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DYNAMICS OF COMPLEX-VALUED MODIFIED KDV SOLITONS WITH APPLICATIONS TO THE STABILITY OF BREATHERS

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We study the long-time dynamics of complex-valued modified Korteweg–de Vries (mKdV) solitons, which are distinguished because they blow up in finite time. We establish stability properties at the H^1 level of regularity, uniformly away from each blow-up point. These new properties are used to prove that mKdV breathers are H^1-stable, improving our previous result [*Comm. Math. Phys.* **324**:1 (2013) 233–262], where we only proved H^2-stability. The main new ingredient of the proof is the use of a Bäcklund transformation which relates the behavior of breathers, complex-valued solitons and small real-valued solutions of the mKdV equation. We also prove that negative energy breathers are asymptotically stable. Since we do not use any method relying on the inverse scattering transform, our proof works even under $L^2(\mathbb{R})$ perturbations, provided a corresponding local well-posedness theory is available.

1. Introduction

Consider the modified Korteweg–de Vries (mKdV) equation on the real line

$$u_t + (u_{xx} + u^3)_x = 0, \tag{1-1}$$

where $u = u(t, x)$ is a complex-valued function and $(t, x) \in \mathbb{R}^2$. Note that (1-1) is not $U(1)$-invariant. In the case of real-valued initial data, the associated Cauchy problem for (1-1) is globally well posed for initial data in $H^s(\mathbb{R})$ for any $s > \frac{1}{4}$; see Kenig, Ponce and Vega [Kenig et al. 1993], and Colliander, Keel, Staffilani, Takaoka and Tao [Colliander et al. 2003]. Additionally, the (real-valued) flow map is not

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uniformly continuous if $s < \frac{1}{4}$ [Kenig et al. 2001].[1] In order to prove this last result, Kenig, Ponce and Vega considered a very particular class of solutions of (1-1) called *breathers*, discovered by Wadati [1973].

Definition 1.1 (see, e.g., [Wadati 1973; Lamb 1980]). Let $\alpha, \beta > 0$ and $x_1, x_2 \in \mathbb{R}$ be fixed parameters. The mKdV breather is a smooth solution of (1-1) given explicitly by the formula

$$\begin{aligned} B = B(t,x;\alpha,\beta,x_1,x_2) &:= 2\sqrt{2}\partial_x\left[\arctan\frac{\beta\sin(\alpha y_1)}{\alpha\cosh(\beta y_2)}\right], \\ &= \frac{2\sqrt{2}\alpha\beta(\alpha\cos(\alpha y_1)\cosh(\beta y_2) - \beta\sin(\alpha y_1)\sinh(\beta y_2))}{\alpha^2\cosh^2(\beta y_2) + \beta^2\sin^2(\alpha y_1)}, \end{aligned} \tag{1-2}$$

where

$$y_1 := x + \delta t + x_1, \quad y_2 := x + \gamma t + x_2, \tag{1-3}$$

and

$$\delta := \alpha^2 - 3\beta^2, \quad \gamma := 3\alpha^2 - \beta^2. \tag{1-4}$$

Breathers are *oscillatory bound states*. They are periodic in time (after a suitable space shift) and localized in space. The parameters α and β are scaling parameters, x_1, x_2 are shifts, and $-\gamma$ represents the *velocity* of a breather. As we will see later, the main difference between *solitons*[2] and breathers is given at the level of the *oscillatory* scaling α, which is not present in the case of solitons. For a detailed account of the physics of breathers, see, e.g., [Lamb 1980; Ablowitz and Clarkson 1991; Aubry 1997; Alejo 2012; Alejo and Muñoz 2013] and references therein.

Numerical computations (see Gorria, Alejo and Vega [Gorria et al. 2013]) showed that breathers are *numerically* stable. Next, in [Alejo and Muñoz 2013] we constructed a Lyapunov functional that controls the dynamics of H^2 perturbations of (1-2). The purpose of this paper is to improve this previous result and show that mKdV breathers are indeed H^1-stable, i.e., stable in the energy space.

Theorem 1.2. *Let $\alpha, \beta > 0$ be fixed scalings. There exist parameters η_0, A_0, depending on α and β only, such that the following holds: Consider $u_0 \in H^1(\mathbb{R})$, and assume that there exists $\eta \in (0, \eta_0)$ such that*

$$\|u_0 - B(0,\cdot;\alpha,\beta,0,0)\|_{H^1(\mathbb{R})} \le \eta. \tag{1-5}$$

Then there exist functions $x_1(t), x_2(t) \in \mathbb{R}$ such that the solution $u(t)$ of the Cauchy problem for the mKdV equation (1-1) *with initial data u_0 satisfies*

$$\sup_{t\in\mathbb{R}} \|u(t) - B(t,\cdot;\alpha,\beta,x_1(t),x_2(t))\|_{H^1(\mathbb{R})} \le A_0\eta, \tag{1-6}$$

$$\sup_{t\in\mathbb{R}} |x_1'(t)| + |x_2'(t)| \le CA_0\eta, \tag{1-7}$$

for some constant $C > 0$.

[1] However, one can construct a solution in L^2; see [Christ et al. 2012].

[2] See (1-8).

The initial condition (1-5) can be replaced by any initial breather profile of the form $B(t_0; \alpha, \beta, x_1^0, x_2^0)$ with t_0, x_1^0, $x_2^0 \in \mathbb{R}$, thanks to the invariance of the equation under translations in time and space.[3] Moreover, using the Miura transform [Miura et al. 1968], one can prove a natural stability property in $L^2(\mathbb{R}; \mathbb{C})$ for an associated *complex-valued* KdV breather.

One can also use the scaling invariance of the equation, $u(t, x) \mapsto \lambda u(\lambda^3 t, \lambda x)$, to reduce the problem to the case where α equals 1 and $\beta > 0$ is arbitrary, but for symmetry reasons we shall not follow this approach.[4]

Additionally, from the proof, the shifts $x_1(t)$ and $x_2(t)$ in Theorem 1.2 can be described almost explicitly[5], which is a substantial improvement with respect to [Alejo and Muñoz 2013], where no exact control on the shift parameters was given. We obtain such a control with no additional decay assumptions on the initial data other than being in $H^1(\mathbb{R})$.

Theorem 1.2 places breathers as stable objects at the same level of regularity as mKdV solitons, even if they are very different in nature. To be more precise, a (real-valued) soliton is a solution of (1-1) of the form

$$u(t, x) = Q_c(x - ct), \quad Q_c(s) := \sqrt{c}\, Q(\sqrt{c} s), \quad c > 0, \tag{1-8}$$

with

$$Q(s) := \frac{\sqrt{2}}{\cosh(s)} = 2\sqrt{2}\partial_s[\arctan(e^s)],$$

and where $Q_c > 0$ satisfies the nonlinear ODE

$$Q_c'' - cQ_c + Q_c^3 = 0, \quad Q_c \in H^1(\mathbb{R}). \tag{1-9}$$

We recall that solitons are H^1-stable (Benjamin [1972], Bona, Souganidis and Strauss [Bona et al. 1987]). See also the works by Grillakis, Shatah and Strauss [Grillakis et al. 1987] and Weinstein [1986] for the nonlinear Schrödinger case.

