentz spaces $L^{(2,\infty)}$ in geometric problems) allowed the successful use of controlled moving frames in the study of harmonic maps and prescribed mean curvature surfaces [Hélein 1996; Müller and Šverák 1995]. We come back to this in Section 2H. Techniques and function spaces related to the moving frame method also apply to the study of the Willmore functional [Rivière 2012] for immersed surfaces.

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The use of controlled gauges especially in relation to Lorentz spaces in dimensions higher than 2 is far less developed. We attempt here a first attack of this completely new area of research, and we obtain some extensions of previous results for the case of Yang–Mills fields on 4-dimensional manifolds.

1A. ***Yang–Mills theory and controlled gauges.*** Yang–Mills theory for 4-manifolds is often associated to the famous result of S. Donaldson [1983] who, using the moduli spaces of anti-selfdual connections, described new invariants of smooth manifolds.

The study of moduli spaces used by Donaldson [1983] starts from the result of K. Uhlenbeck [1982b], who proved that one can find a gauge in which the $W^{1,2}$-norm of the local coordinate expression of the connection is controlled by the L^2-norm of the curvature. Moreover the connection 1-form A can be also made to satisfy the Coulomb condition $d^*A = 0$.

It is easy to construct a Coulomb gauge in which we have just an L^2-control in terms of the curvature (see [Petrache 2013] or [Petrache and Rivière $\geq$ 2014]). This is done by first obtaining any gauge in which

$$\|A\|_{L^2} \leq C\|F\|_{L^2}$$

and then finding the smallest norm coefficients with respect to that gauge on our manifold M:

$$\min\left\{\int_M |g^{-1}dg + g^{-1}Ag|^2\,dx : g \in W^{1,2}(M, \mathrm{SU}(2))\right\}.$$

A unique minimizer will exist by convexity, and it will satisfy the Coulomb equation $d^*A = 0$.

The control of A in the higher norm $W^{1,2}$ is more difficult. A smallness hypothesis on $\|F\|_{L^2(M)}$ is required in order for the control to be achievable:

Theorem 1.1 (controlled Coulomb gauge under assumption of small energy [Uhlenbeck 1982b]). *There exists a constant $\epsilon_0 > 0$ such that if the curvature satisfies $\int_M |F|^2 \leq \epsilon_0$ then there exists a Coulomb gauge $\phi \in W^{2,2}(M, \mathrm{SU}(2))$ such that in that gauge the connection satisfies $\|A_\phi\|_{W^{1,2}(M)} \leq C\|F\|_{L^2(M)}$ with $C > 0$ depending only on the dimension.*

The reason the smallness of the curvature is necessary is that $\|F\|_{L^2(M)}$ being above a certain threshold allows the second Chern number of the bundle to be nontrivial:

$$c_2(E) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge F) \neq 0.$$

If, for such F, the controlled gauge were *global*, i.e., if we had a global trivialization in which the connection of the above F is expressed as $d + A$ with

$$\|A\|_{W^{1,2}(M)} \leq C,$$

then by the Sobolev and Hölder inequalities we would have enough control on the quantities involved to prove the following formal identity for our A:

$$\mathrm{tr}\big[(dA + [A, A]) \wedge (dA + [A, A])\big] = d\,\mathrm{tr}\big(A \wedge dA + \tfrac{2}{3}A \wedge A \wedge A\big).$$

Now the right side, being an exact form, would have integral equal to zero over the boundaryless manifold M, which would contradict $c_2(E) \neq 0$.

M. Atiyah, N. Hitchin, I. Singer [Atiyah et al. 1978] and C. Taubes [1982] constructed instantons with nontrivial Chern numbers behaving as in the above heuristic. To exemplify the phenomena at work consider the simplest instanton, having $c_2(E) = 1$ over $M = \mathbb{S}^4$ (see [Freed and Uhlenbeck 1984, Chapter 6] for notations and details). Recall that we may use quaternion notation due to the isomorphisms $\mathrm{SU}(2) \sim \mathrm{Sp}(1)$ and $\mathrm{su}(2) \sim \mathrm{Im}\,\mathbb{H}$, under which Pauli matrices correspond to quaternion imaginary units. We then have the following local expression of A over $\mathbb{R}^4$ (identified by stereographic projection with $\mathbb{S}^4 \setminus \{p\}$) in a trivialization:

$$A = \mathrm{Im}\left(\frac{x\,d\bar{x}}{1+|x|^2}\right).$$

If Ψ is the inverse stereographic projection then Ψ^*A is smooth away from the pole p, but near p we have $|\Psi^*A|(q) \sim \mathrm{dist}_{\mathbb{S}^4}(p,q)^{-1}$, which is not L^4 in any neighborhood of p.

Such behavior like $1/|x|$ shows that we are in any space L^p for $p < 4$ but not in L^4. The natural space is the weak-L^4 space $L^{4,\infty}$, which is strictly contained between all L^p, $p < 4$, and L^4:

Definition 1.2 [Grafakos 2008]. Let X, μ be a measure space. The space $L^{p,\infty}(X,\mu)$ (also called *weak-L^p* or *Marcinkiewicz* space) is the space of all measurable functions f such that

$$\|f\|^p_{L^{p,\infty}} := \sup_{\lambda>0} \lambda^p \mu\{x : |f(x)| > \lambda\}$$

is finite.

We note immediately that the function $f(x) = 1/|x|$ belongs to $L^{4,\infty}$ on $\mathbb{R}^4$ and the above global gauge gives an $L^{4,\infty}$ 1-form Ψ^*A on $\mathbb{S}^4$. Spaces $L^{p,\infty}$ arise naturally in dealing to the critical exponent estimates for elliptic equations. Indeed, the Green kernel $K_n(x)$ of the Laplacian on $\mathbb{R}^n$ satisfies $\nabla K \in L^{n/(n-1),\infty}$ but not $\nabla K \in L^{n/(n-1)}$. Thus $\Delta u = f$ with $f \in L^1$ implies $\nabla u = \nabla K * f \in L^{n/(n-1),\infty}$ by an extended Young inequality (see [Grafakos 2008]). This is unlike the higher exponent case $f \in L^p$, $p > 1$, which gives the stronger result $\nabla u \in L^p$.

1B. ***Controlled global gauges.*** As shown heuristically by the explicit case of the instanton A above, it is known how to construct $L^{4,\infty}$ global gauges. Our main effort in this work is to obtain *norm-controlled* gauges, mirroring Theorem 1.1 by Uhlenbeck. The main result is the following:

Theorem A. *Let M^4 be a Riemannian 4-manifold. There exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with the following property: Let ∇ be a $W^{1,2}$-connection over an* SU(2)*-bundle over M. Then there exists a* global $W^{1,(4,\infty)}$*-section of the bundle (possibly allowing singularities) over the whole M^4 such that in the corresponding trivialization ∇ is given by $d + A$ with the bound*

$$\|A\|_{L^{(4,\infty)}} \leq f(\|F\|_{L^2(M)}),$$

where F is the curvature form of ∇.

This theorem is related to a second main result of this work, namely the introduction of Lorentz–Sobolev extension theorems for nonlinear maps. This result takes most of our efforts and can be stated as follows:

Theorem B. *There exists a function* $f_1 : \mathbb{R}^+ \to \mathbb{R}^+$ *with the following property*: *Let* $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. *Then there exists an extension* $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ *of* ϕ *such that*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \leq f_1(\|\nabla\phi\|_{L^3}).$$

The originality of Theorem B with respect to the previous results [Bethuel and Demengel 1995; Mucci 2010] is that, whereas the previous works were concerned with the *existence* of an extension, in our case a *control* is provided in terms of the boundary value. We show below that, even under the hypothesis $\deg(\phi) = 0$ — so that a $W^{1,4}$-extension surely exists — no energy control will be available in the (stronger) $W^{1,4}$-norm.

Controlled global gauges as above will probably have many applications in the analysis of gauge theory, for example in simplifying compactness results; see [Petrache 2013]. Controlled global gauges could allow a global control on the Yang–Mills flow provided we obtain also the Coulomb condition, which is however an open question:

Open Problem 1.3. Prove that it is possible to find $L^{4,\infty}$-controlled global Coulomb gauges as in Theorem A. In other words, prove that it is possible to find a gauge as in Theorem A, but with the further requirement that $d^*A = 0$.

1C. ***Strategy of gauge construction.*** The link between Theorems A and B is given by the well-known identification $\mathrm{SU}(2) \simeq \mathbb{S}^3$. Therefore, Theorem B can be rephrased as follows:

Theorem B′. *Fix a trivial* SU(2)*-bundle* E *over the ball* B^4. *There exists a function* $f_1 : \mathbb{R}^+ \to \mathbb{R}^+$ *with the following property*: *if* $g \in W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$ *gives a trivialization of the restricted bundle* $E|_{\partial B^4}$, *then there exists an extension of* g *to a trivialization* $\tilde{g} \in W^{1,(4,\infty)}(B^4, \mathrm{SU}(2))$ *such that*

$$\|\nabla\tilde{g}\|_{L^{4,\infty}(B^4)} \leq f_1(\|\nabla g\|_{L^3(\mathbb{S}^3)}).$$

The proof of Theorem A is by a sequence of gauge extensions along the simplices of a suitable triangulation. We use simplices where Uhlenbeck's Theorem 1.1 holds, i.e., F has energy $\lesssim \epsilon_0$. To ensure a lower bound on the size of simplices we cut areas of energy concentration and use induction on the energy; see the graphical summary (5-1).

1D. ***Extension of Sobolev maps into manifolds.*** We discuss the relevance of our theorem, several possible extensions and related phenomena in Section 2.

Here we point out the main open questions in the area of controlled nonlinear extensions and some analogues of Theorem B. The fundamental group $\pi_m(N)$ is a useful tool to control the topology of N. It is a quotient of $C^0(\mathbb{S}^m, \mathbb{N})$. To say that any map in this space is continuously extendable to B^{m+1} amounts to asserting that $\pi_m(N) = 0$.

We consider here the *controlled* extension problem for maps $\mathbb{S}^m \to \mathbb{S}^n$. As is usually the case, interesting new features appear when smooth maps are not dense in $W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$, in which case we expect topological obstructions to gradually disappear as p decreases. The first facts to note are:

- For extensions of maps from $W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$ to B^{m+1} the natural space given by continuous Sobolev and trace embeddings is $W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ (see Section 2A and 2B).
- For $p < m(n+1)/(m+1)$ the controlled extensions exist (see Section 2A).
- For $p > m$ the extension question reduces to a purely topological problem (see Section 2B).

The open cases when $p < m$ are thus among the following ones:

Open Problem 1.4. Assume that $m(n+1)/(m+1) \leq p < m$ and $m > n$. For which such choices of m, n, p does there exist a finite function $f_{m,n,p} : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\phi \in W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$ there exists an extension $u \in W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ for which the estimate

$$\|u\|_{W^{1,p(m+1)/m}(B^{m+1},\mathbb{S}^n)} \leq f_{m,n,p}(\|\phi\|_{W^{1,p}(\mathbb{S}^m,\mathbb{S}^n)})$$

holds? Does the estimate hold for $p = m$ for the norm $W^{1,(m+1,\infty)}(B^{m+1}, \mathbb{S}^n)$?

Open Problem 1.4 is partially understood or solved just in some cases:

- Due to a relation between extension problems and lifting problems, we answer the above problem for $n = 2 < m$ and $3m/(m+1) \leq p < 4m/(m+1)$; see Proposition 1.7 and Section 2D.
- In particular, we cover all p for the dimensions $m = 3, n = 2$.
- For $n = 1$, $m \geq 3$ and $3m/(m+1) \leq p < m$, it was shown by F. Bethuel and F. Demengel [1995] that no extension exists.

It will be interesting in the future to look at the link between extension and lifting problems in detail. It is possible to do this also in the case of $\mathbb{S}^1$-valued maps and in nonlocal Sobolev spaces, e.g., using the results of J. Bourgain, H. Brezis and P. Mironescu [Bourgain et al. 2000].

In the critical case $p = m$, left aside in Open Problem 1.4, we have the following results:

- Using the Hopf lifts as in the works of R. Hardt and T. Rivière [2003; 2008], we prove Theorem C, which is the solution to the case $p = m = n = 2$ (see Section 3).
- The extension in that case exists but cannot be controlled in the above Sobolev norm, making the Lorentz–Sobolev weakening of Theorem B and of Theorem C below optimal (see Section 2E). This is analogous to the case of global gauges in 4 dimensions pointed out in the introduction.
- We also prove an analogous result for $p = m = n = 1$ (see Theorem 2.5). However this is not the natural space to look at, unlike in higher dimensions. In this case, indeed, the trace space $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ is the natural space to look at, because $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ does not continuously embed in it (we recall a counterexample in Section 2C).

These theorems leave open higher-dimensional cases:

Open Problem 1.5. Assume $n \geq 4$. Prove that there exists a finite function $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for each $\phi \in W^{1,n}(\mathbb{S}^n, \mathbb{S}^n)$, we can find an extension $u \in W^{1,(n+1,\infty)}(B^{n+1}, \mathbb{S}^n)$ for which

$$\|u\|_{W^{1,(n+1,\infty)}(B^{n+1},\mathbb{S}^n)} \leq f_n(\|\phi\|_{W^{1,n}(\mathbb{S}^n,\mathbb{S}^n)}).$$

Unlike in linear Sobolev spaces, not only the topology of the domain must be compared to the Sobolev exponent p, but also the dimension and structure of the constraint (i.e., the target manifold) plays a critical role. This is also related to the topological global obstructions to density results for smooth functions between manifolds found by F. Hang and F.-H. Lin [2001; 2003] and discussed by T. Isobe [2006].

A general tool allowing extensions is the projection trick of Section 2A, which works well for Sobolev exponents smaller than the target dimension plus one. Lifting theorems allow us to increase this dimension and thus to apply the projection trick with higher exponents.

Using the Hopf fibration $H:\mathbb{S}^3\to\mathbb{S}^2$ we construct controlled lifts and apply a version of the projection trick obtaining the following theorem with much less effort than for the 3-dimensional case of Theorem B:

Theorem C (see Section 3). *Suppose $\phi\in W^{1,2}(\mathbb{S}^2,\mathbb{S}^2)$ is given. Then there exists $u\in W^{1,(3,\infty)}(B^3,\mathbb{S}^2)$ such that, in the sense of traces, $u|_{\partial B^3}=\phi$ and such that the following estimate holds, for a constant independent of ϕ:*

$$\|u\|_{W^{1,(3,\infty)}(B^3)}\le C\|\phi\|_{W^{1,2}(\mathbb{S}^2)}(1+\|\phi\|_{W^{1,2}(\mathbb{S}^2)}).$$

The Hopf fibration has a natural structure of $U(1)$-bundle with nontrivial characteristic class, $P\to\mathbb{S}^2$. Lifting a map $\phi:X\to\mathbb{S}^2$ to a map $\tilde{\phi}:X\to\mathbb{S}^3$ for which $H\circ\tilde{\phi}=\phi$ corresponds to giving the trivialization of the pullback bundle ϕ^*P. Analogous lifts are interesting to study for general principal G-bundles, using universal connections. The next case after the one with target $\mathbb{S}^2$ is the SU(2)-bundle of the introduction, which corresponds to the Hopf fibration $\mathbb{S}^7\to\mathbb{S}^4$.

The Hopf lift idea seems to be much more difficult to extend to the case where the target is $\mathbb{S}^3$. We cannot use principal bundles because $\pi_2(G)=0$ for all compact Lie groups G. For other fibrations, the following question is open:

Open Problem 1.6. Is it possible to find a fibration $\pi:E\to\mathbb{S}^3$ with compact fiber M and a constant $C>0$ such that, for each $\phi\in W^{1,3}(\mathbb{R}^3,\mathbb{S}^3)$, there exists a lift $\tilde{\phi}:\mathbb{R}^3\to E$ satisfying the estimate $\|\nabla\tilde{\phi}\|_{L^{(3,\infty)}}\le Cf(\|\nabla\phi\|_{L^3})$ for some finite function $f:\mathbb{R}^+\to\mathbb{R}^+$?

The controlled Hopf lift result for $\mathbb{S}^2$ yields also an answer to Open Problem 1.4 for dimensions $m=3$, $n=2$:

Theorem D. *Assume $\phi\in W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)$. Then there exists a controlled extension $u\in W^{1,(4,\infty)}(B^4,\mathbb{S}^2)$ with the control*

$$\|u\|_{W^{1,(4,\infty)}(B^4,\mathbb{S}^2)}\le C\|\phi\|_{W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)}(1+\|\phi\|_{W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)}).$$

If instead we have $\phi\in W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)$ for $\frac{9}{4}\le p<3$, then there exists an extension $u\in W^{1,\frac{4}{3}p}(B^4,\mathbb{S}^2)$ with

$$\|u\|_{W^{1,\frac{4}{3}p}(B^4,\mathbb{S}^2)}\le C\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}(1+\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}).$$

The same proof allows us to also answer Open Problem 1.4 for $n=2<m$ for some exponents p:

Proposition 1.7. *Assume $n=2$, $m\ge3$ and $3m/(m+1)\le p<4m/(m+1)$ and let $\phi\in W^{1,p}(\mathbb{S}^m,\mathbb{S}^2)$. Then there exists a controlled extension $u\in W^{1,p(m+1)/m}(B^{m+1},\mathbb{S}^2)$ with*

$$\|u\|_{W^{1,p(m+1)/m}(B^4,\mathbb{S}^2)}\le C\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}(1+\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}).$$

1E. ***Ingredients used in the construction of $W^{1,(4,\infty)}(B^4,\mathbb{S}^3)$-extensions.*** The starting new idea was to the use of implicit function theorems and of a limit on the integrability exponent as done in [Uhlenbeck 1982a] for the extension result. Note that the procedure of Appendix A is generalizable to other contexts with no new ingredients, at least as long as a Lie group structure is present.

For the implicit function theorems above, we needed here a new product estimate valid in Sobolev spaces, which is presented in Appendix B, extending partially the results of [Brézis and Mironescu 2001]; cf. [Runst and Sickel 1996; Triebel 1995].

The second idea was to use $L^{(4,\infty)}$ functions such that the L^4-estimate would fail just near a controlled number of points. Such singular points (where "singular" is meant with respect to the L^4-estimates) are introduced via Lemma 4.6 and Theorem 4.3.

The uniform $L^{(4,\infty)}$-control is obtainable just in the case where the boundary value has no large energy "hot spots". To deal with the case where energy concentrates, we use two tools which are available in the particular case of $\mathbb{S}^3 \simeq \mathrm{SU}(2)$: (1) the group operation of SU(2), which gives a continuous product on $W^{1,3}(X,\mathbb{S}^3)$; (2) the Möbius group of $\mathbb{S}^3$ coupled with the conformal invariance of the L^3-norm of the gradient on $\mathbb{S}^3$.

Under a balancing condition on the boundary value ϕ, we can write $\phi = \phi_1\phi_2$, where the product is taken in SU(2), and the energies of ϕ_i, $i = 1, 2$, are strictly less than that of ϕ, allowing an induction on the energy. If the balancing is not valid, we apply a Möbius transformation F_v to $\mathbb{S}^3$ and either reduce to a balanced situation for $F_v \circ \phi$ for some v or provide a substitute $v \in B^4 \mapsto \fint_{\mathbb{S}^3} \phi \circ F_v$ to the harmonic extension of ϕ, to which we can now apply the projection trick. The natural parametrization of the Möbius group of $\mathbb{S}^3$ via vectors in B^4 fits very well in this setting, and we were inspired to use it by the similar use of it in [Marques and Neves 2014].

1F. ***Plan of the paper.*** Section 2 contains a list of positive and negative results concerning phenomena parallel to ours, showing that our results are optimal. Section 3 contains the proof of Theorem C. In Section 4 we prove Theorem B, and in Section 5 we prove Theorem A. Appendix A deals with our new "extension" version of Uhlenbeck's gauge construction and in Appendix B we prove the needed new product inequality. Appendix C contains computations and notation for the Möbius groups of B^{n+1} and $\mathbb{S}^n$.

2. Controlled and uncontrolled nonlinear Sobolev extensions

Classical Sobolev space theory features optimal extension theorems in natural trace norms. For example, if $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $u : \partial\Omega \to \mathbb{R}$ is a $W^{1,n-1}$-function, then there exists an extension $\bar{u} : \Omega \to \mathbb{R}$ such that $\bar{u} \in W^{1,n}$ and the estimate

$$\|\bar{u}\|_{W^{1,n}} \le C\|u\|_{W^{1,n-1}}$$

holds (with C independent of u). This extension theorem is optimal in the sense that for dimensions $n > 2$ the natural trace operator $\bar{u} \in W^{1,n}(\Omega) \mapsto \bar{u}|_{\partial\Omega}$ sends $W^{1,n}$ to the optimal space $W^{1-1/n,n}$ (see [Tartar 2007, Chapter 40] for the natural appearance of this space), and we have the optimal Sobolev continuous embedding $W^{1-1/n,n} \to W^{1,n-1}$ (see [Tartar 2007]) which brings us back to the original space. A similar result still holds if we replace the codomain $\mathbb{R}$ by $\mathbb{R}^m$.

However, for $n = 2$, the space $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ does not continuously embed in $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, making the above reasoning less effective; see Section 2C.

A construction of $\bar{u}$ is possible by imitating the model, valid for $\Omega = \mathbb{R}^n_+ := \{(x_1, \dots, x_n) | x_n \geq 0\}$,

$$\bar{u}(x_1, \dots, x_{n-1}, \epsilon) := (\rho_\epsilon * u)(x_1, \dots, x_{n-1}),$$

where ρ_ϵ is a standard family of radial smooth compactly supported mollifiers.

