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QUANTIZED SLOW BLOW-UP DYNAMICS FOR THE COROTATIONAL ENERGY-CRITICAL HARMONIC HEAT FLOW

PIERRE RAPHAËL AND REMI SCHWEYER

We consider the energy-critical harmonic heat flow from $\mathbb{R}^2$ into a smooth compact revolution surface of $\mathbb{R}^3$. For initial data with corotational symmetry, the evolution reduces to the semilinear radially symmetric parabolic problem

$$\partial_t u - \partial_r^2 u - \frac{\partial_r u}{r} + \frac{f(u)}{r^2} = 0$$

for a suitable class of functions f. Given an integer $L \in \mathbb{N}^*$, we exhibit a set of initial data arbitrarily close to the least energy harmonic map Q in the energy-critical topology such that the corresponding solution blows up in finite time by concentrating its energy

$$\nabla u(t,r) - \nabla Q\left(\frac{r}{\lambda(t)}\right) \to u^* \quad \text{in } L^2$$

at a speed given by the *quantized* rates

$$\lambda(t) = c(u_0)(1+o(1))\frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}},$$

in accordance with the formal predictions of van den Berg et al. (2003). The case $L = 1$ corresponds to the stable regime exhibited in our previous work (*CPAM*, 2013), and the data for $L \geq 2$ leave on a manifold of codimension $L-1$ in some weak sense. Our analysis is a continuation of work by Merle, Rodnianski, and the authors (in various combinations) and it further exhibits the mechanism for the existence of the excited slow blow-up rates and the associated instability of these threshold dynamics.

1. Introduction

The parabolic heat flow. The harmonic heat flow between two embedded Riemannian manifolds (N, g_N), (M, g_M) is the gradient flow associated to the Dirichlet energy of maps from $N \to M$:

$$\begin{cases} \partial_t v = \mathbb{P}_{T_v M}(\Delta_{g_N} v), \\ v_{|t=0} = v_0, \end{cases} \qquad (t,x) \in \mathbb{R} \times N,\ v(t,x) \in M, \tag{1-1}$$

where $\mathbb{P}_{T_v M}$ is the projection onto the tangent space to M at v. The special case $N = \mathbb{R}^2$, $M = \mathbb{S}^2$ corresponds to the harmonic heat flow to the 2-sphere

$$\partial_t v = \Delta v + |\nabla v|^2 v, \quad (t,x) \in \mathbb{R} \times \mathbb{R}^2,\ v(t,x) \in \mathbb{S}^2, \tag{1-2}$$

and is related to the Landau–Lifschitz equation of ferromagnetism; we refer to [van den Berg et al. 2003;

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Angenent et al. 2009; Guan et al. 2009; Gustafson et al. 2010] for a complete introduction to this class of problems. We shall from now on restrict our discussion to the case

$$N = \mathbb{R}^2.$$

Smooth initial data yield unique local-in-time smooth solutions which dissipate the Dirichlet energy

$$\frac{d}{dt}\left\{\int_{\mathbb{R}^2} |\nabla v|^2\right\} = -2\int_{\mathbb{R}^2} |\partial_t v|^2.$$

An essential feature of the problem is that the scaling symmetry

$$u \mapsto u_\lambda(t,x) = u(\lambda^2 t, \lambda x), \quad \lambda > 0,$$

leaves the Dirichlet energy unchanged, and hence the problem is *energy-critical*.

Corotational flows. We restrict our attention in this paper to flows with so-called corotational symmetry. More precisely, let us consider a smooth closed curve in the plane parametrized by arclength

$$u \in [-\pi, \pi] \mapsto \left| \begin{array}{l} g(u) \\ z(u), \end{array} \right. \quad (g')^2 + (z')^2 = 1,$$

where

$$(H) \quad \begin{cases} g \in \mathscr{C}^\infty(\mathbb{R}) \text{ is odd and } 2\pi \text{ periodic}, \\ g(0) = g(\pi) = 0,\ g(u) > 0 \quad \text{for } 0 < u < \pi, \\ g'(0) = 1,\ g'(\pi) = -1. \end{cases} \tag{1-3}$$

Then the revolution surface M with parametrization

$$(\theta, u) \in [0, 2\pi] \times [0, \pi] \mapsto \left| \begin{array}{l} g(u)\cos\theta \\ g(u)\sin\theta \\ z(u) \end{array} \right.$$

is a smooth [1] compact revolution surface of $\mathbb{R}^3$ with metric $(du)^2 + g^2(u)(d\theta)^2$. Given a homotopy degree $k \in \mathbb{Z}^*$, the k-corotational reduction to (1-1) corresponds to solutions of the form

$$v(t,r) = \left| \begin{array}{l} g(u(t,r))\cos(k\theta) \\ g(u(t,r))\sin(k\theta) \\ z(u(t,r)), \end{array} \right. \tag{1-4}$$

which leads to the semilinear parabolic equation

$$\begin{cases} \partial_t u - \partial_r^2 u - \dfrac{\partial_r u}{r} + k^2 \dfrac{f(u)}{r^2} = 0, \\ u_{t=0} = u_0, \end{cases} \qquad f = gg'. \tag{1-5}$$

The k-corotational Dirichlet energy becomes

$$E(u) = \int_0^{+\infty} \left[|\partial_r u|^2 + k^2 \frac{(g(u))^2}{r^2} \right] r\,dr \tag{1-6}$$

[1] See [Gallot et al. 2004], for example.

and is minimized among maps with boundary conditions

$$u(0)=0, \qquad \lim_{r\to+\infty} u(r)=\pi \tag{1-7}$$

onto the least energy harmonic map Q_k, which is the unique, up-to-scaling solution to

$$r\partial_r Q_k = kg(Q_k) \tag{1-8}$$

satisfying (1-7); see for example [Côte 2005]. In the case of $\mathbb{S}^2$ target $g(u)=\sin u$, the harmonic map is explicitly given by

$$Q_k(r)=2\tan^{-1}(r^k). \tag{1-9}$$

The blow-up problem. The question of the existence of blow-up solutions and the description of the associated concentration of energy scenario has attracted considerable attention for the past thirty years. In the pioneering works of Struwe [1985], Ding and Tian [1995], and Qing and Tian [1997] (see [Topping 2004] for a complete history of the problem), it was shown that if occurring, the concentration of energy implies the bubbling off of a nontrivial harmonic map at a finite number of blow-up points

$$v(t_i, a_i+\lambda(t_i)x)\to Q_i, \quad \lambda(t_i)\to 0 \tag{1-10}$$

locally in space. In particular, this shows the existence of a global in time flow on negatively curved targets where no nontrivial harmonic map exists.

For corotational data and homotopy number $k\ge 2$, Guan, Gustaffson, Nakanishi, and Tsai [Guan et al. 2009; Gustafson et al. 2010] proved that the flow is globally defined near the ground state harmonic map. In fact, Q_k is asymptotically stable for $k\ge 3$, and in particular no blow-up will occur. Eternally oscillating solutions and infinite time grow up solutions are exhibited for $k=2$.

In contrast, for $k=1$, the existence of finite time blow-up solutions has been proved in various geometrical settings strongly using the maximum principle; see in particular the work of Chang, Ding, and Ye [Chang et al. 1992], Coron and Ghidaglia [1989], Qing and Tian [1997], and Topping [2004]. Despite some serious efforts and the use of the maximum principle (see in particular [Angenent et al. 2009]), very little was known until recently about the description of the blow-up bubble and the derivation of the blow-up speed, in particular due to the critical nature of the problem.

For the rest of the paper, we focus on the degree

$$k=1$$

case, which generates the least energy, nontrivial harmonic map $Q\equiv Q_1$. For $\mathbb{D}^2$ initial manifold and $\mathbb{S}^2$ target, van den Berg, Hulshof, and King [van den Berg et al. 2003], in continuation of [Herrero and Velázquez 1994], implemented a formal analysis based on the matched asymptotics techniques and predicted the existence of blow-up solutions of the form

$$u(t,r)\sim Q\left(\frac{r}{\lambda(t)}\right) \tag{1-11}$$

with blow-up speed governed by the quantized rates

$$\lambda(t) \sim \frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}}, \quad L \in \mathbb{N}^*.$$

We will further discuss the presence of quantized rates which is reminiscent of the classification of type II blow-up for the supercritical nonlinear heat equation [Mizoguchi 2007].

We completely revisited the blow-up analysis in [Raphaël and Schweyer 2013] by adapting the strategy developed in [Raphaël and Rodnianski 2012; Merle et al. 2011] for the study of wave and Schrödinger maps, with two main new approaches:

- We completely avoid the formal matched asymptotics approach and replace it by an elementary derivation of an explicit and universal system of ODE's which drives the blow-up speed. A similar simplification further occurred in related critical settings; see in particular [Raphaël and Schweyer 2014].
- We designed a robust universal *energy method* to control the solution in the blow-up regime, which applies both to parabolic and dispersive problems. In particular, we aim to make no use of the maximum principle.

These techniques led to [Raphaël and Schweyer 2013] the construction of an *open set* of corotational initial data arbitrarily close to the ground state harmonic map in the energy-critical topology such that the corresponding solution to (1-5) bubbles off a harmonic map according to (1-11) at the speed

$$\lambda(t) \sim \frac{T-t}{|\log(T-t)|^2}, \quad \text{that is, } L = 1.$$

This is the *stable*[2] blow-up regime.

Statement of the result. Our main claim in this paper is that the analysis in [Raphaël and Schweyer 2013] can be further extended to exhibit the unstable modes which are responsible for *a discrete sequence of quantized slow blow-up rates*.

Theorem 1.1 (excited slow blow-up dynamics for the 1-corotational heat flow). *Let $k = 1$ and g satisfy (1-3). Let Q be the least energy harmonic map. Let $L \in \mathbb{N}^*$. Then there exists a smooth corotational initial data $u_0(r)$ such that the corresponding solution to (1-5) blows up in finite time $T = T(u_0) > 0$ by bubbling off a harmonic map*

$$\nabla u(t,r) - \nabla Q\left(\frac{r}{\lambda(t)}\right) \to \nabla u^* \quad \textit{in } L^2 \quad \textit{as } t \to T \tag{1-12}$$

at the excited rate

$$\lambda(t) = c(u_0)(1 + o_{t\to T}(1))\frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}}, \quad c(u_0) > 0. \tag{1-13}$$

Moreover, u_0 can be taken arbitrarily close to Q in the energy-critical topology.

[2] In the presence of corotational symmetry, blow-up dynamics are expected to be unstable by rotation under general perturbations; see [Merle et al. 2011].

Comments on the result. 1. *Regularity of the asymptotic profile.* Arguing as in [Raphaël and Schweyer 2013] and using the estimates of Proposition 3.1, one can directly relate the rate of blow-up (1-13) to the regularity of the remaining excess of energy, in the sense that u^* exhibits an H^{L+1} regularity is some suitable Sobolev sense; see Remark 4.1. See also [Merle and Raphaël 2005b] for a related phenomenon in the dispersive setting.

2. *Stable and excited blow-up rates.* The case $L = 1$ is treated in [Raphaël and Schweyer 2013] and corresponds to stable blow-up. For $L \geq 2$, the set of initial data leading to (1-13) is of codimension $(L-1)$ in the following sense: there exist fixed directions $(\psi_i)_{2\leq i\leq L}$ such that, for any suitable perturbation ε_0 of Q, there exist $(a_i(\varepsilon_0))_{2\leq i\leq L} \in \mathbb{R}^{L-1}$ such that the solution to (1-5) with data

$$Q + \varepsilon_0 + \sum_{i=2}^{L} a_i(\varepsilon_0)\psi_i$$

blows up in finite time with the blow-up speed (1-13). Building a smooth manifold would require proving local uniqueness and smoothness of the flow $\varepsilon_0 \mapsto a_i(\varepsilon_0))_{2\leq i\leq L}$, which is a separate problem; see, for example, [Krieger and Schlag 2009] for an introduction to this kind of issue. The control of the unstable modes relies on a classical soft and powerful Brouwer type topological argument in continuation of [Côte et al. 2011; Côte and Zaag 2013; Hillairet and Raphaël 2012].

3. *On quantized blow-up rates.* There is an important formal and rigorous literature on the existence of quantized blow-up rates for parabolic problems. In the pioneering works [Herrero and Velázquez 1994; Filippas et al. 2000], the authors predicted the existence of a sequence of quantized blow-up rates for the supercritical power nonlinearity heat equation

$$\partial_t u = \Delta u + u^p, \quad x \in \mathbb{R}^d,\ p > p(d),\ d < 7,$$

and this sequence is in one to one correspondence with the spectrum of the linearized operator close to the explicit singular self similar solution. After this formal work, and using the a priori bounds on radial type II blow-up solutions of Matano and Merle [2009; 2004], Mizogushi completely classified the radial data type II blow-up according to these quantized rates. Note that Mizogushi finishes the classification using the Matano–Merle a priori estimates on threshold dynamics, which heavily rely on the maximum principle, but the argument is not constructive. One of the main points of our work is to revisit the formal derivation of the sequence of blow-up rates and to relate it not to a spectral problem, but to the structure of the resonances of the linearized operator H close to Q *and of its iterates*, that is, the growing solutions to

$$H^k T_k = 0, \quad k \in \mathbb{N}^*.$$

In particular, we show how the *dynamics of tails* as initiated in [Raphaël and Rodnianski 2012; Merle et al. 2011] lead to a universal dynamical system driving the blow-up speed, which admits unstable solutions (1-13) corresponding to a codimension $(L-1)$ set of initial data. Another by-product of this analysis is the first explicit construction of type II blow-up for the energy-critical nonlinear heat equation [Schweyer 2012].

4. *Classification of the flow near Q.* The question of the classification of the flow near the harmonic map, and more generally near the ground state solitary wave in nonlinear evolution problems, has attracted considerable attention recently; see, for example, [Raphaël 2013]. This program has been concluded for the mass-critical (gKdV) equation in [Martel et al. 2012a; 2012b; 2012c], where it is shown that, provided the data is taken close enough to the ground state *in a suitable topology* which is strictly smaller than the energy norm, the blow-up dynamics are completely classified. In contrast, arbitrarily slow blow-up can be achieved for large deformations of the ground state in this restricted sense. The existence of such slow blow-up regimes remains however open in many important instances, in particular for the mass-critical NLS equation; see [Merle et al. 2013] for a further introduction to this delicate problem. For energy-critical problems like wave or Schrödinger maps, Krieger et al. [2008] showed that arbitrarily slow blow-up can be achieved, but the known examples so far are never $\mathscr{C}^\infty$ smooth. The structure of the flow near Q is also somewhat mysterious, and various new kinds of global dynamics have been constructed; see [Donninger and Krieger 2013; Bejenaru and Tataru 2014]. One of the new results of our analysis in this paper is to show the essential role played by the control of higher order Sobolev norms, which provide a new topology to measure the distance to the solitary wave which is sharp enough to see all the blow-up regimes (1-13). The control of these norms acts in the energy method as a replacement of the counting of the number of intersections of the solution with the ground state, which, in the parabolic setting, plays an essential role for the classification of the blow-up dynamics [Mizoguchi 2007], but relies in an essential way on maximum principle techniques. We believe that the blow-up solutions we construct in this paper are the building blocks to classify the blow-up dynamics near the ground state in a suitable topology.

5. *Extension to dispersive problems.* We treat in this paper the parabolic problem, but the robustness of our approach has been shown in [Raphaël and Rodnianski 2012; Merle et al. 2011], which treat the dispersive wave and Schrödinger maps with $\mathbb{S}^2$ target. We expect that similar constructions can be performed there as well to produce arbitrarily slow $\mathscr{C}^\infty$ blow-up solutions with quantized rate, hence completing the analysis of these excited regimes, which started in the seminal work [Krieger et al. 2008].

Notations. We introduce the differential operator

$$\Lambda f = y \cdot \nabla f \quad \text{(energy-critical scaling)}.$$

Given a parameter $\lambda > 0$, we let

$$u_\lambda(r) = u(y) \quad \text{with } y = \frac{r}{\lambda}.$$

Given a positive number $b_1 > 0$, we let

$$B_0 = \frac{1}{\sqrt{b_1}}, \qquad B_1 = \frac{|\log b_1|}{\sqrt{b_1}}. \tag{1-14}$$

We let χ be a positive nonincreasing smooth cut-off function with

$$\chi(y) = \begin{cases} 1 & \text{for } y \le 1, \\ 0 & \text{for } y \ge 2. \end{cases}$$

Given a parameter $B > 0$, we denote

$$\chi_B(y) = \chi\left(\frac{y}{B}\right). \tag{1-15}$$

We shall systematically omit the measure in all radial two dimensional integrals and note that

$$\int f = \int_0^{+\infty} f(r) r\, dr.$$

Given a p-uplet $J = (j_1, \dots, j_p) \in \mathbb{N}^p$, we introduce the norms

$$|J|_1 = \sum_{k=1}^{p} j_k, \quad |J|_2 = \sum_{k=1}^{p} k j_k. \tag{1-16}$$

We note that

$$\mathcal{B}_d(R) = \left\{ x \in \mathbb{R}^d,\ |x| = \left(\sum_{i=1}^{d} x_i^2 \right)^{\frac{1}{2}} \le R \right\}.$$

Strategy of the proof. Let us give brief insight into the strategy of the proof of Theorem 1.1.

(i). *Renormalized flow and iterated resonances.* Let us look for a modulated solution $u(t, r)$ of (1-5) in renormalized form

$$u(t, r) = v(s, y), \quad y = \frac{r}{\lambda(t)}, \quad \frac{ds}{dt} = \frac{1}{\lambda^2(t)}, \tag{1-17}$$

which leads to the self-similar equation

$$\partial_s v - \Delta v + b_1 \Lambda v + \frac{f(v)}{y^2} = 0, \quad b_1 = -\frac{\lambda_s}{\lambda}. \tag{1-18}$$

We know from theoretical ground that if blow-up occurs, $v(s, y) = Q(y) + \varepsilon(s, y)$ for some small $\varepsilon(s, y)$, and hence the linear part of the ε flow is governed by the Schrödinger operator

$$H = -\Delta + \frac{f'(Q)}{y^2}.$$

The energy-critical structure of the problem induces an explicit resonance

$$H(\Lambda Q) = 0,$$

where from explicit computation,

$$\Lambda Q \sim \frac{2}{y} \quad \text{as } y \to \infty. \tag{1-19}$$

More generally, the iterates of the kernel of H computed iteratively through the scheme

$$H T_{k+1} = -T_k, \quad T_0 = \Lambda Q, \tag{1-20}$$

display a nontrivial tail at infinity:

$$T_k(y) \sim y^{2k-1}(c_k \log y + d_k) \quad \text{for } y \gg 1. \tag{1-21}$$

(ii). *Tail dynamics*. We now generalize the approach developed in [Raphaël and Rodnianski 2012; Merle et al. 2011] and claim that $(T_k)_{k\geq 1}$ correspond to unstable directions which can be excited in a universal way. To see this, let us look for a slowly modulated solution to (1-18) of the form $v(s, y) = Q_{b(s)}(y)$ with

$$b = (b_1, \dots, b_L), \qquad Q_b = Q(y) + \sum_{i=1}^{L} b_i T_i(y) + \sum_{i=2}^{L+2} S_i(y) \tag{1-22}$$

and with a priori bounds

$$b_i \sim b_1^i, \qquad |S_i(y)| \lesssim b_1^i y^{C_i},$$

so that S_i is in some sense homogeneous of degree i in b_1. Our strategy is the following: choose the universal dynamical system driving the modes $(b_i)_{1\leq i\leq L}$ which generates the *least growing in space solution* S_i. Let us illustrate the procedure.

$\underline{O(b_1)}$. We do not adjust the law of b_1 for the first term.[3] We therefore obtain from (1-18) the equation

$$b_1(HT_1 + \Lambda Q) = 0.$$

$\underline{O(b_1^2, b_2)}$. We obtain

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 HT_2 + HS_2 = b_1^2 NL(T_1, Q),$$

where $NL(T_1, Q)$ corresponds to nonlinear interaction terms. When considering the far away tail (1-21), we have, for y large,

$$\Lambda T_1 \sim T_1, \qquad HT_2 = -T_1,$$

and thus

$$(b_1)_s T_1 + b_1^2 \Lambda T_1 + b_2 HT_2 \sim ((b_1)_s + b_1^2 - b_2) T_1.$$

Hence the leading order growth is canceled by the choice

$$(b_1)_s + b_1^2 - b_2 = 0. \tag{1-23}$$

We then solve for

$$HS_2 = -b_1^2(\Lambda T_1 - T_1) + NL(T_1, Q)$$

and check that $S_2 \ll b_1^2 T_1$ for y large.

$\underline{O(b_1^{k+1}, b_{k+1})}$. At the k-th iteration, we obtain an elliptic equation of the form

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} HT_{k+1} + HS_1 = b_1^{k+1} NL_k(T_1, \dots, T_k, Q).$$

From (1-21), we have, for tails,

$$\Lambda T_k \sim (2k-1) T_k,$$

and therefore

$$(b_k)_s T_k + b_1 b_k \Lambda T_k + b_{k+1} HT_{k+1} \sim ((b_k)_s + (2k-1) b_1 b_k - b_{k+1}) T_k.$$

[3] If $(b_1)_s = -c_1 b_1$, then $-\lambda_s/\lambda \sim b_1 \sim e^{-c_1 s}$, and hence after integration in time, $|\log \lambda| \lesssim 1$ and there is no blow-up.

The cancellation of the leading order growth occurs for

$$(b_k)_s + (2k-1)b_1 b_k - b_{k+1} = 0.$$

We then solve for the remaining S_{k+1} term and check that $S_{k+1} \ll b_1^{k+1} T_{k+1}$ for y large.

(iii). *The universal system of ODE's*. The above approach leads to the universal system of ODE's which we stop after the L-th iterate:

$$(b_k)_s + (2k-1)b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L, \quad b_{L+1} \equiv 0, \quad -\frac{\lambda_s}{\lambda} = b_1. \tag{1-24}$$

It turns out, and this is classical for critical problems, that an additional logarithmic gain related to the growth (1-21) can be captured, and this turns out to be essential for the analysis.[4] This leads to the sharp dynamical system

$$\begin{cases} (b_k)_s + \left(2k - 1 + \dfrac{2}{|\log b_1|}\right) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L,\ b_{L+1} \equiv 0, \\ -\dfrac{\lambda_s}{\lambda} = b_1, \\ \dfrac{ds}{dt} = \dfrac{1}{\lambda^2}. \end{cases} \tag{1-25}$$

It is easily seen (see Lemma 2.14) that (1-25) rewritten in the original t time variable admits solutions such that $\lambda(t)$ touches 0 in finite time T with the asymptotic (1-13). Equivalently in renormalized variables,

$$\lambda(s) \sim \frac{(\log s)^{|d_1|}}{s^{c_1}}, \quad b(s) \sim \frac{c_1}{s} \quad \text{with } c_1 = \frac{L}{2L-1},\ d_1 = \frac{-2L}{(2L-1)^2}. \tag{1-26}$$

Moreover (see Lemma 2.15), the corresponding solution is stable for $L = 1$. This is the stable blow-up regime, and unstable with $(L-1)$ directions of instabilities for $L \ge 2$.

(iv). *Decomposition of the flow and modulation equations*. Let the approximate solution Q_b be given by (1-22), which by construction generates an approximate solution to the renormalized flow (1-18):

$$\Psi_b = \partial_s Q_b - \Delta Q_b + b\Lambda Q_b + \frac{f(Q_b)}{y^2} = \mathrm{Mod}(t) + O(b^{2L+2}),$$

where, roughly,

$$\mathrm{Mod}(t) = \sum_{i=1}^{L} \left[(b_i)_s + \left(2i - 1 + \frac{2}{|\log b_1|}\right) b_1 b_i - b_{i+1} \right] T_i.$$

We localize Q_b in the zone $y \le B_1$ to avoid the irrelevant growing tails for $y \gg 1/\sqrt{b_1}$. We then pick an initial data of the form

$$u_0(y) = Q_b(y) + \varepsilon_0(y), \quad |\varepsilon_0(y)| \ll 1$$

[4] See, for example, [Raphaël and Rodnianski 2012] for further discussion.

in some suitable sense where $b(0)$ is chosen initially close to the exact excited solution to (1-24). From standard modulation argument, we dynamically introduce a modulated decomposition of the flow

$$u(t,r) = (Q_{b(t)} + \varepsilon)\left(t, \frac{r}{\lambda(t)}\right) = (Q_{b(t)})\left(t, \frac{r}{\lambda(t)}\right) + w(t,r), \tag{1-27}$$

where the $L+1$ modulation parameters $(b(t), \lambda(t))$ are chosen in order to manufacture the orthogonality conditions

$$(\varepsilon(t), H^k \Phi_M) = 0, \quad 0 \le k \le M. \tag{1-28}$$

Here $\Phi_M(y)$ is some fixed direction depending on some large constant M which generates an approximation of the kernel of the iterates of H; see (3-7). This orthogonal decomposition, which, for each fixed time t, directly follows from the implicit function theorem, now allows us to compute the modulation equations governing the parameters $(b(t), \lambda(t))$. The Q_b construction is precisely manufactured to produce the expected ODE's:[5]

$$\left|\frac{\lambda_s}{\lambda} + b_1\right| + \sum_{i=1}^{L}\left|(b_i)_s + \left(2i - 1 + \frac{2}{|\log b_1|}\right) b_1 b_i - b_{i+1}\right| \lesssim \|\varepsilon\|_{\text{loc}} + b_1^{L+\frac{3}{2}}, \tag{1-29}$$

where $\|\varepsilon\|_{\text{loc}}$ measures a *local-in-space* interaction with the harmonic map.

(v). *Control of the radiation and monotonicity formula.* According to (1-29), the core of our analysis is now to show that local norms of ε are under control and do not perturb the dynamical system (1-24). This is achieved using high order Sobolev norms adapted to the linear flow, and in particular we claim that the orthogonality conditions (1-28) ensure the Hardy type coercivity of the iterated operator

$$\mathscr{E}_{2k+2} = \int |H^{k+1}\varepsilon|^2 \gtrsim \int \frac{|\varepsilon|^2}{(1+y^{4k+4})(1+|\log y|^2)}, \quad 0 \le k \le L.$$

We now claim the we can control theses norms thanks to an energy estimate *seen on the linearized equation in original variables*, that is, by working with w in (1-27) and not ε, as initiated in [Raphaël and Rodnianski 2012; Merle et al. 2011]. Here the parabolic structure of the problem simplifies the analysis and displays a repulsive property of the renormalized linearized operator; see the proof of (3-48). The outcome is an estimate of the form

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2k+2}}{\lambda^{4k+2}}\right\} \lesssim \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^{c_k}, \tag{1-30}$$

where the right hand side is controlled by the size of the error in the construction of the approximate blow-up profile. Integrating this in time yields two contributions, one from data and one from the error:

$$\mathscr{E}_{2k+2}(s) \lesssim \lambda^{4k+2}(s)\mathscr{E}_{2k+2}(0) + \lambda^{4k+2}(s)\int_{s_0}^{s} \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^{c_k}\, d\sigma.$$

[5] See Lemma 3.3.

The second contribution is estimated in the regime (1-26) using the fundamental algebra

$$(2k+3)-c_1(4k+2)=1+\frac{2(L-k-1)}{2L-1}\begin{cases}\geq 1 & \text{for } k\leq L-1,\\ <1 & \text{for } k=L.\end{cases} \tag{1-31}$$

Hence data dominates for $k \leq L-1$ up to a logarithmic error

$$\lambda^{4k+2}(s)\int_{s_0}^{s}\frac{b_1^{2k+3}}{\lambda^{4k+2}}|\log b_1|^{c_k}\,d\sigma \sim \lambda^{4k+2}(\log s)^C\int_{s_0}^{s}\frac{d\sigma}{\sigma^{2k+3-c_1(4k+2)}}\sim \lambda^{4k+2}(\log s)^C,$$

which yields the bound

$$\mathscr{E}_{2k+2}\lesssim \lambda^{4k+2}|\log s|^C,\quad 0\leq k\leq L-1, \tag{1-32}$$

which simply expresses the boundedness up to a log of w in some Sobolev type H^{k+1} norm. On the other hand, for $k = L$, we can first derive a *sharp logarithmic gain* in (1-30),

$$\frac{d}{ds}\left\{\frac{\mathscr{E}_{2L+2}}{\lambda^{4k+2}}\right\}\lesssim \frac{b_1^{2L+3}}{\lambda^{4L+2}|\log b_1|^2}, \tag{1-33}$$

and then the integral diverges from (1-31) and

$$\lambda^{4k+2}(s)\int_{s_0}^{s}\frac{b_1^{2L+3}}{\lambda^{4L+2}|\log b_1|^2}\,d\sigma\sim\lambda^{4k+2}(s)\int_{s_0}^{s}\frac{1}{\sigma}\frac{b_1^{2L+2}}{\lambda^{4L+2}|\log b_1|^2}\,d\sigma\sim\frac{b_1^{2L+2}}{|\log b_1|^2}\gg\lambda^{4k+2}.$$

We therefore obtain

$$\mathscr{E}_{2L+2}\lesssim\frac{b_1^{2L+2}}{|\log b_1|^2}. \tag{1-34}$$

The difference between the controls (1-32) for $0 \leq k \leq L-1$ and the sharp control (1-34) is an essential feature of the analysis and explains the introduction of an exactly order $L+1$ Sobolev energy.

We can now reinject this bound into (1-29) and, thanks to the logarithmic gain in (1-33), show that ε does not perturb the system (1-25), modulo the control of the associated unstable $L-1$ modes by a further adjusted choice of the initial data. This concludes the proof of Theorem 1.1.

This paper is organized as follows. In Section 2, we construct the approximate self-similar solutions Q_b and obtain sharp estimates on the error term Ψ_b. We also exhibit an explicit solution to the dynamical system (1-25) and show that it displays $(L-1)$ directions of instability. In Section 3, we set up the bootstrap argument in Proposition 3.1 and derive the fundamental monotonicity of the Sobolev-type norm $\|H^{L+1}\varepsilon\|^2_{L^2}$ in Proposition 3.6, which is the heart of the analysis. In Section 4, we close the bootstrap bounds, which easily imply the blow-up statement of Theorem 1.1.

2. Construction of the approximate profile

This section is devoted to the construction of the approximate Q_b blow-up profile and the study of the associated dynamical system for $b = (b_1, \dots, b_L)$.

The linearized Hamiltonian. Let us start by recalling the structure of the harmonic map Q, which is the unique up-to-scaling solution to

$$\Lambda Q = g(Q), \quad Q(0) = 0, \quad \lim_{r\to+\infty} Q(r) = \pi. \tag{2-1}$$

This equation can be integrated explicitly.[6] Q is smooth $Q \in \mathscr{C}^\infty([0, +\infty), [0, \pi))$ and using (1-3) admits a Taylor expansion[7] to all order at the origin,

$$Q(y) = \sum_{i=0}^{p} c_i y^{2i+1} + O(y^{2p+3}) \quad \text{as } y \to 0, \tag{2-2}$$

and at infinity,

$$Q(y) = \pi - \frac{2}{y} - \sum_{i=1}^{p} \frac{d_i}{y^{2i+1}} + O\left(\frac{1}{y^{2p+3}}\right) \quad \text{as } y \to +\infty. \tag{2-3}$$

The linearized operator close to Q displays a remarkable structure. Indeed, let the potentials

$$Z = g'(Q), \quad V = Z^2 + \Lambda Z = f'(Q), \quad \widetilde{V} = (1+Z)^2 - \Lambda Z, \tag{2-4}$$

which, from (2-2),(2-3), satisfy the following behavior at $0, +\infty$:

$$Z(y) = \begin{cases} 1 + \sum_{i=1}^{p} c_i y^{2i} + O(y^{2p+2}) & \text{as } y \to 0, \\ -1 + \sum_{i=1}^{p} \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) & \text{as } y \to +\infty, \end{cases} \tag{2-5}$$

$$V(y) = \begin{cases} 1 + \sum_{i=1}^{p} c_i y^{2i} + O(y^{2p+2}) & \text{as } y \to 0, \\ 1 + \sum_{i=1}^{p} \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) & \text{as } y \to +\infty, \end{cases} \tag{2-6}$$

$$\widetilde{V}(y) = \begin{cases} 4 + \sum_{i=1}^{p} c_i y^{2i} + O(y^{2p+2}) & \text{as } y \to 0, \\ \sum_{i=1}^{p} \frac{c_i}{y^{2i}} + O\left(\frac{1}{y^{2p+2}}\right) & \text{as } y \to +\infty, \end{cases} \tag{2-7}$$

where $(c_i)_{i\ge1}$ stands for some generic sequence of constants which depend on the Taylor expansion of g at $(0, \pi)$. The linearized operator close to Q is the Schrödinger operator

$$H = -\Delta + \frac{V}{y^2}. \tag{2-8}$$

and admits the factorization

$$H = A^* A \tag{2-9}$$

with

$$A = -\partial_y + \frac{Z}{y}, \quad A^* = \partial_y + \frac{1+Z}{y}, \quad Z(y) = g'(Q).$$

[6] See [Raphaël and Schweyer 2013] for more details.
[7] up to scaling

Observe that, equivalently,

$$Au=-\Lambda Q\frac{\partial}{\partial y}\left(\frac{u}{\Lambda Q}\right),\qquad A^*u=\frac{1}{y\Lambda Q}\frac{\partial}{\partial y}(uy\Lambda Q),\tag{2-10}$$

and thus the kernels of A and A^* on $\mathbb{R}_+^*$ are explicit:

$$Au=0\quad\text{if and only if}\quad u\in\operatorname{Span}(\Lambda Q),\qquad A^*u=0\quad\text{if and only if}\quad u\in\operatorname{Span}\left(\frac{1}{y\Lambda Q}\right).\tag{2-11}$$

Hence the kernel of H on $\mathbb{R}_+^*$ is

$$Hu=0\quad\text{if and only if}\quad u\in\operatorname{Span}(\Lambda Q,\Gamma)\tag{2-12}$$

with

$$\Gamma(y)=\Lambda\phi\int_1^y\frac{dx}{x(\Lambda\phi(x))^2}=\begin{cases}O\left(\frac{1}{y}\right) & \text{as } y\to 0,\\ \frac{y}{4}+O\left(\frac{\log y}{y}\right) & \text{as } y\to+\infty.\end{cases}\tag{2-13}$$

In particular, H is a positive operator on $\dot{H}^1_{\text{rad}}$ with a *resonance* ΛQ at the origin induced by the energy-critical scaling invariance. We also introduce the conjugate Hamiltonian

$$\widetilde{H}=AA^*=-\Delta+\frac{\widetilde{V}}{y^2},\tag{2-14}$$

which is definite positive by construction and (2-11); see Lemma B.2.

Admissible functions. Explicit knowledge of the Green's functions allows us to introduce the formal inverse

$$H^{-1}f=-\Gamma(y)\int_0^y f\Lambda Qx\,dx+\Lambda Q(y)\int_0^y f\Gamma x\,dx.\tag{2-15}$$

Given a function f, we introduce the suitable derivatives of f by considering the sequence

$$f_0=f,\qquad f_{k+1}=\begin{cases}A^*f_k & \text{for k odd},\\ Af_k & \text{for k even},\end{cases}\qquad k\geq 0.\tag{2-16}$$

We shall introduce the formal notation

$$f_k=\mathcal{A}^kf.$$

We define a first class of admissible functions which display a suitable behavior both at the origin and infinity.

Definition 2.1 (admissible functions). We say a smooth function $f\in\mathscr{C}^\infty(\mathbb{R}_+,\mathbb{R})$ is admissible of degree $(p_1,p_2)\in\mathbb{N}\times\mathbb{Z}$ if

(i) f admits a Taylor expansion at the origin to all order

$$f(y)=\sum_{k=p_1}^{p}c_ky^{2k+1}+O(y^{2p+3});\tag{2-17}$$

(ii) f and its suitable derivatives admit a bound, for $y \geq 2$,

$$\text{for all } k \geq 0, \quad |f_k(y)| \lesssim \begin{cases} y^{2p_2-k-1}(1+|\log y|) & \text{for } 2p_2-k \geq 1, \\ y^{2p_2-k-1} & \text{for } 2p_2-k \leq 0. \end{cases} \tag{2-18}$$

H naturally acts on the class of admissible functions in the following way.

Lemma 2.2 (action of H and H^{-1} on admissible functions). *Let f be an admissible function of degree (p_1, p_2). Then*

(i) *for all $l \geq 1$, $H^l f$ is admissible of degree*

$$(\max(p_1-l, 0),\, p_2-l), \tag{2-19}$$

(ii) *for all l, $p_2 \geq 0$, $H^{-l} f$ is admissible of degree*

$$(p_1+l,\, p_2+l). \tag{2-20}$$

Proof of Lemma 2.2. This a simple consequence of the expansions (2-2), (2-3).

Let us first show that Hf is admissible of degree at least $(\max(p_1-1, 1), p_2-1)$, which yields (2-19) by induction. We inject the Taylor expansions (2-17), (2-18) into (2-8). Near the origin, the claim directly follows from the Taylor expansion (2-6) and the cancellation $H(y) = cy + O(y^3)$ at the origin. The claim at infinity directly follows from the relation $u_k = f_{k+2}$ by definition.

Now let $p_2 \geq 0$ and $u = H^{-1} f$ be given by (2-15), and let us show that u is admissible of degree at least (p_1+1, p_2+1), which yields (2-20) by induction. From the relation $u_k = f_{k-2}$ for $k \geq 2$, we need only consider $k = 0, 1$. We first observe from the Wronskian relation $\Gamma'(\Lambda Q) - (\Lambda Q)'\Gamma = 1/y$ that

$$A\Gamma = -\Gamma' + \frac{Z}{y}\Gamma = -\Gamma' + \frac{(\Lambda Q)'}{\Lambda Q}\Gamma = -\frac{1}{y\Lambda Q}.$$

Thus, using the cancellation $A\Lambda Q = 0$, we compute

$$Au = -A\Gamma \int_0^y f\Lambda Q x\, dx = \frac{1}{y\Lambda Q}\int_0^y f\Lambda Q x\, dx. \tag{2-21}$$

Moreover, we may invert A using (2-10) and the boundary condition $u = O(y^3)$ from (2-15), which yields

$$u = -\Lambda Q \int_0^y \frac{Au}{\Lambda Q}\, dx = -\Lambda Q(y) \int_0^y \frac{dx}{x(\Lambda Q(x))^2} \int_0^x f(z)\Lambda Q(z) z\, dz. \tag{2-22}$$

Using (2-2), this yields the Taylor expansion near the origin:

$$Au = \sum_{k=p_1}^{p} c_k^{(1)} y^{2k+2} + O(y^{2p+4}), \quad u = -\Lambda Q \int_0^y \frac{Au}{\Lambda Q}\, dx = \sum_{k=p_1}^{p} c_k^{(2)} y^{2k+3} + O(y^{2p+5}),$$

and hence u is of degree at least p_1+1 near the origin. For $y \geq 1$, from (2-21), (2-22), (2-18), we estimate by brute force, for $p_2 \geq 1$,

$$|Au| = |u_1| \lesssim \int_0^y \tau^{2p_2-1}(1+|\log\tau|)\,d\tau \lesssim y^{2p_2}(1+|\log y|),$$
$$|u| \lesssim \frac{1}{y}\int_0^y \tau^{2p_2}(1+|\log\tau|)\tau\,d\tau \lesssim y^{2p_2+1}(1+|\log y|),$$

and, for $p_2 = 0$,

$$|Au| = |u_1| \lesssim \int_1^y \tau^{-1}\,d\tau \lesssim 1+|\log y|,$$
$$|u| \lesssim \frac{1}{y}\int_0^y (1+|\log\tau|)\tau\,d\tau \lesssim y(1+|\log y|).$$

Hence u satisfies (2-18) with $p_2 \to p_2+1$ and $k = 0, 1$. □

Let us give an explicit example of admissible functions which will be essential for the analysis. From (2-2) and the cancellation $A\Lambda Q = 0$, ΛQ is admissible of degree $(0, 0)$, and hence Lemma 2.2 ensures the following.

Lemma 2.3 (generators of the kernel of H^i). *Let the sequence of profiles for $i \geq 1$ be*

$$T_i = (-1)^i H^{-i}\Lambda Q. \tag{2-23}$$

Then T_i is admissible of degree (i, i).

b_1-admissible functions. We will need an extended notion of admissible functions for the construction of the blow-up profile. In the sequel, we consider a small enough $0 < b_1 \ll 1$ and let B_0, χ_{B_0} be given by (1-14), (1-15). Given $l \in \mathbb{Z}$, we let

$$g_l(b_1, y) = \begin{cases} \dfrac{1+|\log(\sqrt{b_1}y)|}{|\log b_1|}\mathbf{1}_{y\leq 3B_0} & \text{for } l \geq 1,\\[2mm] \dfrac{\mathbf{1}_{y\leq 3B_0}}{|\log b_1|} & \text{for } l \leq 0,\end{cases} \tag{2-24}$$

and, similarly,

$$\tilde{g}_l(b_1, y) = \begin{cases} \dfrac{1+|\log y|}{|\log b_1|}\mathbf{1}_{y\leq 3B_0} & \text{for } l \geq 1,\\[2mm] \dfrac{\mathbf{1}_{y\leq 3B_0}}{|\log b_1|} & \text{for } l \leq 0.\end{cases} \tag{2-25}$$

We then define the extended class of b_1-admissible functions.

Definition 2.4 (b_1-admissible functions). We say a smooth function $f \in \mathscr{C}^\infty(\mathbb{R}_+^* \times \mathbb{R}_+, \mathbb{R})$ is b_1-admissible of degree $(p_1, p_2) \in \mathbb{N}\times\mathbb{Z}$ if the following hold:

(i) For $y \leq 1$, f admits a representation

$$f(b_1, y) = \sum_{j=1}^{J} h_j(b_1)\tilde{f}_j(y) \tag{2-26}$$

for some finite order $J \in \mathbb{N}^*$, some smooth functions $\tilde{f}_j(y)$ with a Taylor expansion at the origin to all order, for all $y \le 1$,

$$\tilde{f}_j(y) = \sum_{k=p_1}^{p} c_{k,j} y^{2k+1} + O(y^{2p+3}), \tag{2-27}$$

and some smooth functions $h_j(b_1)$ away from the origin with

$$\text{for all } l \ge 0, \qquad \left| \frac{\partial^l h_j}{\partial b_1^l} \right| \lesssim \frac{1}{b_1^l}. \tag{2-28}$$

(ii) The function f and its suitable derivatives (2-16) satisfy a uniform bound for some constant $c_{p_2} > 0$: for all $y \ge 2$ and all $k \ge 0$,

$$|f_k(b_1, y)| \lesssim y^{2p_2-k-1} g_{2p_2-k}(b_1, y) + y^{2p_2-k-3} |\log y|^{c_{p_2}} + F_{p_2,k,0}(b_1) y^{2p_2-k-3} \mathbf{1}_{y \ge 3B_0}, \tag{2-29}$$

and, for all $l \ge 1$,

$$\left| \frac{\partial^l}{\partial b_1^l} f_k(b_1, y) \right| \lesssim \frac{1}{b_1^l |\log b_1|} \left\{ y^{2p_2-k-1} \tilde{g}_{2p_2-k}(b_1, y) + y^{2p_2-k-3} |\log y|^{c_{p_2}} \right\} + F_{p_2,k,l}(b_1) y^{2p_2-k-3} \mathbf{1}_{y \ge 3B_0}, \tag{2-30}$$

where, for all $l \ge 0$,

$$F_{p_2,k,l}(b_1) = \begin{cases} 0 & \text{for } 2p_2 - k - 3 \le -1, \\ 1/(b_1^{l+1} |\log b_1|) & \text{for } 2p_2 - k - 3 \ge 0. \end{cases} \tag{2-31}$$

Remark 2.5. *Let us consider the solution T_1 to*

$$HT_1 = -\Lambda Q.$$

An explicit computation reveals the growth for y large

$$\Lambda Q \sim \frac{1}{y}, \quad T_1(y) \sim y \log y.$$

The b_1-admissibility corresponds to a $\log b_1$ gain on the growth at ∞, which is an essential feature of the slowly growing tails in the construction of the modulated blow-up profile in Proposition 2.12. Observe for example that (2-29), (2-31) imply the rough bound

$$|f_k| \lesssim (1+y)^{2p_2-1-k}, \quad \left| \frac{\partial^l f_k}{\partial b_1^l} \right| \lesssim \frac{(1+y)^{2p_2-1-k}}{|\log b_1|}, \quad k \ge 0, \ l \ge 1, \tag{2-32}$$

and hence a logarithmic improvement with respect to (2-18). This gain will be measured in a sharp way through the computation of suitable weighted Sobolev bounds; *see Lemma 2.8.*

We claim that H, H^{-1} and the scaling operators naturally act on the class of b_1-admissible functions in the following way.

Lemma 2.6 (action of H, H^{-1} and scaling operators on b_1-admissible functions). *Let f be a b_1-admissible function of degree (p_1, p_2). Then we get:*

(i) *For all $l \geq 1$, $H^l f$ is b_1-admissible of degree*

$$(\max(p_1 - l, 0),\ p_2 - l). \tag{2-33}$$

(ii) *For all l, $p_2 \geq 1$, $H^{-l} f$ is b_1-admissible of degree*

$$(p_1 + l,\ p_2 + l). \tag{2-34}$$

(iii) *$\Lambda f = y\partial_y f$ is admissible of degree (p_1, p_2).*

(iv) *$b_1 \partial f/(\partial b_1)$ is admissible of degree (p_1, p_2).*

Proof of Lemma 2.6. *Proof of* (i). We show that $u = Hf$ is b_1-admissible of degree $(\max(p_1-1, 0), p_2-1)$, which yields (2-33) by induction. Near the origin, the claim directly follows from the Taylor expansion (2-27) with (2-26) and the cancellation $H(y) = cy + O(y^3)$ at the origin. For $y \geq 1$, H is independent of b_1 so that, by definition,

$$\text{for all } l \geq 0, \qquad \frac{\partial^l u_k}{\partial b_1^l} = \frac{\partial^l f_{k+2}}{\partial b_1^l},$$

which satisfies (2-29), (2-30), (2-31) with $p_2 \to p_2 - 1$ and $F_{p_2-1,k,l}(b_1) = F_{p_2,k+2,l}(b_1)$. Equation (2-33) follows.

Proof of (ii). Now let $p_2 \geq 1$ and let us show that $u = H^{-1} f$ is admissible of degree $(p_1 + 1, p_2 + 1)$, which yields (2-34) by induction. Observe that, for $k \geq 2$ and all $l \geq 0$,

$$\frac{\partial^l u_k}{\partial b_1^l} = \frac{\partial^l f_{k-2}}{\partial b_1^l},$$

which satisfies (2-29), (2-30), (2-31) with $p_2 \to p_2 + 1$ and $F_{p_2+1,k,l}(b_1) = F_{p_2,k-2,l}(b_1)$. It thus only remains to estimate u, Au, and their derivatives in b_1.

Estimate for u near the origin. The inversion formulas (2-21), (2-22) ensure the decomposition of variables near the origin

$$u(b_1, y) = \sum_{j=1}^{J} h_j(b_1)\tilde{u}_j(y),$$

where, using (2-2) the Taylor expansion near the origin,

$$A\tilde{u}_j = \sum_{k=p_1}^{p} c_{k,j}^{(1)} y^{2k+2} + O(y^{2p+4}), \qquad \tilde{u}_j = -\Lambda Q \int_0^y \frac{A\tilde{u}_j}{\Lambda Q}\, dx = \sum_{k=p_1}^{p} c_{k,j}^{(2)} y^{2k+3} + O(y^{2p+5}).$$

Hence u is of degree at least $p_1 + 1$ near the origin.

Estimate for $u_1 = Au$ for $y \geq 1$. We use the formula (2-21) and the assumption $p_2 \geq 1$ to estimate, for $1 \leq y \leq 3B_0$,

$$\begin{aligned}|Au| &\lesssim \int_0^y |f|\,d\tau \lesssim \int_0^y [\tau^{2p_2-1} g_{2p_2-1}(b_1,\tau) + \tau^{2p_2-3}|\log\tau|^{c_{p_2}}]\,d\tau \\ &\lesssim \frac{1}{b_1^{p_2}|\log b_1|}\int_0^{\sqrt{b_1}y} \sigma^{2p_2-1}(1+|\log\sigma|)\,d\sigma + O(y^{2p_2-2}|\log y|^{1+c_{p_2}}) \\ &\lesssim y^{2p_2}\frac{1+|\log(\sqrt{b_1 y})|}{|\log b_1|} + y^{2p_2-2}|\log y|^{1+c_{p_2}} \\ &= y^{2(p_2+1)-2} g_{2(p_2+1)-1}(b_1,y) + y^{2(p_2+1)-4}|\log y|^{1+c_{p_2}},\end{aligned}$$

and, for $y \geq 3B_0$,

$$\begin{aligned}|Au| &\lesssim \int_0^y |f|\,d\tau \lesssim \int_0^{3B_0} \tau^{2p_2-1} g_1(b_1,\tau)\,d\tau + \int_{3B_0}^y F_{p_2,0,0}(b_1)\tau^{2p_2-3}\,d\tau + O(y^{2p_2-2}|\log y|^{1+c_{p_2}}) \\ &\lesssim \frac{1}{b_1^{p_2}|\log b_1|} + \int_{3B_0}^y F_{p_2,0,0}(b_1)\tau^{2p_2-3}\,d\tau + O(y^{2p_2-2}|\log y|^{1+c_{p_2}}).\end{aligned}$$

If $p_2 = 1$, which is the borderline case $2p_2 - 3 = -1$, then $F_{p_2,0,0} = 0$, and we thus get the bound, for all $p_2 \geq 1$, $y \geq 3B_0$,

$$\begin{aligned}|Au| &\lesssim y^{2p_2-2}\left(\frac{1}{b_1|\log b_1|} + F_{p_2,0,0}(b_1)\right) + y^{2p_2-2}|\log y|^{1+c_{p_2}} \\ &\lesssim \frac{1}{b_1|\log b_1|} y^{2p_2-2} + y^{2(p_2+1)-4}|\log y|^{1+c_{p_2}},\end{aligned}$$

and (2-31) is satisfied for $(p_2 \to p_2+1, k=1)$ thanks to $2(p_2+1)-1-3 \geq 0$.

We now pick $l \geq 1$. H is independent of b_1, so

$$H\left(\frac{\partial^l u}{\partial b_1^l}\right) = \frac{\partial^l f}{\partial b_1^l},$$

and therefore, from (2-21), we compute

$$\frac{\partial^l u_1}{\partial b_1^l} = \frac{1}{y\Lambda Q}\int_0^y \Lambda Q \frac{\partial^l f}{\partial b_1^l} x\,dx.$$

This yields the bound, for $|y| \leq 3B_0$,

$$\begin{aligned}\left|\frac{\partial^l u_1}{\partial b_1^l}\right| &\lesssim \int_0^y \frac{1}{b_1^l|\log b_1|}\{y^{2p_2-1}\tilde{g}_{2p_2-1}(b_1,y) + y^{2p_2-3}|\log y|^{c_{p_2}}\}\,dy \\ &\lesssim \frac{1}{b_1^l|\log b_1|}[y^{2p_2}\tilde{g}_1(b_1,y) + y^{2p_2-2}|\log y|^{c_{p_2}+1}] \\ &= \frac{1}{b_1^l|\log b_1|}[y^{2(p_2+1)-2}\tilde{g}_{2(p_2+1)-1}(b_1,y) + y^{2(p_2+1)-4}|\log y|^{c_{p_2}+1}],\end{aligned}$$

and, for $|y| \geq 3B_0$,

$$\left|\frac{\partial^l u_1}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^l |\log b_1|}\left[\frac{1}{b_1^{p_2}} + y^{2p_2-2}|\log y|^{c_{p_2}+1}\right] + \int_{3B_0}^{y} F_{p_2,0,l}(b_1) y^{2p_2-3}\, dy.$$

Again, if $p_2 = 1$, then $F_{p_2,0,l} = 0$, and we therefore obtain the bound, for all $p_2 \geq 1$,

$$\begin{aligned}\left|\frac{\partial^l u_1}{\partial b_1^l}\right| &\lesssim y^{2p_2-2}\left[\frac{1}{b_1^{l+1}|\log b_1|} + F_{p_2,0,l}(b_1)\right] + \frac{y^{2p_2-2}|\log y|^{c_{p_2}+1}}{b_1^l|\log b_1|}\\ &\lesssim \frac{y^{2(p_2+1)-1-3}}{b_1^{l+1}|\log b_1|} + \frac{1}{b_1^l|\log b_1|} y^{2(p_2+1)-1-3}|\log y|^{c_{p_2}+1},\end{aligned}$$

and (2-31) is satisfied for $(p_2 \to p_2+1, k=1)$ thanks to $2(p_2+1)-1-3 \geq 0$.

Estimate for u. Now, from the above bounds and (2-22), for $1 \leq y \leq 3B_0$ we estimate

$$\begin{aligned}|u| &\lesssim \frac{1}{y}\int_0^y |Au|\tau\, d\tau \lesssim \frac{1}{y}\int_0^y [\tau^{2p_2+1} g_{2p_2+1}(b_1,\tau) + \tau^{2p_2-1}|\log\tau|^{1+c_{p_2}}]\, d\tau\\ &\lesssim y^{2p_2+1}\frac{1+|\log(\sqrt{b_1}y)|}{|\log b_1|} + y^{2p_2-1}|\log y|^{2+c_{p_2}}\\ &= y^{2(p_2+1)-1} g_{2(p_2+1)}(b_1,y) + y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}}\end{aligned}$$

and for $y \geq 3B_0$ we estimate

$$\begin{aligned}|u| &\lesssim \frac{1}{y}\left[\int_0^{3B_0} \tau^{2p_2+1} g(b_1,\tau)\, d\tau + \int_{3B_0}^{y} F_{p_2+1,1,0}(b_1)\tau^{2p_2-1}\, d\tau\right] + y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}}\\ &\lesssim y^{2p_2-1}\left[\frac{1}{b_1|\log b_1|}\right] + y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}},\end{aligned}$$

which satisfies (2-29) for $(p_2 \to p_2+1, k=0)$ thanks to $2(p_2+1)-3-1 \geq 0$. Finally, for $l \geq 1$, $1 \leq y \leq 3B_0$,

$$\begin{aligned}\left|\frac{\partial^l u}{\partial b_1^l}\right| &\lesssim \frac{1}{y}\int_0^y \left|\frac{\partial^l u_1}{\partial b_1^l}\right|\tau\, d\tau \lesssim \frac{1}{y b_1^l |\log b_1|}\int_0^y [\tau^{2p_2+1}\tilde{g}_{2p_2+1}(b_1,\tau) + \tau^{2p_2-1}|\log\tau|^{1+c_{p_2}}]\, d\tau\\ &\lesssim \frac{1}{b_1^l|\log b_1|}[y^{2(p_2+1)-1}\tilde{g}_{2(p_2+1)}(b_1,y) + y^{2(p_2+1)-3}|\log y|^{c_{p_2}+1}],\end{aligned}$$

and, for $y \geq 3B_0$,

$$\begin{aligned}\left|\frac{\partial^l u}{\partial b_1^l}\right| &\lesssim \frac{1}{y}\left[\int_0^{3B_0} \frac{\tau^{2p_2+1}\tilde{g}_1(b_1,\tau)}{b_1^l|\log b_1|}\, d\tau + \int_{3B_0}^{y} F_{p_2+1,1,l}(b_1)\tau^{2p_2-1}\, d\tau\right] + \frac{y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}}}{b_1^l|\log b_1|}\\ &\lesssim \frac{y^{2(p_2+1)-3}}{b_1^{l+1}|\log b_1|} + \frac{y^{2(p_2+1)-3}|\log y|^{2+c_{p_2}}}{b_1^l|\log b_1|}.\end{aligned}$$

Hence u is b_1-admissible of degree (p_1+1, p_2+1).

Proof of (iii) *and* (iv). The property (iv) is a direct consequence of the definition of b_1 admissible functions (Definition 2.4) and the trivial bound

$$\frac{\tilde{g}_l(b_1, y)}{|\log b_1|}| \lesssim g_l(b_1, y).$$

We now turn to the proof of (iii). First we rewrite the scaling operator as

$$\Lambda = y\partial_y = -Id - yA + (1+Z).$$

Near the origin, the existence of the decomposition (2-26) follows directly from the even parity of the Taylor expansion of Z at the origin (2-5). Far out, let

$$\Lambda f = -f - yf_1 + (1+Z)f.$$

A simple induction argument similar to Lemma D.1 yields the expansion for $k \geq 1$,

$$(yf_1)_k = c_{k+1}yf_{k+1} + c_{k+2}f_k + \sum_{i=1}^{k} P_{k,i}(y)f_i, \tag{2-35}$$

with the improved decay

$$|\partial_y^l P_{k,i}(y)| \lesssim \frac{1}{1+y^{2+l+k-i}} \quad \text{for all } l \geq 0,\ y \geq 1. \tag{2-36}$$

We therefore obtain from (2-32), (2-35), (2-36), (2-5) the bound

$$\begin{aligned}|(\Lambda f)_k| &\lesssim |yf_{k+1}| + |f_k| + \sum_{i=0}^{k} \frac{1}{y^{2+k-i}} y^{2p_2-i-1} \\ &\lesssim y^{2p_2-k-1}(g_{2p_2-(k+1)} + g_{2p_2-k}) + y^{2p_2-k-3}|\log y|^{c_{p_2}} + (F_{p_2,k,0} + F_{p_2,k+1,0})y^{2p_2-k-3}\mathbf{1}_{y\geq 3B_0}.\end{aligned}$$

We now observe the monotonicity $g_{2p_2-k-1} \lesssim g_{2p_2-k}$ from (2-24) and $F_{p_2,k+1,0} \lesssim F_{p_2,k,0}$ from (2-31), and thus $(\Lambda f)_k$ satisfies (2-30), (2-31) for $l=0$. Similarly, for $k \geq 0$, $l \geq 1$, we use the bound, for $y \gtrsim B_0$,

$$y^{2p_2-k-5}|F_{p_2,i,l}(b_1)| \lesssim \frac{y^{2p_2-k-5}}{b_1^{l+1}|\log b_1|} \lesssim \frac{y^{2p_2-k-3}}{b_1^{l}|\log b_1|},$$

to estimate

$$\begin{aligned}\left|\frac{\partial^l(\Lambda f)_k}{\partial b_1^l}\right| &\lesssim \left|y\frac{\partial^l f_{k+1}}{\partial b_1^l}\right| + \left|\frac{\partial^l f_k}{\partial b_1^l}\right| \\ &\qquad + \sum_{i=0}^{k} \frac{1}{y^{k-i+2}}\left\{\frac{1}{b_1^l|\log b_1|}[y^{2p_2-i-1} + y^{2p_2-i-3}|\log y|^{c_{p_2}}] + F_{p_2,i,l}(b_1)y^{2p_2-i-3}\mathbf{1}_{y\geq 3B_0}\right\} \\ &\lesssim \frac{1}{b_1^l|\log b_1|}[y^{2p_2-k-1}\tilde{g}_{2p_2-(k+1)} + \tilde{g}_{2p_2-k}) + y^{2p_2-k-3}|\log y|^{c_{p_2}}] \\ &\qquad\qquad + (F_{p_2,k,l} + F_{p_2,k+1,l})y^{2p_2-k-3}\mathbf{1}_{y\geq 3B_0},\end{aligned}$$

and the bounds $\tilde{g}_{2p_2-k-1} \lesssim \tilde{g}_{2p_2-k}$, $F_{p_2,k+1,l} \lesssim F_{p_2,k,l}$ now ensure (2-30), (2-31) for $l \geq 1$. □

Slowly growing tails. Let us give an example of admissible profiles which will be central in the construction of the leading order slowly modulated blow-up profile. Given $b_1 > 0$ small enough, we let the radiation be

$$\Sigma_{b_1} = H^{-1}\{-c_{b_1}\chi_{B_0/4}\Lambda Q + d_{b_1} H[(1-\chi_{B_0})\Lambda Q]\} \tag{2-37}$$

with

$$c_{b_1} = \frac{4}{\int \chi_{B_0/4}(\Lambda Q)^2}, \quad d_{b_1} = c_{b_1}\int_0^{B_0} \chi_{B_0/4}\Lambda Q \Gamma y\, dy. \tag{2-38}$$

Lemma 2.7 (slowly growing tails). *Let $(T_i)_{i\ge 1}$ be given by (2-23). Then the sequence of profiles for $i \ge 1$*

$$\Theta_i = \Lambda T_i - (2i-1)T_i - (-1)^{i+1}H^{-i+1}\Sigma_{b_1} \tag{2-39}$$

is b_1-admissible of degree (i, i).

Proof of Lemma 2.7. Step 1: *Structure of T_1.* Let us consider $T_1 = -H^{-1}\Lambda Q$, which is admissible of degree $(1, 1)$ from Lemma 2.3. For $y \ge 1$, explicit computation using the expansion (2-3) into (2-15) yields

$$T_1(y) = y\log y + e_0 y + O\left(\frac{|\log y|^2}{y}\right), \quad \Lambda T_1 = y\log y + (1+e_0)y + O\left(\frac{|\log y|^2}{y}\right) \tag{2-40}$$

for some universal constant e_0. Hence we get the essential cancellation

$$\Lambda T_1 - T_1 = y + O\left(\frac{|\log y|^2}{y}\right). \tag{2-41}$$

We now prove that Θ_i is of order (i, i) by induction on i.

Step 2: $i = 1$. By definition,

$$\Sigma_{b_1} = \Gamma(y)\int_0^y c_{b_1}\chi_{B_0/4}(\Lambda Q)^2 x\, dx - \Lambda Q(y)\int_0^y c_{b_1}\chi_{B_0/4}\Gamma\Lambda Q x\, dx + d_{b_1}(1-\chi_{B_0})\Lambda Q(y), \tag{2-42}$$

and thus, by the definition of c_{b_1}, d_{b_1} in (2-38),

$$\Sigma_{b_1} = \begin{cases} c_{b_1} T_1 & \text{for } y \le B_0/4, \\ 4\Gamma & \text{for } y \ge 3B_0. \end{cases} \tag{2-43}$$

In particular, Σ_{b_1} admits a representation (2-26) near the origin with $J = 1$, $h_1(b_1) = c_{b_1}$, and $\tilde f_1(y) = T_1(y)$, and thus an expansion (2-27) of order $p_1 = 1$ from the first step. A direct computation on the formula (2-38) yields the bounds

$$c_{b_1} = \frac{2}{|\log b_1|}\left[1 + O\left(\frac{1}{|\log b_1|}\right)\right], \quad |d_{b_1}| \lesssim \frac{1}{b_1|\log b_1|}, \tag{2-44}$$

and

$$\left|\frac{\partial^l c_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^l|\log b_1|^2}, \quad \left|\frac{\partial^l d_{b_1}}{\partial b_1^l}\right| \lesssim \frac{1}{b_1^{l+1}|\log b_1|} \quad \text{for all } l \ge 1, \tag{2-45}$$

which imply (2-28).

For $y \ge 3B_0$, from (2-13), (2-43), we estimate

$$\Sigma_{b_1}(y) = y + O\left(\frac{\log y}{y}\right), \tag{2-46}$$

and, for $2 \le y \le 3B_0$,

$$\begin{aligned}\Sigma_{b_1}(y) &= c_{b_1}\left(\frac{y}{4} + O\left(\frac{\log y}{y}\right)\right)\left[\int_0^y \chi_{B_0/4}(\Lambda Q)^2 x\,dx\right] - c_{b_1}\Lambda Q(y)\int_1^y O(1)x\,dx \\ &= y\frac{\int_0^y \chi_{B_0/4}(\Lambda Q)^2}{\int \chi_{B_0/4}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b_1|}\right).\end{aligned} \tag{2-47}$$

We thus conclude from (2-41), (2-47) that, for $y \le 3B_0$,

$$\Theta_1(y) = y - y\frac{\int_0^y \chi_{B_0/4}(\Lambda Q)^2}{\int \chi_{B_0/4}(\Lambda Q)^2} + O\left(\frac{1+y}{|\log b_1|}\right) + O\left(\frac{|\log y|^2}{1+y}\right) = O\left(\frac{1+y}{|\log b_1|}(1+|\log(y\sqrt{b_1})|)\right),$$

which, together with the bounds (2-40), (2-46) for $y \ge 3B_0$, yields the bound, for $y \ge 2$,

$$|\Theta_1(y)| \lesssim y g_2(b_1, y) + O\left(\frac{|\log y|^2}{y}\right). \tag{2-48}$$

Now, from (2-21), (2-37), we compute

$$A\Sigma_{b_1} = \frac{1}{y\Lambda Q}\int_0^y \Lambda Q[-c_{b_1}\chi_{B_0/4}\Lambda Q + d_{b_1}H[(1-\chi_{B_0})\Lambda Q]]x\,dx,$$

and from (2-44) we estimate, for $y \le 3B_0$,

$$A\Sigma_{b_1} = -\frac{4}{y\Lambda Q} + \frac{c_{b_1}}{y\Lambda Q}\int_y^{B_0}(\Lambda Q)^2 x\,dx + O\left(\frac{d_{b_1}}{B_0^2}\mathbf{1}_{B_0\le y\le 3B_0}\right) = -2 + O(g_1(b_1, y)) \tag{2-49}$$

and, for $y \ge 3B_0$,

$$A\Sigma_{b_1} = -\frac{4}{y\Lambda Q} = -2 + O\left(\frac{1}{y^2}\right). \tag{2-50}$$

Moreover, a simple rescaling argument yields the formula

$$A(\Lambda u) = Au + \Lambda A u - \frac{\Lambda Z}{y}u$$

and thus, using (2-40), (2-5),

$$A(\Lambda T_1 - T_1) = \Lambda A T_1 - \frac{\Lambda Z}{y}T_1 = \Lambda A T_1 + O\left(\frac{\log y}{y^2}\right).$$

Now, from (2-21), (2-3), we estimate

$$AT_1 = -\left[\frac{1}{y\Lambda Q}\int_0^y (\Lambda Q)^2 x\,dx\right] = -2\log y + O\left(\frac{\log y}{y^2}\right),$$

and, similarly,

$$\Lambda A T_1 = -2 + O\left(\frac{\log y}{y^2}\right),$$

from which

$$A(\Lambda T_1 - T_1) = -2 + O\left(\frac{\log y}{y^2}\right). \tag{2-51}$$

We thus conclude from (2-48), (2-49), (2-50), (2-51) that

$$|A\Theta_1| \lesssim g_1(b_1, y) + O\left(\frac{\log y}{y^2}\right).$$

We now turn to the control of $H\Theta_1$. First, from a simple rescaling argument, we compute

$$H(\Lambda u) = 2Hu + \Lambda Hu - \frac{\Lambda V}{y^2} u, \tag{2-52}$$

which implies

$$H(\Lambda T_1 - T_1) = -\Lambda Q - \Lambda^2 Q + O\left(\frac{\log y}{y^3}\right) = O\left(\frac{\log y}{y^3}\right).$$

Hence, according to (2-24), we get the desired cancellation

$$|H\Theta_1| \lesssim |H(\Lambda T_1 - T_1)| + |H\Sigma_{b_1}| \lesssim \frac{1}{(1+y)|\log b_1|}\mathbf{1}_{y\le 3B_0} + O\left(\frac{\log y}{y^3}\right).$$

The control of higher order suitable derivatives in y now follows by iteration using (2-3), (2-6). Hence Θ_1 satisfies the bound (2-29) with $p_2 = 1$, $l = 0$.

We now take derivatives in b_1, in which case from (2-42), for $l \ge 1$,

$$\frac{\partial^l \Theta_1}{\partial^l b_1} = -\frac{\partial^l \Sigma_{b_1}}{\partial b_1^l} = \Gamma(y)\int_0^y \frac{\partial^l}{\partial b_1^l}\{c_{b_1}\chi_{B_0/4}\}(\Lambda Q)^2 x\,dx - \Lambda Q(y)\int_0^y \frac{\partial^l}{\partial b_1^l}\{c_{b_1}\chi_{B_0/4}\}\Gamma\Lambda Q x\,dx + \frac{\partial^l}{\partial b_1^l}\{d_{b_1}(1-\chi_{B_0})\}\Lambda Q(y),$$

and from (2-43),

$$\frac{\partial^l \Theta_1}{\partial^l b_1} = -\frac{\partial^l \Sigma_{b_1}}{\partial b_1^l}(y) = 0 \quad \text{for } y \ge 3B_0.$$

From (1-14), we estimate by brute force

$$\left|\frac{\partial^l \chi_{B_0}}{\partial b_1^l}\right| \lesssim \frac{\mathbf{1}_{B_0\le y\le 2B_0}}{b_1^l}$$

and thus, from the Leibniz rule and (2-45), for $y \le 3B_0$, we obtain

$$\left|\frac{\partial^l \Theta_1}{\partial b_1^l}\right| \lesssim \frac{y}{b_1^l|\log b_1|^2}(1+|\log y|) + \left[\sum_{k=1}^{l}\frac{1}{b_1^{l-k}b_1^k|\log b_1|^2}\right]y\mathbf{1}_{B_0/2\le y\le 3B_0} + \left[\sum_{k=0}^{l}\frac{1}{b_1^{l-k}b_1^{k+1}|\log b_1|}\right]\frac{\mathbf{1}_{B_0/2\le y\le 3B_0}}{y}$$

$$\lesssim \frac{y(1+|\log y|)}{b_1^l|\log b_1|^2} \lesssim \frac{y\tilde{g}_1}{b_1^l|\log b_1|}.$$

The control of higher suitable derivatives $(\partial^l \mathcal{A}^k \Theta_1/(\partial b_1^l))_{l,k\geq 1}$ follows similarly using the explicit formula (2-37). This concludes the proof of the estimate (2-30) with $p_2 = 1$, and thus Θ_1 is b_1-admissible of degree $(1, 1)$.

Step 3: $i \to i+1$. We assume the claim for Θ_i and prove it for Θ_{i+1}. From (2-23), (2-39), (2-52),

$$\begin{aligned} H\Theta_{i+1} &= H(\Lambda T_{i+1}) - (2i+1)HT_{i+1} - (-1)^i H^{-i+1}\Sigma_{b_1} \\ &= \Lambda HT_{i+1} - (2i-1)HT_{i+1} + (-1)^{i+1}H^{-i+1}\Sigma_{b_1} - \frac{\Lambda V}{y^2}T_{i+1} \\ &= -[(\Lambda T_i - (2i-1)T_i - (-1)^{i+1}H^{-i+1}\Sigma_{b_1}] - \frac{\Lambda V}{y^2}T_{i+1} \\ &= -\Theta_i - \frac{\Lambda V}{y^2}T_{i+1}. \end{aligned}$$

The induction hypothesis ensures that Θ_i is b_1-admissible of order (i, i). Moreover, near the origin, T_{i+1} is from Lemma 2.3 of degree $i+1$ and hence the development (2-6) ensures that $(\Lambda V/y^2)T_{i+1}$ is of degree $i+1$ near the origin. For $y \geq 1$, (2-6) ensures the improved bound

$$\left|\frac{\partial^p}{\partial y^p}\left(\frac{\Lambda V}{y^2}\right)\right| \lesssim \frac{1}{y^{p+4}}, \quad p \geq 0,$$

and since T_{i+1} is of degree $i+1$, we obtain from the Leibniz rule the rough bound, for all $k \geq 0$,

$$\left|\mathcal{A}^k\left[\frac{\Lambda V}{y^2}T_{i+1}\right]\right| \lesssim \sum_{p=0}^{k} \frac{1}{y^{k-p+4}} y^{2(i+1)-p-1}|\log y|^{c_i} \lesssim y^{2i-k-3}|\log y|^{c_i}.$$

Hence $(\Lambda V/y^2)T_{i+1}$, which is independent of b_1, satisfies (2-29) and is b_1-admissible of degree (i, i). We conclude from Lemma 2.6 that Θ_{i+1} is admissible of order $(i+1, i+1)$. □

Sobolev bounds on b_1-admissible functions. The property of b_1-admissibility leads to simple Sobolev bounds with sharp logarithmic gains. We let B_1 be given by (1-14).

Lemma 2.8 (estimate of b_1-admissible function). *Let $i \geq 1$ and f be a b_1-admissible function of degree (i, i). Then*

$$\int_{y\leq 2B_1} |H^k f|^2 \lesssim \frac{|\log b_1|^{4(i-k-1)}}{b_1^{2(i-k)}|\log b_1|^2} \quad \text{for } 0 \leq k \leq i-1, \tag{2-53}$$

$$\int_{y\leq 2B_1} |H^k f|^2 \lesssim 1 \quad \text{for } k \geq i, \tag{2-54}$$

and

$$\int_{y\leq 2B_1} \frac{1+|\log y|^2}{1+y^4}|H^k f|^2 + \int_{y\leq 2B_1} \frac{1+|\log y|^2}{1+y^2}|AH^k f|^2 \lesssim |\log b_1|^3 \quad \text{for } k \geq i-1. \tag{2-55}$$

Remark 2.9. *The boundedness of the Sobolev norm (2-54) in the borderline case $k = i$ is a consequence of the definition (2-24). Indeed,*

$$\int_{y\le 3B_0}\left|\frac{1+|\log\sqrt{b_1}y|}{(1+y)|\log b_1|}\right|^2 \sim |\log b_1|,$$

but

$$\int_{y\le 3B_0}\left|\frac{1}{(1+y)|\log b_1|}\right|^2 \lesssim 1. \tag{2-56}$$

Proof of Lemma 2.8. Let $k \ge 0$. Near the origin, the cancellation $A(y) = y^2 + O(y)$ and the Taylor expansion (2-27) ensure that $H^k f$ is bounded uniformly in $y \le 1$, $|b_1| \le \frac{1}{2}$. For $y \ge 1$, from (2-29) we estimate

$$\int_{y\le 2B_1}|H^k f|^2 = \int |f_{2k}|^2 \lesssim \int_{3B_0\le y\le 2B_1}|F_{i,2k,0}(b_1)y^{2i-2k-3}\mathbf{1}_{y\ge 3B_0}|^2 + \int_{1\le y\le 2B_1}|y^{2i-2k-1}g_{2i-2k}(b,y)+y^{2i-2k-3}|\log y|^{c_i}|^2.$$

For $k \ge i$, $F_{i,2k,0} = 0$ and from (2-24) we estimate (2-56)

$$\int_{y\le 2B_1}|H^k f|^2 \lesssim 1+\int_{1\le y\le 2B_1}\left|y^{2(i-k)-1}\frac{\mathbf{1}_{y\le 3B_0}}{|\log b_1|}+y^{2i-2k-3}|\log y|^{c_i}\right|^2 \lesssim 1.$$

For $k \le i-1$, the growth can be controlled in a sharp way. Indeed, using $F_{i,2k,0} = 0$ for $k = i-1$ precisely to avoid an additional logarithmic error, we estimate

$$\begin{aligned}\int_{y\le 2B_1}|H^k f|^2 &\lesssim \frac{B_1^{4i-4k-4}}{b_1^2|\log b_1|^2}+\frac{1}{|\log b_1|^2}\int_{y\le 3B_0}y^{4(i-k)-2}(1+|\log\sqrt{b_1}y|^2)+B_1^{4(i-k)-4}|\log b_1|^{2c_{p_2}+1}\\ &\lesssim 1+\frac{B_0^{4(i-k)}}{|\log b_1|^2}+\frac{|\log b_1|^{4(i-k-1)}}{b_1^{2(i-k)}|\log b_1|^2}\lesssim\frac{|\log b_1|^{4(i-k-1)}}{b_1^{2(i-k)}|\log b_1|^2}.\end{aligned}$$

Finally, for $k \ge i-1$, using the rough bound (2-32), we estimate

$$\begin{aligned}&\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|H^k f|^2+\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|AH^k f|^2\\ &\quad\lesssim\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|(1+y)^{2i-2k-1}|^2+\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|(1+y)^{2i-2(k+1)-1}|^2\\ &\quad\lesssim|\log b_1|^3.\end{aligned}$$

□

Slowly modulated blow-up profiles. In this section we construct the approximate modulated blow-up profile. Let us start by introducing the notion of homogeneous admissible functions.

Definition 2.10 (homogeneous functions). Given parameters $b=(b_k)_{1\le k\le L}$ and $(p_1,p_2,p_3)\in\mathbb{N}\times\mathbb{Z}\times\mathbb{N}$, we say a function $S(b,y)$ is homogeneous of degree (p_1,p_2,p_3) if it is of the form

$$S(b,y)=\sum_{J=(j_1,\dots,j_L),\,|J|_2=p_3}\left[c_J\left(\prod_{k=1}^{L}b_k^{j_k}\right)\widetilde{S}_J(b_1,y)\right],$$

where

$$J=(j_1,\dots,j_L)\in\mathbb{Z}\times\mathbb{N}^{L-1},\quad |J|_2=\sum_{k=1}^{L}kj_k,$$

and for some b_1-admissible profiles $\widetilde{S}_J$ of degree (p_1,p_2) in the sense of Definition 2.4. We note that

$$\deg(S)=(p_1,p_2,p_3).$$

Remark 2.11. *We allow for negative powers of b_1 only in the above definition. This ensures from Lemma 2.6 that the space of homogeneous functions of a given degree is stable by application of the operator $b_1\partial/(\partial b_1)$. It is also stable by multiplication by c_{b_1} from (2-44), (2-45).*

We may now proceed to the construction of the slowly modulated blow-up profiles.

Proposition 2.12 (construction of the approximate profile). *Let $M>0$ be a large enough universal constant. Then there exists a small enough universal constant $b^*(M)>0$ such that the following holds true. Let there be a $\mathscr{C}^1$ map*

$$b=(b_k)_{1\le k\le L}:[s_0,s_1]\mapsto(-b^*(M),b^*(M))^L$$

with a priori bounds on $[s_0,s_1]$

$$0<b_1<b^*(M),\quad |b_k|\lesssim b_1^k\quad \text{for } 2\le k\le L. \tag{2-57}$$

Let B_1 be given by (1-14) and $(T_i)_{1\le i\le L}$ be given by (2-23). Then there exist homogeneous profiles

$$\begin{cases}S_i(b,y), & 2\le i\le L+2,\\ S_1=0\end{cases}$$

with

$$\begin{cases}\deg(S_i)=(i,i,i),\\ \dfrac{\partial S_i}{\partial b_j}=0 & \text{for } 2\le i\le j\le L\end{cases} \tag{2-58}$$

such that

$$Q_{b(s)}(y)=Q(y)+\alpha_{b(s)}(y),\quad \alpha_b(y)=\sum_{i=1}^{L}b_iT_i(y)+\sum_{i=2}^{L+2}S_i(y) \tag{2-59}$$

generates an approximate solution to the renormalized flow

$$\partial_sQ_b-\Delta Q_b+b_1\Lambda Q_b+\frac{f(Q_b)}{y^2}=\Psi_b+\operatorname{Mod}(t) \tag{2-60}$$

with

$$\mathrm{Mod}(t)=\sum_{i=1}^{L}[(b_i)_s+(2i-1+c_{b_1})b_1b_i-b_{i+1}]\left[T_i+\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right]. \tag{2-61}$$

Here we used the convention

$$b_{L+1}=0,\quad T_0=\Lambda Q,$$

and Ψ_b *satisfies*

(i) *the global weighted bounds*

$$\text{for all } 1\le k\le L,\quad \int_{y\le 2B_1}|H^k\Psi_b|^2\lesssim b_1^{2k+2}|\log b_1|^C, \tag{2-62}$$

$$\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|H^L\Psi_b|^2+\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|AH^L\Psi_b|^2\lesssim\frac{b_1^{2L+3}}{|\log b_1|^2}, \tag{2-63}$$

$$\int_{y\le 2B_1}|H^{L+1}\Psi_b|^2\lesssim\frac{b_1^{2L+4}}{|\log b_1|^2}; \tag{2-64}$$

and

(ii) $$\text{for all } 0\le k\le L+1,\quad \int_{y\le 2M}|H^k\Psi_b|^2\lesssim M^C b_1^{2L+6} \tag{2-65}$$

for some universal constant $C=C(L)>0$ (*improved local control*).

Proof of Proposition 2.12. *Step* 1: *Computation of the error*. From (2-59), (2-60) we compute

$$\partial_s Q_b-\Delta Q_b+b_1\Lambda Q_b+\frac{f(Q_b)}{y^2}=A_1+A_2$$

with

$$A_1=b_1\Lambda Q+\sum_{i=1}^{L}[(b_i)_sT_i+b_iHT_i+b_1b_i\Lambda T_i]+\sum_{i=2}^{L+2}[\partial_sS_i+HS_i+b_1\Lambda S_i],$$

$$A_2=\frac{1}{y^2}[f(Q+\alpha_b)-f(Q)-f'(Q)\alpha_b].$$

Let us rearrange the first sum using the definition (2-23):

$$\begin{aligned}A_1&=b_1\Lambda Q+\partial_sS_{L+2}+b_1\Lambda S_{L+2}+\sum_{i=1}^{L}[(b_i)_sT_i-b_iT_{i-1}+b_1b_i\Lambda T_i]+\sum_{i=2}^{L+1}[\partial_sS_i+b_1\Lambda S_i]+\sum_{i=1}^{L+1}HS_{i+1}\\&=[\partial_sS_{L+2}+b_1\Lambda S_{L+2}]+[HS_{L+2}+\partial_sS_{L+1}+b_1\Lambda S_{L+1}]+\sum_{i=1}^{L}[(b_i)_s+(2i-1+c_{b_1})b_1b_i-b_{i+1}]T_i\\&\quad+\sum_{i=1}^{L}[HS_{i+1}+\partial_sS_i+b_1b_i(\Lambda T_i-(2i-1+c_{b_1})T_i)+b_1\Lambda S_i],\end{aligned}$$

where c_{b_1} is given by (2-38). We now treat the time dependence using the anticipated approximate modulation equation

$$\partial_s S_i = \sum_{j=1}^{L}(b_j)_s \frac{\partial S_i}{\partial b_j} = \sum_{j=1}^{L}((b_j)_s + (2j-1+c_{b_1})b_1 b_j - b_{j+1})\frac{\partial S_i}{\partial b_j} - \sum_{j=1}^{L}((2j-1+c_{b_1})b_1 b_j - b_{j+1})\frac{\partial S_i}{\partial b_j},$$

and thus, using (2-58),

$$\begin{aligned} A_1 = &\left\{ b_1 \Lambda S_{L+2} - \sum_{i=1}^{L}((2i-1+c_{b_1})b_1 b_i - b_{i+1})\frac{\partial S_{L+2}}{\partial b_i} \right\} \\ &+ \left\{ H S_{L+2} + b_1 \Lambda S_{L+1} - \sum_{i=1}^{L}((2i-1+c_{b_1})b_1 b_i - b_{i+1})\frac{\partial S_{L+1}}{\partial b_i} \right\} \\ &+ \sum_{i=1}^{L}\left[H S_{i+1} + b_1 b_i(\Lambda T_i - (2i-1+c_{b_1})T_i) + b_1 \Lambda S_i - \sum_{j=1}^{i-1}((2j-1+c_{b_1})b_1 b_j - b_{j+1})\frac{\partial S_i}{\partial b_j} \right] \\ &+ \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]\left[T_i + \sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i} \right]. \end{aligned}$$

We now expand A_2 using a Taylor expansion:

$$A_2 = \frac{1}{y^2}\left\{ \sum_{j=2}^{L+2}\frac{f^{(j)}(Q)}{j!}\alpha_b^j + R_2 \right\}$$

with

$$R_2 = \frac{\alpha_b^{L+3}}{(L+2)!}\int_0^1 (1-\tau)^{L+2} f^{(L+3)}(Q+\tau\alpha_b)\, d\tau. \tag{2-66}$$

Using the notation (1-16) for the $2L+1$ uplet $J = (i_1, \dots, i_L, j_2, \dots, j_{L+2}) \in \mathbb{N}^{2L+1}$, we sort the Taylor polynomial[8]

$$|J|_1 = \sum_{k=1}^{L} i_k + \sum_{k=2}^{L+2} j_k, \quad |J|_2 = \sum_{k=1}^{L} k i_k + \sum_{k=2}^{L+2} k j_k,$$

and thus

$$\sum_{j=2}^{L+2}\frac{f^{(j)}(Q)}{j!}\alpha_b^j = \sum_{j=2}^{L+2}\frac{f^{(j)}(Q)}{j!}\sum_{|J|_1=j} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k} = \sum_{i=2}^{L+2} P_i + R_1,$$

where

$$P_i = \sum_{j=2}^{L+2}\frac{f^{(j)}(Q)}{j!}\sum_{|J|_1=j, |J|_2=i} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \tag{2-67}$$

$$R_1 = \sum_{j=2}^{L+2}\frac{f^{(j)}(Q)}{j!}\sum_{|J|_1=j, |J|_2\geq L+3} c_J \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}. \tag{2-68}$$

[8]Remember that b_i is order b_1^i from (2-57).

We finally use the definitions (2-23), (2-39), (2-61) to rewrite

$$\Lambda T_i-(2i-1+c_{b_1})T_i=\Theta_i+(-1)^{i+1}H^{-i+1}\Sigma_{b_1}-c_{b_1}T_i=\Theta_i+(-1)^{i+1}H^{-i+1}(\Sigma_{b_1}-c_{b_1}T_1),$$

which, together with (2-60), yields the following expression for the error:

$$\begin{aligned}\Psi_b=&\sum_{i=1}^{L}(-1)^{i+1}b_1b_iH^{-i+1}(\Sigma_{b_1}-c_{b_1}T_1)\\&+\left\{b_1\Lambda S_{L+2}-\sum_{i=1}^{L}((2i-1+c_{b_1})b_1b_i-b_{i+1})\frac{\partial S_{L+2}}{\partial b_i}+\frac{1}{y^2}[R_1+R_2]\right\}\\&+\left\{HS_{L+2}+b_1\Lambda S_{L+1}+\frac{P_{L+2}}{y^2}-\sum_{i=1}^{L}((2i-1+c_{b_1})b_1b_i-b_{i+1})\frac{\partial S_{L+1}}{\partial b_i}\right\}\\&+\sum_{i=1}^{L}\left[HS_{i+1}+b_1b_i\Theta_i+b_1\Lambda S_i+\frac{P_{i+1}}{y^2}-\sum_{j=1}^{i-1}((2j-1+c_{b_1})b_1b_j-b_{j+1})\frac{\partial S_i}{\partial b_j}\right].\end{aligned}\tag{2-69}$$

We now construct iteratively the sequence of profiles $(S_i)_{1\le i\le L+2}$ through the scheme

$$\begin{cases}S_1=0,\\S_i=-H^{-1}\Phi_i,\quad 2\le i\le L+2,\end{cases}\tag{2-70}$$

where, for $1\le i\le L$,

$$\Phi_{i+1}=b_1b_i\Theta_i+b_1\Lambda S_i+\frac{P_{i+1}}{y^2}-\sum_{j=1}^{i-1}((2j-1+c_{b_1})b_1b_j-b_{j+1})\frac{\partial S_i}{\partial b_j},\tag{2-71}$$

$$\Phi_{L+2}=b_1\Lambda S_{L+1}+\frac{P_{L+2}}{y^2}-\sum_{i=1}^{L}((2i-1+c_{b_1})b_1b_i-b_{i+1})\frac{\partial S_{L+1}}{\partial b_i}.\tag{2-72}$$

Step 2: Control of Φ_i, S_i. We claim by induction on i that Φ_i is homogeneous with

$$\deg(\Phi_i)=(i-1,i-1,i)\quad\text{for } 2\le i\le L+2\tag{2-73}$$

and

$$\frac{\partial\Phi_i}{\partial b_j}=0\quad\text{for } 2\le i\le j\le L+2.\tag{2-74}$$

This implies from Lemma 2.6 that S_i given by (2-70) is homogeneous and satisfies (2-58) for $2\le i\le L+2$.

Case 1: $i=1$. We compute explicitly

$$\Phi_2=b_1^2\Theta_1+b_1^2\frac{f''(Q)}{2y^2}T_1^2,$$

which satisfies (2-74). Recall from (1-3) that $f=gg'$ is odd and π periodic so that the expansions (2-2), (2-3) yield, at the origin,

$$\frac{f^{(j)}(Q)}{y^2}=\begin{cases}\sum_{k=-1}^{p}y^{2k+1}+O(y^{2p+3})&\text{for j even,}\\\sum_{k=-1}^{p}y^{2k}+O(y^{2p+2})&\text{for j odd}\end{cases}\tag{2-75}$$

and, at infinity,

$$\frac{f^{(j)}(Q)}{y^2} = \begin{cases} \sum_{k=1}^{p} y^{-2k-1} + O(y^{-2p-3}) & \text{for j even,} \\ \sum_{k=1}^{p} y^{-2k} + O(y^{-2p-2}) & \text{for j odd.} \end{cases} \tag{2-76}$$

From Lemmas 2.3 and 2.7, T_1 and Θ_1 are respectively admissible and b_1-admissible of order $(1, 1)$. In particular, we have the Taylor expansion near the origin

$$\frac{f''(Q)}{2y^2} T_1^2 = \sum_{k=1}^{p} c_k y^{2k+1} + O(y^{2p+3}), \quad p \geq 1,$$

and the bound at infinity

$$\left| \mathcal{A}^k \left(\frac{f''(Q)}{2y^2} T_1^2 \right) \right| \lesssim \frac{1}{y^{3+k}} y^2 |\log y|^2 \lesssim y^{2-k-3} |\log y|^2, \quad k \geq 0.$$

Hence $(f''(Q)/2y^2)T_1^2$ is b_1-admissible of degree $(1, 1)$. We conclude that Φ_2 is homogeneous with

$$\deg(\Phi_2) = (1, 1, 2).$$

Case 2: $i \to i+1$. We estimate all terms in (2-71). Equation (2-74) holds by direct inspection. From Lemma 2.7, $b_1 b_i \Theta_i$ is homogeneous of degree $(i, i, i+1)$. From Lemma 2.6, $b_1 \Lambda S_i$ is homogeneous of degree $(i, i, i+1)$ by induction. For $j \geq 2$, by definition and induction we have that

$$((2j-1+c_{b_1}) b_1 b_j - b_{j+1}) \frac{\partial S_i}{\partial b_j}$$

is homogeneous of degree $(i, i, i+1)$. For $j = 1$, we rewrite the term

$$((1+c_{b_1}) b_1^2 - b_2) \frac{\partial S_i}{\partial b_1} = \left((1+c_{b_1}) b_1 - \frac{b_2}{b_1} \right) \left(b_1 \frac{\partial S_i}{\partial b_1} \right)$$

and, recalling Remark 2.11, conclude that this term is also homogeneous of degree $(i, i, i+1)$. It thus remains to estimate the nonlinear term P_{i+1}/y^2 in (2-71), which, from (2-67), is a linear combination of monomials of the form[9]

$$M_J(y) = \frac{f^{(j)}(Q)}{y^2} \prod_{k=1}^{i} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{i} S_k^{j_k}, \quad |J|_1 = j, \ |J|_2 = i+1, \ 2 \leq j \leq i+1.$$

Using (2-75), (2-76), we conclude that M_J is admissible with the following development at the origin: for $j = 2l$,

$$\begin{aligned} M_J(y) &= y^{-1} y^{\sum_{k\geq 1} i_k(2k+1) + j_k(2k+1)} (c_0 + c_2 y^2 + \cdots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2|J|_2 + j - 1} (c_0 + c_2 y^2 + \cdots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2(i+l)+1} (c_0 + c_2 y^2 + \cdots + c_p y^{2p} + o(y^{2p+1})), \end{aligned}$$

[9] Observe that terms involving $k \geq i+1$ are indeed forbidden in the last product from the constraint $|J|_1 \geq 2$, $|J|_2 = i+1$.

and, for $j = 2l + 1$,

$$\begin{aligned} M_J(y) &= y^{-2} y^{\sum_{k\geq 1} i_k(2k+1)+j_k(2k+1)}(c_0 + c_2y^2 + \ldots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2|J|_2+j-2}(c_0 + c_2y^2 + \ldots + c_p y^{2p} + o(y^{2p+1})) \\ &= y^{2(i+l)+1}(c_0 + c_2y^2 + \ldots + c_p y^{2p} + o(y^{2p+1})). \end{aligned}$$

Now $j \geq 2$ ensures $l \geq 1$, and hence M_J admits a Taylor expansion (2-27) at the origin with $p_1 = i + 1$. For $y \geq 1$, the rough bound (2-32) and (2-18) imply

$$|S_j| \lesssim b_1^j y^{2j-1}, \quad |T_j| \lesssim y^{2j-1}|\log y|^C,$$

which, together with (2-76), yields the control

$$M_J(y) \lesssim |\log y|^C \begin{cases} y^{2|J|_2-j-3} = y^{2(i-l)-1} & \text{for } j = 2l \geq 2, \\ y^{2|J|_2-j-2} = y^{2(i-l)-1} & \text{for } j = 2l+1 \geq 3 \end{cases} \lesssim y^{2i-3}|\log y|^C, \tag{2-77}$$

which is compatible with the degree i control at infinity (2-29). The control of further derivatives in (y, b_1) follows from (2-32) and the Leibniz rule. This concludes the proof of (2-73).

<u>*Step* 3: *Estimate on the error*</u>. From (2-69), we compute

$$\Psi_b = \Psi_b^{(0)} + \Psi_b^{(1)}, \tag{2-78}$$

$$\Psi_b^{(0)} = \sum_{i=1}^{L}(-1)^{i+1} b_1 b_i H^{-i+1}\widetilde{\Sigma}_{b_1} \quad \text{with } \widetilde{\Sigma}_{b_1} = \Sigma_{b_1} - c_{b_1}T_1, \tag{2-79}$$

$$\Psi_b^{(1)} = b_1 \Lambda S_{L+2} - \sum_{i=1}^{L}((2i-1+c_{b_1})b_1 b_i - b_{i+1})\frac{\partial S_{L+2}}{\partial b_i} + \frac{1}{y^2}[R_1 + R_2]. \tag{2-80}$$

Estimates for $\Psi_b^{(0)}$. First observe from (2-43), (2-79) that

$$\operatorname{Supp} \widetilde{\Sigma}_{b_1} \subset \left\{ y \geq \frac{B_0}{4} \right\}. \tag{2-81}$$

We extract from (2-42) the rough bound for $k \geq 0$ and $B_0/4 \leq y \leq 2B_1$

$$|H^{-k}\widetilde{\Sigma}_{b_1}| \lesssim 1 + y^{2k+1}.$$

Thus

$$\int_{y \leq 2B_1} |H^{-k}\widetilde{\Sigma}_{b_1}|^2 \lesssim b_1^{-2k-2}|\log b_1|^C, \quad 0 \leq k \leq L.$$

On the other hand, from (2-37) and the cancellation $H\Lambda Q = 0$, we have

$$|H\widetilde{\Sigma}_{b_1}| \lesssim \frac{1}{|\log b_1|}\left(\frac{1}{1+y}\right)\mathbf{1}_{y \geq B_0/4}, \tag{2-82}$$

$$|H^k\widetilde{\Sigma}_{b_1}| \lesssim \frac{1}{B_0^{2(k-1)}|\log b_1|}\left(\frac{1}{1+y}\right)\mathbf{1}_{B_0 \leq y \leq 3B_0} \quad \text{for } k \geq 2. \tag{2-83}$$

This leads to the bound

$$\int_{y\le 2B_1} |H\widetilde{\Sigma}_{b_1}|^2 \lesssim \frac{1}{|\log b_1|}, \qquad \int |H^k\widetilde{\Sigma}_{b_1}|^2 \lesssim \frac{b_1^{2k-2}}{|\log b_1|^2} \quad \text{for } k\ge 2.$$

Thus from (2-57), for $0\le k\le L$, we estimate

$$\int_{y\le 2B_1} |H^k\Psi_b^{(0)}|^2 \lesssim |\log b_1|^C \sum_{i=1}^{L} b_1^{2+2i} b_1^{2(k-i+1)-2} \lesssim b_1^{2k+2}|\log b_1|^C$$

and the sharp logarithmic gain

$$\int |H^{L+1}\Psi_b^{(0)}|^2 \lesssim \sum_{i=1}^{L} b_1^{2+2i}\|H^{L+2-i}\widetilde{\Sigma}_{b_1}\|_{L^2}^2 \lesssim \frac{1}{|\log b_1|^2}\sum_{i=1}^{L} b_1^{2+2i} b_1^{2(L+1-i+1)-2} \lesssim \frac{b_1^{2L+4}}{|\log b_1|^2}.$$

Similarly, using (2-82), (2-83),

$$\begin{aligned}\int_{y\le 2B_1} \frac{1+|\log y|^2}{1+y^4}|H^L\Psi_b^{(0)}|^2 &\lesssim \sum_{i=1}^{L} b_1^{2+2i}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|H^{L-i+1}\widetilde{\Sigma}_{b_1}|^2 \\ &\lesssim \sum_{i=1}^{L} b_1^{2+2i}\int_{y\ge B_0/4}\frac{1+|\log y|^2}{1+y^4}\,\frac{b_1^{2(L-i+1-1)}}{|\log b_1|^2(1+y^2)} \lesssim b_1^{2L+4},\end{aligned}$$

and

$$\begin{aligned}\int_{y\le 2B_1} \frac{1+|\log y|^2}{1+y^2}|AH^L\Psi_b^{(0)}|^2 &\lesssim \sum_{i=1}^{L} b_1^{2+2i}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}|AH^{L-i+1}\widetilde{\Sigma}_{b_1}|^2 \\ &\lesssim \sum_{i=1}^{L} b_1^{2+2i} b_1^{2(L-i)}\int_{y\ge B_0/4}\frac{1+|\log y|^2}{y^4(1+y^2)} \\ &\lesssim b_1^{2L+4}|\log b_1|^C.\end{aligned}$$

Estimates for $\Psi_b^{(1)}$. By construction, S_{L+2} is homogeneous of degree $(L+2, L+2, L+2)$ and thus so is ΛS_{L+2}. We therefore estimate from (2-53), (2-57), for all $0\le k\le L+1$,

$$\int_{y\le 2B_1} |b_1 H^k\Lambda S_{L+2}|^2 \lesssim \frac{b_1^2 b_1^{2L+4}|\log b_1|^{4(L+2-k-1)}}{b_1^{2(L+2-k)}|\log b_1|^2} = \frac{b_1^{2k+2}}{|\log b_1|^2}|\log b_1|^{4(L+1-k)},$$

and, using the rough bound (2-32),

$$\begin{aligned}&\int_{y\le 2B_1} (1+|\log y|^2)\left[\frac{|b_1H^L\Lambda S_{L+2}|^2}{1+y^4}+\frac{|b_1AH^L\Lambda S_{L+2}|^2}{1+y^2}\right] \\ &\qquad\lesssim b_1^2 b_1^{2L+4}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}(1+y^2)^{2(L+2)-1-2L} \lesssim b_1^{2L+4}|\log b_1|^C.\end{aligned}$$

We now turn to the control of R_1, which, from (2-68), is a linear combination of terms of the form

$$\widetilde{M}_J = \frac{f^{(j)}(Q)}{y^2} \prod_{k=1}^{L} b_k^{i_k} T_k^{i_k} \prod_{k=2}^{L+2} S_k^{j_k}, \quad |J|_1 = j,\ |J|_2 \geq L+3,\ 2 \leq j \leq L+2.$$

At the origin, the homogeneity of S_i and the admissibility of T_i ensure the bound, for $y \leq 1$,

$$|\widetilde{M}_J(y)| \lesssim b_1^{L+3} \begin{cases} y^{2|J|_2+j-1} = y^{2(|J|_2+l)-1} & \text{for } j = 2l, \\ y^{2|J|_2+j-2} = y^{2(|J|_2+l)-1} & \text{for } j = 2l+1 \end{cases} \lesssim b_1^{L+3} y^{2L+6},$$

and similarly for (2-77), for $1 \leq y \leq 2B_1$,

$$|\widetilde{M}_J(y)| \lesssim b_1^{|J|_2} |\log b_1|^C \begin{cases} y^{2|J|_2-j-3} = y^{2(|J|_2-l)-3} & \text{for } j = 2l \\ y^{2|J|_2-j-2} = y^{2(|J|_2-l)-3} & \text{for } j = 2l+1 \end{cases} \lesssim b_1^{|J|_2} y^{2|J|_2-5} |\log b_1|^C,$$

where we used $j \geq 2$, and similarly for higher derivatives. This ensures the control at the origin

$$|H^k \widetilde{M}_J(y)| \lesssim b_1^{L+3} \quad \text{for } 0 \leq k \leq L+1,\ y \leq 1$$

and, for $y \geq 1$,

$$|H^k \widetilde{M}_J(y)| \lesssim b_1^{|J|_2} y^{2(|J|_2-k)-5}, \quad 0 \leq k \leq L+1.$$

Thus, for all $0 \leq k \leq L+1$,

$$\begin{aligned}\int_{y \leq 2B_1} |H^k \widetilde{M}_J|^2 &\lesssim b_1^{2L+6} + b_1^{2|J|_2} |\log b_1|^C \int_{y \leq 2B_1} y^{4(|J|_2-k)-10} \lesssim b_1^{2L+6} + b_1^{2|J|_2} B_1^{4(|J|_2-k)-8} |\log b_1|^C \\ &\lesssim b_1^{2L+6} + b_1^{2k+4} |\log b_1|^C \lesssim b_1^{2k+3}.\end{aligned}$$

Similarly,

$$\begin{aligned}\int_{y \leq 2B_1} (1+|\log y|^2) \left[\frac{|H^L \widetilde{M}_J|^2}{1+y^4} + \frac{|AH^L \widetilde{M}_J|^2}{1+y^2} \right] \\ \lesssim b_1^{2L+6} + |\log b_1|^C \int_{1 \leq y \leq 2B_1} \frac{1+|\log y|^2}{1+y^4} |b_1^{|J|_2} y^{2(|J|_2-L)-5}|^2 \\ \lesssim b_1^{2L+6} + b_1^{2|J|_2} B_1^{4(|J|_2-L)-12} |\log b_1|^C \lesssim b_1^{2L+6} |\log b_1|^C.\end{aligned}$$

It remains to estimate the R_2 term given by (2-66). Near the origin $y \leq 1$, by construction, we have $|\alpha_b| \lesssim b_1 y^3$, and thus, for $0 \leq k \leq L+1$, $y \leq 1$,

$$\left| H^k \left(\frac{R_2}{y^2} \right) \right| \lesssim b_1^{L+3} y^{3(L+3)-2-2k} \lesssim b_1^{L+3}.$$

For $y \geq 1$, we use the rough bound by construction, for $1 \leq y \leq 2B_1$,

$$|\alpha_b| \lesssim b_1 y |\log y|^C,$$

which yields the bound, for $0 \leq k \leq L+1$, $1 \leq y \leq 2B_1$,

$$\left| H^k \left(\frac{R_2}{y^2} \right) \right| \lesssim b_1^{L+3} |\log b_1|^C y^{L+3-2-2k},$$

from which, for $0 \le k \le L+1$,

$$\int_{y\le 2B_1} |H^k\left(\frac{R_2}{y^2}\right)|^2 \lesssim b_1^{2L+6} + b_1^{2L+6}|\log b_1|^C \int_{1\le y\le 2B_1} y^{2L+2-4k}$$
$$\lesssim \begin{cases} b_1^{2L+6} & \text{for } 2L+2-4k<-1, \\ b_1^{2L+6}B_1^{2L+4-4k}|\log b_1|^C = b_1^{2k+L+4}|\log b_1|^C \end{cases}$$
$$\lesssim b_1^{2k+5}|\log b_1|^C.$$

Similarly,

$$\int_{y\le 2B_1}(1+|\log y|^2)\left[\frac{1}{1+y^4}|H^L\left(\frac{R_2}{y^2}\right)|^2 + \frac{1}{1+y^2}|AH^L\left(\frac{R_2}{y^2}\right)|^2\right] \lesssim b_1^{2L+5}.$$

The collection of above estimates yields (2-62), (2-64).

Finally, the local control (2-65) is a simple consequence of the support localization (2-81) and the fact that $\Psi_b^{(1)}$ given by (2-80) satisfies by construction a bound on compact sets:

$$|H^k\Psi_b^{(1)}(y)| \lesssim M^C b_1^{L+3} \quad \text{for all } y\le 2M \ll B_0 \text{ and all } 0\le k\le L+1.$$

This concludes the proof of Proposition 2.12. □

Localization of the profile. We now proceed to a simple localization procedure of the profile Q_b to avoid some irrelevant growth in the region $y \ge 2B_1$.

Proposition 2.13 (localization). *Under the assumptions of Proposition 2.12, assume the a priori bound*

$$|(b_1)_s| \lesssim b_1^2. \tag{2-84}$$

Consider the localized profile

$$\widetilde{Q}_{b(s)}(y) = Q(y) + \tilde{\alpha}_{b(s)}(y), \quad \tilde{\alpha}_b(y) = \sum_{i=1}^{L} b_i\widetilde{T}_i(y) + \sum_{i=2}^{L+2}\widetilde{S}_i(y), \tag{2-85}$$

with

$$\widetilde{T}_i = \chi_{B_1}T_i, \quad \widetilde{S}_i = \chi_{B_1}S_i. \tag{2-86}$$

Then

$$\partial_s\widetilde{Q}_b - \Delta\widetilde{Q}_b + b_1\Lambda\widetilde{Q}_b + \frac{f(\widetilde{Q}_b)}{y^2} = \widetilde{\Psi}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1b_i - b_{i+1}]\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right], \tag{2-87}$$

where $\widetilde{\Psi}_b$ satisfies

(i) *the weighted bounds*

$$\text{for all } 1\le k\le L, \quad \int |H^k\widetilde{\Psi}_b|^2 \lesssim b_1^{2k+2}|\log b_1|^C, \tag{2-88}$$

$$\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}|H^L\Psi_b|^2 + \int\frac{1+|\log y|^2}{1+y^2}|AH^L\widetilde{\Psi}_b|^2 \lesssim \frac{b_1^{2L+3}}{|\log b_1|^2}, \tag{2-89}$$

$$\int |H^{L+1}\widetilde{\Psi}_b|^2 \lesssim \frac{b_1^{2L+4}}{|\log b_1|^2}, \tag{2-90}$$

and

(ii) $$\text{for all } 0 \le k \le L+1, \quad \int_{y \le 2M} |H^k \widetilde{\Psi}_b|^2 \lesssim M^C b_1^{2L+6} \tag{2-91}$$

for some universal constant $C = C(L) > 0$ (*improved local control*).

Proof of Proposition 2.13. From localization we compute

$$\begin{aligned} \partial_s \widetilde{Q}_b - \Delta \widetilde{Q}_b + b_1 \Lambda \widetilde{Q}_b + \frac{f(\widetilde{Q}_b)}{y^2} \\ = \chi_{B_1} \left\{ \partial_s Q_b - \Delta Q_b + b_1 \Lambda Q_b + \frac{f(Q_b)}{y^2} \right\} + (\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1} \partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1} \\ + b_1(1-\chi_{B_1})\Lambda Q + \frac{1}{y^2}\{f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q))\} \end{aligned}$$

so that

$$\widetilde{\Psi}_b = \chi_{B_1} \Psi_b + \widetilde{\Psi}_b^{(0)}$$

with

$$\begin{aligned} \widetilde{\Psi}_b^{(0)} = \frac{1}{y^2}\{f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q))\} \\ + (\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1} \partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1} + b_1(1-\chi_{B_1})\Lambda Q. \end{aligned} \tag{2-92}$$

Note that all terms on the right hand side above are localized in $B_1 \le y \le 2B_1$ except the last one, for which $\mathrm{Supp}((1-\chi_{B_1})\Lambda Q) \subset \{y \ge B_1\}$. Hence (2-65) implies (2-91). The bounds (2-88), (2-89), (2-90) for $\chi_{B_1}\Psi_b$ follow verbatim as in the proof of (2-62), (2-63), (2-64).

To estimate the second error induced by localization in (2-92), first observe from (2-84) the bound

$$|\partial_s \chi_{B_1}| \lesssim \frac{|(b_1)_s|}{b_1} |y \chi'_{B_1}| \lesssim b_1 \mathbf{1}_{B_1 \le y \le 2B_1}.$$

Moreover, from the admissibility of T_i and the b_1-admissibility of S_i, T_i terms dominate for $y \sim B_1$ in α_b, and we estimate from (2-18), for all $k \ge 0$ and $B_1 \le y \le 2B_1$,

$$\left| \frac{\partial^k}{\partial y^k} \alpha_b \right| \lesssim \sum_{i=1}^{L} b_1^i y^{2i-k-1}(1 + |\log b_1|) \lesssim \frac{|\log b_1|}{B_1^{k+1}}. \tag{2-93}$$

This yields, for all $1 \le k \le L$,

$$\begin{aligned} \int \left| H^k \Big((\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1} \partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1} \Big) \right|^2 \\ \lesssim \int_{B_1 \le y \le 2B_1} \left| \frac{b_1 |\log b_1|}{B_1^{2k+1}} + \frac{|\log b_1|}{B_1^{2k+1+2}} \right|^2 \lesssim \frac{b_1^2}{B_1^{4k}} |\log b_1|^C \\ \lesssim b_1^{2k+2} |\log b_1|^C \end{aligned} \tag{2-94}$$

and

$$\int \left| H^{L+1}\Big((\partial_s \chi_{B_1})\alpha_b - 2\partial_y \chi_{B_1} \partial_y \alpha_b - \alpha_b \Delta \chi_{B_1} + b_1 \alpha_b \Lambda \chi_{B_1} \Big) \right|^2 \lesssim \int_{B_1 \le y \le 2B_1} \left| \frac{b_1 |\log b_1|}{B_1^{2(L+1)+1}} + \frac{|\log b_1|}{B_1^{2(L+1)+1+2}} \right|^2 \lesssim \frac{b_1^{2L+4}}{|\log b_1|^2}. \tag{2-95}$$

We next estimate by brute force

$$\left| \frac{d^k}{dy^k} [(1-\chi_{B_1})\Lambda Q] \right| \lesssim \frac{1}{y^{k+1}} \mathbf{1}_{y \ge B_1},$$

from which, for all $1 \le k \le L+1$,

$$\int |H^k (b_1 (1-\chi_{B_1}) \Lambda Q)|^2 \lesssim b_1^2 \int_{B_1 \le y \le 2B_1} \frac{1}{y^{4k+2}} \lesssim \frac{b_1^{2k+2}}{|\log b_1|^{4k}}.$$

It remains to estimate the nonlinear term, for which, using (2-93) and $|f'| \lesssim 1$, we estimate

$$\left| \frac{\partial^k}{\partial y^k} \left\{ \frac{1}{y^2} [f(\widetilde{Q}_b) - f(Q) - \chi_{B_1}(f(Q_b) - f(Q))] \right\} \right| \lesssim \frac{|\log b_1|}{B_1^{k+1}}.$$

The corresponding terms are estimated as in (2-94), (2-95). □

Study of the dynamical system for $b = (b_1, \dots, b_L)$. The essence of the construction of the Q_b profile is to generate according to (2-61) the finite dimensional dynamical system (1-25) for $b = (b_1, \dots, b_L)$:

$$(b_k)_s + \left(2k - 1 + \frac{2}{\log s} \right) b_1 b_k - b_{k+1} = 0, \quad 1 \le k \le L, \; b_{L+1} \equiv 0. \tag{2-96}$$

We show in this section that (2-96) admits exceptional solutions, and that the linearized operator close to these solutions is explicit.

Lemma 2.14 (approximate solution for the b system). *Let $L \ge 2$ and let $s_0 \gg 1$ be a large enough universal constant. We write the sequences*

$$\begin{cases} c_1 = \dfrac{L}{2L-1}, \\ c_{k+1} = -\dfrac{L-k}{2L-1} c_k, \quad 1 \le k \le L-1, \end{cases} \tag{2-97}$$

$$\begin{cases} d_1 = -\dfrac{2L}{(2L-1)^2}, \\ d_{k+1} = -\dfrac{L-k}{2L-1} d_k + \dfrac{4L(L-k)}{(2L-1)^2} c_k, \quad 1 \le k \le L-1. \end{cases} \tag{2-98}$$

Then the explicit choice

$$b_k^e(s) = \frac{c_k}{s^k} + \frac{d_k}{s^k \log s}, \quad 1 \le k \le L, \; b_{L+1}^e \equiv 0 \tag{2-99}$$

generates an approximate solution to (2-96) *in the sense that*

$$(b_k^e)_s+\left(2k-1+\frac{2}{\log s}\right)b_1^e b_k^e-b_{k+1}^e=O\left(\frac{1}{s^{k+1}(\log s)^2}\right),\quad 1\le k\le L. \tag{2-100}$$

The proof of Lemma 2.14 is an explicit computation which is left to the reader. We now claim that this solution corresponds to a codimension $(L-1)$ exceptional manifold.

Lemma 2.15 (linearization). 1. Computation of the linearized system. *Let*

$$b_k(s)=b_k^e(s)+\frac{U_k(s)}{s^k(\log s)^{5/4}},\quad 1\le k\le L,\ b_{L+1}=U_{L+1}\equiv 0, \tag{2-101}$$

and note $U=(U_1,\dots,U_L)$. *Then*

$$(b_k)_s+\left(2k-1+\frac{2}{\log s}\right)b_1b_k-b_{k+1} =\frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(U_k)_s-(A_LU)_k+O\left(\frac{1}{\sqrt{\log s}}+\frac{|U|+|U|^2}{\log s}\right)\right], \tag{2-102}$$

where

$$A_L=(a_{i,j})_{1\le i,j,\le L}\quad \textit{with}\quad \begin{cases} a_{11}=-1/(2L-1), & \\ a_{i,i+1}=1, & 1\le i\le L-1,\\ a_{1,i}=-(2i-1)c_i, & 2\le i\le L,\\ a_{i,i}=(L-i)/(2L-1), & 2\le i\le L,\\ a_{i,j}=0 & \textit{otherwise}.\end{cases} \tag{2-103}$$

2. Diagonalization of the linearized matrix. A_L *is diagonalizable*:

$$A_L=P_L^{-1}D_LP_L,\quad D_L=\operatorname{diag}\left\{-1,\frac{2}{2L-1},\frac{3}{2L-1},\dots,\frac{L}{2L-1}\right\}. \tag{2-104}$$

Proof of Lemma 2.15. Step 1: Linearization. A simple computation from (2-99) ensures

$$(b_k)_s+\left(2k-1+\frac{2}{\log s}\right)b_1b_k-b_{k+1} =\frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(U_k)_s-kU_k+O\left(\frac{|U|}{\log s}\right)\right]+O\left(\frac{1}{s^{k+1}(\log s)^2}\right) +\frac{1}{s^{k+1}(\log s)^{5/4}}\left[(2k-1)c_kU_1+(2k-1)c_1U_k-U_{k+1}+O\left(\frac{|U|+|U|^2}{\log s}\right)\right],$$

and then the relation

$$(2k-1)c_1-k=\frac{(2k-1)L}{2L-1}-k=-\frac{L-k}{2L-1}$$

ensures

$$(b_k)_s+\left(2k-1+\frac{2}{\log s}\right)b_1b_k-b_{k+1} =\frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(U_k)_s+(2k-1)c_kU_1-\frac{L-k}{2L-1}U_k-U_{k+1}+O\left(\frac{1}{\sqrt{\log s}}+\frac{|U|+|U|^2}{\log s}\right)\right],$$

which is equivalent to (2-102), (2-103).

Step 2 Diagonalization. The proof follows by computing the characteristic polynomial. The cases $L = 2, 3$ are done by direct inspection. Let us assume $L \geq 4$. We compute

$$P_L(X) = \det(A_L - X\,\mathrm{Id})$$

by developing on the last row. This yields

$$P_L(x) = (-1)^{L+1}(-1)(2L-1)c_L + (-X)\left\{(-1)^L(-1)(2L-3)c_{L-1} + \left(\frac{1}{2L-1} - X\right)\left[(-1)^{L-1}(-1)(2L-5)c_{L-2} + \left(\frac{2}{2L-1} - X\right)\cdots\right]\right\}.$$

We use the recurrence relation (2-97) to compute explicitly

$$\begin{aligned}(-1)^{L+1}&(-1)(2L-1)c_L \\ &+ (-X)\left\{(-1)^L(-1)(2L-3)c_{L-1} + \left(\frac{1}{2L-1} - X\right)[(-1)^{L-1}(-1)(2L-5)c_{L-2}]\right\} \\ &= (-1)^L\left\{(2L-3)c_{L-1}\left(X - \frac{1}{2L-3}\right) + (2L-5)c_{L-2}\left(X - \frac{1}{2L-1}\right)X\right\}.\end{aligned}$$

We now compute from (2-97), for $1 \leq k \leq L-2$,

$$\begin{aligned}(2L-(2k+1))&c_{L-k}\left(X - \frac{1}{2L-(2k+1)}\right) + (2L-(2k+3))c_{L-(k+1)}X\left(X - \frac{1}{2L-1}\right) \\ &= (2L-(2k+3))c_{L-(k+1)}\left[X\left(X - \frac{1}{2L-1}\right) - \frac{2L-(2k+1)}{2L-(2k+3)}\frac{k+1}{2L-1}\left(X - \frac{1}{2L-(2k+1)}\right)\right] \\ &= (2L-(2k+3))c_{L-(k+1)}\left(X - \frac{k+1}{2L-1}\right)\left(X - \frac{1}{2L-(2k+3)}\right). \qquad (2\text{-}105)\end{aligned}$$

We therefore obtain inductively

$$\begin{aligned}P_L(X) &= (-1)^L\left\{(2L-3)c_{L-1}\left(X - \frac{1}{2L-3}\right) + (2L-5)c_{L-2}\left(X - \frac{1}{2L-1}\right)X\right\} \\ &\quad + (-X)\left(\frac{1}{2L-1} - X\right)\left(\frac{2}{2L-1} - X\right)\left[(-1)^{L-2}(-1)(2L-7)c_{L-3} + \left(\frac{3}{2L-1} - X\right)\cdots\right] \\ &= (-1)^L\left(X - \frac{2}{2L-1}\right)\left\{(2L-5)c_{L-2}\left(X - \frac{1}{2L-5}\right) + (2L-7)c_{L-3}X\left(X - \frac{1}{2L-1}\right)\right\} \\ &\quad + (-X)\left(\frac{1}{2L-1} - X\right)\left(\frac{2}{2L-1} - X\right)\left(\frac{3}{2L-1} - X\right)[(-1)^{L-3}(-1)(2L-9)c_{L-4}\cdots] \\ &= (-1)^L\left(X - \frac{2}{2L-1}\right)\cdots\left(X - \frac{L-2}{2L-1}\right) \\ &\quad \times\left\{3c_2\left(X - \frac{1}{3}\right) + X\left(X - \frac{1}{2L-1}\right)\left(c_1 + X - \frac{L-1}{2L-1}\right)\right\}.\end{aligned}$$

We use (2-105) with $k = L-2$ to compute the last polynomial:

$$
\begin{aligned}
3c_2\left(X-\frac{1}{3}\right)+X\left(X-\frac{1}{2L-1}\right)\left(c_1+X-\frac{L-1}{2L-1}\right)\\
&=\left\{3c_2\left(X-\frac{1}{3}\right)+c_1X\left(X-\frac{1}{2L-1}\right)\right\}+X\left(X-\frac{1}{2L-1}\right)\left(X-\frac{L-1}{2L-1}\right)\\
&=c_1\left(X-\frac{L-1}{2L-1}\right)(X-1)+X\left(X-\frac{1}{2L-1}\right)\left(X-\frac{L-1}{2L-1}\right)\\
&=\left(X-\frac{L-1}{2L-1}\right)\left[\frac{L}{2L-1}(X-1)+X\left(X-\frac{1}{2L-1}\right)\right]\\
&=\left(X-\frac{L-1}{2L-1}\right)\left(X-\frac{L}{2L-1}\right)\left(X+1\right).
\end{aligned}
$$

We have therefore computed

$$
P_L(x)=(-1)^L\left(X-\frac{2}{2L-1}\right)\cdots\left(X-\frac{L-2}{2L-1}\right)\left(X-\frac{L-1}{2L-1}\right)\left(X-\frac{L}{2L-1}\right)\left(X+1\right)
$$

and (2-104) is proved. □

3. The trapped regime

In this section, we introduce the main dynamical tools at the heart of the proof of Theorem 1.1. We start with describing the bootstrap regime in which the blow-up solutions of Theorem 1.1 will be trapped. We then exhibit the Lyapounov type control of H^k norms, which is the heart of our analysis.

Modulation. We describe in this section the set of initial data leading to the blow-up scenario of Theorem 1.1. Let there be a smooth 1-corotational initial data

$$
v(0,x)=\left|\begin{array}{l} g(u_0(r))\cos\theta\\ g(u_0(r))\sin\theta\\ z(u_0(r))\end{array}\right. \quad \text{with} \quad \|\nabla u_0-\nabla Q\|_{L^2}\ll 1, \tag{3-1}
$$

and let $v(t,x)$ be the corresponding smooth solution to (1-1) with life time $0<T<+\infty$. From (A-1), we may decompose on a small time interval

$$
v(t,x)=\left|\begin{array}{l} g(u(t,r))\cos\theta\\ g(u(t,r))\sin\theta\\ z(u(t,r)),\end{array}\right. \tag{3-2}
$$

where

$$
\tilde\varepsilon(t,r)=u(t,r)-Q(r) \quad \text{satisfies (A-4).} \tag{3-3}
$$

Moreover, from a standard argument,

$$
T<+\infty \quad \text{implies } \|\Delta v(t)\|_{L^2}\to+\infty \quad \text{as } t\to T. \tag{3-4}
$$

We now modulate the solution and introduce from a standard argument[10] using the initial smallness (3-1) the unique decomposition of the flow defined on a small time $t \in [0, t_1]$:

$$u(t,r) = (\widetilde{Q}_{b(t)} + \varepsilon(t,r))_{\lambda(t)}, \quad \lambda(t) > 0, \ b = (b_1, \dots, b_L), \tag{3-5}$$

where $\varepsilon(t)$ satisfies the $L+1$ orthogonality conditions

$$(\varepsilon, H^k \Phi_M) = 0, \quad 0 \le k \le L \tag{3-6}$$

and the smallness

$$\|\nabla \varepsilon(t)\|_{L^2} + \left\| \frac{\varepsilon(t)}{y} \right\|_{L^2} + |b(t)| \ll 1.$$

Here, given $M > 0$ large enough, we define

$$\Phi_M = \sum_{p=0}^{L} c_{p,M} H^p (\chi_M \Lambda Q), \tag{3-7}$$

where

$$c_{0,M} = 1, \quad c_{k,M} = (-1)^{k+1} \frac{\sum_{p=0}^{k-1} c_{p,M}(\chi_M H^p(\chi_M \Lambda Q), T_k)}{(\chi_M \Lambda Q, \Lambda Q)}, \quad 1 \le k \le L,$$

is manufactured to ensure the nondegeneracy

$$(\Phi_M, \Lambda Q) = (\chi_M \Lambda Q, \Lambda Q) = 4 \log M (1 + o(1)) \quad \text{as } M \to +\infty \tag{3-8}$$

and the cancellation, for all $1 \le k \le L$,

$$(\Phi_M, T_k) = \sum_{p=0}^{k-1} c_{p,M} (H^p(\chi_M \Lambda Q), T_k) + c_{k,M}(-1)^k (\chi_M \Lambda Q, \Lambda Q) = 0. \tag{3-9}$$

In particular,

$$(H^i T_j, \Phi_M) = (-1)^j (\chi_M \Lambda Q, \Lambda Q) \delta_{i,j}, \quad 0 \le i, j \le L. \tag{3-10}$$

Observe also by induction that

$$\text{for all } 1 \le p \le L, \quad |c_{p,M}| \lesssim M^{2p}, \tag{3-11}$$

from which

$$\int |\Phi_M|^2 \lesssim \int |\chi_M \Lambda Q|^2 + \sum_{p=1}^{L} c_{p,M}^2 \int |H^p(\chi_M \Lambda Q)|^2 \lesssim \log M. \tag{3-12}$$

The existence of the decomposition (3-5) is a standard consequence of the implicit function theorem and the explicit relations

$$\left| \left(\frac{\partial}{\partial \lambda} (\widetilde{Q}_b)_\lambda, \frac{\partial}{\partial b_1} (\widetilde{Q}_b)_\lambda, \dots, \frac{\partial}{\partial b_L} (\widetilde{Q}_b)_\lambda \right) \right|_{\lambda=1,\, b=0} = (\Lambda Q, T_1, \dots, T_L),$$

[10] See, for example, [Martel and Merle 2000; Merle and Raphaël 2005a; Raphaël and Rodnianski 2012] for a further introduction to modulation.

which, using (3-9), imply the nondegeneracy of the Jacobian

$$\left|\left(\frac{\partial}{\partial(\lambda, b_j)}(\widetilde{Q}_b)_\lambda, H^i\Phi_M\right)_{1\le j\le L,\, 0\le i\le L}\right|_{\lambda=1,\, b=0} = (\chi_M \Lambda Q, \Lambda Q)^{L+1} \neq 0.$$

The decomposition (3-5) exists as long as $t < T$ and $\varepsilon(t, r)$ remains small in the energy topology. Observe also from (3-3), (3-5), and the explicit structure of $\widetilde{Q}_b$ that ε satisfies (A-4), and in particular Lemma B.5 applies. In other words, we may measure the regularity of the map through the following coercive norms of ε: the energy norm

$$\|\varepsilon\|_{\mathcal{H}}^2 = \int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2}, \tag{3-13}$$

and higher order Sobolev norms adapted to the linearized operator

$$\mathscr{E}_{2k} = \int |H^k \varepsilon|^2, \quad 1 \le k \le L+1. \tag{3-14}$$

Setting up the bootstrap. We now choose our set of initial data in a more restricted way. More precisely, we pick a large enough time $s_0 \gg 1$ and rewrite the decomposition (3-5) as

$$u(t, r) = (\widetilde{Q}_{b(s)} + \varepsilon)(s, y), \tag{3-15}$$

where we introduce the renormalized variables

$$y = \frac{r}{\lambda(t)}, \quad s(t) = s_0 + \int_0^t \frac{d\tau}{\lambda^2(\tau)} \tag{3-16}$$

and measure time in s, which will be proved to be a global time. We introduce a decomposition (2-101):

$$b_k = b_k^e + \frac{U_k}{s^k(\log s)^{5/4}}, \quad 1 \le k \le L,\ b_{k+1} = U_{k+1} \equiv 0. \tag{3-17}$$

We consider the variable

$$V = P_L U, \tag{3-18}$$

where P_L refers to the diagonalization (2-104) of A_L. We assume that initially

$$|V_1(0)| \le 1, \quad (V_2(0), \dots, V_L(0)) \in \mathcal{B}_{L-1}(2). \tag{3-19}$$

We also assume the explicit initial smallness of the data:

$$\int |\nabla \varepsilon(0)|^2 + \int \left|\frac{\varepsilon(0)}{y}\right|^2 \le b_1^2(0), \tag{3-20}$$

$$|\mathscr{E}_{2k}(0)| \le [b_1(0)]^{10L+4}, \quad 1 \le k \le L+1. \tag{3-21}$$

Note also that, up to a fixed rescaling, we may always assume

$$\lambda(0) = 1. \tag{3-22}$$

Proposition 3.1 (bootstrap). *There exists*

$$(V_2(0), \dots, V_L(0)) \in \mathcal{B}_{L-1}(2)$$

such that the following bounds hold for all $s \geq s_0$:

- *Control of the radiation*:

$$\int |\nabla \varepsilon(s)|^2 + \int \left|\frac{\varepsilon(s)}{y}\right|^2 \leq 10(b_1(0))^{\frac{1}{4}}, \tag{3-23}$$

$$|\mathscr{E}_{2k}(s)| \leq b_1^{(2k-1)2L/(2L-1)}(s)|\log b_1(s)|^K, \quad 1 \leq k \leq L, \tag{3-24}$$

$$|\mathscr{E}_{2L+2}(s)| \leq K \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2}. \tag{3-25}$$

- *Control of the unstable modes*:

$$|V_1(s)| \leq 2, \quad (V_2(s), \dots, V_L(s)) \in \mathscr{B}_{L-1}(2). \tag{3-26}$$

Remark 3.2. *Note that the bounds (3-24) easily imply*[11] *the control of the H^2 norm of the full map (3-2)*

$$\int |\Delta v(s)|^2 < C(s) < +\infty, \quad s < s^*,$$

and therefore the blow-up criterion (3-4) ensures that the map is well defined on $[s, s^*)$.

Equivalently, given $(\varepsilon(0), V(0))$ as above, we introduce the time

$$s^* = s^*(\varepsilon(0), V(0)) = \sup\{s \geq s_0 \text{ such that (3-23), (3-24), (3-25), (3-26) hold on } [s_0, s]\}.$$

Observe that the continuity of the flow and the initial smallness (3-20), (3-21) ensure that $s^* > 0$. We then assume by contradiction that

$$\text{for all } (V_2(0), \dots, V_L(0)) \in \mathscr{B}_{L-1}(2), \quad s^* < +\infty, \tag{3-27}$$

and look for a contradiction. Our main claim is that the a priori control of the unstable modes (3-26) is enough to improve the bounds (3-23), (3-24), (3-25), and then the claim follows from the $(L-1)$ codimensional instability (2-104) of the system (2-96) near the exceptional solution b^e through a standard topological argument à la Brouwer.