Even more surprising is the fact that Theorem 1.2 will arise as a consequence of a suitable stability property of the zero solution and of *complex-valued* mKdV solitons, which are singular solutions.

A complex-valued soliton is a solution of the form (1-8) of (1-1) with a complex-valued scaling and velocity, i.e.,

$$u(t, x) := Q_c(x - ct), \quad \sqrt{c} := \beta + i\alpha, \quad \alpha,\ \beta > 0; \tag{1-10}$$

see Definition 2.1 for a precise interpretation. In Lemma 2.2 we give a detailed description of the singular nature of (1-10). On the other hand, very little is known about mKdV (1-1) when the initial data is complex-valued. For instance, it is known that it has finite-time blow-up solutions, the most important

[3] Indeed, if $u(t, x)$ solves (1-1), then, for any $t_0, x_0 \in \mathbb{R}$ and $c > 0$, $u(t - t_0, x - x_0)$, $c^{1/2}u(c^{3/2}t, c^{1/2}x)$, $u(-t, -x)$ and $-u(t, x)$ are solutions of (1-1).

[4] For example, if (1-6) holds, then $v_0(y) := u_0(y/\alpha)/\alpha$ satisfies

$$\alpha \int_{\mathbb{R}} \left(v_0 - B\left(0, \cdot\,; 1, \frac{\beta}{\alpha}, 0, 0\right)\right)^2 = \int_{\mathbb{R}} (u_0 - B(0, \cdot\,; \alpha, \beta, 0, 0))^2 \le \eta^2.$$

[5] See (7-9).

examples being the complex solitons themselves; see, e.g., Bona, Vento and Weissler [Bona et al. 2013] and references therein for more details. According to [Bona et al. 2013], blow-up in the complex-valued case can be understood as the intersection with the real line $x \in \mathbb{R}$ of a curve of poles of the solution after being extended to the complex plane (i.e., now x is replaced by $z \in \mathbb{C}$). Blow-up in this case seems to have better properties than the corresponding critical blow-up described by Martel and Merle [2002].

Let $H^1(\mathbb{R}; \mathbb{C})$ denote the standard Sobolev space of complex-valued functions $f(x) \in \mathbb{C}$, $x \in \mathbb{R}$. In this paper we prove the following *stability property* for solitons, far away from each blow-up time:

Theorem 1.3. *There exists an open set of initial data in $H^1(\mathbb{R}; \mathbb{C})$ for which the mKdV complex solitons are well-defined and stable in $H^1(\mathbb{R}; \mathbb{C})$ for all times uniformly separated from a countable sequence of finite blow-up times with no limit points. Moreover, one can define a mass and an energy, both invariant* for all time.

We cannot prove an all-time stability result using the $H^1(\mathbb{R}; \mathbb{C})$-norm because even complex solitons leave that space at each blow-up time, and several computations in this paper break down. However, the previous result states that the Cauchy problem is almost globally well-posed around a soliton, and the solution can be continued after (or before) every blow-up time. The novelty with respect to the local Cauchy theory [Kenig et al. 1993] is that now it is possible to define an almost global solution instead of defining a local solution on each subinterval of time defined by two blow-up points, because from the proof we will recognize that the behavior before and after the blow-up time are deeply linked. From this property, the existence and invariance of uniquely well-defined mass and energy will be quite natural. For this particular problem, we answer positively the questions about existence, uniqueness and regularity after blow-up posed by Merle [1992]. See Theorem 4.5 and its corollaries for a more detailed statement.

Lastly, we prove that breathers behaving as standard solitons are asymptotically stable in the energy space. For previous results for the soliton and multisoliton case, see Pego and Weinstein [1994] and Martel and Merle [2005].

Theorem 1.4. *Under the hypotheses of Theorem 1.2, there exists $c_0 > 0$ depending on η, with $c_0(\eta) \to 0$ as $\eta \to 0$, such that the following holds: There exist β^* and α^* (depending on η) close enough to β and α, respectively, for which*

$$\lim_{t\to+\infty} \|u(t) - B(t; \cdot, \alpha^*, \beta^*, x_1(t), x_2(t))\|_{H^1(x\geq c_0 t)} = 0. \tag{1-11}$$

In particular, the asymptotic of $u(t)$ has new and explicit velocity parameters $\delta^* = (\alpha^*)^2 - 3(\beta^*)^2$ and $\gamma^* = 3(\alpha^*)^2 - (\beta^*)^2$ at the leading order.

The previous result is more interesting when $\gamma < 0$; see (1-4). In this case, the breather has negative energy (see [Alejo and Muñoz 2013, p. 9]) and it moves rightwards in space (the so-called physically relevant region). We recall that working in the energy space implies that small solitons moving to the right in a very slow fashion are allowed (the condition $c_0 > 0$ is essential; see, e.g., [Martel and Merle 2005]). Indeed, there are explicit solutions of (1-1) composed of one breather and one very small soliton moving rightwards, which contradicts any sort of *global* asymptotic stability result in the energy space [Lamb 1980]. Additionally, we cannot ensure that the left portion of the real line $\{x < 0\}$ corresponds to radiation only. Following [Lamb 1980], it is possible to construct a solution to (1-1) composed of two breathers, one very small with respect to the other one, the latter with positive velocity and the former with small but still negative velocity (just take the corresponding scaling parameters α and β both small so that $-\gamma < 0$). Such a solution has no radiation at infinity. Of course, working in a neighborhood of the breather using weighted spaces rules out such small perturbations.

The mechanism under which α^* and β^* are chosen is very natural and reflects the power and simplicity of the arguments of the proof: under different scaling parameters, it was impossible to describe the dynamics as in Theorem 1.2. We do indeed have two linked results: in some sense Theorem 1.2 is a consequence of Theorem 1.4 and vice versa.

It is also important to emphasize that (1-1) is a well-known *completely integrable* model [Miura et al. 1968; Ablowitz and Clarkson 1991; Lamb 1980; Lax 1968; Schuur 1986], with infinitely many conserved quantities and a suitable Lax pair formulation. The inverse scattering theory has been applied in [Schuur 1986] to describe the evolution of *rapidly decaying* initial data, by purely algebraic methods. Solutions are shown to decompose into a very particular set of solutions: solitons, breathers and radiation. Moreover, as a consequence of the integrability property, these nonlinear modes interact elastically during the dynamics, and no dispersive effects are present at infinity. In particular, even more complex solutions are present, such as *multisolitons* (explicit solutions describing the interaction of several solitons [Hirota 1972]). Multisolitons for mKdV and several integrable models of Korteweg–de Vries-type are stable in H^1; see Maddocks and Sachs [1993] for the KdV case and in a more general setting see Martel, Merle and Tsai [Martel et al. 2002].

However, the proof of Theorem 1.2 does not involve any method relying on the inverse scattering transform [Miura et al. 1968; Schuur 1986], nor the steepest descent machinery [Deift and Zhou 1993],[6] which allows us to work in the *very large* energy space $H^1(\mathbb{R})$. Note that if the inverse scattering methods are allowed, one could describe the dynamics of very general initial data with more detail. But if this is the case, additional decay and/or spectral assumptions are always needed, and, except with well-prepared initial data, such conditions are difficult to verify. We claim that our proof works even if the initial data is in $L^2(\mathbb{R})$ provided mKdV is locally well-posed at that level of regularity, which remains a very difficult open problem.