An equivalent construction of $\bar{u}$ in terms of function spaces is by harmonic extension. The optimal result is the following:

Proposition 2.1 (harmonic extension; cf. [Gazzola et al. 2010, Chapter 10]). *Assume $q > 1$ and $u \in W^{1-1/q,q}(\partial B^{m+1}, \mathbb{R}^{n+1})$. Then there exists a harmonic extension $\bar{u} \in W^{1,q}(B^{m+1}, \mathbb{R}^{n+1})$ such that*

$$\|\bar{u}\|_{W^{1,q}(B^{m+1}, \mathbb{R}^{n+1})} \leq C_{m,n,q} \|u\|_{W^{1-1/q,q}(\partial B^{m+1}, \mathbb{R}^{n+1})}.$$

By Sobolev embedding, we have the controlled inclusion $W^{1,p} \hookrightarrow W^{1-1/q,q}$ on an m-dimensional bounded open domain (or a compact manifold like ∂B^{m+1}) for $q \leq p(m+1)/m$; therefore, this q is the largest exponent where we can hope to have a control for the extension.

If u is a constrained function with values in a subset of $\mathbb{R}^{n+1}$ (e.g., a curved n-dimensional submanifold like $\mathbb{S}^n$) then averaging even on a very small scale could push the values of $\bar{u}$ quite far from the constraint obeyed by u. This happens in particular for Sobolev exponents that make the dimension "supercritical", i.e., exponents such that $W^{1,q}(B^{m+1})$ is not constituted of continuous functions. We now describe some cases where directly projecting back to $\mathbb{S}^n$ does not destroy the norm control of Proposition 2.1.

2A. *Projection from a well-chosen center.*

We present in this section a trick which probably appeared for the first time in relation to nonlinear Sobolev extensions in R. Hardt, D. Kinderlehrer and F.-H. Lin's works [Hardt et al. 1986; Hardt and Lin 1987]. For a Lorentz space version see Proposition 3.4.

Proposition 2.2 (projection trick). *If $f \in W^{1,q}(\Omega, B^{n+1})$ with $q < n+1$ and Ω is a bounded open simply connected domain of $\mathbb{R}^{m+1}$, then there exists $a \in B^{n+1}_{1/2}$ and a constant C depending only on q, m, n such that if $f_a(x) = \pi_a(f(x))$, where $\pi_a : B^{n+1} \setminus \{a\} \to \mathbb{S}^n$ is the projection which is constant along the segments $[a, \omega]$, $\omega \in \mathbb{S}^n$, then*

$$\|f_a\|_{W^{1,q}(\Omega, \mathbb{S}^n)} \leq C \|f\|_{W^{1,q}(\Omega, B^{n+1})}.$$

Proof. We just have to estimate the gradient of f_a in terms of that of f since in any case the functions themselves are bounded and Ω is assumed of finite measure. We first note that, since $a \in B^{n+1}_{1/2}$ is away from the boundary of B^{n+1}, we have the pointwise estimate

$$|\nabla f_a|(x) \lesssim \frac{|\nabla f|(x)}{|f(x) - a|},$$

where the implicit constant depends only on n. We next consider the following "average" on a:

$$\int_{B^{n+1}_{1/2}}\left(\int_\Omega |\nabla f_a|^q(x)\,dx\right)da \lesssim \int_\Omega |\nabla f|^q(x)\left(\int_{B^{n+1}_{1/2}} \frac{da}{|f(x)-a|^q}\right)dx.$$

We note that the inner integral is of the form

$$I(y) := \int_{B^{n+1}_{1/2}} \frac{da}{|y-a|^q},$$

and

$$\max_y I(y) = I(0) = C_n \int_0^{1/2} r^{n+q}\,dr = C_{n,q} < \infty \quad \text{since } q < n+1;$$

therefore, we obtain

$$\int_{B^{n+1}_{1/2}} \|\nabla f_a\|^q_{L^q}\,da \le C_{n,q}\|\nabla f\|^q_{L^q},$$

and the proof is easily concluded. □

The above proposition together with Proposition 2.1 and the remark on Sobolev exponents following it give the following:

Theorem 2.3 (corollary of the projection trick; cf. [Hardt and Lin 1987, Theorem 6.2]). *Let $m,n \in \mathbb{N}^*$. If $1 \le p < m(n+1)/(m+1)$ then for any $\phi \in W^{1,p}(\partial B^{m+1}, \mathbb{S}^n)$ there exists a nonlinear extension $u \in W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ satisfying the control*

$$\|u\|_{W^{1,p(m+1)/m}(B^{m+1},\mathbb{S}^n)} \le C_{m,n,p}\|\phi\|_{W^{1,p}(\partial B^{m+1},\mathbb{S}^n)}.$$

Remark 2.4. Note that from the same ingredients we obtain also the stronger estimate where for $q := p(m+1)/m < m$ the weaker space $W^{1-1/q,q}(\partial B^{m+1}, \mathbb{S}^n)$ replaces $W^{1,p}(\partial B^{m+1}, \mathbb{S}^n)$. This was done in [Bethuel and Demengel 1995; Hardt and Lin 1987]. We stated Theorem 2.3 as above to emphasize the connection with our Theorems B and C. Indeed, taking $m = n$ we see that those theorems cover the critical exponent $p = n$ for which the projection trick stops working.

2B. ***Large integrability exponents.*** We now consider functions in $W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$ with $p > m$; there is a continuous embedding of $C^{0,1-m/p}(\mathbb{S}^m, \mathbb{S}^n)$ into this space. The candidate extension space $W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ is also composed of $C^{0,1-m/p}$-functions. As described in Section 1D, the extension problem is guaranteed to have a solution as long as $\pi_m(\mathbb{S}^n) = 0$. This is true for $m < n$ but false for many choices of $m > n$ and for $m = n$.

When an extension exists for ϕ representing the identity of the (nontrivial) group $\pi_m(\mathbb{S}^n)$, a controlled extension can be constructed based on the fact that a bound on the $C^{0,\alpha}$-norm for $\alpha > 0$ implies a control on the modulus of continuity.

2C. ***Extension for maps in*** $W^{1,1}(\partial\mathbb{S}^1,\mathbb{S}^1)$. For maps with values in $\mathbb{S}^3$, we are helped by the existence of a well-behaved product structure on $\mathbb{S}^3$, i.e., the one which gives the identification $\mathbb{S}^3\simeq \mathrm{SU}(2)$. This is enough to get the analogous result for $n=1$, as we will see now. It is however well known (see [Hatcher 2009, Section 2.3]) that this is a very unusual case: a group operation exists on $\mathbb{S}^k$ only for $k=1,3$.

We can state a similar extension problem in the 1-dimensional case. This kind of controlled extension result is related to the recent work on Ginzburg–Landau functionals in [Serfaty and Tice 2008].

Here the main structural ingredients present for $\mathbb{S}^3$ are again present: namely, we have a group operation on $\mathbb{S}^1$ (in this case it is the abelian group $U(1)\sim\mathbb{R}/\mathbb{Z}$) and a Möbius structure on D^2 restricting to one on $\mathbb{S}^1$. We follow the strategy of proof described in Section 1D. The result is:

Theorem 2.5 (1-dimensional version of the extension). *There exists a function $g:\mathbb{R}^+\to\mathbb{R}^+$ with the following property: if $\phi\in W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)$ then there exists $u\in W^{1,(2,\infty)}(D^2,\mathbb{S}^1)$ with $u|_{\partial D^2}=\phi$ in the sense of traces and we have the norm control*

$$\|u\|_{W^{1,(2,\infty)}(D^2,\mathbb{S}^1)}\le g(\|\phi\|_{W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)}).$$

We will explain the changes which occur with respect to the proof of Theorem B (see Section 4).

Sketch of proof. The procedure is as in Section 4 and Appendix A; we have just to replace exponents and dimensions 3, 4 with 1, 2. For the analogue of Proposition 4.9 the biharmonic equation (4-36) is replaced by a harmonic equation, while the resulting estimates persist. Perhaps the only significant change is Lemma B.1 of Appendix B. It should be replaced by the following product estimate, valid for $f\in W^{1,1}(D^2)$, $g\in L^\infty\cap W^{1,2}(D^2)$:

$$\|fg\|_{W^{1,1}}\le\|f\|_{W^{1,1}}(\|g\|_{L^\infty}+\|g\|_{W^{1,2}}). \qquad \square$$

We must however note that the naturality of the space $W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)$ in Theorem 2.5 is less evident, since the trace space $H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$ does not continuously embed in it, unlike what happens in higher dimensions. This is seen by considering

$$u_\epsilon(\theta)=\exp\Bigl(i\min\bigl\{1,\epsilon^{-1}\operatorname{dist}_{\mathbb{S}^1}\bigl(\theta,\bigl[-\tfrac12\pi,\tfrac12\pi\bigr]\bigr)\bigr\}\Bigr).$$

It is then clear that $\|\nabla u_\epsilon\|_{L^1(\mathbb{S}^1)}=2$, while we estimate the double integral in θ,θ' giving the $H^{1/2}$-norm by the contribution of the regions $\theta\in\bigl[0,\frac12\pi\bigr]$, $\theta'\in\bigl[\frac12\pi+\epsilon,\pi+\epsilon\bigr]$. Under these choices, $u_\epsilon(\theta)=e^0$, $u_\epsilon(\theta')=e^i$, and their distance in $\mathbb{S}^1$ is 1. Thus,

$$\begin{aligned}\|u_\epsilon\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)}&=\int_{\mathbb{S}^1}\int_{\mathbb{S}^1}\frac{\operatorname{dist}_{\mathbb{S}^1}(u_\epsilon(\theta),u_\epsilon(\theta'))^2}{\operatorname{dist}_{\mathbb{S}^1}(\theta,\theta')^2}\,d\theta\,d\theta'\\&\le\int_0^1\int_0^1\frac{1}{|x+2\epsilon/\pi-y|^2}\,dx\,dy\\&\lesssim|\log\epsilon|+1.\end{aligned}$$

2D. *Using controlled liftings to obtain controlled extensions.* The control obtained for extensions of maps in $W^{1,3}(\mathbb{S}^3,\mathbb{S}^3)$ and $W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)$ is exponential in the norms of these maps. In Section 3 we describe an approach, which works for $\phi \in W^{1,2}(\mathbb{S}^2,\mathbb{S}^2)$, which is completely different than in dimensions 1 and 3 and yields a faster proof and a better control. Such an approach was first considered in [Hardt and Rivière 2003]. This is based on the existence of controlled Hopf lifts. The result (see Corollary 3.3) is that there exists an $L^{2,\infty}$-controlled lifting $\tilde{\phi}:\mathbb{S}^2\to\mathbb{S}^3$, i.e., a function such that $H\circ\tilde{\phi}=\phi$, where $H:\mathbb{S}^3\to\mathbb{S}^2$ is the Hopf fibration and we have the control

$$\|\nabla\tilde{\phi}\|_{L^{2,\infty}}\le C\|\nabla\phi\|_{L^2}(1+\|\nabla\phi\|_{L^2}).$$

The analogous controlled lift exists also for $\phi\in W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)$, whereas for $2\le p<3$ we have a control on the L^p-norm of the lift instead of the $L^{p,\infty}$ one; cf. Proposition 1.7. This lift allows us to prove, along the same lines, Theorems C and D.

The gist of the proof is the following: Once we have the controlled lift, the lifted map takes values into a sphere of a higher dimension. This allows a wider range of application for the projection trick of Proposition 2.2 or of its Lorentz space analogue of Proposition 3.4.

Having extended the lift, reprojecting the extension to $\mathbb{S}^2$ via the Hopf map maintains the gradient estimates. This is due to the fact that the Hopf fibration is a submersion (cf. (3-4)) and our lift can be taken so that the "vertical" component η is also controlled.

Work on the existence of nonlinear liftings has been very active regarding $\mathbb{S}^1$-valued maps (see, e.g., [Bourgain et al. 2000; 2004; Bethuel and Zheng 1988] and the references therein). Looking also at higher-dimensional analogues seems very promising in relation to extension results.

2E. *Small energy extension with estimate.* As for the case of curvatures over bundles with a compact Lie group, the small energy regime allows a kind of linearization of the problem and gives estimates which are better than what is expected in general. We obtain in particular an estimate in $W^{1,4}$ instead of $W^{1,(4,\infty)}$ for the extension, provided that the norm of the boundary trace is small:

Proposition 2.6 (see Theorem 4.4). *There is a constant $\epsilon_0>0$ and a finite constant C such that, if*

$$\int_{\mathbb{S}^3}|\nabla\phi|^3\le\epsilon_0,\quad \phi:\mathbb{S}^3\to\mathbb{S}^3,$$

then there exists $u\in W^{1,4}(B^4,\mathbb{S}^3)$ such that

$$u=\phi\quad\text{on }\partial B^4\text{ in the sense of traces}\qquad\text{and}\qquad \|\nabla u\|_{L^4(B^4)}\le C\|\nabla\phi\|_{L^3(\mathbb{S}^3)}.$$

This is part of our proof of Theorem B and is proved in Section 4B using a method developed in Appendix A in the spirit of [Uhlenbeck 1982b].

2F. *Existence of $W^{1,4}$-extension without norm bounds.* As for the case of global gauges, we can in general obtain $W^{1,4}(B^4,\mathbb{S}^3)$-extensions once we give up the requirement to have a norm control of the extension such as in Theorem B. This phenomenon represents one example of situations in which function spaces have behavior which is more complex than what can be detected by only looking at their norms.

Proposition 2.7. *If* $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$, *then its topological degree is well defined*; *cf.* [Schoen and Uhlenbeck ∼1980; White 1988]. *Suppose that* $\deg \phi = 0$. *Then there exists* $u \in W^{1,4}(B^4, \mathbb{S}^3)$ *such that*

$$u = \phi \quad \text{on } \partial B^4 \text{ in the sense of traces.}$$

Proof. We use the extension as in Section 4A. The construction using Lemma 4.5 is done on a series of domains $B(x_i, \rho_i) \cap B^4$, where $x_i \in \partial B^4$, $\rho_i \in [\rho_F, 2\rho_F]$ for the choice

$$\rho_F := \inf\left\{\rho > 0 : \exists x_0 \in \partial B^4, \int_{B(x_0,2\rho)\cap\partial B^4} |\nabla\phi|^3 \geq \epsilon_0\right\}.$$

Note that we have no a priori control on how small ρ_F could get, but (by absolute continuity of $|\nabla\phi|^3 dx$ and compactness of ∂B^4) it cannot be zero for a fixed ϕ. Then a Lipschitz extension $u : \mathcal{R} \to \mathbb{S}^3$ to a Lipschitz region $\mathcal{R}$ included between $B^4 \setminus B_{1-2\rho_F}$ and $B^4 \setminus B_{1-\rho_F}$ exists as in Section 4A and such a u will also be Lipschitz (with constant bounded by ρ_F^{-1}) and will have degree zero (the preservation of degree follows because the extension used in the construction preserves the homotopy type; cf. [White 1988]). In particular we can do a further Lipschitz (thus $W^{1,4}$) extension to the interior of $B^4 \setminus \mathcal{R}$. This provides the desired u. ☐

The proof of the above proposition is constructive, and no hint that the construction is optimal is available. In the next section we prove that actually *no general bound in* $W^{1,4}$ can be achieved, because of the intervention of the topological degree, much as in the case of SU(2)-instantons.

2G. *Impossibility of $W^{1,4}$ bounds for an extension.*

Proposition 2.8. *There exists no finite function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for each* $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ *there exists a function* $u \in W^{1,4}(B^4, \mathbb{S}^3)$ *satisfying*

$$u = \phi \quad \text{on } \partial B^4 \text{ in the sense of traces} \qquad \text{and} \qquad \|\nabla u\|_{L^4(B^4)} \leq f(\|\nabla\phi\|_{L^3(\mathbb{S}^3)}).$$

Proof. We recall the robustness of degree under strong convergence in $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ (see [Schoen and Uhlenbeck ∼1980; White 1988; Brézis and Nirenberg 1995; 1996]). Consider $\phi = \mathrm{id}_{\mathbb{S}^3}$, which has degree 1. Suppose an extension $u : B^4 \to \mathbb{S}^3$ to ϕ were to exist with $\|u\|_{W^{1,4}} \leq C'$. It would then be possible to approximate u in $W^{1,4}$-norm by functions $u_i \in C^\infty(B^4, \mathbb{S}^3)$, since smooth functions are dense in $W^{1,4}(B^4, \mathbb{S}^3)$. In particular the degrees $\deg(\phi_i)$ of $\phi_i = u_i|_{\partial B^4}$ would have to be zero. This contradicts the fact that $\phi_i \to \phi$ in $W^{1,3}$-norm because the degree of the boundary trace is preserved under strong $W^{1,3}$-convergence.

This proves the absence of a continuous extension operator. To show that boundedness is also impossible, we use a slightly different argument.

Consider $\phi_0 \in W^{1,3} \cap C^\infty(\mathbb{S}^3, \mathbb{S}^3)$ that is a perturbation of the identity equal to the south pole S in a neighborhood N_S of S. Then consider a Möbius transformation $F : \mathbb{S}^3 \to \mathbb{S}^3$ such that $F^{-1}(N_S)$ includes the lower hemisphere, and let $\phi' = \phi_0 \circ F$, $\phi'' = \phi_0 \circ (-F)$. Then, identifying $\mathbb{S}^3 \sim \mathrm{SU}(2)$ so that $S \sim \mathrm{id}_{\mathrm{SU}(2)}$, use the group operation to define $\phi = \phi'\phi''$. Note that $\|\phi\|_{W^{1,3}} \leq 2\|\phi_0\|_{W^{1,3}}$, since the conformal maps F, $-F$ preserve the energy; moreover, ϕ has zero degree.

Let F_n be a family of Möbius transformations symmetric about S that concentrate more and more near S (with the notation of Appendix C we may take $F_n := F_{v_n}$ for $v_n = (1-1/n)S$). Define $\phi'_n := \phi' \circ F_n$ and $\phi_n := \phi'_n \phi''$. It is clear by conformal invariance of the $W^{1,3}$-energy that the ϕ_n have constant energy. They converge weakly to ϕ'' and have degree zero.

Call u_n the extension of ϕ_n and suppose that $\|u_n\|_{W^{1,4}} \leq C$ independent of n. We may suppose that, in $W^{1,4}$-norm, $u_n \rightharpoonup u_\infty \in W^{1,4}(B^4, \mathbb{S}^3)$ and we obtain $u_\infty|_{\partial B^4} = \phi''$ in the sense of traces. We then apply the result of [White 1988] (see also [Schoen and Uhlenbeck ~1980]), which in this case says that the 3-dimensional homotopy class passes to the limit under bounded sequential weak-$W^{1,4}(B^4, \mathbb{S}^3)$ limits. We again obtain a contradiction to boundedness, since $\deg(\phi'') = -1$ whereas the same degree is zero for the maps ϕ_n. □

2H. *Moving frames and their gauges.* We describe here a lifting problem arising in the theory of moving frames on 2-dimensional surfaces, where the Lorentz spaces appear again in the optimal estimates. The model question is as follows:

Question 2.9. Given a map (representing the normal vector of an immersed surface) $\vec{n} \in W^{1,2}(D^2, \mathbb{S}^2)$, does there exist a $W^{1,2}$-*controlled trivialization* $\vec{e} = (\vec{e}_1, \vec{e}_2)$ of the pullback bundle $\vec{n}^{-1}T\mathbb{S}^2$? A trivialization is defined by two vector fields $\vec{e}_1, \vec{e}_2 \in W^{1,2}(D^2, \mathbb{S}^2)$ such that the pointwise constraints $|\vec{e}_1| = |\vec{e}_2| = 1$, $\vec{e}_1 \cdot \vec{e}_2 = 0$ are satisfied almost everywhere and $\vec{n} = \vec{e}_1 \times \vec{e}_2$.

This problem behaves like the one of global controlled gauges; namely for small energy a lift exists and is controlled, and, for large energy, lifts can be found but with no general control. Uhlenbeck's ϵ-regularity estimate is mirrored in the following theorem. This result was proved initially by F. Hélein [1996, Lemma 5.1.4] under the hypothesis $\|\nabla\vec{n}\|_{L^2} \leq C$ and improved by Y. Bernard and T. Rivière, who proved that it is enough to assume a smallness condition in weak-L^2:

Theorem 2.10 [Bernard and Rivière 2014, Lemma IV.3]. *There exists ϵ_0 such that, if $\|\nabla\vec{n}\|_{L^{2,\infty}} \leq \epsilon_0$, then there exists a trivialization with the controls*

$$\|\nabla\vec{e}_1\|_{L^2} + \|\nabla\vec{e}_2\|_{L^2} \leq C\|\nabla\vec{n}\|_{L^2} \quad \textit{and} \quad \|\nabla\vec{e}_1\|_{L^{2,\infty}} + \|\nabla\vec{e}_2\|_{L^{2,\infty}} \leq C\|\nabla\vec{n}\|_{L^{2,\infty}}.$$

Note that, for the improvement above, the L^2-energy might blow up yet still control the energy of the trivialization, as long as we stay small in Lorentz norm. It would be interesting to explore this kind of phenomenon also for curvatures in higher dimensions, like in our setting.

The bad behavior in large energy regimes starts at the energy level 8π (and this is optimal; see [Kuwert and Li 2012]). This number has an evident topological significance because, if $\vec{n}$ is homotopically nontrivial, i.e., parametrizes a noncontractible 2-cell of $\mathbb{S}^2$, then $4\pi = |\mathbb{S}^2| \leq \int_{D^2} u^* \, d\,\mathrm{Vol}_{\mathbb{S}^2} \leq \frac{1}{2}\int_{D^2} |\nabla\vec{n}|^2$, so 8π is the smallest energy of a topologically nontrivial $\vec{n}$.

We also have the following lemma, similar to Section 2G:

Lemma 2.11. *For $\int |\nabla\vec{n}|^2 > 8\pi$ there can be no controlled $W^{1,2}$-trivialization $\vec{e}$.*

Sketch of proof:. We choose $\vec{n}$ mapping a neighborhood $D^2 \setminus B_r := N_1$ for small r to the south pole of $\mathbb{S}^2$ that has degree 1 and equals a conformal map outside a small neighborhood $N_2 \Supset N_1$. Such $\vec{n}$ exists with energy as close as desired to 8π, independently of r, by conformal invariance of the energy.