The rest of this section is devoted to the derivation of the key lemmas for the proof of Proposition 3.1. We will make a systematic implicit use of the interpolation bounds of Lemma C.1, which are a consequence of the coercivity of the $\mathscr{E}_{2k+2}$ energy given by Lemma B.5.

Equation for the radiation. Recall the decomposition of the flow

$$u(t, r) = (\tilde{Q}_{b(t)} + \varepsilon)(s, y) = (Q + \tilde{\alpha}_{b(t)})_{\lambda(s)} + w(t, r).$$

We use the rescaling formulas

$$u(t, r) = v(s, y), \quad y = \frac{r}{\lambda(t)}, \quad \partial_t u = \frac{1}{\lambda^2(t)}\left(\partial_s v - \frac{\lambda_s}{\lambda}\Lambda v\right)_\lambda$$

[11] See [Raphaël and Schweyer 2013] for the full computation.

to derive the equation for ε in renormalized variables,

$$\partial_s \varepsilon - \frac{\lambda_s}{\lambda}\Lambda\varepsilon + H\varepsilon = F - \widetilde{\mathrm{Mod}} = \mathcal{F}. \tag{3-28}$$

Here H is the linearized operator given by (2-8), $\widetilde{\mathrm{Mod}}(t)$ is given by

$$\widetilde{\mathrm{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda \widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right], \tag{3-29}$$

and

$$F = -\widetilde{\Psi}_b + L(\varepsilon) - N(\varepsilon), \tag{3-30}$$

where L is the linear operator corresponding to the error in the linearized operator from Q to $\widetilde{Q}_b$

$$L(\varepsilon) = \frac{f'(Q) - f'(\widetilde{Q}_b)}{y^2}\varepsilon, \tag{3-31}$$

and the remainder term is the purely nonlinear term

$$N(\varepsilon) = \frac{f(\widetilde{Q}_b + \varepsilon) - f(\widetilde{Q}_b) - \varepsilon f'(\widetilde{Q}_b)}{y^2}. \tag{3-32}$$

We also need to write the flow (3-28) in original variables. For this we use the rescaled operators

$$\begin{aligned} A_\lambda &= -\partial_r + \frac{Z_\lambda}{r}, & A_\lambda^* &= \partial_r + \frac{1+Z_\lambda}{r}, \\ H_\lambda &= A_\lambda^* A_\lambda = -\Delta + \frac{V_\lambda}{r^2}, & \widetilde{H}_\lambda &= A_\lambda A_\lambda^* = -\Delta + \frac{\widetilde{V}_\lambda}{r^2}, \end{aligned} \tag{3-33}$$

and the renormalized function

$$w(t,r) = \varepsilon(s,y).$$

Then (3-28) becomes

$$\partial_t w + H_\lambda w = \frac{1}{\lambda^2}\mathcal{F}_\lambda. \tag{3-34}$$

Observe from (2-99) that, for $s < s^*$,

$$|b_k| \lesssim b_1^k, \quad 0 < b_1 \ll 1, \tag{3-35}$$

and hence the a priori bound (2-57) holds.

Modulation equations. Let us now compute the modulation equations for (b, λ) as a consequence of the choice of orthogonality conditions (3-6).

Lemma 3.3 (modulation equations). *We have the bound on the modulation parameters*

$$\left|\frac{\lambda_s}{\lambda}+b_1\right|+\sum_{k=1}^{L-1}|(b_k)_s+(2k-1+c_{b_1})b_1b_k-b_{k+1}|\lesssim b_1^{L+\frac{3}{2}}, \tag{3-36}$$

$$\left|(b_L)_s+(2L-1+c_{b_1})b_1b_L\right|\lesssim\frac{1}{\sqrt{\log M}}\left(\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right). \tag{3-37}$$

Remark 3.4. *Note that this implies in the bootstrap the rough bound*

$$|(b_1)_s|\le 2b_1^2, \tag{3-38}$$

and, *in particular*, (2-84) *holds.*

Proof of Lemma 3.3. *Step* 1: *Law for b_L*. Let

$$D(t)=\left|\frac{\lambda_s}{\lambda}+b_1\right|+\sum_{k=1}^{L}|(b_k)_s+(2k-1+c_{b_1})b_1b_k-b_{k+1}|. \tag{3-39}$$

We take the inner product of (3-28) with $H^L\Phi_M$ and, using the orthogonality (3-6), obtain

$$(\widetilde{\mathrm{Mod}}(t),H^L\Phi_M)=-(\widetilde{\Psi}_b,H^L\Phi_M)-(H^L\varepsilon,H\Phi_M)-\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon-L(\varepsilon)+N(\varepsilon),H^L\Phi_M\right). \tag{3-40}$$

First, from the construction of the profile, (3-29), the localization $\mathrm{Supp}(\Phi_M)\subset[0,2M]$ from (3-7), and the identities (3-8), (3-9), (3-10), we compute

$$\begin{aligned}(H^L(\widetilde{\mathrm{Mod}}(t)),\Phi_M)\\ &=-\left(b_1+\frac{\lambda_s}{\lambda}\right)(H^L\Lambda\widetilde{Q}_b,\Phi_M)+\sum_{i=1}^{L}[(b_i)_s+(2i-1+c_{b_1})b_1b_i-b_{i+1}]\left(\widetilde{T}_i+\chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i},H^L\Phi_M\right)\\ &=(-1)^L(\Lambda Q,\Phi_M)((b_L)_s+(2L-1+c_{b_1})b_1b_L)+O(M^Cb_1|D(t)|).\end{aligned}$$

The linear term in (3-40) is estimated[12] from (3-24), (3-12):

$$|(H^L\varepsilon,H\Phi_M)|\lesssim\|H^{L+1}\varepsilon\|_{L^2}\sqrt{\log M}=\sqrt{\log M\mathscr{E}_{2L+2}}.$$

The remaining nonlinear term is estimated using the Hardy bounds of Appendix A:

$$\left|\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon+L(\varepsilon)+N(\varepsilon),H^L\Phi_M\right)\right|\lesssim M^Cb_1(\sqrt{\mathscr{E}_{2L+2}}+|D(t)|).$$

[12] Observe that we do not use the interpolated bounds of Lemma C.1, but directly the definition (3-14) of $\mathscr{E}_{2L+2}$, and hence the dependence of the constant in M is explicit. This will be crucial for the analysis.

We inject these estimates into (3-40) and conclude from (3-8) and the local estimate (2-91) that

$$\begin{aligned}|(b_L)_s+(2L-1+c_{b_1})b_1b_L| &= \frac{\sqrt{\log M\mathscr{E}_{2L+2}}}{\log M}+M^Cb_1|D(t)|+M^Cb_1^{L+\frac{3}{2}}\\ &\lesssim \frac{1}{\sqrt{\log M}}\left(\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right)+M^Cb_1|D(t)|. \end{aligned}\tag{3-41}$$

Step 2: Degeneracy of the law for λ and $(b_k)_{1\le k\le L-1}$. We now take the inner product of (3-28) with $H^k\Phi_M$, $0\le k\le L-1$ and obtain

$$(\widetilde{\mathrm{Mod}}(t), H^k\Phi_M)=-(\widetilde{\Psi}_b, H^k\Phi_M)-(H^{k+1}\varepsilon, H\Phi_M)-\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon-L(\varepsilon)+N(\varepsilon), H^k\Phi_M\right).\tag{3-42}$$

Note first that the choice of orthogonality conditions (3-6) gets rid of the linear term in ε:

$$\text{for all } 0\le k\le L-1,\quad (H^{k+1}\varepsilon, \Phi_M)=0.$$

Next, from (3-29), the localization $\mathrm{Supp}(\Phi_M)\subset[0, 2M]$ from (3-7), and the identities (3-8), (3-9), (3-10), we compute

$$\begin{aligned}&(H^k(\widetilde{\mathrm{Mod}}(t)), \Phi_M)\\ &=-\left(b+\frac{\lambda_s}{\lambda}\right)(H^k\Lambda\widetilde{Q}_b, \Phi_M)+\sum_{i=1}^{L}[(b_i)_s+(2i-1+c_{b_1})b_1b_i-b_{i+1}]\left(\widetilde{T}_i+\chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}, H^k\Phi_M\right)\\ &=(\Lambda Q, \Phi_M)\begin{cases}-(\lambda_s/\lambda+b_1) & \text{for } k=0,\\ (-1)^k((b_k)_s+(2k-1+c_{b_1})b_1b_k-b_{k+1}) & \text{for } 1\le k\le L-1\end{cases}+O(M^Cb_1|D(t)|).\end{aligned}$$

Nonlinear terms are easily estimated using the Hardy bounds

$$\left|\left(-\frac{\lambda_s}{\lambda}\Lambda\varepsilon+L(\varepsilon)+N(\varepsilon), H^k\Phi_M\right)\right|\lesssim M^Cb_1(\sqrt{\mathscr{E}_{2L+2}}+|D(t)|)\lesssim b_1^{L+\frac{3}{2}}+b_1M^C|D(t)|.$$

Injecting this bound into (3-42) together with the local bound (2-91) yields the first bound,

$$D(t)\lesssim b_1^{L+\frac{3}{2}},\tag{3-43}$$

and (3-36) is proved. Injecting this bound into (3-41) yields (3-37). □

Improved modulation equation for b_L. Observe that (3-37), (3-25) yield the pointwise bound

$$|(b_L)_s+(2L-1+c_{b_1})b_1b_L|\lesssim\frac{1}{\sqrt{\log M}}\left(\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right)\lesssim\frac{b_1^{L+1}}{|\log b_1|},$$

which is worse than (3-36) and critical to close (3-26). We claim that a $|\log b_1|$ is easily gained up to an oscillation in time.

Lemma 3.5 (improved control of b_L). *Let $B_\delta=B_0^\delta$ and*

$$\tilde{b}_L=b_L+\frac{(-1)^L(H^L\varepsilon, \chi_{B_\delta}\Lambda Q)}{4\delta|\log b_1|}.\tag{3-44}$$

Then

$$|\tilde{b}_L - b_L| \lesssim b_1^{L+\frac{1}{2}} \tag{3-45}$$

and $\tilde{b}_L$ satisfies the pointwise differential equation

$$|(\tilde{b}_L)_s + (2L-1+c_{b_1})b_1\tilde{b}_L| \lesssim \frac{C(M)}{\sqrt{|\log b_1|}}\left[\sqrt{\mathscr{E}_{2L+2}} + \frac{b_1^{L+1}}{|\log b_1|}\right]. \tag{3-46}$$

Proof of Lemma 3.5. We commute (3-28) with H^L and take the scalar product with $\chi_{B_\delta}\Lambda Q$ for some small enough universal constant $0 < \delta \ll 1$. This yields

$$\frac{d}{ds}\{(H^L\varepsilon, \chi_{B^\delta}\Lambda Q)\} - (H^L\varepsilon, \Lambda Q\partial_s(\chi_{B^\delta})) \\ = -(H^{L+1}\varepsilon, \chi_{B^\delta}\Lambda Q) + \frac{\lambda_s}{\lambda}(H^L\Lambda\varepsilon, \chi_{B_\delta}\Lambda Q) + (F - \widetilde{\mathrm{Mod}}, H^L\chi_{B_\delta}\Lambda Q).$$

The linear term is estimated by Cauchy–Schwarz:

$$|(H^{L+1}\varepsilon, \chi_{B^\delta}\Lambda Q)| \lesssim C(M)\sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

Using (3-36), we similarly estimate

$$|(H^L\varepsilon, \Lambda Q\partial_s(\chi_{B^\delta}))| + \left|\frac{\lambda_s}{\lambda}(H^L\Lambda\varepsilon, \chi_{B_\delta}\Lambda Q)\right| \\ \lesssim C(M)\frac{|(b_1)_s|}{b_1}\frac{1}{b_1^{C\delta}}\sqrt{\mathscr{E}_{2L+2}} + \frac{b_1}{b_1^{C\delta}}C(M)\sqrt{\mathscr{E}_{2L+2}} \lesssim \sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

The estimate on the error terms easily follows from the Hardy bounds

$$|(L(\varepsilon), H^L\chi_{B^\delta}\Lambda Q)| + (N(\varepsilon), H^L\chi_{B^\delta}\Lambda Q)| \lesssim \frac{b_1}{b_1^{C\delta}}C(M)\sqrt{\mathscr{E}_{2L+2}} \lesssim \sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

From (2-91) we further estimate

$$|(H^L\varepsilon, \tilde{\Psi}_b)| \lesssim \frac{b_1^{L+3}}{b_1^{C\delta}}C(M)\sqrt{\mathscr{E}_{2L+2}} \lesssim \sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}}.$$

From (3-36), (3-29), we now compute

$$-(\widetilde{\mathrm{Mod}}, H^L\chi_{B_\delta}\Lambda Q) \\ = O\left(\frac{b_1^{L+3/2}}{b_1^{C\delta}}\right) + [(b_L)_s + (2L-1+c_{b_1})b_1b_L]\left(H^L\widetilde{T}_L + \sum_{j=L+1}^{L+2} H^L\left[\chi_{B_1}\frac{\partial S_j}{\partial_{b_L}}\right], \chi_{B_\delta}\Lambda Q\right) \\ = (-1)^L[(b_L)_s + (2L-1+c_{b_1})b_1b_L]\left[(\Lambda Q, \chi_{B_\delta}\Lambda Q) + O(b_1^{1-C\delta})\right] + O\left(\frac{b_1^{L+3/2}}{b_1^{C\delta}}\right) \\ = (-1)^L[(b_L)_s + (2L-1+c_{b_1})b_1b_L]4\delta|\log b_1| + O(\sqrt{|\log b_1|}\sqrt{\mathscr{E}_{2L+2}} + b_1^{L+1}).$$

The collection of above bounds yields the preliminary estimate

$$\left|\frac{d}{ds}\{(H^L\varepsilon,\chi_{B^\delta}\Lambda Q)\}+(-1)^L\big[((b_L)_s+(2L-1+c_{b_1}))b_1b_L\big]4\delta|\log b_1|\right| \lesssim C(M)\sqrt{|\log b_1|}\left[\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right] \quad (3\text{-}47)$$

By brute force, from (3-44) we estimate

$$|\tilde{b}_L-b_L|\lesssim|\log b_1|^C b_1^{L+1-C\delta}\lesssim b_1^{L+\frac{1}{2}}$$

and we therefore rewrite (3-47) using (3-38) as

$$\begin{aligned}|(\tilde{b}_L)_s+(2L-1+c_{b_1})b_1\tilde{b}_L| &\lesssim |(H^L\varepsilon,\chi_{B_\delta}\Lambda Q)|\left|\frac{d}{ds}\left\{\frac{1}{4\delta\log b_1}\right\}\right|+\frac{C(M)\sqrt{|\log b_1|}}{|\log b_1|}\left[\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right]\\ &\lesssim b_1^{1-C\delta}\sqrt{\mathscr{E}_{2L+2}}+\frac{C(M)}{\sqrt{|\log b_1|}}\left[\sqrt{\mathscr{E}_{2L+2}}+\frac{b_1^{L+1}}{|\log b_1|}\right]\end{aligned}$$

and (3-46) is proved. □

The Lyapounov monotonicity. We now turn to the core of the argument which is the derivation of a suitable Lyapounov functional for the $\mathscr{E}_{2L+2}$ energy.

Proposition 3.6 (Lyapounov monotonicity). *We have*

$$\frac{d}{dt}\left\{\frac{1}{\lambda^{4L+2}}\left[\mathscr{E}_{2L+2}+O\left(b_1^{\frac{4}{5}}\frac{b_1^{2L+2}}{|\log b_1|^2}\right)\right]\right\}\le C\frac{b_1}{\lambda^{4L+4}}\left[\frac{\mathscr{E}_{2L+2}}{\sqrt{\log M}}+\frac{b_1^{2L+2}}{|\log b_1|^2}+\frac{b_1^{L+1}\sqrt{\mathscr{E}_{2L+2}}}{|\log b_1|}\right] \quad (3\text{-}48)$$

for some universal constant $C>0$ independent of M and of the bootstrap constant K in (3-23), (3-24).

Proof of Proposition 3.6. *Step* 1: *Suitable derivatives*. We define the derivatives of w associated with the linearized Hamiltonian H_λ by

$$w_1=A_\lambda w,\qquad w_{k+1}=\begin{cases}A_\lambda^* w_k & \text{for k odd,}\\ A_\lambda w_k & \text{for k even,}\end{cases}\qquad 1\le k\le 2L+1$$

and we define its renormalized version by

$$\varepsilon_1=A\varepsilon,\qquad \varepsilon_{k+1}=\begin{cases}A^*\varepsilon_k & \text{for k odd,}\\ A\varepsilon_k & \text{for k even,}\end{cases}\qquad 1\le k\le 2L+1.$$

From (3-34), we compute

$$\partial_t w_{2L}+H_\lambda w_{2L}=[\partial_t,H_\lambda^L]w+H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right) \quad (3\text{-}49)$$

$$\partial_t w_{2L+1}+\widetilde{H}_\lambda w_{2L+1}=\frac{\partial_t Z_\lambda}{r}w_{2L}+A_\lambda([\partial_t,H_\lambda^L]w)+A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right). \quad (3\text{-}50)$$

We recall the action of time derivatives on rescaling:

$$\partial_t v_\lambda=\frac{1}{\lambda^2}\left(\partial_s v-\frac{\lambda_s}{\lambda}\Lambda v\right)_\lambda. \quad (3\text{-}51)$$

Step 2: *Modified energy identity*. We compute the energy identity on (3-50) using (3-51):

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\mathscr{E}_{2L+2} &= \frac{1}{2}\frac{d}{dt}\left\{\int \widetilde{H}_\lambda w_{2L+1} w_{2L+1}\right\}\\
&= \int \widetilde{H}_\lambda w_{2L+1}\partial_t w_{2L+1} + \int \frac{\partial_t \widetilde{V}_\lambda}{2r^2} w_{2L+1}^2\\
&= -\int (\widetilde{H}_\lambda w_{2L+1})^2 + b_1 \int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2 - \left(\frac{\lambda_s}{\lambda}+b_1\right)\int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2\\
&\qquad + \int \widetilde{H}_\lambda w_{2L+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L] w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]. \qquad (3\text{-}52)
\end{aligned}
$$

From (3-49), (3-50) we further compute

$$
\begin{aligned}
\frac{d}{dt}\left\{\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1} w_{2L}\right\} &= \int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2L+1} w_{2L}\\
&\quad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L}\left[-\widetilde{H}_\lambda w_{2L+1} + \frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L] w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]\\
&\quad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1}\left[-A_\lambda^* w_{2L+1} + [\partial_t, H_\lambda^L] w + H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right].
\end{aligned}
$$

We now integrate by parts using (2-4) to compute

$$
\begin{aligned}
\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1} A_\lambda^* w_{2L+1} &= \frac{b_1}{\lambda^{4L+4}}\int \frac{\Lambda Z}{y}\varepsilon_{2L+1} A^* \varepsilon_{2L+1}\\
&= \frac{b_1}{\lambda^{4L+4}}\int \frac{2(1+Z)\Lambda Z - \Lambda^2 Z}{2y^2}\varepsilon_{2L+1}^2\\
&= \frac{b_1}{\lambda^{4L+4}}\int \frac{\Lambda \widetilde{V}}{2y^2}\varepsilon_{2L+1}^2 = b_1 \int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2.
\end{aligned}
$$

Injecting this into the energy identity (3-52) yields the modified energy identity

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left\{\mathscr{E}_{2L+2} + 2\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1} w_{2L}\right\}\\
&\quad = -\int (\widetilde{H}_\lambda w_{2L+1})^2 - \left(\frac{\lambda_s}{\lambda}+b_1\right)\int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2L+1}^2 + \int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2L+1} w_{2L}\\
&\qquad + \int \widetilde{H}_\lambda w_{2L+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L] w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]\\
&\qquad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L}\left[-\widetilde{H}_\lambda w_{2L+1} + \frac{\partial_t Z_\lambda}{r} w_{2L} + A_\lambda([\partial_t, H_\lambda^L] w) + A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]\\
&\qquad + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1}\left[[\partial_t, H_\lambda^L] w + H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]. \qquad (3\text{-}53)
\end{aligned}
$$

We now aim at estimating all terms in the right hand side of (3-53). Throughout the proof, we shall make implicit use of the coercivity estimates of Lemma B.2 and Lemma C.1.

Step 3: *Lower order quadratic terms*. We treat the lower order quadratic terms in (3-53) using dissipation. Indeed, from (2-5), (2-6), (3-38), we have the bounds

$$|\partial_t Z_\lambda|+|\partial_t V_\lambda| \lesssim \frac{b_1}{\lambda^2}(|\Lambda Z|+|\Lambda V|)_\lambda \lesssim \frac{b_1}{\lambda^2}\frac{y^2}{1+y^4}. \tag{3-54}$$

We moreover claim the bound

$$\int \frac{([\partial_t, H_\lambda^L]w)^2}{\lambda^2(1+y^2)}+\int |A_\lambda([\partial_t, H_\lambda^L]w)|^2 \lesssim C(M)\frac{b_1^2}{\lambda^{4L+4}}\mathscr{E}_{2L+2}, \tag{3-55}$$

which is proved in Appendix E. From Cauchy–Schwartz, the rough bound (3-38), and Lemma C.1, we conclude

$$\begin{aligned}\int \left|\widetilde{H}_\lambda w_{2L+1}\left[\frac{\partial_t Z_\lambda}{r}w_{2L}+\int A_\lambda([\partial_t, H_\lambda^L]w)\right]\right| + \int |\widetilde{H}_\lambda w_{2L+1}|\left|\frac{b(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L}\right| \\ \le \frac12\int |\widetilde{H}_\lambda w_{2L+1}|^2+\frac{b_1^2}{\lambda^{4L+4}}\left[\int \frac{\varepsilon_{2L}^2}{1+y^6}+C(M)\mathscr{E}_{2L+2}\right] \\ \le \frac12\int |\widetilde{H}_\lambda w_{2L+1}|^2+\frac{b_1}{\lambda^{4L+4}}C(M)b_1\mathscr{E}_{2L+2}.\end{aligned}$$

All other quadratic terms are lower order by a factor b_1 again using (3-38), (3-55), (3-36), and Lemma C.1:

$$\begin{aligned}\left|\frac{\lambda_s}{\lambda}+b_1\right|\int \left|\frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2}w_{2L+1}^2\right| + \int \left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L}\left[\frac{\partial_t Z_\lambda}{r}w_{2L}+A_\lambda([\partial_t, H_\lambda^L]w)\right]\right| \\ +\int \left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L+1}[\partial_t, H_\lambda^L]w\right| + \left|\int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right)w_{2L+1}w_{2L}\right| \\ \lesssim \frac{b_1^2}{\lambda^{4L+4}}\left[\int \frac{\varepsilon_{2L+1}^2}{1+y^4}+\int \frac{\varepsilon_{2L}^2}{1+y^6}+C(M)\mathscr{E}_{2L+2}\right] \lesssim \frac{b_1}{\lambda^{4L+4}}C(M)b_1\mathscr{E}_{2L+2}.\end{aligned}$$

We similarly estimate the boundary term in time using (C-10):

$$\left|\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L+1}w_{2L}\right| \lesssim \frac{b_1}{\lambda^{4L+2}}\left[\int \frac{\varepsilon_{2L+1}^2}{1+y^2}+\int \frac{\varepsilon_{2L}^2}{1+y^4}\right] \lesssim \frac{b_1}{\lambda^{4L+2}}|\log b_1|^C b_1^{2L+2}.$$

We inject these estimates into (3-53) to derive the preliminary bound

$$\begin{aligned}&\frac12\frac{d}{dt}\left\{\frac{1}{\lambda^{4L+2}}\left[\mathscr{E}_{2L+2}+O\left(b_1^{\frac45}\frac{b^{2L+2}}{|\log b|^2}\right)\right]\right\} \\ &\quad\le -\frac12\int (\widetilde{H}_\lambda w_{2L+1})^2+\int \widetilde{H}_\lambda w_{2L+1}A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right) \\ &\qquad +\int H_\lambda^L\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\left[\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L+1}+A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L}\right)\right]+\frac{b_1}{\lambda^{4L+4}}\sqrt{b_1}b_1^{2L+2}\end{aligned} \tag{3-56}$$

with constants independent of M for $|b| < b^*(M)$ small enough.

We now estimate all terms in the right hand side of (3-56).

Step 4: Further use of dissipation. Let us introduce the decomposition from (3-28), (3-30),

$$\mathcal{F} = \mathcal{F}_0 + \mathcal{F}_1, \quad \mathcal{F}_0 = -\widetilde{\Psi}_b - \widetilde{\mathrm{Mod}}(t), \quad \mathcal{F}_1 = L(\varepsilon) - N(\varepsilon). \tag{3-57}$$

The first term in the right hand side of (3-56) is estimated after an integration by parts:

$$\begin{aligned}
&\left|\int \widetilde{H}_\lambda w_{2L+1} A_\lambda H_\lambda^L\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right| \\
&\quad\le \frac{C}{\lambda^{4L+4}}\|A^*\varepsilon_{2L+1}\|_{L^2}\|H^{L+1}\mathcal{F}_0\|_{L^2} + \frac{1}{4}\int|\widetilde{H}_\lambda w_{2L+1}|^2 + \frac{C}{\lambda^{4L+4}}\int|AH^L\mathcal{F}_1|^2 \\
&\quad\le \frac{C}{\lambda^{4L+4}}\left[\|H^{L+1}\mathcal{F}_0\|_{L^2}\sqrt{\mathcal{E}_{2L+2}} + \|AH^L\mathcal{F}_1\|_{L^2}^2\right] + \frac{1}{4}\int|\widetilde{H}_\lambda w_{2L+1}|^2
\end{aligned} \tag{3-58}$$

for some universal constant $C > 0$ independent of M.

The last two terms in (3-56) can be estimated by brute force from Cauchy–Schwarz:

$$\begin{aligned}
\left|\int H_\lambda^L\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L+1}\right| &\lesssim \frac{b_1}{\lambda^{4L+4}}\left(\int\frac{1+|\log y|^2}{1+y^4}|H^L\mathcal{F}|^2\right)^{\frac12}\left(\int\frac{\varepsilon_{2L+1}^2}{y^2(1+|\log y|^2)}\right)^{\frac12} \\
&\lesssim \frac{b_1}{\lambda^{4L+4}}\sqrt{\mathcal{E}_{2L+2}}\left(\int\frac{1+|\log y|^2}{1+y^4}|H^L\mathcal{F}|^2\right)^{\frac12},
\end{aligned} \tag{3-59}$$

where constants are independent of M thanks to the estimate (B-2) for ε_{2L+1}. Similarly,

$$\begin{aligned}
&\left|\int H_\lambda^L\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2L}\right)\right| \\
&\qquad\lesssim \frac{b_1}{\lambda^{4L+4}}\left(\int\frac{1+|\log y|^2}{1+y^2}|AH^L\mathcal{F}|^2\right)^{\frac12}\left(\int\frac{\varepsilon_{2L}^2}{(1+y^4)(1+|\log y|^2)}\right)^{\frac12} \\
&\qquad\lesssim \frac{b_1}{\lambda^{4L+4}}C(M)\sqrt{\mathcal{E}_{2L+2}}\left(\int\frac{1+|\log y|^2}{1+y^2}|AH^L\mathcal{F}_0|^2 + \int|AH^L\mathcal{F}_1|^2\right)^{\frac12}.
\end{aligned} \tag{3-60}$$

We now claim the bounds

$$\int\frac{1+|\log y|^2}{1+y^4}|H^L\mathcal{F}|^2 \lesssim \frac{b_1^{2L+2}}{|\log b_1|^2} + \frac{\mathcal{E}_{2L+2}}{\log M}, \tag{3-61}$$

$$\int\frac{1+|\log y|^2}{1+y^2}|AH^L\mathcal{F}_0|^2 \lesssim \delta(\alpha^*)\left[\frac{b_1^{2L+2}}{|\log b_1|^2} + \mathcal{E}_{2L+2}\right], \tag{3-62}$$

$$\int|H^{L+1}\mathcal{F}_0|^2 \lesssim b_1^2\left[\frac{b_1^{2L+2}}{|\log b_1|^2} + \frac{\mathcal{E}_{2L+2}}{\log M}\right], \tag{3-63}$$

$$\int|AH^L\mathcal{F}_1|^2 \lesssim b_1\left[\frac{b_1^{2L+2}}{|\log b_1|^2} + \frac{\mathcal{E}_{2L+2}}{\log M}\right] \tag{3-64}$$

with all $\lesssim$ constants independent of M for $|b| < \alpha^*(M)$ small enough, and where

$$\delta(\alpha^*) \to 0 \quad \text{as } \alpha^*(M) \to 0.$$

Injecting these bounds together with (3-58), (3-59), (3-60) into (3-56) concludes the proof of (3-48). We now turn to the proof of (3-61), (3-62), (3-63), (3-64).

Step 5: $\widetilde{\Psi}_b$ *terms*. The contribution of $\widetilde{\Psi}_b$ terms to (3-61), (3-62), (3-63) is estimated from (2-89), (2-90), which are at the heart of the construction of $\widetilde{Q}_b$ and yield the desired bounds.

Step 6: $\widetilde{\mathrm{Mod}(t)}$ *terms*. Recall (3-29),

$$\widetilde{\mathrm{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda} + b_1\right)\Lambda \widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1 b_i - b_{i+1}]\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right],$$

and the notation (3-39).

Proof of (3-63) *for* $\widetilde{\mathrm{Mod}}$. We recall that

$$|b_k| \lesssim b_1^k,$$

and, from Lemma 2.8, we estimate

$$\begin{aligned}\int |H^{L+1}\Lambda \widetilde{Q}_b|^2 &\lesssim \sum_{i=1}^{L}\int |H^{L+1} b_i \Lambda \widetilde{T}_i|^2 + \sum_{i=2}^{L+2}\int |H^{L+1}\Lambda \widetilde{S}_i|^2 \\ &\lesssim \sum_{i=1}^{L} b_1^{2i}\int_{y\le 2B_1}\left|\frac{(1+|\log y|^C)y^{2i-1}}{1+y^{2L+2}}\right|^2 + \sum_{i=2}^{L+1} b_1^{2i} + \frac{b_1^{2L+4}}{b_1^2|\log b_1|^2} \lesssim b_1^2.\end{aligned}$$

We then use the cancellation $H^{L+1}T_i = 0$ for $1 \le i \le L$ to estimate

$$\sum_{i=1}^{L}\int |H^{L+1}\widetilde{T}_i|^2 \lesssim \sum_{i=1}^{L}\int_{B_1\le y\le 2B_1}\left|\frac{y^{2i-1}}{y^{2L+2}}\right|^2 \lesssim b_1^2.$$

Then, using Lemma 2.8 again,[13] for $1 \le i \le L$,

$$\sum_{j=i+1}^{L+2}\int\left|H^{L+1}\left[\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right]\right|^2 \lesssim \sum_{j=i+1}^{L+1} b_1^{2(j-i)} + \frac{b_1^{2(L+2-i)}}{b_1^2|\log b_1|^2} \lesssim b_1^2.$$

We thus obtain from Lemma 3.3 the expected bound:

$$\int |H^{L+1}\widetilde{\mathrm{Mod}}|^2 \lesssim b_1^2|D(t)|^2 \lesssim b_1^2\left[\frac{\mathscr{E}_{2L+2}}{|\log M|} + \frac{b_1^{2L+2}}{|\log b_1|^2}\right].$$ □

[13] This is where we used the logarithmic gain (2-54) induced by (2-24).

Proof of (3-61) *for* $\widetilde{\mathrm{Mod}}$. We use Lemma 2.8 to derive the rough bound

$$\begin{aligned}
&\int \frac{1+|\log y|^2}{1+y^4}|H^L\Lambda\widetilde{Q}_b|^2\\
&\quad\lesssim \sum_{i=1}^{L}\int \frac{1+|\log y|^2}{1+y^4}|H^L b_i\Lambda\widetilde{T}_i|^2+\sum_{i=2}^{L+2}\int \frac{1+|\log y|^2}{1+y^4}|H^L\Lambda\widetilde{S}_i|^2\\
&\quad\lesssim \sum_{i=1}^{L}b_1^{2i}\int_{y\le 2B_1}\frac{1+|\log y|^C}{1+y^4}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2+\sum_{i=2}^{L+1}b_1^{2i}|\log b_1|^3+b_1^{2L+4}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}\left|\frac{1+y^{2(L+2)-1}}{1+y^{2L}}\right|^2\\
&\quad\lesssim 1.
\end{aligned}$$

Next,

$$\sum_{j=1}^{L}\frac{1+|\log y|^2}{1+y^4}|H^L\widetilde{T}_i|^2\lesssim \sum_{j=1}^{L}\int_{y\le 2B_1}\frac{1+|\log y|^C}{1+y^4}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2\lesssim 1,$$

and finally, again using Lemma 2.8, for $1\le i\le L$,

$$\begin{aligned}
&\sum_{j=i+1}^{L+2}\int \frac{1+|\log y|^2}{1+y^4}\left|H^L\left[\chi_{B_1}\frac{\partial S_j}{\partial b_i}\right]\right|^2\\
&\qquad\lesssim \sum_{j=i+1}^{L+1}b_1^{2(j-i)}|\log b_1|^2+b_1^{2(L-i)+4}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^4}\left|\frac{1+y^{2(L+2)-1}}{1+y^{2L}}\right|^2\lesssim 1.
\end{aligned}$$

We thus obtain from Lemma 3.3 the expected bound:

$$\int \frac{1+|\log y|^2}{1+y^4}|H^L\widetilde{\mathrm{Mod}}|^2\lesssim |D(t)|^2\lesssim \frac{\mathscr{E}_{2L+2}}{|\log M|}+\frac{b_1^{2L+2}}{|\log b_1|^2}.$$ □

Proof of (3-62) *for* $\widetilde{\mathrm{Mod}}$. We use Lemma 2.8 to estimate

$$\begin{aligned}
&\int \frac{1+|\log y|^2}{1+y^2}|AH^L\Lambda\widetilde{Q}_b|^2\\
&\quad\lesssim \sum_{i=1}^{L}\int \frac{1+|\log y|^2}{1+y^2}|H^L b_i\Lambda\widetilde{T}_i|^2+\sum_{i=2}^{L+2}\int \frac{1+|\log y|^2}{1+y^2}|AH^L\Lambda\widetilde{S}_i|^2\\
&\quad\lesssim \sum_{i=1}^{L}b_1^{2i}\int_{y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2+\sum_{i=2}^{L+1}b_1^{2i}|\log b_1|^3+b_1^{2L+4}\int_{B_1\le y\le 2B_1}\frac{1+|\log y|^2}{1+y^2}\left|\frac{1+y^{2(L+2)-1}}{1+y^{2L+1}}\right|^2\\
&\quad\lesssim b_1^2.
\end{aligned}$$

Next, using the cancellation $AH^LT_i=0$, $1\le i\le L$,

$$\sum_{j=1}^{L}\frac{1+|\log y|^2}{1+y^2}|AH^L\widetilde{T}_i|^2\lesssim \sum_{j=1}^{L}\int_{B_1\le y\le 2B_1}\frac{1+|\log y|^C}{1+y^2}\left|\frac{y^{2i-1}}{1+y^{2L}}\right|^2\lesssim b_1|\log b_1|^C,$$

and finally using Lemma 2.8 again, for $1 \le i \le L$, we have

$$\sum_{j=i+1}^{L+2} \int \frac{1+|\log y|^2}{1+y^2} \left| AH^L \left[\chi_{B_1} \frac{\partial S_j}{\partial b_i} \right] \right|^2 \lesssim \sum_{j=i+1}^{L+1} b_1^{2(j-i)} |\log b_1|^C + b_1^{2(L-i)+4} \int_{y \le 2B_1} \frac{1+|\log y|^2}{1+y^2} \left| \frac{1+y^{2(L+2)-1}}{1+y^{2L+1}} \right|^2 \lesssim b_1.$$

We thus obtain from Lemma 3.3 the desired bound:

$$\int \frac{1+|\log y|^2}{1+y^2} |AH^L \widetilde{\text{Mod}}|^2 \le \sqrt{b_1} |D(t)|^2 \lesssim \delta(\alpha^*) \left[\mathscr{E}_{2L+2} + \frac{b_1^{2L+2}}{|\log b_1|^2} \right]. \qquad \square$$

Step 7: Nonlinear term $N(\varepsilon)$. Control near the origin $y \le 1$. From (3-32) and a Taylor Lagrange formula, we rewrite

$$N(\varepsilon) = z N_0(\varepsilon), \quad z = y \left(\frac{\varepsilon}{y} \right)^2, \quad N_0(\varepsilon) = \frac{1}{y} \int_0^1 (1-\tau) f''(\widetilde{Q}_b + \tau \varepsilon)\, d\tau. \tag{3-65}$$

First observe from (C-2) and the Taylor expansion at the origin of T_i given by (2-39) that

$$z = \frac{1}{y} \left[\sum_{i=1}^{L+1} c_i T_{L+1-i} + r_\varepsilon \right]^2 = \sum_{i=0}^{L} \tilde{c}_i y^{2i+1} + \tilde{r}_\varepsilon, \tag{3-66}$$

where, from (C-3), (C-4),

$$\begin{aligned} &|\tilde{c}_i| \lesssim C(M) \mathscr{E}_{2L+2}, \\ &|\partial_y^k \tilde{r}_\varepsilon| \lesssim y^{2L+1-k} |\log y| C(M) \mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1. \end{aligned} \tag{3-67}$$

We now let $\tau \in [0, 1]$ and

$$v_\tau = \widetilde{Q}_b + \tau \varepsilon,$$

and obtain from Proposition 2.12 and (C-2) the Taylor expansion at the origin

$$v_\tau = \sum_{i=0}^{L} \hat{c}_i y^{2i+1} + \hat{r}_\varepsilon \tag{3-68}$$

with

$$|\hat{c}_i| \lesssim 1, \quad |\partial_y^k \hat{r}_\varepsilon| \lesssim y^{2L+1-k} |\log y|, \quad 0 \le k \le 2L+1. \tag{3-69}$$

Recall that $f \in \mathscr{C}^\infty$ with $f^{2k}(0) = 0$, $k \ge 0$. We therefore obtain a Taylor expansion

$$f''(v_\tau) = \sum_{i=1}^{L+1} \frac{f^{(2i+1)}(0)}{i!} v_\tau^{2i-1} + \frac{v^{2L+2}}{(2L+1)!} \int_0^1 (1-\sigma)^{2L+1} f^{(2L+4)}(\sigma v_\tau)\, d\sigma,$$

which, together with (3-68), ensures an expansion

$$N_0(\varepsilon) = \sum_{i=0}^{L} \hat{\hat{c}}_i y^{2i} + \hat{\hat{r}}_\varepsilon, \quad |\hat{\hat{c}}_i| \lesssim 1, \quad |\partial_y^k \hat{\hat{r}}_\varepsilon| \lesssim y^{2L-k} |\log y|, \quad 0 \le k \le 2L+1.$$

Combining this with (3-66) ensures the expansion

$$N(\varepsilon) = zN_0(\varepsilon) = \sum_{i=0}^{L} \tilde{\tilde{c}}_i y^{2i+1} + \tilde{\tilde{r}}_\varepsilon \tag{3-70}$$

with

$$|\tilde{\tilde{c}}_i| \lesssim C(M)\mathscr{E}_{2L+2}, \quad |\partial_y^k \tilde{\tilde{r}}_\varepsilon| \lesssim y^{2L+1-k}|\log y| C(M)\mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1.$$

Observe that from a direct check this implies the bound

$$|\mathscr{A}^k \tilde{\tilde{r}}_\varepsilon| \lesssim \sum_{i=0}^{k} \frac{\partial_y^i \tilde{\tilde{r}}_\varepsilon}{y^{k-i}} \lesssim C(M)\mathscr{E}_{2L+2} \sum_{i=0}^{k} \frac{|\log y| y^{2L+1-i}}{y^{k-i}} \lesssim y^{2L+1-k}|\log y| C(M)\mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1.$$

We now compute using a simple induction based on the expansions (2-5), (2-6) and the cancellation $A(y) = O(y^2)$ that, for $y \le 1$,

$$\begin{cases} \mathscr{A}^{2k+1}\left(\sum_{i=0}^{L} \tilde{\tilde{c}}_i y^{2i+1}\right) = \sum_{i=k+1}^{L} c_{i,2k+1} y^{2(i-k)} + O(y^{2(L-k)+2}), \\ \mathscr{A}^{2k+2}\left(\sum_{i=0}^{L} \tilde{\tilde{c}}_i y^{2i+1}\right) = \sum_{i=k+1}^{L} c_{i,2k+2} y^{2(i-k)-1} + O(y^{2(L-k)+1}). \end{cases} \tag{3-71}$$

From (3-70), we conclude

$$\|\mathscr{A}^k N(\varepsilon)\|_{L^\infty(y\le 1)} \lesssim C(M)\mathscr{E}_{2L+2}, \quad 0 \le k \le 2L+1, \tag{3-72}$$

and thus, in particular, we get the control near the origin

$$\int_{y\le 1} \frac{1+|\log y|^2}{1+y^4} |H^L N(\varepsilon)|^2 + \int_{y \le 1} |AH^L \mathscr{N}(\varepsilon)|^2 \lesssim C(M)(\mathscr{E}_{2L+2})^2 \lesssim b_1^2 b_1^{2L+2}.$$

Control for $y \ge 1$. We give a detailed proof of (3-64). The proof of (3-61) follows the exact same lines (with more room in fact) and is left to the reader. Let

$$\zeta = \frac{\varepsilon}{y}, \quad N_1(\varepsilon) = \int_0^1 (1-\tau) f''(\tilde{Q}_b + \tau\varepsilon)\, d\tau \quad \text{so that } N(\varepsilon) = \zeta^2 N_1. \tag{3-73}$$

We first estimate from (C-14): for $(i, j) \in \mathbb{N} \times \mathbb{N}$ with $1 \le i+j \le 2L+1$,

$$\left\| \frac{\partial_y^i \zeta}{y^{j-1}} \right\|_{L^\infty(y\ge 1)}^2 \lesssim \sum_{k=0}^{i} \left\| \frac{\partial_y^k \varepsilon}{y^{j+i-k}} \right\|_{L^\infty(y \ge 1)}^2 \lesssim |\log b_1|^C \begin{cases} b_1^{(i+j)2L/(2L-1)} & \text{for } 1 \le i+j \le 2L-1, \\ b_1^{2L+1} & \text{for } i+j = 2L, \\ b_1^{2L+2} & \text{for } i+j = 2L+1. \end{cases} \tag{3-74}$$

Similarly, from (C-12), for $(i, j) \in \mathbb{N} \times \mathbb{N}^*$ with $2 \le i+j \le 2L+2$,

$$\begin{aligned} \int_{y \ge 1} \frac{1+|\log y|^C}{1+y^{2j-2}} |\partial_y^i \zeta|^2 &\lesssim \sum_{k=0}^{i} \int_{y\ge 1} \frac{1+|\log y|^C}{1+y^{2j+2(i-k)}} |\partial_y^k \varepsilon|^2 \\ &\lesssim |\log b_1|^C \begin{cases} b_1^{(i+j-1)2L/(2L-1)} & \text{for } 2 \le i+j \le 2L, \\ b_1^{2L+1} & \text{for } i+j = 2L+1, \\ b_1^{2L+2} & \text{for } i+j = 2L+2. \end{cases} \end{aligned} \tag{3-75}$$

Moreover, from the energy bound (3-23),

$$\int_{y\geq 1} |\zeta|^2 \lesssim 1. \tag{3-76}$$

We now claim the pointwise bound, for $y \geq 1$,

$$\text{for all } 1 \leq k \leq 2L+1, \quad |\partial_y^k N_1(\varepsilon)| \lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + b_1^{a_k/2}\right], \tag{3-77}$$

with

$$a_k = \begin{cases} k2L/(2L-1) & \text{for } 1 \leq k \leq 2L-1 \\ 2L+1 & \text{for } k = 2L, \\ 2L+2 & \text{for } k = 2L+1, \end{cases} \tag{3-78}$$

which is proved below. For $k=0$, we simply need the obvious bound

$$\|N_1(\varepsilon)\|_{L^\infty(y\geq 1)} \lesssim 1. \tag{3-79}$$

Then, by brute force, from (3-73), (3-77), (3-79), we estimate

$$\begin{aligned} |AH^L N(\varepsilon)| &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k N(\varepsilon)|}{y^{2L+1-k}} \lesssim \sum_{k=0}^{2L+1} \frac{1}{y^{2L+1-k}} \sum_{i=0}^{k} |\partial_y^i \zeta^2|\,|\partial_y^{k-i} N_1(\varepsilon)| \\ &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k \zeta^2|}{y^{2L+1-k}} + \sum_{k=1}^{2L+1} \frac{1}{y^{2L+1-k}} \sum_{i=0}^{k-1} |\partial_y^i \zeta^2|\,|\log b_1|^C \left[\frac{1}{y^{k-i+1}} + b_1^{(a_{k-i})/2}\right] \\ &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k \zeta^2|}{y^{2L+1-k}} + |\log b_1|^C \sum_{i=0}^{2L} \frac{|\partial_y^i \zeta^2|}{y^{2L+2-i}} + |\log b_1|^C \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} b_1^{(a_{k-i})/2} \frac{|\partial_y^i \zeta^2|}{y^{2L+1-k}} \\ &\lesssim |\log b_1|^C \left[\sum_{k=0}^{2L+1} \frac{|\partial_y^k \zeta^2|}{y^{2L+1-k}} + \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} b_1^{(a_{k-i})/2} \frac{|\partial_y^i \zeta^2|}{y^{2L+1-k}}\right], \end{aligned}$$

and hence

$$\begin{aligned} &\int_{y\geq 1} |AH^L N(\varepsilon)|^2 \\ &\quad\lesssim |\log b_1|^C \sum_{k=0}^{2L+1} \sum_{i=0}^{k} \int_{y\geq 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{k-i} \zeta|^2}{y^{4L+2-2k}} + |\log b_1|^C \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_1^{a_{k-i}} \int_{y\geq 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j} \zeta|^2}{y^{4L+2-2k}}. \end{aligned}$$

We now claim the bounds

$$\sum_{k=0}^{2L+1} \sum_{i=0}^{k} \int_{y\geq 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{k-i} \zeta|^2}{y^{4L+2-2k}} \lesssim |\log b_1|^C b_1^{\delta(L)} b_1^{2L+3}, \tag{3-80}$$

$$|\log b_1|^C \sum_{k=1}^{2L+1} \sum_{i=0}^{k-1} \sum_{j=0}^{i} b_1^{a_{k-i}} \int_{y\geq 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j} \zeta|^2}{y^{4L+2-2k}} \lesssim |\log b_1|^C b_1^{\delta(L)} b_1^{2L+3} \tag{3-81}$$

for some $\delta(L) > 0$, and this concludes the proof of (3-64) for $N(\varepsilon)$.

Proof of (3-77). We first extract from Proposition 2.12 the rough bound

$$|\partial_y^k \widetilde{Q}_b| \lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + \sum_{i=1}^{2L+2} b_1^i y^{2i-1-k} \mathbf{1}_{y\le 2B_1}\right] \lesssim \frac{|\log b_1|^C}{y^{k+1}}. \tag{3-82}$$

Then let $\tau \in [0,1]$ and $v_\tau = \widetilde{Q}_b + \tau\varepsilon$. From (3-82), (C-14), (3-78), we conclude

$$|\partial_y^k v_\tau| \lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + b_1^{\alpha_k/2}\right], \quad 1 \le k \le 2L+1,\ y \ge 1. \tag{3-83}$$

We therefore estimate N_1 through the formula (3-73) using the rough bound $|\partial_v^i f| \lesssim 1$ and the Faa di Bruno formula: for $1 \le k \le 2L+1$,

$$\begin{aligned}
|\partial_y^k N_1(\varepsilon)| &\lesssim \int_0^1 \sum_{m_1+2m_2+\cdots+km_k=k} |\partial_v^{m_1+\cdots+m_k} f(v_\tau)| \prod_{i=1}^k |\partial_y^i v_\tau|^{m_i}\, d\tau \\
&\lesssim |\log b_1|^C \sum_{m_1+2m_2+\cdots+km_k=k} |\prod_{i=1}^k \left[\frac{1}{y^{i+1}} + b_1^{a_i/2}\right]^{m_i} \\
&\lesssim |\log b_1|^C \left[\frac{1}{y^{k+1}} + b_1^{\alpha_k/2}\right].
\end{aligned}$$

To estimate α_k from the definition (3-78), we observe that for $k \le 2L-1$, $i \le 2L-1$, and thus

$$\alpha_k \ge \sum_{i=0}^k \frac{2iL}{L-1} m_i = \frac{2kL}{L-1} = a_k.$$

For $k = 2L$, we have to treat the boundary term $i = k$, $(m_1, \dots, m_{k-1}, m_k) = (0, \dots, 0, 1) = 1$, which yields

$$\alpha_{2L} \ge \min\left\{\frac{2L(2L)}{2L-1}; 2L+1\right\} = 2L+1.$$

For $k = 2L+1$, we have the two boundary terms $(m_1, m_2, \dots, m_{k-2}, m_{k-1}, m_k) = (1, 0, \dots, 0, 1, 0)$, $(m_1, \dots, , m_{k-1}, m_k) = (0, \dots, 0, 1)$, which yield

$$\alpha_{2L+1} \ge \min\left\{\frac{2L(2L+1)}{2L-1}; 2L+1+\frac{2L}{2L-1}; 2L+2\right\} = 2L+2,$$

and (3-77) is proved. □

Proof of (3-80). Let $0 \le k \le 2L+1$, $0 \le i \le k$. Let $I_1 = k-i$, $I_2 = i$. Then we can pick $J_2 \in \mathbb{N}^*$ such that

$$\max\{1; 2-i\} \le J_2 \le \min\{2L+3-k; 2L+2-i\}$$

and define

$$J_1 = 2L+3-k-J_2.$$

Then, from direct inspection,

$$(I_1, J_1, I_2, J_2) \in \mathbb{N}^3 \times \mathbb{N}^*, \qquad \begin{cases} 1 \le I_1 + J_1 \le 2L+1, \quad 2 \le I_2 + J_2 \le 2L+2, \\ I_1 + I_2 + J_1 + J_2 = 2L+3. \end{cases}$$

Thus

$$A_i = \int_{y\ge 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{k-i}\zeta|^2}{y^{4L+2-2k}} = \int_{y\ge 1} \frac{|\partial_y^{I_1}\zeta|^2 |\partial_y^{I_2}\zeta|^2}{y^{2J_1-2+2J_2-2}} \lesssim \left\| \frac{\partial_y^{I_1}\zeta}{y^{J_1-1}} \right\|^2_{L^\infty(y\ge 1)} \int_{y\ge 1} \frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}} \lesssim |\log b_1|^C b_1^{d_{i,k}},$$

where we now compute the exponent $d_{i,k}$ using (3-74), (3-75):

- for $I_1 + J_1 \le 2L-1$, $I_2 + J_2 \le 2L$,

$$d_{i,k} = \frac{2L}{2L-1}(I_1 + J_1 + I_2 + J_2 - 1) = \frac{2L(2L+2)}{2L-1} > 2L+3;$$

- for $I_1 + J_1 = 2L$, $I_2 + J_2 = 3$,

$$d_{i,k} = 2L+1 + \frac{2L}{2L-1}(3-1) > 2L+3;$$

- for $I_1 + J_1 = 2L+1$, $I_2 + J_2 = 2$,

$$d_{i,k} = 2L+2 + \frac{2L}{2L-1} > 2L+3;$$

- for $I_2 + J_2 = 2L+1$, $I_1 + J_1 = 2$,

$$d_{i,k} = \frac{2(2L)}{2L-1} + 2L+1 > 2L+3;$$

- for $I_2 + J_2 = 2L+2$, $I_1 + J_1 = 1$,

$$d_{i,k} = 2L+2 + \frac{2L}{2L-1} > 2L+3;$$

and (3-80) is proved. □

Proof of (3-81). Let $1 \le k \le 2L+1$, $0 \le j \le i \le k-1$. For $k = 2L+1$ and $0 \le i = j \le 2L$, we use the energy bound (3-76) to estimate

$$b_1^{a_{k-i}} \int_{y\ge 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j}\zeta|^2}{y^{4L+2-2k}} = b_1^{a_{2L+1-i}} \int_{y\ge 1} |\partial_y^i \zeta|^2 |\zeta|^2 \lesssim b_1^{a_{2L+1-i}} \|\zeta\|^2_{L^\infty(y\ge 1)} \int_{y\ge 1} |\partial_y^i \zeta|^2 \lesssim b_1^{d_{i,2L+1}}$$

with

$$d_{i,2L+1} = \begin{cases} \dfrac{2L}{2L-1} + 2L+2 & \text{for } i = 0, \\ \dfrac{2L}{2L-1} + 2L+2 + \dfrac{2L}{2L-1} & \text{for } i = 1, \\ \dfrac{2L}{2L-1} + \dfrac{2L}{2L-1}(i+1-1) + \dfrac{2L}{2L-1}(2L+1-i) & \text{for } 2 \le i \le 2L \end{cases}$$
$$> 2L+3.$$

This exceptional case being treated, we let $I_1 = j$, $I_2 = i - j$, and pick $J_2 \in \mathbb{N}^*$ with

$$\max\{1; 2-(i-j); 2-(k-j)\} \le J_2 \le \min\{2L+3-k; 2L+2-(k-j); 2L+2-(i-j)\}.$$

Let

$$J_1 = 2L+3-k-J_2.$$

Then we can directly check that

$$(I_1, J_1, I_2, J_2) \in \mathbb{N}^3 \times \mathbb{N}^*, \qquad \begin{cases} 1 \le I_1 + J_1 \le 2L+1, \quad 2 \le I_2 + J_2 \le 2L+2, \\ I_1 + I_2 + J_1 + J_2 = 2L+3-(k-i). \end{cases}$$

Hence

$$b_1^{a_{k-i}} \int_{y\ge 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j}\zeta|^2}{y^{4L+2-2k}} = b_1^{a_{k-i}} \int_{y\ge 1} \frac{|\partial_y^{I_1}\zeta|^2|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2+2J_1-2}} \lesssim b_1^{a_{k-i}} \left\| \frac{\partial_y^{I_1}\zeta}{y^{J_1-1}} \right\|^2_{L^\infty(y\ge 1)} \int_{y\ge 1} \frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}}$$
$$\lesssim |\log b_1|^C b_1^{d_{i,j,k}},$$

where we now compute the exponent $d_{i,k}$ using (3-74), (3-75), (3-78):

- for $I_1 + J_1 \le 2L-1$, $I_2 + J_2 \le 2L$, $k - i \le 2L-1$,
$$d_{i,j,k} = (k-i)\frac{2L}{2L-1} + (2L+3-(k-i)-1)\frac{2L}{2L-1} = \frac{2L(2L+2)}{2L-1} > 2L+3;$$
- for $I_1 + J_1 \le 2L-1$, $I_2 + J_2 \le 2L$, $k - i = 2L$,
$$d_{i,j,k} = 2L+1+(2L+3-2L-1)\frac{2L}{2L-1} > 2L+3;$$
- for $I_1 + J_1 = 2L$, $I_2 + J_2 = 3-(k-i) \ge 2$ and thus $k-i = 1$, $I_2 + J_2 = 2$,
$$d_{i,j,k} = \frac{2L}{2L-1} + 2L+1+\frac{2L}{2L_1} > 2L+3;$$
- for $I_2 + J_2 = 2L+1$, $I_1 + J_1 = 2-(k-i) \ge 1$ and thus $k-i = 1$. $I_1 + J_1 = 1$,
$$d_{i,j,k} = \frac{2L}{2L-1} + 2L+1+\frac{2L}{2L-1} > 2L+3.$$

This concludes the proof of (3-81). □

Step 8: *small linear term* $L(\varepsilon)$. Let us rewrite from a Taylor expansion

$$L(\varepsilon) = -\varepsilon N_2(\tilde{\alpha}_b), \quad N_2(\tilde{\alpha}_b) = \frac{f'(Q+\tilde{\alpha}_b) - f'(Q)}{y^2} = \frac{\tilde{\alpha}_b}{y^2}\int_0^1 f''(Q+\tau\tilde{\alpha}_b)\,d\tau. \tag{3-84}$$

Control for $y \le 1$. We use a Taylor expansion with the cancellation $f^{2k}(0) = 0$, $k \ge 0$, and Proposition 2.12 to ensure, for $y \le 1$, a decomposition

$$N_2(\tilde{\alpha}_b) = b_1\left[\sum_{i=0}^{L} \tilde{c}_i y^{2i} + r\right], \quad |\tilde{c}_i| \lesssim 1, \quad |\partial_y^k r| \lesssim y^{2L+2-k}, \quad 0 \le k \le 2L+1.$$

We combine this with (C-2) and obtain the representation, for $y \le 1$,

$$L(\varepsilon) = \left[\sum_{i=1}^{L+1} c_i T_{L+1-i} + r_\varepsilon\right] b_1 \left[\sum_{i=0}^{L} c_i y^{2i} + r\right] = b_1 \left[\sum_{i=1}^{L} \hat{c}_i y^{2i-1} + \hat{r}_\varepsilon\right] \tag{3-85}$$

with bounds

$$|\hat{c}_i| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{3-86}$$

$$|\partial_y^k \hat{r}_\varepsilon| \lesssim y^{2L+1-k} |\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le 2L+1,\ y \le 1. \tag{3-87}$$

We now apply $(\mathscr{A}^k)_{0 \le k \le 2L+1}$ to (3-85) and conclude using (3-71) that

$$\|\mathscr{A}^k L(\varepsilon)\|_{L^\infty(y \le 1)} \lesssim b_1 C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{3-88}$$

from which

$$\int_{y \le 1} \frac{1 + |\log y|^2}{1 + y^4} |H^L L(\varepsilon)|^2 + \int_{y \le 1} |AH^L L(\varepsilon)|^2 \lesssim C(M) b_1^2 \mathscr{E}_{2L+2} \lesssim C(M) b_1^2 b_1^{2L+2}.$$

Control for $y \ge 1$. We give a detailed proof of (3-64). The proof of (3-61) follows the exact same lines and is left to the reader. We claim the pointwise bound, for $y \ge 1$,

$$\text{for all } 0 \le k \le 2L+1, \quad |\partial_y^k N_2(\tilde{\alpha}_b)| \lesssim \frac{b_1 |\log b_1|^C}{y^{k+1}}, \tag{3-89}$$

which is proved below. From the Leibniz rule, this yields

$$|\partial_y^k L(\varepsilon)| \lesssim \sum_{i=0}^{k} \frac{b_1 |\log b_1|^C |\partial_y^i \varepsilon|}{y^{k-i+1}}, \tag{3-90}$$

and thus

$$\begin{aligned}|AH^L L(\varepsilon)| &\lesssim \sum_{k=0}^{2L+1} \frac{|\partial_y^k L(\varepsilon)|}{y^{2L+1-k}} \lesssim \sum_{k=0}^{2L+1} \frac{1}{y^{2L+1-k}} \sum_{i=0}^{k} \frac{b_1 |\log b_1|^C |\partial_y^i \varepsilon|}{y^{k-i+1}} \\ &\lesssim b_1 |\log b_1|^C \sum_{i=0}^{2L+1} \frac{|\partial_y^i \varepsilon|}{y^{2L+2-i}}.\end{aligned}$$

Therefore, from (C-11) with $k = L$, we conclude

$$\int_{y \ge 1} |AH^L L(\varepsilon)|^2 \lesssim b_1^2 |\log b_1|^C \sum_{i=0}^{2L+1} \int_{y \ge 1} \frac{|\partial_y^i \varepsilon|^2}{y^{4L+4-2i}} \lesssim |\log b_1|^C b_1^{2L+4},$$

and (3-64) is proved.

Proof of (3-89). Let

$$N_3 = \int_0^1 f''(Q + \tau \tilde{\alpha}_b)\, d\tau.$$

Letting $\tilde v_\tau = Q + \tau\tilde\alpha_b$, $0 \le \tau \le 1$, from Proposition 2.12 we estimate

$$|\partial_y^k \tilde v_\tau| \lesssim \frac{|\log b_1|^C}{y^{k+1}}, \quad 1 \le k \le 2L+1,\ y \ge 1,$$

and hence, using the Faa di Bruno formula,

$$\begin{aligned}|\partial_y^k N_3(\tilde\alpha_b)| &\lesssim \int_0^1 \sum_{m_1+2m_2+\cdots+km_k=k} |\partial_v^{m_1+\cdots+m_k} f(\tilde v_\tau)| \prod_{i=1}^k |\partial_y^i \tilde v_\tau|^{m_i}\, d\tau \\ &\lesssim |\log b_1|^C \sum_{m_1+2m_2+\cdots+km_k=k} \prod_{i=1}^k \left[\frac{1}{y^{i+1}}\right]^{m_i} \lesssim \frac{|\log b_1|^C}{y^{k+1}}.\end{aligned}$$

This yields in particular the rough bound

$$|\partial_y^k N_3(\tilde\alpha_b)| \lesssim \frac{|\log b_1|^C}{y^k}, \quad y \ge 1,\ 0 \le k \le 2L+1,$$

and hence, from the Leibniz rule,

$$\left|\partial_y^k\left(\frac{N_3(\tilde\alpha_b)}{y^2}\right)\right| \lesssim \frac{|\log b_1|^C}{y^{k+2}}, \quad y \ge 1,\ 0 \le k \le 2L+1. \tag{3-91}$$

From Proposition 2.12, we extract the rough bound

$$|\partial_y^k \tilde\alpha_b| \lesssim \frac{|\log b_1|^C b_1}{y^{k-1}}, \quad 0 \le k \le 2L+1$$

and, from the Leibniz rule, we conclude

$$|\partial_y^k N_2| \lesssim \sum_{i=0}^k |\log b_1|^C \frac{b_1}{y^{i+2} y^{k-i-1}} \lesssim \frac{b_1|\log b_1|^C}{y^{k+1}},$$

which proves (3-89). □

This concludes the proofs of (3-61), (3-62), (3-63), (3-64), and thus of Proposition 3.6. □

4. Closing the bootstrap and proof of Theorem 1.1

We are now in position to close the bootstrap bounds of Proposition 3.1. The proof of Theorem 1.1 will easily follow.

Proof of Proposition 3.1.

Proof. Our aim is first to show that for $s < s^*$, the a priori bounds (3-23), (3-24), (3-25) can be improved, and then the unstable modes $(U_k)_{2\le k\le L}$ will be controlled through a standard topological argument.

Step 1: *improved $\dot H^1$ bound*. First observe from (3-17) and the a priori bound on U_k for $s < s^*$ that

$$|b_k(s)| \lesssim |b_k(0)|. \tag{4-1}$$

The energy bound (3-23) is now a straightforward consequence of the dissipation of energy and the bounds (4-1), (3-26). Indeed, let

$$\tilde{\varepsilon} = \varepsilon + \hat{\alpha}.$$

Then

$$E_0 = \int |\partial_y (Q+\tilde{\varepsilon})|^2 + \int \frac{g^2(Q+\tilde{\varepsilon})}{y^2} = E(Q) + (H\tilde{\varepsilon}, \tilde{\varepsilon}) + \int \frac{1}{y^2}[g^2(Q+\tilde{\varepsilon}) - 2f(Q)\tilde{\varepsilon} - f'(Q)\tilde{\varepsilon}^2]. \tag{4-2}$$

We first use the bound on the profile which is easily extracted from Proposition 2.12

$$\int |\partial_y \hat{\alpha}|^2 + \int \frac{|\hat{\alpha}|^2}{y^2} \lesssim b_1 |\log b_1|^C \le \sqrt{b_1(0)}$$

using (4-1). Using Lemma B.1 ensures the coercivity

$$(H\tilde{\varepsilon}, \tilde{\varepsilon}) \ge c(M)\left[\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2}\right] - \frac{1}{c(M)}(\tilde{\varepsilon}, \Phi_M)^2 \ge c(M)\left[\int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2}\right] - \sqrt{b_1(0)}.$$

The nonlinear term is estimated from a Taylor expansion:

$$\left|\int \frac{1}{y^2}[g^2(Q+\tilde{\varepsilon}) - 2f(Q)\tilde{\varepsilon} - f'(Q)\tilde{\varepsilon}^2]\right| \lesssim \int \frac{|\tilde{\varepsilon}|^3}{y^2} \lesssim \left(\int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2}\right)^{\frac{3}{2}},$$

where we used the Sobolev bound

$$\|\tilde{\varepsilon}\|_{L^\infty}^2 \lesssim \|\partial_y \tilde{\varepsilon}\|_{L^2} \left\|\frac{\tilde{\varepsilon}}{y}\right\|_{L^2}.$$

We inject these bounds into the dissipation of energy (4-2) together with the initial bound (3-20) to estimate

$$\int |\partial_y \varepsilon|^2 + \int \frac{|\varepsilon|^2}{y^2} \lesssim \int |\partial_y \tilde{\varepsilon}|^2 + \int \frac{|\tilde{\varepsilon}|^2}{y^2} + b_1 |\log b_1|^C \le c(M)\sqrt{b_1(0)} \le (b_1(0))^{\frac{1}{4}} \tag{4-3}$$

for $|b_1(0)| \le b_1^*(M)$ small enough.

Step 2: integration of the scaling law. Let us compute explicitly the scaling parameter for $s < s^*$. From (3-36), (3-26), (2-101), (2-99), we have the rough bound

$$-\frac{\lambda_s}{\lambda} = \frac{c_1}{s} - \frac{|d_1|}{\log s} + O\left(\frac{1}{s(\log s)^{5/4}}\right),$$

which we rewrite as

$$\left|\frac{d}{ds}\left\{\frac{s^{c_1}\lambda(s)}{(\log s)^{|d_1|}}\right\}\right| \lesssim \frac{1}{s(\log s)^{5/4}}. \tag{4-4}$$

We integrate this using the initial value $\lambda(0) = 1$ and conclude

$$\frac{s^{c_1}\lambda(s)}{(\log s)^{|d_1|}} = \frac{s_0^{c_1}}{(\log s_0)^{|d_1|}} + O\left(\frac{1}{(\log s_0)^{1/4}}\right). \tag{4-5}$$

Together with the law for b_1 given by (3-26), (2-101), (2-99), this implies

$$b_1(0)^{c_1}|\log b_1(0)|^{|d_1|} \lesssim \frac{b_1^{c_1}(s)|\log b_1|^{|d_1|}}{\lambda(s)} \lesssim b_1(0)^{c_1}|\log b_1(0)|^{|d_1|}. \tag{4-6}$$

<u>*Step* 3: *improved control of* $\mathscr{E}_{2L+2}$.</u> We now improve the control of the high order $\mathscr{E}_{2L+2}$ energy (3-25) by reintegrating the Lyapounov monotonicity (3-48) in the regime governed by (4-5), (2-101). Indeed, we inject the bootstrap bound (3-25) into the monotonicity formula (3-48) and integrate in time s: for all $s \in [s_0, s^*)$,

$$\begin{aligned}\mathscr{E}_{2L+2}(s) \le 2\left(\frac{\lambda(s)}{\lambda(0)}\right)^{4L+2}\left[\mathscr{E}_{2L+2}(0) + Cb_1^{\frac{4}{5}}(0)\frac{b_1^{2L+2}(0)}{|\log b_1(0)|^2}\right] + \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2}\\ + C\left[1 + \frac{K}{\log M} + \sqrt{K}\right]\lambda^{2L+4}(s)\int_{s_0}^{s}\frac{b_1}{\lambda^{4L+2}}\frac{b_1^{2L+2}}{|\log b_1|^2}\,d\sigma\end{aligned} \tag{4-7}$$

for some universal constant $C > 0$ independent of M. We now observe from (4-6) that the integral in the right hand side of (4-7) is divergent, since

$$\frac{b_1}{\lambda^{4L+2}}\frac{b_1^{2L+2}}{|\log b_1|^2} \gtrsim C(b_0)\frac{b_1^{2L+3}}{b_1^{(4L+2)c_1}|\log b_1|^C} \gtrsim \frac{C(b_0)}{(\log s)^C s^{2L+3-(4L+2)L/(2L-1)}} = \frac{C(b_0)}{(\log s)^C s^{(2L-3)/(2L-1)}},$$

and therefore, from (4-6) and $1/s \lesssim b_1 \lesssim 1/s$,

$$\lambda^{4L+2}(s)\int_{s_0}^{s}\frac{b_1}{\lambda^{4L+2}}\frac{b_1^{2L+2}}{|\log b_1|^2}\,d\sigma \lesssim \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2}.$$

We now estimate the contribution of the initial data using (4-6) and the initial bounds (3-21), (3-22):

$$\begin{aligned}&\left(\frac{\lambda(s)}{\lambda(0)}\right)^{4L+2}\left[\mathscr{E}_{2L+2}(0) + Cb_1^{\frac{4}{5}}(0)\frac{b_1^{2L+2}(0)}{|\log b_1(0)|^2}\right]\\ &\quad\lesssim \lambda^{4L+2}(s)b_1^{\frac{4}{5}}(0)\frac{b_1^{2L+2}(0)}{|\log b_1(0)|^2}\\ &\quad\lesssim (b_1(s))^{(4L+2)L/(2L-1)}|\log b_1(s)|^C(b_1(0))^{\frac{4}{5}+2L+2-(4L+2)L/(2L-1)}|\log b_1(0)|^C \lesssim \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2},\end{aligned}$$

where we used the algebra, for $L \ge 2$,

$$0 < \frac{L(4L+2)}{2L-1} - (2L+2) = \frac{2}{2L-1} < \frac{4}{5}.$$

Injecting these bounds into (4-7) yields

$$\mathscr{E}_{2L+2}(s) \lesssim \frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2}\left[1 + \frac{K}{\log M} + \sqrt{K}\right] \le \frac{K}{2}\frac{b_1^{2L+2}(s)}{|\log b_1(s)|^2} \tag{4-8}$$

for K large enough independent of M.

Step 4: *Improved control of* $\mathscr{E}_{2k+2}$, $0 \le k \le L-1$. We now claim the improved bound on the intermediate energies

$$\mathscr{E}_{2k+2} \le b_1^{(2k+1)2L/(2L-1)} |\log b_1|^{C+\sqrt{K}}. \tag{4-9}$$

This follows from the monotonicity formula, for $0 \le k \le L-1$,

$$\frac{d}{dt}\left\{\frac{1}{\lambda^{4k+2}}\left[\mathscr{E}_{2k+2} + O\left(b_1^{\frac{1}{2}} b_1^{(4k+2)2L/(2L-1)}\right)\right]\right\} \lesssim \frac{|\log b_1|^C}{\lambda^{4k+4}}\left[b_1^{2k+3} + b_1^{1+\delta+(2k+1)2L/(2L-1)} + \sqrt{b_1^{2k+4}\mathscr{E}_{2k+2}}\right] \tag{4-10}$$

for some universal constants $C, \delta > 0$ independent of the bootstrap constant K. The proof is similar to that of (4-8) and in fact simpler since we allow for logarithmic losses; details are given in Appendix F.

Using (4-5), we now estimate

$$\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^C \lesssim \frac{(\log s)^{|d_1|}}{s^{(4k+2)c_1}} \int_{s_0}^{s} \frac{(\log \sigma)^C}{\sigma^{2k+3-c_1(4k+2)}}\, d\sigma.$$

From (2-97), we compute

$$(2k+3) - c_1(4k+2) = 1 + \frac{2(L-k-1)}{2L-1}, \tag{4-11}$$

and hence

$$\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{b_1^{2k+3}}{\lambda^{4k+2}} |\log b_1|^C \lesssim \frac{(\log s)^{|d_1|+C}}{s^{(4k+2)c_1}} \lesssim b_1^{(4k+2)c_1} |\log b_1|^C.$$

Similarly, from (4-6),

$$\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{b_1^{1+\delta+(2k+1)2L/(2L-1)}}{\lambda^{4k+2}} |\log b_1|^C \, d\sigma \lesssim \frac{(\log s)^{|d_1|+C}}{s^{(4k+2)c_1}} \int_{s_0}^{s} \frac{(\log \sigma)^C}{\sigma^{1+\delta}}\, d\sigma \lesssim b_1^{(4k+2)c_1} |\log b_1|^C,$$

and, using (4-11), (3-24),

$$\begin{aligned}
\lambda^{4k+2}(s) \int_{s_0}^{s} \frac{|\log b_1|^C}{\lambda^{4k+2}} \sqrt{b_1^{2k+4}\mathscr{E}_{2k+2}}\, d\sigma
&\lesssim \frac{(\log s)^{|d_1|+C}}{s^{(4k+2)c_1}} \int_{s_0}^{s} \frac{(\log \sigma)^{C+\sqrt{K}}}{\sqrt{s^{2k+4-(2k+1)2L/(2L-1)}}}\, d\sigma \\
&\lesssim |\log b_1|^{C+\sqrt{K}} b_1^{(4k+2)c_1} \int_{s_0}^{s} \frac{d\sigma}{\sigma^{1+(L-k-1)/(2L-1)}} \lesssim |\log b_1|^{C+\sqrt{K}} b_1^{(4k+2)c_1}.
\end{aligned}$$

Using the initial smallness (3-21) and (4-6), the time integration of (4-10) from $s = s_0$ to s therefore yields

$$\mathscr{E}_{2k+2}(s) \lesssim \lambda^{4k+2}(s) b_1(0)^{10L+4} + |\log b_1(s)|^{C+\sqrt{K}} b_1^{4k+2}(s) \lesssim |\log b_1(s)|^{C+\sqrt{K}} b_1^{(4k+2)c_1}(s),$$

and (4-9) is proved.

Remark 4.1. *For* $0 \le k \le L-2$, *the above argument shows the bound*

$$\mathscr{E}_{2k+2} \lesssim \lambda^{4k+2},$$

which equivalently corresponds to a uniform high order Sobolev control w. The logarithmic loss for $k = L-1$ could be gained as well with a little more work; see [Raphaël and Schweyer 2013] for the case $L = 1$. This shows that the limiting excess of energy u^* in (1-12) enjoys some suitable high order Sobolev regularity.

Step 5 *contradiction through a topological argument*. Let us consider

$$\tilde{b}_k = b_k \quad \text{for } 1 \le k \le L,\ \tilde{b}_L \text{ given by (3-44)}$$

and the associated variables

$$\tilde{b}_k = b_k^e + \frac{\widetilde{U}_k}{s^k(\log s)^{5/4}}, \quad 1 \le k \le L, \quad \tilde{b}_{k+1} = \widetilde{U}_{k+1} \equiv 0, \quad \widetilde{V} = P_L \widetilde{U}.$$

From (3-45),

$$|V - \widetilde{V}| \lesssim s^L |\log s|^C b_1^{L+\frac{1}{2}} \lesssim \frac{1}{s^{1/4}}. \tag{4-12}$$

Let the associated control of the unstable models be

$$|\widetilde{V}_1(s)| \le 2, \quad (\widetilde{V}_2(s), \dots, \widetilde{V}_L(s)) \in \mathcal{B}_{L-1}\left(\tfrac{1}{2}\right), \tag{4-13}$$

and the slightly modified exit time

$$\tilde{s}^* = \sup\{s \ge s_0 \text{ such that (3-23), (3-24), (3-25), (4-13) hold on } [s_0, s]\}.$$

Then (4-12) and the assumption (3-27) imply

$$\text{for all } (\widetilde{V}_2(0), \dots, \widetilde{V}_L(0)) \in \mathcal{B}_{L-1}\left(\tfrac{1}{2}\right), \quad \tilde{s}^* < +\infty. \tag{4-14}$$

We claim that this contradicts Brouwer's fixed point theorem.

Indeed, first, from (2-102), we estimate

$$(\tilde{b}_k)_s + \left(2k - 1 + \frac{2}{\log s}\right)\tilde{b}_1\tilde{b}_k - \tilde{b}_{k+1} = \frac{1}{s^{k+1}(\log s)^{5/4}}\left[s(\widetilde{U}_k)_s - (A_L\widetilde{U})_k + O\left(\frac{1}{\sqrt{\log s}}\right)\right],$$

and thus, from (3-37), (3-46), (4-8), and (2-44),

$$\begin{aligned} |s(\widetilde{U}_k)_s - (A_L\widetilde{U})_k| &\lesssim \frac{1}{\sqrt{\log s}} + s^{k+1}(\log s)^{\frac{5}{4}}\left|(\tilde{b}_k)_s + \left(2k - 1 + \frac{2}{\log s}\right)\tilde{b}_1\tilde{b}_k - \tilde{b}_{k+1}\right| \\ &\lesssim \frac{1}{\sqrt{\log s}} + s^{k+1}(\log s)^{\frac{5}{4}}\left[b_1^{L+\frac{3}{2}} + \frac{1}{s^{k+1}(\log s)^2} + \frac{b_1^{L+1}}{|\log b_1|^{3/2}}\right] \\ &\lesssim \frac{1}{(\log s)^{1/4}}. \end{aligned}$$

Hence, using the diagonalization (2-104),

$$s(\tilde{V})_s = D_L\widetilde{V}_s + O\left(\frac{1}{(\log s)^{1/4}}\right). \tag{4-15}$$

This first implies the control of the stable mode $\widetilde{V}_1$ from (2-104),

$$|(s\widetilde{V}_1)_s| \lesssim \frac{1}{(\log s)^{1/4}},$$

and thus, from (3-19),

$$|\widetilde{V}_1(s)| \lesssim \frac{1}{s} + \frac{1}{s}\int_{s_0}^{s} \frac{d\tau}{(\log\tau)^{1/4}} \le \tfrac{1}{10}. \tag{4-16}$$

Now, from (4-3), (4-8), (4-9), (4-16), and a standard continuity argument,

$$\sum_{i=2}^{L} \widetilde{V}_i^2(s^*) = \tfrac{1}{4}. \tag{4-17}$$

We then compute from (4-15) the fundamental strict outgoing condition at the exit time $\tilde{s}^*$ defined by (4-17):

$$\begin{aligned}\frac{1}{2}\frac{d}{ds}\left\{\sum_{i=2}^{L}\widetilde{V}_i^2\right\}_{|s=\tilde{s}^*} &= \sum_{i=2}^{L}(\widetilde{V}_i)_s\widetilde{V}_i = \frac{1}{\tilde{s}^*}\left[\sum_{i=2}^{L}\frac{i}{2L-1}\widetilde{V}_i^2(s^*) + O\left(\frac{1}{(\log\tilde{s}^*)^{1/4}}\right)\right] \\ &\ge \frac{1}{s^*}\left[\frac{2}{2L-1}\frac{1}{4} + O\left(\frac{1}{(\log\tilde{s}^*)^{1/4}}\right)\right] > 0.\end{aligned}$$

This implies from a standard argument the continuity of the map

$$(\widetilde{V}_i)_{2\le i\le L} \in \mathcal{B}_{L-1}\left(\tfrac{1}{2}\right) \mapsto \tilde{s}^*[(\widetilde{V}_i)_{2\le i\le L}],$$

and hence the continuous map

$$\begin{aligned}\mathcal{B}_{L-1}\left(\tfrac{1}{2}\right) &\to \mathcal{B}_{L-1}\left(\tfrac{1}{2}\right), \\ (\widetilde{V}_i)_{2\le i\le L} &\mapsto \{\widetilde{V}_i[\tilde{s}^*((\widetilde{V}_i)_{2\le i\le L})]\}\end{aligned}$$

is the identity on the boundary sphere $\mathbb{S}_{L-1}\left(\tfrac{1}{2}\right)$, a contradiction to Brouwer's fixed point theorem. This concludes the proof of Proposition 3.1. □

Proof of Theorem 1.1.

Proof. We pick initial data satisfying the conclusions of Proposition 3.1. In particular, (4-4) implies the existence of $c(u_0) > 0$ such that

$$\lambda(s) = c(u_0)\frac{(\log s)^{|d_1|}}{s^{c_1}}\left[1 + O\left(\frac{1}{(\log s)^{1/4}}\right)\right],$$

and then, from (3-36), (2-101),

$$-\lambda\lambda_t = -\frac{\lambda_s}{\lambda} = b_1 + O\left(\frac{1}{s^L}\right) = \frac{c_1}{s}\left[1 + O\left(\frac{1}{\log s}\right)\right] = \frac{c(u_0)\lambda^{1/c_1}}{|\log\lambda|^{|d_1|/c_1}}\left[1 + O\left(\frac{1}{(\log s)^{1/4}}\right)\right].$$

Hence we get the pointwise differential equation

$$-\lambda^{(-(L-1))/L}|\log\lambda|^{2/(2L-1)}\lambda_t = c(u_0)(1 + o(1)).$$

We easily conclude that λ touches zero at some finite time $T = T(u_0) < +\infty$ with near blow-up time

$$\lambda(t) = c(u_0)(1+o(1))\frac{(T-t)^L}{|\log(T-t)|^{2L/(2L-1)}}(1+o(1)).$$

The strong convergence (1-12) now follows as in [Raphaël and Schweyer 2013]. This concludes the proof of Theorem 1.1. □

Appendix A: Regularity in corotational symmetry

We detail in this appendix the regularity of smooth maps with 1-corotational symmetry.

Lemma A.1 (regularity in corotational symmetry). *Let v be a smooth 1-corotational map*

$$v(y,\theta) = \left| \begin{array}{l} g(u(y))\cos\theta \\ g(u(y))\sin\theta \\ z(u(y)) \end{array} \right. \tag{A-1}$$

with

$$v(0) = e_z, \qquad \lim_{y\to+\infty} v(x) \to -e_z. \tag{A-2}$$

Assume that v is smooth in the Sobolev sense:

$$\sum_{i=1}^{N} \int |(-\Delta)^{i/2} v|^2 < +\infty$$

for some $N \gg L$. Then:

(*i*) *u is a smooth function of y with a Taylor expansion at the origin for $p \le 10L$*:

$$u(y) = \sum_{k=0}^{p} c_k y^{2k+1} + O(y^{2p+3}). \tag{A-3}$$

(*ii*) *Assume that $u(y) = Q(y) + \varepsilon(y)$ with*

$$\|\nabla\varepsilon\|_{L^2} + \left\|\frac{\varepsilon}{y}\right\|_{L^2} \ll 1, \tag{A-4}$$

and consider the sequence of suitable derivatives $\varepsilon_k = \mathscr{A}^k\varepsilon$. Then, for all $1 \le k \le L$,

$$\int |\varepsilon_{2k+2}|^2 + \int \frac{|\varepsilon_{2k+1}|^2}{y^2(1+y^2)} + \sum_{p=0}^{k} \int \left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})} + \frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})} \right] < +\infty. \tag{A-5}$$

Proof of Lemma A.1. Let us consider the rotation matrix

$$R = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{A-6}$$

and rewrite (A-1) as

$$v(r,\theta)=e^{\theta R}w \quad \text{with } w(r)=\left|\begin{array}{l} w_1=g(u)\\ 0\\ w_3=z(u).\end{array}\right. \tag{A-7}$$

<u>*Step* 1: *Control at the origin*.</u> We compute the energy density

$$|\nabla v|^2=|\partial_y w|^2+\frac{|w|^2}{y^2}, \tag{A-8}$$

which is bounded from the smoothness of v, from which

$$\left|\frac{w_1}{y}\right|+|\partial_y w_1|\lesssim 1. \tag{A-9}$$

Similarly,

$$\Delta v=e^{\theta R}\left(\Delta w+R^2\frac{w}{y^2}\right)=e^{\theta R}\left|\begin{array}{l} -\mathbb{H}w_1\\ 0\\ \Delta w_3,\end{array}\right. \tag{A-10}$$

where

$$\mathbb{H}w_1=-\Delta w_1+\frac{w_1}{y^2}=\mathbb{A}^*\mathbb{A}w_1$$

with

$$\mathbb{A}=-\partial_y+\frac{1}{y},\qquad \mathbb{A}^*=\partial_y+\frac{2}{y}.$$

The regularity of v implies

$$|\mathbb{H}w_1|\lesssim 1$$

near the origin, which, together with (A-9), yields

$$\mathbb{A}w_1(y)=\frac{1}{y^2}\int_0^r(\mathbb{H}w_1)\tau^2\,d\tau=O(y). \tag{A-11}$$

We now observe that

$$\mathbb{H}w_1=-\partial_{yy}w_1+\frac{1}{y}\mathbb{A}w_1$$

and conclude

$$|\partial^2_{yy}w_1|\lesssim 1.$$

We now iterate this argument once on (A-10). Indeed, at the origin,

$$|\partial_y\mathbb{H}w_1|^2+\frac{|\mathbb{H}w_1|^2}{y^2}\lesssim|\nabla\Delta v|^2\lesssim 1,\qquad |\mathbb{H}^2w_1|\lesssim|\Delta^2 v|\lesssim 1,$$

and hence

$$|\mathbb{H}w_1|\lesssim y,\qquad |\mathbb{A}\mathbb{H}w_1|\lesssim y,\qquad |\mathbb{H}^2w_1|\lesssim 1.$$

This yields the $\mathscr{C}^3$-regularity of w_1 at the origin and, from (A-11), the improved bound

$$\mathbb{A}w_1(y)=\frac{1}{y^2}\int_0^y(\mathbb{H}w_1)\tau^2\,d\tau=O(y^2).$$

A simple induction now yields for all $k \geq 1$ the $\mathscr{C}^k$-regularity of w_1, and that the sequence

$$(w_1)_0 = w_1, \quad (w_1)_{k+1} = \begin{cases} \mathbb{A}^*(w_1)_k & \text{for k odd,} \\ \mathbb{A}(w_1)_k & \text{for k even,} \end{cases} \qquad k \geq 1$$

satisfies the bound

$$|(w_1)_k| \lesssim \begin{cases} y & \text{for k even,} \\ y^2 & \text{for k odd.} \end{cases} \tag{A-12}$$

We therefore let a Taylor expansion at the origin

$$w_1(y) = \sum_{i=1}^{p} c_i y^i + O(y^{p+1})$$

apply successively the operator $\mathbb{A}$, $\mathbb{A}^*$ and conclude from the relations

$$\mathbb{A}(r^k) = -(k-1)y^{k-1}, \quad \mathbb{A}^*(y^k) = (k+2)y^{k-1}$$

and (A-12) that

$$c_{2k} = 0 \quad \text{for all } k \geq 1.$$

We now recall from (A-7) that $w_1 = g(u)$ and the Taylor expansion (A-3) now follows from the odd parity of g at the origin.

We now claim that this implies the bound (A-5) at the origin. Indeed, ε admits a Taylor expansion (A-3) from (2-2) to which we apply successively the operators A, A^*. We observe from (2-5) the cancellation

$$A(y) = cy^2 + O(y^3),$$

which ensures the bound near the origin

$$|\varepsilon_{2k}| \lesssim y, \quad |\varepsilon_{2k+1}| \lesssim y^2, \tag{A-13}$$

and hence the finiteness of the norms (A-5) at the origin.

Step 2: *control for* $r \geq 1$. We first claim

$$\int \frac{\varepsilon^2}{y^2} + \sum_{k=1}^{L+2} \int (\partial_y^k \varepsilon)^2 < +\infty. \tag{A-14}$$

Indeed, from (A-4),

$$\|\varepsilon\|_{L^\infty} \lesssim \|\nabla \varepsilon\|_{L^2} + \left\| \frac{\varepsilon}{y} \right\|_{L^2} \ll 1. \tag{A-15}$$

From (A-8),

$$\int |\Delta g(u)|^2 \lesssim \int |\Delta v|^2 < +\infty.$$

Now

$$\Delta g(u) = g'(u)\Delta u + (\partial_y u)^2 g'(u) g''(u)$$

and, using Sobolev and the L^∞ control (A-15), we estimate

$$\int ((\partial_y\varepsilon)^2 g'(u)g''(u))^2 \lesssim \int |\Delta\varepsilon|^2 \int |\nabla\varepsilon|^2 \ll \int |\Delta\varepsilon|^2.$$

Moreover, from the smallness (A-15) and the structure of Q,

$$|g'(u)| \gtrsim 1 \quad \text{as } r \to +\infty,$$

from which

$$1 + \int |\Delta v|^2 \gtrsim \int |\Delta\varepsilon|^2.$$

The control of higher order Sobolev norms (A-14) now follows similarly by induction using the Faa di Bruno formula for the computation of $\partial_y^k g(u)$. This is left to the reader. Now (A-14) easily implies

$$\sum_{i=1}^{L+2} \int_{y\geq 1} |\varepsilon_k|^2 < +\infty,$$

and the bound (A-5) is proved far out. □

Appendix B: Coercivity bounds

Given $M \geq 1$, we let Φ_M be given by (3-7). Let us recall the coercivity of the operator H, which is a standard consequence of the knowledge of the kernel of H and the nondegeneracy (3-8).

Lemma B.1 (coercivity of H). *Let $M \geq 1$ be large enough. Then there exists $C(M) > 0$ such that, for all radially symmetric u with*

$$\int \left[|\partial_y u|^2 + \frac{|u|^2}{y^2} \right] < +\infty$$

satisfying

$$(u, \Phi_M) = 0,$$

we have

$$(Hu, u) \geq C(M) \left[\int |\partial_y u|^2 + \frac{|u|^2}{y^2} \right].$$

We now recall the coercivity of $\widetilde{H}$, which is a simple consequence of (2-11) and is proved in [Raphaël and Rodnianski 2012].

Lemma B.2 (coercivity of $\widetilde{H}$). *Let u be such that*

$$\int |\partial_y u|^2 + \int \frac{|u|^2}{y^2} < +\infty. \tag{B-1}$$

Then

$$(\widetilde{H}u, u) = \|A^* u\|_{L^2}^2 \geq c_0 \left[\int |\partial_y u|^2 + \int \frac{|u|^2}{y^2(1+|\log y|^2)} \right] \tag{B-2}$$

for some universal constant $c_0 > 0$.

We now claim the following weighted coercivity bound on H.

Lemma B.3 (coercivity of $\mathscr{E}_2$). *There exists $C(M) > 0$ such that, for all radially symmetric u with*

$$\int \frac{|u|^2}{y^4(1+|\log y|^2)} + \int \frac{|Au|^2}{y^2(1+y^2)} < +\infty \tag{B-3}$$

and

$$(u, \Phi_M) = 0,$$

we have

$$\int |Hu|^2 \geq C(M)\left[\int \frac{|u|^2}{y^4(1+|\log y|^2)} + \int \frac{|Au|^2}{y^2(1+|\log y|^2)}\right]. \tag{B-4}$$

Proof of Lemma B.3. This lemma is proved in [Raphaël and Rodnianski 2012] in the case of the sphere target. Let us briefly recall the argument for the sake of completeness.

Step 1: conclusion using a subcoercivity lower bound. For any u satisfying (B-10), we claim the subcoercivity lower bound

$$\int |Hu|^2 \gtrsim \int \frac{|\partial_y^2 u|^2}{(1+|\log y|^2)} + \int \frac{(\partial_y u)^2}{y^2(1+|\log y|^2)} + \int \frac{|u|^2}{y^4(1+|\log y|^2)} - C\left[\int \frac{(\partial_y u)^2}{1+y^3} + \int \frac{|u|^2}{1+y^5}\right]. \tag{B-5}$$

Let us assume (B-5) and conclude the proof of (B-4). By contradiction, let $M > 0$ be fixed and consider a normalized sequence u_n,

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)} + \int \frac{|Au_n|^2}{y^2(1+|\log y|^2)} = 1, \tag{B-6}$$

satisfying the orthogonality condition

$$(u_n, \Phi_M) = 0 \tag{B-7}$$

and the smallness

$$\int |Hu_n|^2 \leq \frac{1}{n}. \tag{B-8}$$

Note that the normalization condition implies

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)} + \int \frac{|\partial_y u_n|^2}{y^2(1+|\log y|^2)} \lesssim 1,$$

and thus, from (B-5), the sequence u_n is bounded in H^2_{loc}. Hence there exists $u_\infty \in H^2_{\text{loc}}$ such that, up to a subsequence and for any smooth cut-off function ζ vanishing in a neighborhood of $y = 0$, the sequence ζu_n is uniformly bounded in H^2_{loc} and converges to ζu_∞ in H^1_{loc}. Moreover, (B-8) implies

$$Hu_\infty = 0,$$

and by lower semicontinuity of the norm and (B-6),

$$\int \frac{|u_\infty|^2}{y^4(1+|\log y|^2)} < +\infty,$$

which implies from (2-12) that

$$u = \alpha \Lambda Q \quad \text{for some } \alpha \in \mathbb{R}.$$

We may moreover pass to the limit in (B-7) from (B-6) and the local compactness of Sobolev embeddings, and thus

$$(u_\infty, \Lambda Q) = 0, \quad \text{from which } \alpha = 0,$$

where we used the nondegeneracy (3-8). Hence $u_\infty = 0$. Now the subcoercivity lower bound (B-5) together with (B-6), (B-8), and the H^2_{loc} uniform bound imply the existence of $\varepsilon, c > 0$ such that

$$\int_{\varepsilon \le y \le 1/\varepsilon} \left[\frac{|\partial_y u_\infty|^2}{1+y^3} + \frac{|u_\infty|^2}{1+y^5} \right] \ge c > 0,$$

which contradicts the established identity $u_\infty = 0$. Thus (B-4) is proved.

Step 2: *proof of* (B-5). Let us first apply Lemma B.2 to Au, which satisfies (B-1) by assumption, and estimate

$$\int (Hu)^2 = (\widetilde{H} Au, Au) \gtrsim \int |\partial_y (Au)|^2 + \int \frac{|Au|^2}{y^2(1+y^2)}. \tag{B-9}$$

Near the origin, we now recall the logarithmic Hardy inequality

$$\int_{y \le 1} \frac{|v|^2}{y^2(1+|\log y|^2)} \lesssim \int_{1 \le y \le 2} |v|^2 + \int_{y \le 1} |\partial_y v|^2,$$

and thus, using (2-10),

$$\int \frac{|Au|^2}{y^2} = \int \frac{1}{y^2} \left| \Lambda Q \partial_y \left(\frac{u}{\Lambda Q} \right) \right|^2 \gtrsim \int_{y \le 1} \left| \frac{u}{\Lambda Q} \right|^2 \frac{dy}{y^2(1+|\log y|^2)} - \int_{y \le 1} (|\partial_y u|^2 + |u|^2),$$

which, together with (B-9), yields

$$\int |Hu|^2 \gtrsim \int_{y \le 1} \left[\frac{(\partial_y u)^2}{y^2(1+|\log y|^2)} + \int \frac{|u|^2}{y^4(1+|\log y|^2)} \right] - C \int_{y \le 1} (|\partial_y u|^2 + |u|^2).$$

To control the second derivative, we rewrite near the origin

$$Hu = -\partial_y^2 u + \frac{1}{y} \left(-\partial_y u + \frac{u}{y} \right) + \frac{V-1}{y^2} u = -\partial_y^2 u + \frac{Au}{y} + \frac{(V-1)+(1-Z)}{y^2} u$$

and (B-5) follows near the origin.

Away from the origin, let $\zeta(y)$ be a smooth cut-off function with support in $y \ge \frac{1}{2}$ and equal to 1 for $y \ge 1$. We use the logarithmic Hardy inequality

$$\int_{y \ge 1} \frac{|u|^2}{y^2(1+|\log y|^2)} \lesssim \int_{1 \le y \le 2} |u|^2 + \int |\partial_y u|^2$$

to conclude from (B-9) that

$$\int (Hu)^2 \gtrsim \int \zeta \frac{|Au|^2}{y^2(1+|\log y|^2)} - C \int_{1 \le y \le 2} (|u|^2 + |\partial_y u|^2).$$

Now, from (2-5), we estimate

$$\int \zeta \frac{|Au|^2}{y^2(1+|\log y|^2)} = \int \zeta \frac{|-\partial_y u - \frac{u}{y}|^2}{y^2(1+|\log y|^2)} - C\int \zeta \frac{|u|^2}{y^6(1+|\log y|^2)}$$
$$\gtrsim \int \frac{\zeta}{y^2(1+|\log y|^2)}\left[|\partial_y u|^2 + \frac{|u|^2}{y^2}\right] - C\int \left[\frac{|u|^2}{y^5} + \frac{|\partial_y u|^2}{y^3}\right],$$

where we integrated by parts in the last step. The control of the second derivative follows from the explicit expression of H. This concludes the proof of (B-5). □

We now aim at generalizing the coercivity of the $\mathscr{E}_2$ energy of Lemma B.3 to higher order energies. This first requires a generalization of the weighted estimate (B-4).

Lemma B.4 (weighted coercivity bound). *Let $L \geq 1$, $0 \leq k \leq L$, and $M \geq M(L)$ be large enough. Then there exists $C(M) > 0$ such that, for all radially symmetric u with*

$$\int \frac{|u|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|Au|^2}{y^6(1+|\log y|^2)(1+y^{4k+4})} < +\infty \tag{B-10}$$

and

$$(u, \Phi_M) = 0,$$

we have

$$\int \frac{|Hu|^2}{y^4(1+|\log y|^2)(1+y^{4k})}$$
$$\geq C(M)\left\{\int \frac{|u|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|Au|^2}{y^6(1+|\log y|^2)(1+y^{4k})}\right\}. \tag{B-11}$$

Proof of Lemma B.4. *Step 1: subcoercivity lower bound.* For any u satisfying (B-10), we claim the subcoercivity lower bound

$$\int \frac{|Hu|^2}{y^4(1+|\log y|)^2(1+y^{4k})}$$
$$\gtrsim \int \frac{|\partial_y^2 u|^2}{y^4(1+|\log y|^2)(1+y^{4k})} + \int \frac{(\partial_y u)^2}{y^2(1+|\log y|)^2(1+y^{4k+4})}$$
$$+ \int \frac{|u|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} - C\left[\int \frac{(\partial_y u)^2}{1+y^{4k+8}} + \int \frac{|u|^2}{1+y^{4k+10}}\right]. \tag{B-12}$$

Control near the origin. Recall from the finiteness of the norm (B-10) and the formula (2-21) that

$$Au(y) = \frac{1}{y\Lambda Q(y)}\int_0^y \tau \Lambda Q(\tau) Hu(\tau)\, d\tau.$$

Then from Cauchy–Schwarz and Fubini we estimate

$$\int_{y\le 1}\frac{|Au|^2}{y^5(1+|\log y|^2)}\,dy \lesssim \int_{0\le y\le 1}\int_{0\le\tau\le y}\frac{y^5}{y^9(1+|\log y|^2)}|Hu(\tau)|^2\,dy\,d\tau$$
$$\lesssim \int_{0\le\tau\le 1}|Hu(\tau)|^2\left[\int_{\tau\le y\le 1}\frac{dy}{y^4(1+|\log y|^2)}\right]d\tau \lesssim \int_{\tau\le 1}\frac{|Hu(\tau)|^2}{\tau^3(1+|\log\tau|^2)}\,d\tau,$$

and thus

$$\int_{y\le 1}\frac{|Au|^2}{y^6(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)}. \tag{B-13}$$

We now invert A and get from (2-10) the existence of $c(u)$ such that

$$u(y)=c(u)\Lambda Q(y)-\Lambda Q(y)\int_0^y\frac{Au(\tau)}{\Lambda Q(\tau)}\,d\tau.$$

We estimate from Cauchy–Schwarz and (B-13), for $1\le y\le 1$,

$$\left|\int_0^y\frac{Au(\tau)}{\Lambda Q(\tau)}\,d\tau\right|^2 \lesssim y^4(1+|\log y|^2)\int_0^y\frac{|Au|^2}{\tau^5(1+|\log\tau|^2)}\,d\tau \lesssim y^3\int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)},$$

from which

$$|c(u)|^2\lesssim \int_{y\le 1}|u|^2+\int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)}$$

and

$$\int_{y\le 1}\frac{|u|^2}{y^4(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)}+\int_{1\le y\le 2}|u|^2. \tag{B-14}$$

The control of the first derivative follows from (B-13), (B-14), and the definition of A:

$$\int_{y\le 1}\frac{|\partial_y u|^2}{y^2(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Au|^2}{y^2(1+|\log y|^2)}+\int_{y\le 1}\frac{|u|^2}{y^4(1+|\log y|^2)}$$
$$\lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)}+\int_{1\le y\le 2}|u|^2.$$

To control the second derivative, we rewrite near the origin

$$Hu=-\partial_y^2u+\frac{1}{y}\left(-\partial_y u+\frac{u}{y}\right)+\frac{V-1}{y^2}u=-\partial_y^2u+\frac{Au}{y}+\frac{(V-1)+(1-Z)}{y^2}u,$$

which using (B-13), (B-14) and (2-5), (2-6) implies

$$\int_{y\le 1}\frac{|\partial_y^2u|^2}{y^4(1+|\log y|^2)} \lesssim \int_{y\le 1}\frac{|Hu|^2}{y^4(1+|\log y|^2)}+\int_{1\le y\le 2}|u|^2.$$

This concludes the proof of (B-12) near the origin.

Control away from the origin. Let $\zeta(y)$ be a smooth cut-off function with support in $y \geq \frac{1}{2}$ and equal to 1 for $y \geq 1$. We compute

$$\begin{aligned}\int \zeta \frac{|Hu|^2}{y^{4k+4}(1+|\log y|)^2} &= \int \zeta \frac{|-\partial_y(y\partial_y u)+\frac{V}{y}u|^2}{y^{4k+6}(1+|\log y|)^2} \\ &= \int \zeta \frac{|\partial_y(y\partial_y u)|^2}{y^{4k+6}(1+|\log y|)^2} - 2\int \zeta \frac{\partial_y(y\partial_y u)\cdot Vu}{y^{4k+7}(1+|\log y|)^2} + \int \zeta \frac{V^2|u|^2}{y^{4k+8}(1+|\log y|)^2} \\ &= \int \zeta \frac{|\partial_y(y\partial_y u)|^2}{y^{4k+6}(1+|\log y|)^2} + 2\int \zeta \frac{V(\partial_y u)^2}{y^{4k+6}(1+|\log y|)^2} + \int \zeta \frac{V^2|u|^2}{y^{4k+8}(1+|\log y|)^2} \\ &\quad - \int |u|^2 \Delta\left(\frac{\zeta V}{y^{4k+6}(1+|\log y|)^2}\right). \qquad \text{(B-15)}\end{aligned}$$

We now use the two dimensional logarithmic Hardy inequality with best constant:[14] for all $\gamma > 0$,

$$\frac{\gamma^2}{4}\int_{y\geq 1} \frac{|v|^2}{y^{2+\gamma}(1+|\log y|)^2} \leq C_\gamma \int_{1\leq y\leq 2} |v|^2 + \int_{y\geq 1} \frac{|\partial_y v|^2}{y^\gamma(1+|\log y|)^2} \qquad \text{(B-16)}$$

with $\gamma = 4k+6$. We estimate

$$\begin{aligned}\int \zeta \frac{|\partial_y(y\partial_y u)|^2}{y^{4k+6}(1+|\log y|)^2} &\geq \frac{(4k+6)^2}{4}\int_{y\geq 1} \frac{|\partial_y u|^2}{y^{4k+6}(1+|\log y|)^2} - C_k \int_{1\leq y\leq 2} |\partial_y u|^2 \\ &\geq \frac{(4k+6)^4}{16}\int_{y\geq 1} \frac{|u|^2}{y^{4k+8}(1+|\log y|)^2} - C_k \int_{1\leq y\leq 2} [|\partial_y u|^2 + |u|^2].\end{aligned}$$

We now observe that, for $k \geq 0$ and $y \geq 1$,

$$\partial_y^k V(y) = \partial_y^k(1) + O(y^{-2-k}).$$

We compute

$$\Delta\left(\frac{1}{y^{4k+6}}\right) = \frac{(4k+6)^2}{y^{4k+8}}.$$

Injecting these bounds into (B-15) yields the lower bound

$$\int \zeta \frac{|Hu|^2}{y^{4k+4}(1+|\log y|)^2} \geq \left[\frac{(4k+6)^4}{16} - (4k+6)^2\right]\int_{y\geq 1} \frac{|u|^2}{y^{4k+8}(1+|\log y|)^2} - C_k \int \left[\frac{|\partial_y u|^2}{1+y^{4k+8}} + \frac{|u|^2}{1+y^{4k+10}}\right].$$

Note that we can always keep the control of the first two derivatives in these estimates, and the control (B-12) follows away from the origin.

[14] which can be obtained by a simple integration by parts; see [Merle et al. 2011].

Step 2: proof of (B-11). By contradiction, let $M > 0$ be fixed and consider a normalized sequence u_n,

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|Au_n|^2}{y^6(1+|\log y|^2)(1+y^{4k})} = 1, \tag{B-17}$$

satisfying the orthogonality condition

$$(u_n, \Phi_M) = 0 \tag{B-18}$$

and the smallness

$$\int \frac{|Hu_n|^2}{y^4(1+|\log y|^2)(1+y^{4k})} \le \frac{1}{n}. \tag{B-19}$$

Note that the normalization condition implies

$$\int \frac{|u_n|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int \frac{|\partial_y u_n|^2}{y^2(1+|\log y|^2)(1+y^{4k+4})} \lesssim 1, \tag{B-20}$$

and thus, from (B-12), the sequence u_n is bounded in H^2_{loc}. Hence, there exists $u_\infty \in H^2_{\text{loc}}$ such that, up to a subsequence and for any smooth cut-off function ζ vanishing in a neighborhood of $y = 0$, the sequence ζu_n is uniformly bounded in H^2_{loc} and converges to ζu_∞ in H^1_{loc}. Moreover, (B-19) implies

$$Hu_\infty = 0,$$

and, by lower semicontinuity of the norm and (B-17),

$$\int \frac{|u_\infty|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} < +\infty,$$

which implies from (2-12) that

$$u = \alpha \Lambda Q \quad \text{for some } \alpha \in \mathbb{R}.$$

We may moreover pass to the limit in (B-18) from (B-17) and the local compactness embedding, and thus

$$(u_\infty, \Lambda Q) = 0, \quad \text{from which } \alpha = 0,$$

where we used the nondegeneracy (3-8). Hence $u_\infty = 0$.

Now from (B-13), (B-14), (B-19), and (B-17),

$$\int_{y\ge 1} \frac{|u_n|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})} + \int_{y\ge 1} \frac{|\partial_y u_n|^2}{y^6(1+|\log y|^2)(1+y^{4k})} \gtrsim 1,$$

and hence, from (B-12), (B-19),

$$\frac{|\partial_y u_n|^2}{1+y^{4k+8}} + \frac{|u_n|^2}{1+y^{4k+10}} \gtrsim 1,$$

which, from the local compactness of Sobolev embeddings and the a priori bound (B-20), ensures

$$\int \frac{|\partial_y u_\infty|^2}{1+y^{4k+8}} + \int \frac{|u_\infty|^2}{1+y^{4k+10}} \gtrsim 1.$$

This contradicts the established identity $u_\infty = 0$. □

We are now in a position to prove the coercivity of the higher order $(\mathscr{E}_{2k+2})_{0\le k\le L}$ energies under suitable orthogonality conditions. Given a radially symmetric function ε, we recall the definition of suitable derivatives:

$$\varepsilon_{-1}=0,\quad \varepsilon_0=\varepsilon,\quad \varepsilon_{k+1}=\begin{cases}A^*\varepsilon_k & \text{for k odd,}\\ A\varepsilon_k & \text{for k even,}\end{cases}\quad 0\le k\le 2L+1.$$

Lemma B.5 (coercivity of $\mathscr{E}_{2k+2}$). *Let $L\ge 1$, $0\le k\le L$, and $M\ge M(L)$ be large enough. Then there exists $C(M)>0$ such that, for all ε with*

$$\int|\varepsilon_{2k+2}|^2+\int\frac{|\varepsilon_{2k+1}|^2}{y^2(1+y^2)}+\sum_{p=0}^{k}\int\left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})}+\frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})}\right]<+\infty \tag{B-21}$$

satisfying

$$(\varepsilon, H^p\Phi_M)=0,\quad 0\le p\le k, \tag{B-22}$$

we have

$$\mathscr{E}_{2k+2}(\varepsilon)=\int(H^{k+1}\varepsilon)^2\ge C(M)\left\{\int\frac{|\varepsilon_{2k+1}|^2}{y^2(1+|\log y|^2)}+\sum_{p=0}^{k}\int\left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})}+\frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})}\right]\right\}. \tag{B-23}$$

Proof of Lemma B.1. We argue by induction on k. The case $k=0$ is Lemma B.3. We assume the claim for k and prove it for $1\le k+1\le L$. Indeed, let $v=H\varepsilon$. Then $v_p=\varepsilon_{p+2}$, and thus v satisfies (B-21) and[15]

$$\text{for all } 0\le p\le k,\quad (v, H^p\Phi_M)=(\varepsilon, H^{p+1}\Phi_M)=0.$$

We may thus apply the induction claim for k to v and estimate

$$\begin{aligned}\int(H^{k+2}\varepsilon)^2 &=\int(H^{k+1}v)^2\\ &\ge C(M)\left\{\int\frac{|\varepsilon_{2k+3}|^2}{y^2(1+|\log y|^2)}+\sum_{p=0}^{k}\int\left[\frac{|\varepsilon_{2p+1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k-p)})}+\frac{|\varepsilon_{2p+2}|^2}{y^4(1+|\log y|^2)(1+y^{4(k-p)})}\right]\right\}\\ &\ge C(M)\left\{\int\frac{|\varepsilon_{2k+3}|^2}{y^2(1+|\log y|^2)}+\sum_{p=1}^{k+1}\int\left[\frac{|\varepsilon_{2p-1}|^2}{y^6(1+|\log y|^2)(1+y^{4(k+1-p)})}+\frac{|\varepsilon_{2p}|^2}{y^4(1+|\log y|^2)(1+y^{4(k+1-p)})}\right]\right\}.\end{aligned} \tag{B-24}$$

[15] from $k\le L+1$

The orthogonality condition $(\varepsilon, \Phi_M) = 0$ and (B-21) allow us to use Lemma B.4 and to deduce from the weighted bound (B-11) the control

$$\int \frac{|\varepsilon_2|^2}{y^4(1+|\log y|^2)(1+y^{4k})} \gtrsim \int \frac{|\varepsilon|^2}{y^4(1+|\log y|^2)(1+y^{4k+4})},$$

which together with (B-24) concludes the proof of Lemma B.1. □

Appendix C: Interpolation bounds

We derive in this section interpolation bounds on ε in the setting of the bootstrap proposition 3.1, and which are a consequence of the coercivity property of Lemma B.5.

Lemma C.1 (interpolation bounds). (i). Weighted Sobolev bounds for ε_k. *For* $0 \le k \le L$,

$$\sum_{i=0}^{2k+1} \int \frac{|\varepsilon_i|^2}{y^2(1+y^{4k-2i+2})(1+|\log y|^2)} + \int |\varepsilon_{2k+2}|^2 \le C(M)\mathscr{E}_{2k+2}. \tag{C-1}$$

(ii). Development near the origin. *ε admits a Taylor–Lagrange-like expansion*

$$\varepsilon = \sum_{i=1}^{L+1} c_i T_{L+1-i} + r_\varepsilon \tag{C-2}$$

with bounds

$$|c_i| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{C-3}$$

$$|\partial_y^k r_\varepsilon| \lesssim y^{2L+1-k}|\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le 2L+1,\ y \le 1. \tag{C-4}$$

(iii). Bounds near the origin for ε_k. *For* $|y| \le \frac{1}{2}$,

$$|\varepsilon_{2k}| \lesssim C(M) y|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le L, \tag{C-5}$$

$$|\varepsilon_{2k-1}| \lesssim C(M) y^2|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 1 \le k \le L, \tag{C-6}$$

$$|\varepsilon_{2L+1}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}. \tag{C-7}$$

(iv). Bounds near the origin for $\partial_y^k \varepsilon$. *For* $|y| \le \frac{1}{2}$,

$$|\partial_y^{2k}\varepsilon| \lesssim C(M) y|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le k \le L, \tag{C-8}$$

$$|\partial_y^{2k-1}\varepsilon| \lesssim C(M)|\log y|\sqrt{\mathscr{E}_{2L+2}}, \quad 1 \le k \le L+1. \tag{C-9}$$

(v). Lossy bound.

$$\sum_{i=0}^{2k+1} \int \frac{1+|\log y|^C}{1+y^{4k-2i+4}}|\varepsilon_i|^2 \lesssim |\log b_1|^C \begin{cases} b_1^{(4k+2)L/(2L-1)}, & 0 \le k \le L-1, \\ b_1^{2L+2} & \textit{for } k = L, \end{cases} \tag{C-10}$$

$$\sum_{i=0}^{2k+1} \int \frac{1+|\log y|^C}{1+y^{4k-2i+4}}|\partial_y^i\varepsilon|^2 \lesssim |\log b_1|^C \begin{cases} b_1^{(4k+2)L/(2L-1)}, & 0 \le k \le L-1, \\ b_1^{2L+2} & \textit{for } k = L. \end{cases} \tag{C-11}$$

(vi). Generalized lossy bound. *Let* $(i,j)\in\mathbb{N}\times\mathbb{N}^*$ *with* $2\le i+j\le 2L+2$. *Then*

$$\int\frac{1+|\log y|^C}{1+y^{2j}}|\partial_y^i\varepsilon|^2\lesssim|\log b_1|^C\begin{cases}b_1^{(i+j-1)2L/(2L-1)} & \textit{for } 2\le i+j\le 2L,\\ b_1^{2L+1} & \textit{for } i+j=2L+1,\\ b_1^{2L+2} & \textit{for } i+j=2L+2.\end{cases}\tag{C-12}$$

Moreover,

$$\int\frac{|\partial_y^i\varepsilon|^2}{1+|\log y|^2}\lesssim|\log b_1|^C\begin{cases}b_1^{(i-1)2L/(2L-1)}, & 2\le i\le 2L+1,\\ b_1^{2L+2} & \textit{for } i=2L+2,\end{cases}\tag{C-13}$$

(vii). Pointwise bound far away. *Let* $(i,j)\in\mathbb{N}\times\mathbb{N}$ *with* $1\le i+j\le 2L+1$. *Then*

$$\left\|\frac{\partial_y^i\varepsilon}{y^j}\right\|^2_{L^\infty(y\ge1)}\lesssim|\log b_1|^C\begin{cases}b_1^{(i+j)2L/(2L-1)} & \textit{for } 1\le i+j\le 2L-1,\\ b_1^{2L+1} & \textit{for } i+j=2L,\\ b_1^{2L+2} & \textit{for } i+j=2L+1.\end{cases}\tag{C-14}$$

Proof. *Step* 1: *proof of* (i). The estimate (C-1) follows from (B-23) with $0\le k\le L$.

Step 2: *adapted Taylor expansion*. *Initialization*. Recall the boundary condition origin at the origin (A-13), which implies $|\varepsilon_{2L+1}(y)|\le C_{\varepsilon_{2L+1}}y^2$ as $y\to0$. Together with (2-10) and the behavior $\Lambda Q\sim y$ near the origin, this implies

$$r_1=\varepsilon_{2L+1}(y)=\frac{1}{y\Lambda Q}\int_0^y\varepsilon_{2L+2}\Lambda Qx\,dx,\tag{C-15}$$

and this yields the pointwise bound, for $y\le1$,

$$|r_1(y)|\lesssim\frac{1}{y^2}\left(\int_{y\le1}|\varepsilon_{2L+2}|^2x\,dx\right)^{\frac12}\left(\int_0^y x^2x\,dx\right)^{\frac12}\lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}.\tag{C-16}$$

We now remark that there exists $\frac12<a<2$ such that

$$|\varepsilon_{2L+1}(a)|^2\lesssim\int_{|y|\le1}|\varepsilon_{2L+1}|^2\lesssim C(M)\mathscr{E}_{2L+2}$$

from (C-1). We then define

$$r_2=-\Lambda Q\int_a^y\frac{r_1}{\Lambda Q}\,dx$$

and obtain from (C-16) the pointwise bound, for $y\le1$,

$$|r_2|\lesssim y|\log y|C(M)\sqrt{\mathscr{E}_{2L+2}}.\tag{C-17}$$

Now observe that, by construction, using (2-10),

$$Ar_2=r_1=\varepsilon_{2L+1},\qquad Hr_2=A^*\varepsilon_{2L+1}=\varepsilon_{2L+2}=H\varepsilon_{2L}.\tag{C-18}$$

Now, from (B-24),

$$\int_{y\le1}\frac{|\varepsilon_{2L}|^2}{y^4(1+|\log y|^2)}y\,dy<+\infty,$$

and hence $|\varepsilon_{2L}(y_n)| < +\infty$ on some sequence $y_n \to 0$, and from (C-17), (C-18), the explicit knowledge of the kernel of H, and the singular behavior (2-13), we conclude that there exists $c_2 \in \mathbb{R}$ such that

$$\varepsilon_{2L} = c_2 \Lambda Q + r_2. \tag{C-19}$$

Moreover, there exists $\frac{1}{2} < a < 2$ such that

$$|\varepsilon_{2L}(a)|^2 \lesssim \int_{|y|\le 1} |\varepsilon_{2L}|^2 \lesssim C(M)\mathscr{E}_{2L+2}$$

from (C-1), and thus, from (C-17), (C-19),

$$|c_2| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad |\varepsilon_{2L}| \lesssim y|\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}. \tag{C-20}$$

Induction. We now build by induction the sequence

$$r_{2k+1} = \frac{1}{y\Lambda Q}\int_0^y r_{2k}\Lambda Q x\,dx, \quad r_{2k+2} = -\Lambda Q \int_0^y \frac{r_{2k+1}}{\Lambda Q}\,dx, \quad 1 \le k \le L.$$

We claim by induction that, for all $1 \le k \le L+1$, $\varepsilon_{2L+2-2k}$ admits a Taylor expansion at the origin

$$\varepsilon_{2L+2-2k} = \sum_{i=1}^{k} c_{i,k} T_{k-i} + r_{2k}, \quad 1 \le k \le L+1, \tag{C-21}$$

with the bounds, for $|y| \le 1$,

$$|c_{i,k}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}, \tag{C-22}$$

$$|\mathscr{A}^i r_{2k}| \lesssim |\log y| y^{2k-1-i} C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le i \le 2k-1. \tag{C-23}$$

This follows from (C-19), (C-20), (C-17), (C-18) for $k = 1$. We now let $1 \le k \le L$, assume the claim for k, and prove it for $k+1$.

By construction, using (2-10),

$$A r_{2k+2} = r_{2k+1}, \quad H r_{2k+2} = r_{2k}, \tag{C-24}$$

and thus $\mathscr{A}^i r_{2k} = r_{2k-i}$. In particular, for $i \ge 2$, $\mathscr{A}^{i-2} r_{2k+2} = r_{2k-i}$, and therefore the bounds (C-23) for $k+1$ and $2 \le i \le 2k+1$ follow from the induction claim. We now estimate by definition and induction, for $|y| \le 1$,

$$|A r_{2k+2}| = |r_{2k+1}(y)| = \left|\frac{1}{y\Lambda Q}\int_0^y r_{2k}\Lambda Q x\,dx\right| \lesssim \frac{C(M)\sqrt{\mathscr{E}_{2L+2}}}{y^2} y^{3+2k-1}$$

$$|r_{2k+2}| = \left|\Lambda Q \int_0^y \frac{r_{2k+1}}{\Lambda Q}\,dx\right| \lesssim y y^{2k} C(M)\sqrt{\mathscr{E}_{2L+2}},$$

and (C-23) is proved for $k = 1$ and $i = 0, 1$. From the regularity at the origin (A-13), (C-24), the relation

$$H\varepsilon_{2L+2-2(k+1)} = \varepsilon_{2L+2-2k} = \sum_{i=1}^{k} c_{i,k} T_{k-i} + r_{2k},$$

and the bound (C-23), there exists c_{2k+2} such that

$$\varepsilon_{2L+2-2(k+1)} = \sum_{i=1}^{k} c_{i,k} T_{k+1-i} + c_{2k+2}\Lambda Q + r_{2k+2}.$$

We now observe that there exists $\frac{1}{2} < a < 2$ such that

$$|\varepsilon_{2L-2k}(a)|^2 \lesssim \int_{|y|\le 1} |\varepsilon_{2L-2k}|^2 \lesssim C(M)\mathscr{E}_{2L+2}$$

from (C-1), and thus, using (C-23),

$$|c_{2k+2}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}}.$$

This completes the induction claim.

<u>*Step 3*: *proof of* (ii), (iii), and (iv).</u> We obtain from (C-21), (C-3) with $k = L+1$ the Taylor expansion

$$\varepsilon = \sum_{i=1}^{L+1} c_{i,k} T_{k-i} + r_\varepsilon, \quad r_\varepsilon = r_{2L+2}, \quad |c_{i,k}| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}},$$

where, from (C-23),

$$|\mathscr{A}^i r_\varepsilon| \lesssim |\log y| y^{2L+1-i} C(M)\sqrt{\mathscr{E}_{2L+2}}, \quad 0 \le i \le 2L+1.$$

A brute force computation using the expansions (2-5), (2-6) near the origin ensure that, for any function f,

$$\partial_y^k f = \sum_{i=0}^{k} P_{i,k} \mathscr{A}^i f, \quad |P_{i,k}| \lesssim \frac{1}{y^{k-i}}, \tag{C-25}$$

and we therefore estimate, for $0 \le k \le 2L+1$,

$$|\partial_y^k r_\varepsilon| \lesssim C(M)\sqrt{\mathscr{E}_{2L+2}} \sum_{i=0}^{k} \frac{|\log y| y^{2L+1-i}}{y^{k-i}} \lesssim y^{2L+1-k} |\log y| C(M)\sqrt{\mathscr{E}_{2L+2}}.$$

This concludes the proof of (ii). The estimates of (iii), (iv) now directly follow from (ii) using the Taylor expansion of T_i at the origin given by Lemma 2.3, and (C-16) for (C-7).

<u>*Step* 4: *proof of* (v).</u> We first claim that, for $0 \le k \le L$,

$$\sum_{i=0}^{2k+2} \int \frac{|\partial_y^i \varepsilon|^2}{(1+|\log y|^2)(1+y^{4k-2i+4})} \lesssim C(M)(\mathscr{E}_{2k+2} + \mathscr{E}_{2L+2}). \tag{C-26}$$

Observe that this implies (C-13) by taking $i = 2k+2$.

Indeed, from (C-8), (C-9), we estimate

$$\sum_{i=0}^{2k+1} \int_{y \le 1} \frac{1+|\log y|^C}{1+y^{4k-2i+4}} |\partial_y^i \varepsilon|^2 \lesssim C(M)\mathscr{E}_{2L+2}. \tag{C-27}$$

For $y \geq 1$, we recall from the brute force computation (C-25) that

$$|\partial_y^k \varepsilon| \lesssim \sum_{i=0}^{k} \frac{|\varepsilon_i|}{y^{k-i}}, \tag{C-28}$$

and thus, using (C-1), for $0 \leq k \leq L$,

$$\begin{aligned}\sum_{i=0}^{2k+2} \int_{y\geq 1} \frac{|\partial_y^i \varepsilon|^2}{(1+|\log y|^2)(1+y^{4k-2i+4})} &\lesssim \sum_{i=0}^{2k+2}\sum_{j=0}^{i} \int_{y\geq 1} \frac{|\varepsilon_j|^2}{(1+|\log y|^2)(1+y^{4k-2i+4+2i-2j})} \\ &\lesssim \sum_{j=0}^{2k+2} \int \frac{|\varepsilon_j|^2}{(1+|\log y|^2)(1+y^{4k+4-2j})} \lesssim C(M)\mathscr{E}_{2k+2},\end{aligned}$$

and (C-26) is proved. In particular, together with the energy bound (3-23), this yields the rough Sobolev bound

$$\int \frac{|\varepsilon|^2}{y^2} + \sum_{k=1}^{2L+2} \int \frac{|\partial_y^k \varepsilon|^2}{1+|\log y|^2} \lesssim 1.$$

Therefore, again using (C-26), we estimate

$$\begin{aligned}\sum_{i=0}^{2k+1} \int \frac{1+|\log y|^C}{1+y^{4k-2i+4}} |\partial_y^i \varepsilon|^2 &\lesssim \sum_{i=0}^{2k+1} \left[\int_{y\leq B_0^{100L}} \frac{1+|\log y|^C}{1+y^{4k-2i+4}} |\partial_y^i \varepsilon|^2 + \int_{y\geq B_0^{100L}} \frac{1+|\log y|^C}{y^2} |\partial_y^i \varepsilon|^2\right] \\ &\lesssim |\log b_1|^C \mathscr{E}_{2k+2} + \frac{1}{B_0^{10L}}\end{aligned} \tag{C-29}$$

and (C-11) follows. The estimate (C-10) now follows from (C-5), (C-6), (C-7) for $y \leq 1$ with also (1-31), and (C-11) for $y \geq 1$.

Step 5: *proof of* (vi). Let $i \geq 0$, $j \geq 1$ with $2 \leq i + j \leq 2L+2$.

Case 1: $i + j$ *even*. We have

$$i + j = 2(k+1), \quad 0 \leq k \leq L.$$

For $k \leq L-1$, from (C-11) and $0 \leq i = 2k+2-j \leq 2k+1$, we estimate

$$\int \frac{1+|\log y|^C}{1+y^{2j}} |\partial_y^i \varepsilon|^2 = \int \frac{1+|\log y|^C}{1+y^{4k+4-2i}} |\partial_y^i \varepsilon|^2 \lesssim b_1^{(4k+2)L/(2L-1)} |\log b_1|^C \lesssim b_1^{(i+j-1)2L/(2L-1)} |\log b_1|^C.$$

For $k = L$, from (C-11), we have

$$\int \frac{1+|\log y|^C}{1+y^{2j}} |\partial_y^i \varepsilon|^2 = \int \frac{1+|\log y|^C}{1+y^{4k+4-2i}} |\partial_y^i \varepsilon|^2 \lesssim b_1^{2L+2} |\log b_1|^C.$$

Case 2: *$i+j$ odd.* We have $i+j=2k+1$, $1\le k\le L$. Assume $k\le L-1$. If $j\ge 2$, then $i\le 2k+1-j\le 2(k-1)+1$, and thus, from (C-11),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^{2j}}|\partial_y^i\varepsilon|^2 &= \int \frac{1+|\log y|^C}{1+y^{4k+2-2i}}|\partial_y^i\varepsilon|^2\\ &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^{4k+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\left(\int \frac{1+|\log y|^C}{1+y^{4(k-1)+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{L/(2(2L-1))(4k+2+4(k-1)+2)} = b_1^{(i+j-1)2L/(2L-1)}|\log b_1|^C.\end{aligned}$$

For the extremal case $j=1$, $i=2k$, $1\le k\le L-1$, we estimate, from (C-10), (C-26),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^2}|\partial_y^{2k}\varepsilon|^2 &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^4}|\partial_y^{2k}\varepsilon|^2\right)^{\frac12}\left(\int \frac{|\partial_y^{2k}\varepsilon|^2}{1+|\log y|^2}\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{L/(2(2L-1))(4k+2+4(k-1)+2)} = b_1^{(i+j-1)2L/(2L-1)}|\log b_1|^C.\end{aligned}$$

If $k=L$, then for $j\ge 2$, we have $i\le 2k+1-j\le 2(k-1)+1$, and thus, from (C-11),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^{2j}}|\partial_y^i\varepsilon|^2 &= \int \frac{1+|\log y|^C}{1+y^{4k+2-2i}}|\partial_y^i\varepsilon|^2\\ &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^{4k+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\left(\int \frac{1+|\log y|^C}{1+y^{4(k-1)+4-2i}}|\partial_y^i\varepsilon|^2\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{\frac12\left(2L+2+(4(k-1)+2)L/(2L-1)\right)} = b_1^{2L+1}|\log b_1|^C,\end{aligned}$$

and for $j=1$, $i=2L$, from (C-10), (C-13),

$$\begin{aligned}\int \frac{1+|\log y|^C}{1+y^2}|\partial_y^{2L}\varepsilon|^2 &\lesssim \left(\int \frac{1+|\log y|^C}{1+y^4}|\partial_y^{2L}\varepsilon|^2\right)^{\frac12}\left(\int \frac{|\partial_y^{2L}\varepsilon|^2}{1+|\log y|^2}\right)^{\frac12}\\ &\lesssim |\log b_1|^C b_1^{\frac12\left(2L+2+(4(L-1)+2)L/(2L-1)\right)} = b_1^{2L+1}|\log b_1|^C.\end{aligned}$$

<u>*Step* 6: *proof of* (vii).</u> From Cauchy–Schwarz we estimate

$$\left\|\frac{\varepsilon}{y}\right\|^2_{L^\infty(y\ge1)} \lesssim \int_{y\ge1}|\varepsilon\partial_y\varepsilon|\,dy \lesssim \int \frac{(1+|\log y|^2)|\varepsilon|^2}{y^2}+\int\frac{|\partial_y\varepsilon|^2}{1+|\log y|^2}.$$

Let $i,j\ge0$ with $1\le i+j\le 2L+1$. Then $2\le i+j+1\le 2L$, and we conclude from (C-12), (C-13) that

$$\begin{aligned}\left\|\frac{\partial_y^i\varepsilon}{y^j}\right\|^2_{L^\infty(y\ge1)} &\lesssim \int_{y\ge1}\frac{(1+|\log y|^2)|\partial_y^i\varepsilon|^2}{y^{2j+2}}+\int_{y\ge1}\frac{|\partial_y^{i+1}\varepsilon|^2}{y^{2j}(1+|\log y|^2)}\\ &\lesssim |\log b_1|^C\begin{cases} b_1^{(i+j)2L/(2L-1)} & \text{for } 2\le i+j+1\le 2L,\\ b_1^{2L+1} & \text{for } i+j+1=2L+1,\\ b_1^{2L+2} & \text{for } i+j+1=2L+2.\end{cases}\end{aligned}$$

□

Appendix D: Leibniz rule for H^k

Given a smooth function Φ, we prove the following Leibniz rule.