Comparing with [Alejo and Muñoz 2013], where we have proved that mKdV breathers are H^2-stable, now we are not allowed to use the *third* conservation law associated to mKdV,[7]

$$F[u](t) = \frac{1}{2}\int_{\mathbb{R}} u_{xx}^2(t,x)\,dx - \frac{5}{2}\int_{\mathbb{R}} u^2u_x^2(t,x)\,dx + \frac{1}{4}\int_{\mathbb{R}} u^6(t,x)\,dx,$$

nor the elliptic equation satisfied by *any* breather profile,

$$B_{(4x)} - 2(\beta^2-\alpha^2)(B_{xx}+B^3) + (\alpha^2+\beta^2)^2 B + 5BB_x^2 + 5B^2B_{xx} + \tfrac{3}{2}B^5 = 0$$

[6] Note that Deift and Zhou [1993] consider the *defocusing* mKdV equation, which has no smooth solitons or breathers.

[7] See (4-13) and (4-14) for the other two low-regularity conserved quantities.

since the dynamics is no longer in H^2. Moreover, since breathers are bound states, there is no associated decoupling in the dynamics as time evolves as in [Martel et al. 2002], which makes the proof of the H^1 case even more difficult. We need a different method of proof.

We follow a method of proof that is in the spirit of the seminal work by Merle and Vega [2003] (see also Alejo, Muñoz and Vega [Alejo et al. 2013]), where the L^2-stability of KdV solitons has been proved. In those cases, the use of the Miura and Gardner transformations were the new ingredients to prove stability where the standard energy is missing. Recently, the Miura transformation has been studied at very low regularity by Buckmaster and Koch [2014]; using this information, they showed that KdV solitons are orbitally stable under H^{-1} perturbations leading to a $H^n \cap H^{-3/4}$ solution, where $n \geq -1$ is an integer.

More precisely, we will make use of the Bäcklund transformation [Lamb 1980, p. 257] associated to mKdV to obtain new conserved quantities, additional to the mass and energy. Mizumachi and Pelinovsky [2012] and Hoffmann and Wayne [2013] described a similar approach for the NLS and sine-Gordon equations and their corresponding one-solitons. However, unlike those previous works, and in order to control any breather, we use the Bäcklund transformation twice: one to control an associated complex-valued mKdV soliton, and a second one to get almost complete control of the breather.

Indeed, solving the Bäcklund transformation in the vicinity of a breather leads (formally) to the emergence of complex-valued mKdV solitons, which blow up in finite time. A difficult problem arises at the level of the Cauchy theory, and any attempt to prove stability must face the ill-posedness behavior of the complex-valued mKdV equation (1-1). However, after a new use of the Bäcklund transformation around the complex soliton we end up with a small, real-valued $H^1(\mathbb{R})$ solution of mKdV which is stable for all time. The fact that a second application of the Bäcklund transformation leads to a real-valued solution is not trivial and is a consequence of a deep property called the *permutability theorem* [Lamb 1980]. Roughly speaking, that result states that the order under which we perform two inversions of the Bäcklund transformation does not matter. After some work we are able to give a rigorous proof of the following fact: we can invert a breather using Bäcklund towards two particularly well-chosen complex solitons first, and then invert once again to obtain two small solutions — say a and b — and the final result *must be the same*. Even better, one can show that a has to be the conjugate of b, which gives the real character of the solution. Now, the dynamics is real-valued and simple. We use the Kenig–Ponce–Vega theory [Kenig et al. 1993] to evolve the system to any given time. Using this trick we avoid dealing with the blow-up times of the complex soliton — for a while — and at the same time we prove a new stability result for them.

However, unlike [Mizumachi and Pelinovsky 2012; Hoffman and Wayne 2013], we cannot invert the Bäcklund transformation at any given time, and in fact each blow-up time of the complex-valued mKdV soliton is a dangerous obstacle for the breather stability. In order to extend the stability property up to the blow-up times we discard the method involving the Bäcklund transformation. Instead we run a bootstrap argument starting from a fixed time very close to each singular point, using the fact that the real-valued mKdV dynamics is continuous in time. Finally, using energy methods related to the stability of single solitons we are able to extend the uniform bounds in time to any singularity point, with a universal constant A_0 as in Theorem 1.2.

From the proof it will be evident that, even if there is no global well-posedness theory (with uniform bounds in time) below H^s, $s < \frac{1}{4}$, one can prove stability of breathers in spaces of the form $H^1 \cap H^s$, $s < \frac{1}{4}$, following the ideas of Buckmaster and Koch [2014]. We thank Professor Herbert Koch for mentioning to us this interesting property.

Our results apply without significant modifications to the case of the sine-Gordon (SG) equation in $\mathbb{R}_t \times \mathbb{R}_x$,

$$u_{tt} - u_{xx} + \sin u = 0, \quad (u, u_t)(t, x) \in \mathbb{R}^2, \tag{1-12}$$

and its corresponding breather [Lamb 1980, p. 149]. See [Birnir et al. 1994; Denzler 1993; Soffer and Weinstein 1999] for related results. Note that SG is globally well-posed in $L^2 \times H^{-1}$; then we have that breathers are stable under small perturbations in that space. Since the proofs are very similar, and in order to avoid repetition, we skip the details.

Moreover, following our proof it is possible to give a new proof of the global H^1-stability of two-solitons, first proved in [Martel et al. 2002].

We also claim that k-breathers ($k \geq 2$), namely solutions composed of k different breathers, are H^1-stable. Following the proof of Theorem 1.2, one can show by induction that a k-breather can be obtained from a $(k-1)$-breather after two Bäcklund transformations using a fixed set of complex conjugate parameters, as in Lemmas 2.4 and 5.1. After proving this identity, the rest of the proof adapts with no deep modifications.

This paper is organized as follows: In Section 2 we introduce the complex-valued soliton profiles. Section 3 is devoted to the study of the mKdV Bäcklund transformation in the vicinity of a given complex-valued mKdV solution. In Section 4 we apply the previous results to prove Theorem 1.3 (see Theorem 4.5). Section 5 deals with the relation between complex soliton profiles and breathers. In Section 6 we apply the results from Section 3 to the case of a perturbation of a breather solution. Finally, in Sections 7 and 8 we prove Theorems 1.2 and 1.4.