Supposing a trivialization $\vec{e} = (\vec{e}_1, \vec{e}_2)$ exists, on N_1 it will span the "horizontal" 2-plane of $\mathbb{R}^3$ which is perpendicular to $S = (0, 0, -1)$. On circles ∂B_ρ, $\rho > r$, by Fubini's theorem, for almost all ϵ we will have that $\vec{e}_i$, $i = 1, 2$ will be $W^{1,2}$ and thus C^0 and they have values in the equator of $\mathbb{S}^2$. By well-posedness of the topological degree and since $\vec{n}$ is nontrivial in homotopy, we obtain that each e_i will make a full turn on each ∂B_r. This gives that $\int_{\partial B_r} |\nabla \vec{e}_i| \geq 1$ on ∂B_r and by Jensen's inequality we obtain

$$\int_{D^2 \setminus B_r} |\nabla \vec{e}_i|^2 \geq C \int_r^1 \frac{1}{\rho^2} \rho \, d\rho \geq C \left| \log \frac{1}{r} \right| ;$$

since there is no positive lower bound for $r > 0$, we see that we cannot have a controlled trivialization. □

There is an analogue also of our $W^{1,(4,\infty)}$-extension result here, and it corresponds to taking the so-called "Coulomb frames". The result is a general estimate with no restriction on $\vec{n}$, but with the Lorentz norm $L^{(2,\infty)}$ instead of the L^2-norm (this estimate follows from Wente's [1969] inequality using [Adams 1975]):

Proposition 2.12 [Rivière 2012, VII.6.3]. *Let $\vec{n} \in W^{1,2}(D^2, \mathbb{S}^2)$. Then a trivialization $\vec{e}$ belonging to $W^{1,(2,\infty)}$ exists which satisfies the Coulomb condition*

$$\operatorname{div}\langle \vec{e}_1, \nabla \vec{e}_2 \rangle = 0$$

and the control

$$\|\nabla \vec{e}_1\|_{L^{(2,\infty)}} + \|\nabla \vec{e}_2\|_{L^{(2,\infty)}} \lesssim \|\nabla \vec{n}\|_{L^2} + \|\nabla \vec{n}\|_{L^2}^2 .$$

3. The Hopf lift extension

We now prove Theorem C. We consider a fixed $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$ and we need to construct an extension $u \in W^{1,(3,\infty)}(B^3, \mathbb{S}^2)$ such that

$$\|u\|_{W^{1,(3,\infty)}(B^3)} \lesssim \|\phi\|_{W^{1,2}(\mathbb{S}^2)} (1 + \|\phi\|_{W^{1,2}(\mathbb{S}^2)}),$$

where the implicit constant is independent of ϕ.

The strategy of proof uses a construction based on the Hopf fibration which has been introduced in [Hardt and Rivière 2003]. The same strategy was later used in [Bethuel and Chiron 2007] for proving similar lifting results as in [Hardt and Rivière 2003]. In the smooth case we will first lift $\phi : \mathbb{S}^2 \to \mathbb{S}^2$ to $\tilde{\phi} : \mathbb{S}^2 \to \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$, where $H : \mathbb{S}^2 \to \mathbb{S}^3$ is the Hopf fibration. Then we will extend $\tilde{\phi}$ by using a Lorentz analogue of Proposition 2.2, working with similar conditions on dimensions and exponents. Projecting back to $\mathbb{S}^2$ via H will keep the estimates.

Before the proof, we recall some properties of the map H.

3A. ***Facts about the Hopf fibration.*** Identifying $\mathbb{S}^3$ with the unit sphere of $\mathbb{C}^2$ with complex coordinates (Z, W), the Hopf projection is $H(Z, W) = Z/\overline{W}$ and its fibers are great circles. This gives a function with values in $\mathbb{C} \cup \{\infty\} \simeq \mathbb{S}^2$. If we look at $\mathbb{S}^3 \subset \mathbb{R}^4$ with the inherited coordinates (x_1, x_2, x_3, x_4), then we can identify

$$H^*\omega_{\mathbb{S}^2} = d\alpha \quad \text{for } \alpha = \tfrac{1}{2}(x_1dx_2 - x_2dx_1 + x_3dx_4 - x_4dx_3). \tag{3-1}$$

Here $\omega_{\mathbb{S}^2}$ is a constant multiple of the volume form of $\mathbb{S}^2$. Since $\mathbb{S}^1 \sim U(1)$, we can regard $\mathbb{S}^3 \xrightarrow{H} \mathbb{S}^2$ as a principal $U(1)$-bundle $P \to \mathbb{S}^2$.

Let $\phi : \mathbb{C} \to \mathbb{S}^2$ be a smooth function. Then $d(\phi^*\omega_{\mathbb{S}^2}) = 0$, because $\Omega^3(\mathbb{R}^2 \simeq \mathbb{C}) = \{0\}$. Since $H^2_{dR}(\mathbb{C}) = 0$, there exists a 1-form η such that

$$d\eta = \phi^*\omega_{\mathbb{S}^2}\,. \tag{3-2}$$

We also note that for a smooth $\phi : \mathbb{C} \to \mathbb{S}^2$ the pullback of the $U(1)$-bundle P is trivial, since $\mathbb{R}^2$ is contractible. A trivialization of the bundle $\phi^*P \to \mathbb{C}$ can be identified with a lift $\tilde{\phi}$ of ϕ. From (3-1) we can deduce that $d\eta = \tilde{\phi}^*H^*\omega_{\mathbb{S}^2} = \tilde{\phi}^*d\alpha = d(\tilde{\phi}^*\alpha)$ and again there exists a 1-form $\tilde{\eta}$ as in (3-2), defined by

$$\tilde{\eta} = \tilde{\phi}^*\alpha. \tag{3-3}$$

Note that $\tilde{\eta}$ coincides with η up to adding an exact form $d\theta$: we have $\tilde{\phi}^*\alpha - \eta = d\phi$. If we come back to the bundle point of view then $d\theta$ represents the effect of change of coordinates of the trivialization giving $\tilde{\phi}$, i.e., of a change of gauge. We then have $\eta = \tilde{\phi}^*\alpha - d\theta = (e^{-i\theta}\tilde{\phi})^*\alpha$, where the action of $e^{-i\theta}$ is intended as a $U(1)$-gauge change and $\theta : \mathbb{C} \to \mathbb{R}$ is determined up to a constant. Moreover, since DH is an isometry between the orthogonal complement of the tangent space of the fiber $T_pH^{-1}(H(p))$ and $T_p\mathbb{S}^2$, we also obtain the norm identity

$$|D\tilde{\phi}|^2 = |\tilde{\eta}|^2 + |D\phi|^2. \tag{3-4}$$

3B. ***Hopf lift with estimates.*** We start the proof of Theorem C:

Proposition 3.1. *Suppose $\phi \in W^{1,2}(\mathbb{C}, \mathbb{S}^2)$. Then there exists a lifting $\tilde{\phi} : \mathbb{C} \to \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$ and there exists a universal constant C such that*

$$\|\nabla\tilde{\phi}\|_{L^{2,\infty}} \le C\|\nabla\phi\|_{L^2}(1 + \|\nabla\phi\|_{L^2}).$$

Proof. The proof is divided into two steps.

Step 1 (constructions in the smooth case). We have seen that, at least in the smooth case, constructing a 1-form η as in (3-2) is equivalent to constructing a lift $\tilde{\phi} : \mathbb{C} \to \mathbb{S}^3$. We now observe that such a 1-form can in turn be easily constructed by inverting the Laplacian on $\mathbb{C}$ via its Green kernel, which is of the form $K(x) = -\gamma \log|x|$. In particular, $K \in W^{1,(2,\infty)}$, which is the reason why this norm appears. First note that $dd^*(K * \beta) = 0$ for a smooth L^1-integrable 2-form β on $\mathbb{C}$. We can then use this formula for $\beta = \phi^*\omega_{\mathbb{S}^2}$ and, taking into account the fact that ∇K is in $L^{2,\infty}$, by the Lorentz-space Young inequality (see [Grafakos 2008]), we obtain that the 1-form η defined as

$$\eta := d^*[K * (\phi^*\omega_{\mathbb{S}^2})], \quad \eta \to 0 \text{ at infinity}, \tag{3-5}$$

satisfies (3-2) and the estimates

$$\|\eta\|_{L^{2,\infty}} \lesssim \|\phi^*\omega_{\mathbb{S}^2}\|_{L^1} \lesssim \|D\phi\|_{L^2}^2 \|\phi\|_{L^\infty} \simeq \|D\phi\|_{L^2}^2 . \tag{3-6}$$

We have mentioned where to find the proof that η corresponds up to a unitary transformation to a lift $\tilde{\phi}$, and from (3-4) and (3-6) we also obtain the estimate for $\tilde{\phi}$,

$$\|D\tilde{\phi}\|_{L^{2,\infty}} \lesssim \|\eta\|_{L^{2,\infty}} + \|D\phi\|_{L^2} \lesssim \|D\phi\|_{L^2}(1+\|D\phi\|_{L^2}). \tag{3-7}$$

Step 2 (extending the constructions to $W^{1,2}$). The results obtained so far hold for $\phi \in C^\infty(\mathbb{C}, \mathbb{S}^2)$. We use the well-known fact that, while not dense in the strong topology, the functions in $C^\infty(\mathbb{C}, \mathbb{S}^2)$ are instead *dense with respect to weak sequential convergence* (see [Bethuel 1991; Hang and Lin 2003]). The constraint of u_n having values in $\mathbb{S}^2$, as well as the constraint $\tilde{\phi}_n \circ H = \phi_n$ for the $\tilde{\phi}_n$, are pointwise constraints (note indeed that the function H is smooth), so they are preserved under weak convergence $\phi_n \rightharpoonup \phi \in W^{1,2}$. Now we state the only less classical point in the proof in the following lemma:

Lemma 3.2. *$L^{2,\infty}$-estimates are preserved under weak convergence in L^2. In other words, if $f_n \in L^2$ are weakly convergent to $f \in L^2$, then $\|f\|_{L^{2,\infty}} \le \liminf_{n\to\infty} \|f_n\|_{L^{2,\infty}}$.*

Proof. We observe that a positive answer to this question cannot directly and trivially be obtained by interpolation, since the L^∞-norm is not lower semicontinuous with respect to weak convergence in L^2. We thus proceed by duality; namely, we note that

$$L^{(2,\infty)} = (L^{(2,1)})' \quad \text{and} \quad L^{(2,1)} \subset L^2.$$

Therefore $\langle f_n, \phi\rangle \to \langle f, \phi\rangle$ for all $\phi \in L^{(2,1)}$, and by usual Banach space theory we obtain the thesis. □

Applying the lemma, we obtain the desired estimate to conclude the proof of Proposition 3.1 via Bethuel's weak density result [1991]. □

We observe that, given a map $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$, we can obtain a map $u : \mathbb{C} \to \mathbb{S}^2$ having the same norm by composing with the inverse stereographic projection $\Psi^{-1} : \mathbb{C} \to \mathbb{S}^2$; we use the facts that the exponent 2 is equal to the dimension and that Ψ is conformal. In a similar way, having constructed a lift $\tilde{u} : \mathbb{C} \to \mathbb{S}^3$, we obtain automatically a lift $\tilde{\phi}$ of ϕ by composing back with S. The same reasoning using conformality also shows that the $L^{2,\infty}$-norm of the gradient of $\tilde{\phi}$ is preserved. This proves:

Corollary 3.3. *Suppose $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$. Then there exists a lifting $\tilde{\phi} : \mathbb{S}^2 \to \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$ and there exists a universal constant C such that*

$$\|\nabla\tilde{\phi}\|_{L^{2,\infty}} \le C\|\nabla\phi\|_{L^2}(1+\|\nabla\phi\|_{L^2}).$$

3C. *Projection and wise choice of the point.* To proceed in our strategy for the proof of Theorem C, we use a version of the projection trick of Section 2A.

Proposition 3.4 (projection trick 2). *Suppose that $\tilde{\phi} \in W^{1,(2,\infty)}(\mathbb{S}^2, \mathbb{S}^3)$. Then there exists a function $\tilde{u} : B^3 \to \mathbb{S}^3$ such that $\tilde{u}|_{\partial B^3 \setminus \mathbb{S}^2} = \tilde{\phi}$ satisfying the following bound for some universal constant C:*

$$\|\tilde{u}\|_{W^{1,(3,\infty)}(B^3)} \le C\|\tilde{\phi}\|_{W^{1,(2,\infty)}(\mathbb{S}^2)}.$$

Proof. We proceed in two steps: the first one introduces the $W^{1,(3,\infty)}$-norm estimate, and the second one ensures that the constraint of having values in $\mathbb{S}^3$ can be preserved.

Step 1 (harmonic extension). Consider a solution $\tilde{u}$ of the equation

$$\begin{cases}\Delta\tilde{u}=0 & \text{on } B^3,\\ \tilde{u}=\tilde{\phi} & \text{on } \partial B^3.\end{cases} \tag{3-8}$$

By using the Poisson kernel estimates, we obtain that $\tilde{u}\in W^{1,(3,\infty)}(B^3,B^4)$ and

$$\|\nabla\tilde{u}\|_{L^{(3,\infty)}}\lesssim\|\nabla\tilde{\phi}\|_{L^{(2,\infty)}}\,. \tag{3-9}$$

Step 2 (projection in the target). We now correct the fact that $\tilde{u}$ has values not in $\mathbb{S}^3$ but in its convex hull B^4. For $a\in B^4_{1/2}$ we define the radial projection $\pi_a:B^4\to\mathbb{S}^3$ of center a, i.e.,

$$\pi_a(x):=a+t_{a,x}(x-a),\quad \text{where } t_{a,x}\geq 0 \text{ is chosen so that } |\pi_a(x)|=1.$$

In order to estimate the norm of $u_a:=\pi_a\circ\tilde{u}$ we note that

$$|\nabla(\pi_a\circ\tilde{u})|(x)\lesssim\frac{|\nabla\tilde{u}(x)|}{|u(x)-a|}$$

with an implicit constant bounded by 4 as long as $a\in B^4_{1/2}$. We just estimate the L^p-norm of ∇u_a for $p\in[1,4[$. We note that $\int_{B_{1/2}}|\tilde{u}(x)-a|^{-p}\,da$ is bounded for all such p by a number C_p independent of x; therefore, by changing the order of integration and applying Fubini, we obtain

$$\int_{B_{1/2}}\int_{B_1}|\nabla u_a(x)|^p\,dx\,da\leq C_p\int_{B_1}|\nabla\tilde{u}(x)|^p\int_{B_{1/2}}|\tilde{u}(x)-a|^{-p}\,da\leq C_p\|\nabla\tilde{u}\|_p^p\,.$$

In other words, the assignment $a\mapsto u_a$ gives a map whose $L^1_a(B_{1/2},W^{1,p}_x(B^3,\mathbb{S}^3))$-norm is bounded by the L^p-norm of $\nabla\tilde{u}$ for $p\in[1,4[$. First observe that, by Lions–Peetre reiteration (see [Tartar 2007, Chapter 26]), $L^{(3,\infty)}$ is an interpolation between L^{p_0} and L^{p_1} with $3\in]p_0,p_1[\subset]1,4[$. We now use the nonlinear interpolation theorem of Tartar [2007, Chapter 28]. Call $U(a,x):=\nabla\tilde{u}(x)/|\tilde{u}(x)-a|$. We know that the map $u\mapsto U$ is bounded between W^{1,p_i} and L^{p_i} for $i=0,1$. In order to show that it also satisfies

$$\sup_{\lambda>0}\lambda^3\left|\left\{(x,a)\in B_1\times B_{1/2}:\frac{|\nabla u(x)|}{|u(x)-a|}>\lambda\right\}\right|=\|U\|^3_{L^{(3,\infty)}}\lesssim\|\tilde{u}\|^3_{W^{1,(3,\infty)}}\,, \tag{3-10}$$

we will check the local estimate

$$\left\|\frac{\nabla u(x)}{|u(x)-a|}-\frac{\nabla v(x)}{|v(x)-a|}\right\|_{L^{p_1}}\lesssim\|u-v\|_{L^{p_1}}\,.$$

This follows since

$$\int_{B_1}\int_{B_{1/2}}\left|\frac{\nabla u(x)}{|u(x)-a|}-\frac{\nabla v(x)}{|v(x)-a|}\right|^{p_1}\lesssim\int_{B_1}|\nabla u-\nabla v|^{p_1}\int_{B_{1/2}}\big(|u(x)-a|^{-p_1}+|v(x)-a|^{-p_1}\big)\,da\,dx$$

and the same estimates as before apply to the second factor, uniformly in x. Thus (3-10) holds. From (3-10) it easily follows that there exists $a \in B_{1/2}$ for which

$$\|\nabla u_a\|_{L^{(3,\infty)}(B_1)} \lesssim \|\tilde{u}\|_{W^{1,(3,\infty)}} . \tag{3-11}$$

Combining (3-9) and (3-11), we obtain the claim of the proposition for $\hat{u} := u_a$. □

3D. ***End of proof.***

Proof of Theorem C. Apply consecutively Corollary 3.3 and Proposition 3.4. For $\hat{u}$ as in Proposition 3.4, we can then consider $u := H \circ u_a : B^3 \to \mathbb{S}^2$. Since H is Lipschitz, we obtain the pointwise estimate

$$|\nabla u| \lesssim |\nabla u_a|. \tag{3-12}$$

Combining this with the estimates of Corollary 3.3 and Proposition 3.4, we obtain the thesis. □

3E. ***Modification of proof in the case of $W^{1,p}(\mathbb{S}^m, \mathbb{S}^2)$.*** In this section we prove Theorem D and Proposition 1.7.

Proof of Theorem D and of Proposition 1.7. We consider $n = 2 < m$ and $3m/(m+1) \le p < 4m/(m+1)$ as in Proposition 1.7. We will use the fact that such p is always greater than 2. The construction of the 1-form η satisfying (3-3) and (3-4) can be done in a completely analogous way if the domain is $\mathbb{R}^m$, $m \ge 3$. The only difference is that in that case the Laplacian on 2-forms such as $\phi^*\omega_{\mathbb{S}^2}$ has the form $\delta = d^*d + dd^*$, where the first part does not vanish anymore. In this case however we may still solve

$$\begin{cases} d\eta = \phi^*\omega_{\mathbb{S}^2}, \\ d^*\eta = 0, \\ \eta(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

If $\phi \in W^{1,p}(\mathbb{R}^m, \mathbb{S}^2)$ and since $p > 2$, we then have

$$\|d\eta\|_{L^{p/2}(\mathbb{R}^m)} \le C\|\phi^*\omega_{\mathbb{S}^2}\|_{L^{p/2}(\mathbb{R}^m)} \le C\|d\phi\|^2_{L^p(\mathbb{R}^m)}.$$

As before, we have (3-4), from which we also obtain $|D\tilde{\phi}|^p \lesssim |\eta|^p + |D\phi|^p$. Passing to $\mathbb{S}^m$ and noting that in dimension $m \ge p$ we have $W^{1,p/2}(\mathbb{S}^m, \mathbb{S}^2) \hookrightarrow L^{mp/(2m-p)}(\mathbb{S}^m, \mathbb{S}^2) \hookrightarrow L^p(\mathbb{S}^m, \mathbb{S}^2)$, we obtain

$$\|D\tilde{\phi}\|_{L^p(\mathbb{S}^m,\mathbb{S}^2)} \lesssim \|D\phi\|^2_{L^p(\mathbb{S}^m,\mathbb{S}^2)} + \|D\phi\|_{L^p(\mathbb{S}^m,\mathbb{S}^2)}.$$

Harmonic extension and Proposition 2.2 allow us then to obtain an extension $\tilde{u} : B^{m+1} \to \mathbb{S}^2$ of $\tilde{\phi}$ such that

$$\|\nabla\tilde{u}\|_{L^{p(m+1)/m}(B^{m+1},\mathbb{S}^3)} \lesssim \|D\tilde{\phi}\|_{L^p(\mathbb{S}^m,\mathbb{S}^3)}$$

provided $p(m+1)/m < 4$ (which is the condition appearing in Proposition 2.2. Composing with the Hopf map H at most decreases the norm; thus we obtain that $u := H \circ \tilde{u}$ is the desired controlled extension as in Proposition 1.7 and in Theorem D (note that for $m = 3$ the condition $p(m+1)/m < 4$ is equivalent to $p < 3$). □

4. The extension theorem for $W^{1,3}$ maps $\mathbb{S}^3 \to \mathbb{S}^3$

This section is devoted to the proof of the following theorem:

Theorem B″. *There is a constant $C > 0$ such that, if $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$, then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \le C(e^{C\|\nabla\phi\|_{L^3}^9} + \|\nabla\phi\|_{L^3}). \tag{4-1}$$

4A. ***Modulus of integrability estimates.*** In general, during our estimates we indicate by C a positive constant, which may change from line to line and also within the same line. We start by fixing the notation for the main quantity which will be used control the energy concentration of our maps.