Lemma D.1 (Leibniz rule for H^k). *Let $k \geq 1$. Then*

$$\begin{aligned}\mathscr{A}^{2k-1}(\Phi\varepsilon) &= \sum_{i=0}^{k-1} \Phi_{2i,2k-1}\varepsilon_{2i} + \sum_{i=1}^{k} \Phi_{2i-1,2k-1}\varepsilon_{2i-1},\\ \mathscr{A}^{2k}(\Phi\varepsilon) &= \sum_{i=0}^{k} \Phi_{2i,2k}\varepsilon_{2i} + \sum_{i=1}^{k} \Phi_{2i-1,2k}\varepsilon_{2i-1},\end{aligned} \tag{D-1}$$

where $\Phi_{i,k}$ is computed through the recurrence relation

$$\Phi_{0,1} = -\partial_y \Phi, \quad \Phi_{1,1} = \Phi \Phi_{0,2} = -\partial_{yy}\Phi - \frac{1+2Z}{y}\partial_y \Phi, \quad \Phi_{1,2} = 2\partial_y\Phi, \quad \Phi_{2,2} = \Phi, \tag{D-2}$$

$$\begin{cases} \Phi_{2k+2,2k+2} = \Phi_{2k+1,2k+1}, \\ \Phi_{2i,2k+2} = \Phi_{2i-1,2k+1} + \partial_y \Phi_{2i,2k+1} + ((1+2Z)/y)\Phi_{2i,2k+1} & 1 \leq i \leq k, \\ \Phi_{0,2k+2} = \partial_y \Phi_{0,2k+1} + (1+2Z)/y\Phi_{0,2k+1}, \\ \Phi_{2i-1,2k+2} = -\Phi_{2i-2,2k+1} + \partial_y \Phi_{2i-1,2k+1}, & 1 \leq i \leq k+1, \end{cases} \tag{D-3}$$

$$\begin{cases} \Phi_{2k+1,2k+1} = \Phi_{2k,2k}, \\ \Phi_{2i-1,2k+1} = \Phi_{2i-2,2k} + ((1+2Z)/y)\Phi_{2i-1,2k} - \partial_y \Phi_{2i-1,2k}, & 1 \leq i \leq k, \\ \Phi_{2i,2k+1} = -\partial_y \Phi_{2i,2k} - \Phi_{2i-1,2k}, & 1 \leq i \leq k, \\ \Phi_{0,2k+1} = -\partial_y \Phi_{0,2k}. \end{cases} \tag{D-4}$$

Proof. We compute

$$\begin{aligned} A(\Phi\varepsilon) &= \Phi\varepsilon_1 - (\partial_y\Phi)\varepsilon, \\ H(\Phi\varepsilon) &= A^*A\varepsilon = \Phi\varepsilon_2 + \partial_y\Phi\varepsilon_1 - \left(-A + \frac{1+2Z}{y}\right)(\partial_y\Phi\varepsilon) \\ &= \Phi\varepsilon_2 + 2\partial_y\Phi\varepsilon_1 + \left[-\partial_{yy}\Phi - \frac{1+2Z}{y}\partial_y\Phi\right]\varepsilon. \end{aligned}$$

and

$$\begin{aligned} &\mathscr{A}^{2k+1}(\Phi\varepsilon) \\ &= \sum_{i=0}^{k} A[\Phi_{2i,2k}\varepsilon_{2i}] + \sum_{i=1}^{k}\left(-A^* + \frac{1+2Z}{y}\right)\Phi_{2i-1,2k}\varepsilon_{2i-1} \\ &= \sum_{i=0}^{k}\{\Phi_{2i,2k}\varepsilon_{2i+1} - \partial_y\Phi_{2i,2k}\varepsilon_{2i}\} + \sum_{i=1}^{k}\left\{-\Phi_{2i-1,2k}\varepsilon_{2i} + \left[\frac{1+2Z}{y}\Phi_{2i-1,2k} - \partial_y\Phi_{2i-1,2k}\right]\varepsilon_{2i-1}\right\} \\ &= -\partial_y\Phi_{0,2k}\varepsilon + \sum_{i=1}^{k}(-\partial_y\Phi_{2i,2k} - \Phi_{2i-1,2k})\varepsilon_{2i} + \sum_{i=1}^{k}\left\{\Phi_{2i-2,2k} + \frac{1+2Z}{y}\Phi_{2i-1,2k} - \partial_y\Phi_{2i-1,2k}\right\}\varepsilon_{2i-1}, \\ &\qquad + \Phi_{2k,2k}\varepsilon_{2k+1}, \end{aligned}$$

which is (D-4). We then compute

$$
\begin{aligned}
&\mathscr{A}^{2k+2}(\Phi\varepsilon)\\
&= \sum_{i=0}^{k}\Big[-A+\frac{1+2Z}{y}\Big]\{\Phi_{2i,2k+1}\varepsilon_{2i}\}+\sum_{1=i}^{k+1}A^*(\Phi_{2i-1,2k+1}\varepsilon_{2i-1})\\
&= \sum_{i=0}^{k}\Big\{-\Phi_{2i,2k+1}\varepsilon_{2i+1}+\Big[\partial_y\Phi_{2i,2k+1}+\frac{1+2Z}{y}\Phi_{2i,2k+1}\Big]\varepsilon_{2i}\Big\}+\sum_{i=1}^{k+1}\{\Phi_{2i-1,2k+1}\varepsilon_{2i}+\partial_y\Phi_{2i-1,2k+1}\varepsilon_{2i-1}\}\\
&= \Big[\partial_y\Phi_{0,2k+1}+\frac{1+2Z}{y}\Phi_{0,2k+1}\Big]\varepsilon+\Phi_{2k+1,2k+1}\varepsilon_{2k+2}\\
&\qquad+\sum_{i=1}^{k}\Big[\Phi_{2i-1,2k+1}+\partial_y\Phi_{2i,2k+1}+\frac{1+2Z}{y}\Phi_{2i,2k+1}\Big]\varepsilon_{2i}+\sum_{i=1}^{k+1}\big[-\Phi_{2i-2,2k+1}+\partial_y\Phi_{2i-1,2k+1}\big]\varepsilon_{2i-1},
\end{aligned}
$$

which is (D-3). □

Appendix E: Proof of (3-55)

A simple induction argument ensures the formula

$$[\partial_t, H_\lambda^L]w = \sum_{k=0}^{L-1} H_\lambda^k[\partial_t, H_\lambda]H_\lambda^{L-(k+1)}w.$$

We therefore renormalize and explicitly compute

$$[\partial_t, H_\lambda^L]w = \frac{1}{\lambda^{2L+2}}\sum_{k=0}^{L-1} H^k\Big(-\frac{\lambda_s}{\lambda}\frac{\Lambda V}{y^2}H^{L-(k+1)}\varepsilon\Big). \tag{E-1}$$

We now apply the Leibniz rule Lemma D.1 with $\Phi = \Lambda V/y^2$. In view of the expansion (2-6) and the recurrence formula (D-3), we have an expansion at the origin to all orders, for even $k \geq 2$,

$$
\begin{cases}
\Phi_{2i,2k}(y) = \sum_{p=0}^{N} c_{i,k,p}y^{2p} + O(y^{2N+2}), & 0 \leq i \leq k,\\
\Phi_{2i+1,2k}(y) = \sum_{p=0}^{N} c_{i,k,p}y^{2p+1} + O(y^{2N+3}), & 1 \leq i \leq k-1.
\end{cases}
$$

and, for odd $k \geq 1$,

$$
\begin{cases}
\Phi_{2i-1,2k+1}(y) = \sum_{p=0}^{N} c_{i,k,p}y^{2p} + O(y^{2N+2}), & 1 \leq i \leq k+1,\\
\Phi_{2i,2k+1}(y) = \sum_{p=0}^{N} c_{i,k,p}y^{2p+1} + O(y^{2N+3}), & 1 \leq i \leq k-1.
\end{cases}
$$

We also have a bound, for $y \geq 1$,

$$|\Phi_{i,k}| \lesssim \frac{1}{1+y^{4+(2k-i)}}, \quad 0 \leq i \leq 2k.$$

Therefore, from (D-1), we estimate

$$\text{for all } k \geq 0, \quad \left| H^k\left(\frac{\Lambda V}{y^2}\varepsilon\right)\right| \leq \sum_{i=0}^{2k} c_{i,k} \frac{|\varepsilon_i|}{1+y^{4+(2k-i)}}. \tag{E-2}$$

Similarly,

$$\left| AH^k\left(\frac{\Lambda V}{y^2}\varepsilon\right)\right| \lesssim \sum_{i=0}^{2k} c_{i,k} \frac{1}{1+y^{4+(2k-i)}}\left[|\partial_y \varepsilon_i| + \frac{|\varepsilon_i|}{y}\right] \lesssim \sum_{i=0}^{2k+1} c_{i,k} \frac{|\varepsilon_i|}{y(1+y^{4+(2k-i)})}.$$

We now inject (E-2) into (E-1) and obtain using (3-36) the pointwise bound on the commutator

$$|[\partial_t, H_\lambda^L]w| \lesssim \frac{|b_1|}{\lambda^{2L+2}} \sum_{k=0}^{L-1}\sum_{i=0}^{2k} c_{i,k} \frac{|\varepsilon_{2(L-k-1)+i}|}{1+y^{4+(2k-i)}} \lesssim \frac{|b_1|}{\lambda^{2L+2}} \sum_{m=0}^{2L-2} \frac{|\varepsilon_{2L-2-m}|}{1+y^{4+m}} = \frac{|b_1|}{\lambda^{2L+2}} \sum_{m=0}^{2L-2} \frac{|\varepsilon_m|}{1+y^{2+2L-m}}.$$

Hence, after a change of variables in the integral, and using (C-1), we have

$$\int \frac{|[\partial_t, H_\lambda^L]w|^2}{\lambda^2(1+y^2)} \lesssim \frac{|b_1|^2}{\lambda^{4L+4}} \sum_{m=0}^{2L-2} \int \frac{\varepsilon_m^2}{(1+y^2)(1+y^{4+4L-2m})} \lesssim \frac{C(M)b_1^2}{\lambda^{4L+4}} \mathscr{E}_{2L+2},$$

and, similarly,

$$\int |A_\lambda[\partial_t, H_\lambda^L]w|^2 \lesssim \frac{|b_1|^2}{\lambda^{4L+4}} \sum_{m=0}^{2L-1} \int \frac{\varepsilon_m^2}{y^2(1+y^{4+4L-2m})} \lesssim \frac{C(M)b_1^2}{\lambda^{4L+4}} \mathscr{E}_{2L+2},$$

which is (3-55).

Appendix F: Proof of (4-10)

We claim the following Lyapounov monotonicity functional for the $\mathscr{E}_{2k+2}$ energy.

Proposition F.1 (Lyapounov monotonicity for $\mathscr{E}_{2k+2}$). *Let $0 \leq k \leq L-1$. Then we have*

$$\frac{d}{dt}\left\{\frac{1}{\lambda^{4k+2}}\left[\mathscr{E}_{2k+2} + O\left(b_1^{\frac{1}{2}} b_1^{(4k+2)2L/(2L-1)}\right)\right]\right\} \lesssim \frac{|\log b_1|^C}{\lambda^{4k+4}}\left[b_1^{2k+3} + b_1^{1+\delta+(2k+1)2L/(2L-1)} + \sqrt{b_1^{2k+4}\mathscr{E}_{2k+2}}\right] \tag{F-1}$$

for some universal constants $C, \delta > 0$ independent of M and of the bootstrap constant K in (3-23), (3-24).

Proof of Proposition F.1. Step 1: *modified energy identity*. We follow verbatim the algebra of (3-48) with $L \to k$ and obtain the modified energy identity

$$\frac{1}{2}\frac{d}{dt}\left\{\mathscr{E}_{2k+2} + 2\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k+1} w_{2k}\right\} = -\int (\widetilde{H}_\lambda w_{2k+1})^2 - \left(\frac{\lambda_s}{\lambda} + b_1\right)\int \frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2k+1}^2 + \int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2k+1} w_{2k} + \int \widetilde{H}_\lambda w_{2k+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2k} + A_\lambda([\partial_t, H_\lambda^k] w) + A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right] + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k}\left[-\widetilde{H}_\lambda w_{2k+1} + \frac{\partial_t Z_\lambda}{r} w_{2k} + A_\lambda([\partial_t, H_\lambda^k] w) + A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right] + \int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2L+1}\left[[\partial_t, H_\lambda^k] w + H_\lambda^k\left(\frac{1}{\lambda^2}\mathscr{F}_\lambda\right)\right]. \tag{F-2}$$

We now estimate all terms in the right hand side of (F-2).

Step 3: *Lower order quadratic terms*. We treat the lower order quadratic terms in (F-2) using dissipation. The bound

$$\int \frac{([\partial_t, H_\lambda^k] w)^2}{\lambda^2(1+y^2)} + \int |A_\lambda([\partial_t, H_\lambda^k] w)|^2 \lesssim C(M)\frac{b_1^2}{\lambda^{4k+4}}\mathscr{E}_{2k+2} \tag{F-3}$$

follows from (3-55) with $L \to k$. From (3-54), the rough bound (3-38), and Lemma C.1, we estimate

$$\int \left|\widetilde{H}_\lambda w_{2k+1}\left[\frac{\partial_t Z_\lambda}{r} w_{2k} + \int A_\lambda\Big([\partial_t, H_\lambda^k] w\Big)\right]\right| + \int |\widetilde{H}_\lambda w_{2k+1}|\left|\frac{b(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k}\right| \le \frac{1}{2}\int |\widetilde{H}_\lambda w_{2k+1}|^2 + \frac{b_1^2}{\lambda^{4k+4}}\left[\int \frac{\varepsilon_{2k}^2}{1+y^6} + C(M)\mathscr{E}_{2k+2}\right] \le \frac{1}{2}\int |\widetilde{H}_\lambda w_{2k+1}|^2 + \frac{b_1}{\lambda^{4k+4}} C(M) b_1 \mathscr{E}_{2k+2}.$$

All other quadratic terms are lower order by a factor b_1, again using (3-38), (3-55), (3-36), and Lemma C.1:

$$\left|\frac{\lambda_s}{\lambda} + b_1\right|\int\left|\frac{(\Lambda \widetilde{V})_\lambda}{2\lambda^2 r^2} w_{2k+1}^2\right| + \int\left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k}\left[\frac{\partial_t Z_\lambda}{r} w_{2k} + A_\lambda\Big([\partial_t, H_\lambda^k] w\Big)\right]\right| + \int\left|\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k+1}[\partial_t, H_\lambda^L] w\right| + \left|\int \frac{d}{dt}\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}\right) w_{2k+1} w_{2k}\right| \lesssim \frac{b_1^2}{\lambda^{4k+4}}\left[\int \frac{\varepsilon_{2k+1}^2}{1+y^4} + \int \frac{\varepsilon_{2k}^2}{1+y^6} + C(M)\mathscr{E}_{2k+2}\right] \lesssim \frac{b_1}{\lambda^{4k+4}} C(M) b_1 \mathscr{E}_{2k+2}.$$

We similarly estimate the boundary term in time using (C-10):

$$\left|\int \frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r} w_{2k+1} w_{2k}\right| \lesssim \frac{b_1}{\lambda^{4k+2}}\left[\int \frac{\varepsilon_{2k+1}^2}{1+y^2} + \int \frac{\varepsilon_{2k}^2}{1+y^4}\right] \lesssim \frac{b_1}{\lambda^{4k+2}} |\log b_1|^C b_1^{(4k+2)2L/(2L-1)}.$$

We inject these estimates into (3-53) to derive the preliminary bound

$$\frac{1}{2}\frac{d}{dt}\left\{\frac{1}{\lambda^{4k+2}}\left[\mathcal{E}_{2k+2}+O\left(b_1^{\frac{1}{2}}b_1^{(4k+2)2L/(2L-1)}\right)\right]\right\}\leq -\frac{1}{2}\int(\widetilde{H}_\lambda w_{2k+1})^2+\int \widetilde{H}_\lambda w_{2k+1}A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right) + \int H_\lambda^k\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\left[\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k+1}+A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k}\right)\right]+\frac{b_1}{\lambda^{4k+4}}b_1^\delta\mathcal{E}_{2k+2} \tag{F-4}$$

with constants independent of M for $|b|<b^*(M)$ small enough. We now estimate all terms in the right hand side of (F-4).

<u>*Step* 4: *further use of dissipation*</u>. Recall the decomposition (3-57). The first term in the right hand side of (F-4) is estimated after an integration by parts

$$\begin{aligned}\left|\int \widetilde{H}_\lambda w_{2k+1}A_\lambda H_\lambda^k\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\right| &\leq \frac{C}{\lambda^{4k+4}}\|A^*\varepsilon_{2k+1}\|_{L^2}\|H^{k+1}\mathcal{F}_0\|_{L^2}+\frac{1}{4}\int|\widetilde{H}_\lambda w_{2k+1}|^2+\frac{C}{\lambda^{4k+4}}\int|AH^k\mathcal{F}_1|^2\\ &\leq \frac{C}{\lambda^{4k+4}}\left[\|H^{k+1}\mathcal{F}_0\|_{L^2}\sqrt{\mathcal{E}_{2k+2}}+\|AH^k\mathcal{F}_1\|_{L^2}^2\right]+\frac{1}{4}\int|\widetilde{H}_\lambda w_{2k+1}|^2\end{aligned} \tag{F-5}$$

for some universal constant $C>0$ independent of M. The last two terms in (F-4) can be estimated by brute force from Cauchy–Schwarz

$$\begin{aligned}\left|\int H_\lambda^k\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k+1}\right| &\lesssim \frac{b_1}{\lambda^{4k+4}}\left(\int\frac{1+|\log y|^2}{1+y^4}|H^k\mathcal{F}|^2\right)^{\frac{1}{2}}\left(\int\frac{\varepsilon_{2k+1}^2}{y^2(1+|\log y|^2)}\right)^{\frac{1}{2}}\\ &\lesssim \frac{b_1}{\lambda^{4k+4}}\sqrt{\mathcal{E}_{2k+2}}\left(\int\frac{1+|\log y|^2}{1+y^4}|H^k\mathcal{F}|^2\right)^{\frac{1}{2}},\end{aligned} \tag{F-6}$$

where constants are independent of M thanks to the estimate (B-2) for ε_{2k+1}. Similarly,

$$\begin{aligned}&\left|\int H_\lambda^k\left(\frac{1}{\lambda^2}\mathcal{F}_\lambda\right)A_\lambda^*\left(\frac{b_1(\Lambda Z)_\lambda}{\lambda^2 r}w_{2k}\right)\right|\\ &\qquad\lesssim \frac{b_1}{\lambda^{4k+4}}\left(\int\frac{1+|\log y|^2}{1+y^2}|AH^k\mathcal{F}|^2\right)^{\frac{1}{2}}\left(\int\frac{\varepsilon_{2k}^2}{(1+y^4)(1+|\log y|^2)}\right)^{\frac{1}{2}}\\ &\qquad\lesssim \frac{b_1}{\lambda^{4k+4}}C(M)\sqrt{\mathcal{E}_{2k+2}}\left(\int\frac{1+|\log y|^2}{1+y^2}|AH^k\mathcal{F}_0|^2+\int|AH^k\mathcal{F}_1|^2\right)^{\frac{1}{2}}.\end{aligned} \tag{F-7}$$

We now claim the bounds

$$\int\frac{1+|\log y|^2}{1+y^4}|H^k\mathcal{F}|^2\leq b_1^{2k+2}|\log b_1|^C, \tag{F-8}$$

$$\int\frac{1+|\log y|^2}{1+y^2}|AH^k\mathcal{F}_0|^2\leq b_1^{2k+2}|\log b_1|^C, \tag{F-9}$$

$$\int|H^{k+1}\mathcal{F}_0|^2\leq b_1^{2k+4}|\log b_1|^C, \tag{F-10}$$

$$\int|AH^k\mathcal{F}_1|^2\leq b_1^{2k+3}|\log b_1|^C+b_1^{1+\delta+(2k+1)2L/(2L-1)} \tag{F-11}$$

for some universal constants $\delta, C > 0$ independent of M and of the bootstrap constant K in (3-23), (3-24). Injecting these bounds together with (F-5), (F-6), (F-7) into (F-4) concludes the proof of (F-1). We now turn to the proofs of (F-8), (F-9), (F-10), (F-11).

Step 5: $\widetilde{\Psi}_b$ *terms*. From (2-88) we estimate

$$\int \frac{1+|\log y|^2}{1+y^4}|H^k\widetilde{\Psi}_b|^2 \lesssim \int |H^k\widetilde{\Psi}_b|^2 \lesssim b_1^{2k+2}|\log b_1|^C,$$

$$\int \frac{1+|\log y|^2}{1+y^2}|AH^k\widetilde{\Psi}_b|^2 \lesssim \int |AH^k\widetilde{\Psi}_b|^2 = \int H^k\widetilde{\Psi}_b H^{k+1}\widetilde{\Psi}_b \lesssim b_1^{2k+3}|\log b_1|^C,$$

$$\int |H^{k+1}\widetilde{\Psi}_b|^2 \lesssim b_1^{2(k+1)+2}|\log b_1|^C,$$

and (F-8), (F-9), (F-10) are proved for $\widetilde{\Psi}_b$.

Step 6: $\widetilde{\mathrm{Mod}(t)}$ *terms*. Recall (3-29),

$$\widetilde{\mathrm{Mod}}(t) = -\left(\frac{\lambda_s}{\lambda}+b_1\right)\Lambda\widetilde{Q}_b + \sum_{i=1}^{L}[(b_i)_s + (2i-1+c_{b_1})b_1b_i - b_{i+1}]\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right],$$

and the notation (3-39). We will need only the rough bound for b_1-admissible functions (2-32).

Proof of (F-10) *for* $\widetilde{\mathrm{Mod}}$. We estimate from (2-32), for $y \le 2B_1$,

$$|H^{k+1}S_i| + |H^{k+1}\Lambda S_i| + |H^{k+1}b_i\Lambda\widetilde{T}_i| \lesssim b_1^i(1+y)^{2i-1-(2k+2)} \lesssim b_1 b_1^{i-1}(1+y)^{2i-2k-3} \lesssim \frac{b_1|\log b_1|^C}{1+y^{2k+1}},$$

and thus, using $H\Lambda Q = 0$,

$$\int |H^{k+1}\Lambda\widetilde{Q}_b|^2 \lesssim \int_{y\le 2B_1}\frac{b_1^2|\log b_1|^C}{1+y^{4k+2}} \lesssim b_1^2|\log b_1|^C.$$

We also have the rough bound, for $1 \le i \le L$, $i+1 \le j \le L_2$, $y \le 2B_1$,

$$|\widetilde{T}_i| + \left|\chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right| \lesssim |\log b_1|^C\left[y^{2i-1} + y^{2j-1}b_1^{j-i}|\log b_1|^C\right] \lesssim |\log b_1|^C y^{2i-1}, \tag{F-12}$$

and similarly for suitable derivatives, and hence the bound

$$\sum_{i=1}^{L}\int\left|H^{k+1}\left[\widetilde{T}_i + \chi_{B_1}\sum_{j=i+1}^{L+2}\frac{\partial S_j}{\partial b_i}\right]\right|^2 \lesssim |\log b_1|^C\int_{y\le 2B_1}|y^{2L-1-(2k+2)}|^2$$
$$\lesssim |\log b_1|^C B_1^{4(L-k)-4} \lesssim \frac{|\log b_1|^C}{b_1^{2(L-k)-2}}.$$

We therefore obtain from Lemma 3.3 the control

$$\int |H^{k+1}\widetilde{\mathrm{Mod}(t)}|^2 \lesssim C(K)|\log b_1|^C b_1^{2L+2}\left[b_1^2 + \frac{1}{b_1^{2(L-k)-2}}\right] \lesssim C(K)b_1^{2k+4}|\log b_1|^C \lesssim |\log b_1|^C b_1^{2k+4}$$

for $b_1 < b_1^*(M)$ small enough.

Proof of (F-8), (F-9). We estimate

$$|H^k S_i| + |H^k \Lambda S_i| + |H^k b_i \Lambda \widetilde{T}_i| \lesssim b_1^i (1+y)^{2i-1-2k} \lesssim \frac{|\log b_1|^C}{1+y^{2k+1}},$$

and thus

$$\int \frac{1+|\log y|^2}{1+y^4} |H^k \Lambda \widetilde{Q}_b|^2 + \int \frac{1+|\log y|^2}{1+y^2} |AH^k \Lambda \widetilde{Q}_b|^2 \lesssim |\log b_1|^C \int \frac{1}{1+y^{4k+2}} \lesssim |\log b_1|^C.$$

Then, from (F-12), we estimate

$$\sum_{i=1}^{L} \int \frac{1+|\log y|^2}{1+y^4} \left| H^k \left[\widetilde{T}_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2 + \sum_{i=1}^{L} \int \frac{1+|\log y|^2}{1+y^2} \left| AH^k \left[\widetilde{T}_i + \chi_{B_1} \sum_{j=i+1}^{L+2} \frac{\partial S_j}{\partial b_i} \right] \right|^2$$
$$\lesssim |\log b_1|^C \int_{y \le 2B_1} |y^{2L-1-2k-2}|^2 \lesssim \frac{|\log b_1|^C}{b_1^{2(L-k)-2}},$$

and hence, using Lemma 3.3, we have

$$\int \frac{1+|\log y|^2}{1+y^4} |H^k \widetilde{\mathrm{Mod}}|^2 + \int \frac{1+|\log y|^2}{1+y^2} |AH^k \widetilde{\mathrm{Mod}}|^2 \lesssim |\log b_1|^C C(K) b_1^{2L+2} \left[1 + \frac{1}{b_1^{2(L-k)-2}} \right] \lesssim b_1^{2k+2}.$$

Step 7: *nonlinear term $N(\varepsilon)$. Control near the origin $y \le 1$.* The control near the origin follows directly from (3-72).

Control for $y \ge 1$. We detail the proof of the most delicate bound (F-11). The proofs of (F-8) and (F-9) follow similar lines and are left to the reader.

Recall the notations (3-73) and the bounds (3-74), (3-75), (3-76) on ζ. We then have the bounds (3-77), (3-78), (3-79) on $N_1(\varepsilon)$, which yield

$$\begin{aligned} |AH^k N(\varepsilon)| &\lesssim \sum_{p=0}^{2k+1} \frac{|\partial_y^p N(\varepsilon)|}{y^{2k+1-p}} \lesssim \sum_{p=0}^{2k+1} \frac{1}{y^{2k+1-p}} \sum_{i=0}^{p} |\partial_y^i \zeta^2| |\partial_y^{p-i} N_1(\varepsilon)| \\ &\lesssim \sum_{p=0}^{2k+1} \frac{|\partial_y^k \zeta^2|}{y^{2k+1-p}} + \sum_{p=1}^{2k+1} \frac{1}{y^{2k+1-p}} \sum_{i=0}^{p-1} |\partial_y^i \zeta^2| |\log b_1|^C \left[\frac{1}{y^{p-i+1}} + b_1^{a_{p-i}/2} \right] \\ &\lesssim \sum_{p=0}^{2k+1} \frac{|\partial_y^p \zeta^2|}{y^{2k+1-p}} + |\log b_1|^C \sum_{i=0}^{2k} \frac{|\partial_y^i \zeta^2|}{y^{2k+2-i}} + |\log b_1|^C \sum_{p=1}^{2k+1} \sum_{i=0}^{p-1} b_1^{a_{p-i}/2} \frac{|\partial_y^i \zeta^2|}{y^{2k+1-p}} \\ &\lesssim |\log b_1|^C \left[\sum_{p=0}^{2k+1} \frac{|\partial_y^p \zeta^2|}{y^{2k+1-p}} + \sum_{p=1}^{2k+1} \sum_{i=0}^{p-1} b_1^{a_{p-i}/2} \frac{|\partial_y^i \zeta^2|}{y^{2k+1-p}} \right], \end{aligned}$$

and hence

$$\int_{y \ge 1} |AH^k N(\varepsilon)|^2$$
$$\lesssim |\log b_1|^C \sum_{p=0}^{2k+1} \sum_{i=0}^{p} \int_{y \ge 1} \frac{|\partial_y^i \zeta|^2 |\partial_y^{p-i} \zeta|^2}{y^{4L+2-2p}} + |\log b_1|^C \sum_{p=1}^{2k+1} \sum_{i=0}^{p-1} \sum_{j=0}^{i} b_1^{a_{p-i}} \int_{y \ge 1} \frac{|\partial_y^j \zeta|^2 |\partial_y^{i-j} \zeta|^2}{y^{4L+2-2p}}.$$

We now claim the bounds

$$\sum_{p=0}^{2k+1}\sum_{i=0}^{p}\int_{y\geq 1}\frac{|\partial_y^i\zeta|^2|\partial_y^{p-i}\zeta|^2}{y^{4k+2-2p}}\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)}, \tag{F-13}$$

$$|\log b_1|^C\sum_{p=1}^{2k+1}\sum_{i=0}^{p-1}\sum_{j=0}^{i} b_1^{a_{p-i}}\int_{y\geq 1}\frac{|\partial_y^j\zeta|^2|\partial_y^{i-j}\zeta|^2}{y^{4k+2-2p}}\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)} \tag{F-14}$$

for some $\delta > 0$, and this concludes the proof of (F-11) for $N(\varepsilon)$.

Proof of (3-80). Let $0\leq k\leq L-1$, $0\leq p\leq 2k+1$, $0\leq i\leq p$. Let $I_1 = p-i$, $I_2 = i$. Then we can pick $J_2\in\mathbb{N}^*$ such that

$$\max\{1; 2-i\}\leq J_2\leq \min\{2k+3-p; 2k+2-i\}$$

and define

$$J_1 = 2k+3-p-J_2.$$

Then, from direct inspection,

$$(I_1, J_1, I_2, J_2)\in\mathbb{N}^3\times\mathbb{N}^*,\qquad \begin{cases}1\leq I_1+J_1\leq 2k+1\leq 2L-1, & 2\leq I_2+J_2\leq 2k+2\leq 2L,\\ I_1+I_2+J_1+J_2 = 2k+3.\end{cases}$$

Hence, from (3-74), (3-75),

$$\begin{aligned}\int_{y\geq 1}\frac{|\partial_y^i\zeta|^2|\partial_y^{p-i}\zeta|^2}{y^{4k+2-2p}} &\lesssim \left\|\frac{\partial_y^{I_1}\zeta}{y^{J_1-1}}\right\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}\frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}}\\ &\lesssim |\log b_1|^{C(K)} b_1^{(I_1+J_1+I_2+J_2-1)2L/(2L-1)} = |\log b_1|^{C(K)} b_1^{(2k+2)2L/(2L-1)}\\ &\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)}.\end{aligned}$$

□

Proof of (3-81). Let $0\leq k\leq L-1$, $1\leq p\leq 2k+1$, $0\leq j\leq i\leq p-1$. For $p = 2k+1$ and $0\leq i = j\leq 2k$, we use the energy bound (3-76) to estimate

$$\begin{aligned}b_1^{a_{p-i}}\int_{y\geq 1}\frac{|\partial_y^j\zeta|^2|\partial_y^{i-j}\zeta|^2}{y^{4k+2-2p}} &= b_1^{a_{2k+1-i}}\|\zeta\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}|\partial_y^i\zeta|^2\\ &\lesssim b_1^{2L/(2L-1)((2k+1-i)+1+i)}|\log b_1|^{C(K)}\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)}.\end{aligned}$$

This exceptional case being treated, we let $I_1 = j$, $I_2 = i-j$ and pick $J_2\in\mathbb{N}^*$ with

$$\max\{1; 2-(i-j); 2-(p-j)\}\leq J_2\leq \min\{2k+3-p; 2k+2-(p-j); 2k+2-(i-j)\}.$$

Let

$$J_1 = 2k+3-p-J_2.$$

Then we can directly check that

$$(I_1, J_1, I_2, J_2)\in\mathbb{N}^3\times\mathbb{N}^*,\qquad \begin{cases}1\leq I_1+J_1\leq 2k+1, & 2\leq I_2+J_2\leq 2k+2,\\ I_1+I_2+J_1+J_2 = 2k+3-(p-i),\end{cases}$$

and thus

$$\begin{aligned}
b_1^{a_{p-i}}\int_{y\geq 1}\frac{|\partial_y^j\zeta|^2|\partial_y^{i-j}\zeta|^2}{y^{4k+2-2p}} &\lesssim b_1^{a_{p-i}}\left\|\frac{\partial_y^{I_1}\zeta}{y^{J_1-1}}\right\|^2_{L^\infty(y\geq 1)}\int_{y\geq 1}\frac{|\partial_y^{I_2}\zeta|^2}{y^{2J_2-2}}\\
&\lesssim |\log b_1|^{C(K)}b_1^{(p-i+I_1+J_1+I_2+J_2-1)2L/(2L-1)} = |\log b_1|^{C(K)}b_1^{(2k+2)2L/(2L-1)}\\
&\leq b_1 b_1^{\delta} b_1^{(2k+1)2L/(2L-1)}. \qquad \square
\end{aligned}$$

Step 8: *small linear term* $L(\varepsilon)$. We recall the decomposition (3-84).

Control for $y\leq 1$. The control near the origin directly follows from (3-88).

Control for $y\geq 1$. We give a detailed proof of (F-11) and leave (F-8) to the reader. We recall the bound (3-90):

$$|\partial_y^k L(\varepsilon)|\lesssim\sum_{i=0}^{k}\frac{b_1|\log b_1|^C|\partial_y^i\varepsilon|}{y^{k-i+1}}.$$

This implies

$$|AH^kL(\varepsilon)|\lesssim\sum_{p=0}^{2k+1}\frac{|\partial_y^pL(\varepsilon)|}{y^{2k+1-p}}\lesssim\sum_{p=0}^{2k+1}\frac{1}{y^{2k+1-p}}\sum_{i=0}^{p}\frac{b_1|\log b_1|^C|\partial_y^i\varepsilon|}{y^{p-i+1}}\lesssim b_1|\log b_1|^C\sum_{i=0}^{2k+1}\frac{|\partial_y^i\varepsilon|}{y^{2k+2-i}}.$$

We therefore conclude from (C-11) that

$$\begin{aligned}
\int_{y\geq 1}|AH^kL(\varepsilon)|^2 &\lesssim b_1^2|\log b_1|^C\sum_{i=0}^{2k+1}\int_{y\geq 1}\frac{|\partial_y^i\varepsilon|^2}{y^{4k+4-2i}}\\
&\lesssim |\log b_1|^{C(K)}b_1^{2+(2k+1)2L/(2L-1)}\leq b_1^{1+\delta+(2k+1)2L/(2L-1)},
\end{aligned}$$

and (3-64) is proved.

This concludes the proof of (F-8), (F-9), (F-10), (F-11), and thus of Proposition 3.6. □

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PIERRE RAPHAËL: praphael@unice.fr
Laboratoire J. A. Dieudonné, Université de Nice Sophia Antipolis, Institut Universitaire de France, 06000 Nice, France

REMI SCHWEYER: remi.schweyer@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse, Université Paul Sabatier, 31000 Toulouse, France

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EXISTENCE AND ORBITAL STABILITY OF THE GROUND STATES WITH PRESCRIBED MASS FOR THE L^2-CRITICAL AND SUPERCRITICAL NLS ON BOUNDED DOMAINS

BENEDETTA NORIS, HUGO TAVARES AND GIANMARIA VERZINI

Given $\rho > 0$, we study the elliptic problem

$$\text{find } (U, \lambda) \in H_0^1(B_1) \times \mathbb{R} \text{ such that } \begin{cases} -\Delta U + \lambda U = U^p, \\ \int_{B_1} U^2\,dx = \rho, \end{cases} \quad U > 0,$$

where $B_1 \subset \mathbb{R}^N$ is the unitary ball and p is Sobolev-subcritical. Such a problem arises in the search for solitary wave solutions for nonlinear Schrödinger equations (NLS) with power nonlinearity on bounded domains. Necessary and sufficient conditions (about ρ, N and p) are provided for the existence of solutions. Moreover, we show that standing waves associated to least energy solutions are orbitally stable for every ρ (in the existence range) when p is L^2-critical and subcritical, i.e., $1 < p \le 1 + 4/N$, while they are stable for almost every ρ in the L^2-supercritical regime $1 + 4/N < p < 2^* - 1$. The proofs are obtained in connection with the study of a variational problem with two constraints of independent interest: to maximize the L^{p+1}-norm among functions having prescribed L^2- and H_0^1-norms.

1. Introduction

In this paper, we study standing wave solutions of the nonlinear Schrödinger equation (NLS)

$$\begin{cases} i\dfrac{\partial \Phi}{\partial t} + \Delta\Phi + |\Phi|^{p-1}\Phi = 0, & (t, x) \in \mathbb{R} \times B_1, \\ \Phi(t, x) = 0, & (t, x) \in \mathbb{R} \times \partial B_1 \end{cases} \tag{1-1}$$

with B_1 the unitary ball of $\mathbb{R}^N$, $N \ge 1$, and $1 < p < 2^* - 1$, where $2^* = \infty$ if $N = 1, 2$ and $2^* = 2N/(N-2)$ otherwise. In what follows, p is always subcritical for the Sobolev immersion while criticality will be understood in the L^2-sense; see below. The main tool in our investigation will be the analysis of the variational problem

$$\max\left\{ \int_\Omega |u|^{p+1}\,dx : u \in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx = \alpha \right\}$$

and in particular of its asymptotic properties in dependence of the parameter α. As we will show, when the bounded domain $\Omega \subset \mathbb{R}^N$ is chosen to be B_1, the two problems are strongly related.

NLS on bounded domains appear in different physical contexts. For instance, in nonlinear optics, with $N = 2$ and $p = 3$, they describe the propagation of laser beams in hollow-core fibers [Agrawal

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2013; Fibich and Merle 2001]. In Bose–Einstein condensation, when $N \leq 3$ and $p = 3$, they model the presence of an infinite well-trapping potential [Bartsch and Parnet 2014]. When considered in the whole space $\mathbb{R}^N$, this equation admits the L^2-critical exponent $p = 1 + 4/N$; indeed, in the subcritical case $1 < p < 1 + 4/N$, ground state solutions are orbitally stable while in the critical and supercritical one they are always unstable [Cazenave and Lions 1982; Cazenave 2003]. Notice that the exponent $p = 3$ is subcritical when $N = 1$, critical when $N = 2$ and supercritical when $N = 3$. In the case of a bounded domain, only a few papers analyze the effect of boundary conditions on stability, namely [Fibich and Merle 2001] and the more recent [Fukuizumi et al. 2012] by Fukuizumi, Selem and Kikuchi. In these papers, it is proved that also in the critical and supercritical cases there exist standing waves that are orbitally stable (even though a full classification is not provided, even in the subcritical range). This shows that the presence of the boundary has a stabilizing effect.

As is well known, two quantities are conserved along trajectories of (1-1): the energy

$$\mathscr{E}(\Phi) = \int_{B_1} \left(\frac{1}{2} |\nabla \Phi|^2 - \frac{1}{p+1} |\Phi|^{p+1} \right) dx$$

and the mass

$$\mathscr{Q}(\Phi) = \int_{B_1} |\Phi|^2 \, dx.$$

A standing wave is a solution of the form $\Phi(t, x) = e^{i\lambda t} U(x)$, where the real-valued function U solves the elliptic problem

$$\begin{cases} -\Delta U + \lambda U = |U|^{p-1} U & \text{in } B_1, \\ U = 0 & \text{on } \partial B_1. \end{cases} \tag{1-2}$$

In (1-2), one can either consider the chemical potential $\lambda \in \mathbb{R}$ to be given or to be an unknown of the problem. In the latter case, it is natural to prescribe the value of the mass so that λ can be interpreted as a Lagrange multiplier.

Among all possible standing waves, typically the most relevant are ground state solutions. In the literature, the two points of view mentioned above lead to different definitions of ground state; see for instance [Adami et al. 2013]. When λ is prescribed, ground states can be defined as minimizers of the action functional

$$\mathscr{A}_\lambda(\Phi) = \mathscr{E}(\Phi) + \tfrac{1}{2} \lambda \mathscr{Q}(\Phi)$$

among its nontrivial critical points (recall that $\mathscr{A}_\lambda$ is not bounded from below); see for instance [Berestycki and Lions 1983, p. 316]. Equivalently, they can be defined as minimizers of $\mathscr{A}_\lambda$ on the associated Nehari manifold. Even though these solutions of (1-2) are sometimes called least energy solutions, we will refer to them as *least action solutions*. In case λ is not given, one may define the ground states as the minimizers of $\mathscr{E}$ under the mass constraint $\mathscr{Q}(U) = \rho$ for some prescribed $\rho > 0$ [Cazenave and Lions 1982, p. 555]. It is worth noticing that this second definition is fully consistent only in the subcritical case

$$p < 1 + \frac{4}{N}$$

since in the supercritical case $\mathscr{E}|_{\{\mathscr{Q}=\rho\}}$ is unbounded from below [Cazenave 2003]; see also Appendix A.

Remark 1.1. When working on the whole space $\mathbb{R}^N$, the two points of view above are in some sense equivalent. Indeed, in such a situation, it is well known [Kwong 1989] that the problem

$$-\Delta Z + Z = Z^p, \qquad Z \in H^1(\mathbb{R}^N), \quad Z > 0$$

admits a solution $Z_{N,p}$ that is unique (up to translations), radial and decreasing in r. Therefore, both the problem with fixed mass and the one with given chemical potential can be uniquely solved in terms of a suitable scaling of $Z_{N,p}$. On the other hand, NLS on $\mathbb{R}^N$ with a nonhomogeneous nonlinearity cannot be treated in this way, and the fixed mass problem becomes hard to tackle [Bellazzini et al. 2013; Bartsch and de Valeriola 2013; Jeanjean 1997; Jeanjean et al. 2014].

When working on bounded domains, the two papers [Fibich and Merle 2001; Fukuizumi et al. 2012] mentioned above deal with least action solutions. In this paper, we make a first attempt to study the case of prescribed mass. Since we consider p also in the critical and supercritical ranges, we have to restrict the minimization process to constrained critical points of $\mathscr{E}$.

Definition 1.2. Let $\rho > 0$. A positive solution of (1-2) with prescribed L^2-mass ρ is a positive critical point of $\mathscr{E}|_{\{\mathscr{Q}=\rho\}}$, that is, an element of the set

$$\mathscr{P}_\rho = \{U \in H_0^1(B_1) : \mathscr{Q}(U) = \rho,\ U > 0,\ \text{there exists } \lambda \text{ such that } -\Delta U + \lambda U = U^p\}.$$

A positive *least energy solution* is a minimizer of the problem

$$e_\rho = \inf_{\mathscr{P}_\rho} \mathscr{E}.$$

Remark 1.3. When p is subcritical, as we mentioned, the above procedure is equivalent to the minimization of $\mathscr{E}|_{\{\mathscr{Q}=\rho\}}$ with no further constraint. On the other hand, when p is supercritical, the set $\mathscr{P}_\rho$ on which the minimization is settled may be strongly irregular. Contrary to what happens for least action solutions, no natural Nehari manifold seems to be associated to least energy solutions. Furthermore, since we work on a bounded domain, the dependence of $\mathscr{P}_\rho$ on ρ cannot be understood in terms of dilations. As a consequence, no regularized version of the minimization problem defined above seems available.

Remark 1.4. Since $\mathscr{A}_\lambda$ and the corresponding Nehari manifold are even, one can immediately see that least action solutions do not change sign so that they can be chosen to be positive. On the other hand, since $U \in \mathscr{P}_\rho$ does not necessarily imply $|U| \in \mathscr{P}_\rho$, in the previous definition, we require the positivity of U. Nonetheless, this condition can be removed in some cases, for instance when p is subcritical or when it is critical and ρ is small (see also Remark 5.10).

Our main results deal with the existence and orbital stability of the least energy solutions of (1-2) (the definition of orbital stability is recalled at the beginning of Section 6 below).

Theorem 1.5. *Under the above notations, the following hold*:

(1) *If* $1 < p < 1 + 4/N$, *then for every* $\rho > 0$, *the set* $\mathscr{P}_\rho$ *has a unique element*, *which achieves* e_ρ.

(2) *If* $p = 1 + 4/N$, *for* $0 < \rho < \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, *the set* $\mathscr{P}_\rho$ *has a unique element*, *which achieves* e_ρ; *for* $\rho \ge \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, *we have* $\mathscr{P}_\rho = \varnothing$.

(3) *If* $1+4/N < p < 2^*-1$, *there exists* $\rho^* > 0$ *such that* e_ρ *is achieved if and only if* $0 < \rho \le \rho^*$. *Moreover,* $\mathcal{P}_\rho = \varnothing$ *for* $\rho > \rho^*$ *whereas*

$$\#\mathcal{P}_\rho \ge 2 \quad \text{for } 0 < \rho < \rho^*.$$

In this latter case, $\mathcal{P}_\rho$ *contains positive solutions of* (1-2) *that are not least energy solutions.*

Remark 1.6. As a consequence, we have that, for p and ρ as in case (3) of the previous theorem, the problem

$$\text{find } (U,\lambda) \in H_0^1(B_1) \times \mathbb{R} : \begin{cases} -\Delta U + \lambda U = U^p, \\ \int_{B_1} U^2\,dx = \rho \end{cases}$$

admits multiple positive radial solutions.

Concerning the stability, following [Fukuizumi et al. 2012], we apply the abstract results in [Grillakis et al. 1987], which require the local existence for the Cauchy problem associated to (1-1). Since this is not known to hold for all the cases we consider, we take it as an assumption and refer to [Fukuizumi et al. 2012, Remark 1] for further details.

Theorem 1.7. *Suppose that for each* $\Phi_0 \in H_0^1(B_1,\mathbb{C})$ *there exist* $t_0 > 0$, *only depending on* $\|\Phi_0\|$, *and a unique solution* $\Phi(t,x)$ *of* (1-1) *with initial datum* Φ_0 *in the interval* $I = [0, t_0)$.

Let U *denote a least energy solution of* (1-2) *as in Theorem 1.5, and let* $\Phi(t,x) = e^{i\lambda t}U(x)$.

(1) *If* $1 < p \le 1+4/N$, *then* Φ *is orbitally stable.*

(2) *If* $1+4/N < p < 2^*-1$, *then* Φ *is orbitally stable for a.e.* $\rho \in (0,\rho^*]$.

In case (2) of the previous theorem, we expect orbital stability for every $\rho \in (0,\rho^*)$ and instability for $\rho = \rho^*$; see Remark 6.4 ahead.

As we mentioned, [Fibich and Merle 2001; Fukuizumi et al. 2012] consider least action solutions, that is, minimizers associated to

$$a_\lambda = \inf\{\mathcal{A}_\lambda(U) : U \in H_0^1(B_1),\ U \not\equiv 0,\ \mathcal{A}'_\lambda(U) = 0\}.$$

In this situation, the existence and positivity of the least energy solution is not an issue. Indeed, it is well known that problem (1-2) admits a unique positive solution R_λ if and only if $\lambda \in (-\lambda_1(B_1), +\infty)$, where $\lambda_1(B_1)$ is the first eigenvalue of the Dirichlet Laplacian. Such a solution achieves a_λ. Concerning the stability, in the critical case [Fibich and Merle 2001] and in the subcritical one [Fukuizumi et al. 2012], it is proved that $e^{i\lambda t}R_\lambda$ is orbitally stable whenever $\lambda \sim -\lambda_1(B_1)$ and $\lambda \sim +\infty$. Furthermore, stability for all $\lambda \in (-\lambda_1(B_1), +\infty)$ is proved in the second paper in dimension $N=1$ for $1 < p \le 5$ whereas in the first paper numerical evidence of it is provided in the critical case. In this context, our contribution is the following:

Theorem 1.8. *Let us assume local existence as in Theorem 1.7, and let* R_λ *be the unique positive solution of* (1-2). *If* $1 < p \le 1+4/N$, *then* $e^{i\lambda t}R_\lambda$ *is orbitally stable for every* $\lambda \in (-\lambda_1(B_1), +\infty)$.

Remark 1.9. In [Fukuizumi et al. 2012], it is also shown that, in the supercritical case $p > 1+4/N$, the standing wave associated to R_λ is orbitally *unstable* for $\lambda \sim +\infty$. In view of Theorem 1.7(2), this marks a substantial qualitative difference between the two notions of ground state.

Remark 1.10. Working in B_1 allows one to obtain radial symmetry, uniqueness properties and nondegeneracy of solutions (which in turn implies smooth dependence of the solutions on suitable parameters). These properties are not necessary for the existence results of Theorem 1.5, most of which hold also in general bounded domains, but they are crucial in our proof of stability.

As we mentioned, we will prove the above results as a byproduct of the analysis of a different variational problem that we think is of independent interest. The main feature of such a problem is due to the fact that it involves an optimization with two constraints. Let $\Omega \subset \mathbb{R}^N$ be a general bounded domain. For any fixed $\alpha > \lambda_1(\Omega)$, we consider the maximization problem

$$M_\alpha = \sup\left\{\int_\Omega |u|^{p+1}\,dx : u \in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx = \alpha\right\}, \tag{1-3}$$

which is related to the validity of Gagliardo–Nirenberg type inequalities (Appendix A).

Theorem 1.11. *Given $\alpha > \lambda_1(\Omega)$, M_α is achieved by a positive function $u_\alpha \in H_0^1(\Omega)$, and there exist $\mu_\alpha > 0$ and $\lambda_\alpha > -\lambda_1(\Omega)$ such that*

$$-\Delta u_\alpha + \lambda_\alpha u_\alpha = \mu_\alpha u_\alpha^p, \qquad \int_\Omega u_\alpha^2\,dx = 1, \qquad \int_\Omega |\nabla u_\alpha|^2\,dx = \alpha. \tag{1-4}$$

Moreover, as $\alpha \to \lambda_1(\Omega)^+$,

$$u_\alpha \to \varphi_1, \qquad \mu_\alpha \to 0^+, \qquad \lambda_\alpha \to -\lambda_1(\Omega)$$

(φ_1 *denotes the first positive eigenfunction, normalized in* L^2).

As $\alpha \to +\infty$,

$$\frac{\alpha}{\lambda_\alpha} \to \frac{N(p-1)}{N+2-p(N-2)},$$

and

(1) *if $1 < p < 1 + 4/N$, then $\mu_\alpha \to +\infty$,*
(2) *if $p = 1 + 4/N$, then $\mu_\alpha \to \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^{p-1}$ and*
(3) *if $1 + 4/N < p < 2^* - 1$, then $\mu_\alpha \to 0$.*

Furthermore, as $\alpha \to +\infty$, u_α is a one-spike solution, and a suitable scaling of u_α approaches the function $Z_{N,p}$ defined in Remark 1.1.

More detailed asymptotics are provided in Sections 3 and 4. This problem is related to the previous one in the following way. Taking $u > 0$ and $\mu > 0$ as in (1-4), the function $U = \mu^{1/(p-1)}u$ belongs to $\mathcal{P}_\rho$ for $\rho = \mu^{2/(p-1)}$. Incidentally, if one considers the minimization problem

$$m_\alpha = \inf\left\{\int_\Omega |u|^{p+1}\,dx : u \in H_0^1(\Omega),\ \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx = \alpha\right\},$$

then one obtains a solution of (1-4) with $\mu < 0$ and $\lambda < -\lambda_1(\Omega)$. This allows one to recover the well-known theory of ground states for the defocusing Schrödinger equation $i\partial\Phi/\partial t + \Delta\Phi - |\Phi|^{p-1}\Phi = 0$; see Appendix B. Moreover, when $\alpha \sim \lambda_1(\Omega)$, there exist exactly two solutions (u, μ, λ) of (1-4) that achieve M_α and m_α, respectively. More precisely, in the context of Ambrosetti–Prodi theory [1972; 1993], we prove that $(u, \mu, \lambda) = (\varphi_1, 0, -\lambda_1(\Omega))$ is an ordinary singular point for a suitable map, which yields sharp asymptotic estimates as $\alpha \to \lambda_1(\Omega)^+$. On the other hand, the estimates on M_α as $\alpha \to +\infty$ lean on suitable pointwise a priori controls [Esposito and Petralla 2011]: controls of this kind were initiated and performed for the first time for critical nonlinear elliptic problems by Druet, Hebey and Robert [Druet et al. 2004] (see also [Druet et al. 2012]).

We stress that these results about the two-constraints problem hold for a general bounded domain Ω. Going back to the case $\Omega = B_1$, positive solutions for (1-2) have been the object of an intensive study by a number of authors, in particular regarding uniqueness issues; among others, we refer to [Gidas et al. 1979; Kwong 1989; Kwong and Li 1992; Zhang 1992; Kabeya and Tanaka 1999; Korman 2002; Tang 2003; Felmer et al. 2008]. In our framework, we can exploit the synergy with such uniqueness results in order to fully characterize the positive solutions of (1-4). We do this in the following statement, which collects the results of Proposition 5.4 and of Appendix B below:

Theorem 1.12. *Let $\Omega = B_1$ and*

$$\mathcal{S} = \{(u, \mu, \lambda, \alpha) \in H_0^1(\Omega) \times \mathbb{R}^3 : u > 0 \text{ and (1-4) holds}\}.$$

Then

$$\mathcal{S} = \mathcal{S}^+ \cup \mathcal{S}^- \cup \{(\varphi_1, 0, -\lambda_1(B_1), \lambda_1(B_1))\},$$

where both $\mathcal{S}^+$ and $\mathcal{S}^-$ are smooth curves parametrized by $\alpha \in (\lambda_1(B_1), +\infty)$, corresponding to $\mathcal{S} \cap \{\mu > 0\}$ and $\mathcal{S} \cap \{\mu < 0\}$, respectively. In addition, $(u, \mu, \lambda, \alpha) \in \mathcal{S}^+$ ($\mathcal{S}^-$) if and only if u achieves M_α (m_α).

Remark 1.13. As a consequence of the previous theorem, we have that the smooth set $\mathcal{S}^+$ defined through the maximization problem M_α can be used as a surrogate of the Nehari manifold in order to "regularize" the minimization procedure introduced in Definition 1.2.

To conclude, we mention that in [Noris et al. 2014], by exploiting part of the strategy we have described, we were able to find stable solutions with small mass for the cubic Schrödinger system with trapping potential on $\mathbb{R}^N$.

This paper is structured as follows. In Section 2, we address the preliminary study of the two-constraint problems associated to M_α and m_α. Afterwards, in Section 3, we focus on the case where $\alpha \sim \lambda_1(\Omega)$, seen as an Ambrosetti–Prodi-type problem. Section 4 is devoted to the asymptotics as $\alpha \to +\infty$ for M_α, which concludes the proof of Theorem 1.11. In Section 5, we restrict our attention to the case $\Omega = B_1$, proving all the existence results (in particular Theorem 1.5), qualitative properties and more precise asymptotics for the map $\alpha \mapsto (u, \mu, \lambda)$ that parametrizes $\mathcal{S}^+$. In particular, we show that $\mu'(\alpha) > 0$ whenever $p \le 1 + 4/N$ whereas it changes sign in the supercritical case. Relying on such monotonicity properties, the stability issues are addressed in Section 6, which contains the proofs of Theorems 1.7 and 1.8. Finally, in Appendix A, we collect some known results for the reader's convenience, whereas Appendix B is devoted to the study of $\mathcal{S}^-$, which concludes the proof of Theorem 1.12.

2. A variational problem with two constraints

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain, $N \geq 1$. For every $\alpha \geq \lambda_1(\Omega)$ fixed, we consider the variational problems

$$m_\alpha = \inf_{u\in\mathcal{U}_\alpha} \int_\Omega |u|^{p+1}\,dx, \qquad M_\alpha = \sup_{u\in\mathcal{U}_\alpha} \int_\Omega |u|^{p+1}\,dx,$$

where

$$\mathcal{U}_\alpha = \left\{ u \in H_0^1(\Omega) : \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx \leq \alpha \right\}.$$

As we will see, these definitions of M_α and m_α are equivalent to the ones given in the introduction. To start with, we state the following straightforward properties:

Lemma 2.1. *For every fixed $\alpha \geq \lambda_1(\Omega)$,*

(i) $\mathcal{U}_\alpha \neq \varnothing$,

(ii) *$\mathcal{U}_\alpha$ is weakly compact in $H_0^1(\Omega)$,*

(iii) *the functional $u \mapsto \int_\Omega |u|^{p+1}\,dx$ is weakly continuous and bounded in $\mathcal{U}_\alpha$ and*

(iv) *$\|u\|_{L^{p+1}(\Omega)} \geq |\Omega|^{-(p-1)/2(p+1)}$ for every $u \in \mathcal{U}_\alpha$.*

Lemma 2.2. *For every fixed $\alpha > \lambda_1(\Omega)$, the set*

$$\widetilde{\mathcal{U}}_\alpha = \left\{ u \in H_0^1(\Omega) : \int_\Omega u^2\,dx = 1,\ \int_\Omega |\nabla u|^2\,dx = \alpha,\ \int_\Omega u\varphi_1\,dx \neq 0 \right\}$$

is a submanifold of $H_0^1(\Omega)$ of codimension 2.

Proof. Setting $F(u) = \left(\int_\Omega u^2\,dx - 1, \int_\Omega |\nabla u|^2\,dx\right)$, it suffices to prove that, for every $u \in \widetilde{\mathcal{U}}_\alpha$, the range of $F'(u)$ is $\mathbb{R}^2$. We have

$$\tfrac{1}{2}F'(u)[u] = (1,\alpha), \qquad \tfrac{1}{2}F'(u)[\varphi_1] = \int_\Omega u\varphi_1\,dx \cdot (1, \lambda_1(\Omega)),$$

which are linearly independent as $\alpha > \lambda_1(\Omega)$. □

Lemma 2.3. *For every fixed $\alpha > \lambda_1(\Omega)$, there exists $u \in \widetilde{\mathcal{U}}_\alpha$, with $u \geq 0$, such that $m_\alpha = \int_\Omega u^{p+1}\,dx$. Moreover, there exist $\lambda, \mu \in \mathbb{R}$, with $\mu \neq 0$, such that*

$$-\Delta u + \lambda u = \mu u^p \quad \textit{in } \Omega. \tag{2-1}$$

A similar result holds for M_α.

Proof. Let us prove the result for m_α. First, the infimum is attained by a function $u \in \mathcal{U}_\alpha$ by Lemma 2.1; by possibly taking $|u|$, we can suppose that $u \geq 0$. Let us show that $u \in \widetilde{\mathcal{U}}_\alpha$. Notice that, with $u \geq 0$ and $u \not\equiv 0$, it holds that $\int_\Omega u\varphi_1\,dx \neq 0$. Assume by contradiction that $\int_\Omega |\nabla u|^2\,dx < \alpha$; then we have

$$\int_\Omega u^{p+1}\,dx = \inf\left\{ \int_\Omega |v|^{p+1}\,dx : v \in H_0^1(\Omega),\ \int_\Omega v^2\,dx = 1,\ \int_\Omega |\nabla v|^2\,dx < \alpha \right\},$$

and there exists a Lagrange multiplier $\mu \in \mathbb{R}$ so that

$$\int_\Omega u^p z\,dx = \mu \int_\Omega uz\,dx \quad \text{for all } z \in H_0^1(\Omega).$$

Hence, $\mu \equiv u^{p-1} \in H_0^1(\Omega)$, which contradicts the fact that $\int_\Omega u^2\,dx = 1$. Therefore $u \in \widetilde{\mathcal{U}}_\alpha$ so that, by Lemma 2.2, the Lagrange multiplier theorem applies, thus providing the existence of $k_1, k_2 \in \mathbb{R}$ such that

$$\int_\Omega u^p z\,dx = k_1 \int_\Omega \nabla u \cdot \nabla z\,dx + k_2 \int_\Omega uz\,dx \quad \text{for all } z \in H_0^1(\Omega).$$

By the previous argument, we see that $k_1 \neq 0$; hence, setting $\mu = 1/k_1$ and $\lambda = k_2/k_1$, the proposition is proved. □

Proposition 2.4. *Given $\alpha > \lambda_1(\Omega)$, the Lagrange multipliers μ and λ associated to m_α as in Lemma 2.3 satisfy $\mu < 0$ and $\lambda < -\lambda_1(\Omega)$. Similarly, in the case of M_α, it holds that $\mu > 0$ and $\lambda > -\lambda_1(\Omega)$.*

Proof. Let (u, λ, μ) be any triplet associated to m_α as in Lemma 2.3. We will prove that $\mu < 0$. Set

$$w(t) = tu + s(t)\varphi_1,$$

where $t \in \mathbb{R}$ is close to 1, $s(1) = 0$ and $s(t)$ is such that

$$1 = \int_\Omega w(t)^2\,dx = t^2 + 2ts(t) \int_\Omega u\varphi_1\,dx + s(t)^2. \tag{2-2}$$

Since

$$\partial_s \left(t^2 + 2ts \int_\Omega u\varphi_1\,dx + s^2 \right)\bigg|_{(t,s)=(1,0)} = 2 \int_\Omega u\varphi_1\,dx \neq 0,$$

then the implicit function theorem applies, and the map $t \mapsto w(t)$ is of class C^1 in a neighborhood of $t = 1$. Differentiating (2-2) with respect to t at $t = 1$, we obtain

$$0 = \int_\Omega w'(1)w(1)\,dx = \int_\Omega w'(1)u\,dx = 1 + s'(1) \int_\Omega u\varphi_1\,dx,$$

which implies $s'(1) = -1/\int_\Omega u\varphi_1\,dx$ and $w'(1) = u - \varphi_1/\int_\Omega u\varphi_1\,dx$. Thus,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt} \int_\Omega |\nabla w(t)|^2\,dx\bigg|_{t=1} &= \int_\Omega \nabla u \cdot \nabla w'(1)\,dx \\ &= \int_\Omega |\nabla u|^2\,dx - \frac{\int_\Omega \nabla u \cdot \nabla \varphi_1\,dx}{\int_\Omega u\varphi_1\,dx} = \alpha - \lambda_1(\Omega) > 0. \end{aligned} \tag{2-3}$$

In particular, this implies the existence of $\varepsilon > 0$ such that $w(t) \in \mathcal{U}_\alpha$ for $t \in (1-\varepsilon, 1]$. Therefore, by the definition of m_α, $\|w(1)\|_{p+1} \le \|w(t)\|_{p+1}$ for every $t \in (1-\varepsilon, 1]$, and

$$\frac{d}{dt} \int_\Omega |w(t)|^{p+1}\,dx\bigg|_{t=1} \le 0. \tag{2-4}$$

On the other hand, using (2-1) and the fact that $\int_\Omega uw'(1)\,dx = 0$, we have

$$\frac{\mu}{p+1}\frac{d}{dt}\int_\Omega |w(t)|^{p+1}\,dx\bigg|_{t=1} = \mu\int_\Omega u^p w'(1)\,dx = \int_\Omega (-\Delta u + \lambda u)w'(1)\,dx$$
$$= \int_\Omega \nabla u\cdot\nabla w'(1)\,dx = \frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla w(t)|^2\,dx\bigg|_{t=1} > 0$$

by (2-3). By comparing with (2-4), we obtain that $\mu < 0$.

The case of M_α can be handled in the same way, obtaining that in such situation $\mu > 0$. Finally, by multiplying (2-1) by φ_1, we obtain

$$(\lambda_1(\Omega)+\lambda)\int_\Omega u\varphi_1\,dx = \mu\int_\Omega u^p\varphi_1\,dx.$$

As $u, \varphi_1 \geq 0$, we deduce that $\lambda_1(\Omega)+\lambda$ has the same sign as μ. □

We conclude this section with the following boundedness result, which we will need later on:

Lemma 2.5. *Take a sequence $\{(u_n, \mu_n, \lambda_n)\}_n$ such that*

$$\int_\Omega u_n^2\,dx = 1, \qquad \int_\Omega |\nabla u_n|^2\,dx =: \alpha_n \text{ is bounded}$$

and

$$-\Delta u_n + \lambda_n u_n = \mu_n u_n^p. \tag{2-5}$$

Then the sequences $\{\lambda_n\}_n$ and $\{\mu_n\}_n$ are bounded.

Proof. By multiplying (2-5) by u_n, we see that

$$\alpha_n + \lambda_n = \mu_n\int_\Omega u_n^{p+1}\,dx;$$

thus, if one of the sequences $\{\lambda_n\}_n$ or $\{\mu_n\}_n$ is bounded, the other is also bounded. Recall that, by assumption, u_n is bounded in $H_0^1(\Omega)$; hence, it converges in the L^{p+1}-norm to some $u \in H_0^1(\Omega)$ up to a subsequence. Moreover, $u \not\equiv 0$ as $\int_\Omega u^2\,dx = 1$.

For concreteness, suppose without loss of generality that $\mu_n \to +\infty$ and that $\lambda_n \to +\infty$. From the previous identity, we also have that

$$\frac{\lambda_n}{\mu_n} = \int_\Omega u_n^{p+1}\,dx - \frac{\alpha_n}{\mu_n} \to \int_\Omega u^{p+1}\,dx =: \gamma \neq 0$$

up to a subsequence. Now take any $\varphi \in H_0^1(\Omega)$ and use it as test function in (2-5). We obtain

$$\int_\Omega \nabla u_n\cdot\nabla\varphi\,dx = \mu_n\int_\Omega u_n^p\varphi\,dx - \lambda_n\int_\Omega u_n\varphi\,dx$$
$$= \mu_n\biggl(\int_\Omega u_n^p\varphi\,dx - \frac{\lambda_n}{\mu_n}\int_\Omega u_n\varphi\,dx\biggr).$$

As $\mu_n \to +\infty$, we must have

$$\int_\Omega u_n^p\varphi\,dx - \frac{\lambda_n}{\mu_n}\int_\Omega u_n\varphi\,dx \to 0.$$

On the other hand,

$$\int_\Omega u_n^p \varphi\,dx - \frac{\lambda_n}{\mu_n}\int_\Omega u_n\varphi\,dx \to \int_\Omega u^p\varphi\,dx - \gamma\int_\Omega u\varphi\,dx.$$

Thus, we have $u^p \equiv \gamma u$, which is a contradiction. □

3. Asymptotics as $\alpha \to \lambda_1(\Omega)^+$

In this section, we will completely describe the solutions of the problem

$$-\Delta u + \lambda u = \mu u^p, \quad u \in H_0^1(\Omega),\ u > 0, \qquad \int_\Omega u^2\,dx = 1 \tag{3-1}$$

for $\alpha := \int_\Omega |\nabla u|^2\,dx$ in a (right) neighborhood of $\lambda_1(\Omega)$. For that, we will follow the theory presented in [Ambrosetti and Prodi 1993, §3.2], which we now briefly recall.

Definition 3.1. Let X and Y be Banach spaces, $U \subseteq X$ an open set and $\Phi \in C^2(U, Y)$. A point $x \in U$ is said to be *ordinary singular* for Φ if

(a) $\mathrm{Ker}(\Phi'(x))$ is one-dimensional, spanned by a certain $\phi \in X$,

(b) $R(\Phi'(x))$ is closed and has codimension 1 and

(c) $\Phi''(x)[\phi,\phi] \notin R(\Phi'(x))$,

where $\mathrm{Ker}(\Phi'(x))$ and $R(\Phi'(x))$ denote respectively the kernel and the range of the map $\Phi'(x) : X \to Y$.

We will need the following result:

Theorem 3.2 [Ambrosetti and Prodi 1993, §3.2, Lemma 2.5]. *Under the previous notations, let $x^* \in U$ be an ordinary singular point for Φ. Take $y^* = \Phi(x^*)$ and $\phi \in X$ such that* $\mathrm{Ker}(\Phi'(x^*)) = \mathbb{R}\phi$, $\Psi \in Y^*$ *such that $R(\Phi'(x^*)) = \mathrm{Ker}(\Psi)$ and consider $z \in Z$ such that $\Psi(z) = 1$, where $Y = Z \oplus \mathrm{Ker}(\Psi)$. Suppose*

$$\Psi(\Phi''(x^*)[\phi,\phi]) > 0.$$

Then there exist $\varepsilon^*, \delta > 0$ such that the equation

$$\Phi(x) = y^* + \varepsilon z, \qquad x \in B_\delta(x^*),$$

has exactly two solutions for each $0 < \varepsilon < \varepsilon^*$ and no solutions for all $-\varepsilon^* < \varepsilon < 0$. Moreover, there exists $\sigma > 0$ such that the solutions can be parametrized with a parameter $t \in (-\sigma, \sigma)$, $t \mapsto x(t)$ is a C^1 map and

$$x(t) = x^* + t\phi + o(\sqrt{\varepsilon}) \quad \textit{with } t = \pm\sqrt{\frac{2\varepsilon}{\Psi(\Phi''(x^*)[\phi,\phi])}}. \tag{3-2}$$

Let us now set the framework that will allow us to apply the previous results. Given $k > N$, consider $X = \{w \in W^{2,k}(\Omega) : w = 0 \text{ on } \partial\Omega\}$, $Y = L^k(\Omega)$ and $U = \{w \in X : w > 0 \text{ in } \Omega \text{ and } \partial_\nu w < 0 \text{ on } \partial\Omega\}$. Take $\Phi : X \times \mathbb{R}^2 \to L^k(\Omega) \times \mathbb{R}^2$ defined by

$$\Phi(u,\mu,\lambda) = \left(\Delta u - \lambda u + \mu u^p, \int_\Omega u^2\,dx - 1, \int_\Omega |\nabla u|^2\,dx\right). \tag{3-3}$$

Remark 3.3. Note that $\Phi \in C^2(U, Y)$. This is immediate when $p \geq 2$ while for $1 < p < 2$ it can be proved, for instance, along the lines of [Ortega and Verzini 2004, Lemma 4.1].

We start with the following result:

Lemma 3.4. *Let $\alpha_n \to \lambda_1(\Omega)^+$, and suppose there exists (u_n, μ_n, λ_n) such that $\Phi(u_n, \mu_n, \lambda_n) = (0, 0, \alpha_n)$ with $u_n \geq 0$. Then $u_n \to \varphi_1$ in $H_0^1(\Omega)$, $\mu_n \to 0$ and $\lambda_n \to -\lambda_1(\Omega)$. In particular,*

$$\Phi(u, \mu, \lambda) = (0, 0, \lambda_1(\Omega)), \quad u \geq 0, \quad \textit{if and only if } (u, \mu, \lambda) = (\varphi_1, 0, -\lambda_1(\Omega)).$$

Proof. As u_n is bounded in $H_0^1(\Omega)$, up to a subsequence, we have that $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Moreover, $\int_\Omega u^2\,dx = 1$, $u \geq 0$, and by the Poincaré inequality, $\lambda_1(\Omega) \leq \int_\Omega |\nabla u|^2 \leq \liminf \int_\Omega |\nabla u_n|^2\,dx = \lambda_1(\Omega)$, whence $u = \varphi_1$ and the whole sequence u_n converges strongly to φ_1 in $H_0^1(\Omega)$. By Lemma 2.5, we have that μ_n and λ_n are bounded. Denote by μ_∞ and λ_∞ limits of subsequences of each. Then

$$-\Delta\varphi_1 + \lambda_\infty \varphi_1 = \mu_\infty \varphi_1^p,$$

which shows that $\mu_\infty = 0$ and $\lambda_\infty = -\lambda_1(\Omega)$. ☐

Lemma 3.5. *The point $(\varphi_1, 0, -\lambda_1(\Omega)) \in U$ is ordinary singular for Φ. More precisely, for $L := \Phi'(\varphi_1, 0, -\lambda_1(\Omega)) : X \times \mathbb{R}^2 \to L^k(\Omega) \times \mathbb{R}^2$, we have:*

(i) $\operatorname{Ker}(L) = \operatorname{span}\{(\psi, 1, \int_\Omega \varphi_1^{p+1}\,dx)\} =: \operatorname{span}\{\phi\}$, *where $\psi \in X$ is the unique solution of*

$$-\Delta\psi - \lambda_1(\Omega)\psi = \varphi_1^p - \varphi_1 \int_\Omega \varphi_1^{p+1}\,dx \quad \textit{such that} \int_\Omega \psi\varphi_1\,dx = 0. \tag{3-4}$$

(ii) *$R(L) = \operatorname{Ker}(\Psi)$ with $\Psi : L^k(\Omega) \times \mathbb{R}^2 \to \mathbb{R}$ defined by $\Psi(\xi, h, k) = k - \lambda_1(\Omega)h$.*

(iii) $\Psi(\Phi''(\varphi_1, 0, -\lambda_1(\Omega))[\phi, \phi]) > 0$.

Proof. (i) We recall that $-\Delta - \lambda_1(\Omega)\,\mathrm{Id}$ is a Fredholm operator of index 0 with

$$\operatorname{Ker}(-\Delta - \lambda_1(\Omega)\,\mathrm{Id}) = \operatorname{span}\{\varphi_1\},$$
$$R(-\Delta - \lambda_1(\Omega)\,\mathrm{Id}) = \left\{ v \in L^k(\Omega) : \int_\Omega v\varphi_1\,dx = 0 \right\}.$$

Therefore, by the Fredholm alternative, there exists a unique $\psi \in X$ solution of (3-4). Let us check that $\operatorname{Ker}(L) = \operatorname{span}\{(\psi, 1, \int_\Omega \varphi_1^{p+1}\,dx)\}$. We have

$$L(v, m, l) = \left(\Delta v + \lambda_1(\Omega)v - l\varphi_1 + m\varphi_1^p, 2\int_\Omega \varphi_1 v\,dx, 2\int_\Omega \nabla\varphi_1 \cdot \nabla v\,dx \right);$$

thus, $(v, m, l) \in \operatorname{Ker}(L)$ if and only if $l = m\int_\Omega \varphi_1^{p+1}$, $\int_\Omega \varphi_1 v\,dx = \int_\Omega \nabla\varphi_1 \cdot \nabla v\,dx = 0$ and

$$-\Delta v - \lambda_1(\Omega)v = m\left(\varphi_1^p - \varphi_1 \int_\Omega \varphi_1^{p+1}\,dx\right) \quad \text{for some } m \in \mathbb{R}.$$

By the uniqueness of ψ in (3-4), we obtain $v = m\psi$.

(ii) Let us prove that $R(L) = \{(\xi, h, \lambda_1(\Omega)h) : \xi \in L^k(\Omega),\ h \in \mathbb{R}\}$. Recalling the expression for L found in (i), it is clear that $L(v, m, l) = (\xi, h, k)$ implies $k = \lambda_1(\Omega)h$. As for the other inclusion, given any $\xi \in L^k(\Omega)$, let $w \in X$ be the solution of

$$-\Delta w - \lambda_1(\Omega)w = \varphi_1 \int_\Omega \xi\varphi_1\,dx - \xi \quad \text{with} \int_\Omega w\varphi_1\,dx = 0,$$

which exists and is unique again by the Fredholm alternative. Then $L(h\varphi_1/2 + w, 0, \int_\Omega \xi\varphi_1\,dx) = (\xi, h, \lambda_1(\Omega)h)$.

(iii) We have that

$$\Phi''(\varphi_1, 0, -\lambda_1(\Omega))[\phi, \phi] = 2\left(p\varphi_1^{p-1}\psi - \psi\int_\Omega \varphi_1^{p+1}\,dx, \int_\Omega \psi^2\,dx, \int_\Omega |\nabla\psi|^2\,dx\right)$$

with ϕ and ψ defined in (i). Hence,

$$\Psi(\Phi''(\varphi_1, 0, \lambda_1(\Omega))[\phi, \phi]) = \int_\Omega 2(|\nabla\psi|^2 - \lambda_1(\Omega)\psi^2)\,dx > 0 \tag{3-5}$$

since ψ satisfies (3-4). □

Proposition 3.6. *There exists ε^* such that the equation*

$$\Phi(u, \mu, \lambda) = (0, 0, \lambda_1(\Omega) + \varepsilon), \quad (u, \mu, \lambda) \in U \times \mathbb{R}^2,$$

has exactly two positive solutions for each $0 < \varepsilon < \varepsilon^*$ (one with $\mu > 0$ and one with $\mu < 0$). Moreover, such solutions satisfy the asymptotic expansion

$$(u, \mu, \lambda) = (\varphi_1, 0, -\lambda_1(\Omega)) \pm \sqrt{\frac{\varepsilon}{\int_\Omega \varphi_1^p\psi\,dx}}\left(\psi, 1, \int_\Omega \varphi_1^{p+1}\,dx\right) + o(\sqrt{\varepsilon}),$$

where ψ is defined in (3-4). In addition, the L^{p+1}-norm of one of the solutions is equal to $m_{\lambda_1(\Omega)+\varepsilon}$ and the other is equal to $M_{\lambda_1(\Omega)+\varepsilon}$.

Proof. We apply Theorem 3.2 with Φ defined in (3-3), $x^* = (\varphi_1, 0, -\lambda_1(\Omega))$ and $z = (0, 0, 1)$. By the previous lemma, x^* is ordinary singular for Φ, and, moreover, using the notation therein, $\Psi(\Phi''(x^*)[\phi, \phi]) > 0$ and $\Psi(z) = 1$. Therefore, the assumptions of Theorem 3.2 are satisfied, and there exist $\varepsilon^*, \delta > 0$ such that the problem

$$\Phi(u, \mu, \lambda) = (0, 0, \lambda_1(\Omega) + \varepsilon), \quad (u, \mu, \lambda) \in B_\delta(\varphi_1, 0, -\lambda_1(\Omega)),$$

has exactly two solutions for each $0 < \varepsilon < \varepsilon^*$, which can be parametrized using a map $t \mapsto (u(t), \mu(t), \lambda(t))$ of class C^1 in $U \times \mathbb{R}^2$. The asymptotic expansion is obtained by combining (3-2) with the fact (see (3-5))

$$\Psi(\Phi''(\varphi_1, 0, \lambda_1(\Omega))[\phi, \phi]) = 2\int_\Omega \varphi_1^p\psi\,dx.$$

Finally, by possibly choosing a smaller ε^*, $(u(t), \mu(t), \lambda(t))$ are the unique positive solutions in $U \times \mathbb{R}^2$ for $0 < \varepsilon < \varepsilon^*$, as a consequence of Lemma 3.4, and the statement concerning $\int_\Omega u(t)^{p+1}\,dx$ follows from Lemma 2.3 and Proposition 2.4. □

Remark 3.7. From the proof of Proposition 3.6, we deduce an alternative proof of [Fukuizumi et al. 2012, Theorem 17(ii)]; namely, we can show that

$$(\mu^2)'(\lambda_1(\Omega)^+) > 0.$$

This result is relevant when facing stability issues; see Corollary 6.2 ahead.

4. Asymptotics as $\alpha \to +\infty$

In this section, we consider the case when α is large in order to conclude the proof of Theorem 1.11. Since in that case the problems M_α and m_α exhibit different asymptotics, here we only address the study of M_α, and we postpone to Appendix B the complete description of the minimizers corresponding to m_α.

Define, for any $\mu, \lambda \in \mathbb{R}$, the action functional associated to (2-1), namely $J_{\mu,\lambda} : H_0^1(\Omega) \to \mathbb{R}$:

$$J_{\mu,\lambda}(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \frac{\mu}{p+1}\int_\Omega |u|^{p+1}\,dx. \tag{4-1}$$

Lemma 4.1. *For every $\mu > 0$ and $\lambda \in \mathbb{R}$, we have that*

$$u \in \widetilde{\mathcal{U}}_\alpha,\ \int_\Omega |u|^{p+1}\,dx = M_\alpha \quad \Longrightarrow \quad J_{\mu,\lambda}(u) = \inf_{\widetilde{\mathcal{U}}_\alpha} J_{\mu,\lambda}.$$

Proof. By the definition of M_α,

$$\frac{\mu}{p+1}M_\alpha = \sup_{w\in\widetilde{\mathcal{U}}_\alpha}\left\{\frac{\mu}{p+1}\int_\Omega |w|^{p+1}\,dx + \frac{1}{2}\left(\alpha - \int_\Omega |\nabla w|^2\,dx\right) + \frac{\lambda}{2}\left(1 - \int_\Omega w^2\,dx\right)\right\},$$

and hence,

$$J_{\mu,\lambda}(u) = \frac{\alpha+\lambda}{2} - \frac{\mu}{p+1}M_\alpha = \inf_{w\in\widetilde{\mathcal{U}}_\alpha} J_{\mu,\lambda}(w).$$ □

Lemma 4.2. *Fix $\alpha > \lambda_1(\Omega)$, and let $(u, \mu, \lambda) \in \widetilde{\mathcal{U}}_\alpha \times \mathbb{R}^+ \times (-\lambda_1(\Omega), +\infty)$ be any triplet associated to M_α as in Lemma 2.3. Then the Morse index of $J''_{\mu,\lambda}(u)$ is either 1 or 2.*

Proof. If (u, μ, λ) is a triplet associated to M_α, then $\mu > 0$ by Proposition 2.4. Equation (2-1) implies

$$J''_{\mu,\lambda}(u)[u,u] = -(p-1)\mu\int_\Omega u^{p+1}\,dx < 0,$$

so that the Morse index is at least 1. Next we claim that, for such (u, μ, λ),

$$J''_{\mu,\lambda}(u)[\phi,\phi] \geq 0 \quad \text{for every } \phi \in H_0^1(\Omega) \text{ with } \int_\Omega \nabla u\cdot\nabla\phi\,dx = \int_\Omega u\phi\,dx = 0,$$

which implies that the Morse index is at most 2. Indeed, any such ϕ belongs to the tangent space of $\widetilde{\mathcal{U}}_\alpha$ at u; hence, there exists a C^∞ curve $\gamma(t)$ satisfying, for some $\varepsilon > 0$,

$$\gamma : (-\varepsilon, \varepsilon) \to \widetilde{\mathcal{U}}_\alpha, \quad \gamma(0) = u,\ \gamma'(0) = \phi.$$

Lemma 4.1 implies that $J_{\mu,\lambda}(\gamma(t)) - J_{\mu,\lambda}(\gamma(0)) \geq 0$. Hence,

$$0 \leq J_{\mu,\lambda}(\gamma(t)) - J_{\mu,\lambda}(u) = J'_{\mu,\lambda}(u)[\phi]t + J''_{\mu,\lambda}(u)[\phi,\phi]\frac{t^2}{2} + J'_{\mu,\lambda}(u)[\gamma''(0)]\frac{t^2}{2} + \mathrm{o}(t^2).$$

Finally, (2-1) implies that $J'_{\mu,\lambda}(u) \equiv 0$, which concludes the proof. □

Lemma 4.3. *Let $\alpha_n \to +\infty$, and let $u_n \in H_0^1(\Omega)$, $u_n > 0$, satisfy*

$$-\Delta u_n + \lambda_n u_n = \mu_n u_n^p \quad \textit{in } \Omega, \qquad \int_\Omega |\nabla u_n|^2 \, dx = \alpha_n, \qquad \int_\Omega u_n^2 \, dx = 1$$

for some $\mu_n > 0$ and $\lambda_n > -\lambda_1(\Omega)$. Then $\lambda_n \to +\infty$.

Proof. Set

$$L_n := \|u_n\|_{L^\infty(\Omega)} = u_n(x_n).$$

Since $\Delta u_n(x_n) \leq 0$, from the equation for u_n, we obtain $\mu_n L_n^p - \lambda_n L_n \geq 0$, i.e.,

$$-\frac{\lambda_1(\Omega)}{\mu_n L_n^{p-1}} < \frac{\lambda_n}{\mu_n L_n^{p-1}} \leq 1$$

(recall that $\lambda > -\lambda_1(\Omega)$). In particular, since $\mu_n L_n^{p-1} \geq \mu_n \int_\Omega u_n^{p+1} \, dx \geq \alpha_n + \lambda_n \to +\infty$, we have (up to subsequences)

$$\frac{\lambda_n}{\mu_n L_n^{p-1}} \to \lambda^* \in [0, 1]. \tag{4-2}$$

In order to prove that $\lambda_n \to +\infty$, it only remains to show that $\lambda^* \neq 0$. To this aim, we define

$$v_n(x) := \frac{1}{L_n} u_n\left(x_n + \frac{x}{(\mu_n L_n^{p-1})^{1/2}}\right)$$

so that v_n satisfies

$$-\Delta v_n + \frac{\lambda_n}{\mu_n L_n^{p-1}} v_n = v_n^p \quad \text{in } \Omega_n := (\mu_n L_n^{p-1})^{1/2}(\Omega - x_n).$$

Using (4-2) and reasoning as in [Gidas and Spruck 1981b, pp. 887–889], we have that $v_n \to v$ in $(W^{2,p} \cap C^{1,\beta})_{\text{loc}}(\mathbb{R}^N)$ for every $\beta \in (0, 1)$. Moreover, $v \geq 0$, $v(0) = 1$ and

$$-\Delta v + \lambda^* v = v^p \quad \text{in } H,$$

where H is either $\mathbb{R}^N$ or a half-space of $\mathbb{R}^N$ and $v = 0$ on ∂H in case H is the half-space. Since $v \not\equiv 0$, the nonexistence results in [Gidas and Spruck 1981a] imply that $\lambda^* > 0$, and this concludes the proof. □

Next, we use some results from [Esposito and Petralla 2011] in order to show that a suitable rescaling of the solutions converges to the function $Z_{N,p}$ defined in Remark 1.1. Such results rely on pointwise estimates that take fundamental inspiration from the monograph [Druet et al. 2004] (see also [Druet et al. 2012]).

Lemma 4.4. *With the same assumptions as the previous lemma, suppose moreover that the Morse index of $J''_{\mu_n,\lambda_n}(u_n)$ is equal to $k \in \mathbb{N}$ for every n. Then u_k admits k local maxima $P_n^i \in \Omega$, $i = 1, \dots, k$, such that, defining*

$$v_{i,n}(x) = \left(\frac{\mu_n}{\lambda_n}\right)^{1/(p-1)} u_n\left(\frac{x}{\sqrt{\lambda_n}} + P_n^i\right) \tag{4-3}$$

for $x \in \Omega_{i,n} = \sqrt{\lambda_n}(\Omega - P_i^n)$, we have

$$v_{i,n} \to Z_{N,p} \quad \text{in } C^1_{\text{loc}}(\mathbb{R}^N) \quad \text{as } n \to +\infty \text{ for every } i.$$

As a consequence, for every $q \geq 1$,

$$\left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \int_\Omega u_n^q \, dx \to k \int_{\mathbb{R}^N} Z_{N,p}^q \, dx \quad \text{as } n \to +\infty. \tag{4-4}$$

Proof. Since $\lambda_n \to +\infty$ by the previous lemma, we can apply [Esposito and Petralla 2011, Theorem 3.2] to $U_n := \mu_n^{1/(p-1)} u_n$, inferring the existence of k local maxima P_n^i, $i = 1, \dots, k$, such that, for every $i \neq j$,

$$\sqrt{\lambda_n}\, \text{dist}(P_n^i, \partial\Omega) \to +\infty, \qquad \sqrt{\lambda_n}|P_n^i - P_n^j| \to +\infty, \tag{4-5}$$

and for some $C, \gamma > 0$, the following pointwise estimate holds:

$$u_n(x) = \mu_n^{-1/(p-1)} U_n(x) \leq C\left(\frac{\lambda_n}{\mu_n}\right)^{1/(p-1)} \sum_{i=1}^k e^{-\gamma\sqrt{\lambda_n}|x - P_n^i|} \quad \text{for all } x \in \Omega.$$

Furthermore, since $v_{i,n}$ solves $-\Delta v_{i,n} + v_{i,n} = v_{i,n}^p$ in $\Omega_{i,n}$, [Esposito and Petralla 2011, Theorem 3.1] yields that $v_{i,n} \to Z_{N,p}$ in $C^1_{\text{loc}}(\mathbb{R}^N)$, so the only thing that remains to be proved is estimate (4-4).

To this aim, let $R > 0$ be fixed and $r_n = R/\sqrt{\lambda_n}$. Then, if n is sufficiently large, (4-5) implies that, for every $i \neq j$,

$$B_{r_n}(P_n^i) \subset \Omega, \qquad B_{r_n}(P_n^i) \cap B_{r_n}(P_n^j) = \varnothing.$$

We obtain

$$\begin{aligned}
\left|\left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \int_\Omega u_n^q \, dx - \sum_{j=1}^k \int_{B_R(0)} v_{j,n}^q \, dx\right| &= \left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \left|\int_\Omega u_n^q \, dx - \sum_{j=1}^k \int_{B_{r_n}(P_n^j)} u_n^q \, dx\right| \\
&= \left(\frac{\mu_n}{\lambda_n}\right)^{q/(p-1)} \lambda_n^{N/2} \int_{\Omega \setminus \bigcup_{j=1}^k B_{r_n}(P_n^j)} u_n^q \, dx \\
&\leq C\lambda_n^{N/2} \sum_{i=1}^k \int_{\Omega \setminus \bigcup_{j=1}^k B_{r_n}(P_n^j)} e^{-q\gamma\sqrt{\lambda_n}|x - P_n^i|} \, dx \\
&\leq C\lambda_n^{N/2} \sum_{i=1}^k \int_{\mathbb{R}^N \setminus B_{r_n}(P_n^i)} e^{-q\gamma\sqrt{\lambda_n}|x - P_n^i|} \, dx \\
&= Ck \int_{\mathbb{R}^N \setminus B_R(0)} e^{-q\gamma|y|} \, dy \leq C_1 e^{-C_2 R}
\end{aligned}$$

for some positive C_1 and C_2. As $n \to +\infty$, we have, up to subsequences,

$$\left| \lim_n \left(\frac{\mu_n}{\lambda_n} \right)^{q/(p-1)} \lambda_n^{N/2} \int_\Omega u_n^q \, dx - k \int_{B_R(0)} Z_{N,p}^q \, dx \right| \le C_1 e^{-C_2 R},$$

and (4-4) follows by taking $R \to +\infty$. □

Finally, the previous lemma allows us to study the asymptotic behavior of μ as $\alpha \to +\infty$.

Lemma 4.5. *With the same assumptions as the previous lemma, we have that*

(1) *if* $1 < p < 1 + 4/N$, *then* $\mu_n \to +\infty$,

(2) *if* $p = 1 + 4/N$, *then* $\mu_n \to k^{2/N} \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^{4/N}$ *and*

(3) *if* $1 + 4/N < p < 2^* - 1$, *then* $\mu_n \to 0$.

Furthermore,

$$\frac{\alpha_n}{\lambda_n} \to \frac{N(p-1)}{N+2-p(N-2)}.$$

Proof. Exploiting (4-4) with $q = 2$ and $q = p+1$ as well as the relations $\|u_n\|_{L^2}^2 = 1$, $\|\nabla u_n\|_{L^2}^2 = \alpha_n$ and $\alpha_n + \lambda_n = \mu_n \|u_n\|_{L^{p+1}}^{p+1}$, we can write

$$\begin{aligned} \mu_n^{2/(p-1)} \lambda_n^{N/2-2/(p-1)} &\to k \int_{\mathbb{R}^N} Z_{N,p}^2 \, dx, \\ \mu_n^{(p+1)/(p-1)} \lambda_n^{N/2-(p+1)/(p-1)} \int_\Omega u_n^{p+1} \, dx &\to k \int_{\mathbb{R}^N} Z_{N,p}^{p+1} \, dx, \\ \frac{\alpha_n}{\lambda_n} \mu_n^{2/(p-1)} \lambda_n^{N/2-2/(p-1)} &\to k \int_{\mathbb{R}^N} |\nabla Z_{N,p}|^2 \, dx. \end{aligned} \tag{4-6}$$

Now, since $\lambda_n \to +\infty$ (Lemma 4.3) and the exponent $N/2 - 2/(p-1)$ is negative, zero or positive respectively in the subcritical, critical and supercritical cases, the first relation in (4-6) immediately provides the properties for μ_n.

On the other hand, dividing the third relation by the first one, we have

$$\frac{\alpha_n}{\lambda_n} \to \frac{\|\nabla Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2}{\|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2} = \frac{N(p-1)}{N+2-p(N-2)}.$$

The explicit evaluation of this constant can be obtained by the relations

$$\begin{cases} \|\nabla Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 + \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 = \|Z_{N,p}\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}, \\ \frac{N-2}{2} \|\nabla Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 + \frac{N}{2} \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2 = \frac{N}{p+1} \|Z_{N,p}\|_{L^{p+1}(\mathbb{R}^N)}^{p+1}, \end{cases}$$

i.e., by testing the equation for $Z_{N,p}$ either with $Z_{N,p}$ itself or with $x \cdot \nabla Z_{N,p}$ (recall that $Z_{N,p}$ decays exponentially at ∞). The second relation is the well-known Pohozaev identity; see for instance [Berestycki and Lions 1983, §2]. □

End of the proof of Theorem 1.11. The fact that M_α is achieved by a triplet (u, μ, λ) with $\mu > 0$ and $\lambda > -\lambda_1(\Omega)$ is a consequence of Lemma 2.3 and Proposition 2.4. Lemma 3.4 implies the asymptotic behavior as $\alpha \to \lambda_1(\Omega)^+$ while the results as $\alpha \to +\infty$ follow from Lemmas 4.3, 4.4 and 4.5, recalling that u has Morse index k, with k being either 1 or 2, by Lemma 4.2. The only thing that remains to be proved is that both in Lemma 4.4 and in Lemma 4.5(2) k must be equal to 1; in other words, we are left to show that, if u_n achieves M_{α_n}, with α_n large, then its Morse index must be 1 (and not 2).

For easier notation, in the following, we write $Z = Z_{N,p}$. Since u_n achieves M_{α_n}, from (4-6), we infer (up to subsequences)

$$\mu_n \sim k^{(p-1)/2}\|Z\|_{L^2}^{p-1}\lambda_n^{1-N(p-1)/4}, \qquad \alpha_n \sim \frac{\|\nabla Z\|_{L^2}^2}{\|Z\|_{L^2}^2}\lambda_n$$

and

$$\begin{aligned}\frac{M_{\alpha_n}}{\alpha_n^{N(p-1)/4}} = \frac{\int_\Omega u_n^{p+1}\,dx}{\alpha_n^{N(p-1)/4}} &\sim k\|Z\|_{L^{p+1}}^{p+1}\frac{\lambda_n^{(p+1)/(p-1)-N/2}}{\mu_n^{(p+1)/(p-1)}\alpha_n^{N(p-1)/4}}\\ &\to k^{-(p-1)/2}\frac{\|Z\|_{L^{p+1}}^{p+1}}{\|\nabla Z\|_{L^2}^{N(p-1)/2}\|Z\|_{L^2}^{p+1-N(p-1)/2}},\end{aligned} \tag{4-7}$$

where either $k = 1$ or $k = 2$. On the other hand, let us fix $x_0 \in \Omega$ and $\eta \in C_0^\infty(\Omega)$ such that $\eta(x) = 1$ around x_0. It is always possible to find a sequence $a_n \to 0^+$ such that

$$w_n(x) := \eta(x) Z_{N,p}\left(\frac{x - x_0}{a_n}\right), \ \widetilde{w}_n := \frac{w_n}{\|w_n\|_{L_2}} \quad \text{satisfy} \int_\Omega |\nabla \widetilde{w}_n|^2\,dx = \alpha_n$$

(indeed $\alpha_n \to +\infty$ and $\int_\Omega |\nabla \widetilde{w}_n|^2\,dx$ is of order a_n^{-2} as $a_n \to 0$). Then direct calculation yields

$$\frac{M_{\alpha_n}}{\alpha_n^{N(p-1)/4}} \ge \frac{\int_\Omega \widetilde{w}_n^{p+1}\,dx}{\alpha_n^{N(p-1)/4}} \to \frac{\|Z\|_{L^{p+1}}^{p+1}}{\|\nabla Z\|_{L^2}^{N(p-1)/2}\|Z\|_{L^2}^{p+1-N(p-1)/2}},$$

which, together with (4-7), forces $k = 1$. □

Remark 4.6. The previous argument shows that, when α is large, M_α is achieved by a single-peak solution having Morse index 1. This was actually suggested to us by the anonymous referee in his/her report. This also implies the sharper estimate for the asymptotics of μ:

$$\mu_\alpha \sim C\alpha^{1-N(p-1)/4},$$

where C is a constant depending only on N and p (through $Z_{N,p}$).

5. Least energy solutions in the ball

From now on, we will focus on the case

$$\Omega := B_1.$$

To start with, we collect in the following theorem some well-known results about uniqueness and nondegeneracy of positive solutions of (1-2) on the ball:

Theorem 5.1 [Gidas et al. 1979; Kwong 1989; Kwong and Li 1992; Korman 2002; Aftalion and Pacella 2003]. *Let $\lambda \in (-\lambda_1(B_1), +\infty)$ and $\mu > 0$ be fixed. Then the problem*

$$-\Delta u + \lambda u = \mu u^p \quad \text{in } B_1, \qquad u = 0 \quad \text{on } \partial B_1$$

admits a unique positive solution u, which is nondegenerate, radially symmetric and decreasing with respect to the radial variable $r = |x|$.

Proof. The existence easily follows from the mountain pass lemma. The radial symmetry and monotonicity of positive solutions is a direct consequence of [Gidas et al. 1979].

The uniqueness in the case $\lambda > 0$ was proved by Kwong [1989] for $N \geq 2$. For $\lambda \in (-\lambda_1(B_1), 0)$, the uniqueness in dimension $N \geq 3$ was proved by Kwong and Li [1992, Theorem 2] (see also [Zhang 1992]) whereas in dimension $N = 2$ it was proved by Korman [2002, Theorem 2.2]. The case $\lambda = 0$ is treated in Section 2.8 of [Gidas et al. 1979].

As for the nondegeneracy, for $\lambda > 0$, this follows from [Aftalion and Pacella 2003, Theorem 1.1] since we know that u has Morse index 1 as it is a mountain pass solution for $J_{\mu,\lambda}$ (recall that such a functional is defined as in (4-1)). As for $\lambda \in (-\lambda_1(B_1), 0]$, we could not find a precise reference, and for this reason, we present here a proof, following some ideas of [Kabeya and Tanaka 1999].

Assume by contradiction that u is a degenerate solution for some $\lambda \in (-\lambda_1(B_1), 0]$. This means that there exists a solution $0 \neq w \in H_0^1(B_1)$ of

$$-\Delta w + \lambda w = p u^{p-1} w;$$

hence, $w \in H^1_{0,\mathrm{rad}}(B_1)$ and $J''_{\mu,\lambda}(u)[w, \xi] = 0$ for all $\xi \in H_0^1(B_1)$. Moreover, we have that $J''_{\mu,\lambda}(u)[u, u] = -(p-1)\mu \int_{B_1} u^{p+1}\, dx < 0$, and thus,

$$J''_{\mu,\lambda}(u)[h, h] \leq 0 \quad \text{for all } h \in H := \operatorname{span}\{u, w\}.$$

For $\delta > 0$, consider the perturbed functional

$$I_\delta(w) = \int_{B_1} \left(\frac{|\nabla w|^2}{2} + \frac{\lambda + \delta u^{p-1}}{2} w^2 - \frac{\mu + \delta}{p+1} (w^+)^{p+1} \right) dx. \tag{5-1}$$

On the one hand, this functional satisfies, for every $h \in H \setminus \{0\}$,

$$\begin{aligned} I''_\delta(u)[h, h] &= J''_{\mu,\lambda}(u)[h, h] + \int_{B_1} (\delta u^{p-1} h^2 - p\delta u^{p-1} h^2)\, dx \\ &\leq -(p-1)\delta \int_{B_1} u^{p-1} h^2\, dx < 0. \end{aligned} \tag{5-2}$$

On the other hand, I_δ has a mountain pass geometry for δ sufficiently small; hence, it has a critical point of mountain pass type. Every nonzero critical point of I_δ is positive (by the maximum principle), and it solves

$$\begin{cases} -\Delta w = V_\delta(r) w + (\mu + \delta) w^p & \text{in } B_1, \\ w > 0 & \text{in } B_1, \\ w \in H_0^1(B_1) \end{cases}$$

for $V_\delta(r) := -\lambda - \delta u^{p-1}$. Now this problem has a unique radial solution, which is u itself, which is in contradiction to (5-2). The uniqueness of this perturbed problem follows from [Korman 2002, Theorem 2.2] in case $\lambda < 0$ (in fact, $V_\delta(r) > 0$ and $\frac{d}{dr}[r^{2n(1/2-1/(p+1))}V_\delta(r)] \geq 0$) while in case $\lambda = 0$ we can reason exactly as in [Felmer et al. 2008, Proposition 3.1] (the proof there is for the annulus, but the argument also works in the case of a ball). □

Remark 5.2. As we already mentioned, the Morse index of $u > 0$ as a critical point of $J_{\mu,\lambda}$ is 1. Recalling the definition of I_δ in (5-1), we have that also the Morse index of $I''_\delta(u)$ is 1 at least if $\lambda > -\lambda_1(B_1)$ and if $\delta > 0$ is small enough. When $\lambda < 0$, this was shown in the proof of the previous result, where we have dealt also with the case $\lambda = 0$. The proof for $\lambda > 0$ is the same as in the latter case.

Given $k > N$, as before, let us take $X = \{w \in W^{2,k}(B_1) : w = 0 \text{ on } \partial B_1\}$. Let us introduce the map $F : X \times \mathbb{R}^3 \to L^k(B_1) \times \mathbb{R}^2$ defined by

$$F(u, \mu, \lambda, \alpha) = \left(\Delta u - \lambda u + \mu u^p, \int_{B_1} u^2\,dx - 1, \int_{B_1} |\nabla u|^2\,dx - \alpha\right)$$

and its null set restricted to positive u,

$$\mathcal{S} = \{(u, \mu, \lambda, \alpha) \in X \times \mathbb{R}^3 : u > 0,\ F(u, \mu, \lambda, \alpha) = (0, 0, 0)\}.$$

It is immediate to check that $\mathcal{S} \cap \{\alpha \leq \lambda_1(B_1)\} = \{(\varphi_1, 0, -\lambda_1(B_1), \lambda_1(B_1)\}$ so that

$$\mathcal{S}^\pm := \mathcal{S} \cap \{\pm\mu > 0\} \subset \{\alpha > \lambda_1(B_1)\}.$$

We are going to show that $\mathcal{S}^+$ can be parametrized in a smooth way on α, thus proving the part of Theorem 1.12 regarding focusing nonlinearities. As we mentioned, the (easier) study of $\mathcal{S}^-$ is postponed to Appendix B. In view of the application of the implicit function theorem, we have the following:

Lemma 5.3. *Let $(u, \mu, \lambda, \alpha) \in \mathcal{S}^+$. Then the linear bounded operator*

$$F_{(u,\mu,\lambda)}(u, \mu, \lambda, \alpha) : X \times \mathbb{R}^2 \to L^k(B_1) \times \mathbb{R}^2$$

is invertible.

Proof. The lemma is a direct consequence of the Fredholm alternative and of the closed graph theorem once we show that the operator above is injective. Let us suppose by contradiction the existence of $(v, m, l) \neq (0, 0, 0)$ such that $F_{(u,\mu,\lambda)}(u, \mu, \lambda, \alpha)[v, m, l] = (0, 0, 0)$. This explicitly gives

$$\begin{aligned} -\Delta u + \lambda u &= \mu u^p, & \int_{B_1} u^2\,dx &= 1, & \int_{B_1} |\nabla u|^2\,dx &= \alpha, \\ -\Delta v + \lambda v + lu &= p\mu u^{p-1}v + mu^p, & \int_{B_1} uv\,dx &= 0, & \int_{B_1} \nabla u \cdot \nabla v\,dx &= 0. \end{aligned} \tag{5-3}$$

By testing the two differential equations by v, we obtain

$$\int_{B_1} u^p v\,dx = 0, \qquad \int_{B_1} |\nabla v|^2\,dx + \lambda \int_{B_1} v^2\,dx = p\mu \int_{B_1} u^{p-1}v^2\,dx \tag{5-4}$$

so that

$$J''_{\mu,\lambda}(u)[u,u]<0, \qquad J''_{\mu,\lambda}(u)[u,v]=0, \qquad J''_{\mu,\lambda}(u)[v,v]=0.$$

This implies that $J''_{\mu,\lambda}(u)[h,h]\le 0$ for every $h\in H=\operatorname{span}\{u,v\}$. By defining I_δ as in (5-1), for $\delta>0$ small, we obtain $I''_\delta(u)[h,h]<0$ for every $0\ne h\in H$. Since H has dimension 2 ($v=cu$ would imply $c\int_\Omega u^2=0$), this contradicts Remark 5.2. □

Proposition 5.4. *$\mathcal{S}^+$ is a smooth curve, parametrized by a map*

$$\alpha\mapsto(u(\alpha),\mu(\alpha),\lambda(\alpha)), \quad \alpha\in(\lambda_1(B_1),+\infty).$$

In particular, $u(\alpha)$ is the unique maximizer of M_α (as defined in (1-3)).

Proof. To start with, Lemma 2.3 and Proposition 2.4 imply that, for every fixed $\alpha^*>\lambda_1(B_1)$, there exists at least a corresponding point in $\mathcal{S}^+$. If $(u^*,\mu^*,\lambda^*,\alpha^*)$ denotes any such point (not necessarily related to M_{α^*}), then by Lemma 5.3, it can be continued, by means of the implicit function theorem, to an arc $(u(\alpha),\mu(\alpha),\lambda(\alpha))$, defined on a maximal interval $(\underline{\alpha},\overline{\alpha})\ni\alpha^*$, chosen in such a way that $\mu(\alpha)>0$ on this interval. Since $u(\alpha)$ solves the equation, standard arguments involving the maximum principle and Hopf lemma allow one to obtain that $u(\alpha)>0$ (recall that we are using the $W^{2,k}$-topology) along the arc, which consequently belongs to $\mathcal{S}^+$. We want to show that $(\underline{\alpha},\overline{\alpha})=(\lambda_1(B_1),+\infty)$.

Let us assume by contradiction $\underline{\alpha}>\lambda_1(\Omega)$. For $\alpha_n\to\underline{\alpha}^+$, Lemma 2.5 implies that, up to a subsequence,

$$u_n\rightharpoonup\bar u \quad\text{in } H^1_0(\Omega), \qquad \lambda_n\to\bar\lambda, \qquad \mu_n\to\bar\mu.$$

Thus,

$$-\Delta\bar u+\bar\lambda\bar u=\bar\mu\bar u^p \quad\text{in }\Omega,$$

and the convergence $u_n\to\bar u$ is actually strong in $H^2(\Omega)$. Then $\int_\Omega|\nabla\bar u|^2\,dx=\underline{\alpha}>\lambda_1(\Omega)$ so that $\bar\mu>0$. Thus, Lemma 5.3 allows us to reach a contradiction with the maximality of $\underline{\alpha}$, and therefore, $\underline{\alpha}=\lambda_1(\Omega)$. Analogously, we can show that $\overline{\alpha}=+\infty$.

Once we know $\mathcal{S}^+$ is the disjoint union of smooth curves, each parametrized by $\alpha\in(\lambda_1(B_1),+\infty)$, it only remains to show that the curve of solutions is indeed unique. Suppose by contradiction that, for $\alpha_n\to\lambda_1(B_1)$, there exist $(u_1(\alpha_n),\mu_1(\alpha_n),\lambda_1(\alpha_n))\ne(u_2(\alpha_n),\mu_2(\alpha_n),\lambda_2(\alpha_n))$ for every n. Then by Lemma 3.4, both triplets converge to $(\varphi_1,0,-\lambda_1(B_1))$ in contradiction to Proposition 3.6. □

Corollary 5.5. *Writing*

$$\frac{d}{d\alpha}(u(\alpha),\mu(\alpha),\lambda(\alpha))=(v(\alpha),\mu'(\alpha),\lambda'(\alpha)),$$

we have

$$-\Delta v+\lambda' u+\lambda v=p\mu u^{p-1}v+\mu' u^p, \quad v\in H^1_0(B_1)$$

and

$$\int_{B_1}uv\,dx=0, \qquad \int_{B_1}\nabla u\cdot\nabla v\,dx=\tfrac12, \tag{5-5}$$

$$\mu\int_{B_1}u^pv\,dx=\tfrac12, \qquad \mu'\int_{B_1}u^{p+1}\,dx=\lambda'-\frac{p-1}{2}. \tag{5-6}$$

Proof. Direct computations (by differentiating $F(u(\alpha), \mu(\alpha), \lambda(\alpha), \alpha) = 0$ and testing the differential equations by u and v) give the result. □

In the following, we address the study of the monotonicity properties of the map

$$\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha))$$

introduced above, v always denoting the derivative of u with respect to α:

Lemma 5.6. *We have $\lambda'(\alpha) > 0$ for every $\alpha > \lambda_1(B_1)$.*

Proof. Let $(h, k) \in \mathbb{R}^2$, and let us consider the quadratic form

$$J''_{\mu,\lambda}(u)[hu + kv, hu + kv] =: ah^2 + 2bhk + ck^2.$$

Using Corollary 5.5, we obtain

$$\begin{aligned} a &= J''_{\mu,\lambda}(u)[u, u] = \int_{B_1} \left[|\nabla u|^2 + \lambda u^2 - p\mu u^{p+1}\right] dx = -(p-1)\mu \int_{B_1} u^{p+1}\, dx, \\ b &= J''_{\mu,\lambda}(u)[u, v] = \int_{B_1} \left[\nabla u \cdot \nabla v + \lambda uv - p\mu u^p v\right] dx = -\frac{p-1}{2}, \\ c &= J''_{\mu,\lambda}(u)[v, v] = \int_{B_1} \left[|\nabla v|^2 + \lambda v^2 - p\mu u^{p-1} v^2\right] dx = \frac{\mu'}{2\mu}. \end{aligned}$$

Since $J''_{\mu,\lambda}(u)$ has (large) Morse index equal to 1 (Remark 5.2) and $a < 0$, we have that $b^2 - ac > 0$, i.e.,

$$\mu' \int_{B_1} u^{p+1}\, dx > -\frac{p-1}{2}.$$

The lemma follows by comparing to (5-6). □

Lemma 5.7. *If $\omega_N = |\partial B_1|$, then*

$$\mu' \int_{B_1} u^{p+1}\, dx = \frac{p+1}{2(p-1)} \left[\left(-p + 1 + \frac{4}{N} \right) - \frac{4\omega_N}{N} u_r(1) v_r(1) \right].$$

Proof. Recall that both u and v are radial. Since $\int_{B_1} u^2\, dx = 1$, the standard Pohozaev identity gives

$$\left(\frac{N}{2} - 1\right) \int_{B_1} |\nabla u|^2\, dx + \frac{1}{2} \int_{\partial B_1} |\nabla u|^2 (x \cdot \nu)\, d\sigma + \frac{\lambda N}{2} = \frac{\mu N}{p+1} \int_{B_1} |u|^{p+1}\, dx.$$

Inserting the information that u is radial and the equalities $\alpha = \int_{B_1} |\nabla u|^2\, dx$ and $\alpha + \lambda = \mu \int_{B_1} u^{p+1}\, dx$, we obtain

$$\lambda = \frac{2}{N} \frac{p+1}{p-1} \alpha - \alpha - \frac{\omega_N}{N} \frac{p+1}{p-1} u_r(1)^2.$$

Differentiating with respect to α, we have

$$\lambda' = \frac{2}{N} \frac{p+1}{p-1} - 1 - \frac{2\omega_N}{N} \frac{p+1}{p-1} u_r(1) v_r(1).$$

The result follows by recalling relation (5-6). □

The following crucial lemma shows that, if p is subcritical or critical, then μ is an increasing function of α:

Lemma 5.8. *If $p \le 1+4/N$, then $\mu'(\alpha) > 0$ for every $\alpha > \lambda_1(B_1)$.*

Proof. The proof goes by contradiction: suppose that $\mu'(\bar\alpha) \le 0$ for some $\bar\alpha > \lambda_1(B_1)$. In the rest of the proof, all quantities are evaluated at such $\bar\alpha$.

Step 1. Let $v := \frac{d}{d\alpha} u|_{\alpha=\bar\alpha}$; then $v_r(1) < 0$ in case $p < 1+4/N$ and $v_r(1) \le 0$ if $p = 1+4/N$. This is an immediate consequence of Lemma 5.7, since $u_r(1) < 0$ by the Hopf lemma.

Step 2. We claim that, if r is sufficiently close to 1^-, then $v(r) > 0$. Since $v(1) = 0$, this is obvious if $v_r(1) < 0$. Hence, it only remains to consider the case $p = 1+4/N$ and $v_r(1) = 0$.

From the equation for v written in the radial coordinate

$$-v_{rr} - \frac{N-1}{r} v_r + \lambda v + \lambda' u = p\mu u^{p-1} v + \mu' u^p, \quad r \in (0,1),$$

we know (by letting $r \to 1^-$) that $v_{rr}(1) = 0$. Differentiating both sides of the above equation, we can write

$$-v_{rrr} + \frac{N-1}{r^2} v_r - \frac{N-1}{r} v_{rr} + \lambda v_r + \lambda' u_r = p(p-1)\mu u^{p-2} u_r v + p\mu u^{p-1} v_r + p\mu' u^{p-1} u_r;$$

now, if $p \ge 2$, the limit as $r \to 1^-$ yields

$$-v_{rrr}(1) + \lambda' u_r(1) = 0.$$

On the other hand, if $p < 2$, the same identity holds since by the l'Hôpital's rule

$$\lim_{r\to 1^-} u^{p-2} u_r v = \lim_{r \to 1^-} \frac{u_{rr} v + u_r v_r}{(2-p)u^{1-p} u_r} = \frac{u_{rr}(1)v(1) + u_r(1)v_r(1)}{(2-p)u_r(1)} u(1)^{p-1} = 0.$$

Thus, $v_{rrr}(1) < 0$ by Lemma 5.6, and the claim follows.

Step 3. Let $\bar r := \inf\{r : v > 0 \text{ in } (r,1)\}$ ($\bar r > 0$ since $\int_{B_1} uv\,dx = 0$). We claim that $v \le 0$ in $B_{\bar r}$. If not, there would be $0 \le r_1 < r_2 \le \bar r$ with the property that $v > 0$ in (r_1, r_2) and $r_i v(r_i) = 0$. Defining

$$v_1 := v|_{B_{r_2} \setminus B_{r_1}}, \qquad v_2 := v|_{B_1 \setminus B_{\bar r}},$$

we have that $v_i \in H_0^1(B_1)$ and $v_i \ge 0$ for $i = 1, 2$, and v_1 and v_2 are linearly independent. One can use the equation for v in order to evaluate

$$J''_{\mu,\lambda}(u)[v, v_i] = \int_{B_1} (\nabla v \cdot \nabla v_i + (\lambda - p\mu u^{p-1}) v v_i)\,dx = \int_{B_1} (\mu' u^p v_i - \lambda' u v_i)\,dx < 0$$

and obtain

$$J''_{\mu,\lambda}(u)[t_1 v_1 + t_2 v_2, t_1 v_1 + t_2 v_2] < 0 \quad \text{whenever } t_1^2 + t_2^2 \ne 0$$

in contradiction to the fact that the Morse index of u is 1 (Remark 5.2).

Step 4. Once we know that $v \le 0$ in $B_{\bar r}$ and that $v > 0$ in $B_1 \setminus B_{\bar r}$, we can combine the first equations in (5-5) and (5-6), together with the fact that u is monotone decreasing with respect to r, to write

$$\begin{aligned}\frac{1}{2\mu} &= \int_{B_1} u^p v\,dx = \int_{B_1\setminus B_{\bar r}} u^p v\,dx + \int_{B_{\bar r}} u^p v\,dx \\ &\le \Big(\max_{B_1\setminus B_{\bar r}} u^{p-1}\Big)\int_{B_1\setminus B_{\bar r}} uv\,dx + \Big(\min_{B_{\bar r}} u^{p-1}\Big)\int_{B_{\bar r}} uv\,dx \\ &= u^{p-1}(\bar r)\int_{B_1\setminus B_{\bar r}} uv\,dx + u^{p-1}(\bar r)\int_{B_{\bar r}} uv\,dx = 0,\end{aligned}$$

a contradiction. □

Remark 5.9. When $1+4/N < p < 2^*-1$, Lemma 4.5 implies $\mu(+\infty)=0$. Since also $\mu(\lambda_1(B_1)^+)=0$, we deduce that μ' must change sign in the supercritical regime. Numerical experiments suggest that this should happen only once so that μ should have a unique global maximum and be strictly monotone elsewhere; see Remark 6.4 ahead.

We are ready to prove the existence of least energy solutions for (1-2).

Proof of Theorem 1.5. Recalling Definition 1.2, let $\rho > 0$ be fixed, and let $U \in \mathcal{P}_\rho$. Then

$$\int_{B_1} U^2\,dx = \rho, \quad U > 0, \qquad -\Delta U + \lambda U = U^p$$

for some λ. Then, setting $u = \rho^{-1/2}U$, direct calculations yield

$$\int_{B_1} u^2\,dx = 1, \quad u > 0, \qquad -\Delta u + \lambda u = \rho^{(p-1)/2}u^p.$$

Writing $\int_{B_1}|\nabla u|^2\,dx = \alpha$, this amounts to saying that $(u, \rho^{(p-1)/2}, \lambda, \alpha) \in \mathcal{S}^+$. Equivalently,

$$U \in \mathcal{P}_\rho \iff \rho = \mu^{2/(p-1)}, \quad U = \mu^{1/(p-1)}u \quad \text{for some } (u,\mu,\lambda,\alpha) \in \mathcal{S}^+.$$

We divide the end of the proof into three cases.

Case 1: $1 < p < 1+4/N$. By Lemmas 4.5 and 5.8 and Proposition 5.4, we have that, for every ρ, there exists exactly one point in $\mathcal{S}^+$ satisfying $\mu^{2/(p-1)} = \rho$.

Case 2: $p = 1+4/N$. The same as the previous case, taking into account that, by Lemma 4.5, $\mathcal{P}_{\mu^{2/(p-1)}}$ is not empty if and only if $\mu < \|Z_{N,p}\|^{p-1}_{L^2(\mathbb{R}^N)}$.

Case 3: $1+4/N < p < 2^*-1$. Since in this case $\mu(\lambda_1(B_1)) = \mu(+\infty) = 0$ (by Lemma 4.5), then

$$\mu^* = \max_{(\lambda_1(B_1),+\infty)} \mu$$

is well defined and achieved. Furthermore, $\mathcal{P}_{\mu^{2/(p-1)}}$ is empty for $\mu > \mu^*$, and it contains at least two points for $0 < \mu < \mu^*$. It remains to prove that, if $0 < \rho \le \rho^* = (\mu^*)^{(p-1)/2}$, then e_ρ is achieved. This is immediate whenever $\mathcal{P}_\rho$ is finite. Otherwise, let $u_n = u(\alpha_n)$, with $\mu(\alpha_n) = \rho^{(p-1)/2}$, denote a minimizing sequence. Then Lemma 4.5 implies that α_n is bounded, and by continuity, the same is true for λ_n. We deduce that, up to subsequences, $u_n \to u^* \in \mathcal{P}_{\bar\mu}$, and $J_{\bar\mu,0}(u^*) = e_\rho$. □

Remark 5.10. By comparing Theorem 1.5 and Proposition A.1, we have that, when $p \le 1+4/N$ and positive least energy solutions exist, the condition $U > 0$ may be safely removed from Definition 1.2 without altering the problem (in fact, also the condition $-\Delta U + \lambda U = U^{p+1}$ for some λ is not necessary). On the other hand, in other cases, it is essential. For instance, when p is critical, then the set of not necessarily positive solutions with fixed mass

$$\mathcal{P}'_\rho = \{U \in H_0^1(B_1) : \mathfrak{Q}(U) = \rho, \text{ there exists } \lambda \text{ such that } -\Delta U + \lambda U = U^p\}$$

is not empty also when $\rho \ge \|Z_{N,p}\|^2_{L^2(\mathbb{R}^N)}$, as illustrated in [Fibich and Merle 2001, Figure 1].

6. Stability results

In this section, we discuss orbital stability of standing wave solutions $e^{i\lambda t}U(x)$ for the NLS (1-1). We recall that such solutions are called orbitally stable if for each $\varepsilon > 0$ there exists $\delta > 0$ such that, whenever $\Phi_0 \in H_0^1(B_1, \mathbb{C})$ is such that $\|\Phi_0 - U\|_{H_0^1(B_1,\mathbb{C})} < \delta$ and $\Phi(t,x)$ is the solution of (1-1) with $\Phi(0,\cdot) = \Phi_0$ in some interval $[0, t_0)$, then $\Phi(t,\cdot)$ can be continued to a solution in $0 \le t < \infty$ and

$$\sup_{0<t<\infty} \inf_{s\in\mathbb{R}} \|\Phi(t,\cdot) - e^{i\lambda s}U\|_{H_0^1(B_1,\mathbb{C})} < \varepsilon;$$

otherwise, they are called unstable. To do this, we lean on the following result, which expresses in our context the abstract theory developed in [Grillakis et al. 1987]:

Proposition 6.1 [Fukuizumi et al. 2012, Proposition 5]. *Let us assume local existence as in Theorems 1.7 and 1.8, and let R_λ be the unique positive solution of (1-2).*

- *If $\partial_\lambda \|R_\lambda\|^2_{L^2} > 0$, then $e^{i\lambda t}R_\lambda$ is orbitally stable.*
- *If $\partial_\lambda \|R_\lambda\|^2_{L^2} < 0$, then $e^{i\lambda t}R_\lambda$ is unstable.*

Corollary 6.2. *Let $(u(\alpha), \mu(\alpha), \lambda(\alpha), \alpha) \in \mathcal{S}^+$ with $U(\alpha) = \mu^{1/(p-1)}(\alpha)u(\alpha)$ denoting the corresponding solution of (1-2) (with $\lambda = \lambda(\alpha)$).*

- *If $\mu'(\alpha) > 0$, then $e^{i\lambda(\alpha)t}U(\alpha)$ is orbitally stable.*
- *If $\mu'(\alpha) < 0$, then $e^{i\lambda(\alpha)t}U(\alpha)$ is unstable.*

Proof. Taking into account Proposition 5.4 and Lemma 5.6, and reasoning as in the proof of Theorem 1.5, we have that $R_{\lambda(\alpha)} = \mu^{1/(p-1)}(\alpha)u(\alpha)$ so that

$$\partial_\lambda \|R_\lambda\|^2_{L^2} = \frac{(\mu^{2/(p-1)})'(\alpha)}{\lambda'(\alpha)} = \frac{2\mu^{(3-p)/(p-1)}(\alpha)}{(p-1)\lambda'(\alpha)}\mu'(\alpha). \qquad \square$$

We recall that μ' may be negative only when p is supercritical. This case is enlightened by the following lemma:

Lemma 6.3. *Let $p > 1+4/N$, and consider the map $\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha))$ defined as in Proposition 5.4. If $\alpha_1 < \alpha_2$ are such that*

$$\mu(\alpha) > \mu(\alpha_1) = \mu(\alpha_2) =: \bar{\mu} \quad \textit{for every } \alpha \in (\alpha_1, \alpha_2),$$

then

$$J_{\bar\mu,0}(u(\alpha_1)) < J_{\bar\mu,0}(u(\alpha_2)).$$

Proof. Writing $M(\alpha) = M_\alpha = \int_{B_1} u^{p+1}(\alpha)\,dx$, we have that

$$2J_{\bar\mu,0}(\alpha_i) = \alpha_i - \frac{2\bar\mu}{p+1} M(\alpha_i).$$

Now, (5-6) yields $M'(\alpha) = (p+1)\int_{B_1} u^p v\,dx = (p+1)/(2\mu(\alpha))$, where as usual $v := \frac{d}{d\alpha}u$. The Lagrange theorem applied to M forces the existence of $\alpha^* \in (\alpha_1, \alpha_2)$ such that

$$\frac{M(\alpha_2) - M(\alpha_1)}{\alpha_2 - \alpha_1} = \frac{p+1}{2\mu(\alpha^*)} < \frac{p+1}{2\bar\mu},$$

which is equivalent to the desired statement. □

We are ready to give the proofs of our stability results.

Proof of Theorems 1.7 and 1.8. The proof in the subcritical and critical cases is a direct consequence of Lemma 5.8 and Corollary 6.2 (recall that in this case there is a full correspondence between least energy solutions and least action ones). To show Theorem 1.7(2), we prove stability for any $\rho > 0$ such that $\bar\mu = \rho^{(p-1)/2}$ is a regular value of the map $\alpha \mapsto \mu(\alpha)$, the conclusion following by the Sard lemma. Recalling that $\mu(\lambda_1(B_1)) = \mu(+\infty) = 0$, we have that, if $\bar\mu$ is regular, then its counterimage $\{\alpha : \mu(\alpha) = \bar\mu\}$ is the union of a finite number of pairs $\{\alpha_{i,1}, \alpha_{i,2}\}$, each of which satisfies the assumptions of Lemma 6.3, and moreover, $\mu'(\alpha_{i,1}) > 0 > \mu'(\alpha_{i,2})$. Since such a counterimage is in 1-to-1 correspondence with $\mathcal{P}_\rho$ and

$$\mathcal{E}(U(\alpha_{i,j})) = \mathcal{E}(\bar\mu^{1/(p-1)} u(\alpha_{i,j})) = \bar\mu^{2/(p-1)} J_{\bar\mu,0}(u(\alpha_{i,j})),$$

we deduce from Lemma 6.3 that the least energy solution corresponds to $\alpha_{i,1}$, for some i, and the conclusion follows again by Corollary 6.2. □

Remark 6.4. In the supercritical case $p > 1 + 4/N$, we expect orbital stability for every $\rho \in (0, \rho^*)$ and instability for $\rho = \rho^*$. Indeed, in case $N = 3$ and $p = 3$, we have plotted numerically the graph of $\mu(\alpha)$ in Figure 1. The picture suggests that μ has a unique local maximum μ^*, associated to the maximal value of the mass $\rho^* = (\mu^*)^{(p-1)/2}$. For any $\mu < \mu^*$, we have exactly two solutions, and the least energy one corresponds to $\mu'(\alpha) > 0$; hence, it is associated with an orbitally stable standing wave. For $\mu = \mu^*$, we have exactly one solution; in that case, the abstract theory developed in [Grillakis et al. 1987] predicts the corresponding standing wave to be unstable.

Appendix A: Gagliardo–Nirenberg inequalities

It is proved in [Weinstein 1983] that the sharp Gagliardo–Nirenberg inequality

$$\|u\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \le C_{N,p} \|u\|^{p+1-N(p-1)/2}_{L^2(\mathbb{R}^N)} \|\nabla u\|^{N(p-1)/2}_{L^2(\mathbb{R}^N)} \tag{A-1}$$

holds for every $u \in H^1(\mathbb{R}^N)$ and that the best constant $C_{N,p}$ is achieved by (any rescaling of) $Z_{N,p}$.

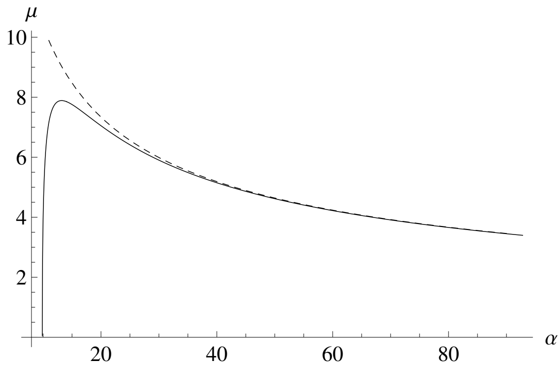

Figure 1. Numerical graph of $\alpha \mapsto \mu(\alpha)$ in the supercritical case $N = 3$ and $p = 3$ (continuous line) and of the map $\alpha \mapsto \alpha^{-1/2} \cdot \sqrt{3} \int_{\mathbb{R}^3} Z_{3,3}^2 \, dx$ (dashed line). The latter is the theoretical asymptotic expansion of $\mu(\alpha)$ as $\alpha \to +\infty$ as predicted by Lemmas 4.4 and 4.5.