2. Complex-valued mKdV soliton profiles

Definition 2.1. Consider parameters α, $\beta > 0$, $x_1, x_2 \in \mathbb{R}$. We introduce the localized profile

$$\tilde{Q} = \tilde{Q}(x; \alpha, \beta, x_1, x_2),$$

defined as

$$\tilde{Q} := 2\sqrt{2} \arctan\left(e^{\beta y_2 + i\alpha y_1}\right), \tag{2-1}$$

where y_1 and y_2 are (re)defined as

$$y_1 := x + x_1, \quad y_2 := x + x_2. \tag{2-2}$$

Note that

$$\lim_{x \to -\infty} \tilde{Q}(x) = 0. \tag{2-3}$$

We define the complex-valued soliton profile as follows:

$$
\begin{aligned}
Q &:= \partial_x \tilde{Q} \\
&= \frac{2\sqrt{2}(\beta + i\alpha)e^{\beta y_2 + i\alpha y_1}}{1 + e^{2(\beta y_2 + i\alpha y_1)}} \qquad (2\text{-}4)\\
&= \sqrt{2}\,\frac{\beta \cosh(\beta y_2)\cos(\alpha y_1) + \alpha \sinh(\beta y_2)\sin(\alpha y_1)}{\cosh^2(\beta y_2) - \sin^2(\alpha y_1)} \\
&\qquad + i\sqrt{2}\,\frac{\alpha \cosh(\beta y_2)\cos(\alpha y_1) - \beta \sinh(\beta y_2)\sin(\alpha y_1)}{\cosh^2(\beta y_2) - \sin^2(\alpha y_1)}. \qquad (2\text{-}5)
\end{aligned}
$$

Finally, we write

$$\tilde{Q}_t := -(\beta + i\alpha)^2 Q, \tag{2-6}$$

and

$$\tilde{Q}_1 := \partial_{x_1} \tilde{Q}, \quad \tilde{Q}_2 := \partial_{x_2} \tilde{Q}. \tag{2-7}$$

Note that Q is complex-valued and is pointwise convergent to the soliton Q_{β^2} as $\alpha \to 0$. A second condition satisfied by $\tilde{Q}$ and Q is the following periodicity property: for all $k \in \mathbb{Z}$,

$$\begin{cases} \tilde{Q}(x; \alpha, \beta, x_1 + k\pi/\alpha, x_2) = (-1)^k \tilde{Q}(x; \alpha, \beta, x_1, x_2), \\ Q(x; \alpha, \beta, x_1 + k\pi/\alpha, x_2) = (-1)^k Q(x; \alpha, \beta, x_1, x_2). \end{cases} \tag{2-8}$$

We remark that, in what follows, $\tilde{Q}$ and Q may blow up in finite time.

Lemma 2.2. *Consider the complex-valued soliton profile defined in* (2-1)–(2-5). *Assume that, for x_2 fixed and some $k \in \mathbb{Z}$,*

$$x_1 = x_2 + \frac{\pi}{\alpha}\big(k + \tfrac{1}{2}\big). \tag{2-9}$$

Then $\tilde{Q}$ and Q cannot be defined at $x = -x_2$. Moreover, if $x_1 = x_2 = 0$, then $Q(\,\cdot\,; \alpha, \beta, 0, 0) \in H^1(\mathbb{R}; \mathbb{C})$.

Remark. We emphasize that, given x_2 fixed, the set of points x_1 of the form (2-9) for some $k \in \mathbb{Z}$ is a countable set of real numbers with no limit points.

Remark. The complex-valued function $\arctan z$ (leading to the definition of $\tilde{Q}$) has two branches of discontinuities of the form im with $m \in \mathbb{R}$, $|m| \geq 1$, appearing from the standard branch of the complex logarithm function $\operatorname{Re} z < 0$, $\operatorname{Im} z = 0$. Such discontinuities may induce singularities on the function Q. Fortunately, both Q and functions of the type sine and cosine of arguments of the form $\tilde{Q}$ are smooth except on the points determined by Lemma 2.2. Throughout this paper we shall work with functions of the latest form instead of the original $\tilde{Q}$.

Proof. Fix $x_2 \in \mathbb{R}$. If (2-9) is satisfied for some $k \in \mathbb{Z}$, we have that, at $x = -x_2$,

$$y_1 = x + x_1 = \frac{\pi}{\alpha}\big(k + \tfrac{1}{2}\big), \quad y_2 = x + x_2 = 0,$$

and

$$\sinh(\beta y_2) = 0, \quad \cos(\alpha y_1) = 0. \tag{2-10}$$

Therefore, under (2-9), we have from (2-1) and (2-5) that $\tilde{Q}$ and Q cannot be defined at $x=-x_2$. Finally, if $x_1=x_2=0$, we have

$$k+\tfrac{1}{2}=0, \quad k\in\mathbb{Z},$$

which is impossible. □

Lemma 2.3. *Fix α, $\beta>0$ and $x_1, x_2\in\mathbb{R}$ such that* (2-9) *is not satisfied. Then we have*

$$Q_{xx}-(\beta+i\alpha)^2Q+Q^3=0 \quad \textit{for all } x\in\mathbb{R}, \tag{2-11}$$

and

$$Q_x^2-(\beta+i\alpha)^2Q^2+\tfrac{1}{2}Q^4=0 \quad \textit{for all } x\in\mathbb{R}. \tag{2-12}$$

Moreover, the previous identities can be extended to any $x_1, x_2\in\mathbb{R}$ by continuity.

Proof. This is direct from the definition. □

Assume that (2-9) does not hold. Consider the sine and cosine functions applied to complex numbers. We have, from (2-1) and (2-4),

$$\begin{aligned}\sin\frac{\tilde{Q}}{\sqrt{2}}&=\sin(2\arctan e^{\beta y_2+i\alpha y_1})\\&=2e^{\beta y_2+i\alpha y_1}\cos^2(\arctan e^{\beta y_2+i\alpha y_1})\\&=\frac{2e^{\beta y_2+i\alpha y_1}}{1+e^{2(\beta y_2+i\alpha y_1)}}=\frac{1}{\beta+i\alpha}\frac{Q}{\sqrt{2}}.\end{aligned}\tag{2-13}$$

Similarly, from this identity we have

$$Q_x-(\beta+i\alpha)\cos\left(\frac{\tilde{Q}}{\sqrt{2}}\right)Q=0, \tag{2-14}$$

so that, from (2-6) and (2-12),

$$\tilde{Q}_t+(\beta+i\alpha)\left[Q_x\cos\frac{\tilde{Q}}{\sqrt{2}}+\frac{Q^2}{\sqrt{2}}\sin\frac{\tilde{Q}}{\sqrt{2}}\right]=-(\beta+i\alpha)^2Q+Q_x^2Q^{-1}+\tfrac{1}{2}Q^3=0.$$

So far, we have proved the following result:

Lemma 2.4. *Let Q be a complex-valued soliton profile with scaling parameters α, $\beta>0$ and shifts $x_1, x_2\in\mathbb{R}$ such that* (2-9) *is not satisfied. Then we have*

$$\frac{Q}{\sqrt{2}}-(\beta+i\alpha)\sin\frac{\tilde{Q}}{\sqrt{2}}\equiv 0, \tag{2-15}$$

and

$$\tilde{Q}_t+(\beta+i\alpha)\left[Q_x\cos\frac{\tilde{Q}}{\sqrt{2}}+\frac{Q^2}{\sqrt{2}}\sin\frac{\tilde{Q}}{\sqrt{2}}\right]\equiv 0, \tag{2-16}$$

where $\sin z$ and $\cos z$ are defined on the complex plane in the usual sense.

We finish this section with a simple computational lemma.