Definition 4.1. If $D \subset \mathbb{R}^4$ and $f : D \to \mathbb{R}$ is measurable then let $E(f, \rho, D)$ denote the (possibly infinite) modulus of integrability of f, which is defined as

$$E(f, \rho, D) = \sup_{x\in D} \int_{B_\rho(x)\cap D} |f|.$$

The modulus of integrability fits into a sort of elliptic estimate as follows:

Proposition 4.2 (integrability modulus estimates). *Let $\phi \in W^{1,3}(\partial B^4, \mathbb{S}^3)$ and assume that u is the solution to the equation*

$$\begin{cases} \Delta u = 0 & \text{on } B^4, \\ u = \phi & \text{on } \partial B^4. \end{cases}$$

Then there exists a constant C_1 independent of ϕ, ρ such that, when $\rho \in \left]0, \tfrac{1}{4}\right[$,

$$E(|\nabla u|^4, \rho, B^4) \le C_1 E(|\nabla\phi|^3, 2\rho, \partial B^4)^{1/3} \int_{\partial B^4} |\nabla\phi|^3. \tag{4-2}$$

Proof. We have to prove that, for all $x_0 \in B^4$,

$$\int_{B_\rho(x_0)\cap B^4} |\nabla u|^4 \le C_1 E(|\nabla\phi|^3, 2\rho, \partial B^4) \int_{\partial B^4} |\nabla\phi|^3. \tag{4-3}$$

Step 1 (the case $x_0 \in \partial B^4$). Let $\eta : \mathbb{S}^3 \to [0, 1]$ be a cutoff function such that $\eta \equiv 1$ on $B_{2\rho}(x_0) \cap \mathbb{S}^3$, $\eta \equiv 0$ on $\mathbb{S}^3 \setminus B_{4\rho}(x_0)$, and $|\nabla\eta| \lesssim \rho^{-1}$. Then write $\phi = \phi_1 + \phi_2$ with $\phi_1 = \eta\phi$, $\eta_2 = (1-\eta)\phi$, and let $u = u_1 + u_2$ with

$$\begin{cases} \Delta u_i = 0 & \text{on } B^4, \\ u_i = \phi_i & \text{on } \partial B^4 \end{cases}$$

for $i = 1, 2$. It suffices to prove (4-3) for each u_i separately. By elliptic theory and by the definition of η,

$$\int_{B_\rho(x_0)\cap B^4} |\nabla u_1|^4 \lesssim \left(\int_{S'} |\nabla\phi|^3 \right)^{4/3}.$$

Poisson's formula gives

$$u_2(x) = C(1 - |x|^2) \int_{\partial B^4} \frac{\phi_2(y)}{|x-y|^4}\, dy;$$

thus, for $x \in B_\rho(x_0) \cap B^4$, $\rho < \frac{1}{4}$,

$$|\nabla u_2|(x) \lesssim \rho \int_{\mathbb{S}^3 \setminus B_{2\rho}(x_0)} \frac{|\nabla \phi|}{|x-y|^4}\, dy + \int_{\mathbb{S}^3 \setminus B_{2\rho}(x_0)} \frac{|\phi|}{|x-y|^4}\, dy \lesssim \rho \int_{\mathbb{S}^3 \setminus B_{2\rho}(x_0)} \frac{|\nabla \phi|}{|x-y|^4}\, dy.$$

Patching together the estimates obtained so far, we write

$$\int_{B_\rho(x_0)\cap B^4} |\nabla u|^4 \lesssim \left(\int_{S'} |\nabla\phi|^3\right)^{4/3} + \rho^8 \left(\int_{S''} \frac{|\nabla \phi|}{|x-y|^4}\right)^4, \tag{4-4}$$

where the factor ρ^8 comes from the pointwise estimate for ∇u_2, keeping in mind that $|B_\rho(x_0)\cap B^4| \lesssim \rho^4$. Let the summands on the right side of (4-4) be I and II respectively. Note that

$$I \le \left(\int_{B_{2\rho}(x_0)\cap \partial B^4} |\nabla\phi|^3\right)^{1/3} \int_{\mathbb{S}^3} |\nabla\phi|^3 \le E(|\nabla\phi|^3, 2\rho, \partial B^4) \int_{\mathbb{S}^3} |\nabla\phi|^3. \tag{4-5}$$

To estimate II, cover $\mathbb{S}^3 \setminus B_{2\rho}(x_0)$ by (finitely many) geodesic balls $B^3_{2\rho}(x_i)$ so that x_i form a maximal 2ρ-net and they are at distance at least 2ρ from x_0. Then

$$\int_{B^3_{2\rho}(x_i)} |\nabla\phi| \le |B^3_{2\rho}| \left(\fint_{B^3_{2\rho}(x_i)} |\nabla\phi|^3\right)^{1/3}.$$

For $y \in B^3_{2\rho}(x_i)$, $x \in B_{2\rho}(x_0)\cap B^4$, we have $|x-y| \sim \operatorname{dist}(x_i, x_0)$. Thus

$$II \lesssim \rho^8 \left(\sum_i \operatorname{dist}^{-4}(x_i, x_0)\rho^3 a_i^{1/3}\right)^4,$$

where $a_i = \fint_{B^3_{2\rho}(x_i)} |\nabla\phi|^3$. By Hölder's inequality we easily obtain

$$II \lesssim \rho^{20} \big(\sup_i a_i^{1/3}\big) \left(\sum_i a_i\right) \left(\sum_i \operatorname{dist}^{-16/3}(x_i, x_0)\right)^3.$$

Now, the first parenthesis is estimated by $\rho^{-1} E(|\nabla\phi|^3, 2\rho, \partial B^4)^{1/3}$, the second one by $\rho^{-3} \int_{\mathbb{S}^3} |\nabla\phi|^3$, and the last one by $\rho^{-16/3}$. Thus we obtain

$$II \lesssim \rho^{20}\rho^{-1} E(|\nabla\phi|^3, 2\rho, \partial B^4)^{1/3} \rho^{-16}\rho^{-3} \int_{\mathbb{S}^3} |\nabla\phi|^3 \lesssim E(|\nabla\phi|^3, 2\rho, \partial B^4)^{1/3} \int_{\mathbb{S}^3} |\nabla\phi|^3. \tag{4-6}$$

By (4-5) and (4-6), we obtain (4-3) for $x_0 \in \partial B^4$.

Step 2. If $|x_0| < 1 - 2\rho$ then we can directly apply the estimates for the term II of (4-4), since now the denominator $|x-y|$ in the Poisson formula will be at least ρ for all $x \in B_\rho(x_0)$.

The estimate of Step 1 also holds for $\rho > \frac{1}{4}$ with the same constant. We can cover the case $|x_0| \in\,]1-2\rho, 1[$ with $\rho < \frac{1}{4}$ by noticing that if $x_0' = x_0/|x_0|$ then $B_{3\rho}(x_0') \supset B_\rho(x_0)$, and that the measures $|\nabla\phi|^3 d\sigma$, $|\nabla u|^4 dx$ are doubling with constants bounded by the packing constants of $\mathbb{S}^3$ and of B^4 respectively, while the function $E(f, \rho, D)$ is increasing in ρ. Therefore the inequality (4-3) also holds for this last choice of x_0 up to changing C_0 by a factor depending only on the above packing constants. □

4B. ***Extension in the case of small energy concentration.*** In small energy concentration regions we utilize the following:

Theorem 4.3 (small concentration extension). *There exists a constant $\delta \in \left]0, \frac{1}{4}\right[$ with the following property: for each $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ such that the local estimate*

$$E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3) \leq \frac{\delta}{C_1 E} \tag{4-7}$$

holds with $\|\nabla\phi\|^3_{L^3(\mathbb{S}^3)} = E$, there exists a function $\tilde{u} \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ which equals ϕ on $\mathbb{S}^3$ in the sense of traces and satisfies

$$\|\nabla\tilde{u}\|_{L^{4,\infty}} \lesssim \frac{\|\nabla\phi\|^2_{L^3}}{\rho} + \|\nabla\phi\|_{L^3}\,. \tag{4-8}$$

Theorem 4.3 follows from several ingredients, the proofs of which are postponed to Appendix A and to the end of Section 4B.

Theorem 4.4 (Uhlenbeck analogue). *There exist two constants $\delta > 0$, $C > 0$ with the following property: Suppose $\psi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ is such that $\|\nabla\psi\|_{L^3(\mathbb{S}^3)} \leq \delta$. Then there exists an extension $v \in W^{1,4}(B^4, \mathbb{S}^3)$ satisfying the estimate*

$$\|v\|_{W^{1,4}(B^4)} \leq C\|\nabla\psi\|_{L^3(\mathbb{S}^3)}.$$

Proof. See Theorem A.2. □

If $u \in W^{1,4}(B^4, \mathbb{R}^4)$ and $\rho \in \left]0, \frac{1}{2}\right[$, $x_0 \in \partial B^4$, then by a mean value argument there exists $\bar{\rho} \in [\rho, 2\rho]$ such that

$$\bar{\rho}\int_{\mathrm{int}(B^4)\cap\partial B_{\bar{\rho}}(x_0)} |\nabla u|^4 \leq C\int_{B^4\cap B_\rho(x_0)} |\nabla u|^4. \tag{4-9}$$

In this case the following lemma will prove useful:

Lemma 4.5 (Courant–Lebesgue analogue). *Fix $\bar{\rho} \in]0, 1[$. There exists a constant $C > 0$ such that, if $u \in W^{1,4}(B^4, \mathbb{R}^4)$ is the extension of $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ and*

$$\bar{\rho}\int_{\mathrm{int}(B^4)\cap\partial B_{\bar{\rho}}(x_0)} |\nabla u|^4 \leq C$$

with $x_0 \in \partial B^4$, then for almost every $x \in \partial(B^4 \cap B_{\bar{\rho}}(x_0))$ we have

$$\mathrm{dist}(u(x), \mathbb{S}^3) \leq \tfrac{1}{8}\,. \tag{4-10}$$

The restriction of u to a smaller ball $B_{1-\rho}$, being harmonic, is smooth. Then we may utilize the following result:

Lemma 4.6 (interior estimate). *Given $u \in W^{1,4} \cap C^1(B^4, B^4)$, there exists a constant C independent of u such that, for half of the points $a \in B^4$,*

$$\left\|\frac{1}{|u-a|}\right\|^4_{L^{4,\infty}(B^4)} \leq C\int_{B^4} |\nabla u|^4.$$

Proof of Theorem 4.3. **Step 1.** We first observe that the harmonic extension u of ϕ satisfies

$$|\nabla u|(x) \lesssim \frac{\|\phi\|_{W^{1,3}(\mathbb{S}^3)}}{\rho} \quad \text{for } x \in B_{1-\rho}\,.$$

A direct way to see this is by estimating via the Poisson formula together with Poincaré's inequality and a good covering by ρ-balls $B_j \subset \mathbb{S}^3$:

$$\begin{aligned}
|\nabla u|(x) &\lesssim \rho\biggl(\int_{\mathbb{S}^3} \frac{\nabla\phi}{|x-y|^4}\,dy + \int_{\mathbb{S}^3} \frac{|\phi|}{|x-y|^4}\,dy\biggr) \\
&\lesssim \sum_j \frac{\fint_{B_j} |\nabla\phi| + |\phi|}{d_j^4}\rho^4, && \text{where } d_j \sim \operatorname{dist}(B_j, x) \\
&\lesssim \sum_j \Bigl(\frac{\rho}{d_j}\Bigr)^4 \fint_{B_j} |\nabla\phi| + 1, && \text{by Poincaré} \\
&\lesssim \biggl(\sum_j \Bigl(\frac{\rho}{d_j}\Bigr)^6\biggr)^{2/3} \biggl(\sum_j \Bigl(\fint_{B_j} |\nabla\phi|\Bigr)^3 + 1\biggr)^{1/3}, && \text{by Hölder} \\
&\lesssim \frac{\|\phi\|_{W^{1,3}(\mathbb{S}^3)}}{\rho}\,.
\end{aligned}$$

To justify the last step we observe that $\operatorname{Card}\{j : d_j \sim 2^j\rho\} \sim 2^{4j}$ and thus the first factor in the penultimate line is bounded by $\bigl(\sum_{j\geq 0} 2^{-2j}\bigr)^{2/3}$, while for the second factor we use Jensen's inequality.

Step 2. We now use Lemma 4.6 and observe that if $\pi_a : B^4 \setminus \{a\} \to \mathbb{S}^3$ is the retraction of center a then

$$|\nabla(\pi_a \circ u)| \leq C\frac{|\nabla u|}{|u-a|}\,.$$

In particular, using Step 1 and Lemma 4.6 we obtain

$$\|\nabla(\pi_a \circ u)\|_{L^{4,\infty}} \leq \|\nabla u\|_{L^\infty} \left\|\frac{1}{|u-a|}\right\|_{L^{4,\infty}} \leq C\frac{\|\nabla\phi\|_{L^3}}{\rho}\|\nabla u\|_{L^4}\,. \tag{4-11}$$

Step 3. Consider a maximal cover $\{B_i\}$ of $\mathbb{S}^3 = \partial B^4$ by 4-dimensional balls of radius ρ and centers on ∂B^4. It is possible to find a constant C, depending only on the dimension, such that the collection of balls of doubled radius $\{2B_i\}$ can be written as a union of C families of disjoint balls $\mathscr{F}_1, \dots, \mathscr{F}_C$.

Then apply (4-9) to each ball $B_i \in \mathscr{F}_1$. This will give a new family of balls $\{B_i' : B_i \in \mathscr{F}_1\}$ with radii between ρ and 2ρ to which it will be possible to apply Lemma 4.5. Thus $\operatorname{dist}(u(x), \partial B^4) < \frac{1}{8}$ on $\partial(B^4 \cap B_i')$ for all B_i'. Because of the choice of $\mathscr{F}_1$ it also follows that the balls B_i' are disjoint.

If we choose a projection π_a from Step 2 so that $\operatorname{dist}(a, \partial B^4) > \frac{1}{4}$, then

$$u_1^i := \pi_a \circ (u|_{\partial((B^4 \cap B_i'))}) \quad \text{satisfies} \quad |\nabla u_1^i| \leq C|\nabla u| \quad \text{on } \partial B_i' \cap B^4$$

by the estimates of Step 2. Note that a will be fixed during the whole construction.

We extend u_1^i (denoting the extension again by u_1^i) inside $B_i' \cap B^4$ via Theorem 4.4, obtaining a new function

$$u_1 := \begin{cases} \pi_a \circ u & \text{on } B^4 \setminus \bigcup B_i', \\ u_1^i & \text{on } B_i'. \end{cases}$$

Theorem 4.4 implies that u_1 satisfies

$$\|\nabla u_1\|_{L^4(B_i')} \le C\biggl(\int_{\partial B_i'} |\nabla u_1|^3\biggr)^{1/3}.$$

We can rewrite this as follows:

$$\begin{aligned}\int_{B_i \cap B^4} |\nabla u_1|^4 &\le C\biggl(\int_{B_i \cap \partial B} |\nabla \phi|^3 + \int_{\mathrm{int}(B)\cap \partial B_i} |\nabla u_1^i|^3\biggr)^{4/3} \\ &\lesssim \biggl(\int_{B_i \cap \partial B} |\nabla \phi|^3\biggr)^{4/3} + \biggl(\int_{\mathrm{int}(B)\cap \partial B_i} |\nabla u_1^i|^3\biggr)^{4/3}. \end{aligned} \tag{4-12}$$

We note that (using Lemma 4.5)

$$\begin{aligned}\biggl(\int_{\partial B_i \cap \mathrm{int}(B)} |\nabla u_1^i|^3\biggr)^{4/3} &\le \mathcal{H}^3(\partial B_i)^{1/3} \int_{\partial B_i \cap \mathrm{int}(B)} |\nabla u_1^i|^4 \\ &\lesssim \rho \int_{\partial B_i \cap \mathrm{int}(B)} |\nabla u|^4 \\ &\lesssim \int_{B_i \cap B^4} |\nabla u|^4; \end{aligned} \tag{4-13}$$

therefore, u_1 still satisfies (4-2) with a constant C_1 which is now changed by a universal factor.

Step 4. It is possible to repeat the same operation starting from the function u_1 and using the balls of the family $\mathcal{F}_2$ to obtain a function u_2, and then do the same iteratively for all the families $\mathcal{F}_2, \dots, \mathcal{F}_C$.

Denote by $\mathcal{R}$ the union of all the perturbed balls B_i' corresponding to the families $\mathcal{F}_1, \dots, \mathcal{F}_C$. Recall that the number of families is equal to the maximal number of overlaps of balls of different families and depends only on the dimension. Then, iterating the estimates (4-12) using (4-13) for all families $\mathcal{F}_i$, we obtain for the last function u_C that

$$\begin{aligned}\int_{\mathcal{R}} |\nabla u_C|^4 &\lesssim E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3)^{1/3} \sum_i \int_{B_i \cap \partial B} |\nabla \phi|^3 + \int_{\mathcal{R}} |\nabla u|^4 \\ &\le \|\nabla\phi\|^3_{L^3(\mathbb{S}^3)}\bigl(E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3)^{1/3} + \|\nabla\phi\|_{L^3(\mathbb{S}^3)}\bigr), \end{aligned} \tag{4-14}$$

where for the last inequality we also used the elliptic estimates for u in terms of ϕ.

Step 5. We now combine the estimate (4-11) for the part $B \setminus \mathcal{R} \subset B_{1-\rho}$ and (4-14). Observe that in general $\|f\|_{L^{4,\infty}} \lesssim \|f\|_{L^4}$ and that the $L^{4,\infty}$-norm satisfies the triangle inequality. We obtain

$$\|\nabla \tilde{u}\|_{L^{4,\infty}} \lesssim \frac{\|\nabla\phi\|^2_{L^3}}{\rho} + \|\nabla\phi\|_{L^3} + \|\nabla\phi\|^{3/4}_{L^3} E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3)^{1/12}. \tag{4-15}$$

Using the trivial estimate $E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3) \le \int_{\mathbb{S}^3} |\nabla\phi|^3$, the desired estimate follows. $\square$

We now proceed to prove the above lemmas.

Proof of Lemma 4.5. The hypotheses $x_0 \in \partial B^4$, $\bar\rho < 1$ have the following two geometric consequences: (1) $\partial B^4 \cap \partial B_{\bar\rho}(x_0)$ has positive measure; (2) $B^4 \cap B_{\bar\rho}(x_0)$ is 2-bi-Lipschitz equivalent to $B_{\bar\rho}$. Therefore, we may just prove that (4-10) holds true on ∂B_ρ for a function u such that

$$\begin{cases} \bar\rho \int_{\partial B_{\bar\rho}} |\nabla u|^4 < C, \\ |\{x : |u|(x) = 1\}| > 0. \end{cases}$$

To do this note that, by definition, $u(x) \in \mathbb{S}^3$ for a.e. $x \in \partial B^4$, then use the Sobolev inequality

$$\|u\|^4_{C^{0,1/4}(\partial B_{\bar\rho})} \lesssim \bar\rho \int_{\partial B_{\bar\rho}} |\nabla u|^4,$$

which is valid in dimension 3. For C small enough we obtain (4-10). □

Proof of Lemma 4.6. By the coarea formula we have

$$|\{x : |u(x) - a|^{-1} > \Lambda\}| = |u^{-1}(B_{\Lambda^{-1}}(a))| = \int_{B_{\Lambda^{-1}}(a)} \mathrm{Card}(u^{-1}(x))\, dx \le C \int_{B^4} |\nabla u|^4.$$

We then observe that the measurable positive function $F_u(x) := \mathrm{Card}(u^{-1}(x))$ belongs to $L^1(B^4)$. The maximal function MF_u has $L^{1,\infty}$-norm bounded by the L^1-norm of F_u, and in particular there exists a constant C independent of u such that for at least half of the points $a \in B^4$ we have

$$\sup_\lambda \frac{1}{\lambda^4} \int_{B_\lambda(a)} F_u \le C \int_{B^4} F_u \le C \int_{B^4} |\nabla u|^4.$$

For such a we have, after the change of notation $\lambda = \Lambda^{-1}$, the desired estimate

$$|\{x : |u(x) - a|^{-1} > \Lambda\}|\Lambda^4 \le C \int_{B^4} |\nabla u|^4. \qquad \square$$

4C. ***The case of large energy concentration.*** Following Theorem 4.3, we are led to divide the set of boundary value functions $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ into two classes, based on whether or not the energy concentrates. Let $L_E := \{\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3) : \|\nabla\phi\|^3_{L^3} \le E\}$ and for $\phi \in L_E$ define $E_\phi := E(|\nabla\phi|^3, \rho_E, \mathbb{S}^3)$. We distinguish between the following two classes of "good" and "bad" boundary value functions:

$$\begin{aligned} \mathcal{G}^E &:= L_E \cap \{\phi : E_\phi \le \bar\delta\}, \\ \mathcal{B}^E &:= L_E \cap \{\phi : E_\phi > \bar\delta\}. \end{aligned} \tag{4-16}$$

We will fix the constants

$$\rho_E = e^{-C \max\{1, E^3\}} \quad \text{and} \quad \bar\delta$$

at the end of Section 4D.

The precise steps of our extension construction are as follows (see also the graphical summary (4-17)):

(1) Theorem 4.3 gives a good estimate for the boundary values in $\mathcal{G}^E$.

(2) If $\phi \in \mathcal{B}^E$ has average close to zero, i.e.,

$$\left|\int_{\mathbb{S}^3} \phi\right| \le \frac{1}{4},$$

then it is possible to write $\phi = \phi_1\phi_2$ with

$$\int_{\mathbb{S}^3} |\nabla\phi_i|^3 \le E - \frac{\bar{\delta}}{2}$$

(the product of $\mathbb{S}^3$-valued functions is pointwise the product on $\mathbb{S}^3 \simeq \mathrm{SU}(2)$).