When dealing with $H_0^1(\Omega)$, $\Omega \neq \mathbb{R}^N$, one can prove that the identity holds with the same best constant: in fact, one inequality is trivial, and the other is obtained by constructing a suitable competitor of the form $u(x) = (hZ_{N,p}(kx) - j)^+$, for suitable h, k and j, and exploiting the exponential decay of Z. Contrary to the previous case, now such a constant cannot be achieved; otherwise, we would contradict [Weinstein 1983]. This is related to the maximization problem (1-3) since

$$C_{N,p} = \sup_{H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|_{L^{p+1}(\Omega)}^{p+1}}{\|u\|_{L^2(\Omega)}^{p+1-N(p-1)/2} \|\nabla u\|_{L^2(\Omega)}^{N(p-1)/2}} = \sup_{\alpha \geq \lambda_1(\Omega)} \frac{M_\alpha}{\alpha^{N(p-1)/4}}.$$

By the above considerations, we deduce that

$$M_\alpha < C_{N,p} \alpha^{N(p-1)/4} \quad \text{for every } \alpha, \qquad \lim_{\alpha \to +\infty} \frac{M_\alpha}{\alpha^{N(p-1)/4}} = C_{N,p} \tag{A-2}$$

in perfect agreement with the estimates at the end of Section 4.

For the reader's convenience, we deduce the following well-known result:

Proposition A.1. *Let $\rho > 0$ be fixed. The infimum*

$$\inf\{\mathscr{E}(U) : U \in H_0^1(\Omega) \textit{ and } \mathscr{Q}(U) = \rho\}$$

(i) *is achieved by a positive function if either $1 < p < 1 + 4/N$ or $p = 1 + 4/N$ and $\rho < \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2$, and*

(ii) *equals $-\infty$ if either $1 + 4/N < p < 2^* - 1$ or $p = 1 + 4/N$ and $\rho > \|Z_{N,p}\|_{L^2(\mathbb{R}^N)}^2$.*

Proof. As usual, writing $u = \rho^{-1/2} U$ and $\bar{\mu} = \rho^{(p-1)/2}$, we have that the above minimization problem is equivalent to

$$\inf\{J_{\bar{\mu},0}(u) : u \in H_0^1(\Omega) \text{ and } \|u\|_{L^2(\Omega)} = 1\},$$

where $J_{\mu,\lambda}$ is defined in (4-1). In turn, this problem can be written as

$$\inf_{\alpha\geq\lambda_1(\Omega)} \tfrac{1}{2}\alpha - \frac{\bar{\mu}}{p+1} M_\alpha .$$

The proposition follows from (A-2), recalling that, when $p = 1+4/N$,

$$C_{N,p} = \left(1+\frac{2}{N}\right)\left(\int_{\mathbb{R}^N} Z_{N,p}^2 \, dx\right)^{-2/N}$$

by the Pohozaev identity. □

Appendix B: The defocusing case $\mu < 0$

In this case, it is not necessary to restrict to spherical domains; therefore, in this appendix, we consider a generic smooth, bounded domain Ω. As in Section 5, we work in the space $X = \{w \in W^{2,k}(\Omega) : w = 0 \text{ on } \partial\Omega\}$, for some $k > N$, and with the map $F : X \times \mathbb{R}^3 \to L^k(\Omega) \times \mathbb{R}^2$ defined by

$$F(u,\mu,\lambda,\alpha) = \left(\Delta u - \lambda u + \mu u^p, \int_\Omega u^2 \, dx - 1, \int_\Omega |\nabla u|^2 - \alpha\right).$$

We aim to provide a full description of the set

$$\mathcal{S}^- = \{(u,\mu,\lambda,\alpha) \in X \times \mathbb{R}^3 : u > 0,\ \mu < 0,\ F(u,\mu,\lambda,\alpha) = (0,0,0)\},$$

thus concluding the proof of Theorem 1.12.

Lemma B.1. *Let $(u,\mu,\lambda,\alpha) \in \mathcal{S}^-$. Then the linear bounded operator*

$$F_{(u,\mu,\lambda)}(u,\mu,\lambda,\alpha) : X \times \mathbb{R}^2 \to L^k(\Omega) \times \mathbb{R}^2$$

is invertible.

Proof. As in the proof of Lemma 5.3, it is sufficient to prove injectivity.

As in that proof, we assume the existence of a nontrivial (v, m, l) such that (5-3) and (5-4) hold. Since $\partial_\nu u < 0$ on $\partial\Omega$, we can test the equation for u by $v^2/u \in H_0^1(\Omega)$, obtaining

$$\begin{aligned}
\int_\Omega (\mu u^{p-1} v^2 - \lambda v^2)\, dx &= \int_\Omega \nabla u \cdot \nabla\left(\frac{v^2}{u}\right) dx = \int_\Omega \nabla u \cdot \left(2\frac{v}{u}\nabla v - \frac{v^2}{u^2}\nabla u\right) dx \\
&= -\int_\Omega \left|\frac{v}{u}\nabla u - \nabla v\right|^2 dx + \int_\Omega |\nabla v|^2 \, dx \\
&\leq \int_\Omega (p\mu u^{p-1} v^2 + m u^p v - luv - \lambda v^2)\, dx \\
&= \int_\Omega (p\mu u^{p-1} v^2 - \lambda v^2)\, dx.
\end{aligned}$$

Therefore, with $\mu < 0$ and $p > 1$, we must have $v \equiv 0$. Finally, by testing the equation for v by u, we deduce that $l = m \int_\Omega u^{p+1} \, dx$, concluding the proof. □

Proposition B.2. *$\mathcal{S}^-$ is a smooth curve, and it can be parametrized by a unique map*

$$\alpha \mapsto (u(\alpha), \mu(\alpha), \lambda(\alpha)), \quad \alpha \in (\lambda_1(\Omega), +\infty).$$

In particular, $u(\alpha)$ is the unique minimizer associated to m_α (as defined in (1-3)). Furthermore, $\mu'(\alpha) < 0$ and $\lambda'(\alpha) < 0$ for every α.

Proof. One can use Lemma B.1 and reason as in the proof of Proposition 5.4 in order to prove that $\mathcal{S}^-$ consists of a unique, smooth curve parametrized by $\alpha \in (\lambda_1(\Omega), +\infty)$ so that $u(\alpha)$ must achieve m_α. Moreover, all the relations contained in Corollary 5.5 are true also in this case.

In order to show the monotonicity of μ and λ, we remark that one can also prove, in a standard way, that u is the global unique minimizer of the related functional $J_{\mu,\lambda}$, which is bounded below and coercive since $\mu < 0$. Since u is nondegenerate (by virtue of Lemma B.1), we obtain that $J''_{\mu,\lambda}(u)[w, w] > 0$ for every nontrivial w. But then one can reason as in the proof of Lemma 5.6: using the corresponding notation, we have that in this case both $c > 0$ and $b^2 - ac < 0$. This, together with (5-6), concludes the proof. □

Remark B.3. By the above results, it is clear that $\mathcal{S}^-$ may be parametrized also with respect to λ (or μ). Under this perspective, uniqueness and continuity for the case $p = 3$ were proved in [Berger and Fraenkel 1970] (for the problem without mass constraint).

We conclude by showing some asymptotic properties of $\mathcal{S}^-$ as $\alpha \to +\infty$ (the case $\alpha \to \lambda_1(\Omega)^+$ has been considered in Section 3). Such properties are well known in the case $p = 3$ since they have been studied in a different context (among others, we cite [Berger and Fraenkel 1970; Bethuel et al. 1993; André and Shafrir 1998; Serfaty 2001]) and the proof can be adapted to general p.

Proposition B.4. *Under the notation of Proposition B.2, we have that, as $\alpha \to +\infty$, $\mu \to -\infty$ and $\lambda \to -\infty$. Furthermore, if $\partial\Omega$ is smooth, then*

$$u \to |\Omega|^{-1/2} \text{ strongly in } L^{p+1}(\Omega), \qquad \frac{\lambda}{\mu} \to |\Omega|^{-(p-1)/2}, \qquad \frac{\alpha}{\lambda} \to 0$$

as $\alpha \to +\infty$.

Proof. Since we know that μ is decreasing and that for each $\mu < 0$ there exists a solution, we must have $\mu(\alpha) \to -\infty$. Moreover, $\lambda \le -\alpha \to -\infty$.

Next we are going to show that, under the assumption that $\partial\Omega$ is smooth,

$$\int_\Omega u^{p+1} \to |\Omega|^{-(p-1)/2}. \tag{B-1}$$

To this aim, notice that, by the uniqueness proved in the previous proposition, u satisfies

$$J_{\mu,0}(u) = \min\left\{ J_{\mu,0}(\varphi) : \varphi \in H^1_0(\Omega),\ \int_\Omega \varphi^2\,dx = 1 \right\}.$$

For $x \in \Omega$, setting $d(x) := \mathrm{dist}(x, \partial\Omega)$, we construct a competitor function for the energy $J_{\mu,0}(u)$ as

$$\varphi_\mu(x) = \begin{cases} k^{-1}|\Omega|^{-1/2} & \text{if } d(x) \ge (-\mu)^{-1/2}, \\ k^{-1}|\Omega|^{-1/2}(-\mu)^{1/2} d(x) & \text{if } 0 \le d(x) \le (-\mu)^{-1/2}, \end{cases}$$

where k is such that $\|\varphi_\mu\|_{L^2(\Omega)} = 1$. With the aid of the coarea formula, and using the fact that $\partial\Omega$ is smooth, it is possible to check that $k = 1 + \mathrm{O}((-\mu)^{-1/2})$, and thus,

$$\int_\Omega |\nabla\varphi_\mu|^2\,dx = \mathrm{O}(\sqrt{-\mu}), \qquad \int_\Omega (\varphi_\mu^q - |\Omega|^{-q/2})\,dx = \mathrm{O}((-\mu)^{-1/2}) \tag{B-2}$$

for every $q > 1$. By rewriting $J_{\mu,0}$ in the form

$$J_{\mu,0}(\varphi) = \int_\Omega \left\{ \frac{|\nabla\varphi|^2}{2} - \frac{\mu}{p+1}(|\varphi|^{p+1} - |\Omega|^{-(p+1)/2}) \right\} dx - \frac{\mu}{p+1}|\Omega|^{-(p-1)/2},$$

and by using the estimates (B-2) with $q = p+1$, we obtain

$$J_{\mu,0}(u) \le J_{\mu,0}(\varphi_\mu) = \mathrm{O}(\sqrt{-\mu}) - \frac{\mu}{p+1}|\Omega|^{-(p-1)/2}$$

so that

$$0 \le \int_\Omega (u^{p+1} - |\Omega|^{-(p+1)/2})\,dx \le \mathrm{O}((-\mu)^{-1/2}) \to 0$$

(by using Lemma 2.1(iv)) so that (B-1) is proved.

Now, for each L^2-normalized φ, we rewrite $J_{\mu,0}(\varphi)$ as

$$J_{\mu,0}(\varphi) = \int_\Omega \left\{ \frac{|\nabla\varphi|^2}{2} - \frac{\mu}{p+1}(|\varphi|^{(p+1)/2} - |\Omega|^{-(p+1)/4})^2 \right\} dx \\ - \frac{2\mu}{p+1}|\Omega|^{-(p+1)/4} \int_\Omega (|\varphi|^{(p+1)/2} - |\Omega|^{-(p+1)/4})\,dx - \frac{\mu}{p+1}|\Omega|^{-(p-1)/2}.$$

Reasoning as before (using this time (B-2) for $q = (p+1)/2$), one shows that

$$\int_\Omega (|u|^{(p+1)/2} - |\Omega|^{-(p+1)/4})^2\,dx + 2|\Omega|^{-(p+1)/2} \int_\Omega (|u|^{(p+1)/2} - |\Omega|^{-(p+1)/2})\,dx \le \mathrm{O}((-\mu)^{-1/2}).$$

If $p \ge 3$, by the Hölder inequality, we have that the second integral in the left-hand side above is nonnegative while for $p < 3$ it tends to 0 as $\alpha \to +\infty$. The latter statement is a consequence of both the Hölder and interpolation inequalities, which yield

$$\int_\Omega u^{(p+1)/2}\,dx \le |\Omega|^{(3-p)/4}, \qquad \|u\|_{L^{(p+1)/2}(\Omega)} \ge \|u\|_{L^{p+1}(\Omega)}^{(p-3)/(p-1)},$$

as well as of (B-1). Thus, we have concluded that

$$u^{(p+1)/2} \to |\Omega|^{-(p+1)/4} \quad \text{in } L^2(\Omega).$$

In particular, up to a subsequence, $u \to |\Omega|^{-1/2}$ a.e., and there exists $h \in L^2$ (independent of α) so that $|u|^{(p+1)/2} \le h$. We can now conclude by applying Lebesgue's dominated convergence theorem.

To proceed with the proof, notice that, from the equality $\alpha + \lambda = \mu \int_\Omega u^{p+1}\,dx$ and Lemma 2.1(iv), we deduce

$$\lambda \le \mu|\Omega|^{-(p-1)/2}. \tag{B-3}$$

On the other hand, we have

$$-\lambda \le (p+1)J_{\mu,0}(u) \le (p+1)J_{\mu,0}(\varphi_\mu) \le C(-\mu)^{1/2} - \mu|\Omega|^{-(p-1)/2}.$$

Dividing the last inequality by $-\mu$ and letting $\mu \to -\infty$, we obtain

$$\limsup \frac{\lambda}{\mu} \le |\Omega|^{-(p-1)/2},$$

which together with (B-3) provides the convergence of μ.

The last part of the statement is obtained by combining the previous asymptotics with the identity $\alpha/\mu = -\mu + \int_\Omega u^{p+1}\,dx$. □

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BENEDETTA NORIS: benedettanoris@gmail.com
INdAM-COFUND Marie Curie Fellow, Laboratoire de Mathématiques, Université de Versailles Saint-Quentin-en-Yvelines, 45 avenue des États-Unis, 78035 Versailles, France

HUGO TAVARES: htavares@ptmat.fc.ul.pt
Universidade de Lisboa, Centro de Matemática e Aplicações Fundamentais and Faculdade de Ciências da Universidade de Lisboa, Avenida Professor Gama Pinto 2, 1649-003 Lisboa, Portugal

GIANMARIA VERZINI: gianmaria.verzini@polimi.it
Dipartimento di Matematica, Politecnico di Milano, Piazza Leonardo da Vinci 32, 20133 Milano, Italy

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BOUNDARY BLOW-UP UNDER SOBOLEV MAPPINGS

AAPO KAURANEN AND PEKKA KOSKELA

We prove that for mappings in $W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$, continuous up to the boundary and with modulus of continuity satisfying a certain divergence condition, the image of the boundary of the unit ball has zero n-Hausdorff measure. For Hölder continuous mappings we also prove an essentially sharp generalised Hausdorff dimension estimate.

1. Introduction

Throughout this paper $\mathcal{B}^n$ denotes the unit ball in $\mathbb{R}^n$ and $W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ is the Sobolev space of $L^n(\mathcal{B}^n, \mathbb{R}^m)$-functions $f : \mathcal{B}^n \to \mathbb{R}^m$ with weak first-order derivatives in $L^n(\mathcal{B}^n)$.

If $f : \mathcal{B}^2 \to \Omega \subset \mathbb{R}^2$ is a conformal mapping, then the boundary of Ω can have positive Lebesgue measure even if f extends continuously up to the boundary of the disk. If one requires more, for example uniform Hölder continuity, then $\partial\Omega$ is necessarily of Lebesgue measure zero. In fact, Jones and Makarov proved [1995, Theorem C.1] that $\partial\Omega$ has measure zero if f satisfies $|f(z) - f(w)| \le \psi(|z - w|)$ in $\mathcal{B}^2$ for $\psi : [0, \infty) \to [0, \infty)$ with

$$\int_0 \left| \frac{\log \psi(t)}{\log t} \right|^2 \frac{\mathrm{d}t}{t} = \infty. \tag{1}$$

This condition is very sharp: if the integral in (1) converges then [Jones and Makarov 1995, Section 6] provides us with a simply connected domain Ω and a conformal mapping $f : \mathcal{B}^2 \to \Omega$ such that the boundary of Ω has positive Lebesgue measure and f has the modulus of continuity ψ.

Our first result gives a surprisingly general extension of the conformal setting; notice that each uniformly continuous conformal mapping $f : \mathcal{B}^2 \to \Omega$ belongs to $W^{1,2}(\mathcal{B}^2, \mathbb{R}^2)$.

Theorem 1.1. *Let $f \in W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ be a continuous mapping that satisfies*

$$|f(z) - f(w)| \le \psi(|z - w|) \tag{2}$$

for all $z, w \in \overline{\mathcal{B}}^n$, where $\psi : (0, \infty) \to (0, \infty)$ is an allowable modulus of continuity with

$$\int_0 \left| \frac{\log \psi(t)}{\log t} \right|^n \frac{dt}{t} = \infty. \tag{3}$$

Then $\mathcal{H}^n(f(\partial\mathcal{B}^n)) = 0$.

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Recall that every uniformly continuous map defined on $\mathcal{B}^n$ has a continuous extension to all of $\overline{\mathcal{B}}^n$. In the above, f on $\partial\mathcal{B}^n$ refers to this extension, $\mathcal{H}^n(A)$ denotes the n-dimensional Hausdorff measure of a set A, and the definition of an *allowable modulus of continuity* is given in Definition 2.2 of Section 2. For example, both $\psi(t) = Ct^\gamma$, $0 < \gamma \le 1$, and

$$\psi_{l,s}(t) = \exp\left(-C\frac{(\log(C_l/t))^{(n-1)/n}}{(\log^{(l)}(C_l/t))^{s/n}\left(\prod_{k=2}^{l-1}\log^{(k)}(C_l/t)\right)^{1/n}}\right)$$

are allowable, where $l \ge 2$ is an integer and $s > 0$. Notice that $\psi_{l,s}$ satisfies (3) if and only if $s \le 1$. Here $C > 0$, $\log^{(k)} t$ is the k-times iterated logarithm and C_l can be any constant with $\log^{(l)}(C_l/2) \ge 1$.

Let us look at the special case $n = m = 2$ of Theorem 1.1 in the Hölder continuous setting: $\psi(t) = Ct^\gamma$, where $0 < \gamma \le 1$. Consider a space-filling (Peano) curve, i.e., a continuous mapping g from the unit circle onto a square. In one of the standard constructions, g is Hölder continuous with exponent $\gamma = \frac{1}{2}$; see, for example, [Buckley 1996, Theorem 3]. If one takes, say, the Poisson extension f of g to the unit disk, then f is also Hölder continuous. It is easy to check by hand that the partial derivatives of f do not belong to $L^2(\mathcal{B}^2)$. By Theorem 1.1, no Hölder continuous (or even continuous with control function satisfying (3)) extension f of a space filling curve can satisfy $|Df| \in L^2(\mathcal{B}^2)$.

In the Hölder continuous case, Jones and Makarov actually proved that the Hausdorff dimension of $f(\partial\mathcal{B}^2)$ is strictly less than two for conformal f. Contrary to the area zero results, this dimension estimate is truly conformal in the following sense:

Example 1.2. Let $p > 1$. There exists a locally Hölder continuous homeomorphism $f : \mathbb{R}^2 \to \mathbb{R}^2$ with $f \in W^{1,2}_{\text{loc}}(\mathcal{B}^2, \mathbb{R}^2)$, which maps $\partial\mathcal{B}^2$ onto a set of positive $\mathcal{H}^g$-measure for the gauge function $g(t) = t^2(\log(1/t))^p$.

This construction can be found in Section 4. Here $\mathcal{H}^g$ denotes the generalised Hausdorff measure with the function g as the dimension gauge. The precise definitions are given in Section 2.

Our second result gives a rather optimal positive result.

Theorem 1.3. *Fix $\gamma \in (0, 1]$, $C > 0$, and let $g(t) = t^n \log(1/t)$. Suppose that $f \in W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ satisfies*

$$|f(z) - f(w)| \le C|z - w|^\gamma$$

for all $z, w \in \mathcal{B}^n$. Then $\mathcal{H}^g(f(\partial\mathcal{B}^n)) = 0$.

Jones and Makarov proved their result via harmonic measure and hence this technique does not work in the setting of Theorem 1.1. An alternate approach, relying on the conformal (quasi)invariance of the (quasi)hyperbolic metric, was given in [Koskela and Rohde 1997]; see also [Nieminen 2006]. Furthermore, Malý and Martio [1995] established Theorem 1.1 in the Hölder continuous case via a technique that we have not been able to push further.

Let us briefly describe the idea of the proof of Theorem 1.1. We consider a Whitney decomposition $\mathcal{W}$ of $\mathcal{B}^n$ and assign to each $Q \in \mathcal{W}$ a vector $f_Q \in \mathbb{R}^m$ and a radius r_Q. The vector f_Q will simply be the "average" of f over Q and r_Q the maximum of $|f_Q - f_{\widetilde{Q}}|$ over all neighbours $\widetilde{Q}$ of Q. Then the n-integrability of the weak derivatives of f guarantees, via the Poincaré inequality, that the sequence

$\{r_Q\}_{Q\in\mathcal{W}}$ belongs to l^n. We realise $f(\partial\mathcal{B}^n)$ as (a part of) the closure of $\{f_Q\}_{Q\in\mathcal{W}}$ in $\mathbb{R}^m$. Those $f(\omega)$, $\omega\in\partial\mathcal{B}^n$, for which one can find a sequence of $Q\in\mathcal{W}$ with $|f_Q-f(\omega)|\lesssim r_Q$ are easily handled. For the remaining $\omega\in\partial\mathcal{B}^n$ we modify our centres f_Q and radii r_Q, while still retaining the l^n-condition, so that suitably blown-up balls cover these points sufficiently many times. This is where the nonintegrability condition (3) kicks in. One cannot fully follow the above idea, and so our proof, given below in Section 3, is more complicated.

Our approach is flexible and applies to many related problems. In order to avoid extra technicalities we do not record such applications here. Let us simply mention that the dimension gap phenomenon from [Hencl et al. 2012] can be shown to extend from conformal mappings to general Sobolev mappings [Koskela and Zapadinskaya 2014].

2. Preliminaries

Let us first agree on some basic notation. Given a number $a>0$, we write $\lfloor a\rfloor$ for the largest integer less than or equal to a. Similarly, $\lceil a\rceil$ is the smallest integer greater than or equal to a. If A is a finite set, $\sharp A$ is the number of elements in A. If $A\subset\mathbb{R}^n$ has finite and strictly positive Lebesgue measure and $f:\mathbb{R}^n\to\mathbb{R}$ is a Lebesgue integrable function, we denote the average $(1/|A|)\int_A f$ of f over the set A by $⨍_A f$ or f_A, where $|A|$ is the n-dimensional Lebesgue measure of the set A. For $f:\mathbb{R}^n\to\mathbb{R}^m$, f_A is then defined via the component functions of f. Given a point $x\in\mathbb{R}^n$ and a nonnegative number r, $B(x,r)$ denotes the open ball with centre x and radius r and $Q(x,r)$ denotes the cube $\{y\in\mathbb{R}^n:\max\{|x_i-y_i|\}_{i=1,2,\dots,n}\le r\}$. If $B=B(x,r)$ is a ball and a is a positive number, the notation aB stands for the ball $B(x,ar)$. We denote the radius of a ball B by $r(B)$. When we write $L=L(\cdot)$, we mean that the positive constant L depends only on the parameters listed inside the parentheses. Finally, C denotes a positive constant, which may depend only on n and m, the dimensions of the domain space and the image space, and may differ from occurrence to occurrence.

We write $\mathcal{H}^h(A)$ for the generalised Hausdorff measure of a set $A\subset\mathbb{R}^n$, given by

$$\mathcal{H}^h(A)=\lim_{\delta\to 0}\mathcal{H}^h_\delta(A),\quad \text{where } \mathcal{H}^h_\delta(A)=\inf\left\{\sum_{i=1}^\infty h(\operatorname{diam}U_i):A\subset\bigcup_{i=1}^\infty U_i,\ \operatorname{diam}U_i\le\delta\right\}$$

and h is a dimension gauge (a nondecreasing function with $\lim_{t\to 0+}h(t)=h(0)=0$ and with $h(t)>0$ for all $t>0$). If $h(t)=t^a$ for some $a\ge 0$ we simply write $\mathcal{H}^a$ for $\mathcal{H}^h$ and call it the a-dimensional Hausdorff measure.

A sequence of pairs $(c_i,U_i)_{i=1}^\infty$, where $c_i\ge 0$ and $U_i\subset\mathbb{R}^n$, that satisfies $\chi_A(x)\le\sum_{i=1}^\infty c_i\chi_{U_i}(x)$ for all $x\in\mathbb{R}^n$ is called a weighted cover of the set A. We also need a generalised weighted Hausdorff content of a set $A\subset\mathbb{R}^n$, given by

$$\lambda^h_\infty(A)=\inf\left\{\sum_{i=1}^\infty c_i h(\operatorname{diam}U_i):(c_i,U_i)_{i=1}^\infty \text{ is a weighted cover of } A\right\}.$$

Here also h is a gauge function. Again we write $\lambda^h_\infty=\lambda^a_\infty$ if $h(t)=t^a$.

Lemma 2.1. *Let $E \subset \mathbb{R}^n$ be bounded. Let h be a continuous gauge function with $h(2t) \le ch(t)$ for some $c > 0$. Then $\mathcal{H}^h_\infty(E) \le c\lambda^h_\infty(E)$.*

Proof. The lemma follows from Corollary 8.2 and the proof of Theorem 9.7 of [Howroyd 1994]; see also [Federer 1969, 2.10.24]. ☐

Recall that for each open subset U of $\mathbb{R}^n$ there exists a Whitney decomposition $\mathcal{W}$ given by $U = \bigcup_{i=1}^\infty Q_i$, where $Q_i \in \mathcal{W}$ are cubes with mutually parallel sides, pairwise disjoint interiors and each of edge-length 2^k for some integer k, such that the relation

$$\frac{1}{4} \le \frac{\operatorname{diam} Q_i}{\operatorname{dist}(Q_i, \partial\Omega)} \le 1 \tag{4}$$

holds for all $i = 1, 2, \ldots$. We write $Q_1 \backsim Q_2$ if the Whitney cubes $Q_1 \ne Q_2$ share at least one point (the so-called neighbour cubes). We have

$$\frac{1}{4} \le \frac{\operatorname{diam} Q}{\operatorname{diam} \widetilde{Q}} \le 4$$

whenever $Q \backsim \widetilde{Q}$. Therefore, the total number $\sharp\{\widetilde{Q} : \widetilde{Q} \backsim Q\}$ of all neighbours of a fixed cube Q does not exceed C. See [Stein 1970] for details.

Let $\omega \in \partial\mathcal{B}^n$. By $(Q_j(\omega))_{j=1}^\infty$ we mean the sequence of all Whitney cubes in a fixed Whitney decomposition of $\mathcal{B}^n$ intersecting the radius $[0, \omega]$. This sequence starts with a central cube and tends to ω. For a point $x \in [0, \omega]$, we denote the number of Whitney cubes intersecting the segment $[0, x]$ by $\sharp q(0, x)$. It is easy to see that

$$c_1 \le \frac{\sharp q(0, x)}{\log(1/(1 - |x|))} \le c_2 \tag{5}$$

whenever $\sharp q(0, x) > c_3$, where $c_i > 0$, $i = 1, 2, 3$ are constants that may depend on n.

Finally, we define the allowable moduli of continuity:

Definition 2.2. A continuously differentiable increasing bijection $\psi : (0, \infty) \to (0, \infty)$ is an *allowable modulus of continuity* if there exists $t_0 < 1$ and $\beta > 0$ such that for every $t \le t_0$ the following conditions hold:

$$\log \frac{1}{\psi^{-1}(t)} \text{ is differentiable and } \frac{(\psi^{-1})'(t)}{\psi^{-1}(t)}\, t \text{ is a decreasing function;} \tag{6}$$

$$\log \frac{1}{\psi^{-1}(t)} \le \beta \log \frac{1}{\psi^{-1}(\sqrt{t})}; \tag{7}$$

$$\frac{(\log \psi(t))' t \log t}{\log \psi(t)} \text{ is a monotone function.} \tag{8}$$

Remark 2.3. (i) One could replace the monotonicity conditions in (6) and (8) with a *pseudomonotonicity* condition (e.g., there exists a constant $C > 0$ such that $u(t) \le Cu(s)$ if $t \le s$). This would only affect the constants in the proofs.

(ii) The conditions (6) and (7) mean that the function $\log(1/\psi^{-1}(t))$ is a function of logarithmic type in the sense of [Nieminen 2006, Definition 4.2].

3. Proofs

Proof of Theorem 1.1. We may assume that $m, n \geq 2$. Let $f \in W^{1,n}(\mathcal{B}^n, \mathbb{R}^m)$ and ψ be as in the statement of Theorem 1.1. Denote $\psi^{-1}(t)$ by $u(t)$. It follows from our assumptions (3), (6), (7), (8) and [Nieminen 2006, Remark 5.3.] that

$$\int_0 \left(\frac{u(t)}{u'(t)}\right)^{n-1} \frac{dt}{t^n} = \infty. \tag{9}$$

We define $\alpha(t) = u(t)/u'(t)$ and $\lambda(k) = 2^{-k}/\alpha(2^{-k})$ for $k \in \mathbb{N}$. By (6), λ is increasing for large k. For simplicity we assume λ to be increasing.

Let $\mathcal{W}$ be a fixed Whitney decomposition of $\mathcal{B}^n$. For each cube $Q \in \mathcal{W}$ we define a corresponding centre f_Q and a corresponding radius $r_Q = \max\{|f_Q - f_{\widetilde{Q}}| : Q \backsim \widetilde{Q}\}$, which determine a family of balls on the image side indexed by $\mathcal{W}$:

$$\mathcal{B} = \{(Q, B(f_Q, r_Q)) : Q \in \mathcal{W},\ r_Q > 0\}.$$

To simplify our notation we abbreviate $(Q, B(f_Q, r_Q))$ to $B(f_Q, r_Q)$ in what follows.

We assign two new weighted collections of balls to each element in $\mathcal{B}$. Given $B = B(x, r) \in \mathcal{B}$, we define concentric subballs $S_i(B) = B(x, r/2^i)$ for all $i \in \mathbb{N}$ and assign the weight $w_{S_i(B)} = 2^i$ to each $S_i(B)$. We set $\mathcal{S}_B = \{S_i(B) : i \in \mathbb{N}\}$. Then

$$\sum_{B' \in \mathcal{S}_B} w_{B'} r(B')^n = \sum_{i=1}^{\infty} w_{S_i(B)} r(S_i(B))^n = \sum_{i=1}^{\infty} 2^i \frac{r(B)^n}{2^{ni}} \leq r(B)^n.$$

The second collection is defined in a similar way. If $B = B(x, r)$ is a ball in $\mathcal{B}$, we choose the smallest number $k_0(r) \in \mathbb{N}$ such that $2^{-k_0(r)} \leq r$. Next, for each $k = k_0(r), k_0(r)+1, \ldots,$ we choose $R_k(B) = B(x, \alpha(2^{-k}))$ and set $\mathcal{R}_B = \{R_k(B) : k = k_0(r), k_0(r)+1, \ldots\}$. The weights we assign this time are $w_{R_k(B)} = \lambda(k)$ for all $k = k_0(r), k_0(r)+1, \ldots$. Similarly to the above,

$$\begin{aligned}\sum_{B' \in \mathcal{R}_B} w_{B'} r(B')^n &= \sum_{k=k_0(r)}^{\infty} w_{R_k(B)} r(R_k(B))^n = \sum_{k=k_0(r)}^{\infty} (\alpha(2^{-k}))^n \lambda(k) \\ &\leq \sum_{k=k_0(r)}^{\infty} (\alpha(2^{-k}))^n \frac{\lambda(k)^n}{\lambda(0)^{n-1}} = \frac{1}{\lambda(0)^{n-1}} \sum_{k=k_0(r)}^{\infty} 2^{-nk} \leq \frac{2 \cdot 2^{-nk_0(r)}}{\lambda(0)^{n-1}} \leq \frac{2}{\lambda(0)^{n-1}} r(B)^n.\end{aligned}$$

Finally, we define our weighted collection of balls by setting $\mathcal{F} = \bigcup_{B \in \mathcal{B}} (\mathcal{S}_B \cup \mathcal{R}_B)$.

Let us now estimate the weighted sums of the n-th powers of the radii of the balls in $\mathcal{F}$. Let $N(Q) = Q \cup \bigcup_{\widetilde{Q} \backsim Q} \widetilde{Q}$ be the union of $Q \in \mathcal{W}$ and all neighbours $\widetilde{Q}$ of Q. For neighbouring cubes Q and $\widetilde{Q}$, we obtain, via the Hölder and Poincaré inequalities, that

$$\begin{aligned}|f_Q - f_{\widetilde{Q}}| &\leq \fint_Q |f - f_{N(Q)}| + \fint_{Q'} |f - f_{N(Q)}| \leq C \fint_{N(Q)} |f - f_{N(Q)}| \leq C \left(\fint_{N(Q)} |f - f_{N(Q)}|^n \right)^{1/n} \\ &\leq C \left(\fint_{N(Q)} |Df|^n \right)^{1/n}.\end{aligned}$$

Hence, we have the estimate

$$r_Q^n = \max\{|f_Q - f_{\widetilde{Q}}|^n : Q \backsim \widetilde{Q}\} \le C \int_{N(Q)} |Df|^n$$

for each $Q \in \mathcal{W}$ and some constant $C > 0$. Next, using the fact that the inequality $\sum_{Q\in\mathcal{W}} \chi_{N(Q)}(y) \le C$ holds for every $y \in \mathbb{R}^n$, we estimate

$$\begin{aligned}\sum_{B\in\mathcal{F}} w_B r(B)^n &\le C(\lambda(0)) \sum_{B\in\mathcal{B}} r(B)^n = C(\lambda(0)) \sum_{Q\in\mathcal{W}} r_Q^n \le C(\lambda(0)) \sum_{Q\in\mathcal{W}} \int_{N(Q)} |Df|^n \\ &\le C_1 \int_{\bigcup_{Q\in\mathcal{W}} N(Q)} |Df|^n \le C_1 \int_{\mathcal{B}^n} |Df|^n < \infty, \end{aligned} \tag{10}$$

where $C_1 > 0$ is some constant depending on n, m and $\lambda(0)$ only.

We may assume that there is at least one $Q \in \mathcal{W}$ with $r_Q > 0$; otherwise $f(\partial\mathcal{B}^n)$ is a singleton. Let $\omega \in \partial\mathcal{B}^n$. We consider the radius $[0,\omega]$ and the sequence $(Q_j(\omega))_{j=1}^\infty$. We fix a large integer $l_0 = l_0(\omega, f) \in \mathbb{N}$ so that there are elements of the sequence $(f_{Q_j(\omega)})_{j=1}^\infty$ outside $B(f(\omega), 2^{-l_0+1})$ if $(f_{Q_j(\omega)})_{j=1}^\infty$ contains at least one element different from $f(\omega)$. If such an integer does not exist there necessarily is some $Q = Q_w \in \mathcal{W}$ with $f_Q = f(\omega)$ and $r_Q > 0$. In this case, we choose $l_0 = l_0(\omega, f) \in \mathbb{N}$ so that $2^{-l_0} < r_{Q_\omega}$. In both cases we also require that $2^{-l_0+1} < t_0$. This allows us to use the properties (6) and (7).

For the purposes of our "porosity argument", we would like to make the number l_0 independent of the point ω. This is done by considering the decomposition

$$\partial\mathcal{B}^n = \bigcup_{l\in\mathbb{N}} E_l, \quad \text{where } E_l = \{\omega \in \partial\mathcal{B}^n : l_0(\omega, f) \le l\}.$$

Setting $F_l = f(E_l)$, we then have $f(\partial\mathcal{B}^n) = \bigcup_{i\in\mathbb{N}} F_l$.

Let us fix $l_0 \in \mathbb{N}$. Our aim is to prove that $\mathcal{H}^n_\infty(F_{l_0}) = 0$.

Fix $x \in F_{l_0}$. Take any $\omega \in E_{l_0}$ such that $x = f(\omega)$ and define the sequence of concentric annuli $A_l(x) = B(x, 2^{-l+1}) \setminus B(x, 2^{-l})$ with $l = l_0, l_0 + 1, \ldots$. Next, we assign a suitable set $P_l(x)$ of cubes from $\mathcal{W}$ to each annulus $A_l(x)$, $l = l_0, l_0 + 1, \ldots$. If $f_{Q_j(\omega)} = x$ for all $j \in \mathbb{N}$, we put $P_l(x) = \{Q_\omega\}$ for each $l \ge l_0$, where Q_ω is the cube defined earlier. Otherwise, all the sets $P_l(x)$ with $l \ge l_0$ consist of elements from $(Q_j(\omega))_{j=1}^\infty$. If an annulus $A_l(x)$ with some $l \ge l_0$ contains no centres from $(f_{Q_j(\omega)})_{j=1}^\infty$ we define $P_l(x) = \{Q_m(\omega)\}$, where an integer $m \in \mathbb{N}$ is chosen so that $f_{Q_{m-1}(\omega)} \notin B(x, 2^{-l+1})$ but $f_{Q_m(\omega)} \in B(x, 2^{-l})$; if, in contrast, there is at least one centre $f_{Q_j(\omega)}$ in $A_l(x)$ we take $P_l(x) = \{Q_k(\omega) : k = m_1, \ldots, m_2\}$, where $m_1, m_2 \in \mathbb{N}$ are such that $f_{Q_{m_1-1}(\omega)} \notin B(x, 2^{-l+1})$, $f_{Q_{m_2+1}(\omega)} \in B(x, 2^{-l})$ and $f_{Q_k(\omega)} \in A_l(x)$ for all $k = m_1, \ldots, m_2$. Moreover, it is possible to choose the sets $P_l(x)$ above so that the inequality $k_1 \le k_2$ is valid whenever $Q_{k_1}(\omega) \in P_{l_1}(x)$, $Q_{k_2}(\omega) \in P_{l_2}(x)$ and $l_1 < l_2$.

Denoting

$$\theta_l(x) = \begin{cases} 1 & \text{if } \sharp P_l(x) \le \tilde{c}_0 \lambda(l), \\ 0 & \text{otherwise} \end{cases}$$

for $l \geq l_0$ and a constant $\tilde{c}_0 > \lambda^{-1}(0)$, which we will specify later, we would like to prove that there exists an integer $l_1 \geq 2l_0$ such that

$$\sum_{k=l_0}^{l} \theta_k(x) \geq \frac{l}{2} \tag{11}$$

for each $l \geq l_1$. In other words, at least half of the annuli do not contain too many centres from $(f_{Q_j(\omega)})_{j=1}^{\infty}$. There is nothing to prove if $f_{Q_j(\omega)} = x$ for all $j \in \mathbb{N}$; otherwise, the proof is by contradiction:

Let us assume that (11) does not hold for some $l \geq 2l_0$. Take the smallest number $J \in \mathbb{N}$ such that $f_{Q_j(\omega)} \in B(x, 2^{-l})$ for all $j > J$ and let $\omega' \in [0, \omega]$ be the point of $Q_J(\omega) \cap [0, \omega]$ which is closest to ω. Now, the assumption on the continuity of f and the properties of our Whitney decomposition imply

$$2^{-l} \leq |f_{Q_J(\omega)} - x| = |f_{Q_J(\omega)} - f(\omega)| \leq \fint_{Q_J} |f(y) - f(\omega)|\, dy \leq \psi(2(1 - |\omega'|)).$$

That is,

$$\frac{u(2^{-l})}{2} \leq 1 - |\omega'|.$$

Next, we connect this estimate to the number of Whitney cubes that precede Q_J in $(Q_j(\omega))_{i=1}^{\infty}$.

Using (5), we observe that

$$\log \frac{2}{u(2^{-l})} \geq \log \frac{1}{1 - |\omega'|} \geq \frac{1}{c_2} \sharp q(0, \omega').$$

In the calculation above we may have to adjust the choice of l_0 to ensure $\sharp q(0, \omega') > c_3$ (see (5)). Finally, we obtain a lower bound for $\sharp q(0, \omega')$ using the assumption that we have at least $\lfloor l/2 \rfloor - l_0 + 2$ annuli $A_k(x)$ with $\theta_k(x) = 0$. We notice that the sets $P_k(x)$ with $\theta_k(x) = 0$ contain different cubes for different k, and if $k \leq l$ then the cubes in $P_k(x)$ precede $Q_J(\omega)$ in $(Q_j(\omega))_{j=1}^{\infty}$. We have

$$\begin{aligned} c_2 \log \frac{2}{u(2^{-l})} &\geq \sharp q(0, \omega') \geq \sum_{\substack{k=l_0,\dots,l \\ \theta_k(x)=0}} \sharp P_k(x) \geq \sum_{k=l_0}^{\lfloor l/2 \rfloor + 1} \tilde{c}_0 \lambda(k) \geq \tilde{c}_0 \sum_{k=l_0}^{\lfloor l/2 \rfloor + 1} \frac{2^{-k} u'(2^{-k})}{u(2^{-k})} \\ &\geq \tilde{c}_0 \left(\log \frac{1}{u(2^{-l/2})} - \log \frac{1}{u(2^{-l_0})} \right) \geq \tilde{c}_0 \beta^{-1} \log \frac{1}{u(2^{-l})} - \tilde{c}_0 \log \frac{1}{u(2^{-l_0})}. \end{aligned}$$

Choosing $\tilde{c}_0 > c_2 \beta$, this cannot hold when l is large enough. Thus there is a number $l_1 = l_1(\tilde{c}_0, l_0, u)$ such that (11) holds for all $l \geq l_1$.

Our next step is to prove that, if $\theta_k(x) = 1$ for some k and $P_k(x) = \{Q_1, \dots, Q_m\}$, then it is possible to find a collection of balls $\{B_1, \dots, B_{m'}\}$ from the families $\mathcal{S}_{B(f_{Q_i}, r_{Q_i})}$ or $\mathcal{R}_{B(f_{Q_i}, r_{Q_i})}$ having radii at least a constant times $\alpha(2^{-k})$ and such that $\sum_{i=1}^{m'} w_{B_i}$ is at least a constant times $\lambda(k)$. Moreover, we choose different balls for different k.

Let us fix $k \geq l_0$ such that $\theta_k(x) = 1$. Suppose first that the annulus $A_k(x)$ contains no centres from $(f_{Q_j(\omega)})_{j=1}^{\infty}$. Then the set $P_k(x)$ consists of a single cube $Q \in \mathcal{W}$ with $f_Q \in B(x, 2^{-k})$. The definitions of r_Q and l_0 imply that $r_Q > 2^{-k}$ and hence $k \geq k_0(r_Q)$. Thus, we may choose the ball $R_k(B(f_Q, r_Q))$, which, by definition, has radius $\alpha(2^{-k})$ and weight $\lambda(k)$. In addition, the centre of this ball lies in $B(x, 2^{-k})$.

Assume now that the annulus $A_k(x)$ contains at least one of the centres from $(f_{Q_j(\omega)})_{j=1}^{\infty}$. Then, by the definitions of $P_k(x)$ and r_Q,

$$\sum_{Q\in P_k(x)} 2r_Q \geq 2^{-k}.$$

Since $\sharp P_k(x) \leq \tilde{c}_0\lambda(k)$, we observe that

$$\sum_{\substack{Q\in P_k(x)\\ 2r_Q\geq \alpha(2^{-k})/2\tilde{c}_0}} 2r_Q \geq \frac{2^{-k}}{2}.$$

For each $Q \in P_k(x)$ with $2r_Q \geq \alpha(2^{-k})/2\tilde{c}_0$ we choose a number $n_Q \in \mathbb{N}$ so that

$$2^{n_Q-1}\frac{\alpha(2^{-k})}{2\tilde{c}_0} \leq 2r_Q < 2^{n_Q}\frac{\alpha(2^{-k})}{2\tilde{c}_0}$$

and pick a ball $\widetilde{B} = S_{n_Q}(B(f_Q, r_Q)) = B(f_Q, r_Q/2^{n_Q}) \in \mathcal{S}_{B(f_Q,r_Q)}$. By the definition of $S_i(B)$, we have $w_{\widetilde{B}} = 2^{n_Q}$ and

$$r(\widetilde{B}) = \frac{r_Q}{2^{n_Q}} \geq \frac{\alpha(2^{-k})}{8\tilde{c}_0}.$$

For the sum of the weights $\sum_Q 2^{n_Q}$ of all the balls obtained in such a manner, we observe that

$$\frac{\alpha(2^{-k})}{2\tilde{c}_0}\sum_{\substack{Q\in P_k(x)\\ 2r_Q\geq \alpha(2^{-k})/2\tilde{c}_0}} 2^{n_Q} > \sum_{\substack{Q\in P_k(x)\\ 2r_Q\geq \alpha(2^{-k})/2\tilde{c}_0}} 2r_Q \geq \frac{2^{-k}}{2}.$$

Hence, we have a collection of balls $\{B_1, \ldots, B_m\} \subset \mathcal{F}$ with weights sum $\sum_{i=1}^m w_{B_i} > \tilde{c}_0\lambda(k)$ and of radii at least $\alpha(2^{-k})/8\tilde{c}_0$. Moreover, all these balls have their centres in the annulus $A_k(x)$ and hence in the ball $B(x, 2^{-k+1})$.

We have proved that there exists a number $l_1 = l_1(l_0, \tilde{c}_0)$ such that, for each $\omega \in E_{l_0}$ and $l \geq l_1$, among the numbers $l_0, \ldots, l$ there are at least $\lceil l/2\rceil$ integers $k \in \{l_0, \ldots, l\}$ such that $\theta_k(x) = 1$. For these k we are able to find a finite collection of balls $\{B_i\}_{i\in I} \subset \mathcal{F}$ with weight-sum $\sum_{i\in I} w_{B_i}$ at least $\lambda(k)$ and of radii at least $\alpha(2^{-k})/8\tilde{c}_0$, so that the centres of the balls B_i, $i \in I$, lie in the ball $B(x, 2^{-k+1})$. Here $\tilde{c}_0$ is a positive constant depending only on β, n and $\lambda(0)$, and the balls are different for a fixed ω and different k.

Fix $l \geq l_1$. We modify our family $\mathcal{F}$ according to l. If $B \in \mathcal{F}$ and there is $k \in \{l_0 + 1, \ldots, l\}$ such that $\alpha(2^{-k})/8\tilde{c}_0 \leq r(B) < \alpha(2^{-k+1})/8\tilde{c}_0$, we replace B with the ball $\widetilde{B} = (\lambda(k)/\lambda(l))B$ and set $w_{\widetilde{B}} = (\lambda(l)/\lambda(k))^n w_B$. The radius of $\widetilde{B}$ satisfies $r(\widetilde{B}) \geq (\lambda(k)/\lambda(l))\alpha(2^{-k})/8\tilde{c}_0 = 2^{-k}/8\tilde{c}_0\lambda(l)$ and the equality $w_{\widetilde{B}}r(\widetilde{B})^n = w_B r(B)^n$ holds. Similarly, we replace a ball B with $r(B) \geq \alpha(2^{-l_0})/8\tilde{c}_0$ with the ball $\widetilde{B} = (\lambda(l_0)/\lambda(l))B$ and set $w_{\widetilde{B}} = (\lambda(l)/\lambda(l_0))^n w_B$. Again, we have $r(\widetilde{B}) \geq 2^{-l_0}/8\tilde{c}_0\lambda(l)$ and $w_{\widetilde{B}}r(\widetilde{B})^n = w_B r(B)^n$. Finally, $\mathcal{F}_l$ is the collection of balls obtained in this manner from the balls in $\mathcal{F}$. For this family of balls, we notice (see (10)) that

$$\sum_{B\in\mathcal{F}_l} w_B r(B)^n \leq \sum_{B\in\mathcal{F}} w_B r(B)^n < \infty. \tag{12}$$

If $\omega \in E_{l_0}$, $x = f(\omega)$ and $k \in \{l_0, \dots, l\}$ is such that $\theta_k(x) = 1$, then there is a collection $\{B_i\}_{i\in I} \subset \mathcal{F}$ with the properties mentioned above. If for some $i \in I$ the ball B_i is replaced by the ball $\widetilde{B}_i = (\lambda(k_i)/\lambda(l))B_i$ while creating $\mathcal{F}_l$, we necessarily have $k_i \le k$. Therefore, the inequalities

$$\sum_{i\in I} w_{\widetilde{B}_i} = \sum_{i\in I}\left(\frac{\lambda(l)}{\lambda(k_i)}\right)^n w_{B_i} \ge \left(\frac{\lambda(l)}{\lambda(k)}\right)^n \sum_{i\in I} w_{B_i} \ge \left(\frac{\lambda(l)}{\lambda(k)}\right)^n \lambda(k) = \lambda(l)^n \frac{1}{\lambda(k)^{n-1}}$$

and $r(\widetilde{B}_i) \ge 2^{-k_i}/(8\tilde{c}_0\lambda(l)) \ge 2^{-k}/(8\tilde{c}_0\lambda(l))$ hold (by (6), λ is increasing). Since, for each $i \in I$, the centre of a ball $\widetilde{B}_i$ is contained in $B(x, 2^{-k+1})$, we have $x \in 16\tilde{c}_0\lambda(l)\widetilde{B}_i$. Hence, we observe that

$$\sum_{B\in\mathcal{F}_l} w_B \chi_{16\tilde{c}_0\lambda(l)B}(y) \ge \sum_{\substack{k=l_0,\dots,l\\ \theta_k(y)=1}} \lambda(l)^n \frac{1}{\lambda(k)^{n-1}} \ge \frac{\lambda(l)^n}{4}\sum_{k=l_1}^{l} \frac{1}{\lambda(k)^{n-1}} \ge \frac{\lambda(l)^n}{4} G_l$$

for each $y \in F_{l_0}$, where $G_l = \sum_{k=l_1}^{l} 1/\lambda(k)^{n-1}$. That is, $(4w_B/(\lambda(l)^n G_l), 16\tilde{c}_0\lambda(l)B)_{B\in\mathcal{F}_l}$ is a weighted cover of the set F_{l_0}. We observe also that the diameters of all balls in this cover are at least 2^{-l}. This information will be used in the proof of Theorem 1.3 below.

Finally, using the weighted cover obtained above and (12), we estimate the weighted Hausdorff n-content $\lambda^n_\infty(F_{l_0})$:

$$\begin{aligned}\lambda^n_\infty(F_{l_0}) &\le \frac{4}{\lambda(l)^n G_l}\sum_{B\in F_l} w_B\big(\operatorname{diam}(16\tilde{c}_0\lambda(l)B)\big)^n \le \frac{4^{2n+1}\tilde{c}_0^n}{G_l}\sum_{B\in F_l} w_B(\operatorname{diam} B)^n\\ &\le \frac{2^{5n+2}\tilde{c}_0^n}{G_l}\sum_{B\in F_l} w_B r(B)^n \le \frac{A}{G_l},\end{aligned}$$

where the constant A depends on β, n, m, $\|f\|_{W^{1,n}(\mathcal{B}^n,\mathbb{R}^m)}$ and $\lambda(0)$ but not on l_0 or l.

Now Lemma 2.1 implies $\mathcal{H}^n_\infty(F_{l_0}) \le CA/G_l$. Here C depends only on the dimension n. Hence, we are done as soon as we can show that $G_l \to \infty$ as $l \to \infty$. Towards this end, we have

$$G_l = \sum_{k=l_1}^{l} \frac{1}{\lambda(k)^{n-1}} = \sum_{k=l_1}^{l} \frac{u(2^{-k})^{n-1}}{2^{-k(n-1)}u'(2^{-k})^{n-1}} \ge \int_{2^{-l}}^{2^{-l_1}} \left(\frac{u(t)}{u'(t)}\right)^{n-1}\frac{dt}{t^n},$$

and the right-hand side diverges as $l \to \infty$ by the assumptions on the modulus of continuity. □

The proof of Theorem 1.3 is similar to the proof of Theorem 1.1. We only point out the required changes.

Proof of Theorem 1.3. Let f be as in statement of the theorem. Our notation will be the same as in previous proof. That is, $\alpha(t) = \gamma t$ and $\lambda(k) = 1/\gamma$.

Fix a small $\varepsilon > 0$. Then there exists a $\delta > 0$ such that

$$\int_{\mathcal{B}^n\setminus B(0,1-\delta)} |Df|^n \le \varepsilon. \tag{13}$$

Let $\mathcal{W}^\delta$ be the set of the cubes in $\mathcal{W}$ which are contained in $\mathcal{B}^n \setminus B(0, 1-\delta)$ and whose neighbour cubes are also contained in $\mathcal{B}^n \setminus B(0, 1-\delta)$. We define our collection of balls to be $\mathcal{B}^\delta = \{B(f_Q, r_Q) : Q \in \mathcal{W}^\delta\}$.

Then, proceeding as in the previous proof, we define $\mathcal{F}^\delta$ analogously to $\mathcal{F}$ and obtain the estimate (see (10))

$$\sum_{B\in\mathcal{F}^\delta} w_B r(B)^n \le C_1\varepsilon. \tag{14}$$

Let $\omega \in \partial\mathcal{B}^n$. We define the number $l_0 = l_0(\omega, f, \delta)$ as in the previous proof, but instead of all cubes in $(Q_j(\omega))_{j=1}^\infty$ we consider only those which are contained in $\mathcal{W}^\delta$. Again, we split $\partial\mathcal{B}^n$ into sets $E_l = \{\omega \in \partial\mathcal{B}^n : l_0(\omega) \le l\}$ and consider a fixed $f(E_{l'})$. With the same method as earlier we find for large l a collection of balls $\mathcal{F}_l^\delta$ with weights such that $(8w_B\gamma/(l-l_1), (16\tilde{c}_0/\gamma)B)_{B\in\mathcal{F}_l^\delta}$ is a weighted cover of the set $f(E_{l'})$, the radii of the balls $(16\tilde{c}_0/\gamma)B$ are at least 2^{-l} and

$$\sum_{B\in\mathcal{F}_l^\delta} w_B r(B)^n \le C_1\varepsilon.$$

We may assume that our $\varepsilon > 0$ is so small that all balls in our weighted cover have radii smaller than $\frac{1}{2}$. With this weighted cover, we obtain

$$\begin{aligned}\lambda_\infty^g(f(E_{l'})) &\le \frac{4\gamma}{l-l_1}\sum_{B\in\mathcal{F}_l^\delta} w_B\left(\operatorname{diam}\left(\frac{16\tilde{c}_0}{\gamma}B\right)\right)^n \log\frac{1}{\operatorname{diam}((16\tilde{c}_0/\gamma)B)}\\ &\le \frac{4\gamma}{l-l_1}\sum_{B\in\mathcal{F}_l^\delta} w_B\left(\operatorname{diam}\left(\frac{16\tilde{c}_0}{\gamma}B\right)\right)^n \log 2^l \le \frac{2^{2+5n}\tilde{c}_0^n}{\gamma^{n-1}}\,\frac{l}{l-l_1}\sum_{B\in\mathcal{F}_l^\delta} w_B r(B)^n \le \frac{2^{3+5n}\tilde{c}_0^n C_1}{\gamma^{n-1}}\,\varepsilon.\end{aligned}$$

Here we assumed l to be so large that $l/(l-l_1) \le 2$. Lemma 2.1 implies $\mathcal{H}_\infty^g(f(E_{l'})) \le A\varepsilon$. Here A depends on γ, n and m but not on l' or l; therefore, we have $\mathcal{H}_\infty^g(f(\partial\mathcal{B}^n)) \le A\varepsilon$; see [Howroyd 1994, Corollary 8.2] or [Federer 1969, 2.10.22]. Letting ε tend to zero gives $\mathcal{H}_\infty^g(f(\partial\mathcal{B}^n)) = 0$, which implies $\mathcal{H}^g(f(\partial\mathcal{B}^n)) = 0$. ☐

4. Example

In this section, we work in $\mathbb{R}^2$ and use the notation $\|x\| = \max\{|x_1|, |x_2|\}$. Let $p > \frac{1}{2}$. We will construct a locally Hölder continuous mapping $f : \mathbb{R}^2 \to \mathbb{R}^2$ that belongs to $W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ and maps $\partial\mathcal{B}^2$ onto a set of positive $\mathcal{H}^g$-measure, where $g(t) = t^2(\log(1/t))^{2p}$.

The mapping is a composition of two locally Hölder continuous mappings. The second mapping is defined in [Herron and Koskela 2003, Proposition 5.1]. It is a homeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ that is the identity mapping outside $[0, 1]^2$ and maps a small Cantor set $\mathcal{C} \subset [0, 1]^2$ onto a large Cantor set $\mathcal{C}' \subset [0, 1]^2$ with positive $\mathcal{H}^g$-measure. It was checked in [Koskela et al. 2009] that this mapping belongs to $W^{1,2}_{\text{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ if $p > \frac{1}{2}$.

Next, we elaborate on the construction of h and prove that it is Hölder continuous in $[0, 1]^2$. Let $\sigma < \frac{1}{2}$. We use the notation $2r_k = \sigma^k$ and $2R_k = \frac{1}{2}\sigma^{k-1}$ for $k \in \mathbb{N}$. The set $\mathcal{C}$ is defined as follows: In the first generation we have one square $Q_0 = [0, 1]^2$ with side length $2r_0$. We split this square into four subsquares P_{1i}, $i = 1, 2, 3, 4$, of side length $2R_1$. We define Q_{1i} to be the square of side length $2r_1$ centred at the centre of P_{1i}. Then P_{1i} and Q_{1i} generate the frame $A_{1i} = P_{1i} \setminus Q_{1i}$. Next, we divide all squares Q_{1i} into squares P_{2j}, $j = 1, \dots, 4^2$. Then we define Q_{2j} and A_{2j} as in the first step. We proceed inductively. Thus, we obtain for all $k \in \mathbb{N}$ sets Q_{ki}, P_{ki} and A_{ki}, where $i = 1, \dots, 2^{2k}$, and we set $\mathscr{C} = \bigcap_k \bigcup_i Q_{ki}$.

The set $\mathscr{C}'$ and sets Q'_{ki}, P'_{ki} and A'_{ki} with $k \in \mathbb{N}$ and $i = 1, \dots, 2^{2k}$ are defined in the same way, using $2r'_1 = \frac{1}{2}(\log 4)^{-p}$, $2R'_2 = r'_1$, and $2r'_k = (\log 4)^{-p}2^{-k}k^{-p}$ and $2R'_k = (\log 4)^{-p}2^{-k}(k-1)^{-p}$ for other $k \in \mathbb{N}$.

The mapping h is defined so that it maps the frame A_{ki} to the frame A'_{ki} via a "radial" stretching and is continuous in $[0, 1]^2$. The radial stretching which maps $A = \{x : r_k \le \|x\| \le R_k\}$ to $A' = \{x : r'_k \le \|x\| \le R'_k\}$ is

$$\rho(x) = (a\|x\| + b)\frac{x}{\|x\|}, \quad \text{where } a = \frac{R'_k - r'_k}{R_k - r_k} \text{ and } b = \frac{R_k r'_k - R'_k r_k}{R_k - r_k}.$$

If $x, y \in A$ then $\|x - y\| \le 2R_k = \frac{1}{2}\sigma^{k-1}$ and

$$a \le \frac{4\sigma}{1 - 2\sigma}(2\sigma)^{-k} \le C(\sigma)\sigma^{-(1-\beta)k} \le C(\sigma)\|x - y\|^{\beta - 1},$$

where $\beta = \log 2 / \log(1/\sigma)$. Similarly,

$$\frac{|b|}{|r_k|} \le \frac{4}{1 - 2\sigma}(2\sigma)^{-k} \le C(\sigma)\|x - y\|^{\beta - 1}.$$

The mapping ρ is Hölder continuous with exponent β, as

$$\|\rho(x) - \rho(y)\| \le Ca\|x - y\| + 2\frac{|b|}{|r_k|}\|x - y\| \le C(\sigma)\|x - y\|^{\beta}.$$

If $x \in A_{ki}$ and $y \in Q_{k+1,j} \subset P_{ki}$, then $\|x - y\| \ge R_{k+1} - r_{k+1} = C(\sigma)\sigma^k$ and $\|h(x) - h(y)\| \le 2R'_k \le 2^{-k}$. These imply

$$\frac{\|h(x) - h(y)\|}{\|x - y\|^{\beta}} \le C(\sigma).$$

The β-Hölder continuity of h easily follows from the continuity estimates obtained above.

The first mapping $G : \mathbb{R}^2 \to \mathbb{R}^2$ is a (locally Hölder continuous) quasiconformal mapping for which $\mathscr{C} \subset G(\partial\mathscr{B}^2)$. Such a mapping was constructed in [Gehring and Väisälä 1973].

Finally, the composition $h \circ G : \mathbb{R}^2 \to \mathbb{R}^2$ is a homeomorphism with $h \circ G(\partial\mathscr{B}^2) \supset \mathscr{C}'$. Moreover, it is locally Hölder continuous and $h \circ G \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}^2, \mathbb{R}^2)$ by quasiconformality of G and the change of variable formula; see, for example, [Astala et al. 2009, Section 3.8].

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AAPO KAURANEN: aapo.p.kauranen@jyu.fi
Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 Jyväskylä, Finland

PEKKA KOSKELA: pekka.j.koskela@jyu.fi
Department of Mathematics and Statistics, University of Jyväskylä, P.O. Box 35 (MaD), FI-40014 Jyväskylä, Finland

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GLOBAL GAUGES AND GLOBAL EXTENSIONS IN OPTIMAL SPACES

MIRCEA PETRACHE AND TRISTAN RIVIÈRE

We consider the problem of extending functions $\phi : \mathbb{S}^n \to \mathbb{S}^n$ to functions $u : B^{n+1} \to \mathbb{S}^n$ for $n = 2, 3$. We assume ϕ belongs to the critical space $W^{1,n}$ and we construct a $W^{1,(n+1,\infty)}$-controlled extension u. The Lorentz–Sobolev space $W^{1,(n+1,\infty)}$ is optimal for such controlled extension. Then we use these results to construct global controlled gauges for L^4-connections over trivial SU(2)-bundles in 4 dimensions. This result is a global version of the local Sobolev control of connections obtained by K. Uhlenbeck.

1. Introduction

The use of Hodge decomposition is by now one of the classical tools in the study of elliptic systems and is related to important breakthroughs such as the famous "div–curl"-type theorems [Coifman et al. 1993]. More recently, in [Rivière 2007], such use allowed the solution of S. Hildebrandt's [1982] conjecture. At the same time, it has helped establish important links to apparently unrelated fields of geometry, such as the study of conformally invariant geometric problems in 2 dimensions [Hélein 1996] and the study of Yang–Mills bundles and gauge theory [Uhlenbeck 1982b], with the introduction of controlled Coulomb gauges.

The study of 2-dimensional problems using controlled gauges has already given its fruits, and in connection to the discovery of H. Wente's inequality (which gave the basis for introducing the Lorentz spaces $L^{(2,\infty)}$ in geometric problems) allowed the successful use of controlled moving frames in the study of harmonic maps and prescribed mean curvature surfaces [Hélein 1996; Müller and Šverák 1995]. We come back to this in Section 2H. Techniques and function spaces related to the moving frame method also apply to the study of the Willmore functional [Rivière 2012] for immersed surfaces.

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The use of controlled gauges especially in relation to Lorentz spaces in dimensions higher than 2 is far less developed. We attempt here a first attack of this completely new area of research, and we obtain some extensions of previous results for the case of Yang–Mills fields on 4-dimensional manifolds.

1A. ***Yang–Mills theory and controlled gauges.*** Yang–Mills theory for 4-manifolds is often associated to the famous result of S. Donaldson [1983] who, using the moduli spaces of anti-selfdual connections, described new invariants of smooth manifolds.

The study of moduli spaces used by Donaldson [1983] starts from the result of K. Uhlenbeck [1982b], who proved that one can find a gauge in which the $W^{1,2}$-norm of the local coordinate expression of the connection is controlled by the L^2-norm of the curvature. Moreover the connection 1-form A can be also made to satisfy the Coulomb condition $d^*A = 0$.

It is easy to construct a Coulomb gauge in which we have just an L^2-control in terms of the curvature (see [Petrache 2013] or [Petrache and Rivière ≥ 2014]). This is done by first obtaining any gauge in which

$$\|A\|_{L^2} \le C\|F\|_{L^2}$$

and then finding the smallest norm coefficients with respect to that gauge on our manifold M:

$$\min\left\{\int_M |g^{-1}dg + g^{-1}Ag|^2\,dx : g \in W^{1,2}(M, \mathrm{SU}(2))\right\}.$$

A unique minimizer will exist by convexity, and it will satisfy the Coulomb equation $d^*A = 0$.

The control of A in the higher norm $W^{1,2}$ is more difficult. A smallness hypothesis on $\|F\|_{L^2(M)}$ is required in order for the control to be achievable:

Theorem 1.1 (controlled Coulomb gauge under assumption of small energy [Uhlenbeck 1982b]). *There exists a constant $\epsilon_0 > 0$ such that if the curvature satisfies $\int_M |F|^2 \le \epsilon_0$ then there exists a Coulomb gauge $\phi \in W^{2,2}(M, \mathrm{SU}(2))$ such that in that gauge the connection satisfies $\|A_\phi\|_{W^{1,2}(M)} \le C\|F\|_{L^2(M)}$ with $C > 0$ depending only on the dimension.*

The reason the smallness of the curvature is necessary is that $\|F\|_{L^2(M)}$ being above a certain threshold allows the second Chern number of the bundle to be nontrivial:

$$c_2(E) = \frac{1}{8\pi^2}\int_M \mathrm{tr}(F \wedge F) \neq 0.$$

If, for such F, the controlled gauge were *global*, i.e., if we had a global trivialization in which the connection of the above F is expressed as $d + A$ with

$$\|A\|_{W^{1,2}(M)} \le C,$$

then by the Sobolev and Hölder inequalities we would have enough control on the quantities involved to prove the following formal identity for our A:

$$\mathrm{tr}\big[(dA + [A,A]) \wedge (dA + [A,A])\big] = d\,\mathrm{tr}\big(A \wedge dA + \tfrac{2}{3}A \wedge A \wedge A\big).$$

Now the right side, being an exact form, would have integral equal to zero over the boundaryless manifold M, which would contradict $c_2(E) \neq 0$.

M. Atiyah, N. Hitchin, I. Singer [Atiyah et al. 1978] and C. Taubes [1982] constructed instantons with nontrivial Chern numbers behaving as in the above heuristic. To exemplify the phenomena at work consider the simplest instanton, having $c_2(E) = 1$ over $M = \mathbb{S}^4$ (see [Freed and Uhlenbeck 1984, Chapter 6] for notations and details). Recall that we may use quaternion notation due to the isomorphisms $\mathrm{SU}(2) \sim \mathrm{Sp}(1)$ and $\mathrm{su}(2) \sim \mathrm{Im}\,\mathbb{H}$, under which Pauli matrices correspond to quaternion imaginary units. We then have the following local expression of A over $\mathbb{R}^4$ (identified by stereographic projection with $\mathbb{S}^4 \setminus \{p\}$) in a trivialization:

$$A = \mathrm{Im}\left(\frac{x\,d\bar{x}}{1+|x|^2}\right).$$

If Ψ is the inverse stereographic projection then Ψ^*A is smooth away from the pole p, but near p we have $|\Psi^*A|(q) \sim \mathrm{dist}_{\mathbb{S}^4}(p,q)^{-1}$, which is not L^4 in any neighborhood of p.

Such behavior like $1/|x|$ shows that we are in any space L^p for $p < 4$ but not in L^4. The natural space is the weak-L^4 space $L^{4,\infty}$, which is strictly contained between all L^p, $p < 4$, and L^4:

Definition 1.2 [Grafakos 2008]. Let X, μ be a measure space. The space $L^{p,\infty}(X,\mu)$ (also called *weak-*L^p or *Marcinkiewicz* space) is the space of all measurable functions f such that

$$\|f\|^p_{L^{p,\infty}} := \sup_{\lambda>0} \lambda^p \mu\{x : |f(x)| > \lambda\}$$

is finite.

We note immediately that the function $f(x) = 1/|x|$ belongs to $L^{4,\infty}$ on $\mathbb{R}^4$ and the above global gauge gives an $L^{4,\infty}$ 1-form Ψ^*A on $\mathbb{S}^4$. Spaces $L^{p,\infty}$ arise naturally in dealing to the critical exponent estimates for elliptic equations. Indeed, the Green kernel $K_n(x)$ of the Laplacian on $\mathbb{R}^n$ satisfies $\nabla K \in L^{n/(n-1),\infty}$ but not $\nabla K \in L^{n/(n-1)}$. Thus $\Delta u = f$ with $f \in L^1$ implies $\nabla u = \nabla K * f \in L^{n/(n-1),\infty}$ by an extended Young inequality (see [Grafakos 2008]). This is unlike the higher exponent case $f \in L^p$, $p > 1$, which gives the stronger result $\nabla u \in L^p$.

1B. ***Controlled global gauges.*** As shown heuristically by the explicit case of the instanton A above, it is known how to construct $L^{4,\infty}$ global gauges. Our main effort in this work is to obtain *norm-controlled* gauges, mirroring Theorem 1.1 by Uhlenbeck. The main result is the following:

Theorem A. *Let M^4 be a Riemannian 4-manifold. There exists a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ with the following property: Let ∇ be a $W^{1,2}$-connection over an* SU(2)*-bundle over M. Then there exists a* global *$W^{1,(4,\infty)}$-section of the bundle (possibly allowing singularities) over the whole M^4 such that in the corresponding trivialization ∇ is given by $d + A$ with the bound*

$$\|A\|_{L^{(4,\infty)}} \leq f(\|F\|_{L^2(M)}),$$

where F is the curvature form of ∇.

This theorem is related to a second main result of this work, namely the introduction of Lorentz–Sobolev extension theorems for nonlinear maps. This result takes most of our efforts and can be stated as follows:

Theorem B. *There exists a function* $f_1 : \mathbb{R}^+ \to \mathbb{R}^+$ *with the following property*: *Let* $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. *Then there exists an extension* $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ *of* ϕ *such that*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \leq f_1(\|\nabla \phi\|_{L^3}).$$

The originality of Theorem B with respect to the previous results [Bethuel and Demengel 1995; Mucci 2010] is that, whereas the previous works were concerned with the *existence* of an extension, in our case a *control* is provided in terms of the boundary value. We show below that, even under the hypothesis $\deg(\phi) = 0$ — so that a $W^{1,4}$-extension surely exists — no energy control will be available in the (stronger) $W^{1,4}$-norm.

Controlled global gauges as above will probably have many applications in the analysis of gauge theory, for example in simplifying compactness results; see [Petrache 2013]. Controlled global gauges could allow a global control on the Yang–Mills flow provided we obtain also the Coulomb condition, which is however an open question:

Open Problem 1.3. Prove that it is possible to find $L^{4,\infty}$-controlled global Coulomb gauges as in Theorem A. In other words, prove that it is possible to find a gauge as in Theorem A, but with the further requirement that $d^*A = 0$.

1C. *Strategy of gauge construction.* The link between Theorems A and B is given by the well-known identification $\mathrm{SU}(2) \simeq \mathbb{S}^3$. Therefore, Theorem B can be rephrased as follows:

Theorem B′. *Fix a trivial* $\mathrm{SU}(2)$*-bundle* E *over the ball* B^4. *There exists a function* $f_1 : \mathbb{R}^+ \to \mathbb{R}^+$ *with the following property*: *if* $g \in W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$ *gives a trivialization of the restricted bundle* $E|_{\partial B^4}$, *then there exists an extension of* g *to a trivialization* $\tilde{g} \in W^{1,(4,\infty)}(B^4, \mathrm{SU}(2))$ *such that*

$$\|\nabla \tilde{g}\|_{L^{4,\infty}(B^4)} \leq f_1(\|\nabla g\|_{L^3(\mathbb{S}^3)}).$$

The proof of Theorem A is by a sequence of gauge extensions along the simplices of a suitable triangulation. We use simplices where Uhlenbeck's Theorem 1.1 holds, i.e., F has energy $\lesssim \epsilon_0$. To ensure a lower bound on the size of simplices we cut areas of energy concentration and use induction on the energy; see the graphical summary (5-1).

1D. *Extension of Sobolev maps into manifolds.* We discuss the relevance of our theorem, several possible extensions and related phenomena in Section 2.

Here we point out the main open questions in the area of controlled nonlinear extensions and some analogues of Theorem B. The fundamental group $\pi_m(N)$ is a useful tool to control the topology of N. It is a quotient of $C^0(\mathbb{S}^m, \mathbb{N})$. To say that any map in this space is continuously extendable to B^{m+1} amounts to asserting that $\pi_m(N) = 0$.

We consider here the *controlled* extension problem for maps $\mathbb{S}^m \to \mathbb{S}^n$. As is usually the case, interesting new features appear when smooth maps are not dense in $W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$, in which case we expect topological obstructions to gradually disappear as p decreases. The first facts to note are:

- For extensions of maps from $W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$ to B^{m+1} the natural space given by continuous Sobolev and trace embeddings is $W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ (see Section 2A and 2B).
- For $p < m(n+1)/(m+1)$ the controlled extensions exist (see Section 2A).
- For $p > m$ the extension question reduces to a purely topological problem (see Section 2B).

The open cases when $p < m$ are thus among the following ones:

Open Problem 1.4. Assume that $m(n+1)/(m+1) \leq p < m$ and $m > n$. For which such choices of m, n, p does there exist a finite function $f_{m,n,p} : \mathbb{R}^+ \to \mathbb{R}^+$ such that for every $\phi \in W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$ there exists an extension $u \in W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ for which the estimate

$$\|u\|_{W^{1,p(m+1)/m}(B^{m+1},\mathbb{S}^n)} \leq f_{m,n,p}(\|\phi\|_{W^{1,p}(\mathbb{S}^m,\mathbb{S}^n)})$$

holds? Does the estimate hold for $p = m$ for the norm $W^{1,(m+1,\infty)}(B^{m+1}, \mathbb{S}^n)$?

Open Problem 1.4 is partially understood or solved just in some cases:

- Due to a relation between extension problems and lifting problems, we answer the above problem for $n = 2 < m$ and $3m/(m+1) \leq p < 4m/(m+1)$; see Proposition 1.7 and Section 2D.
- In particular, we cover all p for the dimensions $m = 3, n = 2$.
- For $n = 1$, $m \geq 3$ and $3m/(m+1) \leq p < m$, it was shown by F. Bethuel and F. Demengel [1995] that no extension exists.

It will be interesting in the future to look at the link between extension and lifting problems in detail. It is possible to do this also in the case of $\mathbb{S}^1$-valued maps and in nonlocal Sobolev spaces, e.g., using the results of J. Bourgain, H. Brezis and P. Mironescu [Bourgain et al. 2000].

In the critical case $p = m$, left aside in Open Problem 1.4, we have the following results:

- Using the Hopf lifts as in the works of R. Hardt and T. Rivière [2003; 2008], we prove Theorem C, which is the solution to the case $p = m = n = 2$ (see Section 3).
- The extension in that case exists but cannot be controlled in the above Sobolev norm, making the Lorentz–Sobolev weakening of Theorem B and of Theorem C below optimal (see Section 2E). This is analogous to the case of global gauges in 4 dimensions pointed out in the introduction.
- We also prove an analogous result for $p = m = n = 1$ (see Theorem 2.5). However this is not the natural space to look at, unlike in higher dimensions. In this case, indeed, the trace space $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$ is the natural space to look at, because $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ does not continuously embed in it (we recall a counterexample in Section 2C).

These theorems leave open higher-dimensional cases:

Open Problem 1.5. Assume $n \geq 4$. Prove that there exists a finite function $f_n : \mathbb{R}^+ \to \mathbb{R}^+$ such that, for each $\phi \in W^{1,n}(\mathbb{S}^n, \mathbb{S}^n)$, we can find an extension $u \in W^{1,(n+1,\infty)}(B^{n+1}, \mathbb{S}^n)$ for which

$$\|u\|_{W^{1,(n+1,\infty)}(B^{n+1},\mathbb{S}^n)} \leq f_n(\|\phi\|_{W^{1,n}(\mathbb{S}^n,\mathbb{S}^n)}).$$

Unlike in linear Sobolev spaces, not only the topology of the domain must be compared to the Sobolev exponent p, but also the dimension and structure of the constraint (i.e., the target manifold) plays a critical role. This is also related to the topological global obstructions to density results for smooth functions between manifolds found by F. Hang and F.-H. Lin [2001; 2003] and discussed by T. Isobe [2006].

A general tool allowing extensions is the projection trick of Section 2A, which works well for Sobolev exponents smaller than the target dimension plus one. Lifting theorems allow us to increase this dimension and thus to apply the projection trick with higher exponents.

Using the Hopf fibration $H:\mathbb{S}^3\to\mathbb{S}^2$ we construct controlled lifts and apply a version of the projection trick obtaining the following theorem with much less effort than for the 3-dimensional case of Theorem B:

Theorem C (see Section 3). *Suppose $\phi\in W^{1,2}(\mathbb{S}^2,\mathbb{S}^2)$ is given. Then there exists $u\in W^{1,(3,\infty)}(B^3,\mathbb{S}^2)$ such that, in the sense of traces, $u|_{\partial B^3}=\phi$ and such that the following estimate holds, for a constant independent of ϕ:*

$$\|u\|_{W^{1,(3,\infty)}(B^3)}\le C\|\phi\|_{W^{1,2}(\mathbb{S}^2)}(1+\|\phi\|_{W^{1,2}(\mathbb{S}^2)}).$$

The Hopf fibration has a natural structure of $U(1)$-bundle with nontrivial characteristic class, $P\to\mathbb{S}^2$. Lifting a map $\phi:X\to\mathbb{S}^2$ to a map $\tilde{\phi}:X\to\mathbb{S}^3$ for which $H\circ\tilde{\phi}=\phi$ corresponds to giving the trivialization of the pullback bundle ϕ^*P. Analogous lifts are interesting to study for general principal G-bundles, using universal connections. The next case after the one with target $\mathbb{S}^2$ is the SU(2)-bundle of the introduction, which corresponds to the Hopf fibration $\mathbb{S}^7\to\mathbb{S}^4$.

The Hopf lift idea seems to be much more difficult to extend to the case where the target is $\mathbb{S}^3$. We cannot use principal bundles because $\pi_2(G)=0$ for all compact Lie groups G. For other fibrations, the following question is open:

Open Problem 1.6. Is it possible to find a fibration $\pi:E\to\mathbb{S}^3$ with compact fiber M and a constant $C>0$ such that, for each $\phi\in W^{1,3}(\mathbb{R}^3,\mathbb{S}^3)$, there exists a lift $\tilde{\phi}:\mathbb{R}^3\to E$ satisfying the estimate $\|\nabla\tilde{\phi}\|_{L^{(3,\infty)}}\le Cf(\|\nabla\phi\|_{L^3})$ for some finite function $f:\mathbb{R}^+\to\mathbb{R}^+$?

The controlled Hopf lift result for $\mathbb{S}^2$ yields also an answer to Open Problem 1.4 for dimensions $m=3$, $n=2$:

Theorem D. *Assume $\phi\in W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)$. Then there exists a controlled extension $u\in W^{1,(4,\infty)}(B^4,\mathbb{S}^2)$ with the control*

$$\|u\|_{W^{1,(4,\infty)}(B^4,\mathbb{S}^2)}\le C\|\phi\|_{W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)}(1+\|\phi\|_{W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)}).$$

If instead we have $\phi\in W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)$ for $\frac{9}{4}\le p<3$, then there exists an extension $u\in W^{1,\frac{4}{3}p}(B^4,\mathbb{S}^2)$ with

$$\|u\|_{W^{1,\frac{4}{3}p}(B^4,\mathbb{S}^2)}\le C\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}(1+\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}).$$

The same proof allows us to also answer Open Problem 1.4 for $n=2<m$ for some exponents p:

Proposition 1.7. *Assume $n=2$, $m\ge 3$ and $3m/(m+1)\le p<4m/(m+1)$ and let $\phi\in W^{1,p}(\mathbb{S}^m,\mathbb{S}^2)$. Then there exists a controlled extension $u\in W^{1,p(m+1)/m}(B^{m+1},\mathbb{S}^2)$ with*

$$\|u\|_{W^{1,p(m+1)/m}(B^4,\mathbb{S}^2)}\le C\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}(1+\|\phi\|_{W^{1,p}(\mathbb{S}^3,\mathbb{S}^2)}).$$

1E. ***Ingredients used in the construction of $W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$-extensions.*** The starting new idea was to the use of implicit function theorems and of a limit on the integrability exponent as done in [Uhlenbeck 1982a] for the extension result. Note that the procedure of Appendix A is generalizable to other contexts with no new ingredients, at least as long as a Lie group structure is present.

For the implicit function theorems above, we needed here a new product estimate valid in Sobolev spaces, which is presented in Appendix B, extending partially the results of [Brézis and Mironescu 2001]; cf. [Runst and Sickel 1996; Triebel 1995].

The second idea was to use $L^{(4,\infty)}$ functions such that the L^4-estimate would fail just near a controlled number of points. Such singular points (where "singular" is meant with respect to the L^4-estimates) are introduced via Lemma 4.6 and Theorem 4.3.

The uniform $L^{(4,\infty)}$-control is obtainable just in the case where the boundary value has no large energy "hot spots". To deal with the case where energy concentrates, we use two tools which are available in the particular case of $\mathbb{S}^3 \simeq \mathrm{SU}(2)$: (1) the group operation of SU(2), which gives a continuous product on $W^{1,3}(X, \mathbb{S}^3)$; (2) the Möbius group of $\mathbb{S}^3$ coupled with the conformal invariance of the L^3-norm of the gradient on $\mathbb{S}^3$.

Under a balancing condition on the boundary value ϕ, we can write $\phi = \phi_1\phi_2$, where the product is taken in SU(2), and the energies of ϕ_i, $i = 1, 2$, are strictly less than that of ϕ, allowing an induction on the energy. If the balancing is not valid, we apply a Möbius transformation F_v to $\mathbb{S}^3$ and either reduce to a balanced situation for $F_v \circ \phi$ for some v or provide a substitute $v \in B^4 \mapsto \fint_{\mathbb{S}^3} \phi \circ F_v$ to the harmonic extension of ϕ, to which we can now apply the projection trick. The natural parametrization of the Möbius group of $\mathbb{S}^3$ via vectors in B^4 fits very well in this setting, and we were inspired to use it by the similar use of it in [Marques and Neves 2014].

1F. ***Plan of the paper.*** Section 2 contains a list of positive and negative results concerning phenomena parallel to ours, showing that our results are optimal. Section 3 contains the proof of Theorem C. In Section 4 we prove Theorem B, and in Section 5 we prove Theorem A. Appendix A deals with our new "extension" version of Uhlenbeck's gauge construction and in Appendix B we prove the needed new product inequality. Appendix C contains computations and notation for the Möbius groups of B^{n+1} and $\mathbb{S}^n$.

2. Controlled and uncontrolled nonlinear Sobolev extensions

Classical Sobolev space theory features optimal extension theorems in natural trace norms. For example, if $\Omega \subset \mathbb{R}^n$ is a bounded smooth domain and $u : \partial\Omega \to \mathbb{R}$ is a $W^{1,n-1}$-function, then there exists an extension $\bar{u} : \Omega \to \mathbb{R}$ such that $\bar{u} \in W^{1,n}$ and the estimate

$$\|\bar{u}\|_{W^{1,n}} \leq C\|u\|_{W^{1,n-1}}$$

holds (with C independent of u). This extension theorem is optimal in the sense that for dimensions $n > 2$ the natural trace operator $\bar{u} \in W^{1,n}(\Omega) \mapsto \bar{u}|_{\partial\Omega}$ sends $W^{1,n}$ to the optimal space $W^{1-1/n,n}$ (see [Tartar 2007, Chapter 40] for the natural appearance of this space), and we have the optimal Sobolev continuous embedding $W^{1-1/n,n} \to W^{1,n-1}$ (see [Tartar 2007]) which brings us back to the original space. A similar result still holds if we replace the codomain $\mathbb{R}$ by $\mathbb{R}^m$.

However, for $n = 2$, the space $W^{1,1}(\mathbb{S}^1, \mathbb{S}^1)$ does not continuously embed in $H^{1/2}(\mathbb{S}^1, \mathbb{S}^1)$, making the above reasoning less effective; see Section 2C.

A construction of $\bar{u}$ is possible by imitating the model, valid for $\Omega = \mathbb{R}^n_+ := \{(x_1, \dots, x_n) | x_n \geq 0\}$,

$$\bar{u}(x_1, \dots, x_{n-1}, \epsilon) := (\rho_\epsilon * u)(x_1, \dots, x_{n-1}),$$

where ρ_ϵ is a standard family of radial smooth compactly supported mollifiers.

An equivalent construction of $\bar{u}$ in terms of function spaces is by harmonic extension. The optimal result is the following:

Proposition 2.1 (harmonic extension; cf. [Gazzola et al. 2010, Chapter 10]). *Assume $q > 1$ and $u \in W^{1-1/q,q}(\partial B^{m+1}, \mathbb{R}^{n+1})$. Then there exists a harmonic extension $\bar{u} \in W^{1,q}(B^{m+1}, \mathbb{R}^{n+1})$ such that*

$$\|\bar{u}\|_{W^{1,q}(B^{m+1}, \mathbb{R}^{n+1})} \leq C_{m,n,q} \|u\|_{W^{1-1/q,q}(\partial B^{m+1}, \mathbb{R}^{n+1})}.$$

By Sobolev embedding, we have the controlled inclusion $W^{1,p} \hookrightarrow W^{1-1/q,q}$ on an m-dimensional bounded open domain (or a compact manifold like ∂B^{m+1}) for $q \leq p(m+1)/m$; therefore, this q is the largest exponent where we can hope to have a control for the extension.

If u is a constrained function with values in a subset of $\mathbb{R}^{n+1}$ (e.g., a curved n-dimensional submanifold like $\mathbb{S}^n$) then averaging even on a very small scale could push the values of $\bar{u}$ quite far from the constraint obeyed by u. This happens in particular for Sobolev exponents that make the dimension "supercritical", i.e., exponents such that $W^{1,q}(B^{m+1})$ is not constituted of continuous functions. We now describe some cases where directly projecting back to $\mathbb{S}^n$ does not destroy the norm control of Proposition 2.1.

2A. *Projection from a well-chosen center.*

We present in this section a trick which probably appeared for the first time in relation to nonlinear Sobolev extensions in R. Hardt, D. Kinderlehrer and F.-H. Lin's works [Hardt et al. 1986; Hardt and Lin 1987]. For a Lorentz space version see Proposition 3.4.

Proposition 2.2 (projection trick). *If $f \in W^{1,q}(\Omega, B^{n+1})$ with $q < n+1$ and Ω is a bounded open simply connected domain of $\mathbb{R}^{m+1}$, then there exists $a \in B^{n+1}_{1/2}$ and a constant C depending only on q, m, n such that if $f_a(x) = \pi_a(f(x))$, where $\pi_a : B^{n+1} \setminus \{a\} \to \mathbb{S}^n$ is the projection which is constant along the segments $[a, \omega]$, $\omega \in \mathbb{S}^n$, then*

$$\|f_a\|_{W^{1,q}(\Omega, \mathbb{S}^n)} \leq C \|f\|_{W^{1,q}(\Omega, B^{n+1})}.$$

Proof. We just have to estimate the gradient of f_a in terms of that of f since in any case the functions themselves are bounded and Ω is assumed of finite measure. We first note that, since $a \in B^{n+1}_{1/2}$ is away from the boundary of B^{n+1}, we have the pointwise estimate

$$|\nabla f_a|(x) \lesssim \frac{|\nabla f|(x)}{|f(x) - a|},$$

where the implicit constant depends only on n. We next consider the following "average" on a:

$$\int_{B^{n+1}_{1/2}} \left(\int_\Omega |\nabla f_a|^q(x)\, dx \right) da \lesssim \int_\Omega |\nabla f|^q(x) \left(\int_{B^{n+1}_{1/2}} \frac{da}{|f(x)-a|^q} \right) dx.$$

We note that the inner integral is of the form

$$I(y) := \int_{B^{n+1}_{1/2}} \frac{da}{|y-a|^q},$$

and

$$\max_y I(y) = I(0) = C_n \int_0^{1/2} r^{n+q}\, dr = C_{n,q} < \infty \quad \text{since } q < n+1;$$

therefore, we obtain

$$\int_{B^{n+1}_{1/2}} \|\nabla f_a\|^q_{L^q}\, da \le C_{n,q} \|\nabla f\|^q_{L^q},$$

and the proof is easily concluded. □

The above proposition together with Proposition 2.1 and the remark on Sobolev exponents following it give the following:

Theorem 2.3 (corollary of the projection trick; cf. [Hardt and Lin 1987, Theorem 6.2]). *Let $m, n \in \mathbb{N}^*$. If $1 \le p < m(n+1)/(m+1)$ then for any $\phi \in W^{1,p}(\partial B^{m+1}, \mathbb{S}^n)$ there exists a nonlinear extension $u \in W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ satisfying the control*

$$\|u\|_{W^{1,p(m+1)/m}(B^{m+1},\mathbb{S}^n)} \le C_{m,n,p} \|\phi\|_{W^{1,p}(\partial B^{m+1},\mathbb{S}^n)}.$$

Remark 2.4. Note that from the same ingredients we obtain also the stronger estimate where for $q := p(m+1)/m < m$ the weaker space $W^{1-1/q,q}(\partial B^{m+1}, \mathbb{S}^n)$ replaces $W^{1,p}(\partial B^{m+1}, \mathbb{S}^n)$. This was done in [Bethuel and Demengel 1995; Hardt and Lin 1987]. We stated Theorem 2.3 as above to emphasize the connection with our Theorems B and C. Indeed, taking $m = n$ we see that those theorems cover the critical exponent $p = n$ for which the projection trick stops working.

2B. ***Large integrability exponents.*** We now consider functions in $W^{1,p}(\mathbb{S}^m, \mathbb{S}^n)$ with $p > m$; there is a continuous embedding of $C^{0,1-m/p}(\mathbb{S}^m, \mathbb{S}^n)$ into this space. The candidate extension space $W^{1,p(m+1)/m}(B^{m+1}, \mathbb{S}^n)$ is also composed of $C^{0,1-m/p}$-functions. As described in Section 1D, the extension problem is guaranteed to have a solution as long as $\pi_m(\mathbb{S}^n) = 0$. This is true for $m < n$ but false for many choices of $m > n$ and for $m = n$.

When an extension exists for ϕ representing the identity of the (nontrivial) group $\pi_m(\mathbb{S}^n)$, a controlled extension can be constructed based on the fact that a bound on the $C^{0,\alpha}$-norm for $\alpha > 0$ implies a control on the modulus of continuity.

2C. *Extension for maps in $W^{1,1}(\partial\mathbb{S}^1,\mathbb{S}^1)$.* For maps with values in $\mathbb{S}^3$, we are helped by the existence of a well-behaved product structure on $\mathbb{S}^3$, i.e., the one which gives the identification $\mathbb{S}^3 \simeq \mathrm{SU}(2)$. This is enough to get the analogous result for $n = 1$, as we will see now. It is however well known (see [Hatcher 2009, Section 2.3]) that this is a very unusual case: a group operation exists on $\mathbb{S}^k$ only for $k = 1, 3$.

We can state a similar extension problem in the 1-dimensional case. This kind of controlled extension result is related to the recent work on Ginzburg–Landau functionals in [Serfaty and Tice 2008].

Here the main structural ingredients present for $\mathbb{S}^3$ are again present: namely, we have a group operation on $\mathbb{S}^1$ (in this case it is the abelian group $U(1) \sim \mathbb{R}/\mathbb{Z}$) and a Möbius structure on D^2 restricting to one on $\mathbb{S}^1$. We follow the strategy of proof described in Section 1D. The result is:

Theorem 2.5 (1-dimensional version of the extension). *There exists a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ with the following property: if $\phi \in W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)$ then there exists $u \in W^{1,(2,\infty)}(D^2,\mathbb{S}^1)$ with $u|_{\partial D^2} = \phi$ in the sense of traces and we have the norm control*

$$\|u\|_{W^{1,(2,\infty)}(D^2,\mathbb{S}^1)} \le g(\|\phi\|_{W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)}).$$

We will explain the changes which occur with respect to the proof of Theorem B (see Section 4).

Sketch of proof. The procedure is as in Section 4 and Appendix A; we have just to replace exponents and dimensions 3, 4 with 1, 2. For the analogue of Proposition 4.9 the biharmonic equation (4-36) is replaced by a harmonic equation, while the resulting estimates persist. Perhaps the only significant change is Lemma B.1 of Appendix B. It should be replaced by the following product estimate, valid for $f \in W^{1,1}(D^2)$, $g \in L^\infty \cap W^{1,2}(D^2)$:

$$\|fg\|_{W^{1,1}} \le \|f\|_{W^{1,1}}(\|g\|_{L^\infty} + \|g\|_{W^{1,2}}). \qquad \square$$

We must however note that the naturality of the space $W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)$ in Theorem 2.5 is less evident, since the trace space $H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)$ does not continuously embed in it, unlike what happens in higher dimensions. This is seen by considering

$$u_\epsilon(\theta) = \exp\Big(i \min\big\{1, \epsilon^{-1} \operatorname{dist}_{\mathbb{S}^1}\big(\theta, \big[-\tfrac{1}{2}\pi, \tfrac{1}{2}\pi\big]\big)\big\}\Big).$$

It is then clear that $\|\nabla u_\epsilon\|_{L^1(\mathbb{S}^1)} = 2$, while we estimate the double integral in θ, θ' giving the $H^{1/2}$-norm by the contribution of the regions $\theta \in \big[0, \frac{1}{2}\pi\big]$, $\theta' \in \big[\frac{1}{2}\pi + \epsilon, \pi + \epsilon\big]$. Under these choices, $u_\epsilon(\theta) = e^0$, $u_\epsilon(\theta') = e^i$, and their distance in $\mathbb{S}^1$ is 1. Thus,

$$\begin{aligned}\|u_\epsilon\|^2_{H^{1/2}(\mathbb{S}^1,\mathbb{S}^1)} &= \int_{\mathbb{S}^1}\int_{\mathbb{S}^1} \frac{\operatorname{dist}_{\mathbb{S}^1}(u_\epsilon(\theta), u_\epsilon(\theta'))^2}{\operatorname{dist}_{\mathbb{S}^1}(\theta,\theta')^2}\, d\theta\, d\theta' \\ &\le \int_0^1\int_0^1 \frac{1}{|x + 2\epsilon/\pi - y|^2}\, dx\, dy \\ &\lesssim |\log\epsilon| + 1.\end{aligned}$$

2D. ***Using controlled liftings to obtain controlled extensions.*** The control obtained for extensions of maps in $W^{1,3}(\mathbb{S}^3,\mathbb{S}^3)$ and $W^{1,1}(\mathbb{S}^1,\mathbb{S}^1)$ is exponential in the norms of these maps. In Section 3 we describe an approach, which works for $\phi \in W^{1,2}(\mathbb{S}^2,\mathbb{S}^2)$, which is completely different than in dimensions 1 and 3 and yields a faster proof and a better control. Such an approach was first considered in [Hardt and Rivière 2003]. This is based on the existence of controlled Hopf lifts. The result (see Corollary 3.3) is that there exists an $L^{2,\infty}$-controlled lifting $\tilde{\phi} : \mathbb{S}^2 \to \mathbb{S}^3$, i.e., a function such that $H \circ \tilde{\phi} = \phi$, where $H : \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf fibration and we have the control

$$\|\nabla\tilde{\phi}\|_{L^{2,\infty}} \leq C\|\nabla\phi\|_{L^2}(1 + \|\nabla\phi\|_{L^2}).$$

The analogous controlled lift exists also for $\phi \in W^{1,3}(\mathbb{S}^3,\mathbb{S}^2)$, whereas for $2 \leq p < 3$ we have a control on the L^p-norm of the lift instead of the $L^{p,\infty}$ one; cf. Proposition 1.7. This lift allows us to prove, along the same lines, Theorems C and D.

The gist of the proof is the following: Once we have the controlled lift, the lifted map takes values into a sphere of a higher dimension. This allows a wider range of application for the projection trick of Proposition 2.2 or of its Lorentz space analogue of Proposition 3.4.

Having extended the lift, reprojecting the extension to $\mathbb{S}^2$ via the Hopf map maintains the gradient estimates. This is due to the fact that the Hopf fibration is a submersion (cf. (3-4)) and our lift can be taken so that the "vertical" component η is also controlled.

Work on the existence of nonlinear liftings has been very active regarding $\mathbb{S}^1$-valued maps (see, e.g., [Bourgain et al. 2000; 2004; Bethuel and Zheng 1988] and the references therein). Looking also at higher-dimensional analogues seems very promising in relation to extension results.

2E. ***Small energy extension with estimate.*** As for the case of curvatures over bundles with a compact Lie group, the small energy regime allows a kind of linearization of the problem and gives estimates which are better than what is expected in general. We obtain in particular an estimate in $W^{1,4}$ instead of $W^{1,(4,\infty)}$ for the extension, provided that the norm of the boundary trace is small:

Proposition 2.6 (see Theorem 4.4). *There is a constant $\epsilon_0 > 0$ and a finite constant C such that, if*

$$\int_{\mathbb{S}^3} |\nabla\phi|^3 \leq \epsilon_0, \quad \phi : \mathbb{S}^3 \to \mathbb{S}^3,$$

then there exists $u \in W^{1,4}(B^4,\mathbb{S}^3)$ such that

$$u = \phi \quad \text{on } \partial B^4 \text{ in the sense of traces} \qquad \text{and} \qquad \|\nabla u\|_{L^4(B^4)} \leq C\|\nabla\phi\|_{L^3(\mathbb{S}^3)}.$$

This is part of our proof of Theorem B and is proved in Section 4B using a method developed in Appendix A in the spirit of [Uhlenbeck 1982b].

2F. ***Existence of $W^{1,4}$-extension without norm bounds.*** As for the case of global gauges, we can in general obtain $W^{1,4}(B^4,\mathbb{S}^3)$-extensions once we give up the requirement to have a norm control of the extension such as in Theorem B. This phenomenon represents one example of situations in which function spaces have behavior which is more complex than what can be detected by only looking at their norms.

Proposition 2.7. *If* $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$, *then its topological degree is well defined*; *cf.* [Schoen and Uhlenbeck ∼1980; White 1988]. *Suppose that* $\deg \phi = 0$. *Then there exists* $u \in W^{1,4}(B^4, \mathbb{S}^3)$ *such that*

$$u = \phi \quad \text{on } \partial B^4 \text{ in the sense of traces.}$$

Proof. We use the extension as in Section 4A. The construction using Lemma 4.5 is done on a series of domains $B(x_i, \rho_i) \cap B^4$, where $x_i \in \partial B^4$, $\rho_i \in [\rho_F, 2\rho_F]$ for the choice

$$\rho_F := \inf\left\{\rho > 0 : \exists x_0 \in \partial B^4, \int_{B(x_0,2\rho)\cap\partial B^4} |\nabla\phi|^3 \geq \epsilon_0\right\}.$$

Note that we have no a priori control on how small ρ_F could get, but (by absolute continuity of $|\nabla\phi|^3 dx$ and compactness of ∂B^4) it cannot be zero for a fixed ϕ. Then a Lipschitz extension $u : \mathcal{R} \to \mathbb{S}^3$ to a Lipschitz region $\mathcal{R}$ included between $B^4 \setminus B_{1-2\rho_F}$ and $B^4 \setminus B_{1-\rho_F}$ exists as in Section 4A and such a u will also be Lipschitz (with constant bounded by ρ_F^{-1}) and will have degree zero (the preservation of degree follows because the extension used in the construction preserves the homotopy type; cf. [White 1988]). In particular we can do a further Lipschitz (thus $W^{1,4}$) extension to the interior of $B^4 \setminus \mathcal{R}$. This provides the desired u. □

The proof of the above proposition is constructive, and no hint that the construction is optimal is available. In the next section we prove that actually *no general bound in* $W^{1,4}$ can be achieved, because of the intervention of the topological degree, much as in the case of SU(2)-instantons.

2G. *Impossibility of* $W^{1,4}$ *bounds for an extension.*

Proposition 2.8. *There exists no finite function* $f : \mathbb{R}^+ \to \mathbb{R}^+$ *such that for each* $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ *there exists a function* $u \in W^{1,4}(B^4, \mathbb{S}^3)$ *satisfying*

$$u = \phi \quad \text{on } \partial B^4 \text{ in the sense of traces} \qquad \text{and} \qquad \|\nabla u\|_{L^4(B^4)} \leq f(\|\nabla\phi\|_{L^3(\mathbb{S}^3)}).$$

Proof. We recall the robustness of degree under strong convergence in $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ (see [Schoen and Uhlenbeck ∼1980; White 1988; Brézis and Nirenberg 1995; 1996]). Consider $\phi = \mathrm{id}_{\mathbb{S}^3}$, which has degree 1. Suppose an extension $u : B^4 \to \mathbb{S}^3$ to ϕ were to exist with $\|u\|_{W^{1,4}} \leq C'$. It would then be possible to approximate u in $W^{1,4}$-norm by functions $u_i \in C^\infty(B^4, \mathbb{S}^3)$, since smooth functions are dense in $W^{1,4}(B^4, \mathbb{S}^3)$. In particular the degrees $\deg(\phi_i)$ of $\phi_i = u_i|_{\partial B^4}$ would have to be zero. This contradicts the fact that $\phi_i \to \phi$ in $W^{1,3}$-norm because the degree of the boundary trace is preserved under strong $W^{1,3}$-convergence.

This proves the absence of a continuous extension operator. To show that boundedness is also impossible, we use a slightly different argument.

Consider $\phi_0 \in W^{1,3} \cap C^\infty(\mathbb{S}^3, \mathbb{S}^3)$ that is a perturbation of the identity equal to the south pole S in a neighborhood N_S of S. Then consider a Möbius transformation $F : \mathbb{S}^3 \to \mathbb{S}^3$ such that $F^{-1}(N_S)$ includes the lower hemisphere, and let $\phi' = \phi_0 \circ F$, $\phi'' = \phi_0 \circ (-F)$. Then, identifying $\mathbb{S}^3 \sim \mathrm{SU}(2)$ so that $S \sim \mathrm{id}_{\mathrm{SU}(2)}$, use the group operation to define $\phi = \phi'\phi''$. Note that $\|\phi\|_{W^{1,3}} \leq 2\|\phi_0\|_{W^{1,3}}$, since the conformal maps F, $-F$ preserve the energy; moreover, ϕ has zero degree.

Let F_n be a family of Möbius transformations symmetric about S that concentrate more and more near S (with the notation of Appendix C we may take $F_n := F_{v_n}$ for $v_n = (1-1/n)S$). Define $\phi'_n := \phi' \circ F_n$ and $\phi_n := \phi'_n \phi''$. It is clear by conformal invariance of the $W^{1,3}$-energy that the ϕ_n have constant energy. They converge weakly to ϕ'' and have degree zero.

Call u_n the extension of ϕ_n and suppose that $\|u_n\|_{W^{1,4}} \le C$ independent of n. We may suppose that, in $W^{1,4}$-norm, $u_n \rightharpoonup u_\infty \in W^{1,4}(B^4, \mathbb{S}^3)$ and we obtain $u_\infty|_{\partial B^4} = \phi''$ in the sense of traces. We then apply the result of [White 1988] (see also [Schoen and Uhlenbeck ~1980]), which in this case says that the 3-dimensional homotopy class passes to the limit under bounded sequential weak-$W^{1,4}(B^4, \mathbb{S}^3)$ limits. We again obtain a contradiction to boundedness, since $\deg(\phi'') = -1$ whereas the same degree is zero for the maps ϕ_n. □

2H. *Moving frames and their gauges.* We describe here a lifting problem arising in the theory of moving frames on 2-dimensional surfaces, where the Lorentz spaces appear again in the optimal estimates. The model question is as follows:

Question 2.9. Given a map (representing the normal vector of an immersed surface) $\vec{n} \in W^{1,2}(D^2, \mathbb{S}^2)$, does there exist a $W^{1,2}$-*controlled trivialization* $\vec{e} = (\vec{e}_1, \vec{e}_2)$ of the pullback bundle $\vec{n}^{-1}T\mathbb{S}^2$? A trivialization is defined by two vector fields $\vec{e}_1, \vec{e}_2 \in W^{1,2}(D^2, \mathbb{S}^2)$ such that the pointwise constraints $|\vec{e}_1| = |\vec{e}_2| = 1$, $\vec{e}_1 \cdot \vec{e}_2 = 0$ are satisfied almost everywhere and $\vec{n} = \vec{e}_1 \times \vec{e}_2$.

This problem behaves like the one of global controlled gauges; namely for small energy a lift exists and is controlled, and, for large energy, lifts can be found but with no general control. Uhlenbeck's ϵ-regularity estimate is mirrored in the following theorem. This result was proved initially by F. Hélein [1996, Lemma 5.1.4] under the hypothesis $\|\nabla\vec{n}\|_{L^2} \le C$ and improved by Y. Bernard and T. Rivière, who proved that it is enough to assume a smallness condition in weak-L^2:

Theorem 2.10 [Bernard and Rivière 2014, Lemma IV.3]. *There exists ϵ_0 such that, if $\|\nabla\vec{n}\|_{L^{2,\infty}} \le \epsilon_0$, then there exists a trivialization with the controls*

$$\|\nabla\vec{e}_1\|_{L^2} + \|\nabla\vec{e}_2\|_{L^2} \le C\|\nabla\vec{n}\|_{L^2} \quad \textit{and} \quad \|\nabla\vec{e}_1\|_{L^{2,\infty}} + \|\nabla\vec{e}_2\|_{L^{2,\infty}} \le C\|\nabla\vec{n}\|_{L^{2,\infty}}.$$

Note that, for the improvement above, the L^2-energy might blow up yet still control the energy of the trivialization, as long as we stay small in Lorentz norm. It would be interesting to explore this kind of phenomenon also for curvatures in higher dimensions, like in our setting.

The bad behavior in large energy regimes starts at the energy level 8π (and this is optimal; see [Kuwert and Li 2012]). This number has an evident topological significance because, if $\vec{n}$ is homotopically nontrivial, i.e., parametrizes a noncontractible 2-cell of $\mathbb{S}^2$, then $4\pi = |\mathbb{S}^2| \le \int_{D^2} u^* d\,\mathrm{Vol}_{\mathbb{S}^2} \le \frac{1}{2}\int_{D^2} |\nabla\vec{n}|^2$, so 8π is the smallest energy of a topologically nontrivial $\vec{n}$.

We also have the following lemma, similar to Section 2G:

Lemma 2.11. *For $\int |\nabla\vec{n}|^2 > 8\pi$ there can be no controlled $W^{1,2}$-trivialization $\vec{e}$.*

Sketch of proof:. We choose $\vec{n}$ mapping a neighborhood $D^2 \setminus B_r := N_1$ for small r to the south pole of $\mathbb{S}^2$ that has degree 1 and equals a conformal map outside a small neighborhood $N_2 \Supset N_1$. Such $\vec{n}$ exists with energy as close as desired to 8π, independently of r, by conformal invariance of the energy.

Supposing a trivialization $\vec{e}=(\vec{e}_1,\vec{e}_2)$ exists, on N_1 it will span the "horizontal" 2-plane of $\mathbb{R}^3$ which is perpendicular to $S=(0,0,-1)$. On circles ∂B_ρ, $\rho>r$, by Fubini's theorem, for almost all ϵ we will have that $\vec{e}_i$, $i=1,2$ will be $W^{1,2}$ and thus C^0 and they have values in the equator of $\mathbb{S}^2$. By well-posedness of the topological degree and since $\vec{n}$ is nontrivial in homotopy, we obtain that each e_i will make a full turn on each ∂B_r. This gives that $\int_{\partial B_r}|\nabla\vec{e}_i|\geq 1$ on ∂B_r and by Jensen's inequality we obtain

$$\int_{D^2\setminus B_r}|\nabla\vec{e}_i|^2\geq C\int_r^1\frac{1}{\rho^2}\rho\,d\rho\geq C\left|\log\frac{1}{r}\right|;$$

since there is no positive lower bound for $r>0$, we see that we cannot have a controlled trivialization. □

There is an analogue also of our $W^{1,(4,\infty)}$-extension result here, and it corresponds to taking the so-called "Coulomb frames". The result is a general estimate with no restriction on $\vec{n}$, but with the Lorentz norm $L^{(2,\infty)}$ instead of the L^2-norm (this estimate follows from Wente's [1969] inequality using [Adams 1975]):

Proposition 2.12 [Rivière 2012, VII.6.3]. *Let $\vec{n}\in W^{1,2}(D^2,\mathbb{S}^2)$. Then a trivialization $\vec{e}$ belonging to $W^{1,(2,\infty)}$ exists which satisfies the Coulomb condition*

$$\operatorname{div}\langle\vec{e}_1,\nabla\vec{e}_2\rangle=0$$

and the control

$$\|\nabla\vec{e}_1\|_{L^{(2,\infty)}}+\|\nabla\vec{e}_2\|_{L^{(2,\infty)}}\lesssim\|\nabla\vec{n}\|_{L^2}+\|\nabla\vec{n}\|_{L^2}^2.$$

3. The Hopf lift extension

We now prove Theorem C. We consider a fixed $\phi\in W^{1,2}(\mathbb{S}^2,\mathbb{S}^2)$ and we need to construct an extension $u\in W^{1,(3,\infty)}(B^3,\mathbb{S}^2)$ such that

$$\|u\|_{W^{1,(3,\infty)}(B^3)}\lesssim\|\phi\|_{W^{1,2}(\mathbb{S}^2)}(1+\|\phi\|_{W^{1,2}(\mathbb{S}^2)}),$$

where the implicit constant is independent of ϕ.

The strategy of proof uses a construction based on the Hopf fibration which has been introduced in [Hardt and Rivière 2003]. The same strategy was later used in [Bethuel and Chiron 2007] for proving similar lifting results as in [Hardt and Rivière 2003]. In the smooth case we will first lift $\phi:\mathbb{S}^2\to\mathbb{S}^2$ to $\tilde{\phi}:\mathbb{S}^2\to\mathbb{S}^3$ such that $H\circ\tilde{\phi}=\phi$, where $H:\mathbb{S}^2\to\mathbb{S}^3$ is the Hopf fibration. Then we will extend $\tilde{\phi}$ by using a Lorentz analogue of Proposition 2.2, working with similar conditions on dimensions and exponents. Projecting back to $\mathbb{S}^2$ via H will keep the estimates.

Before the proof, we recall some properties of the map H.

3A. ***Facts about the Hopf fibration.*** Identifying $\mathbb{S}^3$ with the unit sphere of $\mathbb{C}^2$ with complex coordinates (Z,W), the Hopf projection is $H(Z,W)=Z/\overline{W}$ and its fibers are great circles. This gives a function with values in $\mathbb{C}\cup\{\infty\}\simeq\mathbb{S}^2$. If we look at $\mathbb{S}^3\subset\mathbb{R}^4$ with the inherited coordinates (x_1,x_2,x_3,x_4), then we can identify

$$H^*\omega_{\mathbb{S}^2} = d\alpha \quad \text{for } \alpha = \tfrac{1}{2}(x_1 dx_2 - x_2 dx_1 + x_3 dx_4 - x_4 dx_3). \tag{3-1}$$

Here $\omega_{\mathbb{S}^2}$ is a constant multiple of the volume form of $\mathbb{S}^2$. Since $\mathbb{S}^1 \sim U(1)$, we can regard $\mathbb{S}^3 \xrightarrow{H} \mathbb{S}^2$ as a principal $U(1)$-bundle $P \to \mathbb{S}^2$.

Let $\phi : \mathbb{C} \to \mathbb{S}^2$ be a smooth function. Then $d(\phi^*\omega_{\mathbb{S}^2}) = 0$, because $\Omega^3(\mathbb{R}^2 \simeq \mathbb{C}) = \{0\}$. Since $H^2_{dR}(\mathbb{C}) = 0$, there exists a 1-form η such that

$$d\eta = \phi^*\omega_{\mathbb{S}^2}\,. \tag{3-2}$$

We also note that for a smooth $\phi : \mathbb{C} \to \mathbb{S}^2$ the pullback of the $U(1)$-bundle P is trivial, since $\mathbb{R}^2$ is contractible. A trivialization of the bundle $\phi^*P \to \mathbb{C}$ can be identified with a lift $\tilde{\phi}$ of ϕ. From (3-1) we can deduce that $d\eta = \tilde{\phi}^*H^*\omega_{\mathbb{S}^2} = \tilde{\phi}^*d\alpha = d(\tilde{\phi}^*\alpha)$ and again there exists a 1-form $\tilde{\eta}$ as in (3-2), defined by

$$\tilde{\eta} = \tilde{\phi}^*\alpha. \tag{3-3}$$

Note that $\tilde{\eta}$ coincides with η up to adding an exact form $d\theta$: we have $\tilde{\phi}^*\alpha - \eta = d\phi$. If we come back to the bundle point of view then $d\theta$ represents the effect of change of coordinates of the trivialization giving $\tilde{\phi}$, i.e., of a change of gauge. We then have $\eta = \tilde{\phi}^*\alpha - d\theta = (e^{-i\theta}\tilde{\phi})^*\alpha$, where the action of $e^{-i\theta}$ is intended as a $U(1)$-gauge change and $\theta : \mathbb{C} \to \mathbb{R}$ is determined up to a constant. Moreover, since DH is an isometry between the orthogonal complement of the tangent space of the fiber $T_pH^{-1}(H(p))$ and $T_p\mathbb{S}^2$, we also obtain the norm identity

$$|D\tilde{\phi}|^2 = |\tilde{\eta}|^2 + |D\phi|^2. \tag{3-4}$$

3B. *Hopf lift with estimates.* We start the proof of Theorem C:

Proposition 3.1. *Suppose $\phi \in W^{1,2}(\mathbb{C}, \mathbb{S}^2)$. Then there exists a lifting $\tilde{\phi} : \mathbb{C} \to \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$ and there exists a universal constant C such that*

$$\|\nabla\tilde{\phi}\|_{L^{2,\infty}} \le C\|\nabla\phi\|_{L^2}(1 + \|\nabla\phi\|_{L^2}).$$

Proof. The proof is divided into two steps.

Step 1 (constructions in the smooth case). We have seen that, at least in the smooth case, constructing a 1-form η as in (3-2) is equivalent to constructing a lift $\tilde{\phi} : \mathbb{C} \to \mathbb{S}^3$. We now observe that such a 1-form can in turn be easily constructed by inverting the Laplacian on $\mathbb{C}$ via its Green kernel, which is of the form $K(x) = -\gamma \log|x|$. In particular, $K \in W^{1,(2,\infty)}$, which is the reason why this norm appears. First note that $dd^*(K * \beta) = 0$ for a smooth L^1-integrable 2-form β on $\mathbb{C}$. We can then use this formula for $\beta = \phi^*\omega_{\mathbb{S}^2}$ and, taking into account the fact that ∇K is in $L^{2,\infty}$, by the Lorentz-space Young inequality (see [Grafakos 2008]), we obtain that the 1-form η defined as

$$\eta := d^*[K * (\phi^*\omega_{\mathbb{S}^2})], \quad \eta \to 0 \text{ at infinity}, \tag{3-5}$$

satisfies (3-2) and the estimates

$$\|\eta\|_{L^{2,\infty}} \lesssim \|\phi^*\omega_{\mathbb{S}^2}\|_{L^1} \lesssim \|D\phi\|_{L^2}^2 \|\phi\|_{L^\infty} \simeq \|D\phi\|_{L^2}^2 . \tag{3-6}$$

We have mentioned where to find the proof that η corresponds up to a unitary transformation to a lift $\tilde{\phi}$, and from (3-4) and (3-6) we also obtain the estimate for $\tilde{\phi}$,

$$\|D\tilde{\phi}\|_{L^{2,\infty}} \lesssim \|\eta\|_{L^{2,\infty}} + \|D\phi\|_{L^2} \lesssim \|D\phi\|_{L^2}(1 + \|D\phi\|_{L^2}). \tag{3-7}$$

Step 2 (extending the constructions to $W^{1,2}$). The results obtained so far hold for $\phi \in C^\infty(\mathbb{C}, \mathbb{S}^2)$. We use the well-known fact that, while not dense in the strong topology, the functions in $C^\infty(\mathbb{C}, \mathbb{S}^2)$ are instead *dense with respect to weak sequential convergence* (see [Bethuel 1991; Hang and Lin 2003]). The constraint of u_n having values in $\mathbb{S}^2$, as well as the constraint $\tilde{\phi}_n \circ H = \phi_n$ for the $\tilde{\phi}_n$, are pointwise constraints (note indeed that the function H is smooth), so they are preserved under weak convergence $\phi_n \rightharpoonup \phi \in W^{1,2}$. Now we state the only less classical point in the proof in the following lemma:

Lemma 3.2. *$L^{2,\infty}$-estimates are preserved under weak convergence in L^2. In other words, if $f_n \in L^2$ are weakly convergent to $f \in L^2$, then $\|f\|_{L^{2,\infty}} \le \liminf_{n\to\infty} \|f_n\|_{L^{2,\infty}}$.*

Proof. We observe that a positive answer to this question cannot directly and trivially be obtained by interpolation, since the L^∞-norm is not lower semicontinuous with respect to weak convergence in L^2. We thus proceed by duality; namely, we note that

$$L^{(2,\infty)} = (L^{(2,1)})' \quad \text{and} \quad L^{(2,1)} \subset L^2.$$

Therefore $\langle f_n, \phi\rangle \to \langle f, \phi\rangle$ for all $\phi \in L^{(2,1)}$, and by usual Banach space theory we obtain the thesis. □

Applying the lemma, we obtain the desired estimate to conclude the proof of Proposition 3.1 via Bethuel's weak density result [1991]. □

We observe that, given a map $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$, we can obtain a map $u : \mathbb{C} \to \mathbb{S}^2$ having the same norm by composing with the inverse stereographic projection $\Psi^{-1} : \mathbb{C} \to \mathbb{S}^2$; we use the facts that the exponent 2 is equal to the dimension and that Ψ is conformal. In a similar way, having constructed a lift $\tilde{u} : \mathbb{C} \to \mathbb{S}^3$, we obtain automatically a lift $\tilde{\phi}$ of ϕ by composing back with S. The same reasoning using conformality also shows that the $L^{2,\infty}$-norm of the gradient of $\tilde{\phi}$ is preserved. This proves:

Corollary 3.3. *Suppose $\phi \in W^{1,2}(\mathbb{S}^2, \mathbb{S}^2)$. Then there exists a lifting $\tilde{\phi} : \mathbb{S}^2 \to \mathbb{S}^3$ such that $H \circ \tilde{\phi} = \phi$ and there exists a universal constant C such that*

$$\|\nabla\tilde{\phi}\|_{L^{2,\infty}} \le C\|\nabla\phi\|_{L^2}(1 + \|\nabla\phi\|_{L^2}).$$

3C. *Projection and wise choice of the point.* To proceed in our strategy for the proof of Theorem C, we use a version of the projection trick of Section 2A.

Proposition 3.4 (projection trick 2). *Suppose that $\tilde{\phi} \in W^{1,(2,\infty)}(\mathbb{S}^2, \mathbb{S}^3)$. Then there exists a function $\tilde{u} : B^3 \to \mathbb{S}^3$ such that $\tilde{u}|_{\partial B^3 \setminus \mathbb{S}^2} = \tilde{\phi}$ satisfying the following bound for some universal constant C:*

$$\|\tilde{u}\|_{W^{1,(3,\infty)}(B^3)} \le C\|\tilde{\phi}\|_{W^{1,(2,\infty)}(\mathbb{S}^2)}.$$

Proof. We proceed in two steps: the first one introduces the $W^{1,(3,\infty)}$-norm estimate, and the second one ensures that the constraint of having values in $\mathbb{S}^3$ can be preserved.

Step 1 (harmonic extension). Consider a solution $\tilde{u}$ of the equation

$$\begin{cases} \Delta\tilde{u} = 0 & \text{on } B^3, \\ \tilde{u} = \tilde{\phi} & \text{on } \partial B^3. \end{cases} \tag{3-8}$$

By using the Poisson kernel estimates, we obtain that $\tilde{u} \in W^{1,(3,\infty)}(B^3, B^4)$ and

$$\|\nabla\tilde{u}\|_{L^{(3,\infty)}} \lesssim \|\nabla\tilde{\phi}\|_{L^{(2,\infty)}} . \tag{3-9}$$

Step 2 (projection in the target). We now correct the fact that $\tilde{u}$ has values not in $\mathbb{S}^3$ but in its convex hull B^4. For $a \in B^4_{1/2}$ we define the radial projection $\pi_a : B^4 \to \mathbb{S}^3$ of center a, i.e.,

$$\pi_a(x) := a + t_{a,x}(x - a), \quad \text{where } t_{a,x} \geq 0 \text{ is chosen so that } |\pi_a(x)| = 1.$$

In order to estimate the norm of $u_a := \pi_a \circ \tilde{u}$ we note that

$$|\nabla(\pi_a \circ \tilde{u})|(x) \lesssim \frac{|\nabla\tilde{u}(x)|}{|u(x) - a|}$$

with an implicit constant bounded by 4 as long as $a \in B^4_{1/2}$. We just estimate the L^p-norm of ∇u_a for $p \in [1, 4[$. We note that $\int_{B_{1/2}} |\tilde{u}(x) - a|^{-p}\, da$ is bounded for all such p by a number C_p independent of x; therefore, by changing the order of integration and applying Fubini, we obtain

$$\int_{B_{1/2}}\int_{B_1} |\nabla u_a(x)|^p\, dx\, da \leq C_p \int_{B_1} |\nabla\tilde{u}(x)|^p \int_{B_{1/2}} |\tilde{u}(x) - a|^{-p}\, da \leq C_p \|\nabla\tilde{u}\|_p^p .$$

In other words, the assignment $a \mapsto u_a$ gives a map whose $L^1_a(B_{1/2}, W^{1,p}_x(B^3, \mathbb{S}^3))$-norm is bounded by the L^p-norm of $\nabla\tilde{u}$ for $p \in [1, 4[$. First observe that, by Lions–Peetre reiteration (see [Tartar 2007, Chapter 26]), $L^{(3,\infty)}$ is an interpolation between L^{p_0} and L^{p_1} with $3 \in]p_0, p_1[\subset]1, 4[$. We now use the nonlinear interpolation theorem of Tartar [2007, Chapter 28]. Call $U(a, x) := \nabla\tilde{u}(x)/|\tilde{u}(x) - a|$. We know that the map $u \mapsto U$ is bounded between W^{1,p_i} and L^{p_i} for $i = 0, 1$. In order to show that it also satisfies

$$\sup_{\lambda>0} \lambda^3 \left| \left\{ (x, a) \in B_1 \times B_{1/2} : \frac{|\nabla u(x)|}{|u(x) - a|} > \lambda \right\} \right| = \|U\|^3_{L^{(3,\infty)}} \lesssim \|\tilde{u}\|^3_{W^{1,(3,\infty)}} , \tag{3-10}$$

we will check the local estimate

$$\left\| \frac{\nabla u(x)}{|u(x) - a|} - \frac{\nabla v(x)}{|v(x) - a|} \right\|_{L^{p_1}} \lesssim \|u - v\|_{L^{p_1}} .$$

This follows since

$$\int_{B_1}\int_{B_{1/2}} \left| \frac{\nabla u(x)}{|u(x) - a|} - \frac{\nabla v(x)}{|v(x) - a|} \right|^{p_1} \lesssim \int_{B_1} |\nabla u - \nabla v|^{p_1} \int_{B_{1/2}} \big(|u(x) - a|^{-p_1} + |v(x) - a|^{-p_1} \big)\, da\, dx$$

and the same estimates as before apply to the second factor, uniformly in x. Thus (3-10) holds. From (3-10) it easily follows that there exists $a \in B_{1/2}$ for which

$$\|\nabla u_a\|_{L^{(3,\infty)}(B_1)} \lesssim \|\tilde{u}\|_{W^{1,(3,\infty)}} . \tag{3-11}$$

Combining (3-9) and (3-11), we obtain the claim of the proposition for $\hat{u} := u_a$. □

3D. ***End of proof.***

Proof of Theorem C. Apply consecutively Corollary 3.3 and Proposition 3.4. For $\hat{u}$ as in Proposition 3.4, we can then consider $u := H \circ u_a : B^3 \to \mathbb{S}^2$. Since H is Lipschitz, we obtain the pointwise estimate

$$|\nabla u| \lesssim |\nabla u_a|. \tag{3-12}$$

Combining this with the estimates of Corollary 3.3 and Proposition 3.4, we obtain the thesis. □

3E. ***Modification of proof in the case of $W^{1,p}(\mathbb{S}^m, \mathbb{S}^2)$.*** In this section we prove Theorem D and Proposition 1.7.

Proof of Theorem D and of Proposition 1.7. We consider $n = 2 < m$ and $3m/(m+1) \le p < 4m/(m+1)$ as in Proposition 1.7. We will use the fact that such p is always greater than 2. The construction of the 1-form η satisfying (3-3) and (3-4) can be done in a completely analogous way if the domain is $\mathbb{R}^m$, $m \ge 3$. The only difference is that in that case the Laplacian on 2-forms such as $\phi^*\omega_{\mathbb{S}^2}$ has the form $\delta = d^*d + dd^*$, where the first part does not vanish anymore. In this case however we may still solve

$$\begin{cases} d\eta = \phi^*\omega_{\mathbb{S}^2}, \\ d^*\eta = 0, \\ \eta(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

If $\phi \in W^{1,p}(\mathbb{R}^m, \mathbb{S}^2)$ and since $p > 2$, we then have

$$\|d\eta\|_{L^{p/2}(\mathbb{R}^m)} \le C\|\phi^*\omega_{\mathbb{S}^2}\|_{L^{p/2}(\mathbb{R}^m)} \le C\|d\phi\|^2_{L^p(\mathbb{R}^m)}.$$

As before, we have (3-4), from which we also obtain $|D\tilde{\phi}|^p \lesssim |\eta|^p + |D\phi|^p$. Passing to $\mathbb{S}^m$ and noting that in dimension $m \ge p$ we have $W^{1,p/2}(\mathbb{S}^m, \mathbb{S}^2) \hookrightarrow L^{mp/(2m-p)}(\mathbb{S}^m, \mathbb{S}^2) \hookrightarrow L^p(\mathbb{S}^m, \mathbb{S}^2)$, we obtain

$$\|D\tilde{\phi}\|_{L^p(\mathbb{S}^m,\mathbb{S}^2)} \lesssim \|D\phi\|^2_{L^p(\mathbb{S}^m,\mathbb{S}^2)} + \|D\phi\|_{L^p(\mathbb{S}^m,\mathbb{S}^2)}.$$

Harmonic extension and Proposition 2.2 allow us then to obtain an extension $\tilde{u} : B^{m+1} \to \mathbb{S}^2$ of $\tilde{\phi}$ such that

$$\|\nabla \tilde{u}\|_{L^{p(m+1)/m}(B^{m+1},\mathbb{S}^3)} \lesssim \|D\tilde{\phi}\|_{L^p(\mathbb{S}^m,\mathbb{S}^3)}$$

provided $p(m+1)/m < 4$ (which is the condition appearing in Proposition 2.2. Composing with the Hopf map H at most decreases the norm; thus we obtain that $u := H \circ \tilde{u}$ is the desired controlled extension as in Proposition 1.7 and in Theorem D (note that for $m = 3$ the condition $p(m+1)/m < 4$ is equivalent to $p < 3$). □

4. The extension theorem for $W^{1,3}$ maps $\mathbb{S}^3 \to \mathbb{S}^3$

This section is devoted to the proof of the following theorem:

Theorem B″. *There is a constant $C > 0$ such that, if $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$, then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \le C(e^{C\|\nabla\phi\|_{L^3}^9} + \|\nabla\phi\|_{L^3}). \tag{4-1}$$

4A. ***Modulus of integrability estimates.*** In general, during our estimates we indicate by C a positive constant, which may change from line to line and also within the same line. We start by fixing the notation for the main quantity which will be used control the energy concentration of our maps.