Lemma 2.5. *Fix x_1, x_2 such that* (2-9) *is not satisfied. Then, for all α, $\beta > 0$ we have*

$$\mathcal{N} := \frac{1}{2}\int_{-\infty}^{x} Q^2 = \frac{2(\beta+i\alpha)e^{2(\beta y_2+i\alpha y_1)}}{1+e^{2(\beta y_2+i\alpha y_1)}}, \tag{2-17}$$

and

$$\frac{1}{2}\int_{\mathbb{R}} Q^2 = 2(\beta+i\alpha), \tag{2-18}$$

no matter what x_1, x_2 are. Finally, if we let $L_1 := \log(1+e^{2(\beta x_2+i\alpha x_1)})$,

$$\int_0^x \mathcal{N} = \log(1+e^{2(\beta y_2+i\alpha y_1)}) - L_1. \tag{2-19}$$

Note that the previous formula is well-defined, since x_1 and x_2 do not satisfy (2-9).

Proof. It is not difficult to check that (2-17) is satisfied. Note that

$$\lim_{x\to-\infty}\left|\frac{2(\beta+i\alpha)e^{2(\beta y_2+i\alpha y_1)}}{1+e^{2(\beta y_2+i\alpha y_1)}}\right| = 0.$$

Identity (2-18) is a consequence of the fact that

$$\lim_{x\to+\infty}\frac{2(\beta+i\alpha)e^{2(\beta y_2+i\alpha y_1)}}{1+e^{2(\beta y_2+i\alpha y_1)}} = 2(\beta+i\alpha).$$

Finally, (2-19) is easy to check. □

3. Bäcklund transformation for mKdV

Lemma 2.4 is a consequence of a deeper result. In what follows, we fix a primitive $\tilde{f}$ of f, i.e.,

$$\tilde{f}_x := f, \tag{3-1}$$

where f is assumed only to be in $L^2(\mathbb{R})$. Notice that, even if $f = f(t,x)$ is a solution of mKdV, a corresponding term $\tilde{f}(t,x)$ may be unbounded in space.

Definition 3.1 (see, e.g., [Lamb 1980]). Let

$$(u_a, u_b, v_a, v_b, m) \in H^1(\mathbb{R};\mathbb{C})^2 \times H^{-1}(\mathbb{R};\mathbb{C})^2 \times \mathbb{C}.$$

We set

$$G := (G_1, G_2), \qquad G = G(u_a, u_b, v_a, v_b, m),$$

where

$$G_1(u_a, u_b, v_a, v_b, m) := \frac{u_a - u_b}{\sqrt{2}} - m\sin\frac{\tilde{u}_a+\tilde{u}_b}{\sqrt{2}}, \tag{3-2}$$

and

$$G_2(u_a, u_b, v_a, v_b, m) := v_a - v_b + m\left[((u_a)_x + (u_b)_x)\cos\frac{\tilde{u}_a+\tilde{u}_b}{\sqrt{2}} + \frac{u_a^2+u_b^2}{\sqrt{2}}\sin\frac{\tilde{u}_a+\tilde{u}_b}{\sqrt{2}}\right]. \tag{3-3}$$

For the moment we do not specify the range of $G(u_a, u_b, v_a, v_b, m)$ for data (u_a, u_b, v_a, v_b, m) in $H^1(\mathbb{R};\mathbb{C})^2 \times H^{-1}(\mathbb{R};\mathbb{C})^2 \times \mathbb{C}$. However, thanks to Lemma 2.4, we have the following result:

Lemma 3.2. *Assume that x_1 and x_2 do not satisfy* (2-9). *Then*

$$G(Q, 0, \tilde{Q}_t, 0, \beta + i\alpha) \equiv (0, 0).$$

The previous identity can be extended by zero to the case where x_1 and x_2 satisfy (2-9), in such a form that $G(Q, 0, \tilde{Q}_t, 0, \beta + i\alpha)$, as a function of $(x_1, x_2) \in \mathbb{R}^2$, is now well-defined and continuous everywhere.

In what follows we consider the invertibility of the Bäcklund transformation on complex-valued functions. See [Hoffman and Wayne 2013] for the statement involving the real-valued solitons in the sine-Gordon case and [Mizumachi and Pelinovsky 2012] for the case of nonlinear Schrödinger solitons.

Proposition 3.3. *Let $X^0 := (u_a^0, u_b^0, v_a^0, v_b^0, m^0) \in H^1(\mathbb{R};\mathbb{C})^2 \times H^{-1}(\mathbb{R};\mathbb{C})^2 \times \mathbb{C}$ be such that*

$$\operatorname{Re} m^0 > 0, \tag{3-4}$$

$$G(X^0) = (0, 0), \tag{3-5}$$

$$\sin \frac{\tilde{u}_a^0 + \tilde{u}_b^0}{\sqrt{2}} \in H^1(\mathbb{R};\mathbb{C}), \tag{3-6}$$

$$\textit{and} \quad \lim_{-\infty} (\tilde{u}_a^0 + \tilde{u}_b^0) = 0, \quad \lim_{+\infty} (\tilde{u}_a^0 + \tilde{u}_b^0) = \sqrt{2}\pi. \tag{3-7}$$

Assume additionally that the ODE

$$\mu_x^0 - m^0 \cos\left(\frac{\tilde{u}_a^0 + \tilde{u}_b^0}{\sqrt{2}}\right)\mu^0 = 0, \tag{3-8}$$

has a smooth *solution $\mu^0 = \mu^0(x) \in \mathbb{C}$ satisfying*

$$\mu^0 \in H^1(\mathbb{R};\mathbb{C}), \quad |\mu^0(x)| > 0, \quad \left|\frac{\mu_x^0(x)}{\mu^0(x)}\right| \le C, \tag{3-9}$$

$$\textit{and} \quad \int_{\mathbb{R}} \sin\left(\frac{\tilde{u}_a^0 + \tilde{u}_b^0}{\sqrt{2}}\right)\mu^0 \neq 0. \tag{3-10}$$

Then there exist $\nu_0 > 0$ and $C > 0$ such that the following is satisfied: *For any $0 < \nu < \nu_0$ and any $(u_a, v_a) \in H^1(\mathbb{R};\mathbb{C}) \times H^{-1}(\mathbb{R};\mathbb{C})$ satisfying*

$$\|u_a - u_a^0\|_{H^1(\mathbb{R};\mathbb{C})} < \nu, \tag{3-11}$$

G is well-defined in a neighborhood of X^0 and there exists an unique (u_b, v_b, m) defined in an open *subset of $H^1(\mathbb{R},\mathbb{C}) \times H^{-1}(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ such that*

$$G(u_a, u_b, v_a, v_b, m) \equiv (0, 0), \tag{3-12}$$

$$\|\tilde{u}_a + \tilde{u}_b - \tilde{u}_a^0 - \tilde{u}_b^0\|_{H^2(\mathbb{R};\mathbb{C})} \le C\nu, \tag{3-13}$$

$$\|u_b - u_b^0\|_{H^1(\mathbb{R};\mathbb{C})} + |m - m^0| < C\nu, \tag{3-14}$$

$$\sin \frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}} \in H^1(\mathbb{R};\mathbb{C}), \tag{3-15}$$

$$\textit{and} \quad \lim_{-\infty}(\tilde{u}_a + \tilde{u}_b) = 0, \quad \lim_{+\infty}(\tilde{u}_a + \tilde{u}_b) = \sqrt{2}\pi. \tag{3-16}$$

Proof. Given u_a, u_b, m and v_a well-defined, v_b is uniquely defined from (3-3). We solve for u_b and m now. We will use the implicit function theorem.