(3) If we are not in the two cases above, we use the functions

$$F_v(x) := -v + (1-|v|^2)(x^* - v)^*,$$

where $a^* = a/|a|^2$, $v \in B^4$, which form a subset of the Möbius group of B^4. We have two cases:

(a) For all $v \in B^4$, we have $\left|\int_{\mathbb{S}^3} \phi \circ F_v\right| > \frac{1}{4}$, in which case

$$\tilde{u}(v) := \pi_{\mathbb{S}^3}\left(\int_{\mathbb{S}^3} \phi \circ F_v\right)$$

gives an extension of ϕ with values in $\mathbb{S}^3$ that satisfies

$$\|u\|_{W^{1,4}} \lesssim \|\phi\|_{W^{1,3}}.$$

(b) There exists $v \in B^4$ such that $\left|\int_{\mathbb{S}^3} \phi \circ F_v\right| \le \frac{1}{4}$, in which case we can apply the reasoning of cases (1), (2) above to $\tilde{\phi} := \phi \circ F_v$. Since F_v is conformal and $|\phi| = |\tilde{\phi}| = 1$, we have

$$\|\nabla\phi\|_{L^3} = \|\nabla\tilde{\phi}\|_{L^3}, \quad \|\phi\|_{W^{1,3}} = \|\tilde{\phi}\|_{W^{1,3}}.$$

Again we reason differently in the two cases $\tilde{\phi} \in \mathcal{G}^E$ and $\tilde{\phi} \in \mathcal{B}^E$:

(4) If, in case (3b), $\tilde{\phi} \in \mathcal{B}^E$, then we apply case (2) to $\tilde{\phi}$ and we can express

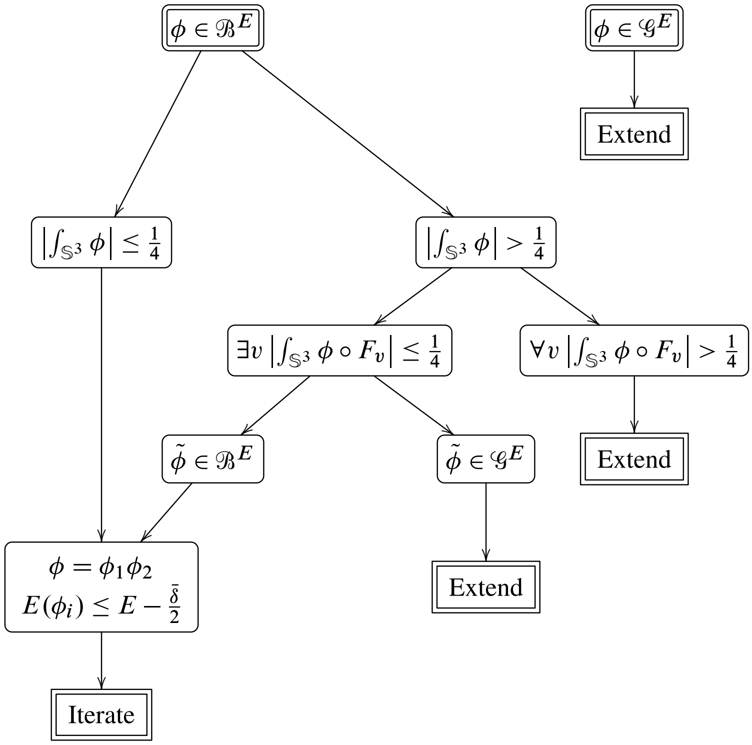
$$\tilde{\phi} = \tilde{\phi}_1\tilde{\phi}_2 \quad \text{and} \quad \phi = (\tilde{\phi}_1 \circ F_v^{-1})(\tilde{\phi}_2 \circ F_v^{-1}).$$

Then $\phi_i := \tilde{\phi}_i \circ F_v^{-1}$ are as in case (2).

(5) If, in case (3b), $\tilde{\phi} \in \mathcal{G}^E$, then we apply case (1) to $\tilde{\phi}$. With a careful study of the relation between the position of $v \in B^4$ relative to ∂B^4 and the parameter ρ_E, we construct

$$u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3) \quad \text{extending} \quad \phi = \tilde{\phi} \circ F_v^{-1},$$

starting from the extension $\tilde{u}$ of $\tilde{\phi}$ given in case (1).



(4-17)

Proposition 4.7 (balancing $\Rightarrow$ splitting). *There exists a constant C with the following property: Suppose that $\phi \in \mathcal{B}^E$ with the notation of (4-16), and assume $\bar{\delta} \le C$ and $\rho_E \le e^{-C\max\{1,E^3\}}$. Further assume that, as a function in $W^{1,3}(\mathbb{S}^3, \mathbb{R}^4)$, ϕ satisfies*

$$\left|\fint_{\mathbb{S}^3} \phi\right| \le \frac{1}{4}.$$

Then, identifying $\mathbb{S}^3 \sim \mathrm{SU}(2)$, there exists a decomposition

$$\phi = \phi_1\phi_2 \tag{4-18}$$

such that, for $i = 1, 2$,

$$\int_{\mathbb{S}^3} |\nabla\phi_i|^3 < E - \frac{\bar{\delta}}{2}. \tag{4-19}$$

Proof. **Step 1.** Fix a concentration ball $B = B^{\mathbb{S}^3}(\rho_E, x_0)$ such that

$$\int_B |\nabla\phi|^3 > \bar{\delta}. \tag{4-20}$$

Step 2. Consider dyadic rings in $\mathbb{S}^3$ defined as $R_i := 2^{i+1}B \setminus 2^iB$, where we denote $2^iB = B^{\mathbb{S}^3}(2^i\rho_E, x_0)$. For an easily computed constant C we can fix $N_E = C|\log_2 \rho_E|$ such that, for $i \le N_E$, it follows that $2^{i+1}\rho_E < \frac{1}{4}$. Since

$$\sum_{i=1}^{N_E} \int_{R_i} |\nabla\phi|^3 < E,$$

by the pigeonhole principle there exists $i_0 \in \{1, \dots, N_E\}$ such that

$$\int_{R_{i_0}} |\nabla\phi|^3 < \frac{E}{N_E}\,.$$

Again by the pigeonhole principle (using the fact that the cubes are dyadic), there therefore exists $t \in [2^{i_0+1}\rho_E, 2^{i_0}\rho_E]$ such that

$$t\int_{\partial B^{\mathbb{S}^3}(t,x_0)} |\nabla\phi|^3 < C\frac{E}{N_E}\,, \tag{4-21}$$

where C is a constant depending only on the geometry of $\mathbb{S}^3$.

Step 3. Denote $B_t = B^{\mathbb{S}^3}(t, x_0)$ as in Step 2. We define the function $\tilde{\phi}_1$ via a suitable harmonic extension outside of B_t by

$$\begin{cases} \tilde{\phi}_1 = \phi & \text{on } \partial B_t\,, \\ \Delta(\tilde{\phi}_1 \circ \Psi) = 0 & \text{on } B_1^{\mathbb{R}^3}, \end{cases}$$

where $\Psi : \mathbb{R}^3 \to \mathbb{S}^3 \setminus \{x_0\}$ is a stereographic projection composed with a dilation of $\mathbb{R}^3$ such that $\Psi(B^{\mathbb{R}^3}(1,0)) = \mathbb{S}^3 \setminus B_t$. On B_t we define $\tilde{\phi}_1 \equiv \phi$. By Hölder's inequality, using elliptic estimates and the conformality of dilations and inverse stereographic projections, we have

$$\begin{aligned} t\int_{\partial B_t} |\nabla\tilde{\phi}_1|^3 \ge C\Big(\int_{\partial B_t} |\nabla\tilde{\phi}_1|^2\Big)^{3/2} = C\Big(\int_{\partial B_1^{\mathbb{R}^3}} |\nabla\tilde{\phi}_1 \circ \Psi|^2\Big)^{3/2} &\ge C\int_{B_1^{\mathbb{R}^3}} |\nabla\tilde{\phi}_1 \circ \Psi|^3 \\ &= C\int_{\mathbb{S}^3\setminus B_t} |\nabla\tilde{\phi}_1|^3. \end{aligned} \tag{4-22}$$

Step 4. We define

$$\phi_1 = \pi_{\mathbb{S}^3} \circ \tilde{\phi}_1.$$

As in Lemma 4.5, there exists a universal constant C such that if

$$\frac{E}{N_E} \le C \tag{4-23}$$

then

$$\operatorname{dist}(\tilde{\phi}_1, \mathbb{S}^3) \le \tfrac{1}{2}.$$

This implies, like in Theorem 4.3, that pointwise a.e. we have the estimate

$$|\nabla\phi_1| \le C|\nabla\tilde{\phi}_1|.$$

By (4-23) it follows that, extending via $\phi_1 = \phi$ on B_t, we obtain $\phi_1 \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ such that

$$\int_{\mathbb{S}^3\setminus B_t} |\nabla\phi_1|^3 \le C\frac{E}{N_E}\,. \tag{4-24}$$

Step 5. We now estimate from below the energy of $\phi|_{\mathbb{S}^3\setminus B_t}$. Denote by $\bar\phi_\Omega$ the average of ϕ on a domain $\Omega\subset\mathbb{S}^3$. First we use the Poincaré inequality on $\mathbb{S}^3\setminus B_t$ and the fact that $|\phi|\equiv 1$ almost everywhere:

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3 \gtrsim \int_{\mathbb{S}^3\setminus B_t}|\phi-\bar\phi_{\mathbb{S}^3\setminus B_t}|^3 \gtrsim \left(\int_{\mathbb{S}^3\setminus B_t}|\phi-\bar\phi_{\mathbb{S}^3\setminus B_t}|\right)^3 \gtrsim \left(|\mathbb{S}^3\setminus B_t|(1-|\bar\phi_{\mathbb{S}^3\setminus B_t}|)\right)^3. \tag{4-25}$$

Using the fact that $|\bar\phi_{\mathbb{S}^3}|\le\frac14$, $|\bar\phi_{B_t}|\le 1$ and the triangle inequality, we have

$$|\mathbb{S}^3\setminus B_t|\,|\bar\phi_{\mathbb{S}^3\setminus B_t}|\le|\bar\phi_{\mathbb{S}^3}|\,|\mathbb{S}^3|+|B_t|\,|\bar\phi_{B_t}|\le\tfrac14|\mathbb{S}^3|+|B_t|. \tag{4-26}$$

Now, (4-25) and (4-26) and the estimate $t<\frac14$ from Step 2 give

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3\gtrsim\left(\tfrac34|\mathbb{S}^3|-2|B_t|\right)^3\ge C. \tag{4-27}$$

From this inequality, if $\bar\delta$ is small enough then we obtain

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3\ge\bar\delta. \tag{4-28}$$

Step 6. We now define $\phi_2:=\phi_1^{-1}\phi$, where the pointwise product uses the group operation on $\mathbb{S}^3\sim SU(2)$. Observe that, since $|\phi|=|\phi_1|=1$ a.e.,

$$|\nabla(\phi_1^{-1}\phi)|=|\phi^{-1}\nabla\phi_1\phi_1^{-1}\phi+\phi_1^{-1}\nabla\phi|\le|\nabla\phi|+|\nabla\phi_1|.$$

Thus, if $C/N_E<1$ in (4-24) (i.e., if $\rho_E=e^{-CN_E}$ is small enough), then

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_2|^3\le\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3+7\left(\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_1|^3\right)^{1/3}\left(\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3\right)^{2/3}.$$

By using (4-28), (4-24) and (4-20) we then obtain

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_2|^3\le\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3+C\frac{E}{N_E^{1/3}}\le E-\bar\delta+C\frac{E}{N_E^{1/3}}. \tag{4-29}$$

Step 7. It is now possible to conclude the proof. The estimate (4-19) for ϕ_2 follows from (4-29) if

$$N_E\ge CE^3\bar\delta^3. \tag{4-30}$$

Similarly, by construction $\phi_1\equiv\phi$ on B_t, and

$$\int_{\mathbb{S}^3}|\nabla\phi_1|^3=\int_{B_t}|\nabla\phi|^3+\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_1|^3\le E-\bar\delta+C\frac{E}{N_E}.$$

Thus we reach (4-19) if

$$N_E\ge CE\bar\delta\,. \tag{4-31}$$

Recall from Step 2 that $N_E=-C\log_2\rho_E$, so (4-30) and (4-31) translate into the requirement that $\rho_E\le e^{-C\max\{E\bar\delta,(E\bar\delta)^3\}}$, which is implied by our hypothesis since $\bar\delta$ is bounded. □

Remark 4.8. The proof of (4-27) in Step 5 gives the following general estimate, valid for bounded Sobolev functions on a compact manifold M and for any Poincaré domain $\Omega \subset M$:

$$\|\nabla\phi\|_{L^p(\Omega)} \geq C_\Omega\big[|M|(\|\phi\|_{L^\infty(M)} - |\bar{\phi}_M|) - 2\|\phi\|_{L^\infty(M)}|M \setminus \Omega|\big], \tag{4-32}$$

where C_Ω is the Poincaré constant of Ω.

Consider now the conformal transformations of the unit ball B^4

$$F_v(x) = -v + (1-|v|^2)(x^* - v)^*, \quad \text{where } v \in B^4 \text{ and } a^* = \frac{a}{|a|^2}.$$

Proposition 4.9 (balancing $\Rightarrow$ extension). *Let $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. Suppose that, for all $v \in B^4$,*

$$\left| \fint_{\mathbb{S}^3} \phi \circ F_v \right| \geq \frac{1}{4}. \tag{4-33}$$

Then the following function $u : B^4 \to \mathbb{S}^3$ extends ϕ:

$$u(v) := \pi_{\mathbb{S}^3}\left(\fint_{\mathbb{S}^3} \phi \circ F_v \right), \quad \textit{where } \pi_{\mathbb{S}^3}(a) = \frac{a}{|a|} \textit{ for } a \in \mathbb{R}^4 \setminus \{0\}. \tag{4-34}$$

Moreover, there exists a constant C independent of ϕ such that

$$\|\nabla u\|_{L^4(B^4)} \leq C\|\nabla\phi\|_{L^3(\mathbb{S}^3)}. \tag{4-35}$$

Proof. **Step 1.** After a change of variable,

$$\fint_{\mathbb{S}^3} \phi \circ F_v(x)\,dx = \fint_{\mathbb{S}^3} \phi(y)|(F_v^{-1})'|^3(y)\,dy,$$

where $|(F_v^{-1})'|$ is the conformal factor of DF_v^{-1}. From Lemma C.1,

$$|(F_v^{-1})'|(y) = |F'_{-v}|(y) = \frac{1-|v|^2}{|y+v|^2};$$

therefore,

$$\fint_{\mathbb{S}^3} \phi \circ F_v = \fint_{\mathbb{S}^3} \phi(y)\left(\frac{1-|v|^2}{|y+v|^2}\right)^3 dy.$$

From [Nicolesco 1936], the function

$$K(x,y) = |\mathbb{S}^3|^{-1}\left[\frac{1-|y|^2}{|x-y|^2}\right]^3$$

is the Poisson kernel for the equation

$$\begin{cases} \Delta^2 u = 0 & \text{on } B^4, \\ \left.\dfrac{\partial u}{\partial v}\right|_{\partial B^4} = 0, \\ u|_{\partial B} = \phi. \end{cases} \tag{4-36}$$

Therefore, the function $\tilde{u}(v) := \fint_{\mathbb{S}^3} \phi \circ F_v$ satisfies (4-36).

Step 2. The following classical estimate holds for (4-36) (see [Gazzola et al. 2010, Chapter 2] for the stronger estimate $\|u\|_{W^{1,4}(\Omega)} \le \|\phi\|_{W^{1-1/4,4}(\partial\Omega)}$):

$$\|\nabla u\|_{L^4(B^4)} \le C\|\nabla\phi\|_{L^3(B^3)}.$$

Step 3. We note that

$$\tfrac{1}{4} \le |\tilde{u}(x)| \le C \quad \text{for all } v \in B^4$$

because of our hypothesis (4-33), $|\phi| \equiv 1$ and by the elementary estimate $\int_{\mathbb{S}^3}((1-|v|^2)/|y+v|^2)^3\,dy \le C$. As in Step 2 of the proof of Theorem 4.3 (with notation $\pi_{\mathbb{S}^3} = \pi_a$ for $a = 0$), we obtain the pointwise estimate

$$|\nabla(\pi_{\mathbb{S}^3} \circ \tilde{u})| \sim |\nabla\tilde{u}|.$$

The estimate (4-35) follows via Step 2. ☐

Consider now the case in which the hypothesis of Proposition 4.9 is false, i.e., that there exists $v \in B^4$ with

$$\left| \fint_{\mathbb{S}^3} \phi \circ F_v \right| \le \frac{1}{4}. \tag{4-37}$$

$F_v|_{\mathbb{S}^3}$ is conformal and bijective (see Appendix C); thus, for $A \subset \mathbb{S}^3$,

$$\int_A |\nabla\tilde{\phi}|^3 = \int_{F_v^{-1}(A)} |\nabla\phi|^3;$$

in particular, $\tilde{\phi} := \phi \circ F_v$ has energy at most E, like ϕ. We observe that Proposition 4.7 applies to $\tilde{\phi}$ directly due to our hypotheses. Therefore, we can find $\tilde{\phi}_1, \tilde{\phi}_2 \in W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$ such that

$$\tilde{\phi} = \tilde{\phi}_1\tilde{\phi}_2, \qquad \int_{\mathbb{S}^3} |\nabla\tilde{\phi}_i|^3 \le E - \frac{\bar{\delta}}{2} \quad \text{for } i = 1, 2.$$

We then precompose with F_v^{-1}, which preserves the pointwise product and the L^3-energy of the gradients, obtaining the same decomposition for ϕ.

The case $\tilde{\phi} \in \mathcal{G}^E$ is a bit more difficult:

Proposition 4.10. *Under the assumption (4-37) and with $\tilde{\phi} := \phi \circ F_v$, suppose that $\tilde{\phi} \in \mathcal{G}^E$. Then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \le \frac{C}{\rho_E}\|\nabla\phi\|^2_{L^3(\mathbb{S}^3)} + \|\nabla\phi\|_{L^3(\mathbb{S}^3)} \tag{4-38}$$

under the assumption that

$$\rho_E \le \tfrac{1}{4}. \tag{4-39}$$

Proof. To simplify notations, let $\rho = \rho_E$ during this proof. We divide the domain B^4 into

$$A := F_v^{-1}(B(0, 1-\rho)), \quad A' := B^4 \setminus A.$$

By Lemma C.2, there exists a constant C dependent only on the dimension and a function $h(v)$ such that, for $x \in A$ and under the condition (4-39),

$$\frac{h(v)}{C} \le |F_v'|(x) \le Ch(v). \tag{4-40}$$

Therefore, we have

$$\begin{aligned} |\{x \in A : |\nabla u|(x) > \Lambda\}| &= \left|\left\{x \in A : |\nabla \tilde{u}|(F_v(x))\,|F_v'|(x) > \Lambda\right\}\right| \\ &\le \left|\left\{x \in A : |\nabla \tilde{u}|(F_v(x)) > \frac{\Lambda}{Ch(v)}\right\}\right| \\ &= \int_{F_v(A)\cap\{y:|\nabla\tilde{u}|(y)>\Lambda/(Ch(v))\}} |F_v'|^{-4}\,dy \\ &\le C^4 h^{-4}(v)\left|\left\{y \in B_{1-\rho} : |\nabla \tilde{u}| > \frac{\Lambda}{Ch(v)}\right\}\right| \\ &\le C^8 \Lambda^{-4} \|\nabla \tilde{u}\|^4_{L^{4,\infty}(B_{1-\rho})}. \end{aligned}$$

By bringing Λ to the other side, it follows that

$$\Lambda^4 |\{x \in A : |\nabla u|(x) > \Lambda\}| \le C^8 \|\nabla \tilde{u}\|_{L^{4,\infty}(B(0,1-\rho))}. \tag{4-41}$$

By conformal invariance, the invertibility of F_v and the usual estimate between $L^{4,\infty}$ and L^4, we have

$$\Lambda^4 |\{x \in A' : |\nabla u|(x) > \Lambda\}| \le C\|\nabla u\|^4_{L^4(A')} = C\|\nabla \tilde{u}\|_{L^4(B\setminus B_{1-\rho})}). \tag{4-42}$$

We now sum (4-41) and (4-42) and we take the supremum on $\Lambda > 0$. It follows that, up to increasing C,

$$[\nabla u]_{L^{4,\infty}(B^4)} \le C(\|\nabla \tilde{u}\|_{L^{4,\infty}(B_{1-\rho})} + \|\nabla \tilde{u}\|_{L^4(B\setminus B_{1-\rho})}). \tag{4-43}$$

The estimate (4-43) together with Theorem 4.3 applied to $\tilde{u}$ gives the desired estimate for the first summand, while for the second summand we proceed as in Step 3 of the proof of Theorem 4.3. On the small concentration regions B_i for $\tilde{\phi}$ we apply Courant's Lemma 4.5, due to which we may project the values of $u := \tilde{u} \circ F_v^{-1}$ on $\mathbb{S}^3$ with little change of the gradient of u. Since F_v^{-1} is conformal, the L^3-energy of $\tilde{u}$ on ∂B_i is the same as the L^3-energy of u on $\partial F_v^{-1}(B_i)$. By Theorem 4.4 applied to $\tilde{u}$ as in Step 3 of the proof of Theorem 4.3, we obtain

$$\|\nabla u\|_{L^4(F_v^{-1}(B\setminus B_{1-\rho}))} = \|\nabla \tilde{u}\|_{L^4(B\setminus B_{1-\rho})} \le C\|\nabla \tilde{\phi}\|_{L^3(\mathbb{S}^3)} = C\|\nabla \phi\|_{L^3(\mathbb{S}^3)}.$$

This and (4-43) conclude the proof. □

4D. ***End of the proof of Theorem B″.*** We refer to the scheme (4-17) for the idea of the proof.

Choice of $\bar{\delta}$. In (4-16), take $\bar{\delta} \le \delta/C_1$ with the notations of Theorem 4.3 and with δ is as in Theorem 4.4. If necessary, shrink $\bar{\delta}$ so that the bound of Proposition 4.7 is also satisfied.

Choice of ρ_E. From Proposition 4.7 with the above choices of $\bar{\delta}$, we get $\rho_E \lesssim e^{-C\max(1,E^3)}$.