Definition 4.1. If $D \subset \mathbb{R}^4$ and $f : D \to \mathbb{R}$ is measurable then let $E(f, \rho, D)$ denote the (possibly infinite) modulus of integrability of f, which is defined as

$$E(f, \rho, D) = \sup_{x\in D} \int_{B_\rho(x)\cap D} |f|.$$

The modulus of integrability fits into a sort of elliptic estimate as follows:

Proposition 4.2 (integrability modulus estimates). *Let $\phi \in W^{1,3}(\partial B^4, \mathbb{S}^3)$ and assume that u is the solution to the equation*

$$\begin{cases} \Delta u = 0 & \text{on } B^4, \\ u = \phi & \text{on } \partial B^4. \end{cases}$$

Then there exists a constant C_1 independent of ϕ, ρ such that, when $\rho \in \left]0, \tfrac14\right[$,

$$E(|\nabla u|^4, \rho, B^4) \le C_1 E(|\nabla\phi|^3, 2\rho, \partial B^4)^{1/3} \int_{\partial B^4} |\nabla\phi|^3. \tag{4-2}$$

Proof. We have to prove that, for all $x_0 \in B^4$,

$$\int_{B_\rho(x_0)\cap B^4} |\nabla u|^4 \le C_1 E(|\nabla\phi|^3, 2\rho, \partial B^4) \int_{\partial B^4} |\nabla\phi|^3. \tag{4-3}$$

Step 1 (the case $x_0 \in \partial B^4$). Let $\eta : \mathbb{S}^3 \to [0, 1]$ be a cutoff function such that $\eta \equiv 1$ on $B_{2\rho}(x_0) \cap \mathbb{S}^3$, $\eta \equiv 0$ on $\mathbb{S}^3 \setminus B_{4\rho}(x_0)$, and $|\nabla\eta| \lesssim \rho^{-1}$. Then write $\phi = \phi_1 + \phi_2$ with $\phi_1 = \eta\phi$, $\eta_2 = (1-\eta)\phi$, and let $u = u_1 + u_2$ with

$$\begin{cases} \Delta u_i = 0 & \text{on } B^4, \\ u_i = \phi_i & \text{on } \partial B^4 \end{cases}$$

for $i = 1, 2$. It suffices to prove (4-3) for each u_i separately. By elliptic theory and by the definition of η,

$$\int_{B_\rho(x_0)\cap B^4} |\nabla u_1|^4 \lesssim \left(\int_{S'} |\nabla\phi|^3\right)^{4/3}.$$

Poisson's formula gives

$$u_2(x) = C(1 - |x|^2) \int_{\partial B^4} \frac{\phi_2(y)}{|x - y|^4}\, dy;$$

thus, for $x \in B_\rho(x_0) \cap B^4$, $\rho < \frac{1}{4}$,

$$|\nabla u_2|(x) \lesssim \rho \int_{\mathbb{S}^3 \setminus B_{2\rho}(x_0)} \frac{|\nabla \phi|}{|x-y|^4}\, dy + \int_{\mathbb{S}^3 \setminus B_{2\rho}(x_0)} \frac{|\phi|}{|x-y|^4}\, dy \lesssim \rho \int_{\mathbb{S}^3 \setminus B_{2\rho}(x_0)} \frac{|\nabla \phi|}{|x-y|^4}\, dy.$$

Patching together the estimates obtained so far, we write

$$\int_{B_\rho(x_0)\cap B^4} |\nabla u|^4 \lesssim \left(\int_{S'} |\nabla \phi|^3\right)^{4/3} + \rho^8 \left(\int_{S''} \frac{|\nabla \phi|}{|x-y|^4}\right)^4, \tag{4-4}$$

where the factor ρ^8 comes from the pointwise estimate for ∇u_2, keeping in mind that $|B_\rho(x_0) \cap B^4| \lesssim \rho^4$. Let the summands on the right side of (4-4) be I and II respectively. Note that

$$I \le \left(\int_{B_{2\rho}(x_0)\cap \partial B^4} |\nabla \phi|^3\right)^{1/3} \int_{\mathbb{S}^3} |\nabla \phi|^3 \le E(|\nabla \phi|^3, 2\rho, \partial B^4) \int_{\mathbb{S}^3} |\nabla \phi|^3. \tag{4-5}$$

To estimate II, cover $\mathbb{S}^3 \setminus B_{2\rho}(x_0)$ by (finitely many) geodesic balls $B^3_{2\rho}(x_i)$ so that x_i form a maximal 2ρ-net and they are at distance at least 2ρ from x_0. Then

$$\int_{B^3_{2\rho}(x_i)} |\nabla \phi| \le |B^3_{2\rho}| \left(⨍_{B^3_{2\rho}(x_i)} |\nabla \phi|^3 \right)^{1/3}.$$

For $y \in B^3_{2\rho}(x_i)$, $x \in B_{2\rho}(x_0) \cap B^4$, we have $|x-y| \sim \operatorname{dist}(x_i, x_0)$. Thus

$$II \lesssim \rho^8 \left(\sum_i \operatorname{dist}^{-4}(x_i, x_0) \rho^3 a_i^{1/3} \right)^4,$$

where $a_i = ⨍_{B^3_{2\rho}(x_i)} |\nabla \phi|^3$. By Hölder's inequality we easily obtain

$$II \lesssim \rho^{20} \big(\sup_i a_i^{1/3}\big) \left(\sum_i a_i\right) \left(\sum_i \operatorname{dist}^{-16/3}(x_i, x_0)\right)^3.$$

Now, the first parenthesis is estimated by $\rho^{-1} E(|\nabla \phi|^3, 2\rho, \partial B^4)^{1/3}$, the second one by $\rho^{-3} \int_{\mathbb{S}^3} |\nabla \phi|^3$, and the last one by $\rho^{-16/3}$. Thus we obtain

$$II \lesssim \rho^{20} \rho^{-1} E(|\nabla \phi|^3, 2\rho, \partial B^4)^{1/3} \rho^{-16} \rho^{-3} \int_{\mathbb{S}^3} |\nabla \phi|^3 \lesssim E(|\nabla \phi|^3, 2\rho, \partial B^4)^{1/3} \int_{\mathbb{S}^3} |\nabla \phi|^3. \tag{4-6}$$

By (4-5) and (4-6), we obtain (4-3) for $x_0 \in \partial B^4$.

Step 2. If $|x_0| < 1 - 2\rho$ then we can directly apply the estimates for the term II of (4-4), since now the denominator $|x-y|$ in the Poisson formula will be at least ρ for all $x \in B_\rho(x_0)$.

The estimate of Step 1 also holds for $\rho > \frac{1}{4}$ with the same constant. We can cover the case $|x_0| \in]1-2\rho, 1[$ with $\rho < \frac{1}{4}$ by noticing that if $x_0' = x_0/|x_0|$ then $B_{3\rho}(x_0') \supset B_\rho(x_0)$, and that the measures $|\nabla \phi|^3 d\sigma$, $|\nabla u|^4 dx$ are doubling with constants bounded by the packing constants of $\mathbb{S}^3$ and of B^4 respectively, while the function $E(f, \rho, D)$ is increasing in ρ. Therefore the inequality (4-3) also holds for this last choice of x_0 up to changing C_0 by a factor depending only on the above packing constants. □

4B. ***Extension in the case of small energy concentration.*** In small energy concentration regions we utilize the following:

Theorem 4.3 (small concentration extension). *There exists a constant $\delta \in \left]0, \frac{1}{4}\right[$ with the following property: for each $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ such that the local estimate*

$$E(|\nabla\phi|^3, 2\rho, \mathbb{S}^3) \leq \frac{\delta}{C_1 E} \tag{4-7}$$

holds with $\|\nabla\phi\|^3_{L^3(\mathbb{S}^3)} = E$, there exists a function $\tilde{u} \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ which equals ϕ on $\mathbb{S}^3$ in the sense of traces and satisfies

$$\|\nabla\tilde{u}\|_{L^{4,\infty}} \lesssim \frac{\|\nabla\phi\|^2_{L^3}}{\rho} + \|\nabla\phi\|_{L^3}\,. \tag{4-8}$$

Theorem 4.3 follows from several ingredients, the proofs of which are postponed to Appendix A and to the end of Section 4B.

Theorem 4.4 (Uhlenbeck analogue). *There exist two constants $\delta > 0$, $C > 0$ with the following property: Suppose $\psi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ is such that $\|\nabla\psi\|_{L^3(\mathbb{S}^3)} \leq \delta$. Then there exists an extension $v \in W^{1,4}(B^4, \mathbb{S}^3)$ satisfying the estimate*

$$\|v\|_{W^{1,4}(B^4)} \leq C\|\nabla\psi\|_{L^3(\mathbb{S}^3)}.$$

Proof. See Theorem A.2. □

If $u \in W^{1,4}(B^4, \mathbb{R}^4)$ and $\rho \in \left]0, \frac{1}{2}\right[$, $x_0 \in \partial B^4$, then by a mean value argument there exists $\bar{\rho} \in [\rho, 2\rho]$ such that

$$\bar{\rho}\int_{\mathrm{int}(B^4)\cap\partial B_{\bar{\rho}}(x_0)} |\nabla u|^4 \leq C\int_{B^4\cap B_\rho(x_0)} |\nabla u|^4. \tag{4-9}$$

In this case the following lemma will prove useful:

Lemma 4.5 (Courant–Lebesgue analogue). *Fix $\bar{\rho} \in]0, 1[$. There exists a constant $C > 0$ such that, if $u \in W^{1,4}(B^4, \mathbb{R}^4)$ is the extension of $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ and*

$$\bar{\rho}\int_{\mathrm{int}(B^4)\cap\partial B_{\bar{\rho}}(x_0)} |\nabla u|^4 \leq C$$

with $x_0 \in \partial B^4$, then for almost every $x \in \partial(B^4 \cap B_{\bar{\rho}}(x_0))$ we have

$$\operatorname{dist}(u(x), \mathbb{S}^3) \leq \tfrac{1}{8}\,. \tag{4-10}$$

The restriction of u to a smaller ball $B_{1-\rho}$, being harmonic, is smooth. Then we may utilize the following result:

Lemma 4.6 (interior estimate). *Given $u \in W^{1,4} \cap C^1(B^4, B^4)$, there exists a constant C independent of u such that, for half of the points $a \in B^4$,*

$$\left\|\frac{1}{|u-a|}\right\|^4_{L^{4,\infty}(B^4)} \leq C\int_{B^4} |\nabla u|^4.$$

Proof of Theorem 4.3. **Step 1.** We first observe that the harmonic extension u of ϕ satisfies

$$|\nabla u|(x) \lesssim \frac{\|\phi\|_{W^{1,3}(\mathbb{S}^3)}}{\rho} \quad \text{for } x \in B_{1-\rho}\,.$$

A direct way to see this is by estimating via the Poisson formula together with Poincaré's inequality and a good covering by ρ-balls $B_j \subset \mathbb{S}^3$:

$$\begin{aligned}
|\nabla u|(x) &\lesssim \rho\Big(\int_{\mathbb{S}^3} \frac{\nabla\phi}{|x-y|^4}\,dy + \int_{\mathbb{S}^3} \frac{|\phi|}{|x-y|^4}\,dy\Big)\\
&\lesssim \sum_j \frac{⨍_{B_j} |\nabla\phi| + |\phi|}{d_j^4}\rho^4, && \text{where } d_j \sim \operatorname{dist}(B_j, x)\\
&\lesssim \sum_j \Big(\frac{\rho}{d_j}\Big)^4 ⨍_{B_j} |\nabla\phi| + 1, && \text{by Poincaré}\\
&\lesssim \Big(\sum_j \Big(\frac{\rho}{d_j}\Big)^6\Big)^{2/3} \Big(\sum_j \Big(⨍_{B_j} |\nabla\phi|\Big)^3 + 1\Big)^{1/3}, && \text{by Hölder}\\
&\lesssim \frac{\|\phi\|_{W^{1,3}(\mathbb{S}^3)}}{\rho}\,.
\end{aligned}$$

To justify the last step we observe that $\operatorname{Card}\{j : d_j \sim 2^j\rho\} \sim 2^{4j}$ and thus the first factor in the penultimate line is bounded by $\big(\sum_{j\geq 0} 2^{-2j}\big)^{2/3}$, while for the second factor we use Jensen's inequality.

Step 2. We now use Lemma 4.6 and observe that if $\pi_a : B^4 \setminus \{a\} \to \mathbb{S}^3$ is the retraction of center a then

$$|\nabla(\pi_a \circ u)| \leq C\frac{|\nabla u|}{|u-a|}\,.$$

In particular, using Step 1 and Lemma 4.6 we obtain

$$\|\nabla(\pi_a \circ u)\|_{L^{4,\infty}} \leq \|\nabla u\|_{L^\infty} \left\| \frac{1}{|u-a|} \right\|_{L^{4,\infty}} \leq C\frac{\|\nabla\phi\|_{L^3}}{\rho}\|\nabla u\|_{L^4}\,. \tag{4-11}$$

Step 3. Consider a maximal cover $\{B_i\}$ of $\mathbb{S}^3 = \partial B^4$ by 4-dimensional balls of radius ρ and centers on ∂B^4. It is possible to find a constant C, depending only on the dimension, such that the collection of balls of doubled radius $\{2B_i\}$ can be written as a union of C families of disjoint balls $\mathcal{F}_1, \dots, \mathcal{F}_C$.

Then apply (4-9) to each ball $B_i \in \mathcal{F}_1$. This will give a new family of balls $\{B_i' : B_i \in \mathcal{F}_1\}$ with radii between ρ and 2ρ to which it will be possible to apply Lemma 4.5. Thus $\operatorname{dist}(u(x), \partial B^4) < \frac{1}{8}$ on $\partial(B^4 \cap B_i')$ for all B_i'. Because of the choice of $\mathcal{F}_1$ it also follows that the balls B_i' are disjoint.

If we choose a projection π_a from Step 2 so that $\operatorname{dist}(a, \partial B^4) > \frac{1}{4}$, then

$$u_1^i := \pi_a \circ (u|_{\partial((B^4\cap B_i'))}) \quad \text{satisfies} \quad |\nabla u_1^i| \leq C|\nabla u| \quad \text{on } \partial B_i' \cap B^4$$

by the estimates of Step 2. Note that a will be fixed during the whole construction.

We extend u_1^i (denoting the extension again by u_1^i) inside $B_i' \cap B^4$ via Theorem 4.4, obtaining a new function

$$u_1 := \begin{cases} \pi_a \circ u & \text{on } B^4 \setminus \bigcup B_i', \\ u_1^i & \text{on } B_i'. \end{cases}$$

Theorem 4.4 implies that u_1 satisfies

$$\|\nabla u_1\|_{L^4(B_i')} \leq C\left(\int_{\partial B_i'} |\nabla u_1|^3\right)^{1/3}.$$

We can rewrite this as follows:

$$\begin{aligned} \int_{B_i \cap B^4} |\nabla u_1|^4 &\leq C\left(\int_{B_i \cap \partial B} |\nabla \phi|^3 + \int_{\operatorname{int}(B)\cap \partial B_i} |\nabla u_1^i|^3\right)^{4/3} \\ &\lesssim \left(\int_{B_i \cap \partial B} |\nabla \phi|^3\right)^{4/3} + \left(\int_{\operatorname{int}(B)\cap \partial B_i} |\nabla u_1^i|^3\right)^{4/3}. \end{aligned} \tag{4-12}$$

We note that (using Lemma 4.5)

$$\begin{aligned} \left(\int_{\partial B_i \cap \operatorname{int}(B)} |\nabla u_1^i|^3\right)^{4/3} &\leq \mathcal{H}^3(\partial B_i)^{1/3} \int_{\partial B_i \cap \operatorname{int}(B)} |\nabla u_1^i|^4 \\ &\lesssim \rho \int_{\partial B_i \cap \operatorname{int}(B)} |\nabla u|^4 \\ &\lesssim \int_{B_i \cap B^4} |\nabla u|^4; \end{aligned} \tag{4-13}$$

therefore, u_1 still satisfies (4-2) with a constant C_1 which is now changed by a universal factor.

Step 4. It is possible to repeat the same operation starting from the function u_1 and using the balls of the family $\mathcal{F}_2$ to obtain a function u_2, and then do the same iteratively for all the families $\mathcal{F}_2, \dots, \mathcal{F}_C$.

Denote by $\mathcal{R}$ the union of all the perturbed balls B_i' corresponding to the families $\mathcal{F}_1, \dots, \mathcal{F}_C$. Recall that the number of families is equal to the maximal number of overlaps of balls of different families and depends only on the dimension. Then, iterating the estimates (4-12) using (4-13) for all families $\mathcal{F}_i$, we obtain for the last function u_C that

$$\begin{aligned} \int_{\mathcal{R}} |\nabla u_C|^4 &\lesssim E(|\nabla \phi|^3, 2\rho, \mathbb{S}^3)^{1/3} \sum_i \int_{B_i \cap \partial B} |\nabla \phi|^3 + \int_{\mathcal{R}} |\nabla u|^4 \\ &\leq \|\nabla \phi\|^3_{L^3(\mathbb{S}^3)}\big(E(|\nabla \phi|^3, 2\rho, \mathbb{S}^3)^{1/3} + \|\nabla \phi\|_{L^3(\mathbb{S}^3)}\big), \end{aligned} \tag{4-14}$$

where for the last inequality we also used the elliptic estimates for u in terms of ϕ.

Step 5. We now combine the estimate (4-11) for the part $B \setminus \mathcal{R} \subset B_{1-\rho}$ and (4-14). Observe that in general $\|f\|_{L^{4,\infty}} \lesssim \|f\|_{L^4}$ and that the $L^{4,\infty}$-norm satisfies the triangle inequality. We obtain

$$\|\nabla \tilde{u}\|_{L^{4,\infty}} \lesssim \frac{\|\nabla \phi\|^2_{L^3}}{\rho} + \|\nabla \phi\|_{L^3} + \|\nabla \phi\|^{3/4}_{L^3} E(|\nabla \phi|^3, 2\rho, \mathbb{S}^3)^{1/12}. \tag{4-15}$$

Using the trivial estimate $E(|\nabla \phi|^3, 2\rho, \mathbb{S}^3) \leq \int_{\mathbb{S}^3} |\nabla \phi|^3$, the desired estimate follows. ☐

We now proceed to prove the above lemmas.

Proof of Lemma 4.5. The hypotheses $x_0 \in \partial B^4$, $\bar\rho < 1$ have the following two geometric consequences: (1) $\partial B^4 \cap \partial B_{\bar\rho}(x_0)$ has positive measure; (2) $B^4 \cap B_{\bar\rho}(x_0)$ is 2-bi-Lipschitz equivalent to $B_{\bar\rho}$. Therefore, we may just prove that (4-10) holds true on ∂B_ρ for a function u such that

$$\begin{cases} \bar\rho \int_{\partial B_{\bar\rho}} |\nabla u|^4 < C, \\ |\{x : |u|(x) = 1\}| > 0. \end{cases}$$

To do this note that, by definition, $u(x) \in \mathbb{S}^3$ for a.e. $x \in \partial B^4$, then use the Sobolev inequality

$$\|u\|^4_{C^{0,1/4}(\partial B_{\bar\rho})} \lesssim \bar\rho \int_{\partial B_{\bar\rho}} |\nabla u|^4,$$

which is valid in dimension 3. For C small enough we obtain (4-10). □

Proof of Lemma 4.6. By the coarea formula we have

$$|\{x : |u(x)-a|^{-1} > \Lambda\}| = |u^{-1}(B_{\Lambda^{-1}}(a))| = \int_{B_{\Lambda^{-1}}(a)} \mathrm{Card}(u^{-1}(x))\,dx \le C \int_{B^4} |\nabla u|^4.$$

We then observe that the measurable positive function $F_u(x) := \mathrm{Card}(u^{-1}(x))$ belongs to $L^1(B^4)$. The maximal function MF_u has $L^{1,\infty}$-norm bounded by the L^1-norm of F_u, and in particular there exists a constant C independent of u such that for at least half of the points $a \in B^4$ we have

$$\sup_\lambda \frac{1}{\lambda^4} \int_{B_\lambda(a)} F_u \le C \int_{B^4} F_u \le C \int_{B^4} |\nabla u|^4.$$

For such a we have, after the change of notation $\lambda = \Lambda^{-1}$, the desired estimate

$$|\{x : |u(x)-a|^{-1} > \Lambda\}|\Lambda^4 \le C \int_{B^4} |\nabla u|^4. \qquad \square$$

4C. ***The case of large energy concentration.*** Following Theorem 4.3, we are led to divide the set of boundary value functions $W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ into two classes, based on whether or not the energy concentrates. Let $L_E := \{\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3) : \|\nabla\phi\|^3_{L^3} \le E\}$ and for $\phi \in L_E$ define $E_\phi := E(|\nabla\phi|^3, \rho_E, \mathbb{S}^3)$. We distinguish between the following two classes of "good" and "bad" boundary value functions:

$$\begin{aligned} \mathcal{G}^E &:= L_E \cap \{\phi : E_\phi \le \bar\delta\}, \\ \mathcal{B}^E &:= L_E \cap \{\phi : E_\phi > \bar\delta\}. \end{aligned} \tag{4-16}$$

We will fix the constants

$$\rho_E = e^{-C\max\{1,E^3\}} \quad \text{and} \quad \bar\delta$$

at the end of Section 4D.

The precise steps of our extension construction are as follows (see also the graphical summary (4-17)):

(1) Theorem 4.3 gives a good estimate for the boundary values in $\mathcal{G}^E$.

(2) If $\phi \in \mathcal{B}^E$ has average close to zero, i.e.,

$$\left|\int_{\mathbb{S}^3} \phi\right| \leq \frac{1}{4},$$

then it is possible to write $\phi = \phi_1\phi_2$ with

$$\int_{\mathbb{S}^3} |\nabla \phi_i|^3 \leq E - \frac{\bar{\delta}}{2}$$

(the product of $\mathbb{S}^3$-valued functions is pointwise the product on $\mathbb{S}^3 \simeq \mathrm{SU}(2)$).

(3) If we are not in the two cases above, we use the functions

$$F_v(x) := -v + (1-|v|^2)(x^* - v)^*,$$

where $a^* = a/|a|^2$, $v \in B^4$, which form a subset of the Möbius group of B^4. We have two cases:

(a) For all $v \in B^4$, we have $\left|\int_{\mathbb{S}^3} \phi \circ F_v\right| > \frac{1}{4}$, in which case

$$\tilde{u}(v) := \pi_{\mathbb{S}^3}\left(\int_{\mathbb{S}^3} \phi \circ F_v\right)$$

gives an extension of ϕ with values in $\mathbb{S}^3$ that satisfies

$$\|u\|_{W^{1,4}} \lesssim \|\phi\|_{W^{1,3}}.$$

(b) There exists $v \in B^4$ such that $\left|\int_{\mathbb{S}^3} \phi \circ F_v\right| \leq \frac{1}{4}$, in which case we can apply the reasoning of cases (1), (2) above to $\tilde{\phi} := \phi \circ F_v$. Since F_v is conformal and $|\phi| = |\tilde{\phi}| = 1$, we have

$$\|\nabla\phi\|_{L^3} = \|\nabla\tilde{\phi}\|_{L^3}, \quad \|\phi\|_{W^{1,3}} = \|\tilde{\phi}\|_{W^{1,3}}.$$

Again we reason differently in the two cases $\tilde{\phi} \in \mathcal{G}^E$ and $\tilde{\phi} \in \mathcal{B}^E$:

(4) If, in case (3b), $\tilde{\phi} \in \mathcal{B}^E$, then we apply case (2) to $\tilde{\phi}$ and we can express

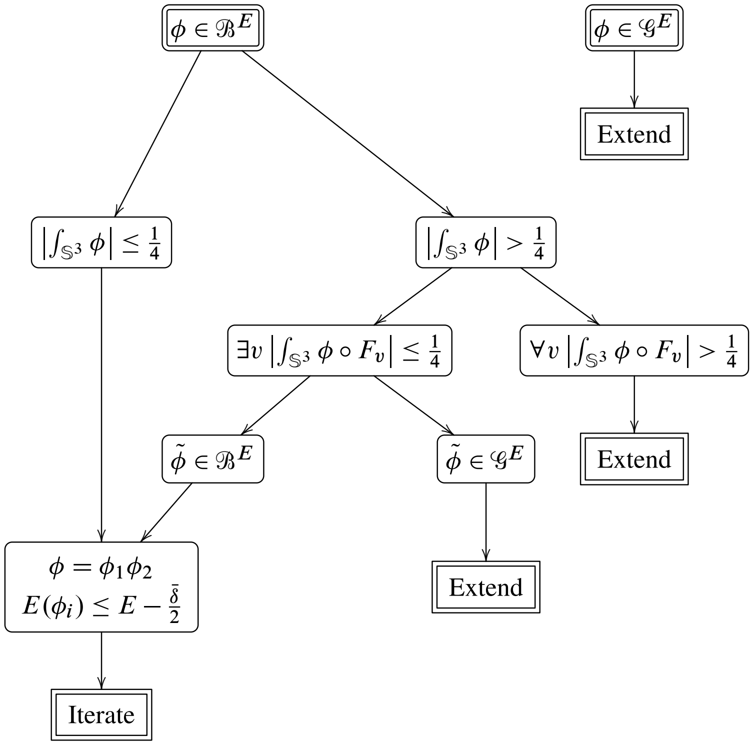
$$\tilde{\phi} = \tilde{\phi}_1\tilde{\phi}_2 \quad \text{and} \quad \phi = (\tilde{\phi}_1 \circ F_v^{-1})(\tilde{\phi}_2 \circ F_v^{-1}).$$

Then $\phi_i := \tilde{\phi}_i \circ F_v^{-1}$ are as in case (2).

(5) If, in case (3b), $\tilde{\phi} \in \mathcal{G}^E$, then we apply case (1) to $\tilde{\phi}$. With a careful study of the relation between the position of $v \in B^4$ relative to ∂B^4 and the parameter ρ_E, we construct

$$u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3) \quad \text{extending} \quad \phi = \tilde{\phi} \circ F_v^{-1},$$

starting from the extension $\tilde{u}$ of $\tilde{\phi}$ given in case (1).



(4-17)

Proposition 4.7 (balancing $\Rightarrow$ splitting). *There exists a constant C with the following property: Suppose that $\phi \in \mathcal{B}^E$ with the notation of (4-16), and assume $\bar{\delta} \le C$ and $\rho_E \le e^{-C\max\{1,E^3\}}$. Further assume that, as a function in $W^{1,3}(\mathbb{S}^3, \mathbb{R}^4)$, ϕ satisfies*

$$\left|\fint_{\mathbb{S}^3} \phi\right| \le \frac{1}{4}.$$

Then, identifying $\mathbb{S}^3 \sim \mathrm{SU}(2)$, there exists a decomposition

$$\phi = \phi_1\phi_2 \tag{4-18}$$

such that, for $i = 1, 2$,

$$\int_{\mathbb{S}^3} |\nabla\phi_i|^3 < E - \frac{\bar{\delta}}{2}. \tag{4-19}$$

Proof. **Step 1.** Fix a concentration ball $B = B^{\mathbb{S}^3}(\rho_E, x_0)$ such that

$$\int_B |\nabla\phi|^3 > \bar{\delta}. \tag{4-20}$$

Step 2. Consider dyadic rings in $\mathbb{S}^3$ defined as $R_i := 2^{i+1}B \setminus 2^i B$, where we denote $2^i B = B^{\mathbb{S}^3}(2^i\rho_E, x_0)$. For an easily computed constant C we can fix $N_E = C|\log_2 \rho_E|$ such that, for $i \le N_E$, it follows that $2^{i+1}\rho_E < \frac{1}{4}$. Since

$$\sum_{i=1}^{N_E} \int_{R_i} |\nabla\phi|^3 < E,$$

by the pigeonhole principle there exists $i_0 \in \{1, \dots, N_E\}$ such that

$$\int_{R_{i_0}} |\nabla\phi|^3 < \frac{E}{N_E}.$$

Again by the pigeonhole principle (using the fact that the cubes are dyadic), there therefore exists $t \in [2^{i_0+1}\rho_E, 2^{i_0}\rho_E]$ such that

$$t\int_{\partial B^{\mathbb{S}^3}(t,x_0)} |\nabla\phi|^3 < C\frac{E}{N_E}, \tag{4-21}$$

where C is a constant depending only on the geometry of $\mathbb{S}^3$.

Step 3. Denote $B_t = B^{\mathbb{S}^3}(t, x_0)$ as in Step 2. We define the function $\tilde{\phi}_1$ via a suitable harmonic extension outside of B_t by

$$\begin{cases} \tilde{\phi}_1 = \phi & \text{on } \partial B_t, \\ \Delta(\tilde{\phi}_1 \circ \Psi) = 0 & \text{on } B_1^{\mathbb{R}^3}, \end{cases}$$

where $\Psi : \mathbb{R}^3 \to \mathbb{S}^3 \setminus \{x_0\}$ is a stereographic projection composed with a dilation of $\mathbb{R}^3$ such that $\Psi(B^{\mathbb{R}^3}(1,0)) = \mathbb{S}^3 \setminus B_t$. On B_t we define $\tilde{\phi}_1 \equiv \phi$. By Hölder's inequality, using elliptic estimates and the conformality of dilations and inverse stereographic projections, we have

$$\begin{aligned} t\int_{\partial B_t} |\nabla\tilde{\phi}_1|^3 \geq C\Big(\int_{\partial B_t} |\nabla\tilde{\phi}_1|^2\Big)^{3/2} = C\Big(\int_{\partial B_1^{\mathbb{R}^3}} |\nabla\tilde{\phi}_1 \circ \Psi|^2\Big)^{3/2} &\geq C\int_{B_1^{\mathbb{R}^3}} |\nabla\tilde{\phi}_1 \circ \Psi|^3 \\ &= C\int_{\mathbb{S}^3\setminus B_t} |\nabla\tilde{\phi}_1|^3. \end{aligned} \tag{4-22}$$

Step 4. We define

$$\phi_1 = \pi_{\mathbb{S}^3} \circ \tilde{\phi}_1.$$

As in Lemma 4.5, there exists a universal constant C such that if

$$\frac{E}{N_E} \leq C \tag{4-23}$$

then

$$\operatorname{dist}(\tilde{\phi}_1, \mathbb{S}^3) \leq \tfrac{1}{2}.$$

This implies, like in Theorem 4.3, that pointwise a.e. we have the estimate

$$|\nabla\phi_1| \leq C|\nabla\tilde{\phi}_1|.$$

By (4-23) it follows that, extending via $\phi_1 = \phi$ on B_t, we obtain $\phi_1 \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ such that

$$\int_{\mathbb{S}^3\setminus B_t} |\nabla\phi_1|^3 \leq C\frac{E}{N_E}. \tag{4-24}$$

Step 5. We now estimate from below the energy of $\phi|_{\mathbb{S}^3\setminus B_t}$. Denote by $\bar{\phi}_\Omega$ the average of ϕ on a domain $\Omega\subset\mathbb{S}^3$. First we use the Poincaré inequality on $\mathbb{S}^3\setminus B_t$ and the fact that $|\phi|\equiv 1$ almost everywhere:

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3\gtrsim\int_{\mathbb{S}^3\setminus B_t}|\phi-\bar{\phi}_{\mathbb{S}^3\setminus B_t}|^3\gtrsim\left(\int_{\mathbb{S}^3\setminus B_t}|\phi-\bar{\phi}_{\mathbb{S}^3\setminus B_t}|\right)^3$$
$$\gtrsim\left(|\mathbb{S}^3\setminus B_t|(1-|\bar{\phi}_{\mathbb{S}^3\setminus B_t}|)\right)^3. \tag{4-25}$$

Using the fact that $|\bar{\phi}_{\mathbb{S}^3}|\le\frac14$, $|\bar{\phi}_{B_t}|\le 1$ and the triangle inequality, we have

$$|\mathbb{S}^3\setminus B_t|\,|\bar{\phi}_{\mathbb{S}^3\setminus B_t}|\le|\bar{\phi}_{\mathbb{S}^3}|\,|\mathbb{S}^3|+|B_t|\,|\bar{\phi}_{B_t}|\le\tfrac14|\mathbb{S}^3|+|B_t|. \tag{4-26}$$

Now, (4-25) and (4-26) and the estimate $t<\frac14$ from Step 2 give

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3\gtrsim\left(\tfrac34|\mathbb{S}^3|-2|B_t|\right)^3\ge C. \tag{4-27}$$

From this inequality, if $\bar{\delta}$ is small enough then we obtain

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3\ge\bar{\delta}. \tag{4-28}$$

Step 6. We now define $\phi_2:=\phi_1^{-1}\phi$, where the pointwise product uses the group operation on $\mathbb{S}^3\sim\mathrm{SU}(2)$. Observe that, since $|\phi|=|\phi_1|=1$ a.e.,

$$|\nabla(\phi_1^{-1}\phi)|=|\phi^{-1}\nabla\phi_1\phi_1^{-1}\phi+\phi_1^{-1}\nabla\phi|\le|\nabla\phi|+|\nabla\phi_1|.$$

Thus, if $C/N_E<1$ in (4-24) (i.e., if $\rho_E=e^{-CN_E}$ is small enough), then

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_2|^3\le\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3+7\left(\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_1|^3\right)^{1/3}\left(\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3\right)^{2/3}.$$

By using (4-28), (4-24) and (4-20) we then obtain

$$\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_2|^3\le\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi|^3+C\frac{E}{N_E^{1/3}}\le E-\bar{\delta}+C\frac{E}{N_E^{1/3}}. \tag{4-29}$$

Step 7. It is now possible to conclude the proof. The estimate (4-19) for ϕ_2 follows from (4-29) if

$$N_E\ge CE^3\bar{\delta}^3. \tag{4-30}$$

Similarly, by construction $\phi_1\equiv\phi$ on B_t, and

$$\int_{\mathbb{S}^3}|\nabla\phi_1|^3=\int_{B_t}|\nabla\phi|^3+\int_{\mathbb{S}^3\setminus B_t}|\nabla\phi_1|^3\le E-\bar{\delta}+C\frac{E}{N_E}.$$

Thus we reach (4-19) if

$$N_E\ge CE\bar{\delta}\,. \tag{4-31}$$

Recall from Step 2 that $N_E=-C\log_2\rho_E$, so (4-30) and (4-31) translate into the requirement that $\rho_E\le e^{-C\max\{E\bar{\delta},(E\bar{\delta})^3\}}$, which is implied by our hypothesis since $\bar{\delta}$ is bounded. □

Remark 4.8. The proof of (4-27) in Step 5 gives the following general estimate, valid for bounded Sobolev functions on a compact manifold M and for any Poincaré domain $\Omega \subset M$:

$$\|\nabla\phi\|_{L^p(\Omega)} \geq C_\Omega\big[|M|(\|\phi\|_{L^\infty(M)} - |\bar{\phi}_M|) - 2\|\phi\|_{L^\infty(M)}|M \setminus \Omega|\big], \tag{4-32}$$

where C_Ω is the Poincaré constant of Ω.

Consider now the conformal transformations of the unit ball B^4

$$F_v(x) = -v + (1-|v|^2)(x^* - v)^*, \quad \text{where } v \in B^4 \text{ and } a^* = \frac{a}{|a|^2}.$$

Proposition 4.9 (balancing $\Rightarrow$ extension). *Let $\phi \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$. Suppose that, for all $v \in B^4$,*

$$\left| \fint_{\mathbb{S}^3} \phi \circ F_v \right| \geq \frac{1}{4}. \tag{4-33}$$

Then the following function $u : B^4 \to \mathbb{S}^3$ extends ϕ:

$$u(v) := \pi_{\mathbb{S}^3}\bigg(\fint_{\mathbb{S}^3} \phi \circ F_v\bigg), \quad \textit{where } \pi_{\mathbb{S}^3}(a) = \frac{a}{|a|} \textit{ for } a \in \mathbb{R}^4 \setminus \{0\}. \tag{4-34}$$

Moreover, there exists a constant C independent of ϕ such that

$$\|\nabla u\|_{L^4(B^4)} \leq C\|\nabla\phi\|_{L^3(\mathbb{S}^3)}. \tag{4-35}$$

Proof. **Step 1.** After a change of variable,

$$\fint_{\mathbb{S}^3} \phi \circ F_v(x)\, dx = \fint_{\mathbb{S}^3} \phi(y)|(F_v^{-1})'|^3(y)\, dy,$$

where $|(F_v^{-1})'|$ is the conformal factor of DF_v^{-1}. From Lemma C.1,

$$|(F_v^{-1})'|(y) = |F'_{-v}|(y) = \frac{1-|v|^2}{|y+v|^2};$$

therefore,

$$\fint_{\mathbb{S}^3} \phi \circ F_v = \fint_{\mathbb{S}^3} \phi(y)\bigg(\frac{1-|v|^2}{|y+v|^2}\bigg)^3 dy.$$

From [Nicolesco 1936], the function

$$K(x,y) = |\mathbb{S}^3|^{-1}\left[\frac{1-|y|^2}{|x-y|^2}\right]^3$$

is the Poisson kernel for the equation

$$\begin{cases} \Delta^2 u = 0 & \text{on } B^4, \\ \dfrac{\partial u}{\partial v}\Big|_{\partial B^4} = 0, \\ u|_{\partial B} = \phi. \end{cases} \tag{4-36}$$

Therefore, the function $\tilde{u}(v) := \fint_{\mathbb{S}^3} \phi \circ F_v$ satisfies (4-36).

Step 2. The following classical estimate holds for (4-36) (see [Gazzola et al. 2010, Chapter 2] for the stronger estimate $\|u\|_{W^{1,4}(\Omega)} \leq \|\phi\|_{W^{1-1/4,4}(\partial\Omega)}$):

$$\|\nabla u\|_{L^4(B^4)} \leq C\|\nabla\phi\|_{L^3(B^3)}.$$

Step 3. We note that

$$\tfrac{1}{4} \leq |\tilde{u}(x)| \leq C \quad \text{for all } v \in B^4$$

because of our hypothesis (4-33), $|\phi| \equiv 1$ and by the elementary estimate $\int_{\mathbb{S}^3}((1-|v|^2)/|y+v|^2)^3\,dy \leq C$. As in Step 2 of the proof of Theorem 4.3 (with notation $\pi_{\mathbb{S}^3} = \pi_a$ for $a = 0$), we obtain the pointwise estimate

$$|\nabla(\pi_{\mathbb{S}^3} \circ \tilde{u})| \sim |\nabla\tilde{u}|.$$

The estimate (4-35) follows via Step 2. □

Consider now the case in which the hypothesis of Proposition 4.9 is false, i.e., that there exists $v \in B^4$ with

$$\left|\fint_{\mathbb{S}^3} \phi \circ F_v\right| \leq \frac{1}{4}. \tag{4-37}$$

$F_v|_{\mathbb{S}^3}$ is conformal and bijective (see Appendix C); thus, for $A \subset \mathbb{S}^3$,

$$\int_A |\nabla\tilde{\phi}|^3 = \int_{F_v^{-1}(A)} |\nabla\phi|^3;$$

in particular, $\tilde{\phi} := \phi \circ F_v$ has energy at most E, like ϕ. We observe that Proposition 4.7 applies to $\tilde{\phi}$ directly due to our hypotheses. Therefore, we can find $\tilde{\phi}_1, \tilde{\phi}_2 \in W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$ such that

$$\tilde{\phi} = \tilde{\phi}_1\tilde{\phi}_2, \qquad \int_{\mathbb{S}^3} |\nabla\tilde{\phi}_i|^3 \leq E - \frac{\bar{\delta}}{2} \quad \text{for } i = 1, 2.$$

We then precompose with F_v^{-1}, which preserves the pointwise product and the L^3-energy of the gradients, obtaining the same decomposition for ϕ.

The case $\tilde{\phi} \in \mathscr{G}^E$ is a bit more difficult:

Proposition 4.10. *Under the assumption (4-37) and with $\tilde{\phi} := \phi \circ F_v$, suppose that $\tilde{\phi} \in \mathscr{G}^E$. Then there exists an extension $u \in W^{1,(4,\infty)}(B^4, \mathbb{S}^3)$ of ϕ such that*

$$\|\nabla u\|_{L^{4,\infty}(B^4)} \leq \frac{C}{\rho_E}\|\nabla\phi\|^2_{L^3(\mathbb{S}^3)} + \|\nabla\phi\|_{L^3(\mathbb{S}^3)} \tag{4-38}$$

under the assumption that

$$\rho_E \leq \tfrac{1}{4}. \tag{4-39}$$

Proof. To simplify notations, let $\rho = \rho_E$ during this proof. We divide the domain B^4 into

$$A := F_v^{-1}(B(0, 1-\rho)), \quad A' := B^4 \setminus A.$$

By Lemma C.2, there exists a constant C dependent only on the dimension and a function $h(v)$ such that, for $x \in A$ and under the condition (4-39),

$$\frac{h(v)}{C} \leq |F_v'|(x) \leq Ch(v). \tag{4-40}$$

Therefore, we have

$$\begin{aligned}
|\{x \in A : |\nabla u|(x) > \Lambda\}| &= \left|\left\{x \in A : |\nabla \tilde{u}|(F_v(x))\,|F_v'|(x) > \Lambda\right\}\right| \\
&\leq \left|\left\{x \in A : |\nabla \tilde{u}|(F_v(x)) > \frac{\Lambda}{Ch(v)}\right\}\right| \\
&= \int_{F_v(A)\cap\{y:|\nabla\tilde{u}|(y)>\Lambda/(Ch(v))\}} |F_v'|^{-4}\,dy \\
&\leq C^4 h^{-4}(v)\left|\left\{y \in B_{1-\rho} : |\nabla \tilde{u}| > \frac{\Lambda}{Ch(v)}\right\}\right| \\
&\leq C^8 \Lambda^{-4} \|\nabla \tilde{u}\|^4_{L^{4,\infty}(B_{1-\rho})}.
\end{aligned}$$

By bringing Λ to the other side, it follows that

$$\Lambda^4 |\{x \in A : |\nabla u|(x) > \Lambda\}| \leq C^8 \|\nabla \tilde{u}\|_{L^{4,\infty}(B(0,1-\rho))}. \tag{4-41}$$

By conformal invariance, the invertibility of F_v and the usual estimate between $L^{4,\infty}$ and L^4, we have

$$\Lambda^4 |\{x \in A' :\, |\nabla u|(x) > \Lambda\}| \leq C\|\nabla u\|^4_{L^4(A')} = C\|\nabla \tilde{u}\|_{L^4(B\setminus B_{1-\rho})}). \tag{4-42}$$

We now sum (4-41) and (4-42) and we take the supremum on $\Lambda > 0$. It follows that, up to increasing C,

$$[\nabla u]_{L^{4,\infty}(B^4)} \leq C(\|\nabla \tilde{u}\|_{L^{4,\infty}(B_{1-\rho})} + \|\nabla \tilde{u}\|_{L^4(B\setminus B_{1-\rho})}). \tag{4-43}$$

The estimate (4-43) together with Theorem 4.3 applied to $\tilde{u}$ gives the desired estimate for the first summand, while for the second summand we proceed as in Step 3 of the proof of Theorem 4.3. On the small concentration regions B_i for $\tilde{\phi}$ we apply Courant's Lemma 4.5, due to which we may project the values of $u := \tilde{u} \circ F_v^{-1}$ on $\mathbb{S}^3$ with little change of the gradient of u. Since F_v^{-1} is conformal, the L^3-energy of $\tilde{u}$ on ∂B_i is the same as the L^3-energy of u on $\partial F_v^{-1}(B_i)$. By Theorem 4.4 applied to $\tilde{u}$ as in Step 3 of the proof of Theorem 4.3, we obtain

$$\|\nabla u\|_{L^4(F_v^{-1}(B\setminus B_{1-\rho}))} = \|\nabla \tilde{u}\|_{L^4(B\setminus B_{1-\rho})} \leq C\|\nabla \tilde{\phi}\|_{L^3(\mathbb{S}^3)} = C\|\nabla \phi\|_{L^3(\mathbb{S}^3)}.$$

This and (4-43) conclude the proof. □

4D. ***End of the proof of Theorem B″.*** We refer to the scheme (4-17) for the idea of the proof.

Choice of $\bar{\delta}$. In (4-16), take $\bar{\delta} \leq \delta/C_1$ with the notations of Theorem 4.3 and with δ is as in Theorem 4.4. If necessary, shrink $\bar{\delta}$ so that the bound of Proposition 4.7 is also satisfied.

Choice of ρ_E. From Proposition 4.7 with the above choices of $\bar{\delta}$, we get $\rho_E \lesssim e^{-C\max(1,E^3)}$.

Estimates for extensions. Consider again the scheme (4-17). Each time we extend some boundary datum ϕ obtained during our constructions via a function $u : B^4 \to \mathbb{S}^3$, we do so with one of the following estimates:

- In the case of the extensions of Theorem 4.3 or of Proposition 4.10 (which in turn actually depends on Theorem 4.3) we have

$$\|\nabla u\|_{L^{4,\infty}} \lesssim \frac{\|\nabla\phi\|_{L^3}^2}{\rho_E} + \|\nabla\phi\|_{L^3}.$$

- In the case of the biharmonic extension of Proposition 4.9, we have the much better

$$\|\nabla u\|_{L^4} \lesssim \|\nabla\phi\|_{L^3}.$$

The number of iterations to be made when we apply the procedure described in scheme (4-17) is bounded by

$$E \big/ \tfrac{1}{2}\bar{\delta} \sim E.$$

Since each iteration creates two new boundary value functions out of one, in the end we may have a decomposition into no more than

$$e^{CE} \text{ boundary value functions.}$$

By the triangle inequality we see that, in this case, there exists an extension of the initial ϕ satisfying

$$\|\nabla u\|_{L^{4,\infty}} \lesssim e^{C\|\nabla\phi\|_{L^3}^9}\|\nabla\phi\|_{L^3}^2 + \|\nabla\phi\|_{L^3}. \tag{4-44}$$

This gives the estimate (4-1) of Theorem B$''$, finishing the proof. □

5. Controlled global gauges

In this section we prove Theorem A.

5A. ***Scheme of the proof.*** We indicate here the sketch of the proof, before going through the details.

Proof. We will denote the L^2-norm of F by E. We may assume that a first guess for A (i.e., a fixed trivialization) is already given and belongs to $W^{1,2}$ (if the bound by ϵ_0 on the energy of F is available, we may assume more by Uhlenbeck's result stated above, namely that one controls the $W^{1,2}$-norm of A by the energy).

It can be seen from the formula of change of gauge that it is equivalent to estimate either the gradient of the trivialization g or the gradient of the connection A in that gauge.

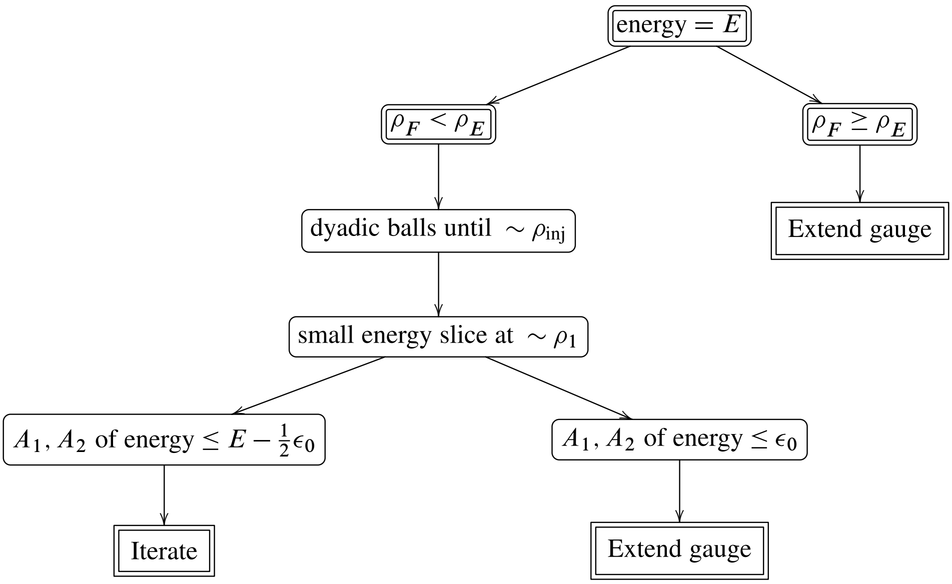
We define f by iteration on the energy bound E. The main steps are as follows (see the scheme (5-1)):

- Uhlenbeck's theorem [1982b] already gives a gauge with an L^4-estimate of the gradient of the trivialization if the energy of F is smaller than ϵ_0.
- Let ρ_F be the largest scale at which no more than $\frac{1}{2}\epsilon_0$ of the L^2-norm of F concentrates.

- In the case $\rho_F \geq \rho_E := C\rho_{\text{inj}}(M)2^{-E/\epsilon_1}$, we iteratively extend our gauge on the small simplices of a triangulation using Theorem B″; see Section 5B. The estimates depend only on M^4.
- The alternative is $\rho_F \leq \rho_E$. Then, consider a point x_0 at which $|F|$ concentrates and look at the geodesic dyadic rings

$$R_k := B(x_0, 2^{k+1}\rho_F) \setminus B(x_0, 2^k\rho_F), \quad k \in \{0, \dots, \lfloor \log_2(C\rho_{\text{inj}}/\rho_F)\rfloor\}.$$

By the pigeonhole principle and by the choice of ϵ_1, we ensure the existence of a small energy slice along a geodesic sphere of radius $t \sim 2^{k_0}\rho_F$. We have extensions of the connections with curvatures of energy smaller than $E - \frac{1}{2}\epsilon_0$. We use Lemma 5.4. To avoid subtleties about traces, we will ensure that these two connections coincide on an open set. The choice of slice is described in Section 5D.
- Then we separately trivialize these two connections. By iterative assumption we then define $f(E)$ based on $f\big(E - \frac{1}{2}\epsilon_0\big)$ and on the function f_1 of Theorem B. The detailed bounds are given in Section 5E.



(5-1)

5B. ***Iterations based on a suitable triangulation.*** Define, for ϵ_0 as in Theorem 5.1, the radius

$$\rho_F := \inf\left\{\rho > 0 : \int_{B_\rho(x_0)} |F|^2 = \tfrac{1}{2}\epsilon_0 \text{ for some } x_0 \in M\right\}, \tag{5-2}$$

where $\rho_E := C\rho_{\text{inj}}(M)2^{-E/\epsilon_1}$ and $\rho_{\text{inj}}(M)$ is the injectivity radius of M. The constant ϵ_1 will be fixed later. Fix a triangulation on M with in-radius $\gtrsim \rho_E$ and size $\lesssim \rho_E$, with implicit constants bounded by 4. We choose $C < 1$ in the definition of ρ_E so that each simplex of the triangulation is contained in a ball of radius $\frac{1}{2}\rho_{\text{inj}}(M)$. In particular, all k-simplices of the triangulation are bi-Lipschitz equivalent to $\mathbb{S}^k$ for $k = 1, \dots, 4$.

We recall here the main result of [Uhlenbeck 1982b]:

Theorem 5.1 (Uhlenbeck gauge). *There exists* $\epsilon_0 > 0$ *such that, if the curvature satisfies* $\int_{B_1} |F|^2 \le \epsilon_0$, *then there is a gauge* $\phi \in W^{2,2}(B_1, \mathrm{SU}(2))$ *in which the connection satisfies* $\|A_\phi\|_{W^{1,2}(B_1)} \le C\|F\|_{L^2(B_1)}$ *with* $C > 0$ *depending only on the dimension.*

Theorem 5.1 gives a trivialization ϕ_i associated to each 4-simplex C_i such that the expression of A in those coordinates,

$$A_i = \phi_i^{-1} d\phi_i + \phi_i^{-1} A \phi_i \quad \text{on } C_i, \tag{5-3}$$

satisfies

$$\|A_i\|_{W^{1,2}(C_i)} \le C\|F\|_{L^2(C_i)}. \tag{5-4}$$

If we call

$$g_{ij} := \phi_j^{-1}\phi_i, \tag{5-5}$$

then $g_{ij}g_{jk} = g_{ik}$, so in particular $g_{ij}^{-1} = g_{ji}$; moreover,

$$A_j = g_{ij}dg_{ji} + g_{ij}A_i g_{ji} \quad \text{on } \partial C_i \cap \partial C_j. \tag{5-6}$$

It follows that $g_{ij} \in W^{1,3}(\partial C_i \cap \partial C_j, \mathrm{SU}(2))$.

Lemma 5.2 (extension on a sphere). *Let* S^3_+ *be the upper hemisphere,* $\mathbb{S}^3 \cap \{x_3 \ge 0\}$. *For any* $g \in W^{1,3}(S^3_+, \mathrm{SU}(2))$, *there exists* $\tilde{g} \in W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$ *such that* $\tilde{g} = g$ *on* $\mathbb{S}^3_+$ *and*

$$\|\nabla\tilde{g}\|_{L^3(\mathbb{S}^3)} \le C\|\nabla g\|_{L^3(S^3_+)}.$$

Proof. Let $\mathbb{S}^3_-$ be a spherical cap of height $t \in \left[\frac{1}{2}, \frac{3}{2}\right]$ such that

$$\left\|g|_{\partial\mathbb{S}^3_-}\right\|_{W^{1,2}(\partial\mathbb{S}^3_-)} \le C\|g\|_{W^{1,3}(\mathbb{S}^3_+)}. \tag{5-7}$$

We observe that $g|_{\partial S^3_+ \simeq \mathbb{S}^2} \in W^{1,2}(\mathbb{S}^2, \mathrm{SU}(2))$, and we desire to extend this trace inside $B^3 \simeq \mathbb{S}^3_-$ with a good norm estimate. Let

$$\begin{cases} \Delta\hat{g} = 0 & \text{on } B^3, \\ \hat{g} = g & \text{on } \partial B^3. \end{cases}$$

Then we have, by the usual elliptic estimates,

$$\|\hat{g}\|_{W^{1,3}(\mathbb{S}^3_-)} \le C\left\|g|_{\partial\mathbb{S}^3_-}\right\|_{W^{1,2}(\partial\mathbb{S}^3_-)}. \tag{5-8}$$

For $a \in B^4_{1/2}$, if g_a is the radial projection of the values of $\hat{g}$ on the boundary with center a, then (as in the projection trick of Section 2A)

$$|\nabla g_a| \le C\frac{|\nabla\hat{g}|}{|\hat{g}-a|} \quad \text{and} \quad \int_{a\in B^4_{1/2}}\int_{B^3}|\nabla g_a|^3 \le C\int_{B^3}|\nabla\hat{g}|^3. \tag{5-9}$$

Therefore, there exists $a \in B^4_{1/2}$ such that

$$\|\nabla g_a\|_{L^3(B^3\simeq\mathbb{S}^3_-)} \le C\|\nabla\hat{g}\|_{L^3(B^3\simeq\mathbb{S}^3_-)}. \tag{5-10}$$

Combining the inequalities (5-7), (5-8), (5-9) and (5-10), we obtain the thesis for $\tilde{g} = g_a$ with a as above. □

Corollary 5.3 (iteration step). *Suppose that on our* 4*-manifold* M *a connection* A *is fixed and an Uhlenbeck gauge* ϕ_j *is defined on a* 4*-simplex* C_j*, i.e.,* (5-4) *holds with notation* (5-3)*. Suppose that a global gauge* ϕ_I *is defined on a finite union of simplices* $C_I := \bigcup_{\alpha\in I} C_{i_\alpha}$ *and that* $\partial C_j \cap C_I^{(3)}$ *(where* $C_I^{(3)}$ *is the simplicial* 3*-skeleton of* C_I*) contains some, but not all,* 3*-faces of* C_j*. It is then possible to extend the gauge change* g_{ij} *of* (5-5) *to* $\tilde g_{ij}$ *defined on the whole of* ∂C_j *with*

$$\|\nabla \tilde g_{ij}\|_{L^3(\partial C_j)} \le C\|\nabla g_{ij}\|_{L^3(\partial C_j\cap C_I^{(3)})},$$

where C *depends only on* M*.*

Proof. $H := (\partial C_j \setminus C_I^{(3)})_\delta$ is bi-Lipschitz to a ball for δ equal to two-thirds of the smallest in-radius of a face of C_j. Here, A_δ is a δ-neighborhood of A inside ∂C_j. Let $H' := (\partial C_j \setminus C_I^{(3)})_{2\delta}$. The triple $(\partial C_j, H, H')$ is C-bi-Lipschitz to $(\mathbb{S}^3, \mathbb{S}^3_-, K)$ where K is the spherical cap of height $\frac{3}{4}$ extending $\mathbb{S}^3_-$. We apply Lemma 5.2 in order to "fill the hole" H extending the gauge g_{ij} with estimates. The bi-Lipschitz constant is bounded by the geometric constraints on our triangulation only. □

Given Lemma 5.2 and Corollary 5.3, we proceed iteratively on the triangulation as follows (the indices labeling the simplices are redefined during the whole procedure in a straightforward way):

- Suppose that we already defined the gauge $\tilde\phi_{j-1}$ on a set of $j-1$ simplices $C_1,\dots,C_{j-1}$ whose union forms a connected set.
- Consider a new simplex C_j extending this connected set. Use Corollary 5.3 to extend g_{ij} to $\tilde g_{ij}$.
- We apply Theorem B″ and extend $\tilde g_{ij}$ to a gauge change h_{ij} defined inside C_j so that

$$\|\nabla h_{ij}\|_{L^{(4,\infty)}(C_j)} \le f(\|\nabla \tilde g_{ij}\|_{L^3(C_j)}) \le C_0, \tag{5-11}$$

  with C_0 depending only on universal constants and on ϵ_0.
- On $\bigcup_{i<j} C_i$ let $\tilde\phi_j = \tilde\phi_{j-1}$, while on C_j we define $\tilde\phi_j = \phi_j h_{ij}$.

Let $\widetilde A_j$ be the local expression corresponding to the gauge $\tilde\phi_j$. Then

$$\|\widetilde A_j\|_{L^{(4,\infty)}(C_j)} \lesssim \|A_j\|_{L^4(C_j)} + \|\nabla h_{ij}\|_{L^{(4,\infty)}(C_j)} \le \epsilon_0 + C_0.$$

Iterating this gauge extension strategy, we obtain a global gauge $\widetilde A$ on the whole of M such that

$$\|\widetilde A\|_{L^{(4,\infty)}(M)} \le C(\text{number of simplices})(C_0+\epsilon_0) \le C\frac{\mathrm{Vol}(M)}{\rho_E^4}. \tag{5-12}$$

The above bound depends on the geometry of M and on the energy E of the curvature only. Note that the above reasoning works only as long as $\rho_F \lesssim \rho_E$. We next consider the case $\rho_F \ge \rho_E$.

5C. ***Extending the connection with small curvature changes.*** Let ϵ_0 be as in Theorem 5.1.

Lemma 5.4 (finding good slices). *There exists a constant* ϵ_1 *with the following properties: If* M *is a fixed* 4*-manifold with a* $W^{1,2}$*-connection* A *and if* $B_{2t}(x_0) \subset M$ *is a geodesic ball such that*

$$t\int_{\partial \widetilde B_t} |F|^2 \le \epsilon_1,$$

then there exists $\hat{A} \in W^{1,2}\big(\bigwedge^1 M, \mathrm{su}(2)\big)$ *such that* $\hat{A} = A$ *on* B_t *and*

$$\int_{M\setminus B_t} |F_{\hat{A}}|^2 \le \frac{\epsilon_0}{4}.$$

Proof. Up to a change of gauge, which does not increase the norm, we may assume the Neumann condition

$$\langle A, \nu\rangle \equiv 0 \quad \text{on } \partial B_t. \tag{5-13}$$

This is obtained, for example, by minimizing $\|g^{-1}dg + g^{-1}Ag\|_{L^2(B_t)}$ among $g \in W^{2,2}(B_t, \mathrm{SU}(2))$.

Extend A to $B_{2t} \setminus B_t$ by $\widetilde{A} := \pi^* i^*{}_{\partial B_t} A$, where $\pi(x) = tx/|x|$ and $i_{\partial B_t}$ is the inclusion. Using the hypothesis, we obtain

$$\int_{B_{2t}\setminus B_t} \big|d\widetilde{A} + \tfrac{1}{2}[\widetilde{A}, \widetilde{A}]\big|^2 \le C\epsilon_1.$$

We apply a change of gauge $g = g(\sigma)$ depending only on the angular variable $\sigma \in \partial B^4$ and such that

$$d^*{}_{\partial B_t} A_g\big|_{\partial B_t} = 0.$$

This preserves (5-13) and gives, as $s \to 0$,

$$C\epsilon_1 \ge \int_{B_s\cap\partial B_t} \big|dA_g + \tfrac{1}{2}[A_g, A_g]\big|^2 \ge \int_{B_s\cap\partial B_t} |dA|^2 - o(s)\int_{B_s\cap\partial B_t} |\nabla A|^2.$$

Therefore, $A_g \in W^{1,2}(\wedge^1 \partial B_t, \mathrm{su}(2))$, $\widetilde{A}_g \in W^{1,2}(\wedge^1 B_{2t} \setminus B_t, \mathrm{su}(2))$, and A_g, $\widetilde{A}_g$ satisfy (5-13). Therefore, $\widetilde{A}_g$ extends by A_g in a neighborhood of ∂B_t, giving still a $W^{1,2}$ gauge. Observe that, by Sobolev embedding,

$$\int_{\partial B_t} |[A, A]|^2 \lesssim \bigg(\int_{\partial B_t} |\nabla A|^2\bigg)^2$$

and, by Hodge decomposition and using $d^*{}_{\partial B_t} A = 0$,

$$\int_{\partial B_t} |\nabla A|^2 \lesssim \int_{\partial B_t} (|dA|^2 + |d^*A|^2) \lesssim \int_{\partial B_t} |F_A|^2 + \bigg(\int_{\partial B_t} |\nabla A|^2\bigg)^2.$$

For $X = \|\nabla A\|^2_{L^2(\partial B_t)}$ we get $X \le \epsilon_1 + X^2$, and thus we may assume that

$$t\int_{\partial B_t} |\nabla A|^2 \le Ct\int_{\partial B_t} |F|^2.$$

Define $\hat{A} := \chi_t A$ for a smooth $[0, 1]$-valued cutoff function χ_t such that $\chi_t \equiv 1$ on B_t and $\chi_t \equiv 0$ outside B_{2t}. We obtain

$$\int_{B_{2t}} |F_{\hat{A}}|^2 \le \int_{B_t} |F_A|^2 + C\epsilon_1$$

and we can extend $\hat{A} \equiv 0$ outside B_{2t}, obtaining the desired estimate for ϵ_1 small enough. □

5D. ***Cutting M by a small energy slice.*** Suppose for this subsection that $\rho_F < \rho_E$. Let C be as in the definition of ρ_E and define

$$\rho_1 := \begin{cases} \inf\{\rho \geq \rho_F : \int_{B_{2\rho}\setminus B_\rho} |F|^2 \leq \frac{1}{4}\epsilon_1\} & \text{if this is less than } C\rho_{\text{inj}}(M), \\ C\rho_{\text{inj}} & \text{otherwise.} \end{cases}$$

Since $\rho_F < \rho_E$ and by the choice of ϵ_1, ρ_1 is rather small and $B_{2\rho_1}$ is bi-Lipschitz to B_1. Thus Lemma 5.4 applies. More precisely, let $t_1 \in \left[\rho_1, \frac{5}{4}\rho_1\right]$, $t_2 \in \left[\frac{7}{4}\rho_1, 2\rho_1\right]$. There exist t_i, $i = 1, 2$, such that

$$t_i \int_{\partial B_{t_i}} |F|^2 \leq \epsilon_1.$$

5E. ***Strategies after cutting.*** Let ϵ_0 be as in Theorem 5.1. We pursue different strategies depending on the energy of F outside $B_{2\rho_1}$.

The case $\int_{M\setminus B_{2\rho_1}} |F|^2 \geq \frac{1}{2}\epsilon_0$. Split to the regions B_{t_2} and $M \setminus B_{t_1}$ and do induction on the energy in order to separately find gauges satisfying our estimates. Lemma 5.4 gives extensions

$$\begin{cases} \hat{A}_1 \equiv A \text{ on } B_{t_2} & \text{s.t. } \int_M |F_{\hat{A}_1}|^2 \leq \int_{B_{t_2}} |F_A|^2 + C\epsilon_1, \\ \hat{A}_2 \equiv A \text{ on } M \setminus B_{t_1} & \text{s.t. } \int_M |F_{\hat{A}_2}|^2 \leq \int_{B_{t_1}} |F_A|^2 + C\epsilon_1. \end{cases} \tag{5-14}$$

In particular, $\hat{A}_1, \hat{A}_2$ are equivalent on $B_{\frac{7}{4}\rho_1} \setminus B_{\frac{5}{4}\rho_1}$ and

$$\int |F_{\hat{A}_i}|^2 \leq \int |F_A|^2 - \tfrac{1}{4}\epsilon_0.$$

If we can find global gauges g_i^∞, $i = 1, 2$, in which $\hat{A}_i$ have expressions $\hat{A}_i^\infty$ with $L^{(4,\infty)}$ bounds as in Theorem B, then it is enough to apply

$$g_{12}^\infty := (g_1^\infty)^{-1} g_2^\infty$$

on $R := B_{\frac{7}{4}\rho_1} \setminus B_{\frac{5}{4}\rho_1}$ in order to obtain

$$A_2^\infty = g_{12}^\infty A_1^\infty (g_{12}^\infty)^{-1} + g_{12}^\infty d(g_{12}^\infty)^{-1} \quad \text{and} \quad \|\nabla g_{12}^\infty\|_{L^{(4,\infty)}(R)} \leq f\big(E - \tfrac{1}{4}\epsilon_0\big).$$

Then there exists $t_3 \in \left[\frac{5}{4}\rho_1, \frac{7}{4}\rho_1\right]$ such that

$$\int_{\partial B_{t_3}} |\nabla g_{12}^\infty|^3 \leq f\big(E - \tfrac{1}{4}\epsilon_0\big).$$

By Theorem B we can find a $W^{1,(4,\infty)}$-extension h_{12}^∞ of g_{12}^∞ to a map from B_{t_3} to SU(2). Thus, if we call f_1 the function of Theorem B, then

$$\|\nabla h_{12}^\infty\|_{L^{(4,\infty)}(B_{t_3})} \leq f_1\big(f\big(E - \tfrac{1}{4}\epsilon_0\big)\big).$$

If we define

$$g^\infty := \begin{cases} g_2^\infty & \text{on } M^4 \setminus B_{t_3}, \\ h_{12}^\infty g_1^\infty & \text{on } B_{t_3}, \end{cases} \tag{5-15}$$

then ∇g^∞ is estimated by an universal constant times

$$f_1\big(f\big(E-\tfrac14\epsilon_0\big)\big)+f\big(E-\tfrac14\epsilon_0\big).$$

The case $\int_{M\setminus B_{2\rho_1}}|F|^2\le\frac12\epsilon_0$. Outside B_{ρ_1} we apply directly Theorem 5.1, while on $B_{2\rho_1}$ we extend the so-obtained gauge via Theorem B″. If we call A_1, A_2 the so-obtained connections on $B_{2\rho_1}$, $M\setminus B_{\rho_1}$ respectively, then there exists $t\in[\rho_1,2\rho_1]$ such that

$$\int_{\partial B_t}(|A_1|^3+|A_2|^3)\le C(f_1(\epsilon_0)+\epsilon_0).$$

As above, the same bound is true also for the gradient of the change of gauge ∇g_{12}. Theorem B gives the extension h_{12} to a gauge in $W^{1,(4,\infty)}(B_t,\mathrm{SU}(2))$ with

$$\|\nabla h_{12}\|_{L^{4,\infty}(B_{t_3})}\le f_1(C(f_1(\epsilon_0)+\epsilon_0)).$$

Then choose

$$g^\infty:=\begin{cases}g_2 & \text{on } M^4\setminus B_{t_3}\,,\\ h_{12}g_1 & \text{on } B_{t_3}\,.\end{cases}\tag{5-16}$$

This g^∞ satisfies an estimate independent on E and dependent only on ϵ_0, again allowing us to define $f(E)$ inductively. □

Appendix A: Uhlenbeck small energy extension

We use the strategy from [Uhlenbeck 1982a] to prove Theorem 4.4. The analogy is in the method of proof more than in the result.

First recall that $W^{1,2}(X,\mathbb{S}^3)=W^{1,2}(X,\mathbb{R}^4)\cap\{u:u(x)\in\mathbb{S}^3\text{ a.e.}\}$ and observe that we attain the infimum

$$\inf\left\{\int_{B^4}|\nabla P|^2 : P\in W^{1,2}(B^4,\mathbb{S}^3),\ P=P_0\ \text{on}\ \partial B^4\right\}.\tag{A-1}$$

Indeed, a minimizing sequence will have a $W^{1,2}$-weakly convergent subsequence, which thus converges pointwise everywhere. By weak lower semicontinuity a minimizer exists, and by convexity it is unique. The minimizer P distributionally verifies

$$\operatorname{div}(P^{-1}\nabla P)=0.\tag{A-2}$$

Lemma A.1 (a priori estimates). *There exists $\epsilon>0$ with the following property: Let P be an extension of $P_0\in W^{1,3}(\mathbb{S}^3,\mathbb{S}^3)$ with $\|P-I\|_{W^{1,4}(B^4)}\le\epsilon$ that satisfies (A-2). We identify $\mathbb{S}^3$ with the Lie group* SU(2). *Then there exists a constant C_ϵ such that*

$$\|P-I\|_{W^{4/3,3}(B^4)}\le C_\epsilon\|\nabla P_0\|_{L^3(\mathbb{S}^3,\mathbb{S}^3)}.\tag{A-3}$$

Proof. By L^2-Hodge decomposition,

$$P^{-1}dP=dU+d^*V,\tag{A-4}$$

where V is the unique minimizer of

$$\min\left\{\int_{B^4}|d^*V-P^{-1}dP|^2,\ *V|_{\partial B^4}=0,\ dV=0\right\};$$

thus,

$$\begin{cases}\Delta V=dd^*V=dP^{-1}\wedge dP,\\ dV=0,\\ *V=0.\end{cases}$$

We claim that

$$\|\nabla V\|_{L^3(\partial B^4)}\lesssim \epsilon\|P-I\|_{W^{1,4}(B^4)}. \tag{A-5}$$

To see this, observe that $d(P^{-1})=P^{-1}\,dP\,P^{-1}$ and $P,\ P^{-1}\in L^\infty$ with norm equal to 1 so, by the elliptic, Hölder and Poincaré estimates,

$$\begin{aligned}\|\nabla V\|_{W^{1,2}(B^4)}\lesssim\|dP^{-1}\wedge dP\|_{L^2(B^4)}&\lesssim\|d(P^{-1})\|_{L^4(B^4)}\|dP\|_{L^4(B^4)}\\&\lesssim\|dP\|_{L^4(B^4)}\|P^{-1}\|^8_{L^\infty}\|\nabla P\|_{L^4(B^4)}\\&\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}.\end{aligned} \tag{A-6}$$

The trace and Sobolev embedding inequalities

$$\|V\|_{L^p(\partial B^4)}\lesssim\|V\|_{W^{1-1/q,q}(\partial B^4)}\lesssim\|V\|_{W^{1,q}(B^4)}$$

are valid for $q=2,\ p=3$. Therefore, we obtain (A-5).

Using the trace of the Hodge decomposition formula (A-4) on the boundary, we obtain from (A-5) that

$$\|dU-P_0^{-1}dP_0\|_{L^3(\partial B^4)}\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}. \tag{A-7}$$

Like for V, for U we have

$$\Delta U=d^*dU=d^*(P^{-1}dP)=0.$$

We apply the elliptic estimates for U to obtain

$$\|dU\|_{W^{1/3,3}(B^4)}\lesssim\|\nabla U\|_{L^3(\partial B^4)}, \tag{A-8}$$

while (A-7), the triangle inequality and the fact that $\|P_0\|_{L^\infty}=1$ give

$$\begin{aligned}\|U\|_{L^3(\partial B^4)}&\lesssim\|dU-P_0^{-1}dP_0\|_{L^3(\partial B^4)}+\|P_0^{-1}dP_0\|_{L^3(\partial B^4)}\\&\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}+\|dP_0\|_{L^3(\partial B^4)}.\end{aligned} \tag{A-9}$$

Using (A-4), the triangle inequality and (A-6), (A-8), (A-9) we obtain

$$\begin{aligned}\|P^{-1}dP\|_{W^{1/3,3}(B^4)}&\lesssim\|d^*V\|_{W^{1/3,3}(B^4)}+\|dU\|_{W^{1/3,3}(B^4)}\\&\lesssim\epsilon\|P-I\|_{W^{1,4}(B^4)}+\|dP_0\|_{L^3(\partial B^4)}.\end{aligned} \tag{A-10}$$

Write $dP=PP^{-1}dP$ and observe that $P\in L^\infty\cap W^{1,4}$ since $\mathbb{S}^3$ is bounded, while $P^{-1}dP\in W^{1/3,3}$ by (A-10). We now use Lemma B.1 for the product fg with $f=P$, $g=P^{-1}dP$ and we obtain

$$\|dP\|_{W^{1/3,3}(B^4)}\lesssim\|P^{-1}dP|\|_{W^{1/3,3}}(\|P\|_{L^\infty}+\|P-I\|_{W^{1,4}(B^4)}). \tag{A-11}$$

Note again that $\|P\|_{L^\infty} = 1$ and deduce then from (A-10), Lemma B.1 and the Poincaré inequality that

$$\|P - I\|_{W^{4/3,3}(B^4)} \leq C\|dP_0\|_{L^3(\mathbb{S}^3)} + C\epsilon\|P - I\|_{W^{1,4}(B^4)}. \tag{A-12}$$

By the Sobolev inequality we can absorb the $\|P - I\|$ term to the left, and we obtain the thesis. □

We are now ready for the proof of Theorem 4.4. We restate the same result with a slight change of notation and more details.

Theorem A.2 (small energy extension). *There exist two constants $\delta > 0$, $C > 0$ with the following property: Suppose $Q \in W^{1,3}(\mathbb{S}^3, \mathbb{S}^3)$ is such that $\|dQ\|_{L^3(\mathbb{S}^3)} \leq \delta$. Then there exists an extension $P \in W^{1,4}(B^4, \mathbb{S}^3)$ satisfying*

$$\|P - I\|_{W^{1,4}(B^4)} \leq C\|dQ\|_{L^3(\mathbb{S}^3)}. \tag{A-13}$$

Proof. Define the following two sets, with $K > 0$ fixed later:

$$\mathscr{G}^\alpha_\epsilon = \{Q \in W^{1,3+\alpha}(\mathbb{S}^3, \mathrm{SU}(2)) : \|\nabla Q\|_{L^3} \leq \epsilon\},$$

$$\mathscr{F}^\alpha_{\epsilon,C} = \big\{Q \in \mathscr{G}^\alpha_\epsilon : \exists P \in W^{1,4+\alpha}(B^4, \mathrm{SU}(2)), \operatorname{div}(P^{-1}\nabla P) = 0 \text{ on } B^4, P = Q \text{ on } \partial B^4,$$
$$\|P - I\|_{W^{1,4}(B^4)} \leq K\|\nabla Q\|_{L^3(\partial B^4)} \|P - I\|_{W^{1,4+\alpha}(B^4)} \leq C\|\nabla Q\|_{L^{3+\alpha}(\partial B^4)}\big\}. \tag{A-14}$$

The claim of our theorem states that a $P \in \mathscr{F}^0_{\epsilon,C}$ can be constructed to extend any $Q \in \mathscr{G}^0_\delta$ when δ is small enough. The strategy of the proof is to use the supercritical spaces $\mathscr{G}^\alpha_\epsilon$, $\alpha > 0$ to approximate $\mathscr{G}^0_\delta$. We divide the proof into five steps, paralleling Uhlenbeck [1982a].

Claim 1. *$\mathscr{G}^\alpha_\epsilon$ is connected for all $\epsilon, \alpha \geq 0$.*

Claim 2. *$\mathscr{F}^\alpha_{\epsilon,C}$ is closed (in $\mathscr{G}^\alpha_\epsilon$) with respect to the $W^{1,3+\alpha}$-norm for $\alpha \geq 0$ and for any $C > 0$.*

Claim 3. *For $\epsilon > 0$ small enough and $\alpha > 0$, there exists $C = C_\alpha$ such that the set $\mathscr{F}^\alpha_{\epsilon,C}$ is open in $\mathscr{G}^\alpha_\epsilon$ with respect to the $W^{1,3+\alpha}$-topology.*

Claim 4. *$\mathscr{G}^0_\epsilon$ is contained in the $W^{1,3}$-closure of $\bigcup_{\alpha>0} \mathscr{G}^\alpha_{2\epsilon}$.*

Proof of Claim 1. This is straightforward, since $\mathscr{G}^\alpha_\epsilon$ embeds in $C^{0,\gamma}(\mathbb{S}^3, \mathrm{SU}(2))$. □

Proof of Claim 2.. Consider a family $Q_j \in \mathscr{F}^\alpha_{\epsilon,C}$ with associated P_j as in (A-14) which converge to Q in $W^{1,3+\alpha}$. We extract a weakly convergent subsequence of the P_j and the estimate passes to the limit by weak lower semicontinuity (and by convergence of the Q_j). Similarly, the equations pass to weak limits, since they are intended in the weak sense. □

Ideas for Claim 3. For the proof we need to study the behavior of solutions to $\operatorname{div}(P^{-1}\nabla P) = 0$, which is regarded here as an equation $\mathcal{N}_\alpha(P) = 0$ with P close to the constant I, which is a zero of $\mathcal{N}_\alpha$. The equation considered is elliptic. The proof of the claim is thus done by linearization of $\mathcal{N}$ near I and by the implicit function theorem. Ellipticity of the equation translates into invertibility of this linearized operator. The estimate of the $W^{1,4}$-norm follows from the a priori estimate of Lemma A.1 once we choose, for example, $K \leq \frac{1}{2}C_\epsilon$. See Lemma A.3 for the complete proof. □

Proof of Claim 4. Consider $Q \in G_\epsilon^0$. There exists a sequence $Q_i \in C^\infty(\mathbb{S}^3, \mathrm{SU}(2))$ such that $Q_i \to Q$ in $W^{1,3}(\mathbb{S}^3, \mathrm{SU}(2))$; see [Bethuel 1991; Hang and Lin 2003] — by the density proofs of these works it follows that we may also assume $Q_i \in \mathscr{G}_{\epsilon_i}^{\alpha_i}$ for some sequence $\alpha_i \to 0^+$. The L^3-norm of a function f can be obtained as

$$\lim_{q \to 3^+} \|f\|_{L^q},$$

so in particular we may assume up to extracting a subsequence that $\epsilon_i \le 2\epsilon$. □

To conclude the proof, consider $Q \in \mathscr{G}_\delta^0$. We use Claim 4 to approximate Q in $W^{1,3}$-norm by $Q_i \in \mathscr{G}_{2\delta}^{\alpha_i}$ with $\alpha_i > 0$. From Claims 1–3 it follows that there exist functions $P_i \in W^{1,4+\alpha_i}(B^4, \mathrm{SU}(2))$ such that

$$\|P_i - I\|_{W^{1,4}(B^4)} \le K \|dQ_i\|_{L^3(\mathbb{S}^3)} \le 2K\delta.$$

The P_i have a weakly convergent subsequence whose limit P satisfies

$$\|P - I\|_{W^{1,4}(B^4)} \le 2K\delta \quad \text{and} \quad \begin{cases} \mathrm{div}(P^{-1}\nabla P) = 0 & \text{on } B^4 \\ P = Q & \text{on } \mathbb{S}^3. \end{cases}$$

Choose $\delta > 0$ such that $2K\delta \le \epsilon$ for ϵ as in Lemma A.1. We can then apply that lemma and obtain that

$$\|P - I\|_{W^{1,4}(B^4)} \le c\|P - I\|_{W^{4/3,3}(B^4)} \le cC_\epsilon \|Q\|_{L^3(\mathbb{S}^3)}.$$ □

We now complete the details of the proof of Claim 3:

Lemma A.3. *There exist $\epsilon > 0$, $K > 0$ such that for all $\alpha > 0$ there exists $C_\alpha > 0$ with the following property: Let $Q_0 \in W^{1,3+\alpha}(\mathbb{S}^3, \mathrm{SU}(2))$ and let $P_0 \in W^{1,4+\alpha}(B^4, \mathrm{SU}(2))$ be an extension of Q_0 which satisfies $\mathrm{div}(P_0^{-1}\nabla P_0) = 0$. If the estimates*

$$\|dQ_0\|_{W^{1,3}(\mathbb{S}^3)} < \epsilon, \tag{A-15}$$

$$\|P_0 - I\|_{W^{1,4}(B^4)} \le K\|dQ_0\|_{W^{1,3}(\mathbb{S}^3)}, \tag{A-16}$$

$$\|P_0 - I\|_{W^{1,4+\alpha}(B^4)} \le C_\alpha \|dQ_0\|_{W^{1,3+\alpha}(\mathbb{S}^3)} \tag{A-17}$$

hold then, for some $\delta > 0$ depending on Q_0, for all Q satisfying

$$\|Q - Q_0\|_{W^{1,3+\alpha}(\mathbb{S}^3, \mathrm{SU}(2))} < \delta \tag{A-18}$$

there exists an extension P of Q satisfying the same equation $\mathrm{div}(P^{-1}\nabla P) = 0$ and such that (A-15), (A-16), (A-17) hold with P, Q in place of P_0, Q_0.

Proof. We fix Q satisfying (A-18) and (A-15). The proof is divided into two parts:

Claim 3.1. *For $\delta > 0$ small enough and for Q satisfying (A-18), there exists an extension P of Q solving $\mathrm{div}(P^{-1}\nabla P) = 0$ and such that (A-17) holds.*

Claim 3.2. *The function P of Claim 3.1 satisfies (A-16).*

Proof of Claim 3.2. This follows directly from Lemma A.1. □

Proof of Claim 3.1. First note that $V = \exp^{-1}(Q_0^{-1}Q)$ is well defined for $\alpha > 0$, because in that case we have an estimate of the form

$$\|Q - Q_0\|_{W^{1,3+\alpha}} \geq c_\alpha \|Q - Q_0\|_{L^\infty} \iff \|Q_0^{-1}Q - I\|_{L^\infty} \leq \frac{\epsilon}{c_\alpha}$$

and $\exp^{-1}$ is well defined in a neighborhood of the identity.

We consider the problem of extending $Q_0 \exp(V)$ inside B^4 to a function $P = P_0 \exp(U)$. Extend V to $\tilde{V}$ such that $\Delta \tilde{V} = 0$ inside B^4.

We look for a P of the form $P_0 \exp(\tilde{V}) \exp(U)$. We thus consider the equation

$$\mathcal{N}(U, V) := d^*\big(\exp(-U)\exp(-\tilde{V})P_0^{-1} d(P_0 \exp(\tilde{V})\exp(U))\big) = 0. \tag{A-19}$$

In order to solve (A-19) it is useful to look at the operator

$$\mathcal{N}(V, U) : W_0^{1,4+\alpha}(B^4, \mathrm{su}(2)) \to W^{-1,4+\alpha}(B^4, \mathrm{su}(2)). \tag{A-20}$$

We have to show that for $\delta > 0$ small enough, for each Q satisfying (A-18) (i.e., for each small enough V), there exists a unique U such that $\mathcal{N}(V, U) = 0$. We prove that $\mathcal{N}(U, V)$ is C^1 near $(U, V) = (0, 0)$ and that $\partial\mathcal{N}/\partial U(0, 0)$ is an isomorphism, given the existence of $\delta > 0$ as desired.

A simple calculation gives

$$\frac{\partial \mathcal{N}}{\partial U} \cdot \eta = \frac{\partial}{\partial t}\bigg|_{t=0} \mathcal{N}(U + t\eta, V) = d^*d\eta - d^*\big[\eta, \exp(-U)\exp(-\tilde{V})P_0^{-1} d(P_0 \exp(\tilde{V}))\exp(U)\big]$$
$$:= \Delta\eta - L\eta.$$

We observe that $d^*d = \Delta$ is an isomorphism between the spaces above, so it will be enough to show that for U, $\tilde{V}$ small enough in the $W^{1,4+\alpha}$-norm the commutator term $L\eta$ is just a small perturbation of Δ (with respect to the norms present in (A-20)). First note that we can write

$$L\eta = [\nabla\eta, X] + [\eta, \operatorname{div} X], \quad \text{where } X := \exp(-U)\exp(-\tilde{V})P_0^{-1} d(P_0 \exp(\tilde{V}))\exp(U).$$

Estimate for $[\nabla\eta, X]$. First note that by the Sobolev, Hölder and triangle inequalities,

$$\|[\nabla\eta, X]\| W^{-1,4+\alpha} \lesssim \|[\nabla\eta, X]\|_{L^{p_\alpha}} \lesssim \|\nabla\eta\|_{L^{4+\alpha}} \|X\|_{L^4},$$

where

$$\frac{1}{p_\alpha} = \frac{1}{4+\alpha} + \frac{1}{4}.$$

We then observe

$$X = \exp(-U)\exp(-\tilde{V})P_0^{-1} d(P_0 \tilde{V})\exp(\tilde{V})\exp(U)$$

and note $|\exp A| = 1$; therefore,

$$\|X\|_{L^4} = \|d(P_0 \tilde{V})\|_{L^4} \lesssim \|dP_0\|_{L^4} + \|d\tilde{V}\|_{L^4} \lesssim \epsilon + \delta.$$

We thus have the first desired estimate,

$$\|[\nabla\eta, X]\|_{W^{-1,4+\alpha}} \lesssim (\epsilon + \delta)\|\eta\|_{W^{1,4+\alpha}}.$$

Estimate for $[\eta, \operatorname{div} X]$**.** Here we start with

$$\|[\eta, \operatorname{div} X]\|_{W^{-1,4+\alpha}} \lesssim \|\eta\|_{L^\infty} \|\operatorname{div} X\|_{L^{p_\alpha}} .$$

Note that $\|\eta\|_{L^\infty} \lesssim \|\eta\|_{W^{1,4+\alpha}}$ by the Sobolev embedding. We start the computations for the second fact or above. Note that

$$\nabla(P_0 \exp \widetilde{V}) = (\nabla P_0) \exp \widetilde{V} + P_0 \nabla(\exp \widetilde{V})$$

and then expand:

$$\begin{aligned}
\operatorname{div} X &= \operatorname{div}[\exp(-U) \exp(-\widetilde{V}) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \exp U] \\
&= \nabla(\exp(-U)) \exp(-\widetilde{V}) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \nabla(\exp(-\widetilde{V})) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) \operatorname{div}\left(P_0^{-1} \nabla P_0\right) \exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) P_0^{-1} P_0 \operatorname{div} \nabla(\exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) P_0^{-1} \nabla P_0 \nabla(\exp \widetilde{V}) \exp U \\
&\quad + \exp(-U) \exp(-\widetilde{V}) P_0^{-1} \nabla(P_0 \exp \widetilde{V}) \nabla(\exp U)
\end{aligned}$$

We have $\operatorname{div}(P_0^{-1} \nabla P_0) = 0$ and $\operatorname{div} \nabla(\exp(\widetilde{V})) = 0$, so two terms cancel. Note also the fact that $\|P_0^{-1} \nabla P_0\|_{L^4} \leq \|\nabla P_0\|_{L^4} \leq \epsilon$. For estimating $\nabla(\exp(\pm\widetilde{V}))$ observe that $\widetilde{V}$ satisfies a Dirichlet boundary value problem, therefore we assume the estimate $\|\widetilde{V}\|_{W^{1,4+\alpha}} \lesssim \delta$, and $\|U\|_{W^{1,4+\alpha}} \lesssim \delta$, which, by the smoothness of exp, imply $\|\nabla(\exp(\pm\widetilde{V}))\|_{L^{4+\alpha}} \lesssim \delta$ and $\|\nabla(\exp(\pm U))\|_{L^{4+\alpha}} \lesssim \delta$. From all this it follows that we can estimate

$$\begin{aligned}
\|\operatorname{div} X\|_{L^{p_\alpha}} &\lesssim \|\nabla(\exp(-U))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} + \|\nabla(\exp(-\widetilde{V}))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} \\
&\quad + \|\nabla P_0\|_{L^4} \|\nabla(\exp(\widetilde{V}))\|_{L^{4+\alpha}} + \|\nabla(\exp(U))\|_{L^{4+\alpha}} \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} \\
&\lesssim \delta \|\nabla(P_0 \exp \widetilde{V})\|_{L^4} + \epsilon\delta \\
&\lesssim \delta(\epsilon + \delta).
\end{aligned}$$

We combine all the estimates and obtain the desired smallness result,

$$\|[\eta, \operatorname{div} X]\|_{W^{-1,4+\alpha}} \lesssim \delta(\epsilon + \delta) \|\eta\|_{W^{1,4+\alpha}} . \qquad \square$$

End of proof. We now have that

$$\|L\eta\|_{W^{-1,4+\alpha}} \lesssim (\delta + 1)(\epsilon + \delta) \|\eta\|_{W^{1,4+\alpha}} ,$$

while

$$\|\Delta\eta\|_{W^{-1,4+\alpha}} \gtrsim \|\eta\|_{W^{1,4+\alpha}} .$$

Therefore, for small enough ϵ, δ we have also

$$\|(\Delta - L)\eta\|_{W^{-1,4+\alpha}} \gtrsim \|\eta\|_{W^{1,4+\alpha}} .$$

This concludes the proof. $\square$

Appendix B: A product estimate with only one bounded factor

Lemma B.1 (cf. [Brézis and Mironescu 2001]). *Let Ω be a smooth compact 4-manifold. If $f \in W^{1/3,3}(\Omega)$ and $g \in W^{1,4} \cap L^\infty(\Omega)$, then we have the following estimate, with the implicit constant depending only on Ω:*

$$\|fg\|_{W^{1/3,3}(\Omega)} \lesssim \|f\|_{W^{1/3,3}(\Omega)}(\|g\|_{L^\infty(\Omega)} + \|g\|_{W^{1,4}(\Omega)}).$$

Proof. The estimates for the nonhomogeneous part of the norms are trivial, so we concentrate on the homogeneous part.

We use the Littlewood–Paley decompositions $f = \sum_{j=0}^\infty f_j$, $g = \sum_{k=0}^\infty g_k$, and we recall that the $W^{s,p}$-norm is equivalent to the Triebel–Lizorkin $\dot F^1_{4,2}$-norm and the $W^{\theta,4}$-norm is equivalent to the $\dot F^s_{p,2}$-norm, where in general the following definition holds:

$$\|f\|_{\dot F^s_{p,q}} = \big\| |2^{ks} f_k(x)|_{\ell^q} \big\|_{L^p}.$$

We use different notations $\|\cdot\|$, $|\cdot|$ for the different norms just to facilitate the reading of formulas. As is usual in the theory of paraproducts, we estimate separately the following three contributions (where $g^k := \sum_{i=0}^k g_k$, and similarly for f^k)

$$fg = \sum_i f_i g^{i-4} + \sum_{|k-l|<4} f_k g_l + \sum_i f^{i-4} g_i =: I + II + III.$$

The support of $(\widehat{f_i g^{i-4}})$ is included in $B_{2^{i+2}} \setminus B_{2^{i-2}}$; thus,

$$\|I\|_{W^{1/3,3}} = \Big\|\sum_i f_i g^{i-4}\Big\|_{W^{1/3,3}} \sim \bigg[\int_\Omega \Big(\sum_i 2^{2i/3} |f_i g^{i-4}|^2\Big)^{3/2}\bigg]^{1/3} \tag{B-1}$$

and analogously for $III = \sum_i f^{i-4} g_i$. Regarding the term II, we will estimate only $II' := \sum_i f_i g_i$ because the same estimate will apply also to the finitely many contributions of the form $\sum_i f_i g_{i+l}$ with $0 < |l| < 4$.

We start with the most difficult term, III. From above we have

$$\begin{aligned}
\|III\|_{W^{1/3,3}} &\sim \bigg[\int \Big(\sum_i 2^{2i/3} |f^{i-4} g_i|^2\Big)^{3/2}\bigg]^{1/3} \\
&\le \bigg[\int \Big(\sum_i 2^{-4i/3} |f^{i-4}|^2\Big)^{3/2} \Big(\sum_i 2^{2i} |g_i|^2\Big)^{3/2}\bigg]^{1/3} \\
&\le \bigg[\int \Big(\sum_i 2^{-4i/3} |f^{i-4}|^2\Big)^{6}\bigg]^{1/12} \bigg[\int \Big(\sum_i 2^{2i} |g_i|^2\Big)^{2}\bigg]^{1/4} \\
&\le \|f\|_{W^{-2/3,12}} \|g\|_{W^{1,4}} \\
&\le \|f\|_{W^{1/3,3}} \|g\|_{W^{1,4}}.
\end{aligned}$$

For the term I we have

$$\|I\|_{W^{1/3,3}} \sim \Big[\int\Big(\sum_i 2^{2i/3}|f_i g^{i-4}|^2\Big)^{3/2}\Big]^{1/3} \lesssim \|g\|_{L^\infty}\|f\|_{W^{1/3,3}}$$

because of the estimate $\|g^{i-4}\|_{L^\infty} \lesssim \|g\|_{L^\infty}$. Finally, we estimate II', as promised. We prove it by duality; namely, we prove that II' is bounded as a linear functional on the unit ball of the dual $W^{-1/3,3/2}$. Consider therefore h in this ball. The support of $(\widehat{f_i g_i})$ is included in $B_{2^{i+2}}$, so some terms cancel:

$$\begin{aligned}\int h\cdot II' &\sim \sum_{k,i}\int h_k f_i g_i = \sum_{k\le i+4}\int h_k f_i f_j = \sum_i \int h^{i+4} f_i g_i\\ &\le \Big|\sum_i\int 2^{-i/3}h^{i+4}2^{i/3} f_i g_i\Big|\\ &\le \|g\|_{B^0_{\infty,\infty}}\int\Big(\sum_i 2^{-2i/3}|h^{i+4}|^2\Big)^{1/2}\Big(\sum_i 2^{2i/3}|f_i|^2\Big)^{1/2}\\ &\le \|g\|_{W^{1,4}}\|h\|_{W^{-1/3,3/2}}\|f\|_{W^{1/3,3}}\,.\end{aligned}$$

The last estimate follows, recalling that

$$\|g\|_{B^0_{\infty,\infty}} := \sup_i \|g_i\|_{L^\infty}$$

and that in dimension 4 we have continuous embeddings

$$W^{1,4} \hookrightarrow \mathrm{BMO} \hookrightarrow B^0_{\infty,\infty}\,.$$

Summing up the different terms, we are done. □

Appendix C: The Möbius group of B^n

We call the Möbius group of $\mathbb{R}^n$ the group $M(\mathbb{R}^n)$ generated by all similarities and the inversion with respect to the unit sphere. Recall that a similarity is an affine map of the form

$$x \mapsto \lambda K x + b \quad \text{with } \lambda > 0,\ K \in O(n),\ b \in \mathbb{R}^n,$$

and the inversion $i_{c,r}$ with respect to the sphere $\partial B(c,r)$ is the map

$$x \mapsto c + r^2\frac{x-c}{|x-c|^2}\,.$$

The formula $i_{c,r} = (r^2\,\mathrm{id}+c)\circ i_{0,1}\circ(\mathrm{id}-c)$ shows that all inversions belong to $M(\mathbb{R}^n)$. We use the abridged notation

$$x^* := i_{1,0}(x) = \frac{x}{|x|^2}\,.$$

The Möbius group of B^{n+1} is the subgroup $M(B^{n+1})$ of all transformations belonging to $M(\mathbb{R}^{n+1})$ which preserve B^{n+1}. Similarly, we define the Möbius group $M(\mathbb{S}^n)$ of the unit sphere $\mathbb{S}^n \subset \mathbb{R}^{n+1}$. The general form of an element $\gamma \in M(B^{n+1})$ is

$$\gamma = K \circ F_v \quad \text{with } K \in O(n),\ v \in B^{n+1},\ F_v := -v + (1-|v|^2)(x^* - v)^*.$$

We use the following basic properties of the functions F_v, which can be found in [Ahlfors 1981, Chapter 2]:

Lemma C.1. • *We have*

$$|F_v|(x) = \frac{1-|v|^2}{[x,v]},$$

where $[x, y] = |x|\,|x^* - y| = |y|\,|y^* - x|$.

- F_v *is conformal. We have* $F_v^{-1} = F_{-v}$, $F_v(0) = -v$ *and* $F_v(v) = 0$.
- *The conformal factor* $|F_v'|(x)$ *is explicitly computed as*

$$|F_v'|(x) = \frac{1-|v|^2}{1+|x|^2|v|^2 - 2x\cdot v} = \frac{|v^*|^2-1}{|x-v^*|^2}.$$

- *The restriction* $F_v|_{\mathbb{S}^n}$ *belongs to* $M(\mathbb{S}^n)$; *in particular*, $F_v|_{\mathbb{S}^n}$ *is a conformal involution and*

$$|(F_v|_{\mathbb{S}^n})'|(x) = \frac{1-|v|^2}{|x-v|^2}.$$

The next lemma gives the estimate needed for the case when v is close to ∂B^{n+1}:

Lemma C.2. *Suppose that*

$$\rho \le \tfrac{1}{4}.$$

Then, on $F_v^{-1}(B_{1-\rho})$, *the following estimate holds with a constant* C *dependent only on the dimension*:

$$\frac{h(v)}{C} \le |F_v'|(x) \le C h(v).$$

Proof. We will calculate

$$\frac{\max\{|F_v'|(y) : y \in F_v^{-1}(B_{1-\rho})\}}{\min\{|F_v'|(y') : y' \in F_v^{-1}(B_{1-\rho})\}} = \max\left\{\frac{|F_v'|(y)}{|F_v'|(y')} : y, y' \in F_v^{-1}(B_{1-\rho})\right\}$$

and we show that this quantity is bounded. The following equalities hold:

$$\begin{aligned}\max\left\{\frac{|F_v'|(x)}{|F_v'|(x')} : x, x' \in B_{1-\rho}\right\} &= \max\left\{\frac{|F_{-v}'|(x)}{|F_{-v}'|(x')} : x, x' \in B_{1-\rho}\right\}\\ &= \max\left\{\frac{|(F_v^{-1})'|(x)}{|(F_v^{-1})'|(x')} : x, x' \in B_{1-\rho}\right\}\\ &= \min\left\{\frac{|F_v'|(F_v^{-1}(x'))}{|F_v'|(F_v^{-1}(x))} : x, x' \in B_{1-\rho}\right\}\\ &= \min\left\{\frac{|F_v'|(y')}{|F_v'|(y)} : y, y' \in F_v^{-1}(B_{1-\rho})\right\}.\end{aligned}$$

From the formula of the previous lemma it follows that

$$\nabla_x |F_v'|(x) = 2\frac{|v^*|^2 - 1}{|v^* - x|^4}(v^* - x);$$

therefore, $|F_v'|$ achieves its extrema on $B_{1-\rho}$ at $\pm(1-\rho)v/|v|$. The maximum M and the minimum m of $|F_v'|$ satisfy

$$M = \frac{1-|v|^2}{1+|v|^2(1-\rho)^2 - 2(1-\rho)|v|} = \frac{1-|v|^2}{(1-(1-\rho)|v|)^2},$$
$$m = \frac{1-|v|^2}{1+|v|^2(1-\rho)^2 + 2(1-\rho)|v|} = \frac{1-|v|^2}{(1+(1-\rho)|v|)^2},$$
$$\frac{M}{m} = \left(\frac{1+(1-\rho)|v|}{1-(1-\rho)|v|}\right)^2 \sim (1-(1-\rho)|v|)^{-2} \sim 1,$$

which finishes the proof. □

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MIRCEA PETRACHE: decostruttivismo@gmail.com
Laboratoire Jacques-Louis Lions, Universitè Pierre et Marie Curie (Paris 6), place Jussieu, 4, 75005 Paris, France

TRISTAN RIVIÈRE: decostruttivismo@gmail.com
Department of Mathematics, ETH Zürich, CH-8092 Zürich, Switzerland

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CONCENTRATION OF SMALL WILLMORE SPHERES IN RIEMANNIAN 3-MANIFOLDS

PAUL LAURAIN AND ANDREA MONDINO

Given a three-dimensional Riemannian manifold (M, g), we prove that, if (Φ_k) is a sequence of Willmore spheres (or more generally area-constrained Willmore spheres) having Willmore energy bounded above uniformly strictly by 8π and Hausdorff converging to a point $\bar{p} \in M$, then $\mathrm{Scal}(\bar{p}) = 0$ and $\nabla \mathrm{Scal}(\bar{p}) = 0$ (respectively, $\nabla \mathrm{Scal}(\bar{p}) = 0$). Moreover, a suitably rescaled sequence smoothly converges, up to subsequences and reparametrizations, to a round sphere in the euclidean three-dimensional space. This generalizes previous results of Lamm and Metzger. An application to the Hawking mass is also established.

1. Introduction

Let Σ be a closed two-dimensional surface and (M, g) a three-dimensional Riemannian manifold. Given a smooth immersion $\Phi : \Sigma \hookrightarrow M$, $W(\Phi)$ denotes the Willmore energy of Φ defined by

$$W(\Phi) := \int_\Sigma H^2 \, d\mathrm{vol}_{\bar{g}}, \tag{1}$$

where $\bar{g} := \Phi^*(g)$ is the pullback metric on Σ (i.e., the metric induced by the immersion), $d\mathrm{vol}_{\bar{g}}$ is the associated volume form, and H is the mean curvature of the immersion Φ (we adopt the convention that $H = \frac{1}{2}\bar{g}^{ij} A_{ij}$, where A_{ij} is the second fundamental form; or in other words, H is the arithmetic mean of the two principal curvatures).

In case the ambient manifold is the euclidean three-dimensional space, the topic is classical and goes back to the works of Blaschke and Thomsen in 1920–1930, who were looking for a conformal invariant theory that included minimal surfaces; the functional was later rediscovered by Willmore [1993] in the 1960s, and from that moment, there has been a flourishing of results (let us mention the fundamental paper of Simon [1993], the work of Kuwert and Schätzle [2001; 2004; 2007], the more recent approach by Rivière [2008; 2014; 2013], etc.) culminating in the recent proof of the Willmore conjecture by Marques and Neves [2014] by min–max techniques (let us mention that partial results towards the Willmore conjecture were previously obtained by Li and Yau [1982], Montiel and Ros [1986], Ros [1999], Topping [2000], etc., and that a crucial role in the proof of the conjecture is played by a result of Urbano [1990]).

On the other hand, the investigation of the Willmore functional in nonconstantly curved Riemannian manifolds is a much more recent topic started in [Mondino 2010] (see also [Mondino 2013] and the

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more recent joint work [Carlotto and Mondino 2014]), where the second author studied existence and nonexistence of Willmore surfaces in a perturbative setting.

Smooth minimizers of the L^2-norm of the second fundamental form among spheres in compact Riemannian 3-manifolds were obtained in collaboration with Kuwert and Schygulla in [Kuwert et al. 2014], where the full regularity theory for minimizers was settled, taking inspiration from the approach of Simon [1993] (see also [Mondino and Schygulla 2014] for minimization in noncompact Riemannian manifolds).

Let us finally mention the work in collaboration with Rivière [Mondino and Rivière 2014; 2013], where using a "parametric approach" inspired by the euclidean theory of [Rivière 2008; 2014; 2013], the necessary tools for studying the calculus of variations of the Willmore functional in Riemannian manifolds (i.e., the definition of the weak objects and related compactness and regularity issues) are settled together with applications; in particular, the existence and regularity of Willmore spheres in homotopy classes is established.

Since — as usual in the calculus of variations — the existence results are obtained by quite general techniques and do not describe the minimizing object, the purpose of the present paper is to investigate the geometric properties of the critical points of W.

More precisely, we investigate the following natural questions. Let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of smooth critical points of the Willmore functional W (or more generally we will also consider critical points under area constraint) converging to a point $\bar{p} \in M$ in Hausdorff distance sense; what can we say about Φ_k? Are they becoming more and more round? Does the limit point $\bar{p}$ have some special geometric property?

These questions have already been addressed in recent articles — below the main known results are recalled for the reader's convenience — but in the present paper we are going to obtain the sharp answers.

Before describing the known and the new results in this direction, let us recall that a critical point of the Willmore functional is called a *Willmore surface* and it satisfies

$$\Delta_{\bar{g}} H + H|A^\circ|^2 + H\,\mathrm{Ric}(\vec{n}, \vec{n}) = 0, \tag{2}$$

where $\Delta_{\bar{g}}$ is the Laplace–Beltrami operator corresponding to the metric $\bar{g}$, $(A^\circ)_{ij} := A_{ij} - H\bar{g}_{ij}$ is the trace-free second fundamental form, $\vec{n}$ is a normal unit vector to Φ, and Ric is the Ricci tensor of the ambient manifold (M, g). Notice that (2) is a fourth-order nonlinear elliptic PDE in the parametrization map Φ.

Throughout the paper, we will consider more generally *area-constrained Willmore surfaces*, i.e., critical points of the Willmore functional under area constraint; the immersion Φ is an area-constrained Willmore surface if and only if it satisfies

$$\Delta_{\bar{g}} H + H|A^\circ|^2 + H\,\mathrm{Ric}(\vec{n}, \vec{n}) = \lambda H \tag{3}$$

for some $\lambda \in \mathbb{R}$ playing the role of Lagrange multiplier.

The first result in the direction of the above questions was achieved in the master degree thesis of Mondino [2010], where it was proved that, if (Φ_k) is a sequence of Willmore surfaces obtained as normal graphs over shrinking geodesic spheres centered at a point $\bar{p}$, then the scalar curvature at $\bar{p}$ must vanish: $\mathrm{Scal}(\bar{p}) = 0$.

In subsequent papers, Lamm and Metzger [2010; 2013] proved that, if $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ is a sequence of area-constrained Willmore surfaces converging to a point $\bar{p}$ in Hausdorff distance sense and such that[1]

$$W(\Phi_k) \leq 4\pi + \varepsilon \quad \text{for some } \varepsilon > 0 \text{ small enough,} \tag{4}$$

then $\nabla \mathrm{Scal}(\bar{p}) = 0$ and, up to subsequences, Φ_k is $W^{2,2}$-asymptotic to a geodesic sphere centered at $\bar{p}$. Moreover in [Lamm and Metzger 2013], using the regularity theory developed in [Kuwert et al. 2014], they showed that, if (M, g) is any compact Riemannian 3-manifold and a_k is any sequence of positive real numbers such that $a_k \downarrow 0$, then there exists a smooth minimizer Φ_k of W under the area constraint $\mathrm{Area}(\Phi_k) = a_k$; moreover, such a sequence (Φ_k) satisfies (4) and therefore $W^{2,2}$-converges to a round critical point of the scalar curvature. Let us mention that the existence of area-constrained Willmore spheres was generalized in [Mondino and Rivière 2013] to any value of the area.

The goal of this paper is multiple. The main achievement is the improvement of the perturbative bound (4) above to the global bound

$$\limsup_k W(\Phi_k) < 8\pi. \tag{5}$$

Secondly, we improve the $W^{2,2}$-convergence above to *smooth* convergence towards a *round* critical point of the scalar curvature; i.e., we show that, if we rescale (M, g) around $\bar{p}$ in such a way that the sequence of surfaces has fixed area equal to 1 (for more details, see Section 2), then the sequence converges smoothly, up to subsequences, to a round sphere centered at $\bar{p}$ and $\bar{p}$ is a critical point of the scalar curvature of (M, g).

Finally we give an application of these results to the Hawking mass.

We believe that the bound (5) is sharp in order to have smooth convergence to a *round* point (in the sense specified above); indeed, if (5) is violated, then the sequence (Φ_k) may degenerate to a couple of bubbles, each one costing almost 4π in terms of Willmore energy.

Now let us state the main results of the present article. The first theorem below concerns the case of a sequence of Willmore immersions and is a consequence of the second more general theorem about area-constrained Willmore immersions.

Theorem 1.1. *Let (M, g) be a three-dimensional Riemannian manifold, and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of Willmore surfaces satisfying the energy bound (5) and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\mathrm{Scal}(\bar{p}) = 0$ and $\nabla \mathrm{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\widetilde{\Phi}_k$ have fixed area equal to 1, then $\widetilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the three-dimensional euclidean space.

Actually, we prove the following more general result about sequences of area-constrained Willmore immersions:

[1] The normalization of the Willmore functional used in [Lamm and Metzger 2010; 2013] differs from our convention by a factor of 2.

Theorem 1.2. *Let (M, g) be a three-dimensional Riemannian manifold, and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of area-constrained Willmore surfaces satisfying the energy bound (5) and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla \operatorname{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\widetilde{\Phi}_k$ have fixed area equal to 1, then $\widetilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the three-dimensional euclidean space.

Of course, Theorem 1.2 implies Theorem 1.1 except the property $\operatorname{Scal}(\bar{p}) = 0$. This fact follows by the aforementioned [Mondino 2010, Theorem 1.3] holding for Willmore graphs over geodesic spheres together with the smooth convergence to a round point ensured by Theorem 1.2.

Now we pass to discuss an application to the Hawking mass m_H, defined for an immersed sphere $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$ by

$$m_H(\Phi) = \frac{\operatorname{Area}_g(\Phi)}{16\pi^{3/2}}(4\pi - W(\Phi)). \tag{6}$$

Of course, the critical points of the Hawking mass under area constraint are exactly the area-constrained Willmore spheres (see [Lamm et al. 2011] and the references therein for more material about the Hawking mass); moreover, it is clear that the inequality $m_H(\Phi) \geq 0$ implies that $W(\Phi) \leq 4\pi$.

Therefore, combining this easy observations with Theorem 1.2, we obtain the following corollary:

Corollary 1.3. *Let (M, g) be a three-dimensional Riemannian manifold, and let $\Phi_k : \mathbb{S}^2 \hookrightarrow M$ be a sequence of critical points of m_H under area constraint having nonnegative Hawking mass and Hausdorff converging to a point $\bar{p} \in M$.*

Then $\nabla \operatorname{Scal}(\bar{p}) = 0$; moreover, if we rescale (M, g) around $\bar{p}$ in such a way that the rescaled immersions $\widetilde{\Phi}_k$ have fixed area equal to 1, then $\widetilde{\Phi}_k$ converges smoothly, up to subsequences and up to reparametrizations, to a round sphere in the three-dimensional euclidean space.

First of all, let us mention that Corollary 1.3 also follows by the analysis performed in [Lamm and Metzger 2010] with the only difference that here we improved the $W^{2,2}$ convergence to the smooth one. Now let us briefly comment on the relevance of Corollary 1.3 despite the triviality of its proof. Recall that, from the note of Christodoulou and Yau [1988], if (M, g) has nonnegative scalar curvature then isoperimetric spheres (and more generally stable CMC spheres) have positive Hawking mass; on the other hand, it is known (see for instance [Druet 2002] or [Nardulli 2009]) that, if M is compact, then small isoperimetric regions converge to geodesic spheres centered at a maximum point of the scalar curvature as the enclosed volume converges to 0 (see also [Mondino and Nardulli 2012] for the noncompact case). Therefore, a link between regions with positive Hawking mass and critical points of the scalar curvature was already present in literature, but Corollary 1.3 expresses this link precisely.

We end the introduction by outlying the structure of the paper and the main ideas of the proof. First of all, as already noticed, it is enough to prove Theorem 1.2 in order to get all the stated results. To prove it, we adopt the blow-up technique taking inspiration from [Laurain 2012], where the first author analyzed the corresponding questions in the context of CMC-surfaces; such technique was introduced in the analysis of the Yamabe problem, which is a second-order scalar problem (for a detailed overview of the method including applications see [Druet et al. 2004]). The technical novelty of [Laurain 2012] was that a second-order *vectorial* problem was considered; the technical originality of the present paper from the point of view of the blow-up method is that we study a *fourth-order vectorial problem*.

More precisely, in Section 2, we consider normal coordinates centered at the limit point $\bar{p}$ and we rescale appropriately the metric g such that the rescaled surfaces all have diameter 1 (or thanks to the monotonicity formula, it is equivalent to fix the area of the rescaled surfaces equal to 1); notice that the rescaled ambient metrics g_k are becoming more and more euclidean.

In Section 2A, by exploiting the divergence form of the Willmore equation established in [Mondino and Rivière 2013], we give a decay estimate on the Lagrange multipliers as k goes to infinity.

Section 3 is devoted to the proof of Theorem 1.2; we start in Section 3A by establishing a fundamental technical result that, under the above working assumptions, the sequence (Φ_k) converges smoothly to a round sphere, up to subsequences and reparametrizations. Let us remark that in the proof we exploit in a crucial way the assumption (5); otherwise, it may be possible for the sequence to degenerate to a couple of bubbles. Once we have smooth convergence to a round sphere ω, we study the remainder given by the difference between Φ_k and ω: in Section 3C, we use the linearized Willmore operator (recalled in the Appendix) in order to give precise asymptotics of such a remainder term, and in the final Section 3D, we refine these estimates and conclude the proof.

2. Notation and preliminaries

Throughout the paper, (M, g) is a Riemannian 3-manifold and $\mathbb{S}^2$ is the round 2-sphere of unit radius in $\mathbb{R}^3$. The Greek indexes α, β, γ, μ, and ν will run from 1 to 3 and will denote quantities in M; Latin indexes will run from 1 to 2 and will denote quantities on $\Phi_k(\mathbb{S}^2)$; we will always use Einstein notation on summation over indexes. Given a smooth immersion $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$, we call $\bar{g} = \Phi^*(g)$ the pullback metric, $d\mathrm{vol}_{\bar{g}}$ the induced area form, and $H_{g,\Phi}$ the mean curvature and

$$W_g(\Phi) := \int_{\mathbb{S}^2} |H_{g,\Phi}|^2 \, d\mathrm{vol}_{\bar{g}}$$

is the Willmore functional.

Now let (Φ_k) be a sequence of smooth immersions from $\mathbb{S}^2$ into M. Under our working assumptions, where $\mathrm{diam}_g(\Omega)$ is the diameter of the subset Ω of M with respect to the metric g, we will always have

$$\varepsilon_k := \mathrm{diam}_g(\Phi_k(\mathbb{S}^2)) \to 0, \tag{7}$$

$$W_g(\Phi_k) := \int_{\mathbb{S}^2} |H_{g,\Phi_k}|^2 \, d\mathrm{vol}_{\bar{g}_k} \leq 8\pi - 2\delta \quad \text{for some } \delta > 0 \text{ independent of } k, \tag{8}$$

where $d\mathrm{vol}_{\bar{g}_k}$ is the area form on $\mathbb{S}^2$ associated to the pullback metric $\bar{g}_k = \Phi_k^*(g)$ and H_{g,Φ_k} is the mean curvature of Φ_k.

Notice that in case M is compact then (7) is sufficient to ensure that, up to subsequences, $\Phi_k(\mathbb{S}^2)$ converges to a point $\bar{p} \in M$ in Hausdorff distance sense; but since there is no further reason to restrict to a compact ambient manifold, we assume the convergence to $\bar{p}$ in the hypothesis of our main results instead of a compactness assumption on M.

In order to efficiently handle the geometric quantities, we need good coordinates; let us now introduce them. Take coordinates (x^μ), $\mu = 1, 2, 3$, around $\bar{p}$, and let $p_k = (p_k^1, p_k^2, p_k^3)$ be the center of mass of $\Phi_k(\mathbb{S}^2)$:

$$p_k^\mu = \frac{1}{\mathrm{Area}_g(\Phi_k)} \int_{\mathbb{S}^2} \Phi_k^\mu \, d\mathrm{vol}_{\bar{g}_k}, \quad \mu = 1, 2, 3,$$

where $\mathrm{Area}_g(\Phi_k) = \int_{\mathbb{S}^2} d\mathrm{vol}_{\bar{g}_k}$ is the area of $\Phi_k(\mathbb{S}^2)$. Clearly, up to subsequences, $p_k \to \bar{p}$.

For every $k \in \mathbb{N}$, consider the exponential normal coordinates centered in p_k and rescale this chart by a factor $1/\varepsilon_k$ with respect to the center of these coordinates. Hence, we get a new sequence of immersions $\widetilde{\Phi}_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$, in the following simply denoted by Φ_k, where the metric g_{ε_k} is defined by

$$g_{\varepsilon_k}(y)(u, v) := g(\varepsilon_k y)(\varepsilon_k^{-1} u, \varepsilon_k^{-1} v). \tag{9}$$

Notice that now we have

$$W_{g_{\varepsilon_k}}(\Phi_k) \le 8\pi - 2\delta, \quad \mathrm{diam}_{g_{\varepsilon_k}}(\Phi_k(\mathbb{S}^2)) = 1, \quad \text{and} \quad \Phi_k(\mathbb{S}^2) \subset B_{g_{\varepsilon_k}}\big(0, \tfrac{3}{2}\big), \tag{10}$$

where the first inequality is a consequence of the invariance under rescaling of the Willmore functional and $B_{g_{\varepsilon_k}}\big(0, \frac{3}{2}\big)$ is the metric ball in $(\mathbb{R}^3, g_{\varepsilon_k})$ of center 0 and radius $\frac{3}{2}$. By the classical expression of the metric in normal coordinates, we get that (see Appendix B in [Laurain 2012])

$$(g_{\varepsilon_k})_{\mu\nu}(y) = \delta_{\mu\nu} + \tfrac{1}{3}\varepsilon_k^2 R_{\alpha\mu\nu\beta}(p_k) y^\alpha y^\beta + \tfrac{1}{6}\varepsilon_k^3 R_{\alpha\mu\nu\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3), \tag{11}$$

the inverse metric is

$$(g_{\varepsilon_k})^{\mu\nu}(y) = \delta_{\mu\nu} - \tfrac{1}{3}\varepsilon_k^2 R_{\alpha\mu\nu\beta}(p_k) y^\alpha y^\beta - \tfrac{1}{6}\varepsilon_k^3 R_{\alpha\mu\nu\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3), \tag{12}$$

the volume form of g_{ε_k} can be written as

$$\sqrt{|g_{\varepsilon_k}|}(y) = 1 - \tfrac{1}{6}\varepsilon_k^2 \,\mathrm{Ric}_{\alpha\beta}(p_k) y^\alpha y^\beta - \tfrac{1}{12}\varepsilon_k^3 \,\mathrm{Ric}_{\alpha\beta,\gamma}(p_k) y^\alpha y^\beta y^\gamma + o(\varepsilon_k^3), \tag{13}$$

and the Christoffel symbols of g_{ε_k} can be expanded as

$$(\Gamma_{\varepsilon_k})^\gamma_{\alpha\beta}(y) = A_{\alpha\beta\gamma\mu}(p_k) y^\mu \varepsilon_k^2 + B_{\alpha\beta\gamma\mu\nu}(p_k) y^\mu y^\nu \varepsilon_k^3 + o(\varepsilon_k^3), \tag{14}$$

where $A_{\alpha\beta\gamma\mu}(p_k) = \frac{1}{3}(R_{\beta\mu\alpha\gamma}(p_k) + R_{\alpha\mu\beta\gamma}(p_k))$ and $B_{\alpha\beta\gamma\mu\nu}(p_k) = \frac{1}{12}(2R_{\beta\mu\alpha\gamma,\nu}(p_k) + 2R_{\alpha\mu\beta\gamma,\nu}(p_k) + R_{\beta\mu\nu\gamma,\alpha}(p_k) + R_{\alpha\mu\nu\gamma,\beta}(p_k) - R_{\alpha\mu\nu\beta,\gamma}(p_k))$.

Since by (11) the metric g_{ε_k} is close to the euclidean metric in the C^∞-norm on $B_{g_0}(0, 2)$, where $B_{g_0}(0, 2)$ is the euclidean ball in $\mathbb{R}^3$ of center 0 and radius 2, recalling (10), we get the following lemma:

Lemma 2.1. *Let g_{ε_k} be the metric defined in (9) having the form (11); let $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ be smooth immersions with $\Phi_k(\mathbb{S}^2) \subset B_{g_{\varepsilon_k}}(0, 2)$ satisfying*

$$W_{g_{\varepsilon_k}}(\Phi_k) \le 8\pi - 2\delta \quad \textit{for some } \delta > 0.$$

Then, for k large enough, we have

$$W_{g_0}(\Phi_k) \le 8\pi - \delta, \quad \tfrac{1}{2} \le \operatorname{diam}_{g_0}(\Phi_k(\mathbb{S}^2)) \le 2, \quad \textit{and} \quad \Phi_k(\mathbb{S}^2) \subset B_{g_0}(0,2), \tag{15}$$

where g_0 is the euclidean metric on $\mathbb{R}^3$, W_{g_0} is the euclidean Willmore functional, and $B_{g_0}(0,2)$ is the euclidean ball of center 0 and radius 2 in $\mathbb{R}^3$. It follows that, for large k, $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ is a smooth embedding and that there exist constants $C_1, C_2 > 0$ such that

$$0 < \frac{1}{C_1} \le \frac{1}{C_2} \operatorname{Area}_{g_0}(\Phi_k) \le \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k) \le C_2 \operatorname{Area}_{g_0}(\Phi_k) \le C_1 < \infty. \tag{16}$$

Proof. The properties expressed in (15) follow from (10) by a direct estimate of the remainders given by the curvature terms of the metric g_{ε_k}; for such estimates, we refer to Lemmas 2.1–2.4 in [Mondino and Schygulla 2014].

It is classically known that, if the Willmore functional of an immersed closed surface in $(\mathbb{R}^3, g_0)$ is strictly below 8π, then the immersion is actually an embedding (see [Li and Yau 1982] or [Simon 1993]), so our second statement follows.

In order to prove (16), let us recall Lemma 1.1 in [Simon 1993] stating that

$$\sqrt{\frac{\operatorname{Area}_{g_0}(\Phi_k)}{W_{g_0}(\Phi_k)}} \le \operatorname{diam}_{g_0} \Phi_k(\mathbb{S}^2) \le C\sqrt{\operatorname{Area}_{g_0}(\Phi_k) W_{g_0}(\Phi_k)} \quad \text{for some universal } C > 0,$$

which, combined with the bound on $\operatorname{diam}_{g_0}(\Phi_k(\mathbb{S}^2))$ and $W_{g_0}(\Phi_k)$ expressed in (15), gives that there exists a constant $C_0 > 0$ such that

$$0 < \frac{1}{C_0} \le \operatorname{Area}_{g_0}(\Phi_k) \le C_0 < \infty;$$

the desired chain of inequalities (16) follows then by estimating the remainders as in Lemma 2.2 in [Mondino and Schygulla 2014]. □

2A. ***The area-constrained Willmore equation and an estimate of the Lagrange multiplier.*** In the rest of the paper, we will work with area-constrained Willmore immersions, i.e., critical points of the Willmore functional under the constraint that the area is fixed. If $\Phi : \mathbb{S}^2 \hookrightarrow (M, g)$ is a smooth area-constraint Willmore immersion, then it satisfies the following PDE (see for instance Section 3 in [Lamm et al. 2011] for the derivation of the equation)

$$\triangle_{\bar{g}} H_{g,\Phi} + H_{g,\Phi} |A^\circ_{g,\Phi}|^2_{\bar{g}} + H_{g,\Phi} \operatorname{Ric}_g(\vec{n}_{g,\Phi}, \vec{n}_{g,\Phi}) = \lambda H_{g,\Phi} \tag{17}$$

for some $\lambda \in \mathbb{R}$, where $\vec{n}_{g,\Phi}$ is a normal unit vector to $\Phi(\mathbb{S}^2) \subset (M, g)$, $(A^\circ_{g,\Phi})_{ij}$ is the traceless second fundamental form $(A^\circ_{g,\Phi})_{ij} = (A_{g,\Phi})_{ij} - \bar{g}_{ij} H_{g,\Phi}$ (of course $(A_{g,\Phi})_{ij}$ is the second fundamental form of Φ in (M, g)), and $|A^\circ_{g,\Phi}|^2_{\bar{g}} = \bar{g}^{ik} \bar{g}^{jl} (A^\circ_{g,\Phi})_{ij} (A^\circ_{g,\Phi})_{kl}$ is its norm with respect to the metric $\bar{g} = \Phi^* g$.

Now let (Φ_k) be a sequence of smooth area-constrained Willmore immersions of $\mathbb{S}^2$ into (M, g) satisfying (7)–(8); perform the rescaling procedure described above, and obtain the immersions $(\widetilde{\Phi}_k)$ of $\mathbb{S}^2$ into $(\mathbb{R}^3, g_{\varepsilon_k})$ (for simplicity denoted again with Φ_k from now on), where g_{ε_k} is defined in (9), satisfying (10). Since the Willmore functional is scale invariant, the rescaled surfaces are still area-constrained Willmore surfaces, so they satisfy the equation

$$\triangle_{\bar{g}_{\varepsilon_k}} H_{g_{\varepsilon_k},\Phi_k} + H_{g_{\varepsilon_k},\Phi_k} |\mathring{A}_{g_{\varepsilon_k},\Phi_k}|^2_{\bar{g}_{\varepsilon_k}} + H_{g_{\varepsilon_k},\Phi_k} \operatorname{Ric}_{g_{\varepsilon_k}}(\vec{n}_{g_{\varepsilon_k},\Phi_k}, \vec{n}_{g_{\varepsilon_k},\Phi_k}) = \lambda_k H_{g_{\varepsilon_k},\Phi_k}. \tag{18}$$

The first step in our arguments is to show that the Lagrange multipliers λ_k are controlled by ε_k^2. Let us mention that this was already proved in [Lamm and Metzger 2013], the idea being to use the invariance under rescaling of the Willmore functional. Here we slightly modify the proof in [Lamm and Metzger 2013] by exploiting the divergence structure of the Willmore equation in Riemannian manifolds discovered in [Mondino and Rivière 2013] (let us stress that the divergence structure of the Willmore equation in euclidean setting was a breakthrough by Rivière [2008]).

Lemma 2.2. *Let (Φ_k) be a sequence of smooth area-constrained Willmore immersions of $\mathbb{S}^2$ into $(\mathbb{R}^3, g_{\varepsilon_k})$, where g_{ε_k} has the form (11) with $\varepsilon_k \to 0$ and $\Phi_k(\mathbb{S}^2) \subset B_{g_0}(0, 2)$, the euclidean ball of center 0 and radius 2.*

Then the Lagrange multipliers λ_k appearing in (18) satisfy

$$\sup_{k\in\mathbb{N}} \frac{|\lambda_k|}{\varepsilon_k^2} < \infty. \tag{19}$$

Proof. Since (Φ_k) are area-constrained Willmore immersions, for every variation vector field $\vec{X}$ on $\mathbb{R}^3$, we have that

$$\delta_{\vec{X}} W_{g_{\varepsilon_k}}(\Phi_k) = \lambda_k \delta_{\vec{X}} \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k), \tag{20}$$

where $\delta_{\vec{X}} W$ and $\delta_{\vec{X}} \operatorname{Area}$ are the first variations of the Willmore and the Area functionals corresponding to the vector field $\vec{X}$. Observe that the vector field corresponding to the dilations in $\mathbb{R}^3$ is the position vector field $\vec{x}$, so the first variation of the euclidean Willmore functional in $\mathbb{R}^3$ with respect to $\vec{x}$ is null: $\delta_{\vec{x}} W_{g_0} = 0$; on the other hand, the first variation of euclidean area with respect to the $\vec{x}$ variation is easy to compute using the tangential divergence formula:

$$\delta_{\vec{x}} \operatorname{Area}_{g_0}(\Phi) = -2 \int_{\mathbb{S}^2} \langle \vec{H}, \vec{x} \rangle_{g_0} \, d\mathrm{vol}_{\bar{g}_0} = \int_{\mathbb{S}^2} \operatorname{div}_{\Phi,g_0} \vec{x} \, d\mathrm{vol}_{\bar{g}_0} = 2 \operatorname{Area}_{g_0}(\Phi),$$

where $\operatorname{div}_{\Phi,g_0}$ is the tangential divergence on $\Phi(\mathbb{S}^2)$ with respect to the euclidean metric. The two euclidean formulas give the well known fact that every area-constraint Willmore surface is actually a Willmore surface.

In the present framework, the ambient metric g_{ε_k} is a perturbation of order ε_k^2 of the euclidean metric g_0, so it is natural to expect that the Lagrange multiplier maybe does not vanish but at least is of order ε_k^2. Let us prove it. First of all, by the expansion of the Christoffel symbols (14), it follows that the covariant derivative in metric g_{ε_k} of the position vector field $\vec{x}$ has the form

$$\nabla^{g_{\varepsilon_k}} \vec{x} = \mathrm{Id} + O(\varepsilon_k^2). \tag{21}$$

It follows that the tangential divergence of $\vec{x}$ on $\Phi_k(\mathbb{S}^2)$ with respect to the metric $\bar{g}_k$ is $\operatorname{div}_{\Phi, g_{\varepsilon_k}} \vec{x} = 2 + O(\varepsilon_k^2)$, and by the tangential divergence formula, we obtain as before

$$\delta_{\vec{x}} \operatorname{Area}_{g_{\varepsilon_k}}(\Phi) = -2 \int_{\mathbb{S}^2} \langle \vec{H}_{\Phi_k, g_{\varepsilon_k}}, \vec{x} \rangle_{g_{\varepsilon_k}} \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = \int_{\mathbb{S}^2} \operatorname{div}_{\Phi_k, g_{\varepsilon_k}} \vec{x} \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = [2 + O(\varepsilon_k^2)] \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k);$$

recalling the uniform area bound given in (16), we get that there exists $C > 0$ such that

$$0 \leq \frac{1}{C} \leq \delta_{\vec{x}} \operatorname{Area}_{g_{\varepsilon_k}}(\Phi) \leq C < \infty. \tag{22}$$

Now let us compute the variation of the Willmore functional with respect to the variation $\vec{x}$:

$$\delta_{\vec{x}} W_{g_{\varepsilon_k}}(\Phi_k) = \int_{\mathbb{S}^2} \langle \vec{x}, \vec{n} \rangle_{g_{\varepsilon_k}} (\triangle_{\bar{g}_{\varepsilon_k}} H + H |A^\circ|^2 + H \operatorname{Ric}(\vec{n}, \vec{n})) \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}, \tag{23}$$

where of course all the quantities are computed on Φ_k and with respect to the metric g_{ε_k}. In order to continue the computations, it is useful to rewrite the first variation of W in divergence form. Up to a reparametrization, we can assume that Φ_k are conformal so that the following identity holds (see Theorem 2.1 in [Mondino and Rivière 2013]):

$$[\triangle_{\bar{g}_{\varepsilon_k}} H \vec{n} + \vec{H} |A^\circ|^2 - R^\perp_\Phi(T\Phi)] \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = D^*\big[\nabla H \vec{n} - \tfrac{1}{2} H D\vec{n} + \tfrac{1}{2} H \star_{g_{\varepsilon_k}} (\vec{n} \wedge D^\perp \vec{n})\big], \tag{24}$$

where $\vec{H} = H\vec{n}$ is the mean curvature vector of the immersion Φ_k, $\star_{g_{\varepsilon_k}}$ is the Hodge operator associated to metric g_{ε_k}, $D \cdot := (\nabla_{\partial_{x_1}\Phi_k} \cdot, \nabla_{\partial_{x_2}\Phi_k} \cdot)$ and $D^\perp \cdot := (-\nabla_{\partial_{x_2}\Phi_k} \cdot, \nabla_{\partial_{x_1}\Phi_k} \cdot)$, and D^* is an operator acting on couples of vector fields $(\vec{V}_1, \vec{V}_2)$ along $(\Phi_k)_*(T\mathbb{S}^2)$ defined as

$$D^*(\vec{V}_1, \vec{V}_2) := \nabla_{\partial_{x_1}\Phi_k} \vec{V}_1 + \nabla_{\partial_{x_2}\Phi_k} \vec{V}_2.$$

Finally $R^\perp_{\Phi_k}(T\Phi_k) := (\operatorname{Riem}(\vec{e}_1, \vec{e}_2)\vec{H})^\perp = \star_{g_{\varepsilon_k}} (\vec{n} \wedge \operatorname{Riem}^h(\vec{e}_1, \vec{e}_2)\vec{H})$, where $\vec{e}_i = \partial_{x_i}\Phi / |\partial_{x_i}\Phi|$ for $i = 1, 2$.