We make a change of variables in order to specify a suitable range for G and to be able to prove (3-16). Define

$$u_c := u_a + u_b - u_c^0, \quad u_c^0 := u_a^0 + u_b^0 \in H^1(\mathbb{R}; \mathbb{C}), \tag{3-17}$$

and similarly for $\tilde{u}_c$ and $\tilde{u}_c^0$:

$$(\tilde{u}_c)_x = u_c, \quad (\tilde{u}_c^0)_x = u_c^0.$$

In what follows, we will look for a suitable $\tilde{u}_c$ with decay, and then we find u_b. Indeed, note that given u_c and u_a, u_b can be easily obtained. Then, with a slight abuse of notation, we consider G defined as follows:

$$G = (G_1, G_2), \quad G = G(u_a, \tilde{u}_c, v_a, v_b, m),$$

and

$$\begin{aligned} G : H^1(\mathbb{R}; \mathbb{C}) \times H^2(\mathbb{R}; \mathbb{C}) \times H^{-1}(\mathbb{R}; \mathbb{C})^2 \times \mathbb{C} &\longrightarrow H^1(\mathbb{R}; \mathbb{C}) \times H^{-1}(\mathbb{R}; \mathbb{C}) \\ (u_a, \tilde{u}_c, v_a, v_b, m) &\longmapsto G(u_a, \tilde{u}_c, v_a, v_b, m), \end{aligned}$$

where, from (3-2),

$$G_1(u_a, \tilde{u}_c, v_a, v_b, m) := \frac{2u_a - u_c^0 - u_c}{\sqrt{2}} - m \sin \frac{\tilde{u}_c^0 + \tilde{u}_c}{\sqrt{2}}, \tag{3-18}$$

and, from (3-3),

$$\begin{aligned} &G_2(u_a, \tilde{u}_c, v_a, v_b, m) \\ &\quad := v_a - v_b + m\left[(u_c^0 + u_c)_x \cos \frac{\tilde{u}_c^0 + \tilde{u}_c}{\sqrt{2}} + \frac{u_a^2 + (u_c^0 + u_c - u_a)^2}{\sqrt{2}} \sin \frac{\tilde{u}_c^0 + \tilde{u}_c}{\sqrt{2}}\right]. \end{aligned} \tag{3-19}$$

Clearly G as in (3-18)–(3-19) defines a C^1 functional in a small neighborhood of X^1 given by

$$X^1 := (u_a^0, 0, v_a^0, v_b^0, m^0) \in H^1(\mathbb{R}; \mathbb{C}) \times H^2(\mathbb{R}; \mathbb{C}) \times H^{-1}(\mathbb{R}; \mathbb{C})^2 \times \mathbb{C}, \tag{3-20}$$

where G is well-defined according to (3-6). Let us apply the implicit function theorem at this point. By (3-18) we have to show that

$$u_c + m^0 \cos\left(\frac{\tilde{u}_c^0}{\sqrt{2}}\right)\tilde{u}_c = f - m \sin \frac{\tilde{u}_c^0}{\sqrt{2}}$$

has a unique solution $(\tilde{u}_c, m)$ such that $\tilde{u}_c \in H^2(\mathbb{R}; \mathbb{C})$ for any $f \in H^1(\mathbb{R}; \mathbb{C})$ with linear bounds. From (3-7), we have

$$\lim_{x \to \pm\infty} \cos \frac{\tilde{u}_c^0}{\sqrt{2}} = \mp 1, \tag{3-21}$$

so that we can assume

$$\mu^0(x) = \exp\left(m^0 \int_0^x \cos \frac{\tilde{u}_c^0}{\sqrt{2}}\right).$$

Note that μ^0 decays exponentially in space as $x \to \pm\infty$. We have

$$\mu^0 u_c + (\mu^0)_x \tilde{u}_c = \mu^0\left[f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right].$$

Using (3-10), we choose $m \in \mathbb{C}$ such that

$$\int_{\mathbb{R}} \mu^0\left[f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right] = 0, \tag{3-22}$$

so that

$$|m| \le C\|f\|_{L^2(\mathbb{R};\mathbb{C})}$$

with $C > 0$ depending on the quantity $\left|\int_{\mathbb{R}} \mu^0 \sin(\tilde{u}_c^0/\sqrt{2})\right| \neq 0$ and $\|\mu^0\|_{L^2(\mathbb{R};\mathbb{C})}$.[8] We get

$$\tilde{u}_c = \frac{1}{\mu^0}\int_{-\infty}^{x} \mu^0\left[f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right]. \tag{3-23}$$

Finally, note that we have $u_c \in H^1(\mathbb{R};\mathbb{C})$. Indeed, first of all, thanks to (3-22), (3-8) and (3-21),

$$\lim_{x\to\pm\infty} \tilde{u}_c = \lim_{x\to\pm\infty} \frac{\mu^0}{\mu_x^0}\left[f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right] = 0.$$

If $s \le x \ll -1$, from (3-21) we get

$$\left|\frac{\mu^0(s)}{\mu^0(x)}\right| = \left|\exp\left(-m^0\int_s^x \cos\frac{\tilde{u}_c^0}{\sqrt{2}}\right)\right| \le Ce^{-\operatorname{Re} m^0(x-s)},$$

so that we have, for $x < 0$ and large,[9]

$$\begin{aligned}|\tilde{u}_c(x)| &\le C\int_{-\infty}^{x} e^{-(\operatorname{Re} m^0)(x-s)}\left|f(s) - m\sin\frac{\tilde{u}_c^0(s)}{\sqrt{2}}\right| ds\\ &\le C\mathbf{1}_{(-\infty,x]}e^{-(\operatorname{Re} m^0)(\cdot)} \star \left|f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right|, \quad \operatorname{Re} m^0 > 0.\end{aligned}$$

A similar result holds for $x > 0$ large, after using (3-22). Therefore, from Young's inequality,

$$\|\tilde{u}_c\|_{L^2(\mathbb{R};\mathbb{C})} \le C\left\|f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right\|_{L^2(\mathbb{R};\mathbb{C})} \le C\|f\|_{L^2(\mathbb{R};\mathbb{C})}, \tag{3-24}$$

as desired. On the other hand,

$$(\tilde{u}_c)_x = \left[f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right] - \frac{\mu_x^0}{(\mu^0)^2}\int_{-\infty}^{x} \mu^0\left[f - m\sin\frac{\tilde{u}_c^0}{\sqrt{2}}\right].$$

Since μ_x^0/μ^0 is bounded (see (3-9)), we have $\tilde{u}_c \in H^1(\mathbb{R};\mathbb{C})$. Finally, it is easy to see that $\tilde{u}_c \in H^2(\mathbb{R};\mathbb{C})$. Note that the constant involving the boundedness of the linear operator $f \mapsto \tilde{u}_c$ depends on the H^1-norm of μ^0, which blows up if (2-9) is satisfied.

[8]Note that $\|\mu^0\|_{L^2(\mathbb{R};\mathbb{C})}$ blows up as (2-9) is attained.

[9]Here the symbol $\star$ denotes *convolution*.

It turns out that we can apply the implicit function theorem to the operator G described in (3-18)–(3-19), so that (3-12) is satisfied, provided (3-11) holds.

First of all, note that (3-15) and (3-16) follow from $\tilde{u}_c \in H^2(\mathbb{R};\mathbb{C})$.