Estimates for extensions. Consider again the scheme (4-17). Each time we extend some boundary datum ϕ obtained during our constructions via a function $u : B^4 \to \mathbb{S}^3$, we do so with one of the following estimates:

- In the case of the extensions of Theorem 4.3 or of Proposition 4.10 (which in turn actually depends on Theorem 4.3) we have

$$\|\nabla u\|_{L^{4,\infty}} \lesssim \frac{\|\nabla \phi\|_{L^3}^2}{\rho_E} + \|\nabla \phi\|_{L^3}.$$

- In the case of the biharmonic extension of Proposition 4.9, we have the much better

$$\|\nabla u\|_{L^4} \lesssim \|\nabla \phi\|_{L^3}.$$

The number of iterations to be made when we apply the procedure described in scheme (4-17) is bounded by

$$E / \tfrac{1}{2}\bar{\delta} \sim E.$$

Since each iteration creates two new boundary value functions out of one, in the end we may have a decomposition into no more than

$$e^{CE} \text{ boundary value functions.}$$

By the triangle inequality we see that, in this case, there exists an extension of the initial ϕ satisfying

$$\|\nabla u\|_{L^{4,\infty}} \lesssim e^{C\|\nabla \phi\|_{L^3}^9} \|\nabla \phi\|_{L^3}^2 + \|\nabla \phi\|_{L^3}. \tag{4-44}$$

This gives the estimate (4-1) of Theorem B$''$, finishing the proof. □

5. Controlled global gauges

In this section we prove Theorem A.

5A. *Scheme of the proof.*

We indicate here the sketch of the proof, before going through the details.

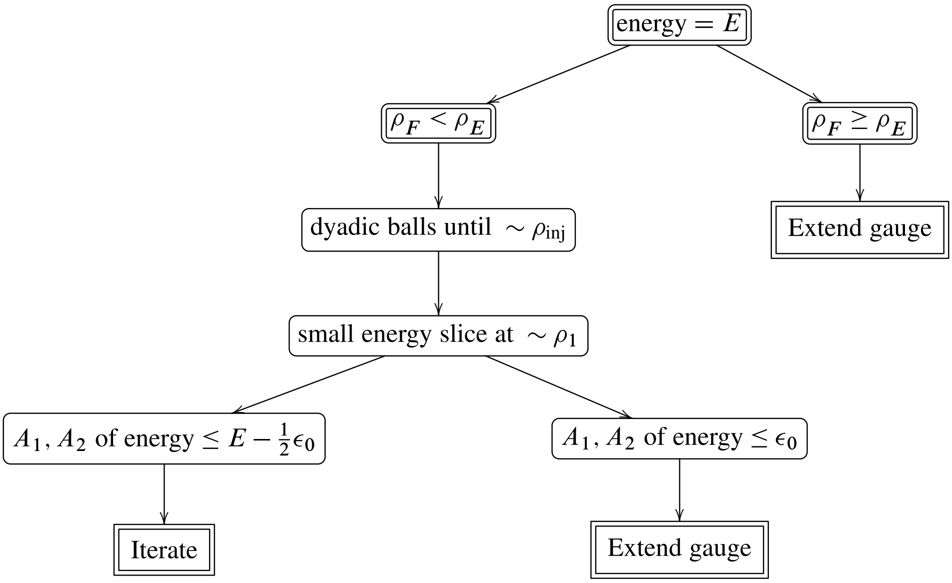
Proof. We will denote the L^2-norm of F by E. We may assume that a first guess for A (i.e., a fixed trivialization) is already given and belongs to $W^{1,2}$ (if the bound by ϵ_0 on the energy of F is available, we may assume more by Uhlenbeck's result stated above, namely that one controls the $W^{1,2}$-norm of A by the energy).

It can be seen from the formula of change of gauge that it is equivalent to estimate either the gradient of the trivialization g or the gradient of the connection A in that gauge.

We define f by iteration on the energy bound E. The main steps are as follows (see the scheme (5-1)):

- Uhlenbeck's theorem [1982b] already gives a gauge with an L^4-estimate of the gradient of the trivialization if the energy of F is smaller than ϵ_0.
- Let ρ_F be the largest scale at which no more than $\frac{1}{2}\epsilon_0$ of the L^2-norm of F concentrates.

- In the case $\rho_F \geq \rho_E := C\rho_{\text{inj}}(M)2^{-E/\epsilon_1}$, we iteratively extend our gauge on the small simplices of a triangulation using Theorem B$''$; see Section 5B. The estimates depend only on M^4.
- The alternative is $\rho_F \leq \rho_E$. Then, consider a point x_0 at which $|F|$ concentrates and look at the geodesic dyadic rings
$$R_k := B(x_0, 2^{k+1}\rho_F) \setminus B(x_0, 2^k\rho_F), \quad k \in \{0, \dots, \lfloor \log_2(C\rho_{\text{inj}}/\rho_F) \rfloor\}.$$
By the pigeonhole principle and by the choice of ϵ_1, we ensure the existence of a small energy slice along a geodesic sphere of radius $t \sim 2^{k_0}\rho_F$. We have extensions of the connections with curvatures of energy smaller than $E - \frac{1}{2}\epsilon_0$. We use Lemma 5.4. To avoid subtleties about traces, we will ensure that these two connections coincide on an open set. The choice of slice is described in Section 5D.
- Then we separately trivialize these two connections. By iterative assumption we then define $f(E)$ based on $f\left(E - \frac{1}{2}\epsilon_0\right)$ and on the function f_1 of Theorem B. The detailed bounds are given in Section 5E.



(5-1)

5B. ***Iterations based on a suitable triangulation.*** Define, for ϵ_0 as in Theorem 5.1, the radius
$$\rho_F := \inf\left\{\rho > 0 : \int_{B_\rho(x_0)} |F|^2 = \tfrac{1}{2}\epsilon_0 \ \text{ for some } \ x_0 \in M\right\}, \tag{5-2}$$
where $\rho_E := C\rho_{\text{inj}}(M)2^{-E/\epsilon_1}$ and $\rho_{\text{inj}}(M)$ is the injectivity radius of M. The constant ϵ_1 will be fixed later. Fix a triangulation on M with in-radius $\gtrsim \rho_E$ and size $\lesssim \rho_E$, with implicit constants bounded by 4. We choose $C < 1$ in the definition of ρ_E so that each simplex of the triangulation is contained in a ball of radius $\frac{1}{2}\rho_{\text{inj}}(M)$. In particular, all k-simplices of the triangulation are bi-Lipschitz equivalent to $\mathbb{S}^k$ for $k = 1, \dots, 4$.

We recall here the main result of [Uhlenbeck 1982b]:

Theorem 5.1 (Uhlenbeck gauge). *There exists $\epsilon_0 > 0$ such that, if the curvature satisfies $\int_{B_1} |F|^2 \le \epsilon_0$, then there is a gauge $\phi \in W^{2,2}(B_1, \mathrm{SU}(2))$ in which the connection satisfies $\|A_\phi\|_{W^{1,2}(B_1)} \le C\|F\|_{L^2(B_1)}$ with $C > 0$ depending only on the dimension.*

Theorem 5.1 gives a trivialization ϕ_i associated to each 4-simplex C_i such that the expression of A in those coordinates,

$$A_i = \phi_i^{-1} d\phi_i + \phi_i^{-1} A\phi_i \quad \text{on } C_i, \tag{5-3}$$

satisfies

$$\|A_i\|_{W^{1,2}(C_i)} \le C\|F\|_{L^2(C_i)}. \tag{5-4}$$

If we call

$$g_{ij} := \phi_j^{-1}\phi_i, \tag{5-5}$$

then $g_{ij}g_{jk} = g_{ik}$, so in particular $g_{ij}^{-1} = g_{ji}$; moreover,

$$A_j = g_{ij}dg_{ji} + g_{ij}A_i g_{ji} \quad \text{on } \partial C_i \cap \partial C_j. \tag{5-6}$$

It follows that $g_{ij} \in W^{1,3}(\partial C_i \cap \partial C_j, \mathrm{SU}(2))$.

Lemma 5.2 (extension on a sphere). *Let S^3_+ be the upper hemisphere, $\mathbb{S}^3 \cap \{x_3 \ge 0\}$. For any $g \in W^{1,3}(S^3_+, \mathrm{SU}(2))$, there exists $\tilde g \in W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$ such that $\tilde g = g$ on $\mathbb{S}^3_+$ and*

$$\|\nabla \tilde g\|_{L^3(\mathbb{S}^3)} \le C\|\nabla g\|_{L^3(S^3_+)}.$$

Proof. Let $\mathbb{S}^3_-$ be a spherical cap of height $t \in \left[\frac{1}{2}, \frac{3}{2}\right]$ such that

$$\left\| g|_{\partial \mathbb{S}^3_-} \right\|_{W^{1,2}(\partial \mathbb{S}^3_-)} \le C\|g\|_{W^{1,3}(\mathbb{S}^3_+)}. \tag{5-7}$$

We observe that $g|_{\partial S^3_+ \simeq \mathbb{S}^2} \in W^{1,2}(\mathbb{S}^2, \mathrm{SU}(2))$, and we desire to extend this trace inside $B^3 \simeq \mathbb{S}^3_-$ with a good norm estimate. Let

$$\begin{cases} \Delta \hat g = 0 & \text{on } B^3, \\ \hat g = g & \text{on } \partial B^3. \end{cases}$$

Then we have, by the usual elliptic estimates,

$$\|\hat g\|_{W^{1,3}(\mathbb{S}^3_-)} \le C\left\| g|_{\partial \mathbb{S}^3_-} \right\|_{W^{1,2}(\partial \mathbb{S}^3_-)}. \tag{5-8}$$

For $a \in B^4_{1/2}$, if g_a is the radial projection of the values of $\hat g$ on the boundary with center a, then (as in the projection trick of Section 2A)

$$|\nabla g_a| \le C\frac{|\nabla \hat g|}{|\hat g - a|} \quad \text{and} \quad \int_{a \in B^4_{1/2}} \int_{B^3} |\nabla g_a|^3 \le C\int_{B^3} |\nabla \hat g|^3. \tag{5-9}$$

Therefore, there exists $a \in B^4_{1/2}$ such that

$$\|\nabla g_a\|_{L^3(B^3 \simeq \mathbb{S}^3_-)} \le C\|\nabla \hat g\|_{L^3(B^3 \simeq \mathbb{S}^3_-)}. \tag{5-10}$$

Combining the inequalities (5-7), (5-8), (5-9) and (5-10), we obtain the thesis for $\tilde g = g_a$ with a as above. ☐

Corollary 5.3 (iteration step). *Suppose that on our 4-manifold M a connection A is fixed and an Uhlenbeck gauge ϕ_j is defined on a 4-simplex C_j, i.e., (5-4) holds with notation (5-3). Suppose that a global gauge ϕ_I is defined on a finite union of simplices $C_I := \bigcup_{\alpha\in I} C_{i_\alpha}$ and that $\partial C_j \cap C_I^{(3)}$ (where $C_I^{(3)}$ is the simplicial 3-skeleton of C_I) contains some, but not all, 3-faces of C_j. It is then possible to extend the gauge change g_{ij} of (5-5) to $\tilde g_{ij}$ defined on the whole of ∂C_j with*

$$\|\nabla \tilde g_{ij}\|_{L^3(\partial C_j)} \le C\|\nabla g_{ij}\|_{L^3(\partial C_j\cap C_I^{(3)})},$$

where C depends only on M.

Proof. $H := (\partial C_j \setminus C_I^{(3)})_\delta$ is bi-Lipschitz to a ball for δ equal to two-thirds of the smallest in-radius of a face of C_j. Here, A_δ is a δ-neighborhood of A inside ∂C_j. Let $H' := (\partial C_j \setminus C_I^{(3)})_{2\delta}$. The triple $(\partial C_j, H, H')$ is C-bi-Lipschitz to $(\mathbb{S}^3, \mathbb{S}^3_-, K)$ where K is the spherical cap of height $\frac{3}{4}$ extending $\mathbb{S}^3_-$. We apply Lemma 5.2 in order to "fill the hole" H extending the gauge g_{ij} with estimates. The bi-Lipschitz constant is bounded by the geometric constraints on our triangulation only. □

Given Lemma 5.2 and Corollary 5.3, we proceed iteratively on the triangulation as follows (the indices labeling the simplices are redefined during the whole procedure in a straightforward way):

- Suppose that we already defined the gauge $\tilde\phi_{j-1}$ on a set of $j-1$ simplices $C_1,\dots,C_{j-1}$ whose union forms a connected set.
- Consider a new simplex C_j extending this connected set. Use Corollary 5.3 to extend g_{ij} to $\tilde g_{ij}$.
- We apply Theorem B″ and extend $\tilde g_{ij}$ to a gauge change h_{ij} defined inside C_j so that
$$\|\nabla h_{ij}\|_{L^{(4,\infty)}(C_j)} \le f(\|\nabla \tilde g_{ij}\|_{L^3(C_j)}) \le C_0, \tag{5-11}$$
with C_0 depending only on universal constants and on ϵ_0.
- On $\bigcup_{i<j} C_i$ let $\tilde\phi_j = \tilde\phi_{j-1}$, while on C_j we define $\tilde\phi_j = \phi_j h_{ij}$.

Let $\widetilde A_j$ be the local expression corresponding to the gauge $\tilde\phi_j$. Then

$$\|\widetilde A_j\|_{L^{(4,\infty)}(C_j)} \lesssim \|A_j\|_{L^4(C_j)} + \|\nabla h_{ij}\|_{L^{(4,\infty)}(C_j)} \le \epsilon_0 + C_0.$$

Iterating this gauge extension strategy, we obtain a global gauge $\widetilde A$ on the whole of M such that

$$\|\widetilde A\|_{L^{(4,\infty)}(M)} \le C(\text{number of simplices})(C_0+\epsilon_0) \le C\frac{\mathrm{Vol}(M)}{\rho_E^4}. \tag{5-12}$$

The above bound depends on the geometry of M and on the energy E of the curvature only. Note that the above reasoning works only as long as $\rho_F \lesssim \rho_E$. We next consider the case $\rho_F \ge \rho_E$.

5C. *Extending the connection with small curvature changes.*

Let ϵ_0 be as in Theorem 5.1.

Lemma 5.4 (finding good slices). *There exists a constant ϵ_1 with the following properties: If M is a fixed 4-manifold with a $W^{1,2}$-connection A and if $B_{2t}(x_0) \subset M$ is a geodesic ball such that*

$$t\int_{\partial \widetilde B_t} |F|^2 \le \epsilon_1,$$

then there exists $\hat{A} \in W^{1,2}\big(\bigwedge^1 M, \mathrm{su}(2)\big)$ *such that* $\hat{A} = A$ *on* B_t *and*

$$\int_{M\setminus B_t} |F_{\hat{A}}|^2 \le \frac{\epsilon_0}{4}\,.$$

Proof. Up to a change of gauge, which does not increase the norm, we may assume the Neumann condition

$$\langle A, \nu\rangle \equiv 0 \quad \text{on } \partial B_t. \tag{5-13}$$

This is obtained, for example, by minimizing $\|g^{-1}dg + g^{-1}Ag\|_{L^2(B_t)}$ among $g \in W^{2,2}(B_t, \mathrm{SU}(2))$.

Extend A to $B_{2t} \setminus B_t$ by $\widetilde{A} := \pi^* i^*{}_{\partial B_t} A$, where $\pi(x) = tx/|x|$ and $i_{\partial B_t}$ is the inclusion. Using the hypothesis, we obtain

$$\int_{B_{2t}\setminus B_t} \big|d\widetilde{A} + \tfrac{1}{2}[\widetilde{A}, \widetilde{A}]\big|^2 \le C\epsilon_1.$$

We apply a change of gauge $g = g(\sigma)$ depending only on the angular variable $\sigma \in \partial B^4$ and such that

$$d^*{}_{\partial B_t} A_g\big|_{\partial B_t} = 0.$$

This preserves (5-13) and gives, as $s \to 0$,

$$C\epsilon_1 \ge \int_{B_s\cap\partial B_t} \big|dA_g + \tfrac{1}{2}[A_g, A_g]\big|^2 \ge \int_{B_s\cap\partial B_t} |dA|^2 - o(s)\int_{B_s\cap\partial B_t} |\nabla A|^2.$$

Therefore, $A_g \in W^{1,2}(\wedge^1 \partial B_t, \mathrm{su}(2))$, $\widetilde{A}_g \in W^{1,2}(\wedge^1 B_{2t} \setminus B_t, \mathrm{su}(2))$, and A_g, $\widetilde{A}_g$ satisfy (5-13). Therefore, $\widetilde{A}_g$ extends by A_g in a neighborhood of ∂B_t, giving still a $W^{1,2}$ gauge. Observe that, by Sobolev embedding,

$$\int_{\partial B_t} |[A, A]|^2 \lesssim \bigg(\int_{\partial B_t} |\nabla A|^2\bigg)^2$$

and, by Hodge decomposition and using $d^*{}_{\partial B_t} A = 0$,

$$\int_{\partial B_t} |\nabla A|^2 \lesssim \int_{\partial B_t} (|dA|^2 + |d^*A|^2) \lesssim \int_{\partial B_t} |F_A|^2 + \bigg(\int_{\partial B_t} |\nabla A|^2\bigg)^2.$$

For $X = \|\nabla A\|^2_{L^2(\partial B_t)}$ we get $X \le \epsilon_1 + X^2$, and thus we may assume that

$$t\int_{\partial B_t} |\nabla A|^2 \le Ct\int_{\partial B_t} |F|^2.$$

Define $\hat{A} := \chi_t A$ for a smooth $[0, 1]$-valued cutoff function χ_t such that $\chi_t \equiv 1$ on B_t and $\chi_t \equiv 0$ outside B_{2t}. We obtain

$$\int_{B_{2t}} |F_{\hat{A}}|^2 \le \int_{B_t} |F_A|^2 + C\epsilon_1$$

and we can extend $\hat{A} \equiv 0$ outside B_{2t}, obtaining the desired estimate for ϵ_1 small enough. $\square$

5D. ***Cutting M by a small energy slice.*** Suppose for this subsection that $\rho_F < \rho_E$. Let C be as in the definition of ρ_E and define

$$\rho_1 := \begin{cases} \inf\{\rho \geq \rho_F : \int_{B_{2\rho}\setminus B_\rho} |F|^2 \leq \frac{1}{4}\epsilon_1\} & \text{if this is less than } C\rho_{\mathrm{inj}}(M), \\ C\rho_{\mathrm{inj}} & \text{otherwise.} \end{cases}$$

Since $\rho_F < \rho_E$ and by the choice of ϵ_1, ρ_1 is rather small and $B_{2\rho_1}$ is bi-Lipschitz to B_1. Thus Lemma 5.4 applies. More precisely, let $t_1 \in \left[\rho_1, \frac{5}{4}\rho_1\right]$, $t_2 \in \left[\frac{7}{4}\rho_1, 2\rho_1\right]$. There exist t_i, $i = 1, 2$, such that

$$t_i \int_{\partial B_{t_i}} |F|^2 \leq \epsilon_1.$$

5E. ***Strategies after cutting.*** Let ϵ_0 be as in Theorem 5.1. We pursue different strategies depending on the energy of F outside $B_{2\rho_1}$.

The case $\int_{M\setminus B_{2\rho_1}} |F|^2 \geq \frac{1}{2}\epsilon_0$. Split to the regions B_{t_2} and $M \setminus B_{t_1}$ and do induction on the energy in order to separately find gauges satisfying our estimates. Lemma 5.4 gives extensions

$$\begin{cases} \hat{A}_1 \equiv A \text{ on } B_{t_2} & \text{s.t. } \int_M |F_{\hat{A}_1}|^2 \leq \int_{B_{t_2}} |F_A|^2 + C\epsilon_1, \\ \hat{A}_2 \equiv A \text{ on } M \setminus B_{t_1} & \text{s.t. } \int_M |F_{\hat{A}_2}|^2 \leq \int_{B_{t_1}} |F_A|^2 + C\epsilon_1. \end{cases} \tag{5-14}$$

In particular, $\hat{A}_1, \hat{A}_2$ are equivalent on $B_{\frac{7}{4}\rho_1} \setminus B_{\frac{5}{4}\rho_1}$ and

$$\int |F_{\hat{A}_i}|^2 \leq \int |F_A|^2 - \tfrac{1}{4}\epsilon_0.$$

If we can find global gauges g_i^∞, $i = 1, 2$, in which $\hat{A}_i$ have expressions $\hat{A}_i^\infty$ with $L^{(4,\infty)}$ bounds as in Theorem B, then it is enough to apply

$$g_{12}^\infty := (g_1^\infty)^{-1} g_2^\infty$$

on $R := B_{\frac{7}{4}\rho_1} \setminus B_{\frac{5}{4}\rho_1}$ in order to obtain

$$A_2^\infty = g_{12}^\infty A_1^\infty (g_{12}^\infty)^{-1} + g_{12}^\infty d(g_{12}^\infty)^{-1} \quad \text{and} \quad \|\nabla g_{12}^\infty\|_{L^{(4,\infty)}(R)} \leq f\left(E - \tfrac{1}{4}\epsilon_0\right).$$

Then there exists $t_3 \in \left[\frac{5}{4}\rho_1, \frac{7}{4}\rho_1\right]$ such that

$$\int_{\partial B_{t_3}} |\nabla g_{12}^\infty|^3 \leq f\left(E - \tfrac{1}{4}\epsilon_0\right).$$

By Theorem B we can find a $W^{1,(4,\infty)}$-extension h_{12}^∞ of g_{12}^∞ to a map from B_{t_3} to SU(2). Thus, if we call f_1 the function of Theorem B, then

$$\|\nabla h_{12}^\infty\|_{L^{(4,\infty)}(B_{t_3})} \leq f_1\left(f\left(E - \tfrac{1}{4}\epsilon_0\right)\right).$$

If we define

$$g^\infty := \begin{cases} g_2^\infty & \text{on } M^4 \setminus B_{t_3}, \\ h_{12}^\infty g_1^\infty & \text{on } B_{t_3}, \end{cases} \tag{5-15}$$

then ∇g^∞ is estimated by an universal constant times

$$f_1\big(f\big(E-\tfrac{1}{4}\epsilon_0\big)\big)+f\big(E-\tfrac{1}{4}\epsilon_0\big).$$

The case $\int_{M\setminus B_{2\rho_1}}|F|^2\leq\frac{1}{2}\epsilon_0$. Outside B_{ρ_1} we apply directly Theorem 5.1, while on $B_{2\rho_1}$ we extend the so-obtained gauge via Theorem B$''$. If we call A_1, A_2 the so-obtained connections on $B_{2\rho_1}$, $M\setminus B_{\rho_1}$ respectively, then there exists $t\in[\rho_1,2\rho_1]$ such that

$$\int_{\partial B_t}(|A_1|^3+|A_2|^3)\leq C(f_1(\epsilon_0)+\epsilon_0).$$

As above, the same bound is true also for the gradient of the change of gauge ∇g_{12}. Theorem B gives the extension h_{12} to a gauge in $W^{1,(4,\infty)}(B_t,\mathrm{SU}(2))$ with

$$\|\nabla h_{12}\|_{L^{4,\infty}(B_{t_3})}\leq f_1(C(f_1(\epsilon_0)+\epsilon_0)).$$

Then choose

$$g^\infty:=\begin{cases}g_2 & \text{on } M^4\setminus B_{t_3},\\ h_{12}g_1 & \text{on } B_{t_3}.\end{cases}\tag{5-16}$$

This g^∞ satisfies an estimate independent on E and dependent only on ϵ_0, again allowing us to define $f(E)$ inductively. □

Appendix A: Uhlenbeck small energy extension

We use the strategy from [Uhlenbeck 1982a] to prove Theorem 4.4. The analogy is in the method of proof more than in the result.