Plugging (24) into (23) and integrating by parts, we obtain

$$\begin{aligned} \delta_{\vec{x}} W_{g_{\varepsilon_k}}(\Phi_k) = \int_{\mathbb{S}^2} \big\langle -D\vec{x}, \nabla H \vec{n} - \tfrac{1}{2} H D\vec{n} + \tfrac{1}{2} H \star_{g_{\varepsilon_k}} (\vec{n} \wedge D^\perp \vec{n}) \big\rangle_{g_{\varepsilon_k}} \, d\mathrm{vol}_{\mathbb{S}^2} \\ + \int_{\mathbb{S}^2} \langle \vec{x}, R^\perp_\Phi(T\Phi_k) + \vec{H} \operatorname{Ric}(\vec{n}, \vec{n}) \rangle_{g_{\varepsilon_k}} \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}. \end{aligned} \tag{25}$$

Since the Riemannian curvature tensor of the metric g_{ε_k} is of order $O(\varepsilon_k^2)$ and both the curvature terms are linear in H, using Schwartz inequality, the integral in the second line can be estimated as

$$\int_{\mathbb{S}^2} \langle \vec{x}, R^\perp_{\Phi_k}(T\Phi_k) + \vec{H} \operatorname{Ric}(\vec{n}, \vec{n}) \rangle_{g_{\varepsilon_k}} \, d\mathrm{vol}_{\bar{g}_{\varepsilon_k}} = O(\varepsilon_k^2)(W_{g_{\varepsilon_k}}(\Phi_k) \operatorname{Area}_{g_{\varepsilon_k}}(\Phi_k))^{1/2} = O(\varepsilon_k^2). \tag{26}$$

The first line of the right hand side of (23) can be written explicitly as

$$\begin{aligned} &\int_{\mathbb{S}^2} \big\langle -\partial_{x^1}\Phi_k - \vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta} (\partial_{x^1}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^1} H)\vec{n} + \tfrac{1}{2} H A_1^j (\partial_{x^j}\Phi_k) + \tfrac{1}{2} H A_2^j \star_{g_{\varepsilon_k}} (\vec{n} \wedge \partial_{x^j}\Phi_k) \big\rangle_{g_{\varepsilon_k}} \, d\mathrm{vol}_{\mathbb{S}^2} \\ &+ \int_{\mathbb{S}^2} \big\langle -\partial_{x^2}\Phi_k - \vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta} (\partial_{x^2}\Phi_k^\alpha)\Phi^\beta, (\partial_{x^2} H)\vec{n} + \tfrac{1}{2} H A_2^j (\partial_{x^j}\Phi_k) - \tfrac{1}{2} H A_1^j \star_{g_{\varepsilon_k}} (\vec{n} \wedge \partial_{x^j}\Phi_k) \big\rangle_{g_{\varepsilon_k}} \, d\mathrm{vol}_{\mathbb{S}^2}. \end{aligned} \tag{27}$$

Recalling that $\star_{g_{\varepsilon_k}}(\vec{n}\wedge\partial_{x^1}\Phi_k)=\partial_{x^2}\Phi_k$ and $\star_{g_{\varepsilon_k}}(\vec{n}\wedge\partial_{x^2}\Phi_k)=-\partial_{x^1}\Phi_k$, we obtain that all terms obtained doing the scalar product with $-\partial_{x^1}\Phi_k$ in the first line and with $-\partial_{x^2}\Phi_k$ in the second line simplify and just the terms containing the Christoffel symbols remain; since $\Phi_k\subset B_{\gamma_{\varepsilon_k}}(0,2)$ and the Christoffel symbols are of order $O(\varepsilon_k^2)$ by (14), (27) can be written as

$$\int_{\mathbb{S}^2}-\sum_{i=1}^{2}\langle\vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta}(\partial_{x^i}\Phi_k^\alpha)\Phi^\beta,(\partial_{x^i}H)\vec{n}\rangle\,d\mathrm{vol}_{\mathbb{S}^2}+O(\varepsilon_k^2)\int_{\mathbb{S}^2}|H_{\Phi_k,g_{\varepsilon_k}}||A_{\Phi_k,g_{\varepsilon_k}}|\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}};\tag{28}$$

using Schwartz inequality, of course, the second summand can be bounded by

$$O(\varepsilon_k^2)\left(\int_{\mathbb{S}^2}|H_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\right)^{1/2}\left(\int_{\mathbb{S}^2}|A_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\right)^{1/2}=O(\varepsilon_k^2),\tag{29}$$

where we used the Gauss equations, Gauss–Bonnet theorem, and area bound (16) to infer that

$$\int_{\mathbb{S}^2}|A_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\le C(W_{g_{\varepsilon_k}}(\Phi_k)+1)\le C_1.$$

In order to estimate the first integral of (28), we integrate by parts the derivative on H and we recall (14), obtaining

$$\begin{aligned}\int_{\mathbb{S}^2}-\sum_{i=1}^{2}\langle\vec{\Gamma}^{g_{\varepsilon_k}}_{\alpha\beta}(\partial_{x^i}\Phi_k^\alpha)\Phi^\beta,(\partial_{x^i}H)\vec{n}\rangle\,d\mathrm{vol}_{\mathbb{S}^2}&=O(\varepsilon_k^2)\int_{\mathbb{S}^2}(|H_{\Phi_k,g_{\varepsilon_k}}|+|H_{\Phi_k,g_{\varepsilon_k}}||A_{\Phi_k,g_{\varepsilon_k}}|)\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\\&=O(\varepsilon_k^2)(W_{g_{\varepsilon_k}}(\Phi_k))^{1/2}\left[(\mathrm{Area}_{g_{\varepsilon_k}}(\Phi_k))^{1/2}+\left(\int_{\mathbb{S}^2}|A_{\Phi_k,g_{\varepsilon_k}}|^2\,d\mathrm{vol}_{\bar{g}_{\varepsilon_k}}\right)^{1/2}\right]=O(\varepsilon_k^2).\end{aligned}\tag{30}$$

Collecting (25)–(30), we obtain that

$$\delta_{\vec{x}}W_{g_{\varepsilon_k}}(\Phi_k)=O(\varepsilon_k^2).$$

Combining the last equation with (22) and (20), we obtain that $\lambda_k=O(\varepsilon_k^2)$ as desired. □

3. The blow-up analysis and the proof of the main theorem

3A. *Existence of just one bubble and convergence.*

Lemma 3.1. *Let g_{ε_k} be the metrics on $\mathbb{R}^3$ defined in (9) having the expression (11), and let (Φ_k) be area-constrained Willmore immersions of $\mathbb{S}^2$ into $(\mathbb{R}^3,g_{\varepsilon_k})$ satisfying (10); without loss of generality, we can assume Φ_k to be conformal with respect to the euclidean metric g_0. Up to a rotation in the domain, we can also assume that, for every $k\in\mathbb{N}$, the north pole $N\in\mathbb{S}^2$ is the maximum point of the quantity $|\nabla\Phi_k|^2+|\nabla^2\Phi_k|$:*

$$\mu_k:=|\nabla\Phi_k|_h^2(N)+|\nabla^2\Phi_k|_h(N)=\max_{\mathbb{S}^2}|\nabla\Phi_k|_h^2+|\nabla^2\Phi_k|_h,$$

where h is the standard round metric of $\mathbb{S}^2$ of constant Gauss curvature equal to 1 and $|\nabla\Phi_k|_h$ and $|\nabla^2\Phi_k|_h$ are the norms evaluated in the h metric.

With $S \in \mathbb{S}^2$ the south pole and $P : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^2$ the stereographic projection, consider the new parametrizations $\widetilde{\Phi}_k$, in the following simply denoted with Φ_k, defined by

$$\widetilde{\Phi}_k(P^{-1}(z)) := \Phi_k\left(P^{-1}\left(\frac{z}{\mu_k^{1/2}}\right)\right) \quad \textit{for all } z \in \mathbb{R}^2.$$

Then $\widetilde{\Phi}_k$, a priori just defined on $\mathbb{S}^2 \setminus \{S\}$, extend to smooth conformal immersions of $\mathbb{S}^2$ into $(\mathbb{R}^3, g_0)$ and converge to a conformal parametrization of a round sphere in the $C^l(\mathbb{S}^2, h)$-norm for every $l \in \mathbb{N}$.

Proof. Step a. There exists a smooth conformal parametrization $\Phi_\infty : \mathbb{S}^2 \to (\mathbb{R}^3, g_0)$ of a round sphere in $\mathbb{R}^3$ endowed with the euclidean metric g_0 such that, up to subsequences, $\widetilde{\Phi}_k \to \Phi_\infty$ in the $C^l_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm for every $l \in \mathbb{N}$.

Denote by u_k the conformal factor associated to $\widetilde{\Phi}_k$, i.e.,

$$\widetilde{\Phi}_k^*(g_0) = e^{2u_k} h,$$

where g_0 is the euclidean metric in $\mathbb{R}^3$. Observe that, by construction, for any compact subset of the form

$$K := \mathbb{S}^2 \setminus B^h_\delta(S) \quad \text{for some } \delta > 0,$$

there holds

$$\sup_{k\in\mathbb{N}} \sup_K (|\nabla \widetilde{\Phi}_k|_h^2 + |\nabla^2 \widetilde{\Phi}_k|_h) < \infty. \tag{31}$$

Then for every compact subset, there exists a constant C_K depending just on K such that, for every $x_0 \in K$ and every $\rho \in (0, \mathrm{dist}(K, S)/2)$,

$$\sup_{k\in\mathbb{N}} \sup_{B^h_\rho(x_0)} |\nabla^2 \widetilde{\Phi}_k|^2 \le C_K,$$

where $B^h_\rho(x_0)$ is the ball of center x_0 and radius ρ in the metric h. By the conformal invariance of the Dirichlet energy, with $\pi_{\tilde{\vec{n}}_k}$ the projection on the normal space to $\widetilde{\Phi}_k$, we infer that for every $\varepsilon_0 > 0$ there exists $\rho_{\varepsilon_0,K} > 0$ (small enough) depending just on K and on ε_0 but not on $k \in \mathbb{N}$ such that, for every $\rho \in (0, \rho_{\varepsilon_0,K})$ and $x_0 \in K$,

$$\begin{aligned}\int_{B^h_\rho(x_0)} |\nabla \tilde{\vec{n}}_k|^2_{\widetilde{\Phi}_k^*(g_0)} \, d\mathrm{vol}_{\widetilde{\Phi}_k^*(g_0)} &= \int_{B^h_\rho(x_0)} |\nabla \tilde{\vec{n}}_k|^2_h \, d\mathrm{vol}_h = \int_{B^h_\rho(x_0)} |\pi_{\tilde{\vec{n}}_k}(\nabla^2 \widetilde{\Phi}_k)|^2_h \, d\mathrm{vol}_h \\ &\le \int_{B^h_\rho(x_0)} |\nabla^2 \widetilde{\Phi}_k|^2_h \, d\mathrm{vol}_h \le C_K \rho^2 \le \varepsilon_0. \end{aligned} \tag{32}$$

Taking $\varepsilon_0 \le \frac{8}{3}\pi$, for any $x_0 \in K$ and $\rho < \rho_{\varepsilon_0,K}$, we can apply the Hélein moving frame method based on Chern construction of conformal coordinates (for more details, see [Rivière 2013, Section 3]) and infer that, up to a reparametrization of $\widetilde{\Phi}_k$ on $B_\rho(x_0)$, with $\bar{u}_k$ the mean value of u_k on $B^h_\rho(x_0)$,

$$\|u_k - \bar{u}_k\|_{L^\infty(B^h_\rho(x_0))} \le \widetilde{C}$$

for some $\widetilde{C} > 0$ independent of $k \in \mathbb{N}$. Covering K by finitely many balls as above, the connectedness of K implies that any two balls of the finite covering are connected by a chain of balls of the same covering and therefore there exists constants $c_{k,K} \in \mathbb{R}$ and $k \in \mathbb{N}$ such that

$$\sup_{k\in\mathbb{N}} \|u_k - c_{k,K}\|_{L^\infty(K)} < \infty. \tag{33}$$

Observe that $\sup_{k\in\mathbb{N}} c_{k,K} < +\infty$; indeed, if $\limsup_k c_{k,K} = +\infty$, then $\limsup_k \operatorname{Area}(\widetilde{\Phi}_k(K)) = +\infty$, contradicting the area bound (16) (here we use that K has positive h-volume). Now let us consider separately the cases $\sup_k |c_{k,K}| < \infty$ and $\liminf_k c_{k,K} = -\infty$.

Case 1: $\sup_k |c_{k,K}| < \infty$. Estimate (33) yields a uniform bound on the conformal factors u_k on the subset K. Since by assumption the immersions $\widetilde{\Phi}_k$ are area-constrained Willmore immersions satisfying (32) with arbitrarily small Lagrange multipliers thanks to Lemma 2.2, then by ε-regularity,[2] we infer that for every $l \in \mathbb{N}$ there exists C_l such that

$$|e^{-lu_k}\nabla^l \widetilde{\Phi}_k|_{L^\infty(B^h_{\rho/2}(x_0))} \le C_l \left(\int_{B^h_\rho(x_0)} |\nabla \tilde{\vec{n}}_k|^2_h \, d\mathrm{vol}_h + 1 \right)^{1/2} \le \widehat{C}_l,$$

and therefore, by the assumed uniform bound on $|u_k|$ and by covering K by finitely many balls, we get

$$\sup_{k\in\mathbb{N}} |\nabla^l \widetilde{\Phi}_k|_{L^\infty(K)} < \infty \quad \text{for all } l \in \mathbb{N}. \tag{34}$$

By the Arzelà–Ascoli theorem and by the estimate on the Lagrange multipliers given in Lemma 2.2, up to subsequences, the maps $\widetilde{\Phi}_k$ converge in the $C^l(K)$-norm, for every $l \in \mathbb{N}$, to a limit Willmore immersion $\widetilde{\Phi}_\infty$ of K into $(\mathbb{R}^3, g_0)$; repeating the above argument to $K = \mathbb{S}^2 \setminus B^h_\delta(S)$, for every $\delta > 0$, we get that, up to subsequences, the maps $\widetilde{\Phi}_k$ converge in the $C^l_{\mathrm{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm, for every $l \in \mathbb{N}$, to a limit Willmore immersion $\Phi_\infty : \mathbb{S}^2 \setminus \{S\} \to \mathbb{R}^3$, a smooth Willmore conformal immersion with finite area and L^2-bounded second fundamental form; therefore, by Lemma A.5 in [Rivière 2014] (let us mention that this result was already present in [Müller and Šverák 1995]; see also [Kuwert and Li 2012]), the map Φ_∞ can be extended up to the south pole S to a possibly branched immersion; i.e., the south pole S is a possible branch point for Φ_∞ and the following expansion around S holds:

$$(C - o(1))|z|^{n-1} \le \left| \frac{\partial \Phi_\infty}{\partial z} \right| \le (C + o(1))|z|^{n-1}, \tag{35}$$

where z is a complex coordinate around the south pole and $n-1$ is the branching order. We claim that the branching order is 0 or in other words that Φ_∞ is unbranched; indeed, by the strong convergence of $\widetilde{\Phi}_k$ to Φ_∞ and the smooth convergence of g_{ϵ_k} to the euclidean metric g_0, we have that

$$W_{g_0}(\Phi_\infty) \le \liminf_k W_{g_{\epsilon_k}}(\widetilde{\Phi}_k) < 8\pi; \tag{36}$$

[2] Note that ε-regularity for Willmore immersions was first proved by Kuwert and Schätzle [2001]. Here we use the ε-regularity theorem proved by Rivière (see Theorem I.5 in [Rivière 2008]; see also Theorem I.1 in [Bernard and Rivière 2014]); to this aim, observe that the ε-regularity theorem was stated for *Willmore immersions*, but the proof can be repeated verbatim to *area-constrained Willmore immersions in metric* g_{ϵ_k}: indeed the Lagrange multiplier $\lambda\vec{H}$ and the Riemannian terms are lower-order terms that can be absorbed in the already present error terms $\vec{g}_1$ and $\vec{g}_2$ in the proof of Theorem I.5 at pp. 24–26 in [Rivière 2008]. Of course, ε-regularity is a consequence of the ellipticity of the equation.

therefore, by the Li–Yau inequality [1982], we get that $n-1=0$, i.e., Φ_∞ is an immersion also at the south pole S. Since Φ_∞ is a smooth Willmore immersion of $\mathbb{S}^2$ into $\mathbb{R}^3$ with energy less than 8π, by the classification of Willmore spheres by Bryant [1984], Φ_∞ is a smooth conformal parametrization of a round sphere in $\mathbb{R}^3$.

Case 2: $\liminf_k c_{k,K} = -\infty$. This cannot happen. In this case, up to subsequences, we have that $\widetilde{\Phi}_k(K) \to \bar{x} \in M$ in Hausdorff distance sense. Consider then the rescaled immersions

$$\widehat{\Phi}_k := e^{-c_{k,K}}\,\widetilde{\Phi}_k \tag{37}$$

of K, and observe that by construction $\sup_k |\hat{u}_{k,K}| < \infty$, where $\hat{u}_{k,K}$ is the conformal factor of $\widehat{\Phi}_k$. Moreover, since the integrals appearing in (32) are invariant under rescaling, estimate (32) holds for $\widehat{\Phi}_k$ as well. Therefore, up to a diagonal extraction, $\widehat{\Phi}_k \to \Phi_\infty$ in the $C^l_{\text{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm. In particular, $\widetilde{\Phi}_k \to 0$ in the $C^2_{\text{loc}}(\mathbb{S}^2 \setminus \{S\})$-norm, which contradicts the fact that

$$|\nabla \widetilde{\Phi}_k|^2_h(N) + |\nabla^2 \widetilde{\Phi}_k|_h(N) = 1.$$

Step b. $\widetilde{\Phi}_k \to \Phi_\infty$ in $C^l(\mathbb{S}^2)$ for every $l \in \mathbb{N}$; namely, the convergence of *Step a* is *on the whole* $\mathbb{S}^2$.

Observe that, if there exists $\bar{\rho} > 0$ such that $\sup_k \sup_{B^h_{\bar{\rho}}(S)} |\nabla \widetilde{\Phi}_k|^2 + |\nabla^2 \widetilde{\Phi}_k| < \infty$, then in *Step a*, we can choose as compact subset K the whole $\mathbb{S}^2$ and the claim of *Step b* follows by the same arguments as *Step a*. So assume by contradiction that there exists a sequence $\rho_k \downarrow 0$ such that, for

$$\bar{\mu}_k := \sup_{B^h_{\rho_k}(\bar{x})} |\nabla \widetilde{\Phi}_k|^2 + |\nabla^2 \widetilde{\Phi}_k|,$$

one has

$$\limsup_k \bar{\mu}_k = +\infty.$$

By a small rotation in the domain $\mathbb{S}^2$, we can assume that, for every $k \in \mathbb{N}$, the maximum of $|\nabla \widetilde{\Phi}_k|^2 + |\nabla^2 \widetilde{\Phi}_k|$ on $B^h_{\rho_k}(S)$ is attained at the south pole S and that, up to subsequences in k,

$$\lim_k \bar{\mu}_k := \lim_k |\nabla \widetilde{\Phi}_k|^2(S) + |\nabla^2 \widetilde{\Phi}_k|(S) = +\infty. \tag{38}$$

Analogously to the above, with $P_N : \mathbb{S}^2 \setminus \{N\} \to \mathbb{R}^2$ the stereographic projection centered at the north pole N, we consider the reparametrized immersions

$$\overline{\Phi}_k(P_N^{-1}(z)) := \widetilde{\Phi}_k\left(P_N^{-1}\left(\frac{z}{\bar{\mu}_k^{1/2}}\right)\right).$$

Observe that, in this way, the compact subsets K considered above are shrinking towards the north pole N and, by the arguments above, their $\overline{\Phi}_k$-images are converging to a round sphere; repeating the arguments above to compact subsets this time containing the south pole S and avoiding the north pole N, we infer that, up to subsequences, $\overline{\Phi}_k$ (or a further rescaling of it) converges smoothly, away the north pole N, to a round sphere, namely a second bubble. Combining the bubble formed in *Step a* and this second bubble, since each bubble contributes 4π of Willmore energy, we infer that

$$\limsup_k W_{g_{\epsilon_k}}(\Phi_k) \geq 8\pi, \tag{39}$$

contradicting the assumption (10). This concludes the proof of the *Step b* and of the lemma. □

3B. *Expansion of the equation.* Recalling that $\Phi_k : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_{\varepsilon_k})$ is a smooth immersion satisfying the area-constrained Willmore equation in metric g_{ε_k} and that g_{ε_k} smoothly converge to the euclidean metric g_0, in the present section, we expand this differential equation with respect to ε_k. Without loss of generality, we can assume that Φ_k is conformal with respect to the metric g_{ε_k}. We will see that curvature terms appear at ε_k^2 order while the derivatives of the curvature appear at ε_k^3 order.

From now on, in order to make the notation a bit lighter, we replace ε_k by ε.

Recall that the area-constrained Willmore equation in metric g_ε has the form

$$\triangle_{\bar{g}_\varepsilon} H_\varepsilon + H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon} + \mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) H_\varepsilon = \lambda_\varepsilon H_\varepsilon. \tag{40}$$

Since $\triangle_{\bar{g}_\varepsilon} = (2/|\nabla\Phi_\varepsilon|^2_{g_\varepsilon})\Delta$, where Δ is the flat laplacian in $\mathbb{R}^2$, multiplying (40) by $|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}/2$, we get

$$\Delta H_\varepsilon + \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon} + \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon \,\mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) = \tfrac{1}{2}\lambda_\varepsilon |\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon. \tag{41}$$

First of all, recalling that $H_\varepsilon = g_\varepsilon(\triangle_{\bar{g}_\varepsilon}\Phi_\varepsilon, \vec{n}_\varepsilon)/2$, we expand H_ε as

$$H_\varepsilon = \frac{1}{|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}} (g_\varepsilon)_{\alpha\beta} \triangle\Phi^\alpha_\varepsilon \sqrt{|g_\varepsilon|} g^{\beta\gamma}_\varepsilon (\vec{\nu}_\varepsilon)_\gamma = \frac{\sqrt{|g_\varepsilon|}}{|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}} \triangle\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}, \tag{42}$$

where $\vec{\nu}_\varepsilon$ is the inward-pointing unit normal with respect to g_0. Using (11) and (13), we get

$$|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} = |\nabla\Phi_\varepsilon|^2 + \tfrac{1}{3}\varepsilon^2 R_{\alpha\beta\gamma\eta}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon \langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle + \tfrac{1}{6}\varepsilon^3 R_{\alpha\beta\gamma\eta,\mu}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\Phi^\mu_\varepsilon \langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle + O(\varepsilon^4)$$

so that

$$\frac{1}{|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}} = \frac{1}{|\nabla\Phi_\varepsilon|^2}\Big(1 - \frac{\varepsilon^2}{3|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle - \frac{\varepsilon^3}{6|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta,\mu}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\Phi^\mu_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle + O(\varepsilon^4)\Big); \tag{43}$$

moreover,

$$\sqrt{|g_\varepsilon|} = 1 - \tfrac{1}{6}\varepsilon^2 \mathrm{Ric}_{\alpha\beta}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon - \tfrac{1}{6}\varepsilon^3 \mathrm{Ric}_{\alpha\beta,\gamma}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon + O(\varepsilon^4). \tag{44}$$

Combining (42) with (43) and (44), we can write

$$H_\varepsilon = \frac{\triangle\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}(1 + \varepsilon^2 S_\varepsilon + \varepsilon^3 T_\varepsilon + O(\varepsilon^4)), \tag{45}$$

where

$$S_\varepsilon := -\frac{1}{3|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle - \tfrac{1}{6}\mathrm{Ric}_{\alpha\beta}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon$$

and

$$T_\varepsilon := -\frac{1}{6|\nabla\Phi_\varepsilon|^2} R_{\alpha\beta\gamma\eta,\mu}(p_k)\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon\Phi^\mu_\varepsilon\langle\nabla\Phi^\alpha_\varepsilon, \nabla\Phi^\eta_\varepsilon\rangle - \tfrac{1}{6}\mathrm{Ric}_{\alpha\beta,\gamma}(p_k)\Phi^\alpha_\varepsilon\Phi^\beta_\varepsilon\Phi^\gamma_\varepsilon.$$

The combination of (44) and (45) gives

$$\mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon)H_\varepsilon = \varepsilon^2 \frac{\Delta\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\,\mathrm{Ric}_g(p_k)(\vec{\nu}_\varepsilon, \vec{\nu}_\varepsilon) + O(\varepsilon^4). \tag{46}$$

Finally, using (45), (46), and (19), we expand (41) up to ε^2 order (the term $H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon}$ will be expanded in the next subsection) as

$$\begin{aligned}\Delta H_\varepsilon &+ \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon} + \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon\,\mathrm{Ric}_{g_\varepsilon}(\vec{n}_\varepsilon, \vec{n}_\varepsilon) - \lambda_\varepsilon H_\varepsilon \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon}\\
&= \Delta\Big(\frac{\Delta\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Big) + \varepsilon^2\Big(\Delta\Big(\frac{\Delta\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Big)S_\varepsilon + 2\Big\langle\nabla\Big(\frac{\Delta\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Big), \nabla S_\varepsilon\Big\rangle + \frac{\Delta\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}}{|\nabla\Phi_\varepsilon|^2}\Delta S_\varepsilon\Big)\\
&\quad + \tfrac{1}{2}|\nabla\Phi_\varepsilon|^2_{g_\varepsilon} H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon} + \tfrac{1}{2}\varepsilon^2 \Delta\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha}\,\mathrm{Ric}_g(p)(\vec{\nu}_\varepsilon, \vec{\nu}_\varepsilon) - \tfrac{1}{2}\lambda_\varepsilon \Delta\Phi^\alpha_\varepsilon \vec{\nu}_{\varepsilon\alpha} + o(\varepsilon^2).\end{aligned} \tag{47}$$

3C. ***Approximated solutions to the area-constrained Willmore equation.*** In this section, we solve (47) up to the ε^2 order. For this, let ω be the inverse of the stereographic projection with respect to the north pole and notice that ω is a solution of the equation when $\varepsilon = 0$. We make the ansatz of looking for a solution up to the order ε^2 of the form $\omega + \varepsilon^2\rho$ for some function ρ. Since $|A^\circ|^2 = 0$ for ω, it is clear that

$$H_\varepsilon |A^\circ_\varepsilon|^2_{\bar{g}_\varepsilon} = O(\varepsilon^4); \tag{48}$$

in particular, since for our arguments it is enough to expand the equation up to ε^3 order, this term will never play a role and therefore will be neglected.

Observing that $\Delta\omega^\alpha\omega_\alpha/|\nabla\omega|^2 \equiv -1$, (47) implies that ρ must solve

$$\begin{aligned}L_\omega(\rho) = \Delta\Big(\frac{1}{3|\nabla\omega|^2} R_{\alpha\beta\gamma\mu}(p_k)\omega^\beta\omega^\gamma\langle\nabla\omega^\alpha, \nabla\omega^\mu\rangle + \tfrac{1}{6}\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta\Big)&\\
- \tfrac{1}{2}|\nabla\omega|^2\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta + \frac{\lambda_\varepsilon}{2\varepsilon^2}|\nabla\omega|^2,&\end{aligned} \tag{49}$$

where L_ω is the linearized Willmore operator at ω; see the Appendix for more details. Using the identity

$$\langle\nabla\omega^\alpha, \nabla\omega^\beta\rangle = (\delta_{\alpha\beta} - \omega^\alpha\omega^\beta)\tfrac{1}{2}|\nabla\omega|^2, \tag{50}$$

(49) reduces to

$$\begin{aligned}L_\omega(\rho) &= \tfrac{1}{3}\Delta(\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta) - \tfrac{1}{2}|\nabla\omega|^2\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta + \frac{\lambda_\varepsilon}{2\varepsilon^2}|\nabla\omega|^2\\
&= \Big(-\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\alpha\omega^\beta + \Big(\frac{\lambda_\varepsilon}{2\varepsilon^2} + \tfrac{1}{3}\,\mathrm{Scal}(p_k)\Big)\Big)|\nabla\omega|^2.\end{aligned} \tag{51}$$

Hence, we easily check that

$$\rho_\varepsilon = \tfrac{1}{3}\,\mathrm{Ric}_{\alpha\beta}(p_k)\omega^\beta + \frac{\lambda_\varepsilon}{\varepsilon^2} f(r)\omega \tag{52}$$

with

$$f(r) = \frac{r^2\ln(r^2/(1+r^2)) - 1 - \ln(1+r^2)}{1+r^2},$$

where $r^2 = x^2 + y^2$, is the desired function. Moreover, it is not difficult to check that this perturbed ω satisfies the conformal conditions up to ε^2 order, that is to say

$$\begin{cases} g_\varepsilon((\omega+\varepsilon^2\rho_\varepsilon)_x, (\omega+\varepsilon^2\rho_\varepsilon)_x) - g_\varepsilon((\omega+\varepsilon^2\rho_\varepsilon)_y, (\omega+\varepsilon^2\rho_\varepsilon)_y) = O(\varepsilon^3), \\ g_\varepsilon((\omega+\varepsilon^2\rho_\varepsilon)_x, (\omega+\varepsilon^2\rho_\varepsilon)_y) = O(\varepsilon^3); \end{cases} \tag{53}$$

a way to prove it is to use the expansion of the metric with the fact that in dimension 3 one has

$$R_{\alpha\beta\gamma\mu} = (g_{\alpha\gamma}\operatorname{Ric}_{\beta\mu} - g_{\alpha\mu}\operatorname{Ric}_{\beta\gamma} + g_{\beta\mu}\operatorname{Ric}_{\alpha\gamma} - g_{\beta\gamma}\operatorname{Ric}_{\alpha\mu}) + \tfrac{1}{2}\operatorname{Scal}(g_{\alpha\mu}g_{\beta\gamma} - g_{\alpha\gamma}g_{\beta\mu}).$$

3D. *Proof of Theorem 1.2.* Let us briefly recall the setting. Let $\Phi_k : \mathbb{S}^2 \hookrightarrow (M, g)$ be conformal Willmore immersions satisfying

$$\varepsilon := \operatorname{diam}_g(\Phi_k(\mathbb{S}^2)) \to 0, \tag{54}$$

$$W_g(\Phi_k) := \int_{\mathbb{S}^2} |H_{g,\Phi_k}|^2 \, d\mathrm{vol}_{\bar{g}_k} \le 8\pi - 2\delta \quad \text{for some } \delta > 0 \text{ independent of } k. \tag{55}$$

Thanks to Lemma 2.2, we associate to Φ_k the new immersion $\Phi^\varepsilon : \mathbb{S}^2 \hookrightarrow (\mathbb{R}^3, g_\varepsilon)$, where $g_\varepsilon(y)(u, v) := g(\varepsilon y)(\varepsilon^{-1}u, \varepsilon^{-1}v)$, which satisfies the area-constrained Willmore equation

$$\triangle_{\bar{g}_\varepsilon} H_{g_\varepsilon,\Phi^\varepsilon} + H_{g_\varepsilon,\Phi^\varepsilon}|A^\circ_{g_\varepsilon,\Phi^\varepsilon}|^2_{\bar{g}_\varepsilon} + H_{g_\varepsilon,\Phi^\varepsilon}\operatorname{Ric}_{g_\varepsilon}(\vec{n}_{g_\varepsilon,\Phi^\varepsilon}, \vec{n}_{g_\varepsilon,\Phi^\varepsilon}) = \lambda_\varepsilon H_{g_\varepsilon,\Phi^\varepsilon} \tag{56}$$

with $\lambda_\varepsilon = O(\varepsilon^2)$. Moreover, by Lemma 3.1, we know that, up to conformal reparametrizations and up to subsequences, we have

$$\Phi^\varepsilon \to \Phi \text{ in } C^2(\mathbb{S}^2),$$

where Φ is a conformal diffeomorphism of $\mathbb{S}^2$. Clearly, up to reparametrizing our sequence, we can assume that $\Phi = \mathrm{Id}$. In the following, we perform all the computations in the chart given by the stereographic projection (which is conformal); we denote by ω the inverse of the stereographic projection.

Before proceeding with the proof, we need to make a small adjustment to the immersions. We claim that there exist $a^\varepsilon \in \mathbb{R}^2$, $b^\varepsilon \in \mathbb{R}^2$, $R^\varepsilon \in \mathrm{SO}(3)$, and $z^\varepsilon \in \mathbb{C}$ satisfying

$$a^\varepsilon = o(1), \quad b^\varepsilon = o(1), \quad |\mathrm{Id} - R^\varepsilon| = o(1), \quad \text{and} \quad z^\varepsilon = o(1) \tag{57}$$

such that, up to replacing Φ^ε by $\Phi^\varepsilon(a^\varepsilon + z^\varepsilon\,\cdot\,)$ and $\Omega^\varepsilon = \omega^\varepsilon + \varepsilon^2\rho^\varepsilon$, where ρ^ε is given by (52), by $R^\varepsilon[\omega(\cdot + b^\varepsilon) + \varepsilon^2\rho^\varepsilon(\cdot + b_\varepsilon)]$, we get

$$|\nabla\Phi^\varepsilon| \text{ and } |\nabla\Omega^\varepsilon| \text{ are maximal at } 0, \quad \mathrm{Vect}\{\Phi^\varepsilon_x(0), \Phi^\varepsilon_y(0)\} = \mathrm{Vect}\{\Omega^\varepsilon_x(0), \Omega^\varepsilon_y(0)\}, \quad \text{and} \quad \Phi^\varepsilon_x(0) = \Omega^\varepsilon_x(0). \tag{58}$$

This is a simple consequence of the $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ convergence of Φ^ε to ω. Indeed, we first choose a^ε and b^ε such that $|\nabla\Phi^\varepsilon|$ and $|\nabla\Omega^\varepsilon|$ are maximal at 0 and then R^ε such that the tangent plane of Φ^ε and $R^\varepsilon\Omega^\varepsilon$ coincide at 0, and finally we find z_ε in order to adjust the first derivatives.

Therefore, from now on, we will assume that (58) is satisfied.

Now we prove Theorem 1.2. We set

$$\Phi^\varepsilon = \Omega^\varepsilon + r^\varepsilon$$

for some function r^ε, and thanks to the computations of Section 3C, we see that r^ε satisfies

$$L_\omega(r^\varepsilon) = O(\varepsilon^3) + o(|\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon|). \tag{59}$$

Moreover, combining (53) and (58), we get that

$$g^\varepsilon(\nabla r^\varepsilon, \nabla r^\varepsilon)(0) = O(\varepsilon^6). \tag{60}$$

Indeed, the error terms of $r^\varepsilon_x(0)$ and $r^\varepsilon_y(0)$ lie in the plane generated by $\Omega^\varepsilon_x(0)$ and $\Omega^\varepsilon_y(0)$. So it suffices to estimate their projection against $\Omega^\varepsilon_x(0)$ and $\Omega^\varepsilon_y(0)$. But this one vanishes up to the ε^3 order thanks to (53). Observe that we also have

$$g^\varepsilon(\nabla^2 r^\varepsilon, \nabla\omega^\varepsilon)(0) = O(\varepsilon^3). \tag{61}$$

Claim. $$\sup_{\mathbb{R}^2} |\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon| = O(\varepsilon^3).$$

Proof of the claim. Let us denote $\mu_\varepsilon := |\nabla r^\varepsilon| + |\nabla^2 r^\varepsilon| + |\nabla^3 r^\varepsilon| + |\nabla^4 r^\varepsilon|$, and assume by contradiction that $\lim \varepsilon^3/\mu_\varepsilon = 0$. Up to a reparametrization, we can assume that this sup is achieved at some point z_ε that is confined in a fixed compact subset of $\mathbb{R}^2$. In fact, we can do a reparametrization in order to make this requirement satisfied before performing the adjustments of the previous page. Then we set

$$\tilde r_\varepsilon = \frac{r_\varepsilon - r_\varepsilon(0)}{\mu^\varepsilon}.$$

By construction, $\tilde r^\varepsilon$ is bounded in the C^4-norm on every compact subset of $\mathbb{R}^2$, and therefore, by the Arzelà–Ascoli theorem, it converges up to subsequences to a limit function $\tilde r$ in C^3_{loc}-topology. Thanks to (59), $\tilde r$ is a solution of the linearized equation (A-1) and, recalling (60)-(61), satisfies (A-2) with $\nabla\tilde r(0) = 0$ and $\langle\nabla^2\tilde r, \nabla\omega\rangle(0) = 0$. Then, applying Lemma A.1, we get that $\nabla\tilde r \equiv 0$, which is in contradiction with the fact that $|\nabla\tilde r| + |\nabla^2\tilde r| + |\nabla^3\tilde r| + |\nabla^4\tilde r| = 1$ at some point at finite distance. This proves the claim. □

Mimicking the proof of the claim above, one can prove that by setting

$$\tilde r_\varepsilon = \frac{r_\varepsilon - r_\varepsilon(0)}{\varepsilon^3}$$

then, up to subsequences, $\tilde r_\varepsilon$ converges to a function $\tilde r$ in $C^3_{\mathrm{loc}}(\mathbb{R}^2)$ that, using (41), (45), and (46), satisfies the linearized Willmore equation

$$L_\omega(\tilde r) = \Delta\Big(\frac{1}{6|\nabla\omega|^2} R_{\alpha\beta\gamma\mu,\nu}(p_k)\omega^\beta\omega^\gamma\omega^\nu\langle\nabla\omega^\alpha, \nabla\omega^\mu\rangle + \tfrac{1}{6}\operatorname{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma\Big).$$

Recalling identity (50), the last equation can be rewritten as

$$L_\omega(\tilde r) = \Delta\big(\tfrac{1}{12}\operatorname{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma\big).$$

Finally, integrating this relation against the ω^α, for $\alpha = 1, \dots, 3$, which are solutions of the linearized equation, we get

$$\int_{\mathbb{R}^2} \Delta\omega\big(\tfrac{1}{12}\operatorname{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma\big)\,dz = 0.$$

Let us note that the integration by parts above has been possible thanks to the decay of ω and its derivatives at infinity. The last identity gives

$$\int_{\mathbb{R}^2} (\mathrm{Ric}_{\alpha\beta,\gamma}(p_k)\omega^\alpha\omega^\beta\omega^\gamma)\tfrac{1}{2}\omega|\nabla\omega|^2\,dz = 0.$$

Then by a change of variable, we get

$$\int_{\mathbb{S}^2} (\mathrm{Ric}_{\alpha\beta,\gamma}(p_k)(p_k)y^\alpha y^\beta y^\gamma)y\,d\mathrm{vol}_h = 0,$$

where h is the standard metric on $\mathbb{S}^2$ and y^α are the position coordinates of $\mathbb{S}^2$ in $\mathbb{R}^3$. Finally, using the relation

$$\int_{S^2} y^\alpha y^\beta y^\gamma y^\mu\,d\mathrm{vol}_h = \tfrac{4}{15}\pi(\delta^{\alpha\beta}\delta^{\mu\gamma}+\delta^{\alpha\mu}\delta^{\beta\gamma}+\delta^{\alpha\gamma}\delta^{\beta\mu})$$

and the second Bianchi identity, we obtain

$$\nabla\,\mathrm{Scal}(\bar{p}) = 0,$$

which proves the theorem. □

Appendix A: The linearized Willmore operator

The aim of this appendix is to derive the linearized Willmore equation and to classify its solution.

The Willmore equation for a conformal immersion Φ into $\mathbb{R}^3$ can be written as

$$W'(\Phi) = \Delta_{\bar{g}}(H) + H|A^\circ|^2_{\bar{g}} = 0,$$

where $\Delta_{\bar{g}} = (2/|\nabla\Phi|^2)\Delta$, H is the mean curvature, and A° is the traceless second fundamental form.

Equivalently, one has

$$H = \tfrac{1}{2}\langle\Delta_{\bar{g}}\Phi, \vec{\nu}\rangle,$$

where $\vec{\nu}$ is the inward-pointing unit normal of the immersion Φ. Hence, by multiplying the first equation by $|\nabla\Phi|^2/2$, we can consider the equivalent equation

$$\widetilde{W}'(\Phi) = \Delta H + \langle\Delta\Phi, \vec{\nu}\rangle\tfrac{1}{2}|A^\circ|^2_{\bar{g}} = 0.$$

Of course, any conformal parametrization, ω, of a round sphere is a solution. Then expanding $\widetilde{W}'(\omega+t\rho)$ for some function ρ and using the fact that $A^\circ \equiv 0$ for a round sphere, we get

$$L_\omega(\rho) := \delta\widetilde{W}_\omega(\rho) = -\Delta\Big(\frac{\langle\Delta\rho,\omega\rangle + 2\langle\nabla\omega,\nabla\rho\rangle}{|\nabla\omega|^2}\Big) = 0. \tag{A-1}$$

Also consider the linearization of the conformality condition, which gives

$$\begin{cases}\langle\omega_x,\rho_x\rangle - \langle\omega_y,\rho_y\rangle = 0,\\ \langle\omega_x,\rho_y\rangle + \langle\omega_y,\rho_x\rangle = 0.\end{cases} \tag{A-2}$$

In the following lemma, we classify the solutions of the linearized operator following the previous work [Laurain 2012] concerning the linearized operator for the constant mean curvature equation:[3]

Lemma A.1. *Let $\rho \in \mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)$ be a solution of the linearized equation (A-1) that satisfies (A-2) and the additional normalizing conditions*

$$\nabla\rho(0) = 0 \quad \textit{and} \quad \langle \nabla^2\rho, \nabla\omega\rangle(0) = 0.$$

Then $\nabla\rho \equiv 0$.

Proof. First we remark that, thanks to the definition of $\mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)$, we have

$$\frac{\langle \Delta\rho, \omega\rangle + 2\langle \nabla\omega, \nabla\rho\rangle}{|\nabla\omega|^2} \in L^2(\mathbb{R}^2).$$

Hence, using Liouville's theorem, we get that

$$\langle \Delta\rho, \omega\rangle + 2\langle \nabla\omega, \nabla\rho\rangle = 0. \tag{A-3}$$

Then thanks to the fact that $(\omega_x, \omega_y, \omega)$ is a basis of $\mathbb{R}^3$ and (A-2), there exist $a, b, c, d : \mathbb{R}^2 \to \mathbb{R}$ such that

$$\begin{cases} \rho_x = a\omega_x + b\omega_y + c\omega, \\ \rho_y = -b\omega_x + a\omega_y + d\omega. \end{cases} \tag{A-4}$$

Then plugging (A-4) into (A-3) and using the relation $\rho_{xy} = \rho_{yx}$, we see that a, b, c, and d satisfy the equations

$$a_y + b_x = d, \tag{A-5}$$

$$b_y - a_x = -c, \tag{A-6}$$

$$c_y - d_x = b|\nabla\omega|^2,$$

$$c_x + d_y = -a|\nabla\omega|^2.$$

These equations imply that a and b satisfy

$$\Delta a = -a|\nabla\omega|^2 \quad \text{and} \quad \Delta b = -b|\nabla\omega|^2.$$

Since $\rho \in \mathring{H}^1(\mathbb{R}^2, \mathbb{R}^3)$, then a and b can be seen as functions in $H^1(S^2)$ satisfying $\Delta\alpha = 2\alpha$; therefore, a and b are linear combinations of the first nonvanishing eigenfunctions of $\Delta_{\mathbb{S}^2}$ (see also Lemma C.1 of [Laurain 2012]); that is to say

$$a = \sum_{i=0}^{2} a_i\psi_i \quad \text{and} \quad b = \sum_{i=0}^{2} b_i\psi_i,$$

where

$$\psi_i(x) = \frac{x_i}{(1+|x|^2)} \quad \text{for } i = 1, 2 \text{ and } \psi_0(x) = \frac{1-|x|^2}{1+|x|^2}.$$

[3] In this statement, $\mathring{H}^2(\mathbb{R}^2, \mathbb{R}^3)$ is the pushforward of $H^2(S^2)$ on $\mathbb{R}^2$ via stereographic projection.

Finally using the facts that $\nabla\rho(0) = 0$ and $\langle\nabla^2\rho, \nabla\omega\rangle(0) = 0$, (A-5), and (A-6), we can conclude that $a \equiv b \equiv c \equiv d \equiv 0$, which proves the lemma. □

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PAUL LAURAIN: laurainp@math.jussieu.fr
Institut de Mathématiques de Jussieu, Paris VII, Bâtiment Sophie Germain, Case 7012, 75205 Paris Cedex 13, France

ANDREA MONDINO: andrea.mondino@math.ethz.ch
Department of Mathematics, ETH, Rämistrasse 101, CH-8092 Zürich, Switzerland

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HOLE PROBABILITIES OF SU($m+1$) GAUSSIAN RANDOM POLYNOMIALS

JUNYAN ZHU

In this paper, we study hole probabilities $P_{0,m}(r, N)$ of SU($m+1$) Gaussian random polynomials of degree N over a polydisc $(D(0,r))^m$. When $r \geq 1$, we find asymptotic formulas and the decay rate of $\log P_{0,m}(r, N)$. In dimension one, we also consider hole probabilities over some general open sets and compute asymptotic formulas for the generalized hole probabilities $P_{k,1}(r, N)$ over a disc $D(0,r)$.

Introduction

Hole probability is the probability that some random field never vanishes over some set. For Gaussian random entire functions we have this (see also [Zrebiec 2007, Theorem 1.2] for a multivariable result):

Theorem [Sodin and Tsirelson 2005, Theorem 1]. *Let $\psi(z) = \sum_{k=0}^{\infty} c_k z^k/\sqrt{k!}$, where the c_k $(k \geq 0)$ are i.i.d. standard complex Gaussian random variables. Then there exist constants $C_1 \geq C_2 > 0$ such that*

$$\exp\{-C_1 r^4\} \leq \mathrm{Prob}\{0 \notin \psi(D(0,r))\} \leq \exp\{-C_2 r^4\}.$$

The case of Gaussian random sections was considered in [Shiffman et al. 2008]: Let M be a compact Kähler manifold with complex dimension m and $(L,h) \to M$ a positive holomorphic line bundle. Let γ_N denote the Gaussian probability measure on $H^0(M, L^N)$ induced by the fiberwise inner product h^N and the polarized volume form $dV_M = \omega_h^m/m! = ((\sqrt{-1}/2\pi)\Theta_h)^m/m!$, where Θ_h is the Chern curvature tensor of (L,h).

Theorem [Shiffman et al. 2008, Theorem 1.4]. *For any nonempty open set $U \subset M$, if there exists s in $H^0(M, L)$ such that s does not vanish on $\overline{U}$, then there exist constants $C_1 \geq C_2 > 0$ such that, for $N \gg 1$,*

$$\exp\{-C_1 N^{m+1}\} \leq \gamma_N\{s_N \in H^0(M, L^N) : 0 \notin s_N(U)\} \leq \exp\{-C_2 N^{m+1}\}.$$

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Therefore, it is natural to ask: can we find sharp constants C_1, C_2 in these two theorems, and is it possible to obtain an asymptotic formula and a decay rate for the hole probability? Using Cauchy's integral estimates, Nishry answered this in the random entire function case as follows (an analogous result for Gaussian random power series is obtained in [Peres and Virág 2005, Corollary 3]).

Theorem [Nishry 2010, Theorem 1]. *Let* $\psi(z) = \sum_{k=0}^{\infty} c_k z^k/\sqrt{k!}$, *where the* c_k $(k \geq 0)$ *are i.i.d. standard complex Gaussian random variables. Then*

$$\operatorname{Prob}\{0 \notin \psi(D(0,r))\} = \exp\{-\tfrac{1}{2}e^2 r^4 + O(r^{\frac{18}{5}})\}.$$

This suggests to us that, for those line bundles with polynomial sections, maybe it is possible to find an asymptotic formula for the hole probability.

If $P_{0,m}(r,N)$ denotes the hole probability of SU$(m+1)$ Gaussian random polynomials over the polydisc $(D(0,r))^m$, $d_m x$ denotes the Lebesgue measure on $\mathbb{R}^m$ and

$$E_r(x) := 2\sum_{i=1}^{m} x_i \log r - \left[\sum_{i=1}^{m} x_i \log x_i + \left(1-\sum_{i=1}^{m} x_i\right)\log\left(1-\sum_{i=1}^{m} x_i\right)\right]$$

is a continuous function defined over the standard simplex $\Sigma_m := \{x = (x_1,\dots,x_m) \in \mathbb{R}^{m+} : \sum_{i=1}^m x_i \leq 1\}$ (here we adopt the convention that $0\log 0 = 0$), we have the following results:

Theorem 0.1. *For* $r \geq 1$,

$$\log P_{0,m}(r,N) = -N^{m+1}\int_{\Sigma_m} E_r(x)\, d_m x + o(N^{m+1}),$$

where

$$\int_{\Sigma_m} E_r(x)\, d_m x = \frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}.$$

Theorem 0.2. *For* $r > 0$,

$$\log P_{0,m}(r,N) \geq -N^{m+1}\int_{x\in\Sigma_m : E_r(x)\geq 0} E_r(x)\, d_m x + o(N^{m+1}),$$

$$\log P_{0,m}(r,N) \leq -N^{m+1}\int_{x\in\mathbb{R}^{m+} : \sum_{i=1}^m x_i \leq \alpha_0} E_r(x)\, d_m x + o(N^{m+1}),$$

where $\alpha_0 = \alpha_0(r,m) > 0$ *is defined by*

$$\alpha_0 = \alpha_0(r,m) = \begin{cases} 1 & \text{if } 2\log r + \sum_{k=2}^m 1/k \geq 0, \\ \text{the nonzero root of } \alpha = \dfrac{\alpha\log\alpha + (1-\alpha)\log(1-\alpha)}{2\log r + \sum_{k=2}^m 1/k} & \text{if } 2\log r + \sum_{k=2}^m 1/k < 0. \end{cases}$$

Here, when $m = 1$, *we take* $\sum_{k=2}^m 1/k = 0$.

Remark 0.3. Theorem 0.1 can be derived from Theorem 0.2 as, when $r \geq 1$, $\{x \in \Sigma_m : E_r(x) \geq 0\} = \Sigma_m$ and $\alpha_0(r,m) = 1$. In fact, we could have proved this general case directly, but the idea of the proof would turn out to be extremely difficult to follow.

Corollary 0.4. *In the case of $m = 1$, the following asymptotic formula for the logarithm of the hole probability over a disc exists for all $r > 0$:*

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x)\,dx + o(N^2);$$

here

$$\int_0^{\alpha_0} E_r(x)\,dx = \tfrac{1}{2}\alpha_0(2\log r + 1 - \log \alpha_0)$$

and $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ is given in Theorem 0.2.

Because of the simplicity of the one-dimensional case, we can obtain more about the hole probability of SU(2) Gaussian random polynomials:

Theorem 0.5. *If $U \subset \mathbb{C}$ is a bounded simply connected domain containing 0 and ∂U is a Jordan curve, let $\phi : D(0, 1) \to U$ be a biholomorphism given by the Riemann mapping theorem such that $\phi(0) = 0$ (thus ϕ is unique up to the composition of a unitary transformation of $\mathbb{C}$). Then the hole probability $P_{0,1}(U, N)$ of SU(2) Gaussian random polynomials of degree N over U satisfies*

$$\log P_{0,1}(U, N) \le -\big(\log|\phi'(0)| + \tfrac{1}{2}\big)N^2 + o(N^2).$$

Also, in dimension one, it makes sense to study the number of zeros in some set. So let the generalized hole probability $P_{k,1}(r, N)$ be the probability that an SU(2) Gaussian random polynomial of degree N has no more than k zeros in $D(0, r)$; then, the following theorem shows that the asymptotic formula of $\log P_{k,1}(r, N)$ exists:

Theorem 0.6. *For all $k \ge 0$ and $r > 0$,*

$$\log P_{k,1}(r, N) = -\tfrac{1}{2}\alpha_0(2\log r + 1 - \log \alpha_0)N^2 + o(N^2),$$

where $\alpha_0 = \alpha_0(r, 1) \in (0, 1]$ is given in Theorem 0.2.

We should remark here that in all the cases we consider, the event that some Gaussian random polynomial has zeros on the boundary of some open set is a null set, i.e., of zero probability. Therefore we do not distinguish between the (generalized) hole probability over an open set and that over its closure.

1. Background

We review in this section some background on SU($m + 1$) Gaussian random polynomials and the definition of our probability measures. Before that, we define two lexicographically ordered sets that will be consistently used as index sets throughout this paper.

Definition 1.1.

$$\Gamma_{m,N} := \{J = (j_1, \dots, j_m) \in [0, N]^m \cap \mathbb{Z}^m : 0 \le j_1 \le \cdots \le j_m \le N\},$$
$$\Lambda_{m,N} := \{K = (k_1, \dots, k_m) \in [0, N]^m \cap \mathbb{Z}^m : |K| = k_1 + \cdots + k_m \le N\}.$$

It is not difficult to show that $|\Gamma_{m,N}| = |\Lambda_{m,N}| = \binom{N+m}{m}$.

The tautological line bundle $\mathcal{O}(-1)$ over the complex projective space $\mathbb{CP}^m$ is a holomorphic line bundle with fibers

$$\mathcal{O}(-1)_{[x]} = \mathbb{C} \cdot x \quad \text{for all } [x] = [x_0 : \cdots : x_m] \in \mathbb{CP}^m.$$

Its dual bundle, denoted by $\mathcal{O}(1)$, is called the hyperplane section bundle, since $\mathcal{O}(1) = \mathcal{O}(H)$, where the divisor

$$H = \{[x] \in \mathbb{CP}^m : x_0 = 0\}$$

is a hyperplane in $\mathbb{CP}^m$. $H^0(\mathbb{CP}^m, \mathcal{O}(N))$, the space of holomorphic sections of the tensor bundle $\mathcal{O}(N) = \mathcal{O}(1)^{\otimes N}$, is isomorphic to ${}^h\mathcal{P}^N_{m+1}$, the space of $(m+1)$-variable homogeneous polynomials of degree N. The Fubini–Study metric h_{FS} on $\mathcal{O}(1)$ can be described in the following way: Over the open subset

$$U_0 = \{[x] = [x_0 : \cdots : x_m] \in \mathbb{CP}^m : x_0 \neq 0\} \subset \mathbb{CP}^m,$$

we have a local frame of $\mathcal{O}(1)$,

$$e([x]) = x_0.$$

Set

$$\|e([x])\|^2_{h_{\mathrm{FS}}} = \frac{|x_0|^2}{\sum_{i=0}^m |x_i|^2} = \frac{|x_0|^2}{\|x\|^2},$$

which is independent of the choice of representative x of $[x]$. In terms of the affine coordinates

$$z = (z_1, \ldots, z_m) = \left(\frac{x_1}{x_0}, \ldots, \frac{x_m}{x_0}\right)$$

over U_0,

$$\|e(z)\|^2_{h_{\mathrm{FS}}} = (1 + \|z\|^2)^{-1} = \left(1 + \sum_{i=1}^m |z_i|^2\right)^{-1},$$

which defines a metric with positive Chern curvature form

$$\omega_{\mathrm{FS}} = -\frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log \|e(z)\|^2_{h_{\mathrm{FS}}} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} \log (1 + |z_1|^2 + \cdots + |z_m|^2).$$

This induces a metric h^N_{FS} on the line bundle $\mathcal{O}(N)$ so that

$$\|e^{\otimes N}(z)\|^2_{h^N_{\mathrm{FS}}} = (1 + \|z\|^2)^{-N}.$$

With the frame $e^{\otimes N}$ over U_0, for any $s \in H^0(\mathbb{CP}^m, \mathcal{O}(N))$, represented as $p(x_0, \ldots, x_m) \in {}^h\mathcal{P}^N_{m+1}$, we have

$$p(x_0, \ldots, x_m) = \frac{p(x_0, \ldots, x_m)}{x_0^N} e^{\otimes N}([x]) = p(1, z_1, \ldots, z_m) e^{\otimes N}([x]),$$

which implies that all the elements in $H^0(\mathbb{CP}^m, \mathcal{O}(N))$ can be viewed over U_0 as polynomials in $(z_1, \ldots, z_m)$ of degree at most N.

Since ω_{FS} is positive over $\mathbb{CP}^m$, we may take it as a polarized metric form on $\mathbb{CP}^m$, and the associated volume form is $dV = \omega_{\text{FS}}^m/m!$. Thus, the metric h_{FS}^N together with the volume form dV induce a Hermitian inner product on the space of holomorphic sections $H^0(\mathbb{CP}^m, \mathbb{O}(N))$: for all $s_1, s_2 \in H^0(\mathbb{CP}^m, \mathbb{O}(N))$,

$$\langle\langle s_1, s_2\rangle\rangle := \int_{\mathbb{CP}^m} \langle s_1, s_2\rangle_{h_{\text{FS}}^N}\, dV.$$

With this inner product, there is an orthonormal basis $\{S_K^N\}_{K=(k_1,\ldots,k_m)\in\Lambda_{m,N}}$ given in local affine coordinates $(z_1, \ldots, z_m)$ over U_0 by

$$S_K^N(z) = \sqrt{(N+1)\cdots(N+m)}\sqrt{\binom{N}{K}}z^K,$$

where we adopt the notations

$$\binom{N}{K} = \frac{N!}{(N-|K|)!k_1!\cdots k_m!}, \quad z^K := z_1^{k_1}\cdots z_m^{k_m}.$$

Thus, $H^0(\mathbb{CP}^m, \mathbb{O}(N))$ is equal to $\left\{s_N = \sum_{K\in\Lambda_{m,N}} c_K S_K^N : c = (c_K)_{K\in\Lambda_{m,N}} \in \mathbb{C}^{\binom{N+m}{m}}\right\}$. Endow $H^0(\mathbb{CP}^m, \mathbb{O}(N))$ with the Gaussian probability measure γ_N defined by

$$d\gamma_N(s_N) := \pi^{-\binom{N+m}{m}} e^{-\|c\|^2}\, d_{2\binom{N+m}{m}}c,$$

where $\|c\|^2 = \sum_{K\in\Lambda_{m,N}}|c_K|^2$ and $d_{2\binom{N+m}{m}}c$ denotes the $2\binom{N+m}{m}$-dimensional Lebesgue measure. Then γ_N is characterized by the property that $\{c_K\}_{K\in\Lambda_{m,N}}$ consists of independent and identically distributed (i.i.d.) standard complex Gaussian random variables. Then $\left(H^0(\mathbb{CP}^m, \mathbb{O}(N)), \gamma_N\right)$ is called the ensemble of SU($m+1$) Gaussian random polynomials of degree N, since the random element s_N is distributionally invariant under SU($m+1$) transformations of $\mathbb{CP}^m$. Its hole probability over the polydisc $(D(0,r))^m \subset \mathbb{C}^m$ is

$$\begin{aligned}
P_{0,m}(r, N) &= \gamma_N\left\{s_N \in H^0(\mathbb{CP}^m, \mathbb{O}(N)) : 0 \notin s_N\left((\bar{D}(0,r))^m\right)\right\}\\
&= \pi^{-\binom{N+m}{m}} \int_{c\in\mathbb{C}^{\binom{N+m}{m}}:0\notin s_N((\bar{D}(0,r))^m)} e^{-\|c\|^2}\, d_{2\binom{N+m}{m}}c\\
&= \pi^{-\binom{N+m}{m}} \int_{c\in\mathbb{C}^{\binom{N+m}{m}}:0\notin \tilde{s}_N((\bar{D}(0,r))^m)} e^{-\|c\|^2}\, d_{2\binom{N+m}{m}}c,
\end{aligned}$$

where $\tilde{s}_N(z) = \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}}z^K$. Hereafter, when considering hole probability, we work on $\tilde{s}_N$ instead of s_N for simplicity.

2. Preliminaries

Definition 2.1.
$$Q_{r,m}(N) := \sum_{K\in\Lambda_{m,N}} \log\left[\binom{N}{K}r^{2|K|}\right].$$

Lemma 2.2. $Q_{r,m}(N) = N^{m+1}\int_{\Sigma_m} E_r(x)\, d_m x + o(N^{m+1}) = \left[\dfrac{2m\log r}{(m+1)!} + \dfrac{1}{m!}\displaystyle\sum_{k=2}^{m+1}\frac{1}{k}\right]N^{m+1} + o(N^{m+1}).$

Proof. We can prove inductively that, for $k \ge 1$,

$$\left(\frac{k}{e}\right)^k \le k! \le \frac{k^{k+1}}{e^{k-1}}$$

or, equivalently,

$$k\log k - k \le \log k! \le (k+1)\log k - (k-1). \tag{2-1}$$

Hence we have

$$-(k+1)\log N + (k-1) \le k\log\frac{k}{N} - \log k! \le -k\log N + k \quad \text{for } 0 \le k \le N. \tag{2-2}$$

For all $K = (k_1, \dots, k_m) \in \Lambda_{m,N}$,

$$\log\Big[\binom{N}{K} r^{2|K|}\Big] - NE_r\Big(\frac{K}{N}\Big)$$
$$= \log N! + \sum_{i=1}^{m}\Big(k_i\log\frac{k_i}{N} - \log k_i!\Big) + \Big[(N-|K|)\log\frac{N-|K|}{N} - \log(N-|K|)!\Big],$$

Applying (2-1) and (2-2), we then get

$$\log\Big[\binom{N}{K} r^{2|K|}\Big] - NE_r\Big(\frac{K}{N}\Big) \ge (N\log N - N) - (N+m+1)\log N + (N-m-1) = -(m+1)(\log N+1),$$
$$\log\Big[\binom{N}{K} r^{2|K|}\Big] - NE_r\Big(\frac{K}{N}\Big) \le [(N+1)\log N - (N-1)] - N\log N + N = \log N + 1.$$

Hence, for all $K \in \Lambda_{m,N}$,

$$\Big|\log\Big[\binom{N}{K} r^{2|K|}\Big] - NE_r\Big(\frac{K}{N}\Big)\Big| \le (m+1)(\log N + 1),$$

so

$$\Big|Q_{r,m}(N) - N\sum_{K\in\Lambda_{m,N}} E_r\Big(\frac{K}{N}\Big)\Big| \le \sum_{K\in\Lambda_{m,N}}\Big|\log\Big[\binom{N}{K} r^{2|K|}\Big] - NE_r\Big(\frac{K}{N}\Big)\Big|$$
$$\le (m+1)(\log N+1)\binom{N+m}{m} = o(N^{m+1}). \tag{2-3}$$

Take

$$\mathring{\Lambda}_{m,N} := \{K \in \Lambda_{m,N} : k_i \ge 1 \text{ for } 1 \le i \le m \text{ and } |K| \le N-m-1\} \subset \Lambda_{m,N}$$

and

$$\mathring{\Sigma}_m(N) := \bigcup_{K\in\mathring{\Lambda}_{m,N}} \Big[\frac{k_1}{N}, \frac{k_1+1}{N}\Big] \times \cdots \times \Big[\frac{k_m}{N}, \frac{k_m+1}{N}\Big] \subset \Sigma_m.$$

Then

$$|\mathring{\Lambda}_{m,N}| = \binom{N-m-1}{m},$$
$$|\Lambda_{m,N} \setminus \mathring{\Lambda}_{m,N}| = \binom{N+m}{m} - \binom{N-m-1}{m} = O(N^{m-1}),$$
$$\mathrm{Vol}_{\mathbb{R}^m}(\Sigma_m \setminus \mathring{\Sigma}_m(N)) = \frac{1}{m!} - N^{-m}\binom{N-m-1}{m} = O(N^{-1}).$$

Over Σ_m we have

$$|E_r| \le 2|\log r| + \frac{m+1}{e} = O(1);$$

hence

$$\left| N \sum_{K\in\Lambda_{m,N}} E_r\left(\frac{K}{N}\right) - N \sum_{K\in\mathring{\Lambda}_{m,N}} E_r\left(\frac{K}{N}\right)\right| \le N|\Lambda_{m,N}\setminus\mathring{\Lambda}_{m,N}| \sup_{\Sigma_m}|E_r| = O(N^m). \tag{2-4}$$

As

$$\sup_{\mathring{\Sigma}_m(N)} \|\nabla E_r\| \le O(\log N),$$

we have

$$\begin{aligned}\Bigg| N \sum_{K\in\mathring{\Lambda}_{m,N}} E_r\left(\frac{K}{N}\right) &- N^{m+1}\int_{\mathring{\Sigma}_m(N)} E_r(x)\,d_m x\Bigg| \\ &\le N^{m+1}\sum_{K\in\mathring{\Lambda}_{m,N}} \int_{\left[\frac{k_1}{N},\frac{k_1+1}{N}\right]\times\cdots\times\left[\frac{k_m}{N},\frac{k_m+1}{N}\right]} \left|E_r\left(\frac{K}{N}\right)-E_r(x)\right| d_m x \\ &\le N^{m+1}\binom{N-m-1}{m} N^{-m} O(\log N) O(N^{-1}) \\ &= O(N^m\log N). \end{aligned}\tag{2-5}$$

Moreover,

$$\left|N^{m+1}\int_{\mathring{\Sigma}_m(N)} E_r(x)\,d_m x - N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x\right| \le N^{m+1}\sup_{\Sigma_m}|E_r|\,\mathrm{Vol}_{\mathbb{R}^m}(\Sigma_m\setminus\mathring{\Sigma}_m(N)) = O(N^m). \tag{2-6}$$

Combining (2-3)–(2-6), we thus obtain

$$\begin{aligned} Q_{r,m}(N) &= N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x + o(N^{m+1}) \\ &= N^{m+1}\int_{\Sigma_m} 2\sum_{i=1}^m x_i\log r - \left[\sum_{i=1}^m x_i\log x_i + \left(1-\sum_{i=1}^m x_i\right)\log\left(1-\sum_{i=1}^m x_i\right)\right] d_m x + o(N^{m+1}) \\ &= N^{m+1}\left[2m\log r\int_{\Sigma_m} x_1\,d_m x - (m+1)\int_{\Sigma_m} x_1\log x_1\,d_m x\right] + o(N^{m+1}) \\ &= \left[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\right] N^{m+1} + o(N^{m+1}). \end{aligned}$$

□

Remark 2.3. The scaled lattice $(1/N)\Lambda_{m,N}\subset\mathbb{R}^m$ tends to Σ_m. Hence Lemma 2.2 is in fact converting a Riemann sum into a Riemann integral and estimating the error. Such procedures will appear several times in this paper.

Remark 2.4. The function $E_r(x)$ in the above lemma can also be written as

$$E_r(x) = -b_{\{x\}}(z_r) + \log(1+\|z_r\|^2),$$

where $z_r = (r, \dots, r) \in \mathbb{R}^m$ and $b_{\{x\}}$ is the exponential decay rate of the expected mass density of random L^2-normalized polynomials with some prescribed Newton polytope (see Theorem 1.2 and (78) in [Shiffman and Zelditch 2004]).

Let $\xi = (\xi_1, \dots, \xi_m)$, where $\xi_i = (\xi_{i,0}, \dots, \xi_{i,N}) \in \mathbb{C}^{N+1}$ for $1 \le i \le m$.

Definition 2.5. $W_{m,N}(\xi)$ is the $\binom{N+m}{m} \times \binom{N+m}{m}$ matrix with rows indexed by $\Gamma_{m,N}$ and columns indexed by $\Lambda_{m,N}$ such that, for all $J = (j_1, \dots, j_m) \in \Gamma_{m,N}$, $K = (k_1, \dots, k_m) \in \Lambda_{m,N}$, the (J, K)-entry of $W_{m,N}(\xi)$ is $\xi_J^K = \xi_{1,j_1}^{k_1} \cdots \xi_{m,j_m}^{k_m}$.

The next lemma gives the formula for a "Vandermonde-type" determinant.

Lemma 2.6.
$$|\det W_{m,N}(\xi)| = \prod_{i=1}^{m} \prod_{0 \le j < k \le N} |\xi_{i,j} - \xi_{i,k}|^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}.$$

Proof. For all $1 \le i \le m$ and $0 \le j < k \le N$, the rows of $W_{m,N}(\xi)$ involving $\xi_{i,j}$ correspond to the set
$$\Gamma_{m,N}^{i,j} = \{(j_1, \dots, j_m) \in \Gamma_{m,N} : j_i = j\},$$
while those rows involving $\xi_{i,k}$ correspond to the set
$$\Gamma_{m,N}^{i,k} = \{(j_1, \dots, j_m) \in \Gamma_{m,N} : j_i = k\}. \tag{2-7}$$
Let
$$\tilde{\Gamma}_{m,N}^{i,j} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \dots \le j_{i-1} \le j \le j_{i+1} \le \dots \le j_m \le N\},$$
$$\tilde{\Gamma}_{m,N}^{i,k} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \dots \le j_{i-1} \le k \le j_{i+1} \le \dots \le j_m \le N\};$$
then
$$|\Gamma_{m,N}^{i,j}| = |\tilde{\Gamma}_{m,N}^{i,j}| = \binom{j+i-1}{i-1}\binom{N-j+m-i}{m-i},$$
$$|\Gamma_{m,N}^{i,k}| = |\tilde{\Gamma}_{m,N}^{i,k}| = \binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i}.$$
Since, for any $1 \le i \le m$,
$$\Gamma_{m,N} = \bigsqcup_{k=0}^{N} \Gamma_{m,N}^{i,k},$$
we have the equality
$$\sum_{k=0}^{N} \binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i} = \binom{N+m}{m}. \tag{2-8}$$
Note that
$$\tilde{\Gamma}_{m,N}^{i,j} \cap \tilde{\Gamma}_{m,N}^{i,k} = \{(j_1, \dots, \hat{j}_i, \dots, j_m) \in [0, N]^{m-1} \cap \mathbb{Z}^{m-1} : 0 \le j_1 \le \dots \le j_{i-1} \le j < k \le j_{i+1} \le \dots \le j_m \le N\}$$
and
$$|\tilde{\Gamma}_{m,N}^{i,j} \cap \tilde{\Gamma}_{m,N}^{i,k}| = \binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i},$$

which means that there are $\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}$ pairs of rows; within each pair the only difference between two rows is $\xi_{i,j}$ instead of $\xi_{i,k}$. Therefore, for all $1 \le i \le m$ and $0 \le j < k \le N$,

$$(\xi_{i,j}-\xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}} \;\Big|\; \det W_{m,N}(\xi),$$

and thus

$$G_{m,N}(\xi) \mid \det W_{m,N}(\xi), \tag{2-9}$$

where

$$G_{m,N}(\xi) := \prod_{i=1}^{m} \prod_{0\le j<k\le N} (\xi_{i,j}-\xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}.$$

Furthermore, for all $1 \le i \le m$,

$$\begin{aligned}
\deg_{\xi_i} G_{m,N}(\xi) &= \sum_{0\le j<k\le N} \binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i} \\
&= \sum_{k=1}^{N}\Bigg[\sum_{j=0}^{k-1}\binom{j+i-1}{i-1}\Bigg]\binom{N-k+m-i}{m-i} \\
&= \sum_{k=1}^{N}\binom{k-1+i}{i}\binom{N-k+m-i}{m-i} \\
&= \sum_{k-1=0}^{N-1}\binom{(k-1)+(i+1)-1}{(i+1)-1}\binom{(N-1)-(k-1)+(m+1)-(i+1)}{(m+1)-(i+1)} \\
&= \binom{(N-1)+(m+1)}{m+1} = \binom{N+m}{m+1},
\end{aligned} \tag{2-10}$$

where the second-to-last equality is due to (2-8). On the other hand, for all $1 \le i \le m$ and $1 \le k \le N$, the number of K in $\Lambda_{m,N}$ with $k_i = k$ is $\binom{N-k+m-1}{m-1}$; hence,

$$\deg_{\xi_i} \det W_{m,N}(\xi) = \sum_{k=1}^{N} k\binom{N-k+m-1}{m-1} = \binom{N+m}{m+1},$$

where the second equality is the special case $i = 1$ in (2-10). Therefore, for all $1 \le i \le m$,

$$\deg_{\xi_i} \det W_{m,N}(\xi) = \deg_{\xi_i} G_{m,N}(\xi). \tag{2-11}$$

By (2-9) and (2-11),

$$\det W_{m,N}(\xi) = C_{m,N} G_{m,N} = C_{m,N} \prod_{i=1}^{m} \prod_{0\le j<k\le N} (\xi_{i,j}-\xi_{i,k})^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}},$$

where $C_{m,N}$ is a constant depending only on m and N. Consider the monomial

$$g_{m,N}(\xi) := \prod_{i=1}^{m}\prod_{k=1}^{N} \xi_{i,k}^{\sum_{j=0}^{k-1}\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}} = \prod_{i=1}^{m}\prod_{k=1}^{N} \xi_{i,k}^{\binom{k+i-1}{i}\binom{N-k+m-i}{m-i}};$$

then

$$G_{m,N}(\xi) = \pm g_{m,N}(\xi) + \cdots .$$

In the Appendix, we show that the coefficient of $g_{m,N}$ in the expansion of $\det W_{m,N}(\xi)$ equals 1, and therefore $C_{m,N} = \pm 1$. □

3. Proof of Theorem 0.1

To prove Theorem 0.1, it suffices to prove separately the lower and upper bounds

$$-N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x + o(N^{m+1}) \le \log P_{0,m}(r,N) \le -N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x + o(N^{m+1})$$

Lower bound.

Proof of the lower bound in Theorem 0.1. Recall that $\tilde{s}_N(z) = \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K$. Hence,

$$|\tilde{s}_N(z)| \ge |c_{(0,\dots,0)}| - \sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} |c_K| \sqrt{\binom{N}{K}} r^{|K|} \quad \text{for all } z = (z_1,\dots,z_m) \in (\bar{D}(0,r))^m. \tag{3-1}$$

Consider the event $\Omega_{r,m,N}$:

(i) $|c_{(0,\dots,0)}| \ge \sqrt{N}$,

(ii) $|c_K| \le \dfrac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|}\binom{|K|+m-1}{m-1}}$, $\quad K \in \Lambda_{m,N}\backslash\{(0,\dots,0)\}$.

Then, if $\Omega_{r,m,N}$ occurs, by (3-1) we have that for all $z = (z_1,\dots,z_m) \in (\bar{D}(0,r))^m$,

$$\begin{aligned}
|\tilde{s}_N(z)| &\ge \sqrt{N} - \sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} \frac{\sqrt{\binom{N}{K}} r^{|K|}}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|}\binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{k=1}^{N} \frac{1}{2\sqrt{N}} = \frac{1}{2}\sqrt{N} > 0;
\end{aligned}$$

hence

$$\begin{aligned}
P_{0,m}(r,N) &\ge \gamma_N(\Omega_{r,m,N}) \\
&= \gamma_N(|c_{(0,\dots,0)}| \ge \sqrt{N}) \prod_{K\in\Lambda_{m,N}\backslash\{(0,\dots,0)\}} \gamma_N\left(|c_K| \le \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}} r^{|K|}\binom{|K|+m-1}{m-1}}\right),
\end{aligned}$$

where $\gamma_N(|c_{(0,\dots,0)}| \ge \sqrt{N}) = e^{-N}$. Recall that for $K \in \Lambda_{m,N} \backslash \{(0,\dots,0)\}$ the standard complex Gaussian random variables c_K satisfy $\gamma_N(|c_K| \le a) \ge \frac{1}{2}a^2$ whenever $a \le 1$. Since

$$\frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}} \le 1$$

if $r \ge 1$, we thus have

$$\gamma_N\left(|c_K| \le \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}}\right) \ge \frac{1}{2}\left[\frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}}\right]^2 = \frac{1}{8N\binom{N}{K}r^{2|K|}\binom{|K|+m-1}{m-1}^2},$$

and

$$\log P_{0,m}(r,N) \ge -N - \sum_{K \in \Lambda_{m,N}\backslash\{(0,\dots,0)\}} \left\{\log 8 + \log N + 2\log\binom{|K|+m-1}{m-1} + \log\left[\binom{N}{K}r^{2|K|}\right]\right\}.$$

Since

$$\log\binom{|K|+m-1}{m-1} \le \log\binom{N+m-1}{m-1} = O(\log N),$$

it follows that

$$\sum_{K \in \Lambda_{m,N}\backslash\{(0,\dots,0)\}} \left[\log 8 + \log N + 2\log\binom{|K|+m-1}{m-1}\right] = \binom{N+m}{m} O(\log N) = o(N^{m+1}).$$

Therefore,

$$\begin{aligned}\log P_{0,m}(r,N) &\ge -\sum_{K \in \Lambda_{m,N}\backslash\{(0,\dots,0)\}} \log\left[\binom{N}{K}r^{2|K|}\right] + o(N^{m+1})\\ &= -Q_{r,m}(N) + o(N^{m+1})\\ &= -N^{m+1}\int_{\Sigma_m} E_r(x)\,d_m x + o(N^{m+1}).\end{aligned}$$ □

Upper bound. Let $\delta > 0$ be small and $\kappa = 1 - \sqrt{\delta}$. We shall first treat δ as a small positive constant and at the end we will let $\delta \to 0+$. For the sake of clarity, all the constants C, the big O and little o terms listed throughout this paper will not depend on δ unless otherwise stated.

Definition 3.1. $z_j(N) := \kappa r e^{2\pi\sqrt{-1}j/N+1}$ for $0 \le j \le N$.

For all $p \in \mathbb{Z}^+$, by division with remainder, $N+1 = q(N)p + l(N)$, where $q(N) \in \mathbb{Z}$, $q(N) \ge 0$ and $0 \le l(N) < p$. For convenience, we drop the dependence on N when there is no chance of confusion. Since $N+1 = l(q+1) + (p-l)q$, for all $1 \le i \le m$, define $\xi_i = (\xi_{i,0},\dots,\xi_{i,N})$ by

$$\xi_{i,sp+t} = \begin{cases} z_{t(q+1)+s} & \text{if } 0 \le t \le l-1,\ 0 \le s \le q,\\ z_{l(q+1)+(t-l)q+s} & \text{if } l \le t \le p-1,\ 0 \le s \le q-1.\end{cases} \tag{3-2}$$

Intuitively, (3-2) gives a way to choose points $\xi_{i,j}$ $(j=0,1,\dots)$ one after another on the circle of radius κr such that the arguments of each two consecutive points differ approximately by $2\pi/p$. Denote the permutation of $N+1$ indices $\{0,\dots,N\}$ given by (3-2) by τ, i.e., $z_j=\xi_{i,\tau(j)}$ for $0\le j\le N$ and $1\le i\le m$.

For $t\in\{0,\dots,p-1\}$, denote

$$I_t=\begin{cases}\{t(q+1),\dots,t(q+1)+q\} & \text{if } 0\le t\le l-1,\\ \{l(q+1)+(t-l)q,\dots,l(q+1)+(t-l)q+(q-1)\} & \text{if } l\le t\le p-1,\end{cases}$$

$$a_t=tq+\min\{t,l\}=\begin{cases}t(q+1) & \text{if } 0\le t\le l-1,\\ l(q+1)+(t-l)q & \text{if } l\le t\le p-1.\end{cases}$$

$I_0,\dots,I_{p-1}$ give a partition of $\{0,\dots,N\}$. Again there is an implicit dependence on N for each term defined above, and we will indicate this dependence explicitly when necessary. Then

$$\tau(j)=(j-a_t)p+t=\begin{cases}[j-t(q+1)]p+t & \text{if } j\in I_t,\ 0\le t\le l-1,\\ [j-l(q+1)-(t-l)q]p+t & \text{if } j\in I_t,\ l\le t\le p-1,\end{cases}$$

and, if $\{j(N)\}_{N=1}^{\infty}$ is a sequence satisfying $j(N)\in I_t(N)$ for all $N\ge 1$, then

$$|\tau_N(j(N))-pj(N)+t(N+1)|\le 2p^2,$$

and therefore

$$\frac{\tau_N(j(N))}{N+1}-\left(p\frac{j(N)}{N+1}-t\right)=O(N^{-1}). \tag{3-3}$$

Lemma 3.2. *With the values of ξ_i given by (3-2),*

$$\log|\det W_{m,N}(\xi)|=m\binom{N+m}{m+1}\log(\kappa r)+\frac{\beta_m}{p}N^{m+1}+o(N^{m+1}),$$

where $\beta_m=(1/(m-1)!)\int_0^1 x^m\log[2\sin(\pi x)]\,dx$, which is finite for each $m\ge 1$ by the comparison test for improper integrals.

Proof. By Lemma 2.6,

$$\begin{aligned}
&\log|\det W_{m,N}(\xi)|\\
&\quad=\log\left[\prod_{i=1}^{m}\prod_{0\le j<k\le N}|\xi_{i,j}-\xi_{i,k}|^{\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}}\right]\\
&\quad=\sum_{i=1}^{m}\sum_{0\le j<k\le N}\binom{j+i-1}{i-1}\binom{N-k+m-i}{m-i}\left\{\log\left|\frac{\xi_{i,j}}{\kappa r}-\frac{\xi_{i,k}}{\kappa r}\right|+\log(\kappa r)\right\}\\
&\quad=\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|\frac{\xi_{i,\tau(j)}}{\kappa r}-\frac{\xi_{i,\tau(k)}}{\kappa r}\right|+m\binom{N+m}{m+1}\log(\kappa r)\\
&\quad=\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right|\\
&\qquad\qquad+m\binom{N+m}{m+1}\log(\kappa r),
\end{aligned}$$

where the second part of the third equality is due to (2-10). Now we are going to show that the first term after the last equality can be approximated by a double integral.

$$\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right|$$
$$=\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\left[\frac{(\tau(j))^{i-1}}{(i-1)!}+o\big((\tau(j))^{i-1}\big)\right]\left[\frac{(N-\tau(k))^{m-i}}{(m-i)!}+o\big((N-\tau(k))^{m-i}\big)\right]$$
$$\times\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|. \tag{3-4}$$

For all $0\le j,k\le N$, $1\le i\le m, 0\le u,v\le p-1$, denote

$$\mathcal{I}_{j,k,N}=\left[\frac{j}{N+1},\frac{j+1}{N+1}\right]\times\left[\frac{k}{N+1},\frac{k+1}{N+1}\right],$$
$$L_{u,v,N}=\{(j,k)\in I_u\times I_v:\tau(j)<\tau(k)\},$$
$$T_{u,v}(N)=\bigcup_{(j,k)\in L_{u,v,N}}\mathcal{I}_{j,k,N},$$
$$\mathring{L}_{u,v,N}=\{(j,k)\in L_{u,v,N}:j-k\neq\pm N \text{ and } j-k\neq\pm1\}\subset L_{u,v,N},$$
$$\mathring{T}_{u,v}(N)=\bigcup_{(j,k)\in\mathring{L}_{u,v,N}}\mathcal{I}_{j,k,N}\subset T_{u,v}(N),$$

and define a function over $\{(x,y)\in(0,1)\times(0,1):x\neq y\}$ by

$$g^i_{u,v}(x,y)=(px-u)^{i-1}[1-(py-v)]^{m-i}\log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right|.$$

Then

$$|L_{u,v,N}\setminus\mathring{L}_{u,v,N}|\le 2N+2, \tag{3-5}$$
$$\mathrm{Vol}_{\mathbb{R}^2}(T_{u,v}(N)\setminus\mathring{T}_{u,v}(N))\le O(N^{-1}), \tag{3-6}$$
$$\frac{1}{N+1}\le\left|\frac{j-k}{N+1}\right|\le\frac{N}{N+1}\qquad\text{for } (j,k)\in L_{u,v,N}, \tag{3-7}$$
$$\frac{1}{N+1}\le|x-y|\le\frac{N}{N+1}\qquad\text{for } (x,y)\in\mathring{T}_{u,v}(N), \tag{3-8}$$
$$|g^i_{u,v}(x,y)|\le O(\log N)\qquad\text{if } \frac{1}{N+1}\le|x-y|\le\frac{N}{N+1}, \tag{3-9}$$
$$\|\nabla g^i_{u,v}(x,y)\|\le O(N^{\frac12})\qquad\text{if } \frac{1}{\sqrt{N+1}}\le|x-y|\le1-\frac{1}{\sqrt{N+1}}. \tag{3-10}$$

From (3-3), we have

$$\sum_{0\le\tau(j)<\tau(k)\le N} (\tau(j))^{i-1}(N-\tau(k))^{m-i}\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$
$$=(N+1)^{m-1}\sum_{0\le u,v\le p-1}\sum_{(j,k)\in L_{u,v,N}}\left[p\frac{j}{N+1}-u+O(N^{-1})\right]^{i-1}\left[1-\left(p\frac{k}{N+1}-v\right)+O(N^{-1})\right]^{m-i}$$
$$\times\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|. \quad (3\text{-}11)$$

For all $0\le u,v\le p-1$, by (3-5), (3-7) and (3-9), we get

$$\sum_{(j,k)\in L_{u,v,N}}\left(p\frac{j}{N+1}-u\right)^{i-1}\left[1-\left(p\frac{k}{N+1}-v\right)\right]^{m-i}\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$
$$=\sum_{(j,k)\in L_{u,v,N}} g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)$$
$$=\sum_{(j,k)\in \mathring{L}_{u,v,N}} g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)+O(N\log N). \quad (3\text{-}12)$$

Moreover,

$$\left|(N+1)^{-2}\sum_{(j,k)\in \mathring{L}_{u,v,N}} g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)-\iint_{\mathring{T}_{u,v}(N)} g^i_{u,v}(x,y)\,dx\,dy\right|$$
$$\le\sum_{(j,k)\in \mathring{L}_{u,v,N}}\iint_{\mathcal{I}_{j,k,N}}\left|g^i_{u,v}(x,y)-g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right|dx\,dy$$
$$=\sum_{\substack{(j,k)\in \mathring{L}_{u,v,N}\\ \frac{1}{\sqrt{N+1}}\le|\frac{j-k}{N+1}|\le 1-\frac{1}{\sqrt{N+1}}}}\iint_{\mathcal{I}_{j,k,N}}\left|g^i_{u,v}(x,y)-g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right|dx\,dy$$
$$+\sum_{\substack{(j,k)\in \mathring{L}_{u,v,N}\\ |\frac{j-k}{N+1}|<\frac{1}{\sqrt{N+1}}\\ \text{or } |\frac{j-k}{N+1}|>1-\frac{1}{\sqrt{N+1}}}}\iint_{\mathcal{I}_{j,k,N}}\left|g^i_{u,v}(x,y)-g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right|dx\,dy. \quad (3\text{-}13)$$

Since

$$\#\left\{(j,k)\in \mathring{L}_{u,v,N}:\frac{1}{\sqrt{N+1}}\le\left|\frac{j-k}{N+1}\right|\le 1-\frac{1}{\sqrt{N+1}}\right\}\le|\mathring{L}_{u,v,N}|=O(N^2),$$
$$\#\left\{(j,k)\in \mathring{L}_{u,v,N}:\left|\frac{j-k}{N+1}\right|<\frac{1}{\sqrt{N+1}}\ \text{or}\ \left|\frac{j-k}{N+1}\right|>1-\frac{1}{\sqrt{N+1}}\right\}\le O(N^{\frac{3}{2}}),$$

(3-10) implies that

$$\sum_{\substack{(j,k)\in\mathring{L}_{u,v,N}\\ \frac{1}{\sqrt{N+1}}\le|\frac{j-k}{N+1}|\le 1-\frac{1}{\sqrt{N+1}}}} \iint_{\mathcal{I}_{j,k,N}} \left| g^i_{u,v}(x,y) - g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right| dx\,dy$$
$$\le O(N^2)\times(N+1)^{-2}\times\frac{\sqrt{2}}{N+1}\times \sup_{\frac{1}{\sqrt{N+1}}\le|x-y|\le 1-\frac{1}{\sqrt{N+1}}} \|\nabla g^i_{u,v}(x,y)\| \le O(N^{-\frac{1}{2}}), \quad (3\text{-}14)$$

and, by (3-8) and (3-9),

$$\sum_{\substack{(j,k)\in\mathring{L}_{u,v,N}\\ |\frac{j-k}{N+1}|<\frac{1}{\sqrt{N+1}}\\ \text{or } |\frac{j-k}{N+1}|>1-\frac{1}{\sqrt{N+1}}}} \iint_{\mathcal{I}_{j,k,N}} \left| g^i_{u,v}(x,y) - g^i_{u,v}\left(\frac{j}{N+1},\frac{k}{N+1}\right)\right| dx\,dy$$
$$\le O(N^{\frac{3}{2}})\times(N+1)^{-2}\times O(\log N) = O(N^{-\frac{1}{2}}\log N). \quad (3\text{-}15)$$

Let $T_{u,v}=\{(x,y)\in\mathbb{R}^2 : 0\le x-u/p\le y-v/p\le 1/p\}$. Since $g^i_{u,v}$ is L^1_{loc}, the measure $g^i_{u,v}(x,y)\,dx\,dy$ is absolutely continuous with respect to the Lebesgue measure. Therefore, by Lemma 3.3 below, we have that

$$\iint_{\mathring{T}_{u,v}(N)} g^i_{u,v}(x,y)\,dx\,dy - \iint_{T_{u,v}} g^i_{u,v}(x,y)\,dx\,dy = o(1) \quad \text{as } N\to\infty. \quad (3\text{-}16)$$

By (3-12)–(3-16),

$$\sum_{(j,k)\in L_{u,v,N}} \left(p\frac{j}{N+1}-u\right)^{i-1}\left[1-\left(p\frac{k}{N+1}-v\right)\right]^{m-i} \log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$
$$= (N+1)^2\iint_{T_{u,v}} g^i_{u,v}(x,y)\,dx\,dy + o(N^2). \quad (3\text{-}17)$$

(3-17) and (3-11) imply that

$$\sum_{0\le\tau(j)<\tau(k)\le N} (\tau(j))^{i-1}(N-\tau(k))^{m-i}\log\left|1-e^{2\pi\sqrt{-1}(\frac{j}{N+1}-\frac{k}{N+1})}\right|$$
$$= (N+1)^{m+1}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}} g^i_{u,v}(x,y)\,dx\,dy + o(N^{m+1}), \quad (3\text{-}18)$$

(3-18) and (3-4) imply

$$\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right|$$
$$=\sum_{i=1}^{m}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}}\frac{g^i_{u,v}(x,y)}{(i-1)!(m-i)!}\,dx\,dy + o(N^{m+1})$$

$$
\begin{aligned}
&=\sum_{i=1}^{m}\sum_{0\le u,v\le p-1}\iint_{T_{u,v}}\frac{[p(x-\frac{u}{p})]^{i-1}}{(i-1)!}\frac{[1-p(y-\frac{v}{p})]^{m-i}}{(m-i)!}\log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right|dx\,dy+o(N^{m+1})\\
&=\sum_{i=1}^{m}\sum_{0\le u,v\le p-1}\iint_{T_{0,0}}\frac{(px)^{i-1}}{(i-1)!}\frac{(1-py)^{m-i}}{(m-i)!}\log\left|1-e^{2\pi\sqrt{-1}(x-y+\frac{u}{p}-\frac{v}{p})}\right|dx\,dy+o(N^{m+1})\\
&=\sum_{i=1}^{m}\sum_{0\le u\le p-1}\iint_{T_{0,0}}\frac{(px)^{i-1}}{(i-1)!}\frac{(1-py)^{m-i}}{(m-i)!}\log\left[\prod_{v=0}^{p-1}\left|e^{2\pi\sqrt{-1}\frac{v}{p}}-e^{2\pi\sqrt{-1}(x-y+\frac{u}{p})}\right|\right]dx\,dy+o(N^{m+1})\\
&=p\sum_{i=1}^{m}\iint_{T_{0,0}}\frac{(px)^{i-1}}{(i-1)!}\frac{(1-py)^{m-i}}{(m-i)!}\log\left|1-e^{2\pi\sqrt{-1}(px-py)}\right|dx\,dy+o(N^{m+1})\\
&=\frac{1}{p}\iint_{T}\sum_{i=1}^{m}\frac{x^{i-1}}{(i-1)!}\frac{(1-y)^{m-i}}{(m-i)!}\log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right|dx\,dy+o(N^{m+1})\\
&=\frac{1}{p(m-1)!}\iint_{T}(1+x-y)^{m-1}\log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right|dx\,dy+o(N^{m+1}),
\end{aligned}
$$

where $T=\{(x,y)\in\mathbb{R}^2:0\le x\le y\le 1\}$. After the change of variables $\tilde{x}=x-y$, $\tilde{y}=y$, T is mapped to $\widetilde{T}=\{(\tilde{x},\tilde{y})\in\mathbb{R}^2:-1\le\tilde{x}\le 0,\ -\tilde{x}\le\tilde{y}\le 1\}$. Then

$$
\begin{aligned}
\frac{1}{(m-1)!}&\iint_{T}(1+x-y)^{m-1}\log\left|1-e^{2\pi\sqrt{-1}(x-y)}\right|dx\,dy\\
&=\frac{1}{(m-1)!}\iint_{\widetilde{T}}(1+\tilde{x})^{m-1}\log\left|1-e^{2\pi\sqrt{-1}\tilde{x}}\right|d\tilde{x}\,d\tilde{y}\\
&=\frac{1}{(m-1)!}\int_{-1}^{0}(1+\tilde{x})^{m}\log\left|1-e^{2\pi\sqrt{-1}\tilde{x}}\right|d\tilde{x}\\
&=\frac{1}{(m-1)!}\int_{0}^{1}x^{m}\log\left|1-e^{2\pi\sqrt{-1}x}\right|dx\\
&=\frac{1}{(m-1)!}\int_{0}^{1}x^{m}\log\left[2\sin(\pi x)\right]dx\\
&=\beta_m;
\end{aligned}
$$

hence,

$$
\sum_{i=1}^{m}\sum_{0\le\tau(j)<\tau(k)\le N}\binom{\tau(j)+i-1}{i-1}\binom{N-\tau(k)+m-i}{m-i}\log\left|e^{2\pi\sqrt{-1}\frac{j}{N+1}}-e^{2\pi\sqrt{-1}\frac{k}{N+1}}\right|
=\frac{\beta_m}{p}N^{m+1}+o(N^{m+1}).
$$

Thus,

$$
\log|\det W_{m,N}(\xi)|=m\binom{N+m}{m+1}\log(\kappa r)+\frac{\beta_m}{p}N^{m+1}+o(N^{m+1}). \qquad \square
$$

Lemma 3.3. $\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle \mathring{T}_{u,v}(N)) = 0$ *for any* $0 \le u, v \le p-1$, *where* $T_{u,v} \triangle \mathring{T}_{u,v}(N)$ *denotes the difference set of* $T_{u,v}$ *and* $\mathring{T}_{u,v}(N)$.

Proof. By (3-6), the statement in the lemma is equivalent to $\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^2}(T_{u,v} \triangle T_{u,v}(N)) = 0$, which follows from $\lim_{N\to\infty}(T_{u,v}(N)\setminus\partial T_{u,v}) = \mathring{T}_{u,v}$, as $T_{u,v}\triangle T_{u,v}(N) = (T_{u,v}\setminus T_{u,v}(N))\cup(T_{u,v}(N)\setminus T_{u,v})$. Since $T_{u,v}\setminus T_{u,v}(N) \subset [\mathring{T}_{u,v}\setminus(T_{u,v}(N)\setminus\partial T_{u,v})]\cup\partial T_{u,v}$,

$$\begin{aligned}\mathrm{Vol}_{\mathbb{R}^2}(T_{u,v}\setminus T_{u,v}(N)) &\le \mathrm{Vol}_{\mathbb{R}^2}(\mathring{T}_{u,v}\setminus(T_{u,v}(N)\setminus\partial T_{u,v})) + \mathrm{Vol}_{\mathbb{R}^2}(\partial T_{u,v}) \\ &= \iint_{\mathbb{R}^2} \mathbb{1}_{\mathring{T}_{u,v}\setminus(T_{u,v}(N)\setminus\partial T_{u,v})}\, dx\, dy \\ &\le \iint_{\mathbb{R}^2} \big|\mathbb{1}_{\mathring{T}_{u,v}} - \mathbb{1}_{T_{u,v}(N)\setminus\partial T_{u,v}}\big|\, dx\, dy;\end{aligned}$$

the last line tends to 0 by Fatou's lemma. A similar proof works for $T_{u,v}(N)\setminus T_{u,v}$. Therefore, it remains to prove $\lim_{N\to\infty}(T_{u,v}(N)\setminus\partial T_{u,v}) = \mathring{T}_{u,v}$.

First we'll show that $\limsup_{N\to\infty} T_{u,v}(N) \subset T_{u,v}$. For all $(x,y)\in\limsup_{N\to\infty} T_{u,v}(N)$, there exists a sequence $\{N_n\}_{n=1}^{\infty}\to\infty$ such that, for any $n\ge 1$, there exists $(j(N_n),k(N_n))\in I_u(N_n)\times I_v(N_n)$ with $\tau_{N_n}(j(N_n)) < \tau_{N_n}(k(N_n))$ and with $(x,y)\in\mathcal{I}_{j(N_n),k(N_n),N_n}$. Then $\lim_{n\to\infty} j(N_n)/(N_n+1) = x$ and $\lim_{n\to\infty} k(N_n)/(N_n+1) = y$. Since $0\le\tau_{N_n}(j(N_n))/(N_n+1) < \tau_{N_n}(k(N_n))/(N_n+1)\le N_n/(N_n+1)$ and $(j(N_n),k(N_n))\in I_u(N_n)\times I_v(N_n)$, (3-3) implies that

$$0 \le p\lim_{n\to\infty} j(N_n)/(N_n+1) - u \le p\lim_{n\to\infty} k(N_n)/(N_n+1) - v \le 1.$$

Hence $0\le px-u\le py-v\le 1$ and $(x,y)\in T_{u,v}$.

Next we will prove $\mathring{T}_{u,v}\subset\liminf_{N\to\infty} T_{u,v}(N)$. For all $(x,y)\in\mathring{T}_{u,v}$, $0<x-u/p<y-v/p<1/p$. Then there exist $0<\epsilon_1,\epsilon_2,\eta_1,\eta_2<1/p$ such that $x = u/p+\epsilon_1 = (u+1)/p-\eta_1$ and $y = v/p+\epsilon_2 = (v+1)/p-\eta_2$. For each $N>0$, define $j(N) = \lfloor(N+1)x\rfloor$ and $k(N) = \lfloor(N+1)y\rfloor$. When N is large enough, $j(N) = \lfloor(N+1)(u/p+\epsilon_1)\rfloor = uq(N)+\lfloor ul(N)/p+\epsilon_1(N+1)\rfloor \ge uq(N)+\min\{u,l(N)\} = a_u$, while

$$\begin{aligned} j(N) = \Big\lfloor (N+1)\Big(\frac{u+1}{p}-\eta_1\Big)\Big\rfloor &= (u+1)q(N) + \Big\lfloor (u+1)\frac{l(N)}{p} - \eta_1(N+1)\Big\rfloor \\ &\le (u+1)q(N)+\min\{u+1,l(N)\}-1 = a_{u+1}-1\end{aligned}$$

for $0\le u<p-1$, which indicates that $j(N)\in I_u(N)$. Similarly, $k(N)\in I_v(N)$ for N large. Moreover, $\lim_{N\to\infty}\tau(j(N))/(N+1) = p\lim_{N\to\infty} j(N)/(N+1)-u = p\lim_{N\to\infty}\lfloor(N+1)x\rfloor/(N+1)-u = px-u$; similarly, $\lim_{N\to\infty}\tau(k(N))/(N+1) = py-v$. And, since $0<px-u<py-v<1$, for N large enough we have $0<\tau(j(N))/(N+1)<\tau(k(N))/(N+1)<1$, so $0<\tau(j(N))<\tau(k(N))\le N$. Thus, by the definition of $j(N)$ and $k(N)$, we have, for N large, $(x,y)\in\mathcal{I}_{j(N),k(N),N}\subset\bigcup_{(j,k)\in L_{u,v,N}}\mathcal{I}_{j,k,N} = T_{u,v}(N)$, which implies that $(x,y)\in\liminf_{N\to\infty} T_{u,v}(N)$.

In conclusion, we have

$$\mathring{T}_{u,v}\subset\liminf_{N\to\infty} T_{u,v}(N)\subset\limsup_{N\to\infty} T_{u,v}(N)\subset T_{u,v},$$

from which

$$\lim_{N\to\infty} (T_{u,v}(N) \setminus \partial T_{u,v}) = \mathring{T}_{u,v}. \qquad \square$$

Let $\zeta = (\zeta_J)^t_{J\in\Gamma_{m,N}} = (\tilde{s}_N(\xi_J))^t_{J\in\Gamma_{m,N}} = (\tilde{s}_N(\xi_{1,j_1}, \dots, \xi_{m,j_m}))^t_{J\in\Gamma_{m,N}}$ be an $\binom{N+m}{m}$-dimensional mean zero complex Gaussian random vector. Let its covariance matrix be Σ; then, for all $J = (j_1, \dots, j_m)$, $J' = (j'_1, \dots, j'_m) \in \Gamma_{m,N}$ and

$$\begin{aligned}
\Sigma_{J,J'} &= \mathbb{E}_N(\zeta_J \bar{\zeta}_{J'}) = \mathbb{E}_N(\tilde{s}_N(\xi_J)\overline{\tilde{s}_N(\xi_{J'})}) \\
&= \sum_{K\in\Lambda_{m,N}} \Bigl[\sqrt{\binom{N}{K}}\xi_J^K\Bigr]\Bigl[\sqrt{\binom{N}{K}}\bar{\xi}_{J'}^K\Bigr] \\
&= \sum_{K\in\Lambda_{m,N}} \binom{N}{K}(\xi_J\bar{\xi}_{J'})^K \\
&= (1+\xi_J\bar{\xi}_{J'})^N \\
&= (1+\xi_{1,j_1}\bar{\xi}_{1,j'_1}+\cdots+\xi_{m,j_m}\bar{\xi}_{m,j'_m})^N,
\end{aligned}$$

where $\mathbb{E}_N$ denotes the expectation with respect to the probability measure γ_N.

Lemma 3.4. *With the assignment of ξ as in (3-2),*

$$\log(\det\Sigma) = Q_{\kappa r,m}(N) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1}).$$

Proof. $\Sigma = V_{m,N}(\xi)V^*_{m,N}(\xi)$, where $V_{m,N}(\xi) = \Bigl(\sqrt{\binom{N}{K}}\xi_J^K\Bigr)_{J\in\Gamma_{m,N},K\in\Lambda_{m,N}}$ is an $\binom{N+m}{m}\times\binom{N+m}{m}$ matrix. Thus

$$\det\Sigma = |\det V_{m,N}(\xi)|^2 = \prod_{K\in\Lambda_{m,N}}\binom{N}{K}|\det W_{m,N}(\xi)|^2.$$

By Lemma 3.2,

$$\begin{aligned}
\log(\det\Sigma) &= \sum_{K\in\Lambda_{m,N}} \log\binom{N}{K} + 2\log|\det W_{m,N}(\xi)| \\
&= \sum_{K\in\Lambda_{m,N}} \log\binom{N}{K} + 2m\binom{N+m}{m+1}\log(\kappa r) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1}) \\
&= \sum_{K\in\Lambda_{m,N}} \log\binom{N}{K} + 2\sum_{K\in\Lambda_{m,N}} |K|\log(\kappa r) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1}) \\
&= Q_{\kappa r,m}(N) + \frac{2\beta_m}{p}N^{m+1} + o(N^{m+1}). \qquad \square
\end{aligned}$$

As $\log|\tilde{s}_N(z)|$ is plurisubharmonic in a neighborhood of $(\bar{D}(0,r))^m$, we have

$$
\begin{aligned}
\log \prod_{J\in\Gamma_{m,N}} |\zeta_J| &= \sum_{J\in\Gamma_{m,N}} \log|\tilde{s}_N(\xi_J)| \\
&\le \sum_{J\in\Gamma_{m,N}} \int\!\cdots\!\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \prod_{i=1}^m P_r(\xi_{i,j_i},u_i)\, d\sigma_r(u_1)\cdots d\sigma_r(u_m) \\
&= (N+1)^m \int\!\cdots\!\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \Bigg[\sum_{J\in\Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} \\
&\qquad\qquad - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Bigg] d\sigma_r(u_1)\cdots d\sigma_r(u_m) \\
&\quad + (N+1)^m \int\!\cdots\!\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\, d\sigma_r(u_1)\cdots d\sigma_r(u_m), \qquad (3\text{-}19)
\end{aligned}
$$

where $P_r(\xi,u) = (r^2-|\xi|^2)/(|u-\xi|^2)$ is the Poisson kernel of $D(0,r)$, $d\sigma_r$ is the Haar measure on $\partial D(0,r)$, $d_m x$ is the Lebesgue measure on $\mathbb{R}^m$, and

$$
H = \bigcup_{0\le t_1,\dots,t_m\le p-1} H_{t_1,\dots,t_m} := \bigcup_{0\le t_1,\dots,t_m\le p-1} \left\{ x=(x_1,\dots,x_m)\in\mathbb{R}^m : 0\le x_1-\frac{t_1}{p}\le\cdots\le x_m-\frac{t_m}{p}\le\frac{1}{p}\right\}.
$$

Let I and II be the two summands on the right-hand side of (3-19). Then

$$
\begin{aligned}
I \le (N+1)^m \max_{u\in(\partial D(0,r))^m} \left| \sum_{J\in\Gamma_{m,N}} \prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x \right| \\
\times \int\!\cdots\!\int_{(\partial D(0,r))^m} \big|\log|\tilde{s}_N(u)|\big|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m). \qquad (3\text{-}20)
\end{aligned}
$$

First we estimate $\int\cdots\int_{(\partial D(0,r))^m} \big|\log|\tilde{s}_N(u)|\big|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m)$.

Lemma 3.5.
$$\gamma_N\Big(\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)|<1\Big) \le e^{-Q_{r,m}(N)}.$$

Proof.
$$\tilde{s}_N(u) = \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} u^K \quad\Rightarrow\quad \frac{\partial^K}{\partial u^K}\tilde{s}_N(0) = K!\sqrt{\binom{N}{K}} c_K,$$

where $\partial^K/\partial u^K$ refers to $(\partial^{k_1}/\partial u_1^{k_1})\cdots(\partial^{k_m}/\partial u_1^{k_m})$ and $K! = k_1!\cdots k_m!$.

By Cauchy's integral formula,

$$\frac{\partial^K}{\partial u^K}\tilde{s}_N(0) = \frac{K!}{(2\pi\sqrt{-1})^m} \int\!\cdots\!\int_{(\partial D(0,r))^m} \frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}}\, du_1\cdots du_m,$$

so

$$c_K = \binom{N}{K}^{-\frac{1}{2}} \frac{1}{(2\pi\sqrt{-1})^m} \int \cdots \int_{(\partial D(0,r))^m} \frac{\tilde{s}_N(u)}{\prod_{i=1}^m u_i^{k_i+1}} \, du_1 \cdots du_m,$$

and thus

$$|c_K| \le \frac{\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)|}{\sqrt{\binom{N}{K}} r^{|K|}} \quad \text{for all } K \in \Lambda_{m,N}.$$

Therefore, $\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)| < 1$ would imply that, for all $K \in \Lambda_{m,N}$,

$$|c_K| \le \left[\binom{N}{K} r^{2|K|}\right]^{-\frac{1}{2}}.$$

Therefore,

$$\begin{aligned}
\gamma_N\big(\sup_{u\in(\partial D(0,r))^m} |\tilde{s}_N(u)| < 1\big) &\le \prod_{K\in\Lambda_{m,N}} \gamma_N\left(|c_K| \le \left[\binom{N}{K} r^{2|K|}\right]^{-\frac{1}{2}}\right) \\
&\le \prod_{K\in\Lambda_{m,N}} \left[\binom{N}{K} r^{2|K|}\right]^{-1} \\
&= e^{-Q_{r,m}(N)}.
\end{aligned}$$

□

The next lemma follows directly from the first part of [Shiffman et al. 2008, Theorem 3.1], but here we provide a self-contained proof without using the language of sections and metrics.

Lemma 3.6. *Given $U \subset \mathbb{C}^m$ open and bounded with $\sup_{z\in\overline{U}} \|z\| = R > 0$, for all $\eta > 0$,*

$$\gamma_N\left\{\sup_{z\in\overline{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\right\} \le e^{-e^{\eta N}} \quad \textit{for } N \gg 1.$$

Proof. By the Cauchy–Schwartz inequality,

$$\begin{aligned}
\sup_{z\in\overline{U}} |\tilde{s}_N(z)| = \sup_{z\in\overline{U}} \left| \sum_{K\in\Lambda_{m,N}} c_K \sqrt{\binom{N}{K}} z^K \right| &\le \|c\| \sup_{z\in\overline{U}} \left[\sum_{K\in\Lambda_{m,N}} \binom{N}{K} |z|^{2K} \right]^{\frac{1}{2}} \\
&= \|c\| \sup_{z\in\overline{U}} (1+\|z\|^2)^{\frac{N}{2}} \\
&= \|c\| (1+R^2)^{\frac{N}{2}},
\end{aligned}$$

so

$$\gamma_N\left\{\sup_{z\in\overline{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\right\} \le \gamma_N\{\|c\| > e^{\eta N}\} = e^{-e^{2\eta N}} \sum_{k=0}^{\binom{N+m}{m}-1} \frac{e^{(2\eta N)k}}{k!};$$

hence,

$$\begin{aligned}
\log \gamma_N\left\{\sup_{z\in\overline{U}} |\tilde{s}_N(z)| > (1+R^2)^{\frac{N}{2}} e^{\eta N}\right\} &\le -e^{2\eta N} + \log \binom{N+m}{m} + (2\eta N)\left[\binom{N+m}{m} - 1\right] \\
&\le -e^{\eta N} \quad \text{for } N \gg 1.
\end{aligned}$$

□

Lemma 3.7. $$\int\cdots\int_{(\partial D(0,r))^m} \big|\log|\tilde{s}_N(u)|\big|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m) \le CN/\delta^m$$

for some constant C outside an event of probability at most $e^{-e^N}+e^{-Q_{\kappa r,m}(N)}$.

Proof. Applying Lemma 3.6 to $U=(D(0,r))^m$, we have

$$\gamma_N\Big\{\sup_{u\in(\partial D(0,r))^m}|\tilde{s}_N(u)|>(1+mr^2)^{\frac{N}{2}}e^{\eta N}\Big\}\le\gamma_N\Big\{\sup_{u\in(\bar{D}(0,r))^m}|\tilde{s}_N(u)|>(1+mr^2)^{\frac{N}{2}}e^{\eta N}\Big\}\le e^{-e^{\eta N}}. \tag{3-21}$$

Therefore, taking $\eta=1$, outside an event of probability at most e^{-e^N} we have

$$\log^+|\tilde{s}_N(u)|\le\tfrac{1}{2}N\log(1+mr^2)+N \quad \text{on } (\partial D(0,r))^m,$$

so

$$\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\le\tfrac{1}{2}N\log(1+mr^2)+N. \tag{3-22}$$

Applying Lemma 3.5 to the distinguished boundary $(\partial D(0,\kappa r))^m$, we have, outside an event of probability at most $e^{-Q_{\kappa r,m}(N)}$, $\sup_{u\in(\partial D(0,\kappa r))^m}|\tilde{s}_N(u)|\ge 1$, i.e., there exists some $\eta\in(\partial D(0,\kappa r))^m$ such that $|\tilde{s}_N(\eta)|\ge 1$ and

$$\begin{aligned}0\le\log|\tilde{s}_N(\eta)|&\le\int\cdots\int_{(\partial D(0,r))^m}\log|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&=\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&\qquad-\int\cdots\int_{(\partial D(0,r))^m}\log^-|\tilde{s}_N(u)|\prod_{i=1}^m P_r(\eta_i,u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m).\end{aligned} \tag{3-23}$$

Since for all $1\le i\le m$, $|\eta_i|=\kappa r=(1-\sqrt{\delta})r$ and $|u_i|=r$ we have $\sqrt{\delta}/2\le P_r(\eta_i,u_i)\le 2/\sqrt{\delta}$, (3-23) implies that, outside an event of probability at most $e^{-Q_{\kappa r,m}(N)}$,

$$\begin{aligned}\Big(\frac{\sqrt{\delta}}{2}\Big)^m&\int\cdots\int_{(\partial D(0,r))^m}\log^-|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&\le\Big(\frac{2}{\sqrt{\delta}}\Big)^m\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m).\end{aligned} \tag{3-24}$$

Combining (3-22) and (3-24), we get that, outside an event of probability at most $e^{-e^N}+e^{-Q_{\kappa r,m}(N)}$,

$$\begin{aligned}&\int\cdots\int_{(\partial D(0,r))^m}\big|\log|\tilde{s}_N(u)|\big|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\\&\quad=\int\cdots\int_{(\partial D(0,r))^m}\log^+|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)+\int\cdots\int_{(\partial D(0,r))^m}\log^-|\tilde{s}_N(u)|\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\end{aligned}$$

$$\leq \Big[1+\Big(\frac{4}{\delta}\Big)^m\Big] \underset{(\partial D(0,r))^m}{\int\cdots\int} \log^+|\tilde{s}_N(u)|\, d\sigma_r(u_1)\cdots d\sigma_r(u_m)$$

$$\leq \Big[1+\Big(\frac{4}{\delta}\Big)^m\Big]\Big[\frac{1}{2}N\log(1+mr^2)+N\Big] = \frac{CN}{\delta^m}. \qquad \square$$

Lemma 3.8. $\displaystyle \max_{u\in(\partial D(0,r))^m}\Big|\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Big| \leq \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}}.$

Proof. For all $u\in(\partial D(0,r))^m$,

$$\begin{aligned}
&\Big|\sum_{J\in\Gamma_{m,N}}\prod_{i=1}^m \frac{P_r(\xi_{i,j_i},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Big| \\
&\quad= \Big|\sum_{\tau(J)\in\Gamma_{m,N}}\prod_{i=1}^m \frac{P_r(\xi_{i,\tau(j_i)},u_i)}{N+1} - \int_H \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Big| \\
&\quad\leq \sum_{0\leq t_1,\dots,t_m\leq p-1}\Big|\sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}}\prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Big| \\
&\quad\leq \sum_{0\leq t_1,\dots,t_m\leq p-1}\Big|\sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}}\prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Big| \\
&\qquad\qquad + \sum_{0\leq t_1,\dots,t_m\leq p-1}\Big|\int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x \\
&\qquad\qquad\qquad - \int_{H_{t_1,\dots,t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Big|, \qquad (3\text{-}25)
\end{aligned}$$

where $H_{t_1,\dots,t_m}(N)=\bigcup_{J\in I_{t_1}\times\cdots\times I_{t_m}:\,\tau(J)\in\Gamma_{m,N}}\Big[\frac{j_1}{N+1},\frac{j_1+1}{N+1}\Big]\times\cdots\times\Big[\frac{j_m}{N+1},\frac{j_m+1}{N+1}\Big]$.

For all $0\leq t_1,\dots,t_m\leq p-1$,

$$\begin{aligned}
&\Big|\sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}}\prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\, d_m x\Big| \\
&\leq \sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}}\int_{\left[\frac{j_1}{N+1},\frac{j_1+1}{N+1}\right]\times\cdots\times\left[\frac{j_m}{N+1},\frac{j_m+1}{N+1}\right]}\Big|\prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i) - \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}\frac{j_i}{N+1}},u_i)\Big|\, d_m x \\
&\leq \frac{(q+1)^m}{(N+1)^m}m \sup_{|\omega|=\kappa r,\,|u|=r}[P_r(\omega,u)]^{m-1}\sup_{|\omega|\leq\kappa r,\,|u|=r}\Big|\frac{\partial P_r(\omega,u)}{\partial\omega}\Big|\frac{2\pi\kappa r}{N+1}
\end{aligned}$$

$$\leq \frac{C}{p^m \delta^{\frac{1}{2}(m+1)}(N+1)},$$

so

$$\sum_{0\leq t_1,\dots,t_m\leq p-1} \left| \sum_{\substack{J\in I_{t_1}\times\cdots\times I_{t_m}\\ \tau(J)\in\Gamma_{m,N}}} \prod_{i=1}^m \frac{P_r(z_{j_i},u_i)}{N+1} - \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x \right| \leq \frac{C}{\delta^{\frac{1}{2}(m+1)}(N+1)} = \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}}. \quad (3\text{-}26)$$

To bound the second term in (3-25), we need the following statement, which can be proved in a similar way as Lemma 3.3:

$$\lim_{N\to\infty} \mathrm{Vol}_{\mathbb{R}^m}(H_{t_1,\dots,t_m}(N) \triangle H_{t_1,\dots,t_m}) = 0 \quad \text{for any } 0\leq t_1,\dots,t_m\leq p-1.$$

Hence,

$$\begin{aligned}\sum_{0\leq t_1,\dots,t_m\leq p-1} &\left| \int_{H_{t_1,\dots,t_m}(N)} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x - \int_{H_{t_1,\dots,t_m}} \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d_m x \right| \\ &\leq \sum_{0\leq t_1,\dots,t_m\leq p-1} \mathrm{Vol}_{R^m}(H_{t_1,\dots,t_m}(N)\triangle H_{t_1,\dots,t_m}) \Big[\sup_{|\omega|=\kappa r,|u|=r} P_r(\omega,u)\Big]^m \\ &\leq \sum_{0\leq t_1,\dots,t_m\leq p-1} o(1)\left(\frac{2}{\sqrt{\delta}}\right)^m = \frac{o(1)}{\delta^{\frac{1}{2}m}}. \end{aligned} \quad (3\text{-}27)$$

This $o(1)$ may depend on p.

By (3-25), (3-26) and (3-27) the lemma is proved. □

Combining (3-20), Lemma 3.7 and Lemma 3.8, we have, outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$,

$$I \leq (N+1)^m \frac{o(1)}{\delta^{\frac{1}{2}(m+1)}} \frac{CN}{\delta^m} = \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.$$

By changing the order of integration,

$$II = (N+1)^m \int_H \int\cdots\int_{(\partial D(0,r))^m} \log|\tilde{s}_N(u)| \prod_{i=1}^m P_r(\kappa r e^{2\pi\sqrt{-1}x_i},u_i)\,d\sigma_r(u_1)\cdots d\sigma_r(u_m)\,d_m x.$$

If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$, $\log|\tilde{s}_N(u)|$ is harmonic in u_i in a neighborhood of $\bar{D}(0,r)$ for each fixed $(u_1,\dots,\hat{u}_i,\dots,u_m)$ in $(\bar{D}(0,r))^{m-1}$. Applying the mean value theorem for harmonic functions, we get

$$II = (N+1)^m \int_H \int\cdots\int_{(\partial D(0,r))^m} \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, u_2, \dots, u_m)| \times \prod_{i=2}^{m} P_r(\kappa r e^{2\pi\sqrt{-1}x_i}, u_i)\, d\sigma_r(u_2)\cdots d\sigma_r(u_m)\, d_m x$$
$$= \cdots = (N+1)^m \int_H \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})|\, d_m x.$$

Define

$$\Xi = \int_H \log |\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})|\, d_m x, \tag{3-28}$$

which is a complex random variable. Thus we have proved:

Lemma 3.9. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then*

$$\log \prod_{J\in\Gamma_{m,N}} |\zeta_J| \le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (N+1)^m\,\Xi$$

outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$.

Replacing $\Gamma_{m,N}$ by $\Gamma^{(\varrho)}_{m,N} = \{J = (j_1,\dots,j_m) \in [0,N]^m \cap \mathbb{Z}^m : 0 \le j_{\varrho(1)} \le \cdots \le j_{\varrho(m)} \le N\}$, where ϱ can be any element in S_m, the permutation group of m letters, similar results hold and we have counterparts for Lemma 3.4 and Lemma 3.9, which we state without proof.

Lemma 3.10. *Denote the covariance matrix of the random vector $(\zeta^{(\varrho)}_J = \tilde{s}_N(\xi_J))^t_{J\in\Gamma^{(\varrho)}_{m,N}}$ by $\Sigma^{(\varrho)}$. Then*

$$\log(\det \Sigma^{(\varrho)}) = Q_{\kappa r,m}(N) + \frac{2\beta_m}{p} N^{m+1} + o(N^{m+1}).$$

For all $\varrho \in S_m$, let

$$H^{(\varrho)} = \bigcup_{0\le t_1,\dots,t_m\le p-1} H^{(\varrho)}_{t_1,\dots,t_m}$$
$$:= \bigcup_{0\le t_1,\dots,t_m\le p-1} \left\{x = (x_1,\dots,x_m) \in \mathbb{R}^m : 0 \le x_{\varrho(1)} - \frac{t_{\varrho(1)}}{p} \le \cdots \le x_{\varrho(m)} - \frac{t_{\varrho(m)}}{p} \le \frac{1}{p}\right\},$$

and define the random variable

$$\Xi^{(\varrho)} = \int_{H^{(\varrho)}} \log \left|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1}, \dots, \kappa r e^{2\pi\sqrt{-1}x_m})\right| d_m x.$$

Then:

Lemma 3.11. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then*

$$\log \prod_{J\in\Gamma^{(\varrho)}_{m,N}} |\zeta^{(\varrho)}_J| \le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (N+1)^m\,\Xi^{(\varrho)}$$

outside an event of probability at most $e^{-e^N} + e^{-Q_{\kappa r,m}(N)}$.

The last ingredient we need to prove the upper bound is the following lemma:

Lemma 3.12 [Nishry 2010, Lemma 4.6]. *Let s, $t > 0$ and $N \in \mathbb{N}^+$ be such that $\log(t^N/s) \geq N$; then*

$$\mathrm{Vol}_{\mathbb{R}^N}\left\{(r_1,\dots,r_N)\in\mathbb{R}^N : 0\le r_j\le t \ \ and \ \prod_{j=1}^{N} r_j \le s\right\} \le \frac{s}{(N-1)!}\log^N\left(\frac{t^N}{s}\right).$$

Proof of the upper bound in Theorem 0.1. If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then, by the mean value property of pluriharmonic functions,

$$\begin{aligned}
\sum_{\varrho\in S_m}\Xi^{(\varrho)} &= \sum_{\varrho\in S_m}\int_{H^{(\varrho)}}\log\big|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})\big|\,d_m x\\
&= \int_{\bigcup_{\varrho\in S_m} H^{(\varrho)}}\log\big|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})\big|\,d_m x\\
&= \int_0^1\!\!\cdots\!\int_0^1\log\big|\tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}x_1},\dots,\kappa r e^{2\pi\sqrt{-1}x_m})\big|\,dx_1\cdots dx_m\\
&= \int\!\!\cdots\!\!\int_{(\partial D(0,\kappa r))^m}\log|\tilde{s}_N(\omega_1,\dots,\omega_m)|\,d\sigma_{\kappa r}(\omega_1)\cdots d\sigma_{\kappa r}(\omega_m)\\
&= \log|\tilde{s}_N(0,\dots,0)|\\
&= \log|c_{(0,\dots,0)}|;
\end{aligned}$$

the second equality holds because, for distinct $\varrho_1,\varrho_2\in S_m$, $H^{(\varrho_1)}\cap H^{(\varrho_2)}$ is of m-dimensional Lebesgue measure zero. Then,

$$\begin{aligned}
P_{0,m}(r,N) &= \gamma_N\big\{0\notin\tilde{s}_N\big((\bar{D}(0,r))^m\big)\big\}\\
&= \gamma_N\big\{(\log|c_{(0,\dots,0)}|>2m!\log N)\cap\big(0\notin\tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\big\}\\
&\qquad\qquad + \gamma_N\big\{(\log|c_{(0,\dots,0)}|\le 2m!\log N)\cap\big(0\notin\tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\big\}\\
&\le \gamma_N(|c_{(0,\dots,0)}|>N^{2m!}) + \gamma_N\left\{\left(\sum_{\varrho\in S_m}\Xi^{(\varrho)}\le 2m!\log N\right)\cap\big(0\notin\tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\right\}\\
&\le e^{-N^{4m!}} + \gamma_N\left\{\bigcup_{\varrho\in S_m}(\Xi^{(\varrho)}\le 2\log N)\cap\big(0\notin\tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\right\}\\
&\le e^{-N^{4m!}} + \sum_{\varrho\in S_m}\gamma_N\big\{(\Xi^{(\varrho)}\le 2\log N)\cap\big(0\notin\tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\big\}.
\end{aligned}$$

Lemma 3.9 implies

$$\begin{aligned}\gamma_N\{(\Xi \le 2\log N)\cap\big(0\notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\}\\ &\le e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N\Big\{\log\prod_{J\in\Gamma_{m,N}}|\zeta_J|\le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}\\ &= e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N\Big\{\prod_{J\in\Gamma_{m,N}}|\zeta_J|\le \exp\Big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}\Big\}.\end{aligned}$$

Define

$$\mathscr{E}_{m,N}=\Big\{\zeta=(\zeta_J)_{J\in\Gamma_{m,N}}\in\mathbb{C}^{\binom{N+m}{m}}:\prod_{J\in\Gamma_{m,N}}|\zeta_J|\le\exp\Big\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}+2(N+1)^m\log N\Big\}\Big\},$$

and

$$\mathscr{F}_{m,N}=\big\{\zeta=(\zeta_J)_{J\in\Gamma_{m,N}}\in\mathscr{E}_{m,N}:|\zeta_J|\le(2+2mr^2)^{\frac{N}{2}}\ \ \forall J\in\Gamma_{m,N}\big\}\subset\mathscr{E}_{m,N},$$

which can both be treated as subsets in $\mathbb{C}^{\binom{N+m}{m}}$ and events in the probability space $\big(H^0(\mathbb{CP}^m,\mathscr{O}(N)),\gamma_N\big)$. Thus,

$$\begin{aligned}\gamma_N\{(\Xi\le 2\log N)\cap\big(0\notin\tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\}\\ &\le e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N(\mathscr{E}_{m,N})\\ &\le e^{-e^N}+e^{-Q_{\kappa r,m}(N)}+\gamma_N(\mathscr{E}_{m,N}\setminus\mathscr{F}_{m,N})+\gamma_N(\mathscr{F}_{m,N})\end{aligned}\tag{3-29}$$

and

$$\begin{aligned}\gamma_N(\mathscr{E}_{m,N}\setminus\mathscr{F}_{m,N})&\le\gamma_N\{|\zeta_J|>(2+2mr^2)^{\frac{N}{2}}\ \text{ for some }\ J\in\Gamma_{m,N}\}\\ &\le\gamma_N\{\sup_{\omega\in(\partial D(0,\kappa r))^m}|\tilde{s}_N(\omega)|>(2+2mr^2)^{\frac{N}{2}}\}\\ &\le\gamma_N\{\sup_{\omega\in(\bar{D}(0,r))^m}|\tilde{s}_N(\omega)|>(1+mr^2)^{\frac{N}{2}}2^{\frac{N}{2}}\}\\ &\le e^{-2^{\frac{N}{2}}},\end{aligned}\tag{3-30}$$

where the last inequality is due to Lemma 3.6. Then Lemma 3.4 gives

$$\begin{aligned}\gamma_N(\mathscr{F}_{m,N})&=\frac{1}{\pi^{\binom{N+m}{m}}\det\Sigma}\int_{\mathscr{F}_{m,N}}e^{-\zeta^*\Sigma^{-1}\zeta}\,d_{2\binom{N+m}{m}}\zeta\\ &\le\exp\Big\{-\Big[Q_{\kappa r,m}(N)+\frac{2\beta_m}{p}N^{m+1}\Big]+o(N^{m+1})\Big\}\pi^{-\binom{N+m}{m}}\operatorname{Vol}_{\mathbb{C}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N}).\end{aligned}$$

Change into polar coordinates and note that

$$
\begin{aligned}
&\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N}) \\
&= \mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}\left\{(x_J)_{J\in\Gamma_{m,N}} \in \left[0,(2+2mr^2)^{\frac{N}{2}}\right]^{\binom{N+m}{m}} : \prod_{J\in\Gamma_{m,N}} x_J \le \exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right\}\right\};
\end{aligned}
$$

then

$$
\begin{aligned}
\gamma_N(\mathscr{F}_{m,N}) &\le 2^{\binom{N+m}{m}} \exp\left\{-\left[Q_{\kappa r,m}(N) + \frac{2\beta_m}{p} N^{m+1}\right] + o(N^{m+1})\right\} \\
&\qquad \times \exp\left\{\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right\} \mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N}) \\
&= 2^{\binom{N+m}{m}} \exp\left\{-\left[Q_{\kappa r,m}(N) + \frac{2\beta_m}{p} N^{m+1}\right] + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\right\} \mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N}).
\end{aligned}
$$

Since $\binom{N+m}{m}\frac{1}{2}N\log(2+2mr^2) - [o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}} + 2(N+1)^m \log N] > \binom{N+m}{m}$ for N large (up to now p, δ are constants), we can apply Lemma 3.12 and get:

$$
\begin{aligned}
&\mathrm{Vol}_{\mathbb{R}^{\binom{N+m}{m}}}(\mathscr{F}_{m,N}) \\
&\quad \le \frac{\exp\{o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}} + 2(N+1)^m \log N\}}{[\binom{N+m}{m} - 1]!} \left\{\binom{N+m}{m}\frac{N}{2}\log(2+2mr^2)\right. \\
&\qquad\qquad \left. - \left[\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N\right]\right\}^{\binom{N+m}{m}} \\
&\quad \le \frac{\exp\{o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}} + 2(N+1)^m \log N\}}{2^{\binom{N+m}{m}}[\binom{N+m}{m} - 1]!} \left[N\binom{N+m}{m}\log(2+2mr^2)\right]^{\binom{N+m}{m}};
\end{aligned}
$$

then,

$$
\begin{aligned}
\gamma_N(\mathscr{F}_{m,N}) &\le \frac{\exp\{o(N^{m+1})/\delta^{\frac{3}{2}m+\frac{1}{2}} + 2(N+1)^m \log N - [Q_{\kappa r,m}(N) + \frac{2\beta_m}{p}N^{m+1}]\}}{[\binom{N+m}{m} - 1]!} \\
&\qquad \times \left[N\binom{N+m}{m}\log(2+2mr^2)\right]^{\binom{N+m}{m}},
\end{aligned}
$$

so

$$
\begin{aligned}
\log\gamma_N(\mathscr{F}_{m,N}) &\le \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + 2(N+1)^m \log N - \left[Q_{\kappa r,m}(N) + \frac{2\beta_m}{p}N^{m+1}\right] \\
&\qquad + \binom{N+m}{m}\log\left[N\binom{N+m}{m}\log(2+2mr^2)\right] - \log\left[\binom{N+m}{m} - 1\right]! \\
&= -Q_{\kappa r,m}(N) - \frac{2\beta_m}{p}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}. \qquad \text{(3-31)}
\end{aligned}
$$

By Lemma 2.2, (3-29), (3-30) and (3-31),

$$\begin{aligned}
&\gamma_N\{(\Xi \le 2\log N)\cap \big(0\notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\} \\
&\quad \le e^{-e^N} + e^{-Q_{\kappa r,m}(N)} + e^{-2^{\frac{N}{2}}} + \exp\Big\{-Q_{\kappa r,m}(N) - \frac{2\beta_m}{p}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\} \\
&\quad \le \exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\}.
\end{aligned}$$

Similarly, for all $\varrho \in S_m$,

$$\begin{aligned}
&\gamma_N\{(\Xi^{(\varrho)} \le 2\log N)\cap \big(0\notin \tilde{s}_N\big((\bar{D}(0,r))^m\big)\big)\} \\
&\quad \le \exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\};
\end{aligned}$$

thus,

$$\begin{aligned}
&P_{0,m}(r,N) \\
&\quad \le e^{-N^{4m!}} \\
&\qquad + m!\exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\} \\
&\quad = \exp\Big\{-\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\Big\},
\end{aligned}$$

so

$$\begin{aligned}
&\log P_{0,m}(r,N) \\
&\qquad \le -\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}N^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}},
\end{aligned}$$

and thus

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}} \le -\min\Big\{\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k},\ \frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k} + \frac{2\beta_m}{p}\Big\}.$$

Let $p\to\infty$; then

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}} \le -\Big[\frac{2m\log(\kappa r)}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\Big].$$

Let $\delta\to 0+$; then $\kappa = 1-\sqrt{\delta}\to 1$, so

$$\limsup_{N\to\infty}\frac{\log P_{0,m}(r,N)}{N^{m+1}} \le -\Big[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\Big].$$

Hence,

$$\log P_{0,m}(r,N) \le -\left[\frac{2m\log r}{(m+1)!} + \frac{1}{m!}\sum_{k=2}^{m+1}\frac{1}{k}\right]N^{m+1} + o(N^{m+1}).$$

Thus, Theorem 0.1 is proved. □

4. Proof of Theorem 0.2

The proof of Theorem 0.2 is quite similar to that of Theorem 0.1. We only need to make some slight modifications in picking "determining exponents" and "sampling points".