On the other hand, the estimate (3-13) is equivalent to

$$\|\tilde{u}_c\|_{H^2(\mathbb{R};\mathbb{C})} \le C\nu.$$

We will obtain this estimate using the almost linear character of the operator G around the point X^1. Since $\tilde{u}_c$ satisfies (3-18), we have

$$\frac{2u_a - (\tilde{u}_c)_x}{\sqrt{2}} - m\sin\frac{\tilde{u}_c^0 + \tilde{u}_c}{\sqrt{2}} = 0.$$

Recall that $\tilde{u}_c$ depends on u_a. Near u_a^0, one has

$$\partial_t \tilde{u}_c[u_a^0 + th]\big|_{t=0} = w[h] + O(h^2),$$

where $w = w[h]$ solves the *derivative* equation

$$w_x + m^0\cos\left(\frac{\tilde{u}_c^0}{\sqrt{2}}\right)w = -2h - m[h]\sin\frac{\tilde{u}_c^0}{\sqrt{2}}.$$

Here $m[h]$ is a constant that makes the right-hand side integrable, just as in (3-23). From (3-11) we know that $\|u_a - u_a^0\|_{H^1(\mathbb{R};\mathbb{C})} < \nu$. We shall use $h := u_a - u_a^0$. Following the computations after (3-23), we obtain the desired conclusion (see, e.g., (3-24)). We conclude that the L^2 norm of $\tilde{u}_c$ is bounded by $C\nu$. For the derivatives of $\tilde{u}_c$, the proof is very similar. □

Later we will need a second invertibility theorem. This time we assume that m is fixed, $u_b \sim u_b^0$ is known and we look for $u_a \sim u_a^0$. Note that the positive sign in front of (3-2) will be essential for the proof, otherwise we cannot take m fixed.

Proposition 3.4. *Let* $X^0 = (u_a^0, u_b^0, v_a^0, v_b^0, m^0) \in H^1(\mathbb{R};\mathbb{C})^2 \times H^{-1}(\mathbb{R},\mathbb{C}) \times \mathbb{C}$ *be such that* (3-4), (3-5), (3-6) *and* (3-7) *are satisfied. Assume additionally that the ODE*

$$(\mu^1)_x + m\cos\left(\frac{\tilde{u}_a^0 + \tilde{u}_b^0}{\sqrt{2}}\right)\mu^1 = 0 \tag{3-25}$$

has a smooth *solution* $\mu^1 = \mu^1(x) \in \mathbb{C}$ *satisfying*

$$|\mu^1(x)| > 0, \qquad \left|\frac{\mu_x^1(x)}{\mu^1(x)}\right| \le C, \qquad \frac{1}{\mu^1} \in H^1(\mathbb{R};\mathbb{C}), \tag{3-26}$$

and G is smooth in a small neighborhood of X^0. Then there exists $\nu_1 > 0$ and a fixed constant $C > 0$ such that for all $0 < \nu < \nu_1$ the following is satisfied: *for any $(u_b, v_b, m) \in H^1(\mathbb{R};\mathbb{C}) \times H^{-1}(\mathbb{R};\mathbb{C}) \times \mathbb{C}$ such that*

$$\|u_b - u_b^0\|_{H^1(\mathbb{R};\mathbb{C})} + |m - m^0| < \nu, \tag{3-27}$$

G is well-defined and there exist unique $(u_a, v_a) \in H^1(\mathbb{R}, \mathbb{C}) \times H^{-1}(\mathbb{R}; \mathbb{C})$ such that

$$G(u_a, u_b, v_a, v_b, m) \equiv (0,0),$$

$$\int_{\mathbb{R}} (u_a - u_b)\left(\frac{1}{\mu^1}\right)_x = 0. \tag{3-28}$$

$$\|\tilde{u}_a + \tilde{u}_b - \tilde{u}_a^0 - \tilde{u}_b^0\|_{H^2(\mathbb{R};\mathbb{C})} \le C\nu, \tag{3-29}$$

$$\lim_{-\infty}(\tilde{u}_a + \tilde{u}_b) = 0, \quad \lim_{+\infty}(\tilde{u}_a + \tilde{u}_b) = \sqrt{2}\pi, \tag{3-30}$$

$$\textit{and} \quad \|u_a - u_a^0\|_{H^1(\mathbb{R};\mathbb{C})} < C\nu. \tag{3-31}$$

Proof. Given u_a, u_b and v_b well-defined, v_a is uniquely defined from (3-3). We solve for u_a now.

We follow the ideas of the proof of Proposition 3.3. However, this time we consider G defined in the opposite sense: using (3-17),

$$\begin{gathered} G = (G_3, G_4), \quad G = G(\tilde{u}_c, u_b, v_a, v_b, m), \\ G : H^2(\mathbb{R};\mathbb{C}) \times H^1(\mathbb{R};\mathbb{C}) \times H^{-1}(\mathbb{R};\mathbb{C})^2 \times \mathbb{C} \longrightarrow H^1(\mathbb{R};\mathbb{C}) \times H^{-1}(\mathbb{R};\mathbb{C}) \\ (\tilde{u}_c, u_b, v_a, v_b, m) \longmapsto G(\tilde{u}_c, u_b, v_a, v_b, m) \end{gathered}$$

with

$$\int_{\mathbb{R}} (\tilde{u}_c)_x \left(\frac{1}{\mu^1}\right)_x = 0, \tag{3-32}$$

where, from (3-2),

$$G_3(\tilde{u}_c, u_b, v_a, v_b, m) := \frac{u_c^0 + u_c - 2u_b}{\sqrt{2}} - m \sin\frac{\tilde{u}_c^0 + \tilde{u}_c}{\sqrt{2}}, \tag{3-33}$$

and, from (3-3),

$$\begin{aligned} &G_4(\tilde{u}_c, u_b, v_a, v_b, m) \\ &\quad := v_a - v_b + m\left[(u_c^0 + u_c)_x \cos\frac{\tilde{u}_c^0 + \tilde{u}_c}{\sqrt{2}} + \frac{(u_c^0 + u_c - u_b)^2 + u_b^2}{\sqrt{2}} \sin\frac{\tilde{u}_c^0 + \tilde{u}_c}{\sqrt{2}}\right]. \end{aligned} \tag{3-34}$$

Clearly G as in (3-33)–(3-34) defines a C^1 functional in a small neighborhood of X^2 given by

$$X^2 := (0, u_b^0, v_a^0, v_b^0, m^0) \in H^2(\mathbb{R};\mathbb{C}) \times H^1(\mathbb{R};\mathbb{C}) \times H^{-1}(\mathbb{R};\mathbb{C})^2 \times \mathbb{C}, \tag{3-35}$$

where G is well-defined according to (3-6) and $G(X^2) = (0,0)$.