First recall that $W^{1,2}(X,\mathbb{S}^3)=W^{1,2}(X,\mathbb{R}^4)\cap\{u:u(x)\in\mathbb{S}^3\text{ a.e.}\}$ and observe that we attain the infimum

$$\inf\left\{\int_{B^4}|\nabla P|^2 : P\in W^{1,2}(B^4,\mathbb{S}^3),\ P=P_0\ \text{on}\ \partial B^4\right\}.\tag{A-1}$$

Indeed, a minimizing sequence will have a $W^{1,2}$-weakly convergent subsequence, which thus converges pointwise everywhere. By weak lower semicontinuity a minimizer exists, and by convexity it is unique. The minimizer P distributionally verifies

$$\operatorname{div}(P^{-1}\nabla P)=0.\tag{A-2}$$

Lemma A.1 (a priori estimates). *There exists $\epsilon>0$ with the following property: Let P be an extension of $P_0\in W^{1,3}(\mathbb{S}^3,\mathbb{S}^3)$ with $\|P-I\|_{W^{1,4}(B^4)}\leq\epsilon$ that satisfies (A-2). We identify $\mathbb{S}^3$ with the Lie group* SU(2). *Then there exists a constant C_ϵ such that*

$$\|P-I\|_{W^{4/3,3}(B^4)}\leq C_\epsilon\|\nabla P_0\|_{L^3(\mathbb{S}^3,\mathbb{S}^3)}.\tag{A-3}$$

Proof. By L^2-Hodge decomposition,

$$P^{-1}dP=dU+d^*V,\tag{A-4}$$

where V is the unique minimizer of

$$\min\left\{\int_{B^4}|d^*V-P^{-1}dP|^2,\ *V|_{\partial B^4}=0,\ dV=0\right\};$$

thus,

$$\begin{cases}\Delta V=dd^*V=dP^{-1}\wedge dP,\\ dV=0,\\ *V=0.\end{cases}$$

We claim that

$$\|\nabla V\|_{L^3(\partial B^4)}\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}. \tag{A-5}$$

To see this, observe that $d(P^{-1})=P^{-1}\,dP\,P^{-1}$ and $P,\ P^{-1}\in L^\infty$ with norm equal to 1 so, by the elliptic, Hölder and Poincaré estimates,

$$\begin{aligned}\|\nabla V\|_{W^{1,2}(B^4)}\lesssim\|dP^{-1}\wedge dP\|_{L^2(B^4)}&\lesssim\|d(P^{-1})\|_{L^4(B^4)}\|dP\|_{L^4(B^4)}\\ &\lesssim\|dP\|_{L^4(B^4)}\|P^{-1}\|^8_{L^\infty}\|\nabla P\|_{L^4(B^4)}\\ &\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}.\end{aligned} \tag{A-6}$$

The trace and Sobolev embedding inequalities

$$\|V\|_{L^p(\partial B^4)}\lesssim\|V\|_{W^{1-1/q,q}(\partial B^4)}\lesssim\|V\|_{W^{1,q}(B^4)}$$

are valid for $q=2,\ p=3$. Therefore, we obtain (A-5).

Using the trace of the Hodge decomposition formula (A-4) on the boundary, we obtain from (A-5) that

$$\|dU-P_0^{-1}dP_0\|_{L^3(\partial B^4)}\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}. \tag{A-7}$$

Like for V, for U we have

$$\Delta U=d^*dU=d^*(P^{-1}dP)=0.$$

We apply the elliptic estimates for U to obtain

$$\|dU\|_{W^{1/3,3}(B^4)}\lesssim\|\nabla U\|_{L^3(\partial B^4)}, \tag{A-8}$$

while (A-7), the triangle inequality and the fact that $\|P_0\|_{L^\infty}=1$ give

$$\begin{aligned}\|U\|_{L^3(\partial B^4)}&\lesssim\|dU-P_0^{-1}dP_0\|_{L^3(\partial B^4)}+\|P_0^{-1}dP_0\|_{L^3(\partial B^4)}\\ &\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}+\|dP_0\|_{L^3(\partial B^4)}.\end{aligned} \tag{A-9}$$

Using (A-4), the triangle inequality and (A-6), (A-8), (A-9) we obtain

$$\begin{aligned}\|P^{-1}dP\|_{W^{1/3,3}(B^4)}&\lesssim\|d^*V\|_{W^{1/3,3}(B^4)}+\|dU\|_{W^{1/3,3}(B^4)}\\ &\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}+\|dP_0\|_{L^3(\partial B^4)}.\end{aligned} \tag{A-10}$$

Write $dP=PP^{-1}dP$ and observe that $P\in L^\infty\cap W^{1,4}$ since $\mathbb{S}^3$ is bounded, while $P^{-1}dP\in W^{1/3,3}$ by (A-10). We now use Lemma B.1 for the product fg with $f=P$, $g=P^{-1}dP$ and we obtain

$$\|dP\|_{W^{1/3,3}(B^4)}\lesssim\|P^{-1}dP|\|_{W^{1/3,3}}(\|P\|_{L^\infty}+\|P-I\|_{W^{1,4}(B^4)}). \tag{A-11}$$

Note again that $\|P\|_{L^\infty}=1$ and deduce then from (A-10), Lemma B.1 and the Poincaré inequality that

$$\|P-I\|_{W^{4/3,3}(B^4)}\le C\|dP_0\|_{L^3(\mathbb{S}^3)}+C\epsilon\|P-I\|_{W^{1,4}(B^4)}. \tag{A-12}$$

By the Sobolev inequality we can absorb the $\|P-I\|$ term to the left, and we obtain the thesis. □

We are now ready for the proof of Theorem 4.4. We restate the same result with a slight change of notation and more details.

Theorem A.2 (small energy extension). *There exist two constants $\delta>0$, $C>0$ with the following property: Suppose $Q\in W^{1,3}(\mathbb{S}^3,\mathbb{S}^3)$ is such that $\|dQ\|_{L^3(\mathbb{S}^3)}\le\delta$. Then there exists an extension $P\in W^{1,4}(B^4,\mathbb{S}^3)$ satisfying*

$$\|P-I\|_{W^{1,4}(B^4)}\le C\|dQ\|_{L^3(\mathbb{S}^3)}. \tag{A-13}$$

Proof. Define the following two sets, with $K>0$ fixed later:

$$\mathscr{G}^\alpha_\epsilon=\{Q\in W^{1,3+\alpha}(\mathbb{S}^3,\mathrm{SU}(2)):\|\nabla Q\|_{L^3}\le\epsilon\},$$

$$\begin{aligned}\mathscr{F}^\alpha_{\epsilon,C}=\big\{Q\in\mathscr{G}^\alpha_\epsilon:\exists P\in W^{1,4+\alpha}(B^4,\mathrm{SU}(2)),\ \mathrm{div}(P^{-1}\nabla P)=0\ \text{ on }\ B^4,\ P=Q\ \text{ on }\ \partial B^4,\\ \|P-I\|_{W^{1,4}(B^4)}\le K\|\nabla Q\|_{L^3(\partial B^4)}\|P-I\|_{W^{1,4+\alpha}(B^4)}\le C\|\nabla Q\|_{L^{3+\alpha}(\partial B^4)}\big\}.\end{aligned} \tag{A-14}$$

The claim of our theorem states that a $P\in\mathscr{F}^0_{\epsilon,C}$ can be constructed to extend any $Q\in\mathscr{G}^0_\delta$ when δ is small enough. The strategy of the proof is to use the supercritical spaces $\mathscr{G}^\alpha_\epsilon$, $\alpha>0$ to approximate $\mathscr{G}^0_\delta$. We divide the proof into five steps, paralleling Uhlenbeck [1982a].

Claim 1. *$\mathscr{G}^\alpha_\epsilon$ is connected for all $\epsilon,\alpha\ge 0$.*

Claim 2. *$\mathscr{F}^\alpha_{\epsilon,C}$ is closed (in $\mathscr{G}^\alpha_\epsilon$) with respect to the $W^{1,3+\alpha}$-norm for $\alpha\ge0$ and for any $C>0$.*

Claim 3. *For $\epsilon>0$ small enough and $\alpha>0$, there exists $C=C_\alpha$ such that the set $\mathscr{F}^\alpha_{\epsilon,C}$ is open in $\mathscr{G}^\alpha_\epsilon$ with respect to the $W^{1,3+\alpha}$-topology.*

Claim 4. *$\mathscr{G}^0_\epsilon$ is contained in the $W^{1,3}$-closure of $\bigcup_{\alpha>0}\mathscr{G}^\alpha_{2\epsilon}$.*

Proof of Claim 1. This is straightforward, since $\mathscr{G}^\alpha_\epsilon$ embeds in $C^{0,\gamma}(\mathbb{S}^3,\mathrm{SU}(2))$. □

Proof of Claim 2.. Consider a family $Q_j\in\mathscr{F}^\alpha_{\epsilon,C}$ with associated P_j as in (A-14) which converge to Q in $W^{1,3+\alpha}$. We extract a weakly convergent subsequence of the P_j and the estimate passes to the limit by weak lower semicontinuity (and by convergence of the Q_j). Similarly, the equations pass to weak limits, since they are intended in the weak sense. □

Ideas for Claim 3. For the proof we need to study the behavior of solutions to $\mathrm{div}(P^{-1}\nabla P)=0$, which is regarded here as an equation $\mathscr{N}_\alpha(P)=0$ with P close to the constant I, which is a zero of $\mathscr{N}_\alpha$. The equation considered is elliptic. The proof of the claim is thus done by linearization of $\mathscr{N}$ near I and by the implicit function theorem. Ellipticity of the equation translates into invertibility of this linearized operator. The estimate of the $W^{1,4}$-norm follows from the a priori estimate of Lemma A.1 once we choose, for example, $K\le\frac12 C_\epsilon$. See Lemma A.3 for the complete proof. □

Proof of Claim 4. Consider $Q \in G^0_\epsilon$. There exists a sequence $Q_i \in C^\infty(\mathbb{S}^3, \mathrm{SU}(2))$ such that $Q_i \to Q$ in $W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$; see [Bethuel 1991; Hang and Lin 2003] — by the density proofs of these works it follows that we may also assume $Q_i \in \mathcal{G}^{\alpha_i}_{\epsilon_i}$ for some sequence $\alpha_i \to 0^+$. The L^3-norm of a function f can be obtained as

$$\lim_{q \to 3^+} \|f\|_{L^q},$$

so in particular we may assume up to extracting a subsequence that $\epsilon_i \leq 2\epsilon$. □

To conclude the proof, consider $Q \in \mathcal{G}^0_\delta$. We use Claim 4 to approximate Q in $W^{1,3}$-norm by $Q_i \in \mathcal{G}^{\alpha_i}_{2\delta}$ with $\alpha_i > 0$. From Claims 1–3 it follows that there exist functions $P_i \in W^{1,4+\alpha_i}(B^4, \mathrm{SU}(2))$ such that

$$\|P_i - I\|_{W^{1,4}(B^4)} \leq K\|dQ_i\|_{L^3(\mathbb{S}^3)} \leq 2K\delta.$$

The P_i have a weakly convergent subsequence whose limit P satisfies

$$\|P - I\|_{W^{1,4}(B^4)} \leq 2K\delta \quad \text{and} \quad \begin{cases} \operatorname{div}(P^{-1}\nabla P) = 0 & \text{on } B^4 \\ P = Q & \text{on } \mathbb{S}^3. \end{cases}$$

Choose $\delta > 0$ such that $2K\delta \leq \epsilon$ for ϵ as in Lemma A.1. We can then apply that lemma and obtain that

$$\|P - I\|_{W^{1,4}(B^4)} \leq c\|P - I\|_{W^{4/3,3}(B^4)} \leq cC_\epsilon\|Q\|_{L^3(\mathbb{S}^3)}. \qquad \square$$

We now complete the details of the proof of Claim 3:

Lemma A.3. *There exist $\epsilon > 0$, $K > 0$ such that for all $\alpha > 0$ there exists $C_\alpha > 0$ with the following property: Let $Q_0 \in W^{1,3+\alpha}(\mathbb{S}^3, \mathrm{SU}(2))$ and let $P_0 \in W^{1,4+\alpha}(B^4, \mathrm{SU}(2))$ be an extension of Q_0 which satisfies $\operatorname{div}(P_0^{-1}\nabla P_0) = 0$. If the estimates*

$$\|dQ_0\|_{W^{1,3}(\mathbb{S}^3)} < \epsilon, \tag{A-15}$$

$$\|P_0 - I\|_{W^{1,4}(B^4)} \leq K\|dQ_0\|_{W^{1,3}(\mathbb{S}^3)}, \tag{A-16}$$

$$\|P_0 - I\|_{W^{1,4+\alpha}(B^4)} \leq C_\alpha\|dQ_0\|_{W^{1,3+\alpha}(\mathbb{S}^3)} \tag{A-17}$$

hold then, for some $\delta > 0$ depending on Q_0, for all Q satisfying

$$\|Q - Q_0\|_{W^{1,3+\alpha}(\mathbb{S}^3,\mathrm{SU}(2))} < \delta \tag{A-18}$$

there exists an extension P of Q satisfying the same equation $\operatorname{div}(P^{-1}\nabla P) = 0$ and such that (A-15), (A-16), (A-17) hold with P, Q in place of P_0, Q_0.

Proof. We fix Q satisfying (A-18) and (A-15). The proof is divided into two parts:

Claim 3.1. *For $\delta > 0$ small enough and for Q satisfying (A-18), there exists an extension P of Q solving $\operatorname{div}(P^{-1}\nabla P) = 0$ and such that (A-17) holds.*

Claim 3.2. *The function P of Claim 3.1 satisfies (A-16).*

Proof of Claim 3.2. This follows directly from Lemma A.1. □

Proof of Claim 3.1. First note that $V = \exp^{-1}(Q_0^{-1}Q)$ is well defined for $\alpha > 0$, because in that case we have an estimate of the form

$$\|Q - Q_0\|_{W^{1,3+\alpha}} \geq c_\alpha \|Q - Q_0\|_{L^\infty} \Longleftrightarrow \|Q_0^{-1}Q - I\|_{L^\infty} \leq \frac{\epsilon}{c_\alpha}$$

and $\exp^{-1}$ is well defined in a neighborhood of the identity.

We consider the problem of extending $Q_0 \exp(V)$ inside B^4 to a function $P = P_0 \exp(U)$. Extend V to $\tilde{V}$ such that $\Delta \tilde{V} = 0$ inside B^4.

We look for a P of the form $P_0 \exp(\tilde{V}) \exp(U)$. We thus consider the equation

$$\mathcal{N}(U, V) := d^*\big(\exp(-U)\exp(-\tilde{V}) P_0^{-1} d(P_0 \exp(\tilde{V})\exp(U))\big) = 0. \tag{A-19}$$

In order to solve (A-19) it is useful to look at the operator

$$\mathcal{N}(V, U) : W_0^{1,4+\alpha}(B^4, \mathrm{su}(2)) \to W^{-1,4+\alpha}(B^4, \mathrm{su}(2)). \tag{A-20}$$

We have to show that for $\delta > 0$ small enough, for each Q satisfying (A-18) (i.e., for each small enough V), there exists a unique U such that $\mathcal{N}(V, U) = 0$. We prove that $\mathcal{N}(U, V)$ is C^1 near $(U, V) = (0, 0)$ and that $\partial\mathcal{N}/\partial U(0, 0)$ is an isomorphism, given the existence of $\delta > 0$ as desired.

A simple calculation gives

$$\begin{aligned}\frac{\partial \mathcal{N}}{\partial U} \cdot \eta = \frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{N}(U + t\eta, V) &= d^*d\eta - d^*\big[\eta, \exp(-U)\exp(-\tilde{V}) P_0^{-1} d(P_0 \exp(\tilde{V}))\exp(U)\big] \\ &:= \Delta\eta - L\eta.\end{aligned}$$

We observe that $d^*d = \Delta$ is an isomorphism between the spaces above, so it will be enough to show that for U, $\tilde{V}$ small enough in the $W^{1,4+\alpha}$-norm the commutator term $L\eta$ is just a small perturbation of Δ (with respect to the norms present in (A-20)). First note that we can write

$$L\eta = [\nabla\eta, X] + [\eta, \operatorname{div} X], \quad \text{where } X := \exp(-U)\exp(-\tilde{V}) P_0^{-1} d(P_0 \exp(\tilde{V}))\exp(U).$$

Estimate for $[\nabla\eta, X]$. First note that by the Sobolev, Hölder and triangle inequalities,

$$\|[\nabla\eta, X]\| W^{-1,4+\alpha} \lesssim \|[\nabla\eta, X]\|_{L^{p_\alpha}} \lesssim \|\nabla\eta\|_{L^{4+\alpha}} \|X\|_{L^4},$$

where

$$\frac{1}{p_\alpha} = \frac{1}{4+\alpha} + \frac{1}{4}.$$

We then observe

$$X = \exp(-U)\exp(-\tilde{V}) P_0^{-1} d(P_0 \tilde{V})\exp(\tilde{V})\exp(U)$$

and note $|\exp A| = 1$; therefore,

$$\|X\|_{L^4} = \|d(P_0 \tilde{V})\|_{L^4} \lesssim \|dP_0\|_{L^4} + \|d\tilde{V}\|_{L^4} \lesssim \epsilon + \delta.$$

We thus have the first desired estimate,

$$\|[\nabla\eta, X]\|_{W^{-1,4+\alpha}} \lesssim (\epsilon + \delta)\|\eta\|_{W^{1,4+\alpha}}.$$

Estimate for $[\eta, \operatorname{div} X]$. Here we start with

$$\|[\eta, \operatorname{div} X]\|_{W^{-1,4+\alpha}} \lesssim \|\eta\|_{L^\infty} \|\operatorname{div} X\|_{L^{p\alpha}} .$$

Note that $\|\eta\|_{L^\infty} \lesssim \|\eta\|_{W^{1,4+\alpha}}$ by the Sobolev embedding. We start the computations for the second fact or above. Note that

$$\nabla(P_0 \exp \widetilde{V}) = (\nabla P_0) \exp \widetilde{V} + P_0 \nabla(\exp \widetilde{V})$$

and then expand:

$$\begin{aligned}
\operatorname{div} X &= \operatorname{div}[\exp(-U) \exp(-\widetilde{V}) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \exp U] \\
&= \nabla(\exp(-U)) \exp(-\widetilde{V}) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \nabla(\exp(-\widetilde{V})) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) \operatorname{div} \big(P_0^{-1} \nabla P_0\big) \exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) P_0^{-1} P_0 \operatorname{div} \nabla(\exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) P_0^{-1} \nabla P_0 \nabla(\exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \nabla(\exp U)
\end{aligned}$$

We have $\operatorname{div}(P_0^{-1} \nabla P_0) = 0$ and $\operatorname{div} \nabla(\exp(\widetilde{V})) = 0$, so two terms cancel. Note also the fact that $\|P_0^{-1} \nabla P_0\|_{L^4} \le \|\nabla P_0\|_{L^4} \le \epsilon$. For estimating $\nabla(\exp(\pm \widetilde{V}))$ observe that $\widetilde{V}$ satisfies a Dirichlet boundary value problem, therefore we assume the estimate $\|\widetilde{V}\|_{W^{1,4+\alpha}} \lesssim \delta$, and $\|U\|_{W^{1,4+\alpha}} \lesssim \delta$, which, by the smoothness of exp, imply $\|\nabla(\exp(\pm \widetilde{V}))\|_{L^{4+\alpha}} \lesssim \delta$ and $\|\nabla(\exp(\pm U))\|_{L^{4+\alpha}} \lesssim \delta$. From all this it follows that we can estimate

$$\begin{aligned}
\|\operatorname{div} X\|_{L^{p\alpha}} &\lesssim \|\nabla(\exp(-U))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} + \|\nabla(\exp(-\widetilde{V}))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} \\
&\qquad + \|\nabla P_0\|_{L^4} \|\nabla(\exp(\widetilde{V}))\|_{L^{4+\alpha}} + \|\nabla(\exp(U))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} \\
&\lesssim \delta \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} + \epsilon\delta \\
&\lesssim \delta(\epsilon + \delta).
\end{aligned}$$

We combine all the estimates and obtain the desired smallness result,

$$\|[\eta, \operatorname{div} X]\|_{W^{-1,4+\alpha}} \lesssim \delta(\epsilon + \delta) \|\eta\|_{W^{1,4+\alpha}} . \qquad \square$$

End of proof. We now have that

$$\|L\eta\|_{W^{-1,4+\alpha}} \lesssim (\delta + 1)(\epsilon + \delta) \|\eta\|_{W^{1,4+\alpha}} ,$$

while

$$\|\Delta\eta\|_{W^{-1,4+\alpha}} \gtrsim \|\eta\|_{W^{1,4+\alpha}} .$$

Therefore, for small enough ϵ, δ we have also

$$\|(\Delta - L)\eta\|_{W^{-1,4+\alpha}} \gtrsim \|\eta\|_{W^{1,4+\alpha}} .$$

This concludes the proof. $\square$

Appendix B: A product estimate with only one bounded factor

Lemma B.1 (cf. [Brézis and Mironescu 2001]). *Let Ω be a smooth compact 4-manifold. If $f \in W^{1/3,3}(\Omega)$ and $g \in W^{1,4} \cap L^\infty(\Omega)$, then we have the following estimate, with the implicit constant depending only on Ω:*

$$\|fg\|_{W^{1/3,3}(\Omega)} \lesssim \|f\|_{W^{1/3,3}(\Omega)}(\|g\|_{L^\infty(\Omega)} + \|g\|_{W^{1,4}(\Omega)}).$$

Proof. The estimates for the nonhomogeneous part of the norms are trivial, so we concentrate on the homogeneous part.