Lower bound.

Definition 4.1.

$$\Lambda_{m,N}(r) := \left\{K \in \Lambda_{m,N} : \binom{N}{K} r^{2|K|} \ge 1\right\} \subset \Lambda_{m,N},$$

$$R_{r,m}(N) := \sum_{K\in\Lambda_{m,N}(r)} \log\left[\binom{N}{K} r^{2|K|}\right].$$

Lemma 4.2.

$$\log P_{0,m}(r,N) \ge -R_{r,m}(N) + o(N^{m+1}).$$

Proof. Consider the following event $\Omega_{r,m,N}$:

(i) $|c_{(0,\ldots,0)}| \ge \sqrt{N}$,

(ii) $|c_K| \le \dfrac{1}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}}, \quad K \in \Lambda_{m,N}(r)\backslash\{(0,\ldots,0)\}$,

(iii) $|c_K| \le \dfrac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}, \quad K \in \Lambda_{m,N}\backslash\Lambda_{m,N}(r)$.

Then, when $\Omega_{r,m,N}$ occurs, for all $z \in (\bar{D}(0,r))^m$,

$$\begin{aligned}
|\tilde{s}_N(z)| &\ge \sqrt{N} - \sum_{K\in\Lambda_{m,N}(r)\backslash\{(0,\ldots,0)\}} \frac{\sqrt{\binom{N}{K}}r^{|K|}}{2\sqrt{N}\sqrt{\binom{N}{K}r^{|K|}\binom{|K|+m-1}{m-1}}} - \sum_{K\in\Lambda_{m,N}\backslash\Lambda_{m,N}(r)} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{K\in\Lambda_{m,N}\backslash\{(0,\ldots,0)\}} \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}} \\
&= \sqrt{N} - \sum_{k=1}^{N} \frac{1}{2\sqrt{N}} \\
&= \tfrac{1}{2}\sqrt{N} > 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
P_{0,m}(r,N) &\geq \gamma_N(\Omega_{r,m,N}) \\
&= \gamma_N(|c_{(0,\ldots,0)}| \geq \sqrt{N}) \prod_{K\in\Lambda_{m,N}(r)\setminus\{(0,\ldots,0)\}} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\sqrt{\binom{N}{K}}r^{|K|}\binom{|K|+m-1}{m-1}}\right) \\
&\qquad \times \prod_{K\in\Lambda_{m,N}\setminus\Lambda_{m,N}(r)} \gamma_N\left(|c_K| \leq \frac{1}{2\sqrt{N}\binom{|K|+m-1}{m-1}}\right) \\
&\geq e^{-N} \prod_{K\in\Lambda_{m,N}(r)\setminus\{(0,\ldots,0)\}} \frac{1}{8N\binom{N}{K}r^{2|K|}\binom{|K|+m-1}{m-1}^2} \prod_{K\in\Lambda_{m,N}\setminus\Lambda_{m,N}(r)} \frac{1}{8N\binom{|K|+m-1}{m-1}^2},
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\log P_{0,m}(r,N) \\
&\quad \geq -N - \sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\ldots,0)\}} \log\left[\binom{N}{K}r^{2|K|}\right] - \sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\ldots,0)\}} \log\left[8N\binom{|K|+m-1}{m-1}^2\right] \\
&\quad = -\sum_{K\in\Lambda_{m,N}(r)\setminus\{(0,\ldots,0)\}} \log\left[\binom{N}{K}r^{2|K|}\right] + o(N^{m+1}) \\
&\quad = -R_{r,m}(N) + o(N^{m+1}). \qquad \square
\end{aligned}$$

Upper bound. For some $\alpha \in (0,1]$, we can define the index sets $\Lambda_{m,\lfloor\alpha N\rfloor}$ and $\Gamma_{m,\lfloor\alpha N\rfloor}$, and the $\binom{\lfloor\alpha N\rfloor+m}{m} \times \binom{\lfloor\alpha N\rfloor+m}{m}$ matrix

$$W_{m,\lfloor\alpha N\rfloor}(\xi) = (\xi_J^K)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,\lfloor\alpha N\rfloor}}.$$

We also assign the values of the variables $(\xi_{i,j})_{0\leq i\leq m,\,0\leq j\leq\lfloor\alpha N\rfloor}$ by the points on $\partial D(0,\kappa r)$ in a way similar to in Section 3, except that we replace N by $\lfloor\alpha N\rfloor$. Then we have the following lemma:

Lemma 4.3. $$\log|\det W_{m,\lfloor\alpha N\rfloor}(\xi)| = m\binom{\lfloor\alpha N\rfloor+m}{m+1}\log(\kappa r) + \frac{\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}).$$

The word $\zeta = (\zeta_J)^t_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}} = (\tilde{s}_N(\xi_J))^t_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}}$ is a dimension-$\binom{\lfloor\alpha N\rfloor+m}{m}$ mean zero complex Gaussian random vector with covariance matrix

$$\Sigma = V_{m,N,\alpha}(\xi)V^*_{m,N,\alpha}(\xi),$$

where $V_{m,N,\alpha}(\xi) = \left(\sqrt{\binom{N}{K}}\xi_J^K\right)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,N}}$ is a $\binom{\lfloor\alpha N\rfloor+m}{m} \times \binom{N+m}{m}$ matrix.

Definition 4.4. $$Q_{r,m,\alpha}(N) := \sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}} \log\left[\binom{N}{K}r^{2|K|}\right].$$

Lemma 4.5. $$\log\det\Sigma \geq Q_{\kappa r,m,\alpha}(N) + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}).$$

Proof. By the Cauchy–Binet identity, summing over the $\binom{\lfloor \alpha N\rfloor+m}{m} \times \binom{\lfloor \alpha N\rfloor+m}{m}$ minors of $V_{m,N,\alpha}(\xi)$,

$$\det \Sigma = \sum_M |\det M|^2 \geq \left|\det\left(\sqrt{\binom{N}{K}}\xi_J^K\right)_{J\in\Gamma_{m,\lfloor\alpha N\rfloor},\ K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\right|^2 = \prod_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}} \binom{N}{K} |\det W_{m,\lfloor\alpha N\rfloor}(\xi)|^2,$$

so

$$\begin{aligned}\log\det\Sigma &\geq \sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}} \log\binom{N}{K} + 2m\binom{\lfloor\alpha N\rfloor+m}{m+1}\log(\kappa r) + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}) \\ &= \sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}} \log\left[\binom{N}{K}(\kappa r)^{2|K|}\right] + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}) \\ &= Q_{\kappa r,m,\alpha}(N) + \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + o(N^{m+1}).\end{aligned}$$ □

The following lemma is a counterpart of Lemma 3.9. The proof is similar.

Lemma 4.6. *If $\tilde{s}_N$ is nonvanishing on $(\bar{D}(0,r))^m$ then*

$$\log \prod_{J\in\Gamma_{m,\lfloor\alpha N\rfloor}} |\zeta_J| \leq \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}} + (\lfloor\alpha N\rfloor+1)^m\,\Xi$$

outside an event of probability at most $e^{-e^N} + e^{-R_{\kappa r,m}(N)}$, where the complex random variable Ξ is defined in (3-28).

By the same trick of permutation as in Section 3, we can get an upper bound estimate for $P_{0,m}(r,N)$:

$$P_{0,m}(r,N) \leq e^{-N^{4m!}} + m!\left\{e^{-e^N} + e^{-R_{\kappa r,m}(N)} + e^{-2^{\frac{N}{2}}} + \exp\left[-Q_{\kappa r,m,\alpha}(N) - \frac{2\beta_m}{p}(\lfloor\alpha N\rfloor)^{m+1} + \frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}\right]\right\}. \quad (4\text{-}1)$$

Punch line of the proof. In order to prove Theorem 0.2, it suffices to compute $R_{r,m}(N)$ and $Q_{r,m,\alpha}(N)$ asymptotically. We follow the same idea as in Lemma 2.2.

The scaled lattice $(1/N)\Lambda_{m,N}(r)$ corresponds to the set

$$\{x = (x_1,\dots,x_m) \in \Sigma_m : E_r(x) \geq 0\}$$

and $(1/N)\Lambda_{r,m,\alpha}(N)$ corresponds to the set

$$\left\{x = (x_1,\dots,x_m) \in \mathbb{R}^{m+} : \sum_{i=1}^m x_i \leq \alpha \leq 1\right\}.$$

So we have

$$R_{r,m}(N)=\sum_{K\in\Lambda_{m,N}(r)}\log\left[\binom{N}{K}r^{2|K|}\right]=N^{m+1}\int_{\substack{x\in\Sigma_m\\ E_r(x)\geq 0}}E_r(x)\,d_mx+o(N^{m+1}),\tag{4-2}$$

$$Q_{r,m,\alpha}(N)=\sum_{K\in\Lambda_{m,\lfloor\alpha N\rfloor}}\log\left[\binom{N}{K}r^{2|K|}\right]=N^{m+1}\int_{\substack{x\in\mathbb{R}^{m+}\\ \sum_{i=1}^m x_i\leq\alpha}}E_r(x)\,d_mx+o(N^{m+1}).\tag{4-3}$$

Moreover, if we go through the proof of Lemma 2.2, we find that the $o(N^{m+1})$ terms in (4-2) and (4-3) are uniform if $r\leq c$ for some constant $c>0$, which implies that, when r is replaced by $\kappa r=(1-\sqrt{\delta})r$, the remainder won't depend on δ.

Proof of Theorem 0.2. The lower bound proof is already implied by Lemma 4.2 and (4-2). To prove the upper bound, by (4-1) and (4-3),

$$\log P_{0,m}(r,N)$$
$$\leq -N^{m+1}\min\left\{\int_{\substack{x\in\Sigma_m\\ E_{\kappa r}(x)\geq 0}}E_{\kappa r}(x)\,d_mx,\int_{\substack{x\in\mathbb{R}^{m+}\\ \sum_{i=1}^m x_i\leq\alpha}}E_{\kappa r}(x)\,d_mx+\frac{2\beta_m\alpha^{m+1}}{p}\right\}+\frac{o(N^{m+1})}{\delta^{\frac{3}{2}m+\frac{1}{2}}}.$$

Similarly as in Section 3, we obtain

$$\log P_{0,m}(r,N)\leq -N^{m+1}\min\left\{\int_{\substack{x\in\Sigma_m\\ E_r(x)\geq 0}}E_r(x)\,d_mx,\int_{\substack{x\in\mathbb{R}^{m+}\\ \sum_{i=1}^m x_i\leq\alpha}}E_r(x)\,d_mx\right\}+o(N^{m+1})$$
$$=-N^{m+1}\int_{\substack{x\in\mathbb{R}^{m+}\\ \sum_{i=1}^m x_i\leq\alpha}}E_r(x)\,d_mx+o(N^{m+1}).$$

Now we must find a proper $\alpha_0=\alpha_0(r,m)\in(0,1]$ which maximizes $\int_{x\in\mathbb{R}^{m+},\,\sum_{i=1}^m x_i\leq\alpha}E_r(x)\,d_mx$. For this purpose, we consider the function defined on $(0,1]$ by

$$\Upsilon(\alpha):=\int_{x\in\mathbb{R}^{m+}:\sum_{i=1}^m x_i\leq\alpha}E_r(x)\,d_mx.$$

Then

$$\Upsilon(\alpha)=2m\log r\int_{\substack{x\in\mathbb{R}^{m+}\\ \sum_{i=1}^m x_i\leq\alpha}}x_1\,d_mx-m\int_{\substack{x\in\mathbb{R}^{m+}\\ \sum_{i=1}^m x_i\leq\alpha}}x_1\log x_1\,d_mx$$
$$-\int_{\substack{x\in\mathbb{R}^{m+}\\ \sum_{i=1}^m x_i\leq\alpha}}\left(1-\sum_{i=1}^m x_i\right)\log\left(1-\sum_{i=1}^m x_i\right)d_mx$$
$$=2m\log r\frac{\alpha^{m+1}}{(m+1)!}-m\frac{\alpha^{m+1}}{(m+1)!}\left[\log\alpha-\sum_{k=2}^{m+1}\frac{1}{k}\right]-\frac{1}{(m-1)!}\int_0^\alpha(1-x)x^{m-1}\log(1-x)\,dx,$$

$$\Upsilon'(\alpha)=\frac{\alpha^{m-1}}{(m-1)!}\left\{\left(2\log r+\sum_{k=2}^m\frac{1}{k}\right)\alpha-[\alpha\log\alpha+(1-\alpha)\log(1-\alpha)]\right\},$$

where we take $\sum_{k=2}^{m} 1/k = 0$ when $m = 1$. So, if $2\log r + \sum_{k=2}^{m} 1/k \geq 0$, $\Upsilon'(\alpha) \geq 0$ over $(0, 1]$, thus $\max_{(0,1]} \Upsilon = \Upsilon(1)$ and therefore $\alpha_0 = 1$.

If $2\log r + \sum_{k=2}^{m} 1/k < 0$, $\big(2\log r + \sum_{k=2}^{m} 1/k\big)\alpha = \alpha\log\alpha + (1-\alpha)\log(1-\alpha)$ has a unique nonzero root $\alpha_0 \in (0, 1)$, and

$$\max_{(0,1]} \Upsilon = \Upsilon(\alpha_0) = \int_{\substack{x\in\mathbb{R}^{m+} \\ \sum_{i=1}^{m} x_i \leq \alpha_0}} E_r(x)\, d_m x = \frac{1}{(m+1)!}\left[(1-\alpha_0^m)\log(1-\alpha_0) + \sum_{k=1}^{m} \frac{\alpha_0^k}{k}\right]. \tag{4-4}$$

This concludes the proof. □

Remark 4.7. The proofs of Theorems 0.1 and 0.2 also work for a general polydisc $\prod_{i=1}^{m} D(0, r_i)$. For example, if $r = (r_1, \dots, r_m) \in [1, \infty)^m$, the function E_r in Theorem 0.1 would be

$$E_r(x) = 2\sum_{i=1}^{m} x_i \log r_i - \left[\sum_{i=1}^{m} x_i \log x_i + \left(1 - \sum_{i=1}^{m} x_i\right)\log\left(1 - \sum_{i=1}^{m} x_i\right)\right]$$

and $\int_{\Sigma_m} E_r(x)\, d_m x$ would equal $(2/(m+1)!) \sum_{i=1}^{m} \log r_i + (1/m!) \sum_{k=2}^{m+1} 1/k$.

5. Hole probability of SU(2) polynomials

Proof of Corollary 0.4. When $r \geq 1$, $\alpha_0 = 1$. The result follows from Theorem 0.1.

When $0 < r < 1$, for $x \in \mathbb{R}^+$,

$$E_r(x) = 2x\log r - [x\log x + (1-x)\log(1-x)] \geq 0 \quad\Longleftrightarrow\quad 0 \leq x \leq \alpha_0.$$

By Theorem 0.2,

$$\log P_{0,1}(r, N) = -N^2 \int_0^{\alpha_0} E_r(x)\, dx + o(N^2),$$

where the value of the integral in the corollary is due to (4-4) and the fact that

$$2\alpha_0 \log r = \alpha_0 \log\alpha_0 + (1-\alpha_0)\log(1-\alpha_0).$$ □

Proof of Theorem 0.5. As ∂U is a Jordan curve, by Carathéodory's theorem ϕ can be extended to a homeomorphism $\bar{D}(0, 1) \to \bar{U}$. We still use ϕ to denote the extended map. Thus, $\tilde{s}_N(z) = \sum_{k=0}^{N} c_k \binom{N}{k}^{1/2} z^k$ is nonvanishing over $\bar{U}$ if and only if $t_N(\omega) := \sum_{k=0}^{N} c_k \binom{N}{k}^{1/2} (\phi(\omega))^k$ is nonvanishing over $\bar{D}(0, 1)$, where $t_N \in \mathcal{O}(D(0, 1)) \cap \mathscr{C}(\bar{D}(0, 1))$.

Since

$$\begin{bmatrix} t_N(0) \\ t_N'(0) \\ \vdots \\ t_N^{(N)}(0) \end{bmatrix} = A \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_N \end{bmatrix},$$

where A is an $(N+1)\times(N+1)$ lower triangular matrix with diagonal entries $\left\{k!\sqrt{\binom{N}{k}}(\phi'(0))^k\right\}_{0\le k\le N}$, $(t_N(0)\dots\ t_N^{(N)}(0))^t$ is Gaussian with covariance matrix AA^*. Then

$$\det(AA^*) = |\det A|^2 = \prod_{k=0}^{N}\left[k!^2\binom{N}{k}|\phi'(0)|^{2k}\right] \neq 0 \tag{5-1}$$

because ϕ is a biholomorphism.

We again define $\kappa = 1-\sqrt{\delta}$. Then, if $\sup_{\partial D(0,\kappa)}|t_N| < 1$, for $0\le k\le N$,

$$|t_N^{(k)}(0)| = \left|\frac{k!}{2\pi\sqrt{-1}}\int_{\partial D(0,\kappa)}\frac{t_N(u)}{u^{k+1}}\,du\right| \le \frac{k!}{\kappa^k}.$$

Therefore,

$$\begin{aligned}\gamma_N(\sup_{\partial D(0,\kappa)}|t_N|<1) &\le \gamma_N\left\{(t_N(0),\dots,t_N^{(N)}(0))\in\prod_{k=0}^{N}\bar{D}\left(0,\frac{k!}{\kappa^k}\right)\right\}\\ &= \frac{1}{\pi^{N+1}\det(AA^*)}\int_{\prod_{k=0}^{N}\bar{D}(0,k!/\kappa^k)}\exp\{-\eta^*(AA^*)^{-1}\eta\}\,d_{2(N+1)}\eta\\ &\le \frac{\pi^{N+1}\prod_{k=0}^{N}(k!/\kappa^k)^2}{\pi^{N+1}\det(AA^*)}.\end{aligned}$$

By (5-1),

$$\begin{aligned}\gamma_N\Big(\sup_{\partial D(0,\kappa)}|t_N|<1\Big) &\le \frac{\prod_{k=0}^{N}(k!/\kappa^k)^2}{\prod_{k=0}^{N}\left[k!^2\binom{N}{k}|\phi'(0)|^{2k}\right]}\\ &= \left\{\prod_{k=0}^{N}\left[\binom{N}{k}(\kappa|\phi'(0)|)^{2k}\right]\right\}^{-1}\\ &= \exp\{-Q_{\kappa|\phi'(0)|,1}(N)\}\\ &= \exp\left\{-\left(\log|\phi'(0)|+\log\kappa+\tfrac{1}{2}\right)N^2+o(N^2)\right\},\end{aligned}$$

where the last equality is due to Lemma 2.2.

Similarly as in Lemma 3.9, we can show that if $t_N|_{\bar{D}(0,1)}\neq 0$ then, outside an event of probability at most $e^{-e^N}+\exp\{-Q_{\kappa|\phi'(0)|,1}(N)\} = \exp\left\{-\left(\log|\phi'(0)|+\log\kappa+\tfrac{1}{2}\right)N^2+o(N^2)\right\}$,

$$\log\prod_{j=0}^{N}|t_N(z_j)| \le \frac{o(N^2)}{\delta^2}+(N+1)\log|c_0|,$$

where $z_j = \kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}}$, $0\le j\le N$.

Now, $(t_N(z_0)\ \cdots\ t_N(z_N))^t$ is complex Gaussian with covariance matrix

$$\Sigma = \big(\mathbb{E}_N(t_N(z_j)\overline{t_N(z_j)})\big)_{0\le i,j\le N} = \left(\sum_{k=0}^{N}\binom{N}{k}(\phi(z_i))^k(\overline{\phi(z_j)})^k\right)_{0\le i,j\le N}$$

$$= \begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix} \begin{bmatrix} \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_0) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_0))^N \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\binom{N}{0}} & \sqrt{\binom{N}{1}}\phi(z_N) & \cdots & \sqrt{\binom{N}{N}}(\phi(z_N))^N \end{bmatrix}^*$$

and

$$\det\Sigma = \prod_{k=0}^{N}\binom{N}{k}\prod_{0\le i<j\le N}|\phi(z_i)-\phi(z_j)|^2,$$

so

$$\log\det\Sigma = \sum_{k=0}^{N}\log\binom{N}{k} + 2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)|. \tag{5-2}$$

Next we will show that

$$2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)| = N^2\iint_{(\partial D(0,\kappa))^2}\log|\phi(u_1)-\phi(u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2)+o_\delta(N^2), \tag{5-3}$$

where $o_\delta(N^2)$ denotes a lower-order term depending on δ.

Since

$$2\sum_{0\le i<j\le N}\log|\phi(z_i)-\phi(z_j)| = 2(N+1)^2\sum_{0\le i<j\le N}\frac{1}{(N+1)^2}\log\left|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\right|$$

and

$$\begin{aligned}\iint_{(\partial D(0,\kappa))^2}\log|\phi(u_1)-\phi(u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2) &= \int_0^1\int_0^1\log\left|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\right|dx\,dy\\ &= 2\iint_{0\le x\le y\le 1}\log\left|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\right|dx\,dy,\end{aligned}$$

it suffices to show that

$$\left|\sum_{0\le i<j\le N}\frac{1}{(N+1)^2}\log\left|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\right| - \iint_{0\le x\le y\le 1}\log\left|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\right|dx\,dy\right| = o_\delta(1).$$

Since ϕ is a biholomorphism, we set

$$\inf_{\bar{D}(0,\kappa)} |\phi'| = a(\delta) > 0.$$

And, by Cauchy's inequality, we have

$$\sup_{\bar{D}(0,\kappa)} |\phi'| \leq O(\delta^{-1}).$$

For each N, define

$$\Delta(N) = \{(i, j) \in \mathbb{Z}^2 : 0 \leq i < j \leq N\},$$

the "far from diagonal" indices $FD(N)$ to be the set of those $(i, j) \in \Delta(N)$ such that

$$\begin{cases} \lfloor\sqrt{N+1}\rfloor + i \leq j \leq N - \lfloor\sqrt{N+1}\rfloor + i & \text{if } 0 \leq i \leq \lfloor\sqrt{N+1}\rfloor, \\ \lfloor\sqrt{N+1}\rfloor + i \leq j \leq N & \text{if } \lfloor\sqrt{N+1}\rfloor < i \leq N - \lfloor\sqrt{N+1}\rfloor, \\ j \in \varnothing & \text{if } i > N - \lfloor\sqrt{N+1}\rfloor, \end{cases}$$

with

$$\mathcal{FD}(N) = \bigcup_{(i,j)\in FD(N)} \mathcal{I}_{i,j,N}$$

(recall the definition of $\mathcal{I}_{i,j,N}$ on page 1935), and the "near diagonal" indices to be

$$D(N) = \Delta(N) \setminus FD(N).$$

Then

$$|D(N)| = O(N^{\frac{3}{2}})$$

and, for $(i, j) \in FD(N)$,

$$\frac{i}{N+1} - \frac{j}{N+1} \geq (N+1)^{-\frac{1}{2}} \mod 1.$$

So,

$$\Bigg| \sum_{0\leq i<j\leq N} \frac{1}{(N+1)^2} \log \left|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}}) - \phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\right| \\ - \iint_{0\leq x\leq y\leq 1} \log \left|\phi(\kappa e^{2\pi\sqrt{-1}x}) - \phi(\kappa e^{2\pi\sqrt{-1}y})\right| dx\, dy \Bigg|$$

$$\le \sum_{(i,j)\in D(N)} \frac{1}{(N+1)^2}\Big|\log\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\Big|$$

$$+\sum_{(i,j)\in FD(N)}\int_{\frac{j}{N+1}}^{\frac{j+1}{N+1}}\int_{\frac{i}{N+1}}^{\frac{i+1}{N+1}}\Big|\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|$$

$$-\log\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\Big|\,dx\,dy$$

$$+\Big|\iint_{\mathscr{F}\mathscr{D}(N)}\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\,dx\,dy$$

$$-\iint_{0\le x\le y\le 1}\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\,dx\,dy\Big|.$$

Let I, II and III be the summands of the last expression.

For all $(i,j)\in D(N)$,

$$\frac{a(\delta)}{N+1}\le\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\le O(1),$$

so

$$\Big|\log\big|\phi(\kappa e^{2\pi\sqrt{-1}\frac{i}{N+1}})-\phi(\kappa e^{2\pi\sqrt{-1}\frac{j}{N+1}})\big|\Big|\le|\log a(\delta)|+\log(N+1),$$

and thus

$$I\le\frac{O(N^{\frac{3}{2}})}{N^2}[|\log a(\delta)|+\log(N+1)]=o_\delta(1).$$

Since

$$\sup_{x-y\ge(N+1)^{-\frac{1}{2}}\ \mathrm{mod}\ 1}\big\|\nabla\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\big\|\le\frac{O(\delta^{-1})}{a(\delta)(N+1)^{-\frac{1}{2}}}=\frac{O(N^{\frac{1}{2}})}{\delta a(\delta)},$$

we have

$$II\le\frac{N^2}{(N+1)^2}\sup_{x-y\ge(N+1)^{-\frac{1}{2}}\ \mathrm{mod}\ 1}\big\|\nabla\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|\big\|O(N^{-1})$$

$$\le\frac{O(N^{-\frac{1}{2}})}{\delta a(\delta)}=o_\delta(1).$$

By a similar argument as in Lemma 3.3, we have

$$\lim_{N\to\infty}\mathrm{Vol}_{\mathbb{R}^2}\big(\mathscr{F}\mathscr{D}(N)\,\triangle\,\{(x,y)\in\mathbb{R}^2:0\le x\le y\le 1\}\big)=0.$$

Furthermore, (5-4) and (5-5) below indicate that the function $\log\big|\phi(\kappa e^{2\pi\sqrt{-1}x})-\phi(\kappa e^{2\pi\sqrt{-1}y})\big|$ is L^1 over $[0,1]^2$, so

$$III\le o_\delta(1).$$

Thus, we have proved (5-3).

For $u_1, u_2 \in D(0,1)$, define

$$\psi(u_1,u_2) = \begin{cases} \dfrac{\phi(u_1)-\phi(u_2)}{u_1-u_2} & \text{if } u_1 \neq u_2, \\ \phi'(u_1) & \text{if } u_1 = u_2. \end{cases}$$

Then ψ is continuous and nonzero in $D(0,1)\times D(0,1)$. Moreover, by the removable singularity theorem, ψ is holomorphic in u_1 as well as u_2. Therefore, $\log|\psi|$ is pluriharmonic in $D(0,1)\times D(0,1)$. By the mean value equality,

$$\begin{aligned} &\int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)} \log|\phi(u_1)-\phi(u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2) \\ &\quad = \int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)} \log|\psi(u_1,u_2)|\,d\sigma_\kappa(u_1)\,d\sigma_\kappa(u_2) + \int_{\partial D(0,\kappa)}\int_{\partial D(0,\kappa)} \log|u_1-u_2|\,d\sigma_\kappa(u_1)d\sigma_\kappa(u_2) \\ &\quad = \log|\psi(0,0)| + \log\kappa + \int_{\partial D(0,1)}\int_{\partial D(0,1)} \log|u_1-u_2|\,d\sigma_1(u_1)\,d\sigma_1(u_2) \\ &\quad = \log|\phi'(0)| + \log\kappa + \int_{\partial D(0,1)}\int_{\partial D(0,1)} \log|u_1-u_2|\,d\sigma_1(u_1)\,d\sigma_1(u_2), \end{aligned} \tag{5-4}$$

and

$$\begin{aligned} \int_{\partial D(0,1)}\int_{\partial D(0,1)} \log|u_1-u_2|\,d\sigma_1(u_1)\,d\sigma_1(u_2) &= \int_0^1\int_0^1 \log\left|e^{2\pi\sqrt{-1}x} - e^{2\pi\sqrt{-1}y}\right|dx\,dy \\ &= \int_0^1 \log\left|1-e^{2\pi\sqrt{-1}x}\right|dx \\ &= \int_{\partial D(0,1)} \log|1-z|\,d\sigma_1(z) \\ &= 0, \end{aligned} \tag{5-5}$$

where the last equality is due to Lebesgue's dominated convergence theorem.

Equations (5-2)–(5-5) show that

$$\begin{aligned} \log\det\Sigma &= \sum_{k=0}^{N} \log\binom{N}{k} + (\log|\phi'(0)| + \log\kappa)N^2 + o_\delta(N^2) \\ &= \left(\log|\phi'(0)| + \log\kappa + \tfrac{1}{2}\right)N^2 + o_\delta(N^2). \end{aligned}$$

The remaining part is similar to Section 3. □

Remark 5.1. For $U = D(0,r)$, ϕ is a rotation composed with a scaling by r, so $|\phi'(0)| = r$. Thus, the upper bound in Theorem 0.5 is $-\left(\log r + \frac{1}{2}\right)N^2 + o(N^2)$, which agrees with Corollary 0.4 in the case $r \geq 1$.

6. Generalized hole probabilities of SU(2) polynomials

If $n(r, N)$ denotes the number of zeros of $\tilde{s}_N(z)$ in $\bar{D}(0, r)$, counting multiplicity, then the hole probability $P_{0,1}(r, N)$ is just the first term of a sequence of probabilities

$$P_{k,1}(r, N) = \gamma_N\{n(r, N) \le k\}, \quad k \ge 0.$$

We call $P_{k,1}(r, N)$ a generalized hole probability because, compared with the large degree or total number of zeros in $\mathbb{C}$ of the polynomial $\tilde{s}_N$, any finite number k is negligible. It has the status of almost having no zeros in $D(0, r)$. And, by Theorem 0.6, it turns out that the generalized hole probabilities are numerically almost equal to the regular one.

Proof of Theorem 0.6. Equation (3-21) implies that, for all $\eta > 0$,

$$\gamma_N\left\{\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u) > \frac{N}{2}\log(1+r^2) + \eta N\right\} \le e^{-e^{\eta N}} \quad \text{for } N \gg 1. \tag{6-1}$$

We follow the notations in Section 4, except this time $m = 1$ and we take the number of partitions to be $p = 1$. The corresponding restatement of Lemma 4.6 is

$$\gamma_N\left\{\log\prod_{j=0}^{\lfloor\alpha_0 N\rfloor}|\zeta_j| > \frac{o(N^2)}{\delta^2} + (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u)\right\} \le e^{-e^N} + e^{-R_{\kappa r,1}(N)},$$

where $\zeta_j = \tilde{s}_N(\kappa r e^{2\pi\sqrt{-1}j/(\lfloor\alpha_0 N\rfloor+1)})$, $0 \le j \le \lfloor\alpha_0 N\rfloor$. Here we do not need to assume $0 \notin \tilde{s}_N(\bar{D}(0, r))$ as we do in Lemma 4.6; the counterpart of II in (3-19) is

$$II = (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)| \int_H P_r(\kappa r e^{2\pi\sqrt{-1}x}, u)\,dx\,d\sigma_r(u).$$

Since $m = 1$ and $p = 1$, $H = [0, 1] \subset \mathbb{R}$, so

$$\begin{aligned} II &= (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)| \int_0^1 P_r(\kappa r e^{2\pi\sqrt{-1}x}, u)\,dx\,d\sigma_r(u) \\ &= (\lfloor\alpha_0 N\rfloor + 1)\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u). \end{aligned}$$

Therefore, for all $\eta > 0$ small enough,

$$\begin{aligned} &\gamma_N\left\{\int_{\partial D(0,r)} \log|\tilde{s}_N(u)|\,d\sigma_r(u) \le \frac{N}{2}\log(1+r^2) - \eta N\right\} \\ &\le e^{-e^N} + e^{-R_{\kappa r,1}(N)} + \gamma_N\left\{\prod_{j=0}^{\lfloor\alpha_0 N\rfloor}|\zeta_j| \le \exp\left\{\frac{o(N^2)}{\delta^2} + (\lfloor\alpha_0 N\rfloor + 1)\left[\frac{N}{2}\log(1+r^2) - \eta N\right]\right\}\right\}. \end{aligned} \tag{6-2}$$

Following the steps (3-29)–(3-31), we can show that

$$\log \gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)|\, d\sigma_r(u) \le \frac{N}{2} \log(1+r^2) - \eta N \right\}$$
$$\le N(\lfloor \alpha_0 N \rfloor + 1)[\log(1+r^2) - 2\eta] - Q_{\kappa r,1,\alpha_0}(N) - 2\beta_1 \alpha_0^2 N^2 + \frac{o(N^2)}{\delta^2},$$

$$Q_{\kappa r,1,\alpha_0}(N) \sim N^2 \int_0^{\alpha_0} E_r(x)\, dx = \tfrac{1}{2}\alpha_0[2\log \kappa r + 1 - \log \alpha_0] N^2,$$

$$\begin{aligned}
\beta_1 &= \int_0^1 x \log[2\sin(\pi x)]\, dx \\
&= \int_0^1 (x - \tfrac{1}{2}) \log[2\sin(\pi x)]\, dx + \frac{1}{2}\int_0^1 \log[2\sin(\pi x)]\, dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} x \log\left[2\sin\pi\left(x+\tfrac{1}{2}\right)\right] dx + \frac{1}{2}\int_0^1 \log[2\sin(\pi x)]\, dx \\
&= \int_{-\frac{1}{2}}^{\frac{1}{2}} x \log[2\cos(\pi x)]\, dx + \frac{1}{2}\int_0^1 \log[2\sin(\pi x)]\, dx.
\end{aligned}$$

Since $\int_{-\frac{1}{2}}^{0} x \log[2\cos(\pi x)]\, dx$ and $\int_0^{\frac{1}{2}} x \log[2\cos(\pi x)]\, dx$ both converge and $x \log[2\cos(\pi x)]$ is odd,

$$\beta_1 = \frac{1}{2}\int_0^1 \log[2\sin(\pi x)]\, dx = \frac{1}{2}\int_{\partial D(0,1)} \log|1-z|\, d\sigma_1(z),$$

which equals 0 as in (5-5). Thus

$$\log \gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)|\, d\sigma_r(u) \le \frac{N}{2} \log(1+r^2) - \eta N \right\}$$
$$\le -\frac{1}{2}\alpha_0[1 + 2\log(\kappa r) - \log \alpha_0 - 2\log(1+r^2) + 4\eta] N^2 + \frac{o(N^2)}{\delta^2}. \tag{6-3}$$

On the other hand,

$$R_{\kappa r,1}(N) \sim N^2 \int_{E_{\kappa r}(x) \ge 0} E_{\kappa r}(x)\, dx. \tag{6-4}$$

Combining (6-2)–(6-4) and letting $\delta \to 0+$, we get

$$\log \gamma_N \left\{ \int_{\partial D(0,r)} \log |\tilde{s}_N(u)|\, d\sigma_r(u) \le \frac{N}{2} \log(1+r^2) - \eta N \right\}$$
$$\le -\min\left\{ \tfrac{1}{2}\alpha_0[1 + 2\log r - \log \alpha_0 - 2\log(1+r^2) + 4\eta],\ \tfrac{1}{2}\alpha_0[1 + 2\log r - \log \alpha_0] \right\} N^2 + o(N^2)$$
$$= -\tfrac{1}{2}\alpha_0[1 + 2\log r - \log \alpha_0 - 2\log(1+r^2) + 4\eta] N^2 + o(N^2), \tag{6-5}$$

for $0 < \eta < \frac{1}{2}\log(1+r^2)$. Since

$$\int_{E_r(x) \ge 0} E_r(x)\, dx = \tfrac{1}{2}\alpha_0[1 + 2\log r - \log \alpha_0] > 0 \quad \text{and thus} \quad 1 + 2\log r - \log \alpha_0 > 0,$$

we can choose $0<\eta<\frac{1}{2}\log(1+r^2)$ close to $\frac{1}{2}\log(1+r^2)$ such that

$$1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta>0.$$

Therefore, (6-5) makes sense. Denote

$$F_\eta(r)=\tfrac{1}{2}\alpha_0[1+2\log r-\log\alpha_0-2\log(1+r^2)+4\eta];$$

then we have, for $0<\eta<\frac{1}{2}\log(1+r^2)$,

$$\gamma_N\left\{\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)\le\frac{N}{2}\log(1+r^2)-\eta N\right\}\le e^{-F_\eta(r)N^2+o(N^2)}. \tag{6-6}$$

Let $\rho>1$, to be determined. By discarding a null set, we may assume $\tilde{s}_N(0)\neq 0$, $0\notin\tilde{s}_N(\partial D(0,r))$ and $0\notin\tilde{s}_N(\partial D(0,\rho^{-1}r))$. So, by Jensen's formula (cf. [Hough et al. 2009, (7.2.11)]), almost surely,

$$\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)=\log|c_0|+\int_0^r\frac{n(t,N)}{t}\,dt, \tag{6-7}$$

$$\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)=\log|c_0|+\int_0^{\rho^{-1}r}\frac{n(t,N)}{t}\,dt. \tag{6-8}$$

Since $n(r,N)$ is increasing with respect to r, (6-7) and (6-8) imply

$$\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)-\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)=\int_{\rho^{-1}r}^r\frac{n(t,N)}{t}\,dt\le n(r,N)\log\rho,$$

and thus

$$n(r,N)\ge\frac{1}{\log\rho}\left[\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)-\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)\right]. \tag{6-9}$$

By (6-1), for $\eta_1>0$, outside an event of probability at most $e^{-e^{\eta_1 N}}$,

$$\int_{\partial D(0,\rho^{-1}r)}\log|\tilde{s}_N(u)|\,d\sigma_{\rho^{-1}r}(u)\le\frac{N}{2}\log(1+\rho^{-2}r^2)+\eta_1 N, \tag{6-10}$$

By (6-6), for $0<\eta_2<\frac{1}{2}\log(1+r^2)$, outside an event of probability at most $e^{-F_{\eta_2}(r)N^2+o(N^2)}$,

$$\int_{\partial D(0,r)}\log|\tilde{s}_N(u)|\,d\sigma_r(u)\ge\frac{N}{2}\log(1+r^2)-\eta_2 N. \tag{6-11}$$

By (6-9)–(6-11), outside an event of probability at most $e^{-e^{\eta_1 N}}+e^{-F_{\eta_2}(r)N^2+o(N^2)}$,

$$n(r,N)\ge\frac{N}{\log\rho}\left[\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho^{-2}r^2)-(\eta_1+\eta_2)\right].$$

Therefore,

$$\gamma_N\left\{n(r,N)<\frac{N}{\log\rho}\left[\tfrac{1}{2}\log(1+r^2)-\tfrac{1}{2}\log(1+\rho^{-2}r^2)-(\eta_1+\eta_2)\right]\right\}\le e^{-e^{\eta_1 N}}+e^{-F_{\eta_2}(r)N^2+o(N^2)},$$

where the right-hand side is independent of ρ. We need to choose proper ρ, η_1 and η_2.

For all $\tau > 0$, we set

$$\frac{1}{\log\rho}\Big[\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho^{-2}r^2)-(\eta_1+\eta_2)\Big]=\tau,$$

$$\eta_1+\eta_2=\eta_\tau(\rho):=\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho^{-2}r^2)-\tau\log\rho.$$

If $\tau > 0$ is small enough, let $\rho_0(\tau) := \sqrt{(1-\tau)/\tau}\, r > 1$; then

$$\eta_\tau'(\rho)=\frac{\rho^{-3}r^2}{1+\rho^{-2}r^2}-\frac{\tau}{\rho}=\frac{(1-\tau)r^2-\tau\rho^2}{\rho(\rho^2+r^2)}\begin{cases}>0 & \text{when } 1<\rho<\rho_0,\\ =0 & \text{when } \rho=\rho_0,\\ <0 & \text{when } \rho>\rho_0,\end{cases}$$

and thus

$$\begin{aligned}(\eta_1+\eta_2)_{\max}&=\eta_\tau(\rho_0(\tau))\\ &=\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log\Big(1+\frac{\tau}{1-\tau}\Big)-\tau\Big[\frac{1}{2}\log(1-\tau)-\frac{1}{2}\log\tau+\log r\Big]\\ &=\tfrac{1}{2}\log(1+r^2)+\tfrac{1}{2}\log(1-\tau)-\tfrac{1}{2}\tau\log(1-\tau)+\tfrac{1}{2}\tau\log\tau-\tau\log r\\ &=\tfrac{1}{2}\log(1+r^2)+\tfrac{1}{2}[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r].\end{aligned}$$

For a fixed $r > 0$, we can choose smaller $\tau > 0$ if necessary so that

$$-\tfrac{1}{2}\log(1+r^2)<\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r<0.$$

This is possible since

$$\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r<0 \quad \text{if } 0<\tau<\alpha_0$$

and

$$\lim_{\tau\to 0+}[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]=0.$$

Thus, for such τ and the corresponding $\rho_0 = \rho_0(\tau)$,

$$\tfrac{1}{4}\log(1+r^2)<\eta_1+\eta_2=\eta_\tau(\rho_0)<\tfrac{1}{2}\log(1+r^2).$$

In this case, for all $0 < \eta_1 < \frac{1}{4}\log(1+r^2)$,

$$0<\eta_2=\tfrac{1}{2}\log(1+r^2)+\tfrac{1}{2}[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]-\eta_1<\tfrac{1}{2}\log(1+r^2),$$

$$\begin{aligned}\gamma_N\{n(r,N)<\tau N\}&=\gamma_N\Big\{n(r,N)<\frac{N}{\log\rho_0}\Big[\frac{1}{2}\log(1+r^2)-\frac{1}{2}\log(1+\rho_0^{-2}r^2)-(\eta_1+\eta_2)\Big]\Big\}\\ &\le e^{-e^{\eta_1 N}}+e^{-F_{\eta_2}(r)N^2+o(N^2)}.\end{aligned}$$

Fix any $k \geq 0$; when N is large enough, $k < \tau N$,

$$\begin{aligned}&\exp\big\{-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)N^2+o(N^2)\big\}=P_{0,1}(r,N)\le P_{k,1}(r,N)\le\gamma_N\{n(r,N)<\tau N\}\\ &\le e^{-e^{\eta_1 N}}+\exp\big\{-\tfrac{1}{2}\alpha_0\big\{(1+2\log r-\log\alpha_0)+2[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]-4\eta_1\big\}N^2+o(N^2)\big\}.\end{aligned}$$

Therefore,

$$-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)\le\liminf_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}\le\limsup_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}$$
$$\le-\tfrac{1}{2}\alpha_0\big\{(1+2\log r-\log\alpha_0)+2[\tau\log\tau+(1-\tau)\log(1-\tau)-2\tau\log r]-4\eta_1\big\}.$$

Let $\eta_1\to 0+$ and then $\tau\to 0+$; we have

$$\lim_{N\to\infty}\frac{\log P_{k,1}(r,N)}{N^2}=-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)$$

or, equivalently,

$$\log P_{k,1}(r,N)\sim-\tfrac{1}{2}\alpha_0(1+2\log r-\log\alpha_0)N^2. \qquad \square$$

Appendix

We now prove the following lemma:

Lemma A.1. *The coefficient of $g_{m,N}(\xi)$ in* $\det W_{m,N}(\xi)$ *equals* 1.

Proof. Let $\mathcal{S}_{m,N}$ be the set of bijections from $\Gamma_{m,N}$ to $\Lambda_{m,N}$ and, for all $\sigma\in\mathcal{S}_{m,N}$, $J\in\Gamma_{m,N}$, write $\sigma(J)=(\sigma_1(J),\dots,\sigma_m(J))$. Then

$$\det W_{m,N}(\xi)=\sum_{\sigma\in\mathcal{S}_{m,N}}\operatorname{sgn}(\sigma)\prod_{J\in\Gamma_{m,N}}\xi_J^{\sigma(J)}=\sum_{\sigma\in\mathcal{S}_{m,N}}\operatorname{sgn}(\sigma)\prod_{J\in\Gamma_{m,N}}\xi_{1,j_1}^{\sigma_1(J)}\cdots\xi_{m,j_m}^{\sigma_m(J)}.$$

To find those $\sigma\in\mathcal{S}_{m,N}$ ending up with $g_{m,N}(\xi)$, it is equivalent to find σ satisfying, for all $1\le i\le m$,

$$\sum_{J\in\Gamma_{m,N}^{i,k}}\sigma_i(J)=\begin{cases}\binom{k+i-1}{i}\binom{N-k+m-i}{m-i} & 1\le k\le N,\\ 0 & k=0,\end{cases}\tag{A-1}$$

where the set $\Gamma_{m,N}^{i,k}$ is defined in (2-7). We are going to prove by induction that

$$\sigma(J)=(j_1,j_2-j_1,\dots,j_m-j_{m-1})\quad\text{for all } J\in\Gamma_{m,N}.\tag{A-2}$$

First of all, similarly to $\Gamma_{m,N}^{i,k}$, we introduce

$$\Lambda_{m,N}^{i,k}=\{(k_1,\dots,k_m)\in\Lambda_{m,N}:k_1+\dots+k_i=k\};$$

then

$$\Lambda_{m,N}=\bigsqcup_{k=0}^{N}\Lambda_{m,N}^{i,k}\quad\text{for all } 1\le i\le m,$$

and

$$|\Lambda_{m,N}^{i,k}|=\binom{k+i-1}{i-1}\binom{N-k+m-i}{m-i}=|\Gamma_{m,N}^{i,k}|.$$

When $i = 1$, (A-1) shows that, for $0 \le k \le N$,

$$\sum_{J \in \Gamma_{m,N}^{1,k}} \sigma_1(J) = k\binom{N-k+m-1}{m-1}, \tag{A-3}$$

where the number of terms in the summation on the left is $|\Gamma_{m,N}^{1,k}| = \binom{N-k+m-1}{m-1} = |\Lambda_{m,N}^{1,k}|$ for all $0 \le k \le N$. Then

$$\begin{aligned}
k = 0 \text{ in (A-3)} &\implies \sigma(\Gamma_{m,N}^{1,0}) = \Lambda_{m,N}^{1,0} \implies \sigma\Big(\bigsqcup_{k=1}^{N} \Gamma_{m,N}^{1,k}\Big) = \bigsqcup_{k=1}^{N} \Lambda_{m,N}^{1,k},\\
k = 1 \text{ in (A-3)} &\implies \sigma(\Gamma_{m,N}^{1,1}) = \Lambda_{m,N}^{1,1} \implies \sigma\Big(\bigsqcup_{k=2}^{N} \Gamma_{m,N}^{1,k}\Big) = \bigsqcup_{k=2}^{N} \Lambda_{m,N}^{1,k},\\
&\vdots\\
k = N \text{ in (A-3)} &\implies \sigma(\Gamma_{m,N}^{1,N}) = \Lambda_{m,N}^{1,N},
\end{aligned}$$

so

$$\sigma_1(J) = j_1 \quad \text{for all } J \in \Gamma_{m,N}.$$

Now assume, for some $1 \le i \le m-1$, that $(\sigma_1 + \cdots + \sigma_i)(J) = j_i$ for all $J \in \Gamma_{m,N}$. Then, for any $1 \le k \le N$,

$$\begin{aligned}
\sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1 + \cdots + \sigma_{i+1})(J) &= \sum_{J \in \Gamma_{m,N}^{i+1,k}} [j_i + \sigma_{i+1}(J)]\\
&= \sum_{j=0}^{k} j\,|\Gamma_{m,N}^{i,j} \cap \Gamma_{m,N}^{i+1,k}| + \binom{k+i}{i+1}\binom{N-k+m-i-1}{m-i-1}\\
&= \sum_{j=0}^{k} j\binom{j+i-1}{i-1}\binom{N-k+m-i-1}{m-i-1} + \binom{k+i}{i+1}\binom{N-k+m-i-1}{m-i-1}\\
&= k\binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1},
\end{aligned}$$

where the second term on the second line of the calculation comes from (A-1). And, for $k = 0$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,0}} (\sigma_1 + \cdots + \sigma_{i+1})(J) = \sum_{J \in \Gamma_{m,N}^{i+1,0}} [j_i + \sigma_{i+1}(J)] = 0.$$

So, for all $0 \le k \le N$,

$$\sum_{J \in \Gamma_{m,N}^{i+1,k}} (\sigma_1 + \cdots + \sigma_{i+1})(J) = k\binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1}, \tag{A-4}$$

where the number of terms in the summation on the left is $|\Gamma_{m,N}^{i+1,k}| = \binom{k+i}{i}\binom{N-k+m-i-1}{m-i-1} = |\Lambda_{m,N}^{i+1,k}|$ for all $0 \le k \le N$.

$$k=0 \text{ in (A-4)} \implies \sigma(\Gamma_{m,N}^{i+1,0}) = \Lambda_{m,N}^{i+1,0} \implies \sigma\Big(\bigsqcup_{k=1}^{N} \Gamma_{m,N}^{i+1,k}\Big) = \bigsqcup_{k=1}^{N} \Lambda_{m,N}^{i+1,k},$$
$$k=1 \text{ in (A-4)} \implies \sigma(\Gamma_{m,N}^{i+1,1}) = \Lambda_{m,N}^{i+1,1} \implies \sigma\Big(\bigsqcup_{k=2}^{N} \Gamma_{m,N}^{i+1,k}\Big) = \bigsqcup_{k=2}^{N} \Lambda_{m,N}^{i+1,k},$$
$$\vdots$$
$$k=N \text{ in (A-4)} \implies \sigma(\Gamma_{m,N}^{i+1,N}) = \Lambda_{m,N}^{i+1,N},$$

so

$$(\sigma_1 + \cdots + \sigma_{i+1})(J) = j_{i+1} \quad \text{for all } J \in \Gamma_{m,N}.$$

Thus, (A-2) is proved. And it is trivial to check that the σ defined in (A-2) satisfies all the equations in (A-1). This means that there is only one $\sigma \in \mathcal{S}_{m,N}$ that ends up with $g_{m,N}(\xi)$, and it turns out to be order-preserving. Therefore,

$$\det W_{m,N}(\xi) = g_{m,N}(\xi) + \cdots . \qquad \square$$

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JUNYAN ZHU: jyzhu@math.jhu.edu
Department of Mathematics, Johns Hopkins University, 3400 N. Charles St., Krieger 404, Baltimore, MD 21218, United States

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STOCHASTIC HOMOGENIZATION OF VISCOUS HAMILTON–JACOBI EQUATIONS AND APPLICATIONS

SCOTT N. ARMSTRONG AND HUNG V. TRAN

We present stochastic homogenization results for viscous Hamilton–Jacobi equations using a new argument that is based only on the subadditive structure of maximal subsolutions (i.e., solutions of the "metric problem"). This permits us to give qualitative homogenization results under very general hypotheses: in particular, we treat nonuniformly coercive Hamiltonians that satisfy instead a weaker averaging condition. As an application, we derive a general quenched large deviation principle for diffusions in random environments and with absorbing random potentials.

1. Introduction

1A. ***Motivation and informal summary of results.*** In this paper, we consider the *qualitative* stochastic homogenization of second-order, "viscous" Hamilton–Jacobi equations. We present a new, short and self-contained argument that yields homogenization under very general and essentially optimal hypotheses. Our framework includes a class of equations for which the homogenization result has an equivalent formulation in probabilistic terms as a quenched large deviation principle (LDP) for diffusions in random environments (and/or with random obstacles), and so a corollary of our analysis is a very general LDP for such problems that generalizes many previous results on the topic.

In its time-dependent form, the viscous Hamilton–Jacobi equation we consider is

$$u_t^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big) D^2 u^\varepsilon\Big) + H\Big(Du^\varepsilon, \frac{x}{\varepsilon}\Big) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty). \tag{1-1}$$

Here $D\phi$ and $D^2\phi$ denote the gradient and Hessian of a real-valued function ϕ, and $\operatorname{tr} B$ is the trace of a d-by-d matrix B. The coefficients A and H are called the diffusion matrix and the Hamiltonian, respectively, and are assumed to be stationary-ergodic random fields. That is, they are randomly selected from the set of all such equations by an underlying probability measure that is stationary and ergodic with respect to $\mathbb{R}^d$-translations. The essential structural hypotheses on the coefficients are that A takes values in the nonnegative definite matrices (and in particular may be degenerate or even vanish) and H is *convex* and *growing superlinearly* in its first variable. See below for some important examples of the equations that fit into our framework.

The presence of the ε factor in the diffusion term of (1-1) gives the equation a critical scaling, and it turns out that it behaves like a first-order Hamilton–Jacobi equation in the limit $\varepsilon \to 0$. Indeed, rather

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than providing any useful regularizing effect, the diffusion term actually makes the analysis more difficult compared to the pure first-order case by destroying localization effects (such as the finite speed of propagation). Also notice that, while we choose to write the principal part of (1-1) in nondivergence form, thanks to the scaling of the equation, our study also covers the case of equations with principal part in divergence form. Indeed, we may rewrite an equation with principal part divergence form, at least in the case that the diffusion matrix is sufficiently smooth (on the microscopic scale) in the form of (1-1) by simply expanding out the divergence, observing that the ε's cancel, and absorbing the new first-order drift term into the Hamiltonian.

The archetypical result of almost-sure, qualitative homogenization for (1-1) is that there exists a *deterministic*, constant-coefficient equation

$$u_t + \overline{H}(Du) = 0 \quad \text{in } \mathbb{R}^d \times (0, \infty) \tag{1-2}$$

such that, subject to an appropriate initial condition, u^ε converges locally uniformly, as $\varepsilon \to 0$ and with probability one, to the solution u of (1-2). The nonlinearity $\overline{H}$, called the effective Hamiltonian, depends on $\mathbb{P}$ but is a deterministic quantity. It inherits convexity and superlinearity from the heterogeneous Hamiltonian. Its fine qualitative properties encode information regarding the behavior of solutions of the heterogeneous equation (1-1). In the particular case corresponding to quenched large deviation principles for diffusions in random environments, $\overline{H}$ is, up to a constant, the Legendre–Fenchel transform of the rate function (see below for more details).

The first qualitative homogenization results of this type for second-order equations, asserting that (1-1) homogenizes to a limiting equation of the form of (1-2), were proved independently by Kosygina, Rezakhanlou and Varadhan [Kosygina et al. 2006] and Lions and Souganidis [2005]. Earlier homogenization results for first-order equations (i.e., $A \equiv 0$) in the random setting are due to Souganidis [1999] and Rezakhanlou and Tarver [2000], and subsequent work can be found in [Armstrong and Souganidis 2012; Lions and Souganidis 2010]. We also refer the reader to the nice survey article of Kosygina [2007].

In this paper, we present a new proof of homogenization that applies to a wider class of equations. The idea is to apply the subadditive ergodic theorem to certain *maximal subsolutions* (these are the functions m_μ in Section 2), thereby obtaining a deterministic limit (which we denote $\overline{m}_\mu$) and hence a candidate for $\overline{H}$ (by the formula (3-2)) and then recovering the full homogenization result by deterministic comparison arguments (presented in Sections 4 and 5). The approach is simple and more or less self-contained (the reader may consult our recent paper [Armstrong and Tran 2014] for the necessary deterministic PDE theory) and yields a very general qualitative homogenization theorem under essentially optimal hypotheses. In addition to recovering all of the known cases, including the results mentioned above, we can also treat for the first time general Hamiltonians that are not necessarily uniformly coercive. An essential characteristic of (1-1) is that $p \mapsto H(p, y)$ exhibits superlinear growth in p, and this is typically assumed to be uniform in x. Here we can handle Hamiltonians satisfying an *averaged* coercivity condition that is not uniform in x.

The most important feature of the method is that, unlike previous approaches, our proof of homogenization is quantifiable, as demonstrated in [Armstrong and Cardaliaguet 2015]. Much recent effort has been put into obtaining quantitative stochastic homogenization results, for example, estimates for the difference $u^\varepsilon - u$, rigorous bounds for numerical methods for computing effective coefficients and so on. For first-order Hamilton–Jacobi equations, quantitative stochastic homogenization results were recently obtained by Armstrong, Cardaliaguet and Souganidis [Armstrong et al. 2014], who quantified the convergence proof in [Armstrong and Souganidis 2013]. Unfortunately, the method of this last paper is not applicable in the viscous case without new ideas, as the presence of the diffusion term generates significant additional difficulties. From this point of view, the results in this paper can be considered as the completion of the idea that originated in [Armstrong and Souganidis 2013].

1B. *Statement of the main results.* We begin by defining “the set of all equations” by specifying some structural conditions on the coefficients. We work with parameters $q > 1$, $n \in \mathbb{N}$, $\Lambda_1 \geq 1$ and $\Lambda_2 \geq 0$, which are fixed throughout the paper.

We require the coefficients to be functions $A : \mathbb{R}^d \to \mathbb{S}^d$ (here $\mathbb{S}^d$ denotes the set of d-by-d real symmetric matrices) and $H : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ satisfying the following conditions. First, the diffusion matrix has a Lipschitz square root. Precisely, we assume that there exists a function $\sigma : \mathbb{R}^d \to \mathbb{R}^{n\times d}$ such that

$$A = \tfrac{1}{2}\sigma^t\sigma,$$

where σ is bounded and Lipschitz: for every $y, z \in \mathbb{R}^d$,

$$|\sigma(y)| \leq \Lambda_2, \tag{1-3}$$

$$|\sigma(y) - \sigma(z)| \leq \Lambda_2|y - z|. \tag{1-4}$$

(Here $\mathbb{R}^{n\times d}$ is the set of real n-by-d matrices.) Regarding the Hamiltonian, we assume that, for every $y \in \mathbb{R}^d$,

$$p \mapsto H(p, y) \quad \text{is convex} \tag{1-5}$$

and, for every $R > 0$, there exist constants $0 < a_R \leq 1$ and $M_R \geq 1$ such that, for every $p, \hat{p} \in \mathbb{R}^d$ and $y, z \in B_R$,

$$a_R|p|^q - M_R \leq H(p, y) \leq \Lambda_1(|p|^q + 1), \tag{1-6}$$

$$|H(p, y) - H(p, z)| \leq (\Lambda_1|p|^q + M_R)|y - z|, \tag{1-7}$$

$$|H(p, y) - H(\hat{p}, y)| \leq \Lambda_1(|p| + |\hat{p}| + 1)^{q-1}|p - \hat{p}|. \tag{1-8}$$

We define the probability space Ω to be the set of ordered pairs (σ, H) satisfying the above conditions:

$$\Omega := \{(\sigma, H) : \sigma \text{ and } H \text{ satisfy (1-3), (1-4), (1-5), (1-6), (1-7), and (1-8)}\}.$$

We may write $\Omega = \Omega(q, n, \Lambda_1, \Lambda_2)$ if we wish to emphasize the dependence of Ω on the parameters.

We endow the set Ω with

$$\mathcal{F} := \sigma\text{-algebra generated by } (\sigma, H) \mapsto \sigma(y) \text{ and } (\sigma, H) \mapsto H(p, y) \text{ with } p, y \in \mathbb{R}^d.$$

The random environment is modeled by a probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$. The expectation with respect to $\mathbb{P}$ is denoted by $\mathbb{E}$. We assume that $\mathbb{P}$ is stationary and ergodic with respect to the action of $\mathbb{R}^d$

on Ω given by translation. To be precise, we let $\{\tau_z\}_{z\in\mathbb{R}^d}$ be the group action of translation on Ω defined by

$$\tau_z(\sigma, H) := (\tau_z\sigma, \tau_z H), \quad \text{where } (\tau_z\sigma)(y) := \sigma(y+z) \text{ and } (\tau_z H)(p, y) := H(p, y+z).$$

We extend this to $\mathcal{F}$ by setting, for every event $F \in \mathcal{F}$,

$$\tau_z F := \{\tau_z\omega : \omega \in F\}.$$

The stationary-ergodic hypothesis is that

$$\text{for all } y \in \mathbb{R}^d \text{ and } F \in \mathcal{F}, \quad \mathbb{P}[\tau_y F] = \mathbb{P}[F] \quad \text{(stationarity)} \tag{1-9}$$

and, for all $F \in \mathcal{F}$,

$$\bigcap_{z\in\mathbb{R}^d} \tau_z F = F \quad \text{implies that} \quad \mathbb{P}[F] \in \{0, 1\} \quad \text{(ergodicity)}. \tag{1-10}$$

The final assumption we impose on $\mathbb{P}$ is a *weak coercivity* condition: there exists an exponent $\alpha > d$ such that

$$\mathbb{E}\left[\left(\frac{\Lambda_2}{a_1}\right)^{2\alpha/(q-1)} + \left(\frac{M_1}{a_1}\right)^{\alpha/q}\right] < +\infty. \tag{1-11}$$

It is important to note that $\Lambda_2 \geq 0$ is a constant but $0 < a_1 \leq 1$ and $M_1 \geq 1$ are random variables in the above condition.

Remark 1.1. We emphasize that, in contrast to q, n, Λ_1 and Λ_2, the positive constants a_R and M_R in the assumptions (1-6) and (1-7) depend on H itself; that is, they are random variables on Ω. To make this precise, for each $\omega = (\sigma, H) \in \Omega$, we redefine $M_R(\omega)$ to be the smallest constant not smaller than 1 for which (1-7) holds in B_R; we then redefine $a_R(\omega)$ to be the largest constant not larger than 1 for which (1-6) holds in B_R. We denote

$$a_R(x, \omega) := a_R(\tau_x\omega) \quad \text{and} \quad M_R(x, \omega) := M_R(\tau_x\omega).$$

We drop the dependence on ω from the notation where possible, e.g., $a_R(x, \omega) = a_R(x)$.

We present our main homogenization result in terms of the initial-value problem

$$\begin{cases} u_t^\varepsilon - \varepsilon \operatorname{tr}\left(A\left(\frac{x}{\varepsilon}\right)D^2u^\varepsilon\right) + H\left(Du^\varepsilon, \frac{x}{\varepsilon}\right) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u^\varepsilon = g & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{1-12}$$

Here the initial data g is a given element of $\mathrm{BUC}(\mathbb{R}^d)$, the set of bounded and uniformly continuous real-valued functions on $\mathbb{R}^d$, and the unknown function u^ε depends on (x, t) as well as g and the coefficients $\omega = (\sigma, H)$. We typically write $u^\varepsilon(x, t, g, \omega)$ or often simply $u^\varepsilon(x, t, g)$ or $u^\varepsilon(x, t)$. As explained in Section 5, under our assumptions, the problem (1-12) has a unique viscosity solution (subject to an appropriate growth condition) almost surely with respect to $\mathbb{P}$. In fact, it is defined by formula (5-2) below. We remark that all differential equations and inequalities in this paper, including the ones above, are interpreted in the viscosity sense; see Section 1D.

In our main result, we identify a continuous, convex $\bar{H} : \mathbb{R}^d \to \mathbb{R}$ and show that, as $\varepsilon \to 0$, the solutions u^ε of (1-12) converge, $\mathbb{P}$-almost surely, to the unique solution of

$$\begin{cases} u_t + \bar{H}(Du) = 0 & \text{in } \mathbb{R}^d \times (0, \infty), \\ u = g & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{1-13}$$

That the latter has a unique solution is a consequence of the properties of $\bar{H}$ summarized in Lemma 3.1 (see Section 5 for more details).

We now state our main homogenization theorem.

Theorem 1. *Let $(\Omega, \mathcal{F})$ be defined as above for fixed constants $q > 1$ and $\Lambda_1, \Lambda_2 > 0$. Suppose that $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ satisfying (1-9), (1-10) and (1-11). Then there exists a convex $\bar{H} \in C(\mathbb{R}^d)$ satisfying, for some constants $C, c > 0$,*

$$c(|p|^q - C) \le \bar{H}(p) \le C(|p|^q + 1)$$

with the following property: with $u^\varepsilon(x, t, g, \omega)$ defined by (5-2), and denoting by $u = u(x, t, g)$ the unique solution of (1-13), we have

$$\mathbb{P}\Big[\forall g \in \mathrm{BUC}(\mathbb{R}^d),\ \forall R > 0,\ \limsup_{\varepsilon \to 0} \sup_{(x,t) \in B_R \times [0,R)} |u^\varepsilon(x, t, g) - u(x, t, g)| = 0\Big] = 1.$$

Let us say a few words regarding the role of the weak coercivity assumption. The first thing to notice about (1-11) is that a particular case occurs when $\mathbb{P}$ is supported on the set of (σ, H) for which H satisfies (1-6) and (1-7) for constants $a_R > 0$ and $M_R > 1$ that are independent of R. We call this a *uniform coercivity* condition, and it is the traditional hypothesis under which homogenization results for viscous Hamilton–Jacobi equations have been obtained. From the PDE point of view, it is important because it provides uniform Lipschitz estimates for solutions, which is a starting point for the analysis. The condition (1-11) can then be seen as a relaxation of the uniform coercivity condition, replacing it by an averaging condition. We remark that we expect the averaging condition stated here to be optimal in terms of the range of the exponent α. The result should not hold if we only have (1-11) for $\alpha = d$.

There are few homogenization results in the random setting without uniform coercivity. Armstrong and Souganidis [2012] recently proved such a result under a less general averaging condition (essentially (1-11) with a_1 bounded below). They also assumed the random environment satisfied a strong mixing condition with an algebraic mixing rate assumed to be sufficiently fast, depending on the exponent α. Similar results stated in probabilistic terms were obtained at about the same time by Rassoul-Agha, Seppäläinen and Yilmaz [Rassoul-Agha et al. 2013]. In contrast to these results, we do not require any mixing condition here, merely that the environment be stationary-ergodic.

We next present a model equation that fits into our framework.

Example 1.2. Consider the particular case of the Hamiltonian

$$H(p, y) = a(y)|p|^q - V(y), \tag{1-14}$$

where $q > 1$, the functions a and V are stationary-ergodic random fields that are almost surely locally Lipschitz, $V \ge 0$ and a is positive and uniformly Lipschitz on $\mathbb{R}^d$. Assume also that A satisfies the usual assumption stated above. This of course fits under our framework since given such a random function H (together with σ) we simply take $\mathbb{P}$ to be the law of (σ, H). The weak coercivity condition is satisfied in this case provided that, for some $\alpha > d$,

$$\mathbb{E}\left[\left(\frac{1}{a(0)}\right)^{2\alpha/(q-1)} + \left(\frac{\|V\|_{C^{0,1}(B_1)}}{a(0)}\right)^{\alpha/q}\right] < +\infty.$$

If the diffusion matrix A vanishes, we only need that, for some $\alpha > d$,

$$\mathbb{E}\left[\left(\frac{\|V\|_{C^{0,1}(B_1)}}{a(0)}\right)^{\alpha/q}\right] < +\infty.$$

In the case that V is bounded and uniformly Lipschitz, we need simply that $a^{-1} \in L^p(\Omega)$ for some $p > 2d/(q-1)$; if in addition there is no diffusion ($A = 0$), then we just need $p > d/q$. Even in these relatively simple situations, the homogenization result we obtain is completely new. In the case that a is bounded below, then we just need that $\mathbb{E}[\|V\|^p_{C^{0,1}(B_1)}] < +\infty$ for some $p > d/q$, which is better than the condition $p \geq d$ assumed in [Armstrong and Souganidis 2012].

Remark 1.3. It is customary in the homogenization literature to hide the specifics of the probability space Ω by introducing the "dummy variable" ω and expressing σ and H as maps $\sigma : \mathbb{R}^d \times \Omega \to \mathbb{S}^d$ and $H : \mathbb{R}^d \times \mathbb{R}^d \times \Omega \to \mathbb{R}$ by identifying $\sigma(\cdot, \omega)$ and $H(\cdot, \cdot, \omega)$ with $\tilde{\sigma}$ and $\widetilde{H}$, respectively, where $\omega = (\tilde{\sigma}, \widetilde{H})$. Viewed this way, the functions A and H are stationary with respect to the translation group action $\{\tau_z\}_{z\in\mathbb{R}^d}$ in the sense that, for every $p, y, z \in \mathbb{R}^d$ and $\omega \in \Omega$,

$$\sigma(y, \tau_z\omega) = \sigma(y + z, \omega) \quad \text{and} \quad H(p, y, \tau_z\omega) = H(p, y + z, \omega).$$

While this is evidently equivalent to the formulation here, we feel that writing ω everywhere is both unsightly and unnecessary, and so we avoid it wherever possible. The meaning of expressions such as $\mathbb{P}[\cdots]$ and $\mathbb{E}[\cdots]$ are always quite clear from the context. Meanwhile, measurability issues are already set up by the definition of $\mathcal{F}$ and become, in our opinion, more rather than less confusing if we display explicit dependence on ω.

1C. ***A quenched LDP for diffusions in random environments.*** In order to state the main probabilistic application of Theorem 1, we require some additional notation. We begin with another example of a Hamilton–Jacobi equation with random coefficients that is contained in the framework of Theorem 1.

Example 1.4. With $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ as described in the hypotheses (with $n = d$) and given a random vector field b and potential $V \geq 0$, we define the Hamiltonian

$$H(p, y) = \tfrac{1}{2}|\sigma p|^2 + b(y) \cdot p - V(y) = p \cdot Ap + b(y) \cdot p - V(y), \tag{1-15}$$

where as usual $A = \frac{1}{2}\sigma^t\sigma$, which is precisely the given diffusion matrix. The weak coercivity condition is satisfied provided there exists $\alpha > d$ such that

$$\mathbb{E}\left[\left(\frac{1}{\lambda_1(A(0))}\right)^{2\alpha} + \left(\frac{\|V\|_{C^{0,1}(B_1)}}{\lambda_1(A(0))}\right)^{\alpha/2}\right] < +\infty, \tag{1-16}$$

where $\lambda_1(A) = \frac{1}{2}\min_{|z|=1}|\sigma z|^2$ is the smallest eigenvalue of A. If this random variable is bounded below, we say that A is *uniformly elliptic*, and in this case, we need only that the potential V has a finite q-th moment for some $q > d/2$.

Throughout the rest of this subsection, we take σ, A, b and V to be as in Example 1.4. In this situation, we may identify the probability space Ω with ordered triples (σ, b, V).

We denote by $t \mapsto X_t$ the canonical process on $C(\mathbb{R}_+, \mathbb{R}^d)$. Recall that the martingale problem corresponding to σ and b has a unique solution [Stroock and Varadhan 1979]. This means that, for each $x \in \mathbb{R}^d$ and $\omega = (\sigma, b, V) \in \Omega$, there exists a unique probability measure $P_{x,\omega}$ on $C(\mathbb{R}_+, \mathbb{R}^d)$ such that, under $P_{x,\omega}$, the canonical process $X = \{X_t\}_{t\geq 0}$ satisfies the stochastic differential equation

$$\begin{cases} dX_t = \sigma(X_t, \omega)\, dB_t + b(X_t, \omega)\, dt, \\ P_{x,\omega}[X_0 = x] = 1, \end{cases}$$

where $\{B_t\}_{t\geq 0}$ is a d-dimensional Brownian motion with respect to $P_{x,\omega}$.

The main object of interest is the quenched path measure of the diffusion $t \mapsto X_t$ in the random potential $V(\,\cdot\,, \omega)$, which is defined, for each $x \in \mathbb{R}^d$, $\omega \in \Omega$ and $t > 0$, by

$$Q_{t,x,\omega}(dv) := \frac{1}{S(t, x, \omega)} \exp\left(- \int_0^t V(X_s, \omega)\, ds \right) P_{x,\omega}(dv),$$

where the normalizing factor $S(t, x, \omega)$, called the *quenched partition function*, is given by

$$S(t, x, \omega) := E_{x,\omega}\left[\exp\left(- \int_0^t V(X_s, \omega)\, ds \right) \right]. \tag{1-17}$$

Note that $Q_{t,x,\omega}$ is a probability measure on the path space $C(\mathbb{R}_+; \mathbb{R}^d)$.

The physical interpretation of the quenched path measures is that $Q_{t,x,\omega}$ describes the behavior of the diffusion X in an "absorbing" potential (in this interpretation, the half-life of a particle at position x is $\log 2/V(x, \omega)$) conditioned on the (exponentially unlikely event) that X is not absorbed up to time t; the probability that the particle lives until time t is precisely $S_{t,x,\omega}$. We note that the case $V \equiv 0$ is also of interest, in which case $Q_{t,x,\omega} = P_{x,\omega}$ and our results below describe the quenched large deviations of $P_{x,\omega}$, that is, of the diffusion in the random medium with no absorption. We also remark that we may allow V to take negative values, provided that V is uniformly bounded below; in the particle interpretation, negative values of V correspond to the creation of particles.

A central task in the study of diffusions in random environments is to obtain statistical information about the typical sample paths under $Q_{t,tx,\omega}$. Here we are interested in information regarding the large deviations of $Q_{t,tx,\omega}$ in the asymptotic limit $t \to \infty$.

Corollary 2. *Let $\mathbb{P}$ be a probability measure on Ω (which is identified with ordered triples (σ, b, V) as explained above) satisfying (1-9), (1-10) and (1-16). Let $\overline{H}$ be as in the statement of Theorem 1 corresponding to the Hamiltonian H given in (1-15), and let $\overline{L}$ be the Legendre–Fenchel transform of $\overline{H}$, defined for $z \in \mathbb{R}^d$ by*

$$\overline{L}(z) := \sup_{p\in\mathbb{R}^d} (p \cdot z - \overline{H}(p)).$$

Then there exists $\Omega_0 \in \mathcal{F}$ *with* $\mathbb{P}[\Omega_0] = 1$ *such that, for every* $\omega \in \Omega_0$, *we have the following*:

(i) *For every closed set* $K \subseteq \mathbb{R}^d$ *and* $x \in \mathbb{R}^d$,

$$\liminf_{t\to\infty} -\frac{1}{t}\log Q_{t,tx,\omega}[X_t \in tK] \geq \inf_{y\in K} \bar{L}(x-y) + \bar{H}(0). \tag{1-18}$$

(ii) *For every open set* $U \subseteq \mathbb{R}^d$ *and* $x \in \mathbb{R}^d$,

$$\limsup_{t\to\infty} -\frac{1}{t}\log Q_{t,tx,\omega}[X_t \in tU] \leq \inf_{y\in U} \bar{L}(x-y) + \bar{H}(0). \tag{1-19}$$

The proof that Theorem 1 implies Corollary 2 is presented in Section 6, and it follows along similar lines as the ones that previously appeared for example in [Lions and Souganidis 2005; Kosygina 2007].

Sznitman [1994] was the first to prove a quenched large deviations result like this in dimensions larger than one. Precisely, he proved Corollary 2 in the special case that $\sigma = I_d$ is the identity matrix, $b(y,\omega) = b_0 \in \mathbb{R}^d$ is a constant vector and the potential V is Poissonian; i.e.,

$$V(y,\omega) = \int_{\mathbb{R}^d} W(y-z)\,d\rho(z),$$

where $W \in C_c^\infty(\mathbb{R}^d)$ and the locally finite measure ρ has a Poissonian law (see [Sznitman 1998, Theorem 4.7]). In particular, such a random potential has a finite range of dependence and bounded finite moments.

In fact, the first phase of the strategy followed in this paper to homogenize the Hamilton–Jacobi equation is analogous in many respects to the probabilistic approach Sznitman used to obtain the large deviation principle. His proof relied on an application of the subadditive ergodic theorem to certain quantities, essentially equivalent to the maximal subsolutions considered here (the m_μ's), to obtain deterministic limits, the *Lyapunov exponents*, which are precisely the $\bar{m}_\mu$'s we encounter in the next section. See also the discussion preceding Proposition 2.5.

Let us check that the rate function in Corollary 2 agrees with the one in [Sznitman 1998]. First note that $\min_{\mathbb{R}^d} \bar{H} = \bar{H}(0) = 0$ in Sznitman's case. The effective Lagrangian $\bar{L}$ may thus be expressed in terms of the $\bar{m}_\mu$'s as follows:

$$\begin{aligned}
\bar{L}(z) &= \sup_{z\in\mathbb{R}^d} (p\cdot z - \bar{H}(p)) && \text{(definition of } \bar{L}) \\
&= \sup_{\mu>0} \sup\{p\cdot z - \bar{H}(p) : \bar{H}(p) \leq \mu\} && \text{(by } 0 = \min \bar{H}) \\
&= \sup_{\mu>0} \sup\{p\cdot z - \mu : \bar{H}(p) \leq \mu\} \\
&= \sup_{\mu>0} (\bar{m}_\mu(z) - \mu) && \text{(by (3-3) below).}
\end{aligned}$$

In the absorption-free case $V \equiv 0$, Zerner [1998] proved a result similar to Corollary 2 for random walks on the lattice $\mathbb{Z}^d$ with i.i.d. transition probabilities at each lattice point. He required (loosely translated into our notation) that A be "almost" uniformly elliptic:

$$\mathbb{E}[-\log \lambda_1(A(0,\omega))^d] < \infty. \tag{1-20}$$

This condition is much weaker than our (1-16) but is compensated for by the much stronger independence assumption on the random environment.

The subject of large deviations of random walks in random environments continues to receive much attention, and the works of Sznitman and Zerner have been subsequently extended to more general settings, and properties of the rate function have been studied in more depth; in particular, we refer to [Varadhan 2003; Rassoul-Agha 2004]. See also the more recent work of Yilmaz [2009], who proves a discrete version of Corollary 2 with no absorption, $V = 0$, in a quite general stationary-ergodic framework like ours with a slight strengthening of (1-20). Finally, a large deviation result for random walks in the case of absorption, $V \not\equiv 0$, was proved recently by Rassoul-Agha et al. [2013] under the assumptions that the random environment is strongly mixing. Admitting the proof of Corollary 2 from Theorem 1, the results of [Rassoul-Agha et al. 2013] may be compared to those of [Armstrong and Souganidis 2012].

Finally, we mention that the connection between large deviations and viscosity solutions of Hamilton–Jacobi equations was observed by Evans and Ishii [1985], who studied large deviations of the occupation times of small random perturbations of ODEs.

1D. *Disclaimer on viscosity solutions.* Throughout the paper, all differential equalities and inequalities are understood in the viscosity sense. For a general introduction to viscosity solutions, we refer to [Crandall et al. 1992]. Many of the fundamental PDE results we need here are proved in [Armstrong and Tran 2014], which we cite many times below. Recall that the natural function space for viscosity subsolutions on a domain X is the space $\mathrm{USC}(X)$ of upper semicontinuous functions on X and, for supersolutions, it is $\mathrm{LSC}(X)$, the set of lower semicontinuous functions on X.

1E. *Outline of the paper.* In the next section, we introduce the maximal subsolutions and homogenize them using the subadditive ergodic theorem. In Section 3, we construct the effective Hamiltonian and study some of its basic properties. In Section 4, we give the proof of an intermediate homogenization result and finally prove Theorem 1 in Section 5. The quenched large deviation principle is shown in Section 6 to be a consequence of the homogenization result.

2. The shape theorem: homogenization of the maximal subsolutions

In this section, we homogenize the *maximal subsolutions* of the inequality

$$-\operatorname{tr}(A(y)D^2 w) + H(Dw, y) \le \mu \quad \text{in } \mathbb{R}^d. \tag{2-1}$$

In subsequent sections, we show with comparison arguments that homogenizing these maximal subsolutions is enough to imply Theorem 1. As we will see, the reason that the maximal subsolutions are easier to homogenize is due to their subadditive structure.

The maximal subsolutions are defined, for each $\mu \in \mathbb{R}$ and $y, z \in \mathbb{R}^d$, by

$$m_\mu(y, z) := \sup\Big\{ w(y) - \sup_{\bar{B}_1(z)} w : w \in \mathrm{USC}(\mathbb{R}^d) \text{ satisfies (2-1)} \Big\}. \tag{2-2}$$

If the admissible class in the supremum above is empty, then we take $m_\mu(y,z) \equiv -\infty$. We denote, for every $\omega = (\sigma, H) \in \Omega$, the critical parameter $h(\omega)$ for which m_μ is finite by

$$h := \inf\{\mu : \text{there exists } w \in \mathrm{USC}(\mathbb{R}^d) \text{ satisfying (2-1)}\}. \tag{2-3}$$

According to (1-6), we have $h(\omega) \le \Lambda_1$. It is sometimes convenient to work with the quantity

$$\widetilde{m}_\mu(y,z) := \sup_{B_1(y)} m_\mu(\cdot\,,z). \tag{2-4}$$

Some deterministic properties of the maximal subsolutions are summarized in the following proposition, which is proved in [Armstrong and Tran 2014]. See Proposition 3.1 and Section 5 of that paper. The estimate (2-7) below is particularly important in our analysis and comes from the explicit Lipschitz estimates proved in [Armstrong and Tran 2014, Proposition 3.1].

Proposition 2.1 [Armstrong and Tran 2014]. *Fix $\omega = (\sigma, H) \in \Omega$ and $\mu \ge h(\omega)$. Then, for every $z \in \mathbb{R}^d$, the function $m_\mu(\cdot\,,z)$ belongs to $C^{0,1}_{\mathrm{loc}}(\mathbb{R}^d \setminus \bar{B}_1(z)) \cap \mathrm{USC}(\mathbb{R}^d)$ and satisfies*

$$-\operatorname{tr}(A(y)D^2 m_\mu) + H(Dm_\mu, y) \le \mu \quad \text{in } \mathbb{R}^d \tag{2-5}$$

as well as

$$-\operatorname{tr}(A(y)D^2 m_\mu) + H(Dm_\mu, y) = \mu \quad \text{in } \mathbb{R}^d \setminus \bar{B}_1(z). \tag{2-6}$$

There exists a constant $C > 0$, depending only on d and q, such that, for every $y, z \in \mathbb{R}^d$,

$$\operatorname*{osc}_{B_1(y)} m_\mu(\cdot\,,z) \le C\left[\left(\frac{(1+\Lambda_1)^{1/2}\|\sigma\|_{C^{0,1}(B_2(y))}}{a_2(y)}\right)^{2/(q-1)} + \left(\frac{M_2(y)+\mu}{a_2(y)}\right)^{1/q}\right]. \tag{2-7}$$

For every $\lambda \in [0,1]$, $\mu, \nu \ge h(\omega)$ and $y, z \in \mathbb{R}^d$,

$$m_{\lambda\mu+(1-\lambda)\nu}(y,z) \ge \lambda m_\mu(y,z) + (1-\lambda) m_\nu(y,z). \tag{2-8}$$

Finally, for every $x, y, z \in \mathbb{R}^d$, we have

$$\widetilde{m}_\mu(y,z) \le \widetilde{m}_\mu(y,x) + \widetilde{m}_\mu(x,z). \tag{2-9}$$

We define $K_\mu(y)$ to be the random variable on the right side of (2-7), that is,

$$K_\mu(y) := C\left[\left(\frac{(1+\Lambda_1)^{1/2}\|\sigma\|_{C^{0,1}(B_2(y))}}{a_2(y)}\right)^{2/(q-1)} + \left(\frac{M_2(y)+\mu}{a_2(y)}\right)^{1/q}\right],$$

so that we can write the bound (2-7) as

$$\operatorname*{osc}_{B_1(y)} m_\mu(\cdot\,,z) \le K_\mu(y). \tag{2-10}$$

We also denote $K_\mu = K_\mu(0)$. The primary use of the weak coercivity hypothesis (1-11) is that it implies that the α-th moment of K_μ, which we denote by $\bar{K}^\alpha_\mu$, is finite for some $\alpha > d$:

$$\bar{K}_\mu := \mathbb{E}[K^\alpha_\mu]^{1/\alpha} < +\infty. \tag{2-11}$$

Note that we have used (1-11) with a_2 and M_2 replacing a_1 and M_1, respectively, which is seen to be equivalent to (1-11) by an easy covering argument.

As far as the dependence of $\bar{K}_\mu$ on μ, we use $M_2 \geq 1$ to check that

$$\bar{K}_\mu \leq \bar{K}_0(1+\mu^{1/q}). \tag{2-12}$$

We next use ergodicity to show that the random variable h defined in (2-3) is, up to an event of probability zero, a deterministic constant.

Lemma 2.2. *Assume that $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ satisfying (1-9) and (1-10). Then there exists a constant $\bar{H}_* \in \mathbb{R}$, depending on $\mathbb{P}$, such that*

$$\mathbb{P}\big[\bar{H}_* = \inf\{\mu \in \mathbb{R} : \textit{there exists } w \in \mathrm{USC}(\mathbb{R}^d) \textit{ satisfying } (2\text{-}1)\}\big] = 1. \tag{2-13}$$

Proof. Let us see that h defined in (2-3) is finite. We have already seen that $h \leq \Lambda_1$ by (1-6). To argue that $h(\omega) > -\infty$ for every $\omega = (\sigma, H) \in \Omega$, we use the test function

$$\phi(y) := k(1-|y|^2)^{-1/(q-1)}.$$

If $k > 1$ and $C > 1$ are sufficiently large, depending only on Λ_2 and the constants a_1 and M_1 in (1-6) for H, then ϕ is a smooth solution of

$$-\operatorname{tr}(A(y)D^2\phi) + H(D\phi, y) > -C \quad \text{in } B_1.$$

Now consider an arbitrary element $w \in \mathrm{USC}(\mathbb{R}^d)$. Since $\phi(y) \to +\infty$ as $y \to \partial B_1$, there exists $x_0 \in B_1$ such that $w - \phi$ has a local maximum at x_0. In view of the differential inequality for ϕ, we obtain that w cannot be a subsolution of (1-6) for any $\mu \geq -C$.

It is immediate from its definition that h is invariant under the translation group action $\{\tau_y\}_{y \in \mathbb{R}^d}$. By the ergodicity assumption, this implies that $\mathbb{P}$ assigns each of the events $\{h > \lambda\}$ and $\{h < \lambda\}$, for every $\lambda \in \mathbb{R}$, probability either zero or one. This implies that h is $\mathbb{P}$-almost surely a constant. Taking this constant to be $\bar{H}_*$ yields the lemma. □

Our main interest lies in the asymptotic behavior of $m_\mu(y, z)$ for $|y - z| \simeq |z| \gg 1$. In the next lemma, we use Morrey's inequality together with the local oscillation bound (2-10) and the ergodic theorem to prove the large-scale oscillation bound $\operatorname{osc}_{B_R(Ry)} m_\mu(\cdot, z) \lesssim R$ uniformly in $z \in \mathbb{R}^d$ for $R \gg 1$. Recall that Morrey's inequality [Evans 1998, Section 5.6.2] states that, for any $R > 0$, $u \in C^1(B_R)$ and $\beta > d$, there exists $C(\beta, d) > 1$ such that

$$\operatorname*{osc}_{B_R} u \leq CR\left(\fint_{B_R} |Du(x)|^\beta \, dx\right)^{1/\beta}. \tag{2-14}$$

Therefore, we can control the oscillation of a function in terms of "averaged pointwise oscillation bounds". Thus, it is natural to attempt to control the large-scale oscillation of $m_\mu(\cdot, z)$ in terms of the average of a power of its local oscillation with the hope of using (2-10), (2-11) and the ergodic theorem to control the latter.

Lemma 2.3. *Assume that $\mathbb{P}$ is a probability measure on $(\Omega, \mathscr{F})$ satisfying (1-9), (1-10) and (1-11). Then there exists $C > 0$, depending only on d and α, such that*

$$\mathbb{P}\left[\forall \mu \geq \overline{H}_*,\ \forall x \in \mathbb{R}^d,\ \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d} \frac{1}{R} \operatorname*{osc}_{B_R(Rx)} m_\mu(\,\cdot\,, z) \leq C\overline{K}_\mu\right] = 1. \tag{2-15}$$

Proof. It is convenient to mollify the functions in order to put the local oscillation bounds into a pointwise form suitable for the application of Morrey's inequality. We first observe that, owing to Lemma 2.2, we may assume that m_μ is finite for all $\mu \geq \overline{H}_*$ by removing an event of zero probability.

We now fix $\mu \geq \overline{H}_*$ and $z \in \mathbb{R}^d$ and take a nonnegative $\eta \in C_c^\infty(\mathbb{R}^d)$ with support in $B_{1/2}$ and unit mass, $\int_{\mathbb{R}^d} \eta(y)\,dy = 1$, and set

$$\widehat{m}_\mu(y) := \int_{\mathbb{R}^d} \eta(y-x) m_\mu(x,z)\,dx. \tag{2-16}$$

Then $\widehat{m}_\mu$ is smooth, and using (2-7), we have, for every $y \in \mathbb{R}^d$,

$$\begin{aligned} |\widehat{m}_\mu(y) - m_\mu(y,z)| &\leq \int_{\mathbb{R}^d} \eta(y-x)|m_\mu(x,z) - m_\mu(y,z)|\,dx \\ &\leq \operatorname*{osc}_{B_{1/2}(y)} m_\mu(\,\cdot\,,z) \leq \inf_{B_{1/2}(y)} K_\mu(\,\cdot\,) \end{aligned} \tag{2-17}$$

and

$$|D\widehat{m}_\mu(y)| = \left| \int_{\mathbb{R}^d} D\eta(y-x)(m_\mu(x,z) - m_\mu(y,z))\,dx \right| \leq CK_\mu(y). \tag{2-18}$$

Applying (2-14) and then using (2-18), we deduce the existence of $C(d,\alpha) > 1$ such that, for every $x \in \mathbb{R}^d$,

$$\operatorname*{osc}_{B_R(x)} \widehat{m}_\mu \leq CR\left(\fint_{B_R(x)} |D\widehat{m}_\mu(y)|^\alpha\,dy\right)^{1/\alpha} \leq CR\left(\fint_{B_R(x)} K_\mu^\alpha(y)\,dy\right)^{1/\alpha}. \tag{2-19}$$

Next, we return to (2-17) and observe that

$$\begin{aligned} \sup_{y\in B_R(x)} |\widehat{m}_\mu(y) - m_\mu(y,z)| &\leq \sup_{y\in B_R(x)} \inf_{x\in B_{1/2}(y)} K_\mu(x) \leq \left(\sup_{y\in B_R(x)} \fint_{B_{1/2}(y)} K_\mu^\alpha(x)\,dx\right)^{1/\alpha} \\ &\leq C\left(\int_{B_{R+1}(x)} K_\mu^\alpha(x)\,dx\right)^{1/\alpha} \leq C(R+1)^{d/\alpha}\left(\fint_{B_{R+1}(x)} K_\mu^\alpha(x)\,dx\right)^{1/\alpha}. \end{aligned}$$

Making note of the fact that $d/\alpha < 1$ and combining the above inequality with (2-19), we deduce that, for every $R > 1$ and $x, z \in \mathbb{R}^d$,

$$\frac{1}{R} \operatorname*{osc}_{B_R(x)} m_\mu(\,\cdot\,,z) \leq C\left(\fint_{B_{R+1}(x)} K_\mu^\alpha(y)\,dy\right)^{1/\alpha}. \tag{2-20}$$

According to the ergodic theorem [Becker 1981],

$$\mathbb{P}\left[\lim_{R\to\infty} \left(\fint_{B_{R+1}(Rx)} K_\mu^\alpha(y)\,dy\right)^{1/\alpha} = \mathbb{E}[K_\mu^\alpha]^{1/\alpha}\right] = 1.$$

In view of the definition of $\overline{K}_\mu$, the last two lines yield that, for every $\mu \ge \overline{H}_*$,

$$\mathbb{P}\Big[\forall x \in \mathbb{R}^d,\ \limsup_{R\to\infty} \sup_{z\in\mathbb{R}^d} \frac{1}{R} \operatorname*{osc}_{B_R(Rx)} m_\mu(\,\cdot\,, z) \le C\overline{K}_\mu\Big] = 1.$$

Using the monotonicity of $\mu \to m_\mu$ and the continuity of $\mu \mapsto \overline{K}_\mu$ and intersecting the events corresponding to all rational μ and $\mu = \overline{H}_*$, we obtain (2-15). □

The following lemma is an abstract tool that allows us to obtain uniform convergence, with respect to the translation group $\{\tau_y\}_{y\in\mathbb{R}^d}$, for sequences of random variables that converge almost surely and satisfy appropriate oscillation bounds. The argument follows an (unpublished) idea attributed to Varadhan, using a combination of Egoroff's theorem and the ergodic theorem.

Lemma 2.4. *Assume $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ satisfying (1-9) and (1-10). Suppose that $\{X_t\}_{t>0}$ is a family of $\mathcal{F}$-measurable random variables on Ω such that*

$$\mathbb{P}\Big[\limsup_{t\to\infty} X_t(0) \le 0\Big] = 1.$$

Denote $X_t(y,\omega) := X_t(\tau_y\omega)$, and suppose that

$$\mathbb{P}\Big[\forall z \in \mathbb{R}^d,\ \limsup_{r\to 0} \limsup_{t\to\infty} \operatorname*{osc}_{y\in B_{tr}(tz)} X_t(y,\,\cdot\,) = 0\Big] = 1.$$

Then

$$\mathbb{P}\Big[\forall R > 0,\ \limsup_{t\to\infty} \sup_{y\in B_{tR}} X_t(y,\,\cdot\,) \le 0\Big] = 1.$$

Proof. We first notice that, after a routine covering argument, the second hypothesis can be rewritten in a slightly stronger way as

$$\mathbb{P}\Big[\forall R > 0,\ \limsup_{r\to 0} \limsup_{t\to\infty} \sup_{z\in B_R} \operatorname*{osc}_{y\in B_{tr}(tz)} X_t(y,\,\cdot\,) = 0\Big] = 1. \tag{2-21}$$

By the first hypothesis, for each $\varepsilon > 0$, there exists $T_\varepsilon > 0$ sufficiently large that

$$\mathbb{P}\Big[\sup_{t\ge T_\varepsilon} X_t(0,\,\cdot\,) \le \varepsilon\Big] \ge 1 - \tfrac{1}{2}\varepsilon^d. \tag{2-22}$$

Denote this event by $D_\varepsilon := \{\omega \in \Omega : \sup_{t\ge T_\varepsilon} X_t(0,\omega) \le \varepsilon\}$. According to the multiparameter ergodic theorem [Becker 1981], for each $\varepsilon > 0$, there exists an event $\widetilde{\Omega}_\varepsilon \in \mathcal{F}$ with $\mathbb{P}[\widetilde{\Omega}_\varepsilon] = 1$ such that, for every $\omega \in \widetilde{\Omega}_\varepsilon$,

$$\lim_{r\to\infty} \fint_{B_r} \mathbb{1}_{D_\varepsilon}(\tau_x\omega)\,dx = \mathbb{P}[D_\varepsilon] \ge 1 - \tfrac{1}{2}\varepsilon^d. \tag{2-23}$$

Here $\mathbb{1}_E$ denotes the indicator function of an event $E \in \mathcal{F}$. It follows that, for each $\omega \in \widetilde{\Omega}_\varepsilon$, there exists $r_\varepsilon > 0$ sufficiently large (and depending on ω in addition to ε) that

$$\inf_{r\ge r_\varepsilon} \fint_{B_r} \mathbb{1}_{D_\varepsilon}(\tau_x\omega)\,dx > 1 - \varepsilon^d. \tag{2-24}$$

Notice that (2-24) implies that, for $r \geq r_\varepsilon(\omega)$,

$$|\{x \in B_r : \tau_x\omega \in D_\varepsilon\}| > (1-\varepsilon^d)|B_r|. \tag{2-25}$$

In particular, if $r \geq r_\varepsilon(\omega)$, then no ball of radius $r\varepsilon$ is contained in $\{x \in B_r : \tau_x\omega \notin D_\varepsilon\}$.

Let $\widetilde{\Omega}$ be the intersection of $\widetilde{\Omega}_\varepsilon$ over all $\varepsilon \in \mathbb{Q}_+$. Fix $R, \varepsilon > 0$ with $\varepsilon \in \mathbb{Q}_+$, $\omega \in \widetilde{\Omega}$ such that ω also belongs to the event inside the probability in (2-21), $t \geq R^{-1}\max\{r_\varepsilon(\omega), T_\varepsilon\}$ and $y \in B_{tR}$. Then there exists $z \in B_R$ such that $\tau_{tz}\omega \in D_\varepsilon$ and $|y-tz| \leq tR\varepsilon$. Note that $\tau_{tz}\omega \in D_\varepsilon$ is equivalent to $X_t(tz,\omega) \leq \varepsilon$. We deduce that

$$X_t(y,\omega) \leq X_t(tz,\omega) + \operatorname*{osc}_{x \in B_{tR\varepsilon}(tz)} X_t(x,\omega) \leq \varepsilon + \sup_{z' \in B_R} \operatorname*{osc}_{x \in B_{tR\varepsilon}(tz')} X_t(x,\omega).$$

This holds for all $y \in B_{tR}$; hence,

$$\sup_{y \in B_{tR}} X_t(y,\omega) \leq \varepsilon + \sup_{z' \in B_R} \operatorname*{osc}_{x \in B_{tR\varepsilon}(tz')} X_t(x,\omega).$$

We have shown that, for all $\varepsilon \in \mathbb{Q}$ such that $\varepsilon > 0$, we have

$$\limsup_{t\to\infty} \sup_{y \in B_{tR}} X_t(y,\omega) \leq \varepsilon + \limsup_{t\to\infty} \sup_{z' \in B_R} \operatorname*{osc}_{x \in B_{tR\varepsilon}(tz')} X_t(x,\omega).$$

Sending $\varepsilon \to 0$, using that ω belongs to the event inside the probability in (2-21), we obtain

$$\limsup_{t\to\infty} \sup_{y \in B_{tR}} X_t(y,\omega) \leq 0.$$

This conclusion applies for every $R > 0$ and ω belonging to the intersection of $\widetilde{\Omega}$ and the event in (2-21), which has probability one. □

We next employ the subadditive ergodic theorem [Akcoglu and Krengel 1981] and the subadditivity of m_μ to get the following result, which asserts that, for large $t > 0$, we have $m_\mu(ty,tz) \approx t\overline{m}_\mu(y-z)+o(t)$ for some deterministic function $\overline{m}_\mu$. The key ingredients in the proof are subadditivity (2-9) and the local oscillation estimate (2-15).

The terminology "shape theorem" originated in first-passage percolation, and "shape" refers to the sublevel sets of m_μ. In particular, the result here generalizes [Sznitman 1998, Theorem 5.2.5] and also covers the case that $A \equiv 0$ and the Hamiltonian has the specific form $H(p,x) = a(x)|p|$ where $a > 0$ is an appropriate random field, which is a continuum analogue of the first passage percolation model.

Proposition 2.5 (the shape theorem). *Assume $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ satisfying (1-9), (1-10) and (1-11). Then there exists a family $\{\overline{m}_\mu : \mu \geq \overline{H}_*\} \subseteq C(\mathbb{R}^d)$ of convex, positively homogeneous functions such that*

$$\mathbb{P}\Big[\forall \mu \geq \overline{H}_*,\ \forall R > 0,\ \limsup_{t\to\infty} \sup_{y,z \in B_R} \Big|\frac{m_\mu(ty,tz)}{t} - \overline{m}_\mu(y-z)\Big| = 0\Big] = 1. \tag{2-26}$$

Proof. We break the argument into five steps. In the first step, we construct $\overline{m}_\mu$ using the subadditive ergodic theorem and, in Step 2, derive some of its basic properties. In Step 3, we prove (2-26) for $z = 0$, and in the fourth step, we remove this restriction. For the first four steps, we fix $\mu \geq \bar{H}_*$. The universal quantifier over $\mu \geq \bar{H}_*$ will be moved inside the probability in the final step.

Before commencing with the argument, we make a reduction. With $\widetilde{m}_\mu$ defined as in (2-4), we observe that

$$0 \leq \widetilde{m}_\mu(y,z) - m_\mu(y,z) = \sup_{\xi \in B_1(y)} (m_\mu(\xi,z) - m_\mu(y,z)) \leq \operatorname*{osc}_{B_1(y)} m_\mu(\,\cdot\,,z).$$

Using this together with Lemma 2.3, we find that

$$\begin{aligned}\mathbb{P}\Big[\forall \mu \geq \bar{H}_*,\ \forall R > 0,\ \limsup_{t\to\infty} \sup_{y,z\in B_R} \frac{1}{t}|m_\mu(ty,tz) - \widetilde{m}_\mu(ty,tz)| = 0\Big] \\ \geq \mathbb{P}\Big[\forall \mu \geq \bar{H}_*,\ \forall R > 0,\ \limsup_{t\to\infty} \sup_{y,z\in B_R} \frac{1}{t} \operatorname*{osc}_{B_1(ty)} m_\mu(\,\cdot\,,tz) = 0\Big] \\ \geq \mathbb{P}\Big[\forall \mu \geq \bar{H}_*,\ \forall R,\delta > 0,\ \limsup_{t\to\infty} \sup_{z\in\mathbb{R}^d} \sup_{y\in B_R} \frac{1}{t} \operatorname*{osc}_{B_{t\delta}(ty)} m_\mu(\,\cdot\,,z) \leq C\bar{K}_\mu \delta\Big] = 1.\end{aligned}$$

Therefore, it suffices to prove the proposition with $\widetilde{m}_\mu$ in place of m_μ.

Step 1. We apply the subadditive ergodic theorem to construct $\bar{m}_\mu$. Note that it is immediate from the definitions that both m_μ and $\widetilde{m}_\mu$ are jointly stationary in (y,z). Precisely, we mean that, using the notation $m_\mu(y,z,\omega)$ and $\widetilde{m}_\mu(y,z,\omega)$ to denote dependence on $\omega \in \Omega$, then with respect to the translation group action $\{\tau_x\}_{x\in\mathbb{R}^d}$, we have

$$m_\mu(y,z,\tau_x\omega) = m_\mu(y+x,z+x,\omega) \quad \text{and} \quad \widetilde{m}_\mu(y,z,\tau_x\omega) = \widetilde{m}_\mu(y+x,z+x,\omega).$$

Note that $\widetilde{m}_\mu$ is subadditive by (2-9) and $\mathbb{P}$-integrable on Ω since (2-20) implies

$$\begin{aligned}\mathbb{E}[\widetilde{m}_\mu(y,z)] &\leq \mathbb{E}\Big[\sup_{B_{|y-z|+1}(z)} m_\mu(\,\cdot\,,z)\Big] \\ &\leq C(|y-z|+1)\mathbb{E}\bigg[\bigg(\fint_{B_{|y-z|+2}} K_\mu^\alpha(x)\,dx\bigg)^{1/\alpha}\bigg] \leq C\bar{K}_\mu(|y-z|+1),\end{aligned} \tag{2-27}$$

where the last inequality follows by Jensen's inequality. We have checked that $\widetilde{m}_\mu$ verifies the hypothesis of the subadditive ergodic theorem [Akcoglu and Krengel 1981], and we obtain, for each fixed $y \in \mathbb{R}^d$, a random variable $\bar{m}_\mu(y)$ such that

$$\mathbb{P}\Big[\lim_{t\to\infty} \frac{1}{t}\widetilde{m}_\mu(ty,0) = \bar{m}_\mu(y)\Big] = 1. \tag{2-28}$$

However, it turns out that $\bar{m}_\mu(y)$ is constant $\mathbb{P}$-almost surely, that is,

$$\mathbb{P}\big[\bar{m}_\mu(y) = \mathbb{E}[\bar{m}_\mu(y)]\big] = 1. \tag{2-29}$$

This follows from the ergodic hypothesis and the fact that $\overline{m}_\mu(y)$ is invariant under translations. To see this, we write $\widetilde{m}_\mu(y, z, \omega)$ and $\overline{m}_\mu(y, \omega)$ to denote dependence on $\omega \in \Omega$ and observe that, for every $z \in \mathbb{R}^d$,

$$\begin{aligned}\overline{m}_\mu(y, \tau_z\omega) &= \lim_{t\to\infty} \frac{1}{t}\widetilde{m}_\mu(ty+z, z, \omega)\\ &\le \lim_{t\to\infty} \frac{1}{t}\big(\widetilde{m}_\mu(ty+z, ty, \omega) + \widetilde{m}_\mu(ty, 0, \omega) + \widetilde{m}_\mu(0, z, \omega)\big)\\ &\le \lim_{t\to\infty} \frac{1}{t}\widetilde{m}_\mu(ty, 0, \omega) + \limsup_{t\to\infty} \frac{1}{t}\Big(\operatorname*{osc}_{B_{|z|+1}(ty)} m_\mu(\,\cdot\,, ty, \omega) + \operatorname*{osc}_{B_1(0)} m_\mu(\,\cdot\,, z, \omega)\Big)\\ &= \lim_{t\to\infty} \frac{1}{t}\widetilde{m}_\mu(ty, 0, \omega) = \overline{m}_\mu(y, \omega).\end{aligned}$$

Here we used stationarity, followed by (2-9), the definition of $\widetilde{m}_\mu$ and Lemma 2.3. We deduce that $\overline{m}_\mu(y, \tau_z\omega) = \overline{m}_\mu(y, \omega)$ for all $\omega \in \Omega$ and $z \in \mathbb{R}^d$, which, in view of (1-10), implies that each of the events $\{\omega \in \Omega : \overline{m}_\mu(y, \omega) > \mathbb{E}[\overline{m}_\mu(y, \,\cdot\,)]\}$ and $\{\omega \in \Omega : \overline{m}_\mu(y, \omega) < \mathbb{E}[\overline{m}_\mu(y, \,\cdot\,)]\}$ has probability either zero or one. So both must be of probability zero, and (2-29) holds.

We henceforth identify $\overline{m}_\mu(y)$ and the deterministic quantity $\mathbb{E}[\overline{m}_\mu(y, \,\cdot\,)]$. With this identification, we may combine (2-28) and (2-29) to write

$$\mathbb{P}\Big[\limsup_{t\to\infty}\Big|\frac{\widetilde{m}_\mu(ty, 0)}{t} - \overline{m}_\mu(y)\Big| = 0\Big] = 1. \tag{2-30}$$

This holds for all $y \in \mathbb{R}^d$. By intersecting the events in (2-30) over all $y \in \mathbb{Q}^d$, we get

$$\mathbb{P}\Big[\forall y \in \mathbb{Q}^d,\ \limsup_{t\to\infty}\Big|\frac{\widetilde{m}_\mu(ty, 0)}{t} - \overline{m}_\mu(y)\Big| = 0\Big] = 1. \tag{2-31}$$

Step 2. We next verify that $\overline{m}_\mu : \mathbb{R}^d \to \mathbb{R}$ is continuous, convex and positively homogeneous. It is immediate from (2-27) that

$$|\overline{m}_\mu(y)| \le C\overline{K}_\mu|y|. \tag{2-32}$$

The stationarity and subadditivity of $\widetilde{m}_\mu$ yield that $\overline{m}_\mu$ is sublinear. Indeed, for every $y, z \in \mathbb{R}^d$,

$$\begin{aligned}\overline{m}_\mu(y+z) = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}[\widetilde{m}_\mu(t(y+z), 0)] &\le \lim_{t\to\infty}\frac{1}{t}\mathbb{E}[\widetilde{m}_\mu(t(y+z), tz) + \widetilde{m}_\mu(tz, 0)]\\ &= \lim_{t\to\infty}\frac{1}{t}\mathbb{E}[\widetilde{m}_\mu(ty, 0)] + \lim_{t\to\infty}\frac{1}{t}\mathbb{E}[\widetilde{m}_\mu(tz, 0)] = \overline{m}_\mu(y) + \overline{m}_\mu(z).\end{aligned} \tag{2-33}$$

Combining (2-32) and (2-33) yields

$$\overline{m}_\mu(y) - \overline{m}_\mu(z) \le \overline{m}_\mu(y-z) \le C\overline{K}_\mu|y-z|.$$

By interchanging y and z, we get

$$|\overline{m}_\mu(y) - \overline{m}_\mu(z)| \le C\overline{K}_\mu|y-z|, \tag{2-34}$$

and so $\overline{m}_\mu$ is Lipschitz with constant $C\overline{K}_\mu$. It is immediate from the form of the limit (2-28) that $\overline{m}_\mu$ is positively homogeneous, and from this and (2-33), we deduce that $\overline{m}_\mu$ is convex. For future reference, we observe that $\mu \mapsto \overline{m}_\mu(y)$ is concave by (2-8). Since this map is nondecreasing, it must also be continuous.

Step 3. We next upgrade assertion (2-31) to

$$\mathbb{P}\Big[\forall R>0,\ \lim_{t\to\infty}\sup_{y\in B_R}\Big|\frac{1}{t}\widetilde{m}_\mu(ty,0,\omega)-\overline{m}_\mu(y)\Big|=0\Big]=1. \tag{2-35}$$

Observe that, for every $y\in\mathbb{R}^d$ and $z\in\mathbb{Q}^d$, we have

$$\begin{aligned}\Big|\frac{1}{t}\widetilde{m}_\mu(ty,0)-\overline{m}_\mu(y)\Big| &\le \frac{1}{t}|\widetilde{m}_\mu(ty,0)-\widetilde{m}_\mu(tz,0)|+\Big|\frac{1}{t}\widetilde{m}_\mu(tz,0)-\overline{m}_\mu(z)\Big|+|\overline{m}_\mu(y)-\overline{m}_\mu(z)|\\ &\le \frac{1}{t}\operatorname*{osc}_{B_{t|y-z|+2}(tz)} m_\mu(\,\cdot\,,0)+\frac{1}{t}|\widetilde{m}_\mu(tz,0)-\overline{m}_\mu(z)|+C\overline{K}_\mu|y-z|.\end{aligned}$$

Fix $R>0$. Let $\delta>0$, and select finitely many $z_1,\dots,z_k\in\mathbb{Q}^d\cap B_R$ such that the union of the balls $B(z_i,\delta)$ covers B_R. Then from the above inequality, we find that

$$\sup_{y\in B_R}\Big|\frac{1}{t}\widetilde{m}_\mu(ty,0)-\overline{m}_\mu(y)\Big|\le \sup_{y\in B_R}\sup_{i\in\{1,\dots,k\}}\frac{1}{t}\operatorname*{osc}_{B_{t\delta+2}(tz_i)} m_\mu(\,\cdot\,,0)+\sup_{i\in\{1,\dots,k\}}\frac{1}{t}|\widetilde{m}_\mu(tz_i,0)-\overline{m}_\mu(z_i)|+C\overline{K}_\mu\delta.$$

Now taking the lim sup as $t\to\infty$, we deduce from (2-15) and (2-31) that, for every $R,\delta>0$,

$$\mathbb{P}\Big[\lim_{t\to\infty}\sup_{y\in B_R}\Big|\frac{1}{t}\widetilde{m}_\mu(ty,0,\omega)-\overline{m}_\mu(y)\Big|\le 2C\overline{K}_\mu\delta\Big]=1.$$

We recover (2-35) after intersecting over all the events corresponding to $\delta\in\mathbb{Q}_+$ and then over all of the resulting events corresponding to $R\in\mathbb{N}^*$.

Step 4. We next release the vertex point using Lemma 2.4 with

$$X_t:=\sup_{y\in B_{2R}}\Big|\frac{1}{t}\widetilde{m}_\mu(ty,0)-\overline{m}_\mu(y)\Big|,\qquad t>0.$$

Lemma 2.3 and (2-35) give the hypotheses of Lemma 2.4 for X_t, and so an application of the lemma yields, for every $R>0$,

$$\mathbb{P}\Big[\lim_{t\to\infty}\sup_{y,z\in B_R}\Big|\frac{1}{t}\widetilde{m}_\mu(ty,tz)-\overline{m}_\mu(y-z)\Big|=0\Big]\ge\mathbb{P}\Big[\lim_{t\to\infty}\sup_{z\in B_R}\sup_{y\in B_{2R}(z)}\Big|\frac{1}{t}\widetilde{m}_\mu(ty+tz,tz)-\overline{m}_\mu(y)\Big|=0\Big]=1.$$

Intersecting the events corresponding to $R=1,2,\dots,$ we obtain

$$\mathbb{P}\Big[\forall R>0,\ \lim_{t\to\infty}\sup_{y,z\in B_R}\Big|\frac{1}{t}\widetilde{m}_\mu(ty,tz)-\overline{m}_\mu(y-z)\Big|=0\Big]=1. \tag{2-36}$$

Step 5. We immediately obtain (2-26) from (2-36) by the monotonicity of $\mu\mapsto m_\mu(y,z)$, the continuity of $\mu\mapsto\overline{m}_\mu(y)$ (see the end of Step 2) and intersecting the events corresponding to each rational $\mu>\overline{H}_*$ as well as to $\mu=\overline{H}_*$. □

Remark 2.6. For future reference, we note that $m_\mu(y,z)\ge\beta|y-z|$ for any $\beta>0$ and $\mu\ge\Lambda_1(\beta^q+1)$. Indeed, in view of the monotonicity of $\mu\mapsto m_\mu(y,z)$, it is enough to check that the cone function $\phi(y):=\beta\max\{0,|y-z|-1\}$ is a subsolution of (2-1) for $\mu=\Lambda_1(\beta^q+1)$. This is easy to obtain from (1-6), using $|D\phi| \le \beta$ and the fact that the diffusion term has a helpful sign due to the convexity of ϕ. This also yields

$$\mu \ge \Lambda_1(\beta^q + 1) \quad \Longrightarrow \quad \text{for all } y \in \mathbb{R}^d,\ \overline{m}_\mu(y) \ge \beta|y|. \tag{2-37}$$

In view of the concavity of $\mu \mapsto \overline{m}_\mu(y)$, which was obtained in Step 2 of the proof above, we get the following: there exists $c > 0$ such that, for every $\mu \ge \nu \ge \overline{H}_*$ and $y, z \in \mathbb{R}^d$,

$$\overline{m}_\mu(y) \ge \overline{m}_\nu(y) + c\mu^{-(q-1)/q}(\mu - \nu)|y|.$$

(This remark is needed in the proof (3-4) and to check that $\overline{H}$ is well defined.)

3. Identification of the effective Hamiltonian

In this section, we define $\overline{H}$ in terms of the family $\{\overline{m}_\mu : \mu \ge \overline{H}_*\}$ of homogenized maximal subsolutions and proceed to study some of its basic properties. Throughout this section, we assume that $\mathbb{P}$ is a given probability measure satisfying (1-9), (1-10) and (1-11).

We begin with an informal heuristic that leads to a guess for what $\overline{H}$ should be, thinking in terms of an inverse problem. Write the metric problem in the "theatrical scaling" by introducing a parameter $\varepsilon > 0$ and defining

$$m_\mu^\varepsilon(x) := \varepsilon m_\mu\Big(\frac{x}{\varepsilon}, 0\Big).$$

At this scale, Proposition 2.5 asserts that $m_\mu^\varepsilon \to \overline{m}_\mu$ locally uniformly in $\mathbb{R}^d$ and $\mathbb{P}$-almost surely, as $\varepsilon \to 0$, and we may write (2-6) as

$$-\varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2 m_\mu^\varepsilon\Big) + H\Big(Dm_\mu^\varepsilon, \frac{x}{\varepsilon}\Big) = \mu \quad \text{in } \mathbb{R}^d \setminus \overline{B}_\varepsilon(0).$$

By formally passing to the limit $\varepsilon \to 0$ in this equation (and in the rescaled version of (2-5)) under the assumption that it homogenizes, this suggests that we should obtain

$$\overline{H}(D\overline{m}_\mu) \le \mu \quad \text{in } \mathbb{R}^d \quad \text{and} \quad \overline{H}(D\overline{m}_\mu) = \mu \quad \text{in } \mathbb{R}^d \setminus \{0\}. \tag{3-1}$$

That is, we expect that $\overline{m}_\mu$ is the maximal subsolution of $\overline{H}$ with respect to μ and the gradient of this positively homogeneous function should prescribe the μ-level set of $\overline{H}$; the image of its subdifferential should be the μ-sublevel set of $\overline{H}$.

In view of this discussion, we simply *define* $\overline{H}$ in such a way that this is so:

$$\overline{H}(p) := \inf\{\mu \ge \overline{H}_* : \forall y \in \mathbb{R}^d,\ \overline{m}_\mu(y) \ge p \cdot y\}. \tag{3-2}$$

Note that, since $\overline{m}_\mu$ is convex and positively homogeneous, the subdifferential $\partial m_\mu(0)$ is actually the closed convex hull of the image of $\mathbb{R}^d$ under $D\overline{m}_\mu$. Recall that the subdifferential $\partial\phi(x)$ of a convex function $\phi : \mathbb{R}^d \to \mathbb{R}$ at a point x is defined by

$$\partial\phi(x) := \{p \in \mathbb{R}^d : \forall y \in \mathbb{R}^d,\ \phi(y) \ge \phi(x) + p \cdot (y - x)\}.$$

We expect $\partial\overline{m}_\mu(0)$ to be the μ-sublevel set of $\overline{H}$ and the image of $\mathbb{R}^d$ under $D\overline{m}_\mu$ to be the μ-level set of $\overline{H}$. This indeed follows from (3-2), and we may invert this formula to write $\overline{m}_\mu$ in terms of $\overline{H}$:

$$\overline{m}_\mu(y) = \sup\{p \cdot y : \overline{H}(p) \le \mu\}. \tag{3-3}$$

That is, $\overline{m}_\mu$ is simply the support function of the μ-sublevel set of $\overline{H}$. So the definition (3-2) is formally in accord with (3-1), and once we have verified that $\overline{H}$ is convex (which we do below in Lemma 3.1), checking the latter in the viscosity sense is simply a routine exercise. Since here we do not actually use this fact, we omit the argument, but the reader may consult for example [Armstrong and Souganidis 2013] or else argue directly that the maximal subsolutions of a constant-coefficient convex Hamiltonian are the support functions of the sublevel sets.

We need to check that the quantity $\overline{H}(p)$ is well defined (and finite). In view of the monotonicity of $\mu \mapsto \overline{m}_\mu$, we need only show that, for every $p \in \mathbb{R}^d$, there exists $\mu > \overline{H}_*$ sufficiently large that the graph of $\overline{m}_\mu$ is above the plane $y \mapsto p \cdot y$. But this is immediate from (2-37), which in fact gives the estimate

$$\overline{H}_* \le \overline{H}(p) \le \Lambda_1(|p|^q + 1). \tag{3-4}$$

We collect some more basic properties of the effective Hamiltonian $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ in the following lemma:

Lemma 3.1. *The function $\overline{H} : \mathbb{R}^d \to \mathbb{R}$ is continuous, convex and satisfies $\overline{H}_* = \min_{p\in\mathbb{R}^d} \overline{H}(p)$. Moreover, there exist $C, c > 0$, depending only on d, such that*

$$c\overline{K}_0^{-q}(|p| - C\overline{K}_0)^q \le \overline{H}(p) \le \Lambda_1(|p|^q + 1). \tag{3-5}$$

Proof. By definition, $\overline{H}(\cdot) \ge \overline{H}_*$. On the other hand, take $\delta > 0$, and set $\mu := \overline{H}_* + \delta$. Since $\overline{m}_\mu$ is convex, we may select $p_0 \in \partial m_\mu(0)$. This implies that $\overline{m}_\mu(y) \ge p_0 \cdot y$ for every $y \in \mathbb{R}^d$. Thus,

$$\min_{p\in\mathbb{R}^d} \overline{H}(p) \le \overline{H}(p_0) \le \mu = \overline{H}_* + \delta.$$

Since $\delta > 0$ was arbitrary, we obtain the first assertion that $\overline{H}_* = \min_{p\in\mathbb{R}^d} \overline{H}(p)$.

The upper bound for $\overline{H}$ was proved already in (3-4). The lower bound follows from (2-12) and (2-32) and the definition of $\overline{H}$ after an easy computation. □

An immediate consequence of the convexity of $\overline{H}$ is that, with the possible exception of the minimal level set $\{\overline{H} = \overline{H}_*\}$, each of the level sets of $\overline{H}$ is the boundary of the corresponding sublevel set. That is, for every $p \in \mathbb{R}^d$,

$$\overline{H}(p) > \overline{H}_* \quad \text{implies that} \quad p \in \partial\{\hat{p} \in \mathbb{R}^d : \overline{H}(\hat{p}) \le \overline{H}(p)\}. \tag{3-6}$$

To prove the main homogenization result, we need further geometric information, summarized in the following lemma, relating the level sets of $\overline{H}$ and the maximal subsolutions.

Recall that, if $K \subseteq \mathbb{R}^d$ is closed and convex, an *exposed point* of K is a point $p \in K$ such that there exists a linear functional $l : \mathbb{R}^d \to \mathbb{R}$ such that $l(p) > l(\hat{p})$ for every $\hat{p} \in K \setminus \{p\}$. The set of exposed points is, for a general bounded convex subset K of $\mathbb{R}^d$, a subset of the set of extreme points of K. However, Straszewicz's theorem [Rockafellar 1970, Theorem 18.6] asserts that every extreme point is a limit of exposed points.

Lemma 3.2. *Let $\mu \geq \bar{H}_*$ and $p \in \partial\{\hat{p} \in \mathbb{R}^d : \bar{H}(\hat{p}) \leq \mu\}$. There exists a unit vector $e \in \partial B_1$ such that*

$$\bar{m}_\mu(e) - p \cdot e = 0 = \inf_{y \in \mathbb{R}^d} (\bar{m}_\mu(y) - p \cdot y). \tag{3-7}$$

If in addition p is an exposed point of $\{\hat{p} \in \mathbb{R}^d : \bar{H}(\hat{p}) \leq \mu\}$, then e can be chosen in such a way that $\bar{m}_\mu$ is differentiable at e with $p = D\bar{m}_\mu(e)$.

Proof. Set $S := \{\hat{p} \in \mathbb{R}^d : \bar{H}(\hat{p}) \leq \mu\}$. By elementary convex separation, there exists a linear functional $l : \mathbb{R}^d \to \mathbb{R}$ such that $l(p) = 0$ and $l(\hat{p}) \leq 0$ for every $\hat{p} \in S$. If p is an exposed point, then we also take l so that $l(\hat{p}) < 0$ for every $\hat{p} \in S \setminus \{p\}$. There exists $e \in \mathbb{R}^d \setminus \{0\}$ such that $l(x) = e \cdot (x - p)$. By normalizing, we may assume that $|e| = 1$. We deduce that, for every $y \in \mathbb{R}^d$,

$$\bar{m}_\mu(e) - p \cdot e = \sup\{(\hat{p} - p) \cdot e : \hat{p} \in S\} = 0 \leq \sup\{(\hat{p} - p) \cdot y : \hat{p} \in S\} = \bar{m}_\mu(y) - p \cdot y. \tag{3-8}$$

This is (3-7). Since $\bar{m}_\mu$ is positively homogeneous, we see that $p \in \partial\bar{m}_\mu(e)$. In fact, if we repeat (3-8) with an arbitrary element of S in place of p, we find that

$$\partial\bar{m}_\mu(e) \subseteq \{\hat{p} \in S : l(\hat{p}) = 0\}. \tag{3-9}$$

Thus, if p is an exposed point of S, then we have $\partial\bar{m}_\mu(e) = \{p\}$ by our choice of l. This implies that $\bar{m}_\mu$ is differentiable at e and $D\bar{m}_\mu(e) = p$. □

Remark 3.3. We can express $\bar{H}$ via the following "min-max" formula:

$$\bar{H}(p) = \inf\Big\{\mu \in \mathbb{R} : \text{there exists } w \in C^{0,1}_{\text{loc}}(\mathbb{R}^d) \text{ satisfying (2-1) and } \liminf_{|y|\to\infty} \frac{w(y) - p \cdot y}{|y|} \geq 0\Big\}. \tag{3-10}$$

Indeed, if $w \in \text{USC}(\mathbb{R}^d)$ satisfies (2-1), then

$$\bar{m}_\mu(y) - p \cdot y \geq \liminf_{t\to\infty} \frac{w(ty) - p \cdot (ty)}{t}.$$

If the latter is nonnegative for all $y \in \mathbb{R}^d$, then $\bar{H}(p) \leq \mu$ by definition. This yields "$\leq$" in (3-10). To obtain the reverse inequality, we use m_μ with $\mu = \bar{H}(p)$ and observe that

$$\liminf_{|y|\to\infty} \frac{m_\mu(y) - p \cdot y}{|y|} = \liminf_{t\to\infty} \inf_{|y|=1} \Big(\frac{m_\mu(ty)}{t} - p \cdot y\Big) = \inf_{|y|=1} (\bar{m}_\mu(y) - p \cdot y) \geq 0.$$

The reason that we call (3-10) a "min-max" representation is that it can be formally written

$$\bar{H}(p) = \inf_{w \in \mathscr{L}_p} \sup_{y \in \mathbb{R}^d} \big(-\operatorname{tr}(A(y)D^2 w(y)) + H(Dw(y), y)\big), \tag{3-11}$$

where

$$\mathscr{L}_p := \Big\{w \in C^{0,1}_{\text{loc}}(\mathbb{R}^d) : \liminf_{|y|\to\infty} \frac{w(y) - p \cdot y}{|y|} \geq 0\Big\}.$$

The expression inside the infimum on the right of (3-11) does not make sense since w may not have enough regularity. It must therefore be interpreted in the viscosity sense, and this leads precisely to (3-10).