Fix m close enough to m^0. Now we have to show that

$$u_c - m\cos\left(\frac{\tilde{u}_c^0}{\sqrt{2}}\right)\tilde{u}_c = f \tag{3-36}$$

has a unique solution $\tilde{u}_c$ such that $u_c \in H^2(\mathbb{R};\mathbb{C})$ for any $f \in H^1(\mathbb{R};\mathbb{C})$. Indeed, consider μ^1 given by (3-25). It is not difficult to check that (see conditions (3-4), (3-27) and (3-7))

$$\operatorname{Re} m > 0, \quad \lim_{\pm\infty} \cos\frac{\tilde{u}_c^0}{\sqrt{2}} = \mp 1, \quad \text{and} \quad \mu^1 = \exp\left(-m\int_0^x \cos\frac{\tilde{u}_c^0}{\sqrt{2}}\right). \tag{3-37}$$

Note that, by (3-37) and (3-4), $|\mu^1(x)|$ is exponentially growing in space as $x \to \pm\infty$. From (3-36),

$$(\mu^1 \tilde{u}_c)_x = \mu^1 f,$$

so that, thanks to (3-26),

$$\tilde{u}_c = \frac{1}{\mu^1}\mu^1(0)\tilde{u}_c(0) + \frac{1}{\mu^1}\int_0^x \mu^1 f.$$

Clearly $\lim_{\pm\infty} \tilde{u}_c = 0$ for $f \in H^1(\mathbb{R}; \mathbb{C})$. In order to ensure uniqueness, we seek $\tilde{u}_c$ satisfying

$$\int_{\mathbb{R}} u_c \left(\frac{1}{\mu^1}\right)_x = 0,$$

which is nothing but (3-32) and (3-28), which is justified by (3-26). Let us show that $\tilde{u}_c \in L^2(\mathbb{R}; \mathbb{C})$. We have, for $x > 0$ large,

$$|\tilde{u}_c(x)| \le C \int_0^x e^{-(\operatorname{Re} m)(x-s)} |f(s)|\, ds = Ce^{-(\operatorname{Re} m)(\cdot)} \star |f|, \quad \operatorname{Re} m > 0.$$

A similar estimate can be established if $x < 0$. Therefore, using Young's inequality,

$$\|\tilde{u}_c\|_{L^2(\mathbb{R};\mathbb{C})} \le C\|f\|_{L^2(\mathbb{R};\mathbb{C})},$$

as desired. Now we check that $u_c \in H^1(\mathbb{R}; \mathbb{C})$. Indeed, we have

$$u_c = f - \frac{\mu^1_x}{(\mu^1)^2}\int_0^x \mu^1 f.$$

Since μ^1_x/μ^1 is bounded, we have proven that $u_c \in L^2(\mathbb{R}; \mathbb{C})$. A new iteration proves that $u_c \in H^1(\mathbb{R}; \mathbb{C})$. Estimates (3-29)–(3-31) are consequences of the implicit function theorem and can be proved as in the previous proposition. The proof is complete. □

We finish this section by pointing out the role played by the Bäcklund transformation in the mKdV dynamics. We recall the following standard result:

Theorem 3.5. *Let* $m \in \mathbb{C}$ *be a* fixed *parameter, and* $I \subset \mathbb{R}$ *an open time interval. Assume that* $u_b \in C(I; H^1(\mathbb{R}; \mathbb{C}))$ *solves* (1-1), *i.e.*,

$$(u_b)_t + ((u_b)_{xx} + u_b^3)_x = 0, \tag{3-38}$$

in the H^1*-sense. Assume, moreover that* u_b *is close to* u_b^0 *and that* (3-25) *and* (3-26) *hold. Define* $v_b := -((u_b)_{xx} + u_b^3)$ *as a distribution in* $H^{-1}(\mathbb{R}; \mathbb{C})$*. Then, for each* $t \in I$*, the corresponding solution* $(u_a(t), v_a(t))$ *of* $G_1 = G_2 = 0$ *for* m *fixed, obtained in the space* $H^1(\mathbb{R}; \mathbb{C}) \times H^{-1}(\mathbb{R}; \mathbb{C})$*, satisfies the following*:

(1) $u_a \in C(I; H^1(\mathbb{R}; \mathbb{C}))$;

(2) $(u_a)_t := (v_a)_x$ *is well-defined in* $H^{-2}(\mathbb{R}; \mathbb{C})$; *and*

(3) u_a *solves* (1-1) *in the* H^1*-sense.*

Proof. The first step is an easy consequence of the continuous character of the solution map given by the implicit function theorem. By density we can assume $u_b(t) \in H^3(\mathbb{R}; \mathbb{C})$. From (3-2) we have

$$(u_a)_x - (u_b)_x = m \cos\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right)(u_a + u_b), \tag{3-39}$$

and

$$(u_a)_{xx} - (u_b)_{xx} = m \cos\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right)((u_a)_x + (u_b)_x) - \frac{m}{\sqrt{2}} \sin\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right)(u_a + u_b)^2.$$

Therefore, from (3-3) and (3-2),

$$\begin{aligned} v_a - v_b &= -((u_a)_{xx} - (u_b)_{xx}) - \frac{m}{\sqrt{2}} \sin\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right)[(u_a + u_b)^2 + (u_a^2 + u_b^2)] \\ &= -((u_a)_{xx} - (u_b)_{xx}) - (u_a - u_b)(u_a^2 + u_a u_b + u_b^2) \\ &= -((u_a)_{xx} + u_a^3) - ((u_b)_{xx} + u_b^3). \end{aligned}$$

We have from (3-38) that $(v_b)_x + ((u_b)_{xx} + u_b^3)_x = 0$. Therefore,

$$(v_a)_x + ((u_a)_{xx} + u_a^3)_x = 0. \tag{3-40}$$

Finally, if $(u_a)_t = (v_a)_x$, we have that u_a solves (1-1). In order to prove this result, we compute the time derivative in (3-2): we get

$$(u_a)_t - (u_b)_t = m \cos\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right)((\tilde{u}_a)_t + (\tilde{u}_b)_t). \tag{3-41}$$

Note that, given u_b, the solution u_a is uniquely defined, thanks to the implicit function theorem. Additionally, from (3-3),

$$\begin{aligned} (v_a)_x - (v_b)_x + m\Bigg[&((u_a)_{xx} + (u_b)_{xx}) \cos \frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}} - \frac{1}{\sqrt{2}}((u_a)_x + (u_b)_x)(u_a + u_b) \sin \frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}} \\ &+ \sqrt{2}(u_a (u_a)_x + u_b (u_b)_x) \sin \frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}} + \frac{(u_a^2 + u_b^2)}{2}(u_a + u_b) \cos \frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\Bigg] = 0. \end{aligned}$$

We use (3-2) and (3-3) in the previous identity, and get

$$(v_a)_x - (v_b)_x + \left[m((u_a)_{xx} + (u_b)_{xx}) \cos \frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}} + (u_a^2 - u_a u_b + u_b^2)((u_a)_x - (u_b)_x)\right] = 0.$$

Finally, we use (3-39) to obtain

$$(v_a)_x - (v_b)_x + m \cos\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right)((u_a)_{xx} + u_a^3 + (u_b)_{xx} + u_b^3) = 0,$$

so (3-38) and (3-40) imply

$$(v_a)_x - (v_b)_x = m \cos\left(\frac{\tilde{u}_a + \tilde{u}_b}{\sqrt{2}}\right)(v_a + v_b),$$

so that from (3-41) and the uniqueness we are done. $\square$

4. Dynamics of complex-valued mKdV solitons

In what follows we will apply the results from the previous section in a neighborhood of the complex soliton at time zero. Define (