We use the Littlewood–Paley decompositions $f = \sum_{j=0}^\infty f_j$, $g = \sum_{k=0}^\infty g_k$, and we recall that the $W^{s,p}$-norm is equivalent to the Triebel–Lizorkin $\dot{F}^1_{4,2}$-norm and the $W^{\theta,4}$-norm is equivalent to the $\dot{F}^s_{p,2}$-norm, where in general the following definition holds:

$$\|f\|_{\dot{F}^s_{p,q}} = \big\||2^{ks} f_k(x)|_{\ell^q}\big\|_{L^p}.$$

We use different notations $\|\cdot\|$, $|\cdot|$ for the different norms just to facilitate the reading of formulas. As is usual in the theory of paraproducts, we estimate separately the following three contributions (where $g^k := \sum_{i=0}^k g_k$, and similarly for f^k)

$$fg = \sum_i f_i g^{i-4} + \sum_{|k-l|<4} f_k g_l + \sum_i f^{i-4} g_i =: I + II + III.$$

The support of $(\widehat{f_i g^{i-4}})$ is included in $B_{2^{i+2}} \setminus B_{2^{i-2}}$; thus,

$$\|I\|_{W^{1/3,3}} = \Big\|\sum_i f_i g^{i-4}\Big\|_{W^{1/3,3}} \sim \bigg[\int_\Omega \Big(\sum_i 2^{2i/3} |f_i g^{i-4}|^2\Big)^{3/2}\bigg]^{1/3} \tag{B-1}$$

and analogously for $III = \sum_i f^{i-4} g_i$. Regarding the term II, we will estimate only $II' := \sum_i f_i g_i$ because the same estimate will apply also to the finitely many contributions of the form $\sum_i f_i g_{i+l}$ with $0 < |l| < 4$.

We start with the most difficult term, III. From above we have

$$\begin{aligned}
\|III\|_{W^{1/3,3}} &\sim \bigg[\int \Big(\sum_i 2^{2i/3} |f^{i-4} g_i|^2\Big)^{3/2}\bigg]^{1/3} \\
&\le \bigg[\int \Big(\sum_i 2^{-4i/3} |f^{i-4}|^2\Big)^{3/2} \Big(\sum_i 2^{2i} |g_i|^2\Big)^{3/2}\bigg]^{1/3} \\
&\le \bigg[\int \Big(\sum_i 2^{-4i/3} |f^{i-4}|^2\Big)^{6}\bigg]^{1/12} \bigg[\int \Big(\sum_i 2^{2i} |g_i|^2\Big)^{2}\bigg]^{1/4} \\
&\le \|f\|_{W^{-2/3,12}} \|g\|_{W^{1,4}} \\
&\le \|f\|_{W^{1/3,3}} \|g\|_{W^{1,4}}.
\end{aligned}$$

For the term I we have

$$\|I\|_{W^{1/3,3}} \sim \left[\int \left(\sum_i 2^{2i/3} |f_i g^{i-4}|^2 \right)^{3/2} \right]^{1/3} \lesssim \|g\|_{L^\infty} \|f\|_{W^{1/3,3}}$$

because of the estimate $\|g^{i-4}\|_{L^\infty} \lesssim \|g\|_{L^\infty}$. Finally, we estimate II', as promised. We prove it by duality; namely, we prove that II' is bounded as a linear functional on the unit ball of the dual $W^{-1/3,3/2}$. Consider therefore h in this ball. The support of $(\widehat{f_i g_i})$ is included in $B_{2^{i+2}}$, so some terms cancel:

$$\begin{aligned}
\int h \cdot II' &\sim \sum_{k,i} \int h_k f_i g_i = \sum_{k \le i+4} \int h_k f_i f_j = \sum_i \int h^{i+4} f_i g_i \\
&\le \left| \sum_i \int 2^{-i/3} h^{i+4} 2^{i/3} f_i g_i \right| \\
&\le \|g\|_{B^0_{\infty,\infty}} \int \left(\sum_i 2^{-2i/3} |h^{i+4}|^2 \right)^{1/2} \left(\sum_i 2^{2i/3} |f_i|^2 \right)^{1/2} \\
&\le \|g\|_{W^{1,4}} \|h\|_{W^{-1/3,3/2}} \|f\|_{W^{1/3,3}} .
\end{aligned}$$

The last estimate follows, recalling that

$$\|g\|_{B^0_{\infty,\infty}} := \sup_i \|g_i\|_{L^\infty}$$

and that in dimension 4 we have continuous embeddings

$$W^{1,4} \hookrightarrow \mathrm{BMO} \hookrightarrow B^0_{\infty,\infty} .$$

Summing up the different terms, we are done. □

Appendix C: The Möbius group of B^n

We call the Möbius group of $\mathbb{R}^n$ the group $M(\mathbb{R}^n)$ generated by all similarities and the inversion with respect to the unit sphere. Recall that a similarity is an affine map of the form

$$x \mapsto \lambda K x + b \quad \text{with } \lambda > 0,\ K \in O(n),\ b \in \mathbb{R}^n,$$

and the inversion $i_{c,r}$ with respect to the sphere $\partial B(c, r)$ is the map

$$x \mapsto c + r^2 \frac{x - c}{|x - c|^2} .$$

The formula $i_{c,r} = (r^2 \operatorname{id} + c) \circ i_{0,1} \circ (\operatorname{id} - c)$ shows that all inversions belong to $M(\mathbb{R}^n)$. We use the abridged notation

$$x^* := i_{1,0}(x) = \frac{x}{|x|^2} .$$

The Möbius group of B^{n+1} is the subgroup $M(B^{n+1})$ of all transformations belonging to $M(\mathbb{R}^{n+1})$ which preserve B^{n+1}. Similarly, we define the Möbius group $M(\mathbb{S}^n)$ of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The general form of an element $\gamma \in M(B^{n+1})$ is

$$\gamma = K \circ F_v \quad \text{with } K \in O(n),\ v \in B^{n+1},\ F_v := -v + (1-|v|^2)(x^* - v)^*.$$

We use the following basic properties of the functions F_v, which can be found in [Ahlfors 1981, Chapter 2]:

Lemma C.1. • *We have*

$$|F_v|(x) = \frac{1-|v|^2}{[x, v]},$$

where $[x, y] = |x|\,|x^* - y| = |y|\,|y^* - x|$.

- F_v *is conformal. We have* $F_v^{-1} = F_{-v}$, $F_v(0) = -v$ *and* $F_v(v) = 0$.
- *The conformal factor* $|F_v'|(x)$ *is explicitly computed as*

$$|F_v'|(x) = \frac{1-|v|^2}{1+|x|^2|v|^2 - 2x\cdot v} = \frac{|v^*|^2 - 1}{|x - v^*|^2}.$$

- *The restriction* $F_v|_{\mathbb{S}^n}$ *belongs to* $M(\mathbb{S}^n)$; *in particular*, $F_v|_{\mathbb{S}^n}$ *is a conformal involution and*

$$|(F_v|_{\mathbb{S}^n})'|(x) = \frac{1-|v|^2}{|x - v|^2}.$$

The next lemma gives the estimate needed for the case when v is close to ∂B^{n+1}:

Lemma C.2. *Suppose that*

$$\rho \le \tfrac{1}{4}.$$

Then, on $F_v^{-1}(B_{1-\rho})$, *the following estimate holds with a constant* C *dependent only on the dimension*:

$$\frac{h(v)}{C} \le |F_v'|(x) \le C h(v).$$

Proof. We will calculate

$$\frac{\max\{|F_v'|(y) : y \in F_v^{-1}(B_{1-\rho})\}}{\min\{|F_v'|(y') : y' \in F_v^{-1}(B_{1-\rho})\}} = \max\left\{\frac{|F_v'|(y)}{|F_v'|(y')} : y, y' \in F_v^{-1}(B_{1-\rho})\right\}$$

and we show that this quantity is bounded. The following equalities hold:

$$\begin{aligned}
\max\left\{\frac{|F_v'|(x)}{|F_v'|(x')} : x, x' \in B_{1-\rho}\right\} &= \max\left\{\frac{|F_{-v}'|(x)}{|F_{-v}'|(x')} : x, x' \in B_{1-\rho}\right\}\\
&= \max\left\{\frac{|(F_v^{-1})'|(x)}{|(F_v^{-1})'|(x')} : x, x' \in B_{1-\rho}\right\}\\
&= \min\left\{\frac{|F_v'|(F_v^{-1}(x'))}{|F_v'|(F_v^{-1}(x))} : x, x' \in B_{1-\rho}\right\}\\
&= \min\left\{\frac{|F_v'|(y')}{|F_v'|(y)} : y, y' \in F_v^{-1}(B_{1-\rho})\right\}.
\end{aligned}$$

From the formula of the previous lemma it follows that

$$\nabla_x |F_v'|(x) = 2\frac{|v^*|^2-1}{|v^*-x|^4}(v^*-x);$$

therefore, $|F_v'|$ achieves its extrema on $B_{1-\rho}$ at $\pm(1-\rho)v/|v|$. The maximum M and the minimum m of $|F_v'|$ satisfy

$$\begin{aligned} M &= \frac{1-|v|^2}{1+|v|^2(1-\rho)^2-2(1-\rho)|v|} = \frac{1-|v|^2}{(1-(1-\rho)|v|)^2}, \\ m &= \frac{1-|v|^2}{1+|v|^2(1-\rho)^2+2(1-\rho)|v|} = \frac{1-|v|^2}{(1+(1-\rho)|v|)^2}, \\ \frac{M}{m} &= \left(\frac{1+(1-\rho)|v|}{1-(1-\rho)|v|}\right)^2 \sim (1-(1-\rho)|v|)^{-2} \sim 1, \end{aligned}$$

which finishes the proof. □

References

[Adams 1975] D. R. Adams, "A note on Riesz potentials", *Duke Math. J.* **42**:4 (1975), 765–778. MR 56 #16361 Zbl 0336.46038

[Ahlfors 1981] L. V. Ahlfors, *Möbius transformations in several dimensions*, University of Minnesota, School of Mathematics, Minneapolis, MN, 1981. MR 84m:30028 Zbl 0517.30001

[Atiyah et al. 1978] M. F. Atiyah, N. J. Hitchin, and I. M. Singer, "Self-duality in four-dimensional Riemannian geometry", *Proc. Roy. Soc. London Ser. A* **362**:1711 (1978), 425–461. MR 80d:53023 Zbl 0389.53011

[Bernard and Rivière 2014] Y. Bernard and T. Rivière, "Energy quantization for Willmore surfaces and applications", *Ann. of Math.* (2) **180**:1 (2014), 87–136. MR 3194812 Zbl 06316067 arXiv 1106.3780

[Bethuel 1991] F. Bethuel, "The approximation problem for Sobolev maps between two manifolds", *Acta Math.* **167**:3-4 (1991), 153–206. MR 92f:58023 Zbl 0756.46017

[Bethuel and Chiron 2007] F. Bethuel and D. Chiron, "Some questions related to the lifting problem in Sobolev spaces", pp. 125–152 in *Perspectives in nonlinear partial differential equations*, edited by H. Berestycki et al., Contemporary Mathematics **446**, American Mathematical Society, Providence, RI, 2007. MR 2009j:46075 Zbl 1201.46029

[Bethuel and Demengel 1995] F. Bethuel and F. Demengel, "Extensions for Sobolev mappings between manifolds", *Calc. Var. Partial Differential Equations* **3**:4 (1995), 475–491. MR 97j:46026 Zbl 0846.46021

[Bethuel and Zheng 1988] F. Bethuel and X. Zheng, "Density of smooth functions between two manifolds in Sobolev spaces", *J. Funct. Anal.* **80**:1 (1988), 60–75. MR 89i:58015 Zbl 0657.46027

[Bourgain et al. 2000] J. Bourgain, H. Brézis, and P. Mironescu, "Lifting in Sobolev spaces", *J. Anal. Math.* **80** (2000), 37–86. MR 2001h:46044 Zbl 0967.46026

[Bourgain et al. 2004] J. Bourgain, H. Brézis, and P. Mironescu, "$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg–Landau equation", *Publ. Math. Inst. Hautes Études Sci.* **99** (2004), 1–115. MR 2005k:58017 Zbl 1051.49030

[Brézis and Mironescu 2001] H. Brézis and P. Mironescu, "Gagliardo–Nirenberg, composition and products in fractional Sobolev spaces", *J. Evol. Equ.* **1**:4 (2001), 387–404. MR 2002k:46073 Zbl 1023.46031

[Brézis and Nirenberg 1995] H. Brézis and L. Nirenberg, "Degree theory and BMO, I: Compact manifolds without boundaries", *Selecta Math.* (*N.S.*) **1**:2 (1995), 197–263. MR 96g:58023 Zbl 0852.58010

[Brézis and Nirenberg 1996] H. Brézis and L. Nirenberg, "Degree theory and BMO, II: Compact manifolds with boundaries", *Selecta Math.* (*N.S.*) **2**:3 (1996), 309–368. MR 98a:58024 Zbl 0868.58017

[Coifman et al. 1993] R. R. Coifman, P.-L. Lions, Y. Meyer, and S. Semmes, "Compensated compactness and Hardy spaces", *J. Math. Pures Appl.* (9) **72**:3 (1993), 247–286. MR 95d:46033 Zbl 0864.42009

[Donaldson 1983] S. K. Donaldson, "An application of gauge theory to four-dimensional topology", *J. Differential Geom.* **18**:2 (1983), 279–315. MR 85c:57015 Zbl 0507.57010

[Freed and Uhlenbeck 1984] D. S. Freed and K. K. Uhlenbeck, *Instantons and four-manifolds*, Mathematical Sciences Research Institute Publications **1**, Springer, New York, 1984. MR 86c:57031 Zbl 0559.57001

[Gazzola et al. 2010] F. Gazzola, H.-C. Grunau, and G. Sweers, *Polyharmonic boundary value problems: positivity preserving and nonlinear higher order elliptic equations in bounded domains*, Lecture Notes in Mathematics **1991**, Springer, Berlin, 2010. MR 2011h:35001 Zbl 1239.35002

[Grafakos 2008] L. Grafakos, *Classical Fourier analysis*, 2nd ed., Graduate Texts in Mathematics **249**, Springer, New York, 2008. MR 2011c:42001 Zbl 1220.42001

[Hang and Lin 2001] F. Hang and F.-H. Lin, "Topology of Sobolev mappings", *Math. Res. Lett.* **8**:3 (2001), 321–330. MR 2002i:58009 Zbl 1049.46018

[Hang and Lin 2003] F. Hang and F.-H. Lin, "Topology of Sobolev mappings, II", *Acta Math.* **191**:1 (2003), 55–107. MR 2005m:58023 Zbl 1061.46032

[Hardt and Lin 1987] R. Hardt and F.-H. Lin, "Mappings minimizing the L^p norm of the gradient", *Comm. Pure Appl. Math.* **40**:5 (1987), 555–588. MR 88k:58026 Zbl 0646.49007

[Hardt and Rivière 2003] R. Hardt and T. Rivière, "Connecting topological Hopf singularities", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **2**:2 (2003), 287–344. MR 2004j:58009 Zbl 1150.58004

[Hardt and Rivière 2008] R. Hardt and T. Rivière, "Connecting rational homotopy type singularities", *Acta Math.* **200**:1 (2008), 15–83. MR 2010h:58016 Zbl 1148.58003

[Hardt et al. 1986] R. Hardt, D. Kinderlehrer, and F.-H. Lin, "Existence and partial regularity of static liquid crystal configurations", *Comm. Math. Phys.* **105**:4 (1986), 547–570. MR 88a:35207 Zbl 0611.35077

[Hatcher 2009] A. Hatcher, "Vector bundles and K-theory", preprint, 2009, Available at http://www.math.cornell.edu/~hatcher/VBKT/VBpage.html.

[Hélein 1996] F. Hélein, *Applications harmoniques, lois de conservation et repères mobiles*, Diderot, Paris, 1996. Translated as *Harmonic maps, conservation laws and moving frames*, 2nd ed., Cambridge Tracts in Mathematics **150**, Cambridge University Press, 2002. MR 2003g:58024 Zbl 0920.58022

[Hildebrandt 1982] S. Hildebrandt, "Nonlinear elliptic systems and harmonic mappings", pp. 481–615 in *Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations* (Beijing, 1980), vol. 1, edited by S. S. Chern and W.-t. Wu, Science Press, Beijing, 1982. MR 85k:35078 Zbl 0515.58012

[Isobe 2006] T. Isobe, "On global singularities of Sobolev mappings", *Math. Z.* **252**:4 (2006), 691–730. MR 2007b:46142 Zbl 1099.46052

[Kuwert and Li 2012] E. Kuwert and Y. Li, "$W^{2,2}$-conformal immersions of a closed Riemann surface into $\mathbb{R}^n$", *Comm. Anal. Geom.* **20**:2 (2012), 313–340. MR 2928715 Zbl 1271.53010 arXiv 1007.3967

[Marques and Neves 2014] F. C. Marques and A. Neves, "Min-max theory and the Willmore conjecture", *Ann. of Math.* (2) **179**:2 (2014), 683–782. MR 3152944 Zbl 06284347 arXiv 1202.6036

[Mucci 2010] D. Mucci, "The homological singularities of maps in trace spaces between manifolds", *Math. Z.* **266**:4 (2010), 817–849. MR 2011m:49093 Zbl 1200.49044

[Müller and Šverák 1995] S. Müller and V. Šverák, "On surfaces of finite total curvature", *J. Differential Geom.* **42**:2 (1995), 229–258. MR 97b:53007 Zbl 0853.53003

[Nicolesco 1936] M. Nicolesco, *Les fonctions polyharmoniques*, Actualités Scientifiques et Industrielles **331**, Hermann, Paris, 1936. Exposés sur la théorie des fonctions, IV. Zbl 0015.15905

[Petrache 2013] M. Petrache, *Weak bundle structures and a variational approach to Yang–Mills Lagrangians in supercritical dimensions*, thesis, Eidgenössische Technische Hochschule, Zürich, 2013, Available at http://www.math.ethz.ch/~riviere/papers/petrache-thesis.pdf.

[Petrache and Rivière ≥ 2014] M. Petrache and T. Rivière, "Weak closure of weak L^2 curvatures in supercritical dimension", preprint. In preparation.

[Rivière 2007] T. Rivière, "Conservation laws for conformally invariant variational problems", *Invent. Math.* **168**:1 (2007), 1–22. MR 2008d:58010 Zbl 1128.58010

[Rivière 2012] T. Rivière, "Conformally invariant variational problems", preprint, 2012. arXiv 1206.2116

[Runst and Sickel 1996] T. Runst and W. Sickel, *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, de Gruyter Series in Nonlinear Analysis and Applications **3**, de Gruyter, Berlin, 1996. MR 98a:47071 Zbl 0873.35001

[Schoen and Uhlenbeck ∼1980] R. Schoen and K. K. Uhlenbeck, "Approximation theorems for Sobolev mappings", unpublished manuscript, ∼1980.

[Serfaty and Tice 2008] S. Serfaty and I. Tice, "Lorentz space estimates for the Ginzburg–Landau energy", *J. Funct. Anal.* **254**:3 (2008), 773–825. MR 2008k:58045 Zbl 1159.26004

[Tartar 2007] L. Tartar, *An introduction to Sobolev spaces and interpolation spaces*, Lecture Notes of the Unione Matematica Italiana **3**, Springer, Berlin, 2007. MR 2008g:46055 Zbl 1126.46001

[Taubes 1982] C. H. Taubes, "Self-dual Yang–Mills connections on non-self-dual 4-manifolds", *J. Differential Geom.* **17**:1 (1982), 139–170. MR 83i:53055 Zbl 0484.53026

[Triebel 1995] H. Triebel, *Interpolation theory, function spaces, differential operators*, 2nd ed., Johann Ambrosius Barth, Heidelberg, 1995. MR 96f:46001 Zbl 0830.46028

[Uhlenbeck 1982a] K. K. Uhlenbeck, "Removable singularities in Yang–Mills fields", *Comm. Math. Phys.* **83**:1 (1982), 11–29. MR 83e:53034 Zbl 0491.58032

[Uhlenbeck 1982b] K. K. Uhlenbeck, "Connections with L^p bounds on curvature", *Comm. Math. Phys.* **83**:1 (1982), 31–42. MR 83e:53035 Zbl 0499.58019

[Wente 1969] H. C. Wente, "An existence theorem for surfaces of constant mean curvature", *J. Math. Anal. Appl.* **26** (1969), 318–344. MR 39 #4788 Zbl 0181.11501

[White 1988] B. White, "Homotopy classes in Sobolev spaces and the existence of energy minimizing maps", *Acta Math.* **160**:1-2 (1988), 1–17. MR 89a:58031 Zbl 0647.58016

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MIRCEA PETRACHE: decostruttivismo@gmail.com
Laboratoire Jacques-Louis Lions, Universitè Pierre et Marie Curie (Paris 6), place Jussieu, 4, 75005 Paris, France

TRISTAN RIVIÈRE: decostruttivismo@gmail.com
Department of Mathematics, ETH Zürich, CH-8092 Zürich, Switzerland

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