4. Homogenization of the approximate cell problem

In this section, we show using a comparison argument that Proposition 2.5 implies a homogenization result for a special time-independent problem. The particularities of this argument are new here, even for uniformly coercive Hamiltonians or first-order equations.

Throughout, we assume $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ satisfying (1-9), (1-10) and (1-11).

For each fixed $p \in \mathbb{R}^d$, we consider the problem

$$w^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2 w^\varepsilon\Big) + H\Big(p + Dw^\varepsilon, \frac{x}{\varepsilon}\Big) = 0 \quad \text{in } \mathbb{R}^d. \tag{4-1}$$

We will show that (4-1) has a unique bounded-below solution with probability one that we denote by $w^\varepsilon(\,\cdot\,, p)$. We argue that

$$\mathbb{P}\Big[\forall p \in \mathbb{R}^d,\ \forall R > 0,\ \limsup_{\varepsilon\to 0} \sup_{x\in B_R} |w^\varepsilon(x, p) + \bar{H}(p)| = 0\Big] = 1. \tag{4-2}$$

Recall that (4-1), often written at the microscopic ("nontheatrical") scale (as in (4-5) below), is often called the *approximate cell problem* and homogenizing it (by which we mean proving (4-2)) is the key step in the derivation of Theorem 1 from Proposition 2.5. To see why we expect $w^\varepsilon(\,\cdot\,, p)$ to converge locally uniformly to the constant $-\bar{H}(p)$ as $\varepsilon \to 0$, observe that the (unique) solution of

$$w + \bar{H}(p + Dw) = 0 \quad \text{in } \mathbb{R}^d \tag{4-3}$$

is precisely the constant function $w \equiv -\bar{H}(p)$. Thus, (4-2) can be understood roughly as the assertion that "(4-1) homogenizes to (4-3)".

4A. ***Basic properties of* (4-1).** In order to prove (4-2), we must first establish some fundamental properties of (4-1) including wellposedness. In the uniformly coercive case, it is straightforward (and classical) to show that the Perron method and the comparison principle yield a unique bounded solution of (4-1) given by the formula

$$w^\varepsilon(x, p) := \sup\{v(x) : v \in \mathrm{USC}(\mathbb{R}^d) \text{ is a subsolution of (4-1)}\}. \tag{4-4}$$

Wellposedness in the general weakly coercive setting is more nontrivial because it is less easy to show a priori that $w^\varepsilon(\,\cdot\,, p)$ satisfies a suitable growth condition at infinity for the application of the comparison principle.

We take (4-4) to be the *definition* of the function $w^\varepsilon(x, p)$ and continue with a discussion of some elementary properties of w^ε. First, we remark that it is often convenient to consider (4-1) at the microscopic scale in order to use the stationarity of the environment. The rescaled equation is

$$\varepsilon v - \operatorname{tr}(A(y)D^2 v) + H(p + Dv, y) = 0 \quad \text{in } \mathbb{R}^d, \tag{4-5}$$

and we rescale w^ε by introducing

$$v^\varepsilon(y, p) := \frac{1}{\varepsilon} w^\varepsilon(\varepsilon y, p) = \sup\{v(x) : v \in \mathrm{USC}(\mathbb{R}^d) \text{ is a subsolution of (4-5)}\}. \tag{4-6}$$

The second equality in (4-6) follows from the definition of w^ε and a rescaling of (4-1). Note that it is immediate from (4-6) that $v^\varepsilon(x, p)$ is stationary with respect to the translation action. According to [Armstrong and Tran 2014, Theorem 6.1], for every $\varepsilon > 0$, $p \in \mathbb{R}^d$ and choice of coefficients $(\sigma, H) \in \Omega$, the function $v^\varepsilon(\cdot, p)$ defined in (4-6) belongs to $C^{0,1}_{\mathrm{loc}}(\mathbb{R}^d)$ and is a solution of (4-5). It follows immediately from reversing the scaling that $w^\varepsilon(\cdot, p) \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^d)$ is a solution of (4-1). Uniqueness is a separate issue addressed below; see (4-16).

Next, we observe that $w^\varepsilon(\cdot, p)$ is bounded below uniformly in ε. Indeed, for all $p \in \mathbb{R}^d$,

$$\inf_{x\in\mathbb{R}^d} w^\varepsilon(x, p) \geq -\Lambda_1(|p|^q + 1). \tag{4-7}$$

This follows from the definition of w^ε and the fact that the right side of this inequality is a subsolution of (4-1), according to (1-6), as we have already seen in (2-37). Using this bound for the equation at the microscopic scale, we obtain that $v^\varepsilon(\cdot, p)$ is a solution of the inequality

$$-\operatorname{tr}(A(y)D^2 v^\varepsilon) + H(p + Dv^\varepsilon, y) \leq \Lambda_1(|p|^q + 1) \quad \text{in } \mathbb{R}^d.$$

Then according to the definition of m_μ with $\mu = \Lambda_1(|p|^q + 1)$, we obtain the estimate

$$v^\varepsilon(y, p) - \sup_{x\in B_1(z)} v^\varepsilon(x, p) \leq m_\mu(y, z) \quad \text{for every } \mu \geq \Lambda_1(|p|^q + 1). \tag{4-8}$$

Note that this inequality holds uniformly in ε.

Lemma 4.1. *For every $\varepsilon > 0$, $x \in \mathbb{R}^d$ and $(\sigma, H) \in \Omega$,*

$$p \mapsto w^\varepsilon(x, p) \quad \textit{is concave}. \tag{4-9}$$

Proof. Observe that, if $v_1, v_2 \in \mathrm{USC}(\mathbb{R}^d)$ are subsolutions of (4-1) with $p = p_1$ and $p = p_2$, respectively, and $\lambda \in [0, 1]$, then the function $\lambda v_1 + (1-\lambda)v_2$ is a subsolution of (4-1) with $p = \lambda p_1 + (1-\lambda)p_2$. This follows formally from the convexity of the Hamiltonian, and for a rigorous proof, we refer to the argument of [Armstrong and Tran 2014, Lemma 2.4]. In view of the definition of w^ε in (4-4), this observation gives the lemma. □

An immediate consequence of (4-7) and Lemma 4.1 is that the map $p \mapsto \max\{k, w^\varepsilon(x, p)\}$ is uniformly continuous for every $k > 0$. Indeed, we obtain that, for all $p, \hat{p} \in \mathbb{R}^d$ with $|p - \hat{p}| < 1$,

$$w^\varepsilon(x, p) \geq (1 - |p - \hat{p}|)w^\varepsilon(x, \hat{p}) - \Lambda_1(|p|^q + 1)|p - \hat{p}|. \tag{4-10}$$

We next show that $w^\varepsilon(x, p)$ satisfies, almost surely with respect to $\mathbb{P}$, an appropriate sublinear growth condition uniformly in ε and for bounded $|p|$. This is required both in order to establish w^ε as the *unique* bounded-below solution of (4-1) and is also needed in the proof of (4-2). Note that this estimate is trivial for uniformly coercive Hamiltonians since in that case $w^\varepsilon(x, p)$ is bounded above uniformly for $x \in \mathbb{R}^d$, $p \in B_R$ and $0 < \varepsilon \leq 1$. In the general case, it is a consequence of the averaged coercivity condition (1-11) and its proof uses the ergodic theorem, which is the reason we expect it to hold only almost surely with respect to $\mathbb{P}$.

Lemma 4.2. *We have*

$$\mathbb{P}\Big[\forall R>0,\ \limsup_{|x|\to\infty}\ \sup_{|p|\le R}\ \sup_{0<\varepsilon\le 1}\frac{|w^\varepsilon(x,p)|}{|x|}=0\Big]=1. \tag{4-11}$$

Proof. In view of (4-7), we need only prove upper bounds for w^ε. For most of the argument, we work at the microscopic scale. It clearly suffices to prove the lemma for fixed $R>0$ since we obtain the general case by intersecting the events corresponding to all positive integers R.

It is convenient to work with the random fields

$$V^\varepsilon(y):=\sup_{|p|\le R}\ \sup_{z\in B_1(y)} v^\varepsilon(z,p).$$

Note that V^ε is stationary with respect to the translation group action. According to [Armstrong and Tran 2014, Theorem 4.2], the family $\{V^\varepsilon\}_{\varepsilon>0}$ is locally equi-Lipschitz continuous in $\mathbb{R}^d$ for every realization $\omega=(\sigma,H)\in\Omega$ of the coefficients.

Step 1. We begin from the estimate from [Armstrong and Tran 2014] that, for $C>0$ depending only on d and q,

$$\mathbb{P}\Bigg[\forall\varepsilon\in(0,1],\ \varepsilon V^\varepsilon(0)\le M_2(1+\Lambda_1R^q)+C\bigg(\frac{\Lambda_2^2}{a_2}\bigg)^{1/(q-1)}\Bigg]=1. \tag{4-12}$$

This is shown by exhibiting explicit, smooth supersolutions. See for example [Armstrong and Tran 2014, Lemma 3.2, Remark 4.5], which handles the case $R=0$, and note that the estimate for $R>0$ can be reduced to the former by using (1-8).

Let ξ denote the random variable

$$\xi:=M_2(1+\Lambda_1R^q)+C\bigg(\frac{\Lambda_2^2}{a_2}\bigg)^{1/(q-1)},$$

and let I denote its essential infimum (with respect to $\mathbb{P}$):

$$I:=\inf\{\lambda\in\mathbb{R}:\mathbb{P}[\xi<\lambda]>0\}<\infty.$$

We eventually apply Lemma 2.4 to the sequence of random fields defined by

$$X_t(y):=\frac{1}{t}\inf_{z\in B_t(y)}\ \sup_{0<\varepsilon\le 1}\Big(V^\varepsilon(z)-\frac{2}{\varepsilon}I\Big),\qquad t>0.$$

In the next few steps, we check that the hypotheses of Lemma 2.4 hold for X_t.

Step 2. We show that

$$\mathbb{P}\Big[\limsup_{t\to\infty}X_t(0)\le 0\Big]=1. \tag{4-13}$$

According to the ergodic theorem,

$$\mathbb{P}\Bigg[\lim_{s\to\infty}\fint_{B_s}\mathbb{1}_{\{\xi(\cdot)\le 2I\}}(y)\,dy=\mathbb{P}[\xi(0)\le 2I]\Bigg]=1.$$

Note that $\mathbb{P}[\xi(0) \le 2I] > 0$ by the definition of I and that, if $\mathbb{1}_{\{\xi(\cdot)\le 2I\}}(y)$ does not vanish identically in B_t, then $X_t(0) \le 0$ by (4-12). This yields (4-13).

Step 3. We show that

$$\mathbb{P}\Big[\limsup_{r\to 0}\limsup_{t\to\infty}\frac{1}{t}\sup_{y\in B_t}\sup_{0<\varepsilon\le 1}\operatorname*{osc}_{B_{rt}(y)} V^\varepsilon = 0\Big] = 1. \tag{4-14}$$

To see this, observe that (4-8) implies that, for every $\varepsilon > 0$ and $y, z \in \mathbb{R}^d$,

$$V^\varepsilon(y) - V^\varepsilon(z) \le \widetilde{m}_\mu(y, z) \quad \text{with } \mu := \Lambda_1(R^q + 1).$$

We therefore obtain (4-14) from (2-15). As a consequence of (4-14), we get

$$\mathbb{P}\Big[\limsup_{r\to 0}\limsup_{t\to\infty}\sup_{y\in B_t}\operatorname*{osc}_{B_{rt}(y)} X_t = 0\Big] = 1. \tag{4-15}$$

Step 4. We complete the argument. In view of (4-13) and (4-15), we may apply Lemma 2.4 to conclude that

$$\mathbb{P}\Big[\forall K > 0,\ \limsup_{t\to\infty}\sup_{y\in B_{Kt}} X_t(y) \le 0\Big] = 1.$$

Using the definition of X_t, replacing Kt by t and setting $r = 1/K$, this gives

$$\mathbb{P}\Big[\forall r > 0,\ \limsup_{t\to\infty}\frac{1}{t}\sup_{y\in B_t}\inf_{z\in B_{rt}(y)}\sup_{0<\varepsilon\le 1}\Big(V^\varepsilon(z) - \frac{2}{\varepsilon}I\Big) \le 0\Big] = 1.$$

Using again (4-14), we obtain

$$\mathbb{P}\Big[\limsup_{t\to\infty}\frac{1}{t}\sup_{y\in B_t}\sup_{0<\varepsilon\le 1}\Big(V^\varepsilon(y) - \frac{2}{\varepsilon}I\Big) \le 0\Big] = 1.$$

Using the definition of V^ε and rewriting the expression in terms of w^ε, we get

$$\mathbb{P}\Big[\limsup_{t\to\infty}\sup_{0<\varepsilon\le 1}\sup_{|p|\le R}\sup_{x\in B_{\varepsilon t}}\frac{w^\varepsilon(x, p) - 2I}{\varepsilon t} \le 0\Big] = 1.$$

This is actually stronger than (4-11). Indeed,

$$\begin{aligned}\limsup_{t\to\infty}\sup_{0<\varepsilon\le 1}\sup_{|p|\le R}\sup_{x\in B_{\varepsilon t}}\frac{w^\varepsilon(x, p) - 2I}{\varepsilon t} &= \lim_{s\to\infty}\sup_{t\ge s}\sup_{0<\varepsilon\le 1}\sup_{|p|\le R}\sup_{x\in B_{\varepsilon t}}\frac{w^\varepsilon(x, p) - 2I}{\varepsilon t}\\ &\ge \limsup_{s\to\infty}\sup_{0<\varepsilon\le 1}\sup_{|p|\le R}\sup_{x\in B_s}\frac{w^\varepsilon(x, p) - 2I}{s}\\ &\ge \limsup_{|x|\to\infty}\sup_{0<\varepsilon\le 1}\sup_{|p|\le R}\frac{w^\varepsilon(x, p)}{|x|}.\end{aligned}$$

Note that the inequality on the second line was obtained by reversing the first two suprema and then taking $t = s/\varepsilon$ in the supremum over t. This completes the proof. ∎

It follows from Lemma 4.2 and [Armstrong and Tran 2014, Theorem 2.1] that, with probability one, $w^\varepsilon(\cdot\,, p)$ is the unique bounded-below solution of (4-1) for every fixed $\varepsilon > 0$ and $p \in \mathbb{R}^d$. That is,

$$\mathbb{P}\Big[\forall p \in \mathbb{R}^d,\ \forall \varepsilon > 0,\ w^\varepsilon(\cdot\,, p) \text{ belongs to } C^{0,1}_{\text{loc}}(\mathbb{R}^d) \text{ and is the unique solution of (4-1), which is bounded below on } \mathbb{R}^d\Big] = 1. \tag{4-16}$$

4B. ***The proof of* (4-2).** The next lemma is the first step in the direction of (4-2). For the argument, we again use Lemma 2.4.

Lemma 4.3. *We have*

$$\mathbb{P}\Big[\forall p \in \mathbb{R}^d,\ \forall R > 0,\ \limsup_{\varepsilon\to 0}\ \sup_{x\in B_R} w^\varepsilon(x, p) \le -\overline{H}_*\Big] = 1. \tag{4-17}$$

Proof. Here we employ a soft compactness argument using the rescaled functions v^ε defined in (4-6). Let

$$E := \big\{(\sigma, H) \in \Omega : \overline{H}_* = \inf\{\mu \in \mathbb{R} : \text{there exists } w \in \operatorname{USC}(\mathbb{R}^d) \text{ satisfying (2-1)}\}\big\}.$$

Recall from Lemma 2.2 that $\mathbb{P}[E] = 1$.

Step 1. We first show that, for all $p \in \mathbb{R}^d$ and $\omega \in E$,

$$\limsup_{\varepsilon\to 0}\ \sup_{z\in B_1} \varepsilon v^\varepsilon(z, p) \le -\overline{H}_*. \tag{4-18}$$

Suppose on the contrary that there exist $\eta > 0$ and a subsequence $\varepsilon_k \to 0$ such that, for every $k \in \mathbb{N}$,

$$\varepsilon_k \sup_{z\in B_1} v^{\varepsilon_k}(z, p) \ge -\overline{H}_* + \eta.$$

Define the function

$$\tilde{v}^\varepsilon(y, p) := p \cdot y + v^\varepsilon(y, p) - \sup_{z\in B_1} v^\varepsilon(z, p).$$

According to the local Lipschitz estimates [Armstrong and Tran 2014, Proposition 3.1] and (4-11), the family $\{\tilde{v}^\varepsilon\}_{\varepsilon>0}$ is uniformly bounded in $C^{0,1}(B_s)$ for every $s > 0$. By taking a further subsequence of $\{\varepsilon_k\}$, we may suppose that $\tilde{v}^{\varepsilon_k}$ converges locally uniformly on $\mathbb{R}^d$ to a function $\tilde{v} \in C^{0,1}_{\text{loc}}(\mathbb{R}^d)$. In view of the fact that $\tilde{v}^\varepsilon$ satisfies the equation

$$\varepsilon\tilde{v}^\varepsilon - \operatorname{tr}(A(y)D^2\tilde{v}^\varepsilon) + H(D\tilde{v}^\varepsilon, y) = -\varepsilon \sup_{z\in B_1} v^\varepsilon(z, p) \quad \text{in } \mathbb{R}^d,$$

we obtain, by the stability of viscosity solutions under local uniform convergence, that $\tilde{v}$ satisfies

$$-\operatorname{tr}(A(y)D^2\tilde{v}) + H(D\tilde{v}, y) \le \overline{H}_* - \eta \quad \text{in } \mathbb{R}^d.$$

This contradicts the assumption that $\omega = (\sigma, H) \in E$ and completes the proof of (4-18). As a consequence, we obtain

$$\mathbb{P}\Big[\forall p \in \mathbb{R}^d,\ \limsup_{\varepsilon\to 0}\ \sup_{z\in B_1} \varepsilon v^\varepsilon(z, p) \le -\overline{H}_*\Big] = 1. \tag{4-19}$$

Step 2. To obtain the conclusion of the lemma from (4-18), we apply Lemma 2.4 to the family of random variables

$$X_t := \sup_{z\in B_1} \varepsilon v^{\varepsilon}(z, p) \quad \text{with } t = \varepsilon^{-1}.$$

The first hypothesis of Lemma 2.4 is satisfied by (4-18), and the second hypothesis is confirmed by (4-8) and (2-15). The conclusion of Lemma 2.4 yields that, for every $p \in \mathbb{R}^d$,

$$\mathbb{P}\Big[\forall R > 0,\ \limsup_{\varepsilon\to 0} \sup_{z\in B_{R/\varepsilon}} \varepsilon v^{\varepsilon}(z, p) \le -\bar{H}_*\Big] = 1.$$

Using (4-10) and intersecting over all events corresponding to rational p, we obtain

$$\mathbb{P}\Big[\forall p \in \mathbb{R}^d,\ \forall R > 0,\ \limsup_{\varepsilon\to 0} \sup_{z\in B_{R/\varepsilon}} \varepsilon v^{\varepsilon}(z, p) \le -\bar{H}_*\Big] = 1.$$

This is equivalent to (4-17). □

We now show that (4-1) homogenizes to (4-3).

Proposition 4.4. *The assertion (4-2) holds.*

Proof. The argument is deterministic and based on the comparison principle. To give an overview of the proof, we introduce the following events:

$$E_1 := \Big\{(\sigma, H) \in \Omega : \forall \mu \ge \bar{H}_*,\ \forall R > 0,\ \limsup_{t\to\infty} \sup_{y,z\in B_R} \Big|\frac{m_\mu(ty, tz)}{t} - \bar{m}_\mu(y - z)\Big| = 0\Big\},$$

$$E_2 := \Big\{(\sigma, H) \in \Omega : \forall R > 0,\ \limsup_{|x|\to\infty} \sup_{|p|\le R} \sup_{0<\varepsilon\le 1} \frac{|w^{\varepsilon}(x, p)|}{|x|} = 0\Big\},$$

$$E_3 := \Big\{(\sigma, H) \in \Omega : \forall p \in \mathbb{R}^d,\ \limsup_{\varepsilon\to 0} \sup_{x\in B_R} w^{\varepsilon}(x, p) \le -\bar{H}_*\Big\},$$

$$E_4 := \Big\{(\sigma, H) \in \Omega : \forall p \in \mathbb{R}^d,\ \forall R > 0,\ \limsup_{\varepsilon\to 0} \sup_{x\in B_R} |w^{\varepsilon}(x, p) + \bar{H}(p)| = 0\Big\}.$$

According to Proposition 2.5, Lemma 4.2 and Lemma 4.3, we have

$$\mathbb{P}[E_1 \cap E_2 \cap E_3] = 1.$$

To obtain $\mathbb{P}[E_4] = 1$, it therefore suffices to demonstrate that

$$E_1 \cap E_2 \cap E_3 \subseteq E_4. \tag{4-20}$$

Thus, for the remainder of the proof, we fix $p \in \mathbb{R}^d$, $R > 0$ and $(\sigma, H) \in E_1 \cap E_2 \cap E_3$ and argue that

$$\limsup_{\varepsilon\to 0} \sup_{x\in B_R} |w^{\varepsilon}(x, p) + \bar{H}(p)| = 0. \tag{4-21}$$

The proof of (4-21) is broken into two steps.

Step 1. We show that

$$\liminf_{\varepsilon\to 0} \inf_{z\in B_R} w^{\varepsilon}(z, p) \ge -\bar{H}(p). \tag{4-22}$$

We begin with some reductions. By the concavity of the map $\hat{p} \mapsto w^\varepsilon(x, \hat{p})$, we may assume without loss of generality that p is an extreme point of $\{\hat{p} : \bar{H}(\hat{p}) \le \bar{H}(p)\}$. Second, by (4-10), we may also suppose that $\bar{H}(p) > \bar{H}_*$. Next, Straszewicz's theorem [Rockafellar 1970, Theorem 18.6] and (4-10) permit us to further suppose that p is an *exposed* point of $\{\hat{p} : \bar{H}(\hat{p}) \le \bar{H}(p)\}$. This is useful in view of (3-6) and Lemma 3.2, which imply the existence of $e \in \partial B_1$ such that $\overline{m}_\mu(e) = e \cdot p$ and $\overline{m}_\mu$ is differentiable at e with $p = D\overline{m}_\mu(e)$, where as usual we have set $\mu := \bar{H}(p)$ for convenience. In view of the limit (2-26), this forces the function $m_\mu(\,\cdot\,, z - te)$, with $t > 0$ very large, to be very "flat" in large balls centered at z, as we will see. This is what allows us to use this function as an "approximate subcorrector" in order to bound w^ε from below.

We proceed with the demonstration of (4-22) by supposing that $-\bar{H}(p) - w^\varepsilon(z, p) \ge \delta > 0$ for some $z \in B_R$ and deriving a contradiction if $0 < \varepsilon \le 1$ is too small. The idea is to compare $w^\varepsilon(\,\cdot\,, p)$ in the ball $B_s(z)$, for a large enough but fixed $s > 0$, to the function $x \mapsto -p\cdot(x-z+te)+\varepsilon m_\mu(x/\varepsilon, (z-te)/\varepsilon)$ for $t \gg s$. We argue that the former is a strict supersolution of the equation solved by the latter, and then we derive a contradiction by showing that their difference has a local minimum. To ensure that we can touch the first function from below by the second, we use the fact that both functions are expected to be "flat" near z (for the second function, this is due to the fact that $p = D\overline{m}_\mu(e)$), and we add a small linearly growing perturbative term made possible by the positivity of δ.

In order to prepare $w^\varepsilon(\,\cdot\,, p)$ for comparison, we take $c > 0$ and $\lambda > 1$ to be selected below and define the auxiliary function

$$W^\varepsilon(x) := \lambda(w^\varepsilon(x, p) - w^\varepsilon(z, p)) + c\delta((1 + |x - z|^2)^{1/2} - 1).$$

Since $\omega \in E_2$, there exists an $s > 0$, which does not depend on z or $\varepsilon > 0$, such that

$$U_\varepsilon := \left\{x \in \mathbb{R}^d : W^\varepsilon(x) \le \tfrac{1}{4}\delta\right\} \subseteq B_s(z).$$

We claim that, by choosing λ sufficiently close to 1 and $c > 0$ sufficiently small depending on λ, then we have

$$-\operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2W^\varepsilon\Big) + H\Big(p + DW^\varepsilon, \frac{x}{\varepsilon}\Big) \ge \bar{H}(p) + \tfrac{1}{2}\delta \quad \text{in } U_\varepsilon. \tag{4-23}$$

In order to verify (4-23), take any smooth test function φ such that $v^\varepsilon - \varphi$ has a strict local minimum at $x_0 \in U_\varepsilon$. Set $\psi(x) := (1 + |x - z|^2)^{1/2}$. Then $w^\varepsilon - \lambda^{-1}(\varphi + c\delta\psi)$ has a strict local minimum at x_0. Using the equation satisfied by w^ε and the definition of viscosity supersolution, we obtain

$$w^\varepsilon(x_0) - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x_0}{\varepsilon}\Big)\lambda^{-1}D^2(\varphi + c\delta\psi)(x_0)\Big) + H\Big(p + \lambda^{-1}D(\varphi + c\delta\psi)(x_0), \frac{x_0}{\varepsilon}\Big) \ge 0.$$

The convexity of H gives

$$H\Big(p+\lambda^{-1}D(\varphi+c\delta\psi)(x_0), \frac{x_0}{\varepsilon}\Big) \le \lambda^{-1}H\Big(p+D\varphi(x_0), \frac{x_0}{\varepsilon}\Big)+(1-\lambda^{-1})H\Big(p+(\lambda-1)^{-1}c\delta D\psi(x_0), \frac{x_0}{\varepsilon}\Big).$$

Combining the above computations and using $x_0 \in U_\varepsilon$, we deduce that, for λ sufficiently close to 1 and $c > 0$ sufficiently small depending on λ,

$$-\operatorname{tr}\Big(A\Big(\frac{x_0}{\varepsilon}, \omega\Big)D^2\varphi(x_0)\Big) + H\Big(p + D\varphi(x_0), \frac{x_0}{\varepsilon}, \omega\Big) \ge \bar{H}(p) + \tfrac{1}{2}\delta.$$

This completes the proof of (4-23).

We may now apply the comparison principle [Armstrong and Tran 2014, Theorem 2.2] to conclude that, for every $t \geq s+1$,

$$\inf_{x\in U_\varepsilon}\left(W^\varepsilon(x)+p\cdot(x-z+te)-\varepsilon m_\mu\left(\frac{x}{\varepsilon},\frac{z-te}{\varepsilon}\right)\right) = \inf_{x\in \partial U_\varepsilon}\left(W^\varepsilon(x)+p\cdot(x-z+te)-\varepsilon m_\mu\left(\frac{x}{\varepsilon},\frac{z-te}{\varepsilon}\right)\right). \tag{4-24}$$

Estimating the infimum on the left side of (4-24) by taking $x=z$ and recalling that $W^\varepsilon(z)=0$ and the term on the right side by using that $W^\varepsilon\equiv\delta/4$ on ∂U_ε and $\partial U_\varepsilon\subseteq B_s(z)$, we conclude after a rearrangement that, for every $t\geq s+1$,

$$\inf_{x\in B_s(z)}\left(p\cdot(x-z)+\varepsilon m_\mu\left(\frac{z}{\varepsilon},\frac{z-te}{\varepsilon}\right)-\varepsilon m_\mu\left(\frac{x}{\varepsilon},\frac{z-te}{\varepsilon}\right)\right)\leq -\tfrac14\delta. \tag{4-25}$$

This holds for every $z\in B_R$ and $\varepsilon>0$ for which $-\overline{H}(p)-w^\varepsilon(z,p)\geq\delta>0$. So if $-\overline{H}(p)-w^{\varepsilon_j}(z_j,p)\geq\delta$ along subsequences $\{z_j\}_{j\in\mathbb{N}}\subseteq B_R$ and $\varepsilon_j\to 0$, then by passing to limits in (4-25), using (2-26), we obtain, for every $t\geq s+1$,

$$\inf_{x\in B_s}\left(p\cdot x+\overline{m}_\mu(te)-\overline{m}_\mu(x+te)\right)\leq -\tfrac14\delta.$$

This contradicts the fact that $p=D\overline{m}_\mu(e)$ since the latter implies, in view of the positive homogeneity of $\overline{m}_\mu$, that

$$\lim_{t\to\infty}\sup_{x\in B_s}|\overline{m}_\mu(x+te)-\overline{m}_\mu(te)-p\cdot x|=0. \tag{4-26}$$

This completes the proof of (4-22).

Step 2. We demonstrate that

$$\limsup_{\varepsilon\to 0}\sup_{z\in B_R} w^\varepsilon(z,p)\leq -\overline{H}(p). \tag{4-27}$$

We may suppose that $\overline{H}(p)>\overline{H}_*$ since otherwise the claim follows from $\omega\in E_3$.

The argument is similar to one introduced in [Armstrong and Souganidis 2013], relying on the limit (2-26) and using m_μ as a supercorrector. Here it is a bit simpler than Step 1 since we do not need to use Straszewicz's theorem or to restrict our attention to exposed points of the sublevel set of $\overline{H}$. Applying Lemma 3.2 in view of (3-6) and the assumption that $\overline{H}(p)>\overline{H}_*$, we may select $e\in\partial B_1$ such that $p\in\partial\overline{m}_\mu(e)$ and $\overline{m}_\mu(e)=e\cdot p$, where as usual we set $\mu:=\overline{H}(p)$. The reason we do not need $p=D\overline{m}_\mu(e)$ is because m_μ will be used as a supercorrector; so the fact that it may not be flat and rather "bends upward" like a cone can only help in the comparison argument.

We consider a point $z\in B_s$ and $\varepsilon,\delta>0$ such that $w^\varepsilon(z,p,\omega)+\overline{H}(p)\geq\delta>0$. With $c>0$ and $\lambda<1$ to be selected, we consider the auxiliary function

$$W^\varepsilon(x):=\lambda(w^\varepsilon(x,p)-w^\varepsilon(z,p))-c\delta(1+|x-z|^2)^{1/2}+c\delta. \tag{4-28}$$

Since $\omega \in E_2$, there exists $s > 0$, which does not depend on z or ε, such that

$$U_\varepsilon := \left\{x \in \mathbb{R}^d : W^\varepsilon(x) \geq -\tfrac{1}{4}\delta\right\} \subseteq B_s(z). \tag{4-29}$$

Choosing λ sufficiently close to 1 and $c > 0$ sufficiently small depending on λ and after similar computations arguments as in the demonstration of (4-23), we find that

$$-\operatorname{tr}\left(A\left(\frac{x}{\varepsilon}\right)D^2W^\varepsilon\right) + H\left(p + DW^\varepsilon, \frac{x}{\varepsilon}\right) \leq \bar{H}(p) - \tfrac{1}{2}\delta \quad \text{in } U_\varepsilon. \tag{4-30}$$

The comparison principle yields

$$\begin{aligned}\inf_{x\in U_\varepsilon}\left(\varepsilon m_\mu\left(\frac{x}{\varepsilon}, \frac{z-(s+1)e}{\varepsilon}\right) - W^\varepsilon(x) - p\cdot(x - z + (s+1)e)\right)&\\ = \inf_{x\in\partial U_\varepsilon}\left(\varepsilon m_\mu\left(\frac{x}{\varepsilon}, \frac{z-(s+1)e}{\varepsilon}\right) - W^\varepsilon(x) - p\cdot(x - z + (s+1)e)\right).&\end{aligned} \tag{4-31}$$

Using that $W^\varepsilon(z) = 0$ and $W^\varepsilon \equiv -\delta/4$ on $\partial U_\varepsilon \subseteq B_s(z)$ and rearranging, we obtain

$$\inf_{x\in B_s(z)}\left(\varepsilon m_\mu\left(\frac{x}{\varepsilon}, \frac{z-(s+1)e}{\varepsilon}\right) - \varepsilon m_\mu\left(\frac{z}{\varepsilon}, \frac{z-(s+1)e}{\varepsilon}\right) - p\cdot(x-z)\right) \leq -\tfrac{1}{4}\delta. \tag{4-32}$$

To obtain a contradiction, we suppose that $w^{\varepsilon_j}(z_j, p) + \bar{H}(p) \geq \delta > 0$ for sequences $\{z_j\}_{j\in\mathbb{N}} \subseteq B_R$ and $\varepsilon_j \to 0$. Applying (4-32) and sending $j \to \infty$ yields, in light of (2-26),

$$\inf_{x\in B_s}\left(\bar{m}_\mu(x + (s+1)e) - \bar{m}_\mu((s+1)e) - p\cdot x\right) \leq -\tfrac{1}{4}\delta. \tag{4-33}$$

Since $\bar{m}_\mu((s+1)e) = (s+1)e\cdot p$, we conclude that, for some $x \in B_s$,

$$\bar{m}_\mu(x + (s+1)e) - p\cdot(x + (s+1)e) \leq -\tfrac{1}{8}\delta. \tag{4-34}$$

This contradicts that $p \in \partial\bar{m}_\mu(e)$ and finishes Step 2 and the proof of the proposition. □

Remark 4.5. The reader may object to the proof of Theorem 1 on the grounds that several steps in the proof are not as “quantifiable” as promised in the introduction. In particular, it seems at first glance impossible to quantify (i) the limit in (4-26) without extra information about the shape of the level sets of $\bar{H}$ (which is not easy to obtain) and (ii) Lemma 4.3 since it is obtained by a compactness argument.

About (i): this step is actually quantifiable because we can approximate the level sets of $\bar{H}$ by nice sets with positive curvature. Rather than the exposed points of the sublevel sets of $\bar{H}$, we may instead consider “points of positive curvature” of the boundary of the level set, that is, points that also lie on the boundary of a large ball that contains the level set. The radius of this ball controls the rate of the limit (4-26), and the error this introduces is relatively small. The details will appear in [Armstrong and Cardaliaguet 2015].

The second objection is more serious, but the phenomenon we encounter here is not artificial or a limitation of the method. Indeed, it was shown already in the first-order case [Armstrong et al. 2014] that the rate of convergence in the limit in Lemma 4.3 may be arbitrarily slow (even with a finite range of dependence quantifying the ergodicity assumption). In this sense, the proof above seems to optimally capture the underlying phenomena driving the homogenization of Hamilton–Jacobi equations in random media.

5. Homogenization: the proof of Theorem 1

In this section, we present the proof of our main result, Theorem 1. The convergence result is obtained from the classical perturbed test function argument, suitably modified to handle the lack of uniform Lipschitz estimates for weakly coercive Hamiltonians. The argument can be seen as a method for showing that the homogenization result of (4-2), which is a special case of Theorem 1, is actually strong enough to imply the theorem.

As in the previous section, we assume throughout that $\mathbb{P}$ is a probability measure on $(\Omega, \mathcal{F})$ satisfying (1-9), (1-10) and (1-11).

5A. ***Wellposedness and basic properties.*** Before giving the proof of homogenization, we first consider the question of wellposedness of solutions of the time-dependent initial-value problem

$$\begin{cases} u_t^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\dfrac{x}{\varepsilon}\Big) D^2 u^\varepsilon\Big) + H\Big(Du^\varepsilon, \dfrac{x}{\varepsilon}\Big) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u^\varepsilon(\,\cdot\,, 0) = g \in \mathrm{BUC}(\mathbb{R}^d). \end{cases} \tag{5-1}$$

For each $\varepsilon > 0$, $g \in \mathrm{BUC}(\mathbb{R}^d)$ and $(x,t) \in \mathbb{R}^d \times (0,\infty)$, we define the random variable

$$u^\varepsilon(x,t,g) := \sup\Big\{ w(x,t) : w \in \mathrm{USC}(\mathbb{R}^d \times [0,t]) \text{ is a subsolution of (1-1) in } \mathbb{R}^d \times [0,t), \\ \limsup_{|x|\to\infty} \sup_{0<s\le t} \frac{w(x,s)}{|x|} = 0 \text{ and } w(\,\cdot\,,0) \le g \text{ on } \mathbb{R}^d \Big\}. \tag{5-2}$$

This is the candidate for the unique solution of (5-1). Observe that we have

$$u^\varepsilon(x,t,g) \ge -\Lambda_1 t + \inf_{\mathbb{R}^d} g \tag{5-3}$$

since the function on the right belongs to the admissible class in (5-2) by (1-6) and (1-3).

Similar to the situation for the approximate cell problem, checking that $(x,t) \mapsto u^\varepsilon(x,t,g)$ does indeed solve (5-1) reduces to proving a sublinear growth condition at infinity (uniformly in time). We remark that this is of interest only in the nonuniformly coercivity case since otherwise wellposedness of (5-1) is classical.

Lemma 5.1. *We have*

$$\mathbb{P}\Big[\forall T > 0,\ \forall g \in \mathrm{BUC}(\mathbb{R}^d),\ \limsup_{|x|\to\infty} \sup_{0<t\le T} \sup_{0<\varepsilon\le 1} \frac{|u^\varepsilon(x,t,g)|}{|x|} = 0\Big] = 1.$$

Proof. In view of (5-3), we may focus only on obtaining upper bounds for u^ε. By definition, $g \mapsto u^\varepsilon(x,t,g)$ is monotone nondecreasing, and so we may suppose that g is constant. Since $g \mapsto u^\varepsilon(x,t,g)$ also commutes with constants, it suffices therefore to prove the sublinear growth estimate for $g \equiv 0$. That is, we need to show only the following:

$$\mathbb{P}\Big[\forall T > 0,\ \limsup_{|x|\to\infty} \sup_{0<t\le T} \sup_{0<\varepsilon\le 1} \frac{|u^\varepsilon(x,t,0)|}{|x|} = 0\Big] = 1.$$

We proceed by exhibiting an explicit supersolution and appealing to the comparison principle. The supersolution is

$$V^\varepsilon(x,t) := e^t w^\varepsilon(x,0) + e^t \Lambda_1,$$

where $w^\varepsilon(x,p)$ is, as in the previous section, the solution of (4-1). The convexity of H and (1-6) imply that, for every $p \in \mathbb{R}^d$ and $\lambda \geq 1$,

$$\lambda^{-1} H(\lambda p, y) \geq H(p,y) - (1-\lambda^{-1}) H(0,y) \geq H(p,y) - (1-\lambda^{-1})\Lambda_1.$$

Using this with $\lambda = e^t$, we find that, for each $t > 0$, the function $w^\varepsilon(\,\cdot\,,0)$ satisfies the inequality

$$w^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big) D^2 w^\varepsilon\Big) + e^{-t} H\Big(e^t Dw^\varepsilon, \frac{x}{\varepsilon}\Big) \geq -(1-e^{-t})\Lambda_1 \quad \text{in } \mathbb{R}^d.$$

From this, it follows that V^ε satisfies

$$V^\varepsilon_t - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big) D^2 V^\varepsilon\Big) + H\Big(DV^\varepsilon, \frac{x}{\varepsilon}\Big) \geq 0 \quad \text{in } \mathbb{R} \times [0,\infty).$$

Since V^ε is bounded below by 0 uniformly in $\mathbb{R}^d \times [0,\infty)$, by comparing V^ε to any function in the admissible class in (5-2) using the comparison principle, we find that, for all $(x,t) \in \mathbb{R}^d \times [0,\infty)$ and every realization of the coefficients,

$$u^\varepsilon(x,t,0) \leq V^\varepsilon(x,t).$$

According to Lemma 4.2,

$$\mathbb{P}\Big[\forall T > 0,\ \limsup_{|x|\to\infty} \sup_{0<t\leq T} \sup_{0<\varepsilon\leq 1} \frac{|V^\varepsilon(x,t)|}{|x|} = 0\Big] = 1.$$

This yields the lemma. □

By Lemma 5.1, the lower bound (5-3), the comparison principle [Armstrong and Tran 2014, Theorem 2.3] and the classical Perron argument, we obtain

$$\mathbb{P}\Big[\forall \varepsilon > 0,\ \forall g \in \mathrm{BUC}(\mathbb{R}^d),\ (x,t) \mapsto u^\varepsilon(x,t,g) \text{ belongs to } C(\mathbb{R}^d \times (0,\infty)) \text{ and, for all } T > 0, \text{ is the unique bounded-below solution of (5-1) in } \mathbb{R}^d \times [0,T]\Big] = 1. \tag{5-4}$$

5B. *Homogenization.* In this subsection, we complete the proof of Theorem 1. We let $u(x,t,g)$ denote the unique solution of the homogenized problem

$$\begin{cases} u_t + \overline{H}(Du) = 0 & \text{in } \mathbb{R}^d \times (0,\infty), \\ u = g & \text{on } \mathbb{R}^d \times \{0\}. \end{cases} \tag{5-5}$$

In view of the growth condition (3-5), the problem (5-5) indeed possesses a unique solution, and it is given by the *Hopf–Lax formula*

$$u(x,t,g) := \inf_{y\in\mathbb{R}^d} \Big(t\overline{L}\Big(\frac{x-y}{t}\Big) + g(y)\Big),$$

where $\bar{L} : \mathbb{R}^d \to \mathbb{R}$ is the Legendre–Fenchel transform of $\bar{H}$, that is,

$$\bar{L}(z) := \sup_{p \in \mathbb{R}^d} (p \cdot z - \bar{H}(p)).$$

Note that $\bar{L}$ is continuous, convex and satisfies $|z|^{-1}\bar{L}(z) \to +\infty$ as $|z| \to \infty$ [Evans 1998].

A proof that the Hopf–Lax formula defines a viscosity solution of (5-5) can be found for example in [Evans 1998, Theorem 3 in Section 10.3.4] under the assumption that $g \in C^{0,1}_{\text{loc}}(\mathbb{R}^d)$. It is easy to extend this to the case that $g \in \text{BUC}(\mathbb{R}^d)$ using the monotonicity of the Hopf–Lax formula in g and the stability of viscosity solutions under local uniform convergence. The uniqueness of this solution follows from classical comparison principles for first-order equations.

We now present the proof of the main result.

Proof of Theorem 1. The theorem follows from Proposition 4.4 by a variation of the classical perturbed test function argument first introduced by Evans [1992]. This comparison argument is entirely deterministic. The fact that the functions u^ε are not uniformly equi-Lipschitz continuous causes a technical difficulty that is overcome by the use of the parameter λ in Step 1, an idea which first appeared in [Armstrong and Souganidis 2012].

To set up the argument, we let the events E_2 and E_4 be defined as in the proof of Proposition 4.4 and set

$$E_5 := \Big\{(\sigma, H) \in \Omega : \forall g \in \text{BUC}(\mathbb{R}^d),\ \forall R > 0,\ \limsup_{\varepsilon \to 0} \sup_{(x,t) \in B_R \times [0,R)} |u^\varepsilon(x,t,g) - u(x,t,g)| = 0\Big\}.$$

We claim that

$$E_2 \cup E_4 \subseteq E_5. \tag{5-6}$$

Since $\mathbb{P}[E_2 \cap E_4] = 1$ by Lemma 4.2 and Proposition 4.4, the theorem follows from (5-6).

For the rest of the argument, we fix $(\sigma, H) \in E_2 \cap E_4$, $g \in \text{BUC}(\mathbb{R}^d)$ and $R > 0$ and argue that

$$\limsup_{\varepsilon \to 0} \sup_{(x,t) \in B_R \times [0,R)} |u^\varepsilon(x,t,g) - u(x,t,g)| = 0.$$

By the comparison principle [Armstrong and Tran 2014, Theorem 2.3], the flow $g \mapsto u^\varepsilon(\cdot, t, g)$ is monotone nondecreasing as well as a contraction mapping on $L^\infty(\mathbb{R}^d)$. We may therefore assume without loss of generality that $g \in C^{1,1}(\mathbb{R}^d)$. For notational convenience, we henceforth drop the dependence of u and u^ε on g.

We first argue that

$$U(x,t) := \limsup_{\varepsilon \to 0} u^\varepsilon(x,t) \le u(x,t). \tag{5-7}$$

By the comparison principle, it suffices to check that U is a subsolution of the limiting equation and $U(\cdot, 0) \le g$. We handle these claims in the next two steps.

Step 1. To check that U is a subsolution of the limiting equation, take a smooth test function $\psi \in C^\infty(\mathbb{R}^d \times (0,\infty))$ and a point $(x_0, t_0) \in \mathbb{R}^d \times (0,\infty)$ so that

$$U - \psi \quad \text{has a strict local maximum at } (x_0, t_0). \tag{5-8}$$

We must show that

$$\psi_t(x_0,t_0)+\overline{H}(D\psi(x_0,t_0))\le 0. \tag{5-9}$$

Arguing by contradiction, we suppose on the contrary that

$$\eta := \psi_t(x_0,t_0)+\overline{H}(D\psi(x_0,t_0))>0. \tag{5-10}$$

With $p_0 := D\psi(x_0,t_0)$ and $\lambda>1$ a constant to be selected below, we introduce the perturbed test function

$$\psi^\varepsilon(x,t) := \psi(x,t)+\lambda w^\varepsilon(x,p_0),$$

where w^ε is the solution of the approximate cell problem (4-1). It is appropriate to compare ψ^ε to u^ε, and to this end, we must check that, for $\varepsilon, r>0$ sufficiently small, ψ^ε is a solution of the inequality

$$\psi_t^\varepsilon-\varepsilon\,\mathrm{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2\psi^\varepsilon\Big)+H\Big(D\psi^\varepsilon,\frac{x}{\varepsilon}\Big)\ge \tfrac{1}{6}\eta \quad \text{in } B(x_0,r)\times(t_0-r,t_0+r). \tag{5-11}$$

Let us admit the claim (5-11) for the moment and show that it allows us to obtain the desired contraction, completing the proof that U is a subsolution of the limiting equation. Applying the comparison principle [Armstrong and Tran 2014, Theorem 2.3], in view of (5-11) and the equation satisfied by u^ε, we deduce that

$$\sup_{B(x_0,r)\times(t_0-r,t_0+r)}(u^\varepsilon-\psi^\varepsilon)=\sup_{\partial(B(x_0,r)\times(t_0-r,t_0+r))}(u^\varepsilon-\psi^\varepsilon).$$

This holds for all sufficiently small $r>0$ and $\varepsilon>0$, and by passing to the limit $\varepsilon\to 0$, using that by $(\sigma,H)\in E_4$ we have that $w^\varepsilon(\cdot\,,p_0)$ converges to the constant $-\overline{H}(p_0)$ uniformly on compact subsets of $\mathbb{R}^d$ as $\varepsilon\to 0$, we find that

$$\sup_{B(x_0,r)\times(t_0-r,t_0+r)}(U-\psi)=\sup_{\partial(B(x_0,r)\times(t_0-r,t_0+r))}(U-\psi).$$

This holds for all sufficiently small $r>0$, which contradicts the assumption (5-8).

To check that (5-11) holds in the viscosity sense, we take a smooth test function φ and a point $(x_1,t_1)\in B(x_0,r)\times(t_0-r,t_0+r)$ such that

$$\psi^\varepsilon-\varphi \quad \text{has a strict local minimum at } (x_1,t_1).$$

Rewriting this using the definition of ψ^ε, we get

$$(x,t)\mapsto w^\varepsilon(x,p_0)-\lambda^{-1}(\varphi-\psi)(x,t) \quad \text{has a strict local minimum at } (x_1,t_1).$$

Using the equation for w^ε, we find that

$$w^\varepsilon(x_1,p_0)-\varepsilon\,\mathrm{tr}\Big(A\Big(\frac{x_1}{\varepsilon}\Big)\lambda^{-1}D^2(\varphi-\psi)(x_1,t_1)\Big)+H\Big(p_0+\lambda^{-1}D(\varphi-\psi),\frac{x_1}{\varepsilon}\Big)\ge 0. \tag{5-12}$$

Using that $(\sigma,H)\in E_4$ and ψ is smooth, we may select $\varepsilon>0$ sufficiently small and λ sufficiently close to 1 so that

$$|\lambda w^\varepsilon(x_1,p_0)+\overline{H}(p_0)|+\Big|\varepsilon\,\mathrm{tr}\Big(A\Big(\frac{x_1}{\varepsilon}\Big)D^2\psi(x_1,t_1)\Big)\Big|\le \tfrac{1}{3}\eta. \tag{5-13}$$

Next, by selecting $r > 0$ small enough, depending on λ and ψ, we obtain

$$(\lambda-1)^{-1}|\lambda p_0 - D\psi(x_1,t_1)| \le |p_0| + (\lambda-1)^{-1}|p_0 - D\psi(x_1,t_1)| \le 2|p_0|.$$

Using the convexity of H together with the previous line and (1-6), we discover that

$$\begin{aligned}\lambda H\Big(p_0+\lambda^{-1}D(\varphi-\psi)(x_1,t_1),\frac{x_1}{\varepsilon}\Big) &\le H\Big(D\varphi(x_1,t_1),\frac{x_1}{\varepsilon}\Big)+(\lambda-1)H\Big(\frac{\lambda p_0-D\psi(x_1,t_1)}{\lambda-1},\frac{x_1}{\varepsilon}\Big)\\ &\le H\Big(D\varphi(x_1,t_1),\frac{x_1}{\varepsilon}\Big)+\Lambda_1(\lambda-1)(2^q|p_0|^q+1).\end{aligned}$$

Taking $\lambda > 1$ closer to 1, if necessary, we obtain

$$\lambda H\Big(p_0+\lambda^{-1}D(\varphi-\psi)(x_1,t_1),\frac{x_1}{\varepsilon}\Big) \le H\Big(D\varphi(x_1,t_1),\frac{x_1}{\varepsilon}\Big)+\tfrac{1}{3}\eta. \tag{5-14}$$

Combining (5-12), (5-13) and (5-14) yields

$$-\overline{H}(p_0)-\varepsilon\,\mathrm{tr}\Big(A\Big(\frac{x_1}{\varepsilon}\Big)D^2\varphi(x_1,t_1)\Big)+H\Big(D\varphi,\frac{x_1}{\varepsilon}\Big)\ge -\tfrac{2}{3}\eta, \tag{5-15}$$

and then combining (5-10) and (5-15) gives

$$\psi_t(x_0,t_0)-\varepsilon\,\mathrm{tr}\Big(A\Big(\frac{x_1}{\varepsilon}\Big)D^2\varphi(x_1,t_1)\Big)+H\Big(D\varphi,\frac{x_1}{\varepsilon}\Big)\ge \tfrac{1}{3}\eta.$$

By making $r > 0$ smaller, if necessary, and using $\varphi_t(x_1,t_1)=\psi_t(x_1,t_1)$, we obtain

$$\varphi_t(x_1,t_1)-\varepsilon\,\mathrm{tr}\Big(A\Big(\frac{x_1}{\varepsilon}\Big)D^2\varphi(x_1,t_1)\Big)+H\Big(D\varphi,\frac{x_1}{\varepsilon}\Big)\ge \tfrac{1}{6}\eta.$$

This completes the proof of (5-11) and thus that of Step 1.

Step 2. We next show that $U(\cdot,0)\le g$ or, more precisely, that for every $R > 0$,

$$\limsup_{t\to 0}\ \sup_{x\in B_R}\,(U(x,t)-g(x))\le 0. \tag{5-16}$$

To accomplish this, we must construct supersolution barriers from above and apply the comparison principle. Note that this is very easy to do in the uniformly coercive case; we simply use the map $(x,t)\mapsto g(x)+kt$ where $k>0$ is a large constant depending on the constants in the hypotheses and $\|g\|_{C^{1,1}(\mathbb{R}^d)}$. Unfortunately, this function is not a supersolution in the nonuniformly coercive case, and so we need to consider a more elaborate barrier function. Rather than construct a barrier from scratch, we build it from the functions w^ε and use the fact that these homogenize.

For each fixed $x_0\in\mathbb{R}^d$, the functions we consider have the form

$$V^\varepsilon(x,t) := 2W^\varepsilon(x,t)-\phi(x,t),$$

where

$$W^\varepsilon(x,t) := e^t w^\varepsilon\big(x,\tfrac12 Dg(x_0)\big)+\overline{H}\big(\tfrac12 Dg(x_0)\big)+\tfrac12 g(x_0)+\tfrac12 Dg(x_0)\cdot(x-x_0)$$

and

$$\phi(x,t) := -2(1+\|g\|_{C^{1,1}(\mathbb{R}^d)})((1+|x-x_0|^2)^{1/2}-1)-k(e^t-1)$$

and $k > 0$ is a constant depending only on g, x_0, and other structural constants defined by

$$k := 2\Lambda_2(1 + \|g\|_{C^{1,1}(\mathbb{R}^d)}) + \Lambda_1(2^q(1 + \|g\|_{C^{1,1}(\mathbb{R}^d)})^q + 1) + 2\Lambda_1(2^{-q}|Dg(x_0)|^q + 1).$$

We next derive a supersolution inequality for W^ε. The convexity of H and (1-6) imply that, for every $p, \hat{p} \in \mathbb{R}^d$ and $\lambda \geq 1$,

$$\lambda^{-1}H(\lambda p + \hat{p}, y) \geq H(p + \hat{p}, y) - (1 - \lambda^{-1})H(\hat{p}, y) \geq H(p + \hat{p}, y) - (1 - \lambda^{-1})\Lambda_1(|\hat{p}|^q + 1).$$

Using this with $\hat{p}$ fixed and $\lambda = e^t$, we find that, for each $t > 0$, the function $w^\varepsilon(\,\cdot\,, \hat{p})$ satisfies the inequality

$$w^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2 w^\varepsilon\Big) + e^{-t} H\Big(\hat{p} + e^t Dw^\varepsilon, \frac{x}{\varepsilon}\Big) \geq -(1 - e^{-t})\Lambda_1(|\hat{p}|^q + 1) \quad \text{in } \mathbb{R}^d.$$

From this, we see that W^ε satisfies the inequality

$$W_t^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2 W^\varepsilon\Big) + H\Big(DW^\varepsilon, \frac{x}{\varepsilon}\Big) \geq -(e^t - 1)\Lambda_1(2^{-q}|Dg(x_0)|^q + 1) \quad \text{in } \mathbb{R}^d \times (0, \infty).$$

On the other hand, we see by a routine calculation, using the definition of k, (1-6) and (1-3), that ϕ is a (smooth) subsolution of the inequality

$$\phi_t - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2 \phi\Big) + H\Big(D\phi, \frac{x}{\varepsilon}\Big) \leq -2e^t \Lambda_1(2^{-q}|Dg(x_0)|^q + 1) \quad \text{in } \mathbb{R}^d \times (0, \infty).$$

The definition of k has been split into three terms, and we see from (1-6) that the first two terms take care of the contributions from spatial derivatives of ϕ and the third term is responsible for the right-hand side.

We may now apply [Armstrong and Tran 2014, Lemma 2.5 and Remark 2.6 with $\lambda = 1$] to find that V^ε is a supersolution of

$$V_t^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon}\Big)D^2 V^\varepsilon\Big) + H\Big(DV^\varepsilon, \frac{x}{\varepsilon}\Big) \geq 0 \quad \text{in } \mathbb{R}^d \times (0, \infty).$$

Therefore, the comparison principle implies that, for every $\varepsilon > 0$,

$$u^\varepsilon \leq V^\varepsilon - \inf_{x \in \mathbb{R}^d} (V^\varepsilon(x, 0) - g(x)) \quad \text{in } \mathbb{R}^d \times [0, \infty). \tag{5-17}$$

Since w^ε is bounded below (see (4-7)) and g is bounded, the linearly growing term in ϕ ensures that $V^\varepsilon(\,\cdot\,, 0)$ is larger than g outside a ball of fixed radius and centered at x_0. But due to the fact that $\omega = (\sigma, H)$ belongs to E_4, we have that, for every $R > 0$,

$$\lim_{\varepsilon \to 0} \sup_{x \in B_R} \sup_{0 \leq t \leq R} |V^\varepsilon(x, t) - V(x, t)| = 0,$$

where

$$V(x, t) := 2(e^t - 1)\overline{H}\big(\tfrac{1}{2}Dg(x_0)\big) + g(x_0) + Dg(x_0) \cdot (x - x_0) + 2(1 + \|g\|_{C^{1,1}(\mathbb{R}^d)})((1 + |x - x_0|^2)^{1/2} - 1) + k(e^t - 1).$$

It is routine to check that, for every $x \in \mathbb{R}^d$,

$$g(x) \leq g(x_0) + Dg(x_0) \cdot (x - x_0) + 2(1 + \|g\|_{C^{1,1}(\mathbb{R}^d)})((1 + |x - x_0|^2)^{1/2} - 1) = V(x, 0).$$

We deduce that

$$\limsup_{\varepsilon\to 0} \inf_{x\in\mathbb{R}^d} (V^\varepsilon(x,0) - g(x)) \ge 0.$$

Since $V(x_0, 0) = g(x_0)$ and V is uniformly Lipschitz continuous on $\mathbb{R}^d \times [0, 1)$ with a constant that is bounded above independently of x_0, this inequality combined with (5-17) yields (5-16).

Step 3. We complete the proof by arguing that

$$\liminf_{\varepsilon\to 0} u^\varepsilon(x,t) \ge u(x,t). \tag{5-18}$$

The argument here is similar to the demonstration of (5-7). We omit the proof that the left side of (5-18) is a supersolution of the limiting equation since this part is essentially identical to Step 1 (except that we remark that it is necessary to take $0 < \lambda < 1$ in contrast to $\lambda > 1$ as we did above). The second step, which is the analogue of Step 2, is actually much easier because we may produce a single smooth function that is a subsolution of the heterogeneous equation for all $\varepsilon > 0$. Indeed, since $H(p, x)$ is uniformly bounded above for bounded $|p|$, we may take $k > 0$ large enough, depending only on Λ_1, Λ_2 and $\|g\|_{C^{1,1}(\mathbb{R}^d)}$, such that $(x,t) \mapsto g(x) - kt$ is a subsolution of (5-1). Thus, $u^\varepsilon(x,t) \ge g(x) - kt$ for all $\varepsilon > 0$, giving us the desired lower bound at the initial time. □

6. The proof of the quenched large deviation principle

In this section, we give the proof of Corollary 2 and study some properties of the rate function $\bar{L}$. To our knowledge, the argument is originally due to Varadhan (communicated orally and unpublished) and also appeared later in [Lions and Souganidis 2005] and well as in [Kosygina 2007].

Before giving the demonstration of Corollary 2, let us see how the viscous Hamilton–Jacobi equation arises by considering the asymptotics of the partition function. According to the Feynman–Kac formula, for each $\omega \in \Omega$, the map $(x,t) \mapsto S(t,x,\omega)$ defined in (1-17) is a solution of the equation

$$S_t - \operatorname{tr}(A(y,\omega)D^2S) - b(y,\omega)\cdot DS + V(y,\omega)S = 0 \quad \text{in } \mathbb{R}^d \times \mathbb{R}_+$$

and we have $S(0,\cdot,\omega) \equiv 1$. If we take the (inverse) Hopf–Cole transform of S, setting

$$U(x,t,\omega) := -\log S(t,x,\omega),$$

then we check that $(x,t) \mapsto U(x,t,\omega)$ is the unique viscosity solution of the initial-value problem

$$\begin{cases} U_t - \operatorname{tr}(A(y,\omega)D^2U) + DU\cdot A(y,\omega)DU + b(y,\omega)\cdot DU - V(y,\omega) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}_+, \\ U(\cdot,0,\omega) \equiv 0 & \text{on } \mathbb{R}^d. \end{cases}$$

This suggests the definition (1-15) of H. Rescale by setting

$$u^\varepsilon(x,t,\omega) := \varepsilon U\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega\Big), \tag{6-1}$$

and observe that u^ε is the solution of (5-1) with $g \equiv 0$. An application of Theorem 1 yields

$$\mathbb{P}\Big[\lim_{t\to\infty} \frac{1}{t} U(tx,t,\omega) = \lim_{\varepsilon\to 0} u^\varepsilon(x,1,\omega) = -\bar{H}(0) \text{ locally uniformly in } x \in \mathbb{R}^d\Big] = 1.$$

This gives the approximate likelihood that a particle survives for a very long time:

$$\sup_{|x|\le Rt} e^{-\bar{H}(0)t} S(t, tx, \omega) = \exp(o(t)) \quad \text{as } t \to \infty. \tag{6-2}$$

(Note that in this context we have $\bar{H}(0) \le 0$ as can be seen from the fact that $w^\varepsilon \ge 0$ since the zero function is a subsolution of (4-1).) In fact, we have just proved Corollary 2 in the case $K = U = \mathbb{R}^d$ since, by the duality of the Legendre transform,

$$\inf_{y\in\mathbb{R}^d} \bar{L}(y) = -\bar{H}(0).$$

It turns out that by varying the initial condition g in Theorem 1 (taking it to be approximately the characteristic function of K or U) and using the Hopf–Lax formula for the solution of the limiting equation, this argument yields a proof of the large deviation principle. Here it is:

Proof of Corollary 2. Fix an element $\omega \in \Omega$ belonging to the event inside the probability in the conclusion of Theorem 1. We prove only the upper bound since the argument for the lower bound is similar (except that in the latter case we have to approximate initial data that is $-\infty$ from below, but this technicality can be handled by recalling the monotonicity of the solutions with respect to the data and using an approximation argument). Select a positive, uniformly continuous function g on $\mathbb{R}^d$ such that $g \le 1$ in $\mathbb{R}^d$ and $g \equiv 1$ on K, and observe that

$$-\log Q_{t,x,\omega}[X_t \in sK] \ge \underbrace{-\log E_{x,\omega}\Big[g(X_t/s)\exp\Big(-\int_0^t V(X_s,\omega)\,ds\Big)\Big]}_{=:\,U(x,t,\omega;\,s)} + \log S(t,x,\omega). \tag{6-3}$$

The limit of the second term on the right side is given by (6-2):

$$\lim_{t\to\infty} \frac{1}{t}\log S(t, tx, \omega) = \bar{H}(0).$$

Therefore, we concentrate on the first term on the right of (6-2). By the Feynman–Kac formula and an inverse Hopf–Cole change of variables, the function U defined in (6-3) is a solution of the initial-value problem

$$\begin{cases} U_t - \operatorname{tr}(A(y,\omega)D^2U) + DU\cdot A(y,\omega)DU + b(y,\omega)\cdot DU - V(y,\omega) = 0 & \text{in } \mathbb{R}^d\times\mathbb{R}_+,\\ U(\,\cdot\,,0,\omega;s) = -\log g(\,\cdot\,/s) & \text{on } \mathbb{R}^d.\end{cases}$$

Rescale by introducing

$$u^\varepsilon(x,t,\omega) := \varepsilon U\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \omega; \frac{1}{\varepsilon}\Big),$$

and notice that u^ε satisfies the rescaled equation

$$u_t^\varepsilon - \varepsilon \operatorname{tr}\Big(A\Big(\frac{x}{\varepsilon},\omega\Big)D^2u^\varepsilon\Big) + Du^\varepsilon\cdot A\Big(\frac{x}{\varepsilon},\omega\Big)Du^\varepsilon + b\Big(\frac{x}{\varepsilon},\omega\Big)\cdot Du^\varepsilon - V\Big(\frac{x}{\varepsilon},\omega\Big) = 0 \quad \text{in } \mathbb{R}^d\times\mathbb{R}_+$$

with the initial condition $u^\varepsilon(\,\cdot\,,0,\omega) = -\log g$ on $\mathbb{R}^d$.

Since ω belongs to the event in the conclusion of Theorem 1, we have

$$\lim_{t\to\infty}\frac{1}{t}U(tx,t,\omega;t) = \lim_{\varepsilon\to 0} u^\varepsilon(x,1,\omega) = u(x,1),$$

where $u = u(x,t)$ is the unique solution of the deterministic problem

$$\begin{cases} u_t + \bar{H}(Du) = 0 & \text{in } \mathbb{R}^d \times \mathbb{R}_+, \\ u(\,\cdot\,,0) = -\log g & \text{on } \mathbb{R}^d. \end{cases}$$

According to the Hopf–Lax formula, we have

$$u(x,t) = \inf_{y\in\mathbb{R}^d} \Bigl(t\bar{L}\Bigl(\frac{x-y}{t}\Bigr) - \log g(y) \Bigr).$$

Combining the last few lines, we obtain

$$\lim_{t\to\infty} \frac{1}{t} U(tx,t,\omega;t) = \inf_{y\in\mathbb{R}^d} (\bar{L}(x-y) - \log g(y)).$$

Inserting into (6-3), we obtain

$$\lim_{t\to\infty} -\frac{1}{t} \log Q_{t,tx,\omega}[X_t \in tK] \ge \inf_{y\in\mathbb{R}^d} (\bar{L}(x-y) - \log g(y)) + \bar{H}(0).$$

Using the continuity of $\bar{L}$ and taking g to approximate the characteristic function of K, we obtain

$$\lim_{t\to\infty} -\frac{1}{t} \log Q_{t,tx,\omega}[X_t \in tK] \ge \inf_{y\in K} \bar{L}(x-y) + \bar{H}(0). \qquad \square$$

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SCOTT N. ARMSTRONG: armstrong@ceremade.dauphine.fr
Centre de Recherche en Mathématiques de la Décision, Université Paris-Dauphine, Place du Maréchal de Lattre de Tassigny, 75775 Paris 16, France

HUNG V. TRAN: hung@math.uchicago.edu
Department of Mathematics, The University of Chicago, 5734 South University Avenue, Chicago, IL 60637, United States

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GLOBAL REGULARITY FOR A SLIGHTLY SUPERCRITICAL HYPERDISSIPATIVE NAVIER–STOKES SYSTEM

DAVID BARBATO, FRANCESCO MORANDIN AND MARCO ROMITO

We prove global existence of smooth solutions for a slightly supercritical hyperdissipative Navier–Stokes under the optimal condition on the correction to the dissipation. This proves a conjecture formulated by Tao.

1. Introduction

Let $d \geq 3$ and consider the generalized Navier–Stokes system

$$\begin{cases} \partial u/\partial t + (u \cdot \nabla)u + \nabla p + D_0^2 u = 0, \\ \nabla \cdot u = 0, \\ \int_{[0,2\pi]^d} u(t,x)\,dx = 0, \end{cases} \tag{1-1}$$

on $[0, 2\pi]^d$ with periodic boundary conditions, where D_0 is a Fourier multiplier with nonnegative symbol m. The Navier–Stokes system is recovered when $m(k) = |k|$. If

$$m(k) \geq c \frac{|k|^{(d+2)/4}}{G(|k|)}, \tag{1-2}$$

where $G : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that

$$\int_1^\infty \frac{ds}{sG(s)^4} = \infty, \tag{1-3}$$

and

$$\frac{G(x)}{|x|^{(d+2)/4}} \text{ is eventually nonincreasing,} \tag{1-4}$$

then in [Tao 2009] it is proved[1] that (1-1) has a global smooth solution for every smooth initial condition. The result has been extended to the two-dimensional case in [Katz and Tapay 2012].

A heuristic argument developed in [Tao 2009] and based on the comparison between the speed of propagation of a (possible) blow-up and the rate of dissipation suggests that regularity should still hold under the weaker condition

$$\int_1^\infty \frac{ds}{sG(s)^2} = \infty. \tag{1-5}$$

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[1] The proof of that result is given in $\mathbb{R}^d$, but it can be easily extended to the periodic setting; see [Tao 2009, Remark 2.1].

The main result of this paper, contained in the following theorem, is a complete proof of this conjecture.

Theorem 1.1. *Let $d \geq 2$ and assume conditions (1-2), (1-4) and (1-5) hold for a nondecreasing function $G : [0, \infty) \to [0, \infty)$. Then (1-1) has a global smooth solution for every smooth initial condition.*

A simple version of this conjecture, when reformulated on a toy model, has been proved for the dyadic model in [Barbato et al. 2014]. Actually, for that model one could prove regularity in the full supercritical regime, with $m(k) = |k|$, as was done in [Barbato et al. 2011], but it was natural to develop there some of the main ideas on which also this paper is based. In fact, here we prove that the equations for the velocity can be reduced to a suitable dyadic-like model, but with infinitely many interactions. A more sophisticated version of the arguments of [Barbato et al. 2014] ensures regularity of this dyadic model and, in turn, of the solution of problem (1-1).

Our technique for proving Theorem 1.1 is flexible enough to include an additional critical parameter. Consider the generalized Leray α-model,

$$\begin{cases} \partial v/\partial t + (u \cdot \nabla) v + \nabla p + D_1 v = 0, \\ v = D_2 u, \\ \nabla \cdot v = 0, \\ \int_{[0,2\pi]^d} v(t,x)\,dx = \int_{[0,2\pi]^d} u(t,x)\,dx = 0, \end{cases} \tag{1-6}$$

where D_1 and D_2 are Fourier multipliers with nonnegative symbols m_1 and m_2.

Theorem 1.2. *Let $d \geq 2$ and $\alpha, \beta \geq 0$, and assume*

$$m_1(k) \geq c\frac{|k|^\alpha}{g(|k|)}, \quad m_2(k) \geq c|k|^\beta, \quad \alpha + \beta \geq \frac{d+2}{2},$$

where $g : [0, \infty) \to [0, \infty)$ is a nondecreasing function such that $x^{-\alpha} g(x)$ is eventually nonincreasing and

$$\int_1^\infty \frac{ds}{s g(s)} = \infty. \tag{1-7}$$

Then (1-6) has a global smooth solution for every smooth initial condition.

Under the assumptions of Theorem 1.1, if $\beta = 0$, $\alpha = (d+2)/2$, $g(x) = G(x)^2$, $m_2(k) = 1$, and $m_1(k) = m(k)^2$, then the assumptions of Theorem 1.2 are met. Therefore Theorem 1.1 follows immediately from Theorem 1.2, and it is sufficient to prove only the second result.

Our results hold as well when the problems are considered in $\mathbb{R}^d$, since in our method large scales play no significant role (see Remark 2.9).

The model (1-6) with $g \equiv 1$ was introduced by Olson and Titi [2007]. They proposed the idea that a weaker nonlinearity and a stronger viscous dissipation could work together to yield regularity. Their statement uses the stronger hypothesis $\alpha + \beta/2 \geq (d+2)/2$ though, and this result was later logarithmically improved in [Yamazaki 2012] with condition (1-3).

Our results are also relevant in view of the analysis in [Tao 2014, Remark 5.2], since they confirm that the condition (1-7) is optimal when general nonlinear terms with the same scaling are considered.

The proof of the above theorem is based on two crucial ideas. The first idea is that smoothness of (1-6) can be reduced to the smoothness of a suitable shell model, obtained by averaging the energy of a solution of (1-6) over dyadic shells in Fourier space. We believe that this reduction may be interesting beyond the scope of this paper. The second idea is that the overall contribution of energy and dissipation over large shells satisfies a recursive inequality. Under condition (1-7), dissipation significantly dumps the flow of energy towards small scales and ensures smoothness. This is a more sophisticated version of the result obtained in [Barbato et al. 2014], due to the larger number of interactions between shells.

The paper is organized as follows. In Section 2 we derive the *shell approximation* of a solution of (1-6). The recursive formula is obtained in Section 3. In Section 4 we deduce exponential decays of shell modes by the recursive formula. The Appendix contains a standard existence and uniqueness result for the sake of completeness.

2. From the generalized Fourier Navier–Stokes to the dyadic equation

This section contains one of the crucial steps in our approach. We show that the proof of Theorem 1.2 can be reduced to a proof of the decay of solutions of a suitable shell model. For simplicity and without loss of generality, from now on we assume that

$$m_1(k) = \frac{|k|^\alpha}{g(|k|)}, \qquad m_2(k) \geq |k|^\beta.$$

The shell approximation. The dynamics of our generalized version of the Navier–Stokes equation in Fourier decomposition are

$$\begin{cases} v_k' = -\dfrac{|k|^\alpha}{g(|k|)} v_k - i \displaystyle\sum_{h \in \mathbb{Z}^d \setminus \{0\}} \frac{\langle v_h, k\rangle}{|h|^\beta} P_k(v_{k-h}), \\ \langle v_k, k\rangle = 0, \\ v_{-k} = \overline{v_k}, \end{cases} \tag{2-1}$$

for $k \in \mathbb{Z}^d \setminus \{0\}$, where $P_k(w) := w - (\langle w, k\rangle / |k|^2)k$ and $v_0 = 0$. A solution is a family $(v_k)_{k \in \mathbb{Z}^d \setminus \{0\}}$ where each $v_k = v_k(t)$ is a differentiable map from $[0, \infty)$ to $\mathbb{C}^d$ satisfying (2-1) for all times.

As is common in Littlewood–Paley theory, let $\Phi : [0, \infty) \to [0, 1]$ be a smooth function such that $\Phi \equiv 1$ on $[0, 1]$, $\Phi \equiv 0$ on $[2, \infty)$, and Φ is strictly decreasing on $[1, 2]$. For $x \geq 0$, let $\psi(x) := \Phi(x) - \Phi(2x)$, so that ψ is a smooth bump function supported on $\left(\frac{1}{2}, 2\right)$ satisfying

$$\sum_{n=0}^{\infty} \psi\left(\frac{x}{2^n}\right) = 1 - \Phi(2x) \equiv 1, \qquad x \geq 1.$$

Notice that it is elementary to show that $\sqrt{\psi}$ is Lipschitz continuous.

Let $\mathbb{N}_0$ denote the set of nonnegative integers. For all $n \in \mathbb{N}_0$, we introduce the radial maps $\psi_n : \mathbb{R}^d \to [0, 1]$ defined by $\psi_n(x) = \psi(2^{-n}|x|)$. Notice that

$$\sum_{n \in \mathbb{N}_0} \psi_n(x) \equiv 1, \qquad x \in \mathbb{Z}^d \setminus \{0\}.$$

In Littlewood–Paley theory, one typically defines ψ_n for all $n \in \mathbb{Z}$, introduces objects like

$$P_n(x) := \sum_{k \in \mathbb{Z}^d} \psi_n(k) v_k e^{i\langle k,x\rangle},$$

and then proves that $u = \sum_n P_n$. Since these P_n are not orthogonal[2] this does not give a nice decomposition of energy, as

$$\sum_{n \in \mathbb{Z}} \|P_n\|_{L^2}^2 \neq \sum_{k \in \mathbb{Z}^d} |v_k|^2 = \|u\|_{L^2}^2.$$

Thus, instead of $P_n(x)$, we introduce a sort of *square-averaged* Littlewood–Paley decomposition. Let

$$X_n(t) := \Big(\sum_{k \in \mathbb{Z}^d} \psi_n(k) |v_k(t)|^2\Big)^{\frac{1}{2}}, \qquad n \in \mathbb{N}_0,\ t \geq 0. \tag{2-2}$$

Then clearly

$$\sum_{n \in \mathbb{N}_0} X_n^2 = \sum_{k \in \mathbb{Z}^d} |v_k|^2 = \|u\|_{L^2}^2.$$

Remark 2.1. One major difference with respect to the usual Littlewood–Paley theory is that it is impossible to recover v from these X_n (as it was with the components $P_n(x)$), since they are averaged both in the physical space and over one shell of the frequency space.

We will denote by H^γ the Hilbert–Sobolev space of periodic functions with differentiation index γ, namely

$$H^\gamma = \Big\{ v = (v_k)_{k \in \mathbb{Z}^d} \ :\ \sum (1 + |k|^2)^\gamma |v_k|^2 < \infty \Big\}. \tag{2-3}$$

Definition 2.2. If (2-2) holds, we say that $X = (X_n(t))_{n \in \mathbb{N}_0, t \geq 0}$ is the *shell approximation* of v.

If $v \in H^\gamma$ and X is its shell approximation, then

$$\sum_n 2^{2\gamma n} X_n^2 = \sum_k \Big(\sum_n 2^{2\gamma n} \psi_n(k)\Big) |v_k|^2 \approx \sum_k |k|^{2\gamma} |v_k|^2 = \|v\|_{H^\gamma}^2. \tag{2-4}$$

Hence, $v(t) \in C^\infty$ if and only if $\sup_n 2^{\gamma n} X_n < \infty$ for every $\gamma > 0$. In view of Theorem A.1, Theorem 1.2 follows if we can prove:

Theorem 2.3. *Under the assumptions of* Theorem 1.2, *let* $v(0)$ *be smooth and periodic and let* $m \geq 2 + d/2$. *If* v *is a solution of* (1-6) *in* H^m *on its maximal interval of existence* $[0, T_\star)$, X *is its shell approximation and*

$$\sup_{[0,T_\star)} \sum 2^{2mn} X_n^2 < \infty,$$

then $T_\star = \infty$.

[2]They are in fact *almost orthogonal*, in the sense that $\langle P_n, P_m\rangle_{L^2} = 0$ whenever $|m - n| \geq 2$.

The shell solution. We want to write a system of equations for the shell approximation of a solution of (1-6). We give a more formal connection between (1-6) and its shell equation because we believe the notion will turn out to be useful beyond the scopes of the present work.

Define the set I to be those $(l, m, n) \in \mathbb{N}_0^3$ for which the difference between the two largest integers among l, m and n is at most 2.

We are now ready to introduce the shell model ODE for the energy of each shell (Equation (2-5)).

Definition 2.4 (shell solution). Let $X = (X_n)_{n\in\mathbb{N}_0}$ be a sequence of real-valued maps $X_n : [0, \infty) \to \mathbb{R}$. We say that X is a *shell solution* if there are two families of real-valued maps $\chi = (\chi_n)_{n\in\mathbb{N}_0}$ and $\phi = (\phi_{(l,m,n)})_{(l,m,n)\in I}$ such that

$$\frac{d}{dt}X_n^2(t) = -\chi_n(t)X_n^2(t) + \sum_{\substack{l,m\in\mathbb{N}_0\\(l,m,n)\in I}} \phi_{(l,m,n)}(t)X_l(t)X_m(t)X_n(t) \tag{2-5}$$

for all $n \in \mathbb{N}_0$ and $t > 0$, where the sum above is understood as absolutely convergent, and χ, ϕ satisfy the following:

(1) The family ϕ is *antisymmetric*, in the sense that

$$\phi_{(l,m,n)}(t) = -\phi_{(l,n,m)}(t), \quad (l, m, n) \in I,\ t \geq 0.$$

(2) There exist two positive constants c_1 and c_2 for which

$$\chi_n(t) \geq c_1\frac{2^{\alpha n}}{g(2^{n+1})} \quad \text{and} \quad |\phi_{(l,m,n)}(t)| \leq c_2 2^{(d/2+1-\beta)\min\{l,m,n\}} \tag{2-6}$$

for all $(l, m, n) \in I$ and $t \geq 0$.

Remark 2.5. We will prove below that the shell approximation of a solution of (1-6) is a shell solution. It is easy to check that the dissipation term is local, as expected, due to the way the shell components of a solution interact in the model's dynamics. As for the nonlinear term, it turns out that the set I of the triples of indices (l, m, n) for which there may be interaction between the shell components l, m and n is quite small. This is basically because, in the Fourier space, three components may interact only if they are the sides of a triangle, and by the triangle inequality their lengths cannot be in three shells far away from each other.

Remark 2.6. To ensure that the sum in (2-5) is absolutely convergent, it is sufficient to assume that the sequence $(X_n(t))_{n\in\mathbb{N}_0}$ is square-summable (this will be a consequence of the energy inequality; see Definition 3.1). Indeed, if n is not the smallest index, then the sum is extended to a finite number of indices. Otherwise, $\phi_{(l,m,n)}$ is constant with respect to l, m.

Remark 2.7. The antisymmetric property is what makes the nonlinearity of (2-5) *formally* conservative. In fact, using antisymmetry, a change of variable ($m' = n$ and $n' = m$) and the fact that $(l, m', n') \in I$ if and only if $(l, n', m') \in I$, one could formally write

$$-\sum_{\substack{l,m,n\in\mathbb{N}_0\\(l,m,n)\in I}} \phi_{(l,m,n)} X_l X_m X_n = \sum_{\substack{l,m,n\in\mathbb{N}_0\\(l,m,n)\in I}} \phi_{(l,n,m)} X_l X_m X_n = \sum_{\substack{l,m',n'\in\mathbb{N}_0\\(l,n',m')\in I}} \phi_{(l,m',n')} X_l X_{m'} X_{n'}$$
$$= \sum_{\substack{l,m',n'\in\mathbb{N}_0\\(l,m',n')\in I}} \phi_{(l,m',n')} X_l X_{m'} X_{n'}.$$

If these sums are absolutely convergent, this would prove indeed that the expression itself is equal to zero.

Since these are infinite sums, these computations are not rigorous unless we know, for instance, that $\sum_n 2^{2\gamma n} X_n^2 < \infty$ with $\gamma \geq \frac{1}{3}\big(\frac{1}{2}d + 1 - \beta\big)$, as can be verified by an elementary computation.

The shell model as a shell approximation. The bounds on the coefficients given in Definition 2.4 are in the correct direction to prove regularity results (and hence Theorem 2.3). The following theorem, which is the main result of this section, shows that they capture the natural scaling of the shell interactions for the *physical* solutions.

Theorem 2.8. *If v is a solution of (1-6) on $[0, T]$ and X is its shell approximation, then X is a shell solution.*

Remark 2.9. At this stage it is easy to realize that our main results hold also in $\mathbb{R}^d$ with minimal changes. Indeed when passing to the shell approximation, all large frequencies are considered together in the first element of the shell model.

The proof of Theorem 2.8 can be found at the end of this section. It is based on Propositions 2.10–2.11 below, which give the actual definitions of χ and ϕ and prove their properties.

Proposition 2.10. *Let X be the shell approximation of a solution v. Define $\chi_n(t)$ for $n \in \mathbb{N}_0$ and $t \geq 0$ by*

$$\chi_n(t) := \begin{cases} \dfrac{2}{X_n^2(t)} \displaystyle\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k) \frac{|k|^\alpha}{g(|k|)} |v_k(t)|^2 & \text{if } X_n(t) \neq 0, \\ \dfrac{2^{\alpha n - \alpha + 1}}{g(2^{n+1})} & \text{if } X_n(t) = 0. \end{cases} \tag{2-7}$$

Then

$$\chi_n(t) \geq \frac{2^{\alpha n - \alpha + 1}}{g(2^{n+1})}, \qquad n \in \mathbb{N}_0,\ t \geq 0.$$

Proof. Fix $n \in \mathbb{N}_0$ and $t \geq 0$. The map ψ_n is supported on $\{x \in \mathbb{Z}^d : 2^{n-1} < |x| < 2^{n+1}\}$ and g is nondecreasing, so

$$\sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k) \frac{|k|^\alpha}{g(|k|)} |v_k(t)|^2 \geq \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k) \frac{2^{(n-1)\alpha}}{g(2^{n+1})} |v_k(t)|^2 = \frac{2^{(n-1)\alpha}}{g(2^{n+1})} X_n^2(t),$$

where we used (2-2). By (2-7) we get the result. ∎

We finally turn our attention to the antisymmetry property and an upper bound for $\phi_{(l,m,n)}(t)$:

Proposition 2.11. *Let X be the shell approximation of a solution v. Define $\phi_{(l,m,n)}(t)$ for all $l, m, n \in \mathbb{N}_0$ and $t \geq 0$ as*

$$\phi_{(l,m,n)}(t) := \frac{2}{X_l(t)X_m(t)X_n(t)} \sum_{\substack{h,k\in\mathbb{Z}^d \\ h\neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h(t), k\rangle\langle v_{k-h}(t), v_k(t)\rangle\}}{|h|^\beta} \tag{2-8}$$

(unless $X_l(t)X_m(t)X_n(t) = 0$, in which case $\phi_{(l,m,n)}(t) := 0$). Then:

(1) *$\phi_{(l,m,n)}(t) = 0$ for all $(l, m, n) \notin I$ and all $t \geq 0$.*

(2) *$\phi_{(l,m,n)}(t) = -\phi_{(l,n,m)}(t)$ for all $l, m, n \in \mathbb{N}_0$ and all $t \geq 0$.*

(3) *For any $\beta \geq 0$ there exists a constant $c_3 > 0$ depending only on d, β and ψ such that*

$$|\phi_{(l,m,n)}(t)| \leq c_3 2^{(d/2+1-\beta)\min\{l,m,n\}}, \quad (l, m, n) \in I,\ t \geq 0. \tag{2-9}$$

For the proof we need a couple of lemmas:

Lemma 2.12. *Suppose $v = (v_k)_{k\in\mathbb{Z}^d}$ is a complex field over $\mathbb{Z}^d$ such that, for all $k \in \mathbb{Z}^d$, $\langle k, v_k\rangle = 0$ and $\overline{v_k} = v_{-k}$. Then, for all $h \in \mathbb{Z}^d$,*

$$\sum_{k\in\mathbb{Z}^d} \psi_m(k-h)\psi_n(k) \operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\} = -\sum_{k\in\mathbb{Z}^d} \psi_m(k)\psi_n(k-h) \operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\}.$$

Proof. Consider the left-hand side. By performing the change of variable $k' = h - k$, we obtain

$$\begin{aligned} \psi_m(k-h) &= \psi_m(-k') = \psi_m(k'), \\ \psi_n(k) &= \psi_n(h-k') = \psi_n(k'-h), \\ \langle v_h, k\rangle &= \langle v_h, h-k'\rangle = -\langle v_h, k'\rangle, \\ \langle v_{k-h}, v_k\rangle &= \langle v_{-k'}, v_{h-k'}\rangle = \langle \overline{v_{k'}}, \overline{v_{k'-h}}\rangle = \langle v_{k'-h}, v_{k'}\rangle. \end{aligned}$$

The sum for $k \in \mathbb{Z}^d$ is equivalent to the sum for $k' \in \mathbb{Z}^d$, and this concludes the proof. □

Lemma 2.13. *Let v be a solution and X its shell approximation. Then, for all $a, b, c \in \mathbb{N}_0$ and all $t \geq 0$,*

$$\sum_{h\in\mathbb{Z}^d} \psi_a(h)|v_h(t)| \sum_{k\in\mathbb{Z}^d} \sqrt{\psi_b(k)\psi_c(k-h)}\,|v_k(t)|\,|v_{k-h}(t)| \leq 2^{d(a+3)/2} X_a(t)X_b(t)X_c(t).$$

Proof. By the Cauchy–Schwarz inequality and formula (2-2), we have that, for all $h \in \mathbb{Z}^d$,

$$\sum_{k\in\mathbb{Z}^d} \sqrt{\psi_b(k)\psi_c(k-h)}\,|v_k(t)|\,|v_{k-h}(t)| \leq X_b(t)X_c(t).$$

Then, let S_a denote the intersection of $\mathbb{Z}^d$ and the support of ψ_a. By inscribing S_a in a cube, we can bound its cardinality by $|S_a| \leq (2^{a+2}+1)^d \leq 2^{(a+3)d}$, so

$$\sum_{k\in\mathbb{Z}^d} \psi_a(k)|v_k(t)| \leq \left(|S_a| \sum_{k\in S_a} \psi_a^2(k)v_k^2(t)\right)^{\frac{1}{2}} \leq (2^{(a+3)d})^{1/2} X_a(t),$$

where we used the fact that $\psi_a(k) \le 1$. □

Proof of Proposition 2.11. Consider Equation (2-8), the definition of $\phi_{(l,m,n)}$. By applying Lemma 2.12, for fixed t we immediately conclude that

$$\phi_{(l,n,m)} = -\phi_{(l,m,n)}, \qquad l, m, n \in \mathbb{N}_0,$$

and in particular that $\phi_{(l,m,m)} = 0$.

Moreover, for all choices of h and k, the arguments of ψ_l, ψ_m and ψ_n are the sides of a triangle in $\mathbb{R}^d$, so by the triangle inequality the size of the largest (without loss of generality k) is at most twice the size of the second largest (without loss of generality h). On the other hand, for all $j \in \mathbb{N}_0$ the support of ψ_j is $\{x \in \mathbb{R}^d : 2^{j-1} < |x| < 2^{j+1}\}$. Thus, whenever $\psi_l(h)\psi_n(k) \neq 0$, necessarily $n \le l+2$, since

$$2^{n-1} < |k| \le 2|h| < 2^{l+2}.$$

This proves that $\phi_{(l,m,n)} = 0$ outside the set I defined before Definition 2.4.

Finally, we prove inequality (2-9) for $(l, m, n) \in I$ with $m < n$. We will consider separately the two cases $n - m > 2$ and $n - m \in \{1, 2\}$, starting with the former.

Case 1. Since $m < n - 2$ and $(l, m, n) \in I$, we have $m = \min\{l, m, n\}$ and $|l - n| \le 2$. This means in particular that typically $|k - h| < |k|$ for all the nonzero terms of the sum in (2-8), so it is convenient to substitute $\langle v_h, k\rangle = \langle v_h, k - h\rangle$ in the equation to obtain the bound

$$|\phi_{(l,m,n)}| \le \frac{2}{X_l X_m X_n} \sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k)\frac{|v_h|\,|k-h|\,|v_{k-h}|\,|v_k|}{|h|^\beta}.$$

By the definition of ψ_l, either $\psi_l(h) = 0$ or $|h| \ge 2^{l-1} \ge 2^m$. Applying this and the change of variable $k' = k - h$, one gets

$$|\phi_{(l,m,n)}| \le \frac{2^{1-\beta m}}{X_l X_m X_n} \sum_{k'\in\mathbb{Z}^d} \psi_m(k')|k'|\,|v_{k'}| \sum_{h\in\mathbb{Z}^d} \psi_l(h)\psi_n(k'+h)|v_h|\,|v_{k'+h}|.$$

In the same way, we can substitute $|k'| \le 2^{m+1}$ and apply Lemma 2.13 (recall that $\psi \le 1$, so $\psi \le \sqrt{\psi}$) to get

$$|\phi_{(l,m,n)}| \le 2^{1-\beta m+m+1+d(m+3)/2}.$$

Since in the present case $\min\{l, m, n\} = m$, this proves inequality (2-9) with $c_3 = 2^{2+3d/2}$.

Case 2. Suppose now that $n - m \in \{1, 2\}$ and $(l, m, n) \in I$; then $l \le n+2$ and $\min\{l, m, n\} \ge l - 4$. In this case it is l that can be small with respect to m and n, so we take the terms in l and h outside the internal sum:

$$|\phi_{(l,m,n)}| \le \frac{2}{X_l X_m X_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \frac{\psi_l(h)}{|h|^\beta}\left|\sum_{k\in\mathbb{Z}^d} \psi_m(k-h)\psi_n(k)\,\mathrm{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\}\right|.$$

The idea is to exploit the cancellations in the sum over k that happen when $k-h$ and k are switched. By Lemma 2.12 and the bound $|k| \le 2^{n+1}$ for k in the support of ψ_m or ψ_n,

$$|\phi_{(l,m,n)}| \le \frac{2}{X_l X_m X_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \frac{\psi_l(h)}{|h|^\beta}\frac{1}{2}\left|\sum_{k\in\mathbb{Z}^d}\big(\psi_m(k-h)\psi_n(k)-\psi_m(k)\psi_n(k-h)\big)\,\mathrm{Im}\{\langle v_h,k\rangle\langle v_{k-h},v_k\rangle\}\right|$$

$$\le \frac{2^{n+1}}{X_l X_m X_n} \sum_{h\in\mathbb{Z}^d\setminus\{0\}} \frac{\psi_l(h)|v_h|}{|h|^\beta}\sum_{k\in\mathbb{Z}^d}\big|\psi_m(k-h)\psi_n(k)-\psi_m(k)\psi_n(k-h)\big||v_{k-h}||v_k|.$$

We turn our attention to the term $\psi_m(k-h)\psi_n(k)-\psi_m(k)\psi_n(k-h)$ and show that it is small. Let L denote the Lipschitz constant of the function $\psi^{1/2}$. Then, for all $h, k \in \mathbb{Z}^d$ and all $m, n \in \mathbb{N}_0$ such that $m \ge n-2$,

$$\begin{aligned}
&\left|\sqrt{\psi_m(k-h)\psi_n(k)}-\sqrt{\psi_m(k)\psi_n(k-h)}\right| \\
&\qquad = \left|\sqrt{\psi_m(k-h)\psi_n(k)}-\sqrt{\psi_m(k)\psi_n(k)}+\sqrt{\psi_m(k)\psi_n(k)}-\sqrt{\psi_m(k)\psi_n(k-h)}\right| \\
&\qquad \le L\frac{|h|}{2^m}\sqrt{\psi_n(k)}+L\frac{|h|}{2^n}\sqrt{\psi_m(k)} \le L\frac{|h|}{2^{n-3}}.
\end{aligned}$$

Moreover, by symmetry with respect to m and n,

$$\sum_{k\in\mathbb{Z}^d}\left(\sqrt{\psi_m(k-h)\psi_n(k)}+\sqrt{\psi_m(k)\psi_n(k-h)}\right)|v_{k-h}||v_k| = 2\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}\,|v_{k-h}||v_k|,$$

so that

$$|\phi_{(l,m,n)}| \le \frac{2^5 L}{X_l X_m X_n}\sum_{h\in\mathbb{Z}^d\setminus\{0\}}|h|^{1-\beta}\psi_l(h)|v_h|\sum_{k\in\mathbb{Z}^d}\sqrt{\psi_m(k-h)\psi_n(k)}\,|v_{k-h}||v_k|.$$

By the usual bound $2^{l-1} \le |h| \le 2^{l+1}$ and since $\beta \ge 0$, we see that $|h|^{1-\beta} \le 2^{l(1-\beta)+1+\beta}$ so, by Lemma 2.13,

$$|\phi_{(l,m,n)}| \le 2^5 2^{(1-\beta)l+1+\beta}2^{(l+3)d/2}L \le 2^{(d/2+1-\beta)(l-4)+9-3\beta+11d/2}L.$$

Since in the present case $\min\{l,m,n\} \ge l-4$, this proves inequality (2-9) with $c_3 = 2^{9+11d/2-3\beta}L$. □

Finally we have all the ingredients to prove the main theorem of this section:

Proof of Theorem 2.8. A direct computation using (2-2) and (2-1) shows that

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}X_n^2 &= \mathrm{Re}\sum_{k\in\mathbb{Z}^d}\psi_n(k)\langle v_k', v_k\rangle \\
&= -\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\alpha}{g(|k|)}|v_k|^2 + \mathrm{Im}\sum_{k\in\mathbb{Z}^d}\sum_{h\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{\langle v_h,k\rangle}{|h|^\beta}\langle P_k(v_{k-h}),v_k\rangle \\
&= -\sum_{k\in\mathbb{Z}^d\setminus\{0\}}\psi_n(k)\frac{|k|^\alpha}{g(|k|)}|v_k|^2 + \sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}}\psi_n(k)\frac{\mathrm{Im}\{\langle v_h,k\rangle\langle v_{k-h},v_k\rangle\}}{|h|^\beta}.
\end{aligned}$$

To deal with the first sum, define χ as in Proposition 2.10. By applying (2-7) for $X_n(t) \neq 0$ and (2-2) for $X_n(t) = 0$, we see that in both cases

$$2 \sum_{k\in\mathbb{Z}^d\setminus\{0\}} \psi_n(k) \frac{|k|^\alpha}{g(|k|)} |v_k|^2 = \chi_n(t) X_n^2(t).$$

Now consider the second sum. Since the terms with $h = k$ give no contribution, we can apply

$$\sum_{l\in\mathbb{N}_0} \psi_l(h) = \sum_{m\in\mathbb{N}_0} \psi_m(k-h) = 1, \qquad h, k \in \mathbb{Z}^d,\ 0 \neq h \neq k,$$

to get

$$\begin{aligned}\sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}} \psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\}}{|h|^\beta} &= \sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}} \sum_{l,m\in\mathbb{N}_0} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\}}{|h|^\beta} \\ &= \sum_{l,m\in\mathbb{N}_0} \sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\}}{|h|^\beta},\end{aligned}$$

where it was possible to exchange the order of summation because the middle expression is clearly absolutely convergent.

Now define ϕ as in Proposition 2.11. By applying (2-8) or (2-2), depending on $X_l(t)X_m(t)X_n(t)$ being positive or zero, we see that, for all $l, m, n \in \mathbb{N}_0$ and $t \geq 0$,

$$2 \sum_{\substack{h,k\in\mathbb{Z}^d\\ h\neq 0}} \psi_l(h)\psi_m(k-h)\psi_n(k) \frac{\operatorname{Im}\{\langle v_h, k\rangle\langle v_{k-h}, v_k\rangle\}}{|h|^\beta} = \phi_{(l,m,n)}(t) X_l(t) X_m(t) X_n(t).$$

Putting it all together we get

$$\frac{d}{dt} X_n^2(t) = -\chi_n(t) X_n^2(t) + \sum_{l,m\in\mathbb{N}_0} \phi_{(l,m,n)}(t) X_l(t) X_m(t) X_n(t), \qquad n \in \mathbb{N}_0,\ t \geq 0.$$

Finally, recalling by Proposition 2.11 that $\phi \equiv 0$ outside I, we may restrict the scope of the sum and obtain (2-5). The required properties of the coefficients χ and ψ follow again from Propositions 2.10–2.11. □

3. From the dyadic equation to the recursive inequality

In view of the results of the previous section, we can now concentrate on shell solutions and forget (1-6). In this section we proceed as in [Barbato et al. 2014] and deduce a recursive inequality between the tails of energy and dissipation. Clearly here, due to the more complex nonlinear interaction, the relation is less trivial than in [Barbato et al. 2014].

Definition 3.1. A shell solution X satisfies the *energy inequality* on $[0, T]$ if $\sum_n X_n^2(0)$ is finite and

$$\sum_{n\in\mathbb{N}_0} X_n^2(t) + \int_0^t \sum_{n\in\mathbb{N}_0} \chi_n(s) X_n^2(s)\, ds \leq \sum_{n\in\mathbb{N}_0} X_n^2(0), \qquad t \in [0, T]. \tag{3-1}$$

Definition 3.2. Let X be a shell solution and define the sequences of real-valued maps $(F_n)_{n\in\mathbb{N}_0}$ and $(d_n)_{n\in\mathbb{N}_0}$ for $t \geq 0$ by

$$F_n(t) := \sum_{k\geq n} X_k^2(t), \quad d_n(t) := \left(F_n(t) + \sum_{h\geq n} \int_0^t \chi_h(s) X_h^2(s)\, ds\right)^{\frac{1}{2}}.$$

We will call $(F_n)_{n\in\mathbb{N}_0}$ the *tail* of X and $(d_n)_{n\in\mathbb{N}_0}$ the *energy bound* of X.

The recursive inequality between the tails and the energy bound is given in the next result.

Proposition 3.3. *Let X be a shell solution that satisfies the energy inequality on a time interval $[0, t]$, let $(d_n)_{n\in\mathbb{N}_0}$ be its sequence of energy bounds, and set $\lambda = 2^\alpha$.*

Then there is a positive constant $c_4 > 0$, not depending on t, such that, for all $n \in \mathbb{N}_0$,

$$d_n^2(t) \leq F_n(0) + c_4 \sum_{l=0}^{n-1} \frac{\bar{d}_l}{\lambda^{n-l}} \sum_{m\geq n-2} \frac{g(2^{m+1})}{\lambda^{m-n}} (d_m^2(t) - d_{m+1}^2(t)), \tag{3-2}$$

where $\bar{d}_l := \max_{s\in[0,t]} d_l(s)$.

Proof. Fix $n \in \mathbb{N}_0$. Differentiate $\sum_{h=0}^{n-1} X_h^2$ using (2-5):

$$\frac{d}{dt} \sum_{h=0}^{n-1} X_h^2 = -\sum_{h=0}^{n-1} \chi_h X_h^2 + \sum_{\substack{l,m,h\in\mathbb{N}_0 \\ (l,m,h)\in I \\ h\leq n-1}} \phi_{(l,m,h)} X_l X_m X_h.$$

Apply Lemma 3.4 below to the second sum and integrate on $[0, t]$ to obtain

$$\sum_{h=0}^{n-1} X_h^2(t) - \sum_{h=0}^{n-1} X_h^2(0) = -\int_0^t \sum_{h=0}^{n-1} \chi_h X_h^2\, ds - \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} \phi_{(l,m,h)} X_l X_m X_h\, ds$$

so that, by the energy inequality (3-1),

$$F_n(t) + \int_0^t \sum_{h\geq n} \chi_h(s) X_h^2(s)\, ds \leq F_n(0) + \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} \phi_{(l,m,h)} X_l(s) X_m(s) X_h(s)\, ds,$$

where the F_n are the tails of X and $F_n(0) < \infty$ by hypothesis. Thus, by the definition of d_n (Definition 3.2),

$$d_n^2(t) \leq F_n(0) + \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} \phi_{(l,m,h)} X_l(s) X_m(s) X_h(s)\, ds.$$

Recall that $\alpha + \beta \geq \frac{1}{2}d + 1$, hence the bound (2-6) for ϕ yields $\phi_{(l,m,h)} \leq c_2 \lambda^{\min\{l,m,h\}}$. Therefore

$$d_n^2(t) \leq F_n(0) + \int_0^t \sum_{\substack{(l,m,h)\in I \\ m<n\leq h}} c_2 \lambda^{\min\{l,m\}} |X_l(s) X_m(s) X_h(s)|\, ds.$$

It is convenient to split the set over which the sum is taken into the sets $\{l < m\}$ and $\{m \le l\}$:

$$\begin{aligned}\sum_{\substack{(l,m,h)\in I\\ m<n\le h}} \lambda^{\min\{l,m\}}|X_l X_m X_h| &\le \sum_{\substack{(l,m,h)\in I\\ l<m<n\le h}} \lambda^l |X_l X_m X_h| + \sum_{\substack{(l,m,h)\in I\\ m<n\le h\\ m\le l}} \lambda^m |X_l X_m X_h| \\ &\le \sum_{\substack{(l,m,h)\in I\\ l<m<n\le h}} \lambda^l |X_l X_m X_h| + \sum_{\substack{(l,m,h)\in I\\ l<n\le h\\ l\le m}} \lambda^l |X_l X_m X_h| \\ &\le 2 \sum_{\substack{(l,m,h)\in I\\ l<n\le h\\ l\le m}} \lambda^l |X_l X_m X_h| \le 2\sum_{l=0}^{n-1} \lambda^l \bar{d}_l \sum_{h\ge n} \sum_{m=h-2}^{h+2} |X_m X_h|.\end{aligned}$$

Apply the Cauchy–Schwarz inequality to get

$$2\sum_{h\ge n} \sum_{m=h-2}^{h+2} |X_h X_m| \le \sum_{h\ge n} \sum_{m=h-2}^{h+2} (X_h^2 + X_m^2) \le 10 \sum_{m\ge n-2} X_m^2.$$

Then by the bound on χ in (2-6), on all $[0, t]$,

$$\sum_{m\ge n-2} X_m^2 \le c_1^{-1} \sum_{m\ge n-2} \frac{g(2^{m+1})}{\lambda^m} \chi_m X_m^2.$$

Finally the integral of $\chi_m X_m^2$ can be bounded as follows, since $F_m(t)$ is nonincreasing with respect to m:

$$d_m^2(t) - d_{m+1}^2(t) = F_m(t) - F_{m+1}(t) + \int_0^t \chi_m(s) X_m^2(s)\, ds \ge \int_0^t \chi_m(s) X_m^2(s)\, ds.$$

Putting it all together we obtain

$$d_n^2(t) \le F_n(0) + 10\frac{c_2}{c_1} \sum_{l=0}^{n-1} \frac{\bar{d}_l}{\lambda^{-l}} \sum_{m\ge n-2} \frac{g(2^{m+1})}{\lambda^m} (d_m^2(t) - d_{m+1}^2(t)),$$

thus proving (3-2) with $c_4 = 10c_2/c_1$. □

Lemma 3.4. *Let X be a shell solution; then, for all $n \in \mathbb{N}_0 \setminus \{0\}$ and $s \in [0, t]$,*

$$\sum_{\substack{(l,m,h)\in I\\ h\le n-1}} \phi_{(l,m,h)} X_l X_m X_h = - \sum_{\substack{(l,m,h)\in I\\ m\le n-1<h}} \phi_{(l,m,h)} X_l X_m X_h. \tag{3-3}$$

Proof. By using (2-6) and noticing that $\min(l, m, h) \le n - 1$, we see that by the definition of shell solutions (Definition 2.4) the left-hand side of (3-3) is an absolutely convergent sum. Therefore we can exploit the cancellations due to the antisymmetry of ϕ, as in Remark 2.7. Indeed

$$\sum_{\substack{(l,m,h)\in I\\ h\le n-1}} \phi_{(l,m,h)} X_l X_m X_h = \sum_{\substack{(l,m,h)\in I\\ m<h\le n-1}} \phi_{(l,m,h)} X_l X_m X_h + \sum_{\substack{(l,m,h)\in I\\ h\le n-1\\ m>h}} \phi_{(l,m,h)} X_l X_m X_h \tag{3-4}$$

and

$$\sum_{\substack{(l,m,h)\in I\\ h\le n-1\\ m>h}} \phi_{(l,m,h)}X_lX_mX_h = -\sum_{\substack{(l,m,h)\in I\\ h\le n-1\\ m>h}} \phi_{(l,h,m)}X_lX_mX_h = -\sum_{\substack{(l,h',m')\in I\\ m'\le n-1\\ h'>m'}} \phi_{(l,m',h')}X_lX_{m'}X_{h'}$$
$$= -\sum_{\substack{(l,m',h')\in I\\ m'\le n-1\\ m'<h'}} \phi_{(l,m',h')}X_lX_{m'}X_{h'}. \tag{3-5}$$

By substituting (3-5) into (3-4) the conclusion follows. □

4. Solving the recursion

In this section we complete the proof of our main result. In the previous section we have shown a recursive inequality involving the energy bounds of a shell solution. The following theorem shows that shell solutions are smooth. By Theorem 2.8, the shell approximation of a solution of (1-6) is a shell solution; hence Theorem 2.3 holds, and in turn Theorem 1.2 holds as well.

Theorem 4.1. *Let X be a shell solution satisfying the energy inequality on $[0,t)$. If $\sup_n 2^{mn}|X_n(0)|<\infty$ for every $m\ge 1$, then*

$$\sup_{s\in[0,t]}\sup_n 2^{mn}|X_n(s)|<\infty \quad \textit{for every } m\ge 1.$$

Let $b_n=g(2^{n+1})^{-1}$, $n\ge 0$; then the assumptions of Theorem 1.2 for g, in terms of the sequence b, are

- $(b_n)_{n\in\mathbb{N}_0}$ is nonincreasing,
- $(\lambda^n b_n)_{n\in\mathbb{N}_0}$ is nondecreasing, and
- $\sum_n b_n=\infty$.

Let X be a shell solution as in the statement of Theorem 4.1, denote by $(d_n)_{n\in\mathbb{N}_0}$ and $(F_n)_{n\in\mathbb{N}_0}$ the energy bound and the tail of X (see Definition 3.2), and set $\bar d_n=\sup_{[0,t]}d_n(t)$ for every n. Set

$$Q_n=\sum_{j=0}^{n-1}\frac{\bar d_j}{\lambda^{n-j}} \quad\text{and}\quad R_n(t)=\sum_{j\ge n}\frac{d_j(t)^2-d_{j+1}(t)^2}{\lambda^{j-n}b_j},$$

where $\lambda=2^\alpha$ as in the previous section. We recall that, by Proposition 3.3,

$$d_n(t)^2\le F_n(0)+c_4Q_nR_{n-2}(t). \tag{4-1}$$

We now collect some properties of the quantities R_n, Q_n, $\bar d_n$ that will be crucial in the proof of Theorem 4.1.

Lemma 4.2. (1) *For every $1\le m_1\le m_2$ and $t>0$,*

$$\min\{R_{m_1}(t),R_{m_1+1}(t),\dots,R_{m_2}(t)\}\le\frac{\lambda}{\lambda-1}\frac{d_{m_1}(t)^2}{\sum_{n=m_1}^{m_2}b_n}. \tag{4-2}$$

(2) *For every $t>0$, $\liminf_n R_n(t)=0$.*

(3) $\bar{d}_n \downarrow 0$ *as* $n \to \infty$.

(4) $Q_n \to 0$ *as* $n \to \infty$.

(5) $(Q_n)_{n\geq 1}$ *is eventually nonincreasing.*

Proof. Since $\lambda^n b_n$ is nondecreasing, we know that $b_n - \lambda^{-1} b_{n-1} \geq 0$. Hence, by exchanging the sums,

$$\sum_{n=m_1}^{\infty} (b_n - \lambda^{-1} b_{n-1}) R_n(t) = \sum_{k=m_1}^{\infty} \frac{d_k(t)^2 - d_{k+1}(t)^2}{\lambda^k b_k} \sum_{n=m_1}^{k} \left(\lambda^n b_n - \lambda^{n-1} b_{n-1}\right) \leq \sum_{k=m_1}^{\infty} (d_k(t)^2 - d_{k+1}(t)^2) \leq d_{m_1}(t)^2.$$

If $m_2 \geq m_1$, since $(b_n)_{n\geq 1}$ is nonincreasing,

$$\sum_{n=m_1}^{m_2} (b_n - \lambda^{-1} b_{n-1}) R_n(t) \geq \min\{R_{m_1}(t), \ldots, R_{m_2}(t)\} \sum_{n=m_1}^{m_2} (b_n - \lambda^{-1} b_{n-1}) \geq \frac{\lambda - 1}{\lambda} \left(\sum_{n=m_1}^{m_2} b_n \right) \min\{R_{m_1}(t), \ldots, R_{m_2}(t)\}.$$

The claim $\liminf_n R_n(t) = 0$ follows from (4-2), since $d_n(t) \leq d_1(t)$ for every n, and since, by the assumptions on $(b_n)_{n\geq 1}$, we can find a sequence $(m_k)_{k\geq 1}$ such that $\sum_{n=m_k}^{m_{k+1}-1} b_n \uparrow \infty$.

To prove that $\bar{d}_n \downarrow 0$, we notice that the sequence $(m_k)_{k\geq 1}$ mentioned above does not depend on t; hence, using the monotonicity of $(d_n(t))_{n\geq 1}$ and formula (4-2), we can prove that $\liminf_n \bar{d}_n = 0$, and hence $\bar{d}_n \downarrow 0$ by monotonicity. Once we know that $\bar{d}_n \downarrow 0$, an easy and standard argument proves that $Q_n \to 0$.

To prove that $(Q_n)_{n\geq 1}$ is eventually nonincreasing, we notice that, since $(\bar{d}_n)_{n\geq 1}$ is nonincreasing,

$$(Q_{n+1} - Q_n) = \frac{1}{\lambda}(Q_n - Q_{n-1}) + \frac{1}{\lambda}(\bar{d}_n - \bar{d}_{n-1}) \leq \frac{1}{\lambda}(Q_n - Q_{n-1}).$$

In view of the above inequality, it is sufficient to show that for some m the difference $Q_m - Q_{m-1}$ is nonpositive. This is true because otherwise the sequence $(Q_n)_{n\geq 1}$ would be nondecreasing, in contradiction with $Q_n \to 0$ and $Q_n \geq 0$. □

Given $\theta > 0$ and $n_0 \geq 1$, define by recursion the sequence

$$n_{k+1} = 2 + \min\left\{ n \geq n_k - 1 : \sum_{j=n_k-1}^{n} b_j \geq \theta \lambda^{-k/4} \right\}. \tag{4-3}$$

The definition of Q_n and the fact that the sequence $(\bar{d}_n)_{n\geq 1}$ is nonincreasing yield the following recursive formula for Q_{n_k}:

$$Q_{n_{k+1}} = \frac{1}{\lambda^{n_{k+1}-n_k}} Q_{n_k} + \sum_{j=n_k}^{n_{k+1}-1} \frac{\bar{d}_j}{\lambda^{n_{k+1}-j}} \leq \frac{1}{\lambda} Q_{n_k} + c\bar{d}_{n_k}, \tag{4-4}$$

for a constant $c > 0$ depending only on λ. Moreover, if we choose n_0 large enough that $(Q_n)_{n\geq 0}$ is nonincreasing,

$$d_{n_{k+1}}(t)^2 \leq d_n(t)^2 \leq F_n(0) + c_4 Q_n R_{n-2}(t) \leq F_{n_k}(0) + c_4 Q_{n_k} R_{n-2}(t)$$

for each $n \in \{n_k + 1, \dots, n_{k+1}\}$; hence, by formula (4-2) and the definition of the sequence $(n_k)_{k\geq 1}$,

$$\begin{aligned} d_{n_{k+1}}(t)^2 &\leq F_{n_k}(0) + c_4 Q_{n_k} \min\{R_{n_k-1}, \dots, R_{n_{k+1}-2}\} \\ &\leq F_{n_k}(0) + c Q_{n_k} \frac{d_{n_k-1}(t)^2}{\sum_{n_k-1}^{n_{k+1}-2} b_j} \leq F_{n_k}(0) + c \frac{\lambda^{k/4}}{\theta} Q_{n_k} d_{n_k-1}(t)^2 \end{aligned}$$

and, in conclusion,

$$\bar{d}^2_{n_{k+1}} \leq F_{n_k}(0) + c \frac{\lambda^{k/4}}{\theta} Q_{n_k} \bar{d}^2_{n_k-1}. \tag{4-5}$$

Lemma 4.3 (initial step of the cascade). *Given $M > 0$, there are $n_0 \geq 1$ and $\theta > 0$ such that*

$$Q_{n_k} \leq \lambda^{-k/2} \quad \textit{and} \quad \bar{d}^2_{n_k} \leq \lambda^{-Mk},$$

for all $k \geq 0$.

Proof. Without loss of generality we can choose M large (depending only on the value of λ; see the end of the proof). Choose n_0 large enough that $(Q_n)_{n\geq n_0}$ is nonincreasing and

$$Q_{n_0-i} \leq \epsilon, \quad \bar{d}_{n_0-i} \leq \epsilon, \quad i = 0, 1, \qquad \text{and} \qquad \lambda^{Mn} F_n(0) \leq \epsilon, \quad n \geq n_0,$$

for a number $\epsilon \in (0, 1)$ suitably chosen below. We will prove by induction that

$$Q_{n_{k-i}} \leq \lambda^{-(k-i)/2}, \quad \bar{d}^2_{n_{k-i}} \leq \lambda^{-M(k-i)}, \quad i = 0, 1, \ k \geq 1. \tag{4-6}$$

For the initial step of the induction ($k = 1$), we notice that, by (4-4) and (4-5),

$$\begin{aligned} Q_{n_1} &\leq \frac{1}{\lambda} Q_{n_0} + c\bar{d}_{n_0} \leq \frac{\epsilon}{\lambda} + c\epsilon \leq \frac{1}{\lambda^{1/2}}, \\ \bar{d}^2_{n_1} &\leq F_{n_0}(0) + \frac{c}{\theta} Q_{n_0} \bar{d}^2_{n_0-1} \leq \epsilon + \frac{c}{\theta}\epsilon^3 \leq \lambda^{-M}, \end{aligned}$$

if we choose ϵ small enough, depending on the values of λ, M and θ.

Assume now that (4-6) holds for some $k \geq 1$, and let us prove that the same holds for $k+1$. To this end it is sufficient to give the estimate for $Q_{n_{k+1}}$ and $\bar{d}^2_{n_{k+1}}$. Again by (4-4), (4-5) and the induction hypothesis, and since $(n_k)_{k\geq 0}$ is increasing by definition,

$$\begin{aligned} Q_{n_{k+1}} &\leq \frac{1}{\lambda} Q_{n_k} + c\bar{d}_{n_k} \leq \lambda^{-k/2-1} + c\lambda^{-Mk/2} \leq \lambda^{-(k+1)/2}, \\ \bar{d}^2_{n_{k+1}} &\leq F_{n_k}(0) + c\frac{\lambda^{k/4}}{\theta} Q_{n_k} \bar{d}^2_{n_k-1} \leq \epsilon\lambda^{-Mk} + \frac{c}{\theta}\lambda^{-k/4}\lambda^{-M(k-1)} \leq \lambda^{-M(k+1)}, \end{aligned}$$

if M is large (depending on λ), and ϵ is small and θ is large (depending only on M, λ). □

Before giving the last step of the proof of Theorem 4.1, we show a property of the sequence $(n_k)_{k\geq 0}$. The proof is the same as [Barbato et al. 2014, Lemma 11]; we give the details for completeness.

Lemma 4.4. *Given $n_0 \geq 1$ and $\theta > 0$, consider the sequence defined in (4-3). For infinitely many k, $n_{k+1} = n_k + 1$. In particular, $b_{n_k-1} \geq \theta\lambda^{-k/4}$ for all such k.*

Proof. Assume by contradiction that there is r such that $n_{k+1} \geq n_k + 2$ for $k \geq r$. On the one hand

$$\sum_{j=n_k-1}^{n_{k+1}-3} b_j \leq \theta\lambda^{-k/4},$$

and summing up over $k \geq r$ yields

$$\sum_{k\geq r}\sum_{j=n_k-1}^{n_{k+1}-3} b_j < \infty \quad \Longrightarrow \quad \sum_k b_{n_k-2} = \infty.$$

On the other hand, $b_{n_k-2} \leq b_{n_k-3} \leq \theta\lambda^{-(k-1)/4}$ and the series $\sum_k b_{n_k-2}$ converges. □

Lemma 4.5 (cascade recursion). *For every $M > 0$ there is $c_M > 0$ such that*

$$\bar{d}_n^2 \leq c_M\lambda^{-Mn}, \qquad Q_n \leq c_M\lambda^{-n}.$$

Proof. There is no loss of generality if we assume M is large. Let n_0, θ be the values provided by Lemma 4.3. By Lemma 4.3 and Lemma 4.4 there are infinitely many $k \geq 1$ such that

$$b_{n_k-1} \geq \theta\lambda^{-k/4}, \qquad Q_{n_k} \leq \lambda^{-k/2}, \qquad \bar{d}_{n_k}^2 \leq \lambda^{-Mk}. \tag{4-7}$$

Let k_0 be one such index, taken sufficiently large (the size of k_0 will be chosen at the end of the proof). We will prove by induction that

$$\bar{d}_{n_{k_0}+m}^2 \leq c\lambda^{-Mm}, \qquad Q_{n_{k_0}+m} \leq c'\lambda^{-m}, \qquad b_{n_{k_0}-1+m} \geq \theta\lambda^{-k_0/4-m}, \tag{4-8}$$

for a suitable choice of the constants $c > 0$, $c' > 0$. We first notice that there is nothing to prove concerning $b_{n_{k_0}-1+m}$, since this is a straightforward consequence of the choice of k_0 and the monotonicity of $(\lambda^n b_n)_{n\geq 1}$.

The initial step $m = 0$ holds, since the inequalities in (4-7) hold for the index k_0. For $m = 1$,

$$\bar{d}_{n_{k_0}+1}^2 \leq \bar{d}_{n_{k_0}}^2 \leq c\lambda^{-M},$$
$$Q_{n_{k_0}+1} = \frac{1}{\lambda}Q_{n_{k_0}} + \frac{1}{\lambda}\bar{d}_{n_{k_0}} \leq \frac{1}{\lambda}(\lambda^{-k_0/2} + \lambda^{-Mk_0/2}) \leq \frac{c'}{\lambda},$$

if $c = \lambda^{-M(k_0-1)}$ and $c' \geq \lambda^{-k_0/2} + \lambda^{-Mk_0/2}$.

Assume that (4-8) holds for $1, \dots, m$, for some $m \geq 1$. By definition,

$$\begin{aligned} Q_{n_{k_0}+m+1} = Q_{n_{k_0}}\lambda^{-(m+1)} + \sum_{j=n_{k_0}}^{n_{k_0}+m} \frac{\bar{d}_j}{\lambda^{n_{k_0}+m+1-j}} &\leq \lambda^{-k_0/2-(m+1)} + \sqrt{c}\lambda^{-(m+1)}\sum_{j=0}^{m}\lambda^{-(M/2-1)j} \\ &\leq \left(\lambda^{-k_0/2} + \frac{\lambda}{\lambda-1}\sqrt{c}\right)\lambda^{-(m+1)} \\ &\leq c'\lambda^{-(m+1)} \end{aligned}$$

if $c' = \lambda^{-k_0/2} + \lambda(\lambda-1)^{-1}\sqrt{c}$ (the previous constraint on c' is satisfied by this choice).

By (4-1) and (4-2) we have that, for every $n \geq 2$,

$$d_{n+1}(t)^2 \leq F_{n+1}(0) + c_4 Q_{n+1} R_{n-1}(t) \leq F_{n+1}(0) + c_4 Q_{n+1} \frac{\bar{d}_{n-1}^2}{b_{n-1}};$$

hence, using the inequality for $Q_{n_{k_0}+m+1}$ already proved and the induction hypothesis,

$$\begin{aligned}\bar{d}^2_{n_{k_0}+m+1} &\leq F_{n_{k_0}+m+1}(0) + c_4 Q_{n_{k_0+m+1}} \frac{\bar{d}^2_{n_{k_0}+m-1}}{b_{n_{k_0}+m-1}} \\ &\leq c\lambda^{-M(m+1)}\Big(\lambda^{M(n_{k_0}+m+1)} F_{n_{k_0}+m+1}(0) + \frac{c_4}{\theta} c' \lambda^{2M+k_0/4}\Big) \\ &\leq c 2^{-M(m+1)},\end{aligned}$$

where the last inequality follows if k_0 is large enough since $\lambda^n F_n(0) \to 0$ by assumption, and by our choice of c, c' we have that $\lambda^{k_0/4} c' \to 0$ as $k_0 \to \infty$. □

Appendix A: Local existence and uniqueness

Consider the generalized system (1-6), under the same assumptions of Theorem 1.2. Assume[3] for simplicity that $m_1(k) = |k|^\alpha / g(|k|)$. Denote by V_m the subspace of H^m (see (2-3)) of divergence-free vector fields with mean zero. Our main theorem on local existence and uniqueness for (1-6) is as follows:

Theorem A.1. *Let $m \geq 2 + \frac{1}{2}d$ and $v_0 \in V_m$. Then there are $T > 0$ and a unique solution v of (1-6) on $[0, T]$ with initial condition v_0 such that*

$$v \in L^\infty([0, T]; V_m) \cap \mathrm{Lip}([0, T]; V_{m-\alpha}) \cap C([0, T]; V_m^{\mathrm{weak}}), \qquad \int_0^T \|D_1^{1/2} v\|_m^2 \, dt < \infty, \tag{A-1}$$

where V_m^{weak} is the space V_m with the weak topology. Moreover, v is right-continuous with values in V_m for the strong topology.

If $T_\star$ is the maximal time of existence of the solution starting from v_0, then either $T_\star = \infty$ or

$$\limsup_{t \uparrow T_\star} \|v(t)\|_m = \infty.$$

The proof of the theorem is based on a proof of existence of a local unique solution for the Euler equation taken from [Majda and Bertozzi 2002, Section 3.2]. The idea is that we cannot use the D_1 operator as a replacement for the Laplacian, since in general D_1 may not have smoothing properties (indeed, it is easy to adapt the counterexample in [Barbato et al. 2014, Remark 15] to D_1 on $\mathbb{R}^d$ or on the d-dimensional torus). Likewise we do not use any smoothing properties of D_2, so that our proof includes the case $\beta = 0$. The result is by no means optimal, but fits the needs of our paper.

[3]Existence and uniqueness can be proved also in the general case $m_1(k) \geq |k|^\alpha g(|k|)^{-1}$. A simple assumption that keeps our proof almost unchanged is a control from above, say $m(k) \leq |k|^\beta$ for some $\beta \geq \alpha$.

We work on the torus $[0, 2\pi]^d$, although the proof, essentially unchanged, works in $\mathbb{R}^d$. Denote by H the projection of $L^2([0, 2\pi]^d)$ onto divergence-free vector fields, and, for every $s > 0$, denote by V_s the projection of the Sobolev space $H^s([0, 2\pi]^d)$ onto divergence-free vector fields. We will denote by $\|\cdot\|_H$ and $\langle\cdot,\cdot\rangle_H$ the norm and the scalar product in H, and by $\|\cdot\|_s$ and $\langle\cdot,\cdot\rangle_s$ the norm and the scalar product in V_s.

We denote by $\widehat{B}(v_1, v_2)$ the (Leray) projection of the nonlinearity, namely

$$\widehat{B}(v_1, v_2) = \Pi_{\text{Leray}}[(D_2^{-1} v_1 \cdot \nabla)v_2].$$

Since $\beta \geq 0$, $\|D_2^{-1} v\|_s \leq \|v\|_s$ for every $s \in \mathbb{R}$. Hence (see for instance [Kato 1972] or [Constantin and Foiaş 1988]), for every $m \geq 1 + [d/2]$, there exists $c_m > 0$ such that

$$\|\widehat{B}(v_1, v_2)\|_m \leq c_m \|v_1\|_m \|v_2\|_{m+1},$$
$$\langle \widehat{B}(v_1, v_2), v_2\rangle_m \leq c_m \|v_1\|_m \|v_2\|_m^2.$$

In the rest of the section we briefly outline the proof of Theorem A.1, following [Majda and Bertozzi 2002, Section 3.2]. The proof of the following result is a slight modification of the arguments to prove [Majda and Bertozzi 2002, Theorem 3.4].

Proposition A.2. *Given an integer $m \geq 2 + d/2$, there exists a number $c_\star > 0$ such that for every $v_0 \in V_m$, if $T < c_\star/\|v_0\|_m$, there is a unique solution of (1-6) with initial condition v_0. Moreover, $v_\epsilon \to v$ in $C([0, T]; V_{m'})$ for $m' < m$ and in $C([0, T]; V_m^{\text{weak}})$, the inequalities in (A-1) hold for v, and for any $\epsilon > 0$,*

$$\sup_{[0,T]} \|v_\epsilon\|_m \leq \frac{\|v_0\|_m}{1 - c_\star T \|v_0\|_m}. \tag{A-2}$$

Unfortunately, at this stage, we cannot prove the analog of [Majda and Bertozzi 2002, Theorem 3.5] for our v, namely that v is continuous in time for the strong topology of V_m. The reason is that their proof uses either the reversibility of the Euler equation (which we do not have due to the presence of D_1), or the smoothing of the Laplace operator, which we do not have here either (as already mentioned). On the other hand, we can prove right-continuity:

Lemma A.3. *The solution v from Proposition A.2 is right-continuous with values in V_m for the strong topology, and dv/dt is right-continuous with values in $V_{m-\alpha}$.*

Proof. Given $t \in [0, T]$, the same computations leading to (A-2) yield

$$\sup_{[0,t]} \|v(s)\|_m \leq \|v_0\|_m + \frac{c_\star t \|v_0\|_m^2}{1 - c_\star t \|v_0\|_m};$$

therefore $\limsup_{t\downarrow 0} \|v(t)\|_m \leq \|v_0\|_m$. On the other hand, by weak continuity, $\|v_0\|_m \leq \liminf_{t\downarrow 0} \|v(t)\|_m$ and v is right-continuous at 0. Uniqueness for (1-6) and the same argument applied to $t \in (0, T]$ yield right-continuity in t. □

Nevertheless, we can still define a maximal solution and a maximal time of existence. Given $v_0 \in V_m$, let $T_\star$ be the maximal time of existence of the solution starting from v_0, that is the supremum over all $T > 0$ such that there exists a solution v of (1-6) on $[0, T]$ with $v(0) = u_0$, v right-continuous with values in V_m, continuous with values in V_m^{weak} and with dv/dt right-continuous with values in $V_{m-\alpha}$. Due to uniqueness, any two such solutions coincide on the common interval of definition.

Proposition A.4. *Given $v_0 \in V_m$, if $T_\star$ is the maximal time of existence of the solution starting from v_0, then either $T_\star = \infty$ or*

$$\limsup_{t \uparrow T_\star} \|v(t)\|_m = \infty.$$

Proof. Assume by contradiction that $T_\star < \infty$ and that $M := \sup_{t<T_\star} \|v(t)\|_m < \infty$. Let $T_0 = T_\star - c_\star/(4M)$, and start a solution with initial condition $v(T_0)$ at time T_0. By Proposition A.2 there is a solution of (1-6) on a time span of length at least $c_\star/(2\|v(T_0)\|_m) \geq c_\star/(2M)$, hence at least up to time $T_0 + c_\star/(2M) > T_\star$. By uniqueness, this solution is equal to v up to time $T_\star$. □

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DAVID BARBATO: barbato@math.unipd.it
Dipartimento di Matematica, Università di Padova, Via Trieste 63, I-35121 Padova, Italy

FRANCESCO MORANDIN: francesco.morandin@unipr.it
Dipartimento di Matematica e Informatica, Università di Parma, Parco Area delle Scienze, 53/A, I-43121 Parma, Italy

MARCO ROMITO: romito@dm.unipi.it
Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, I-56127 Pisa, Italy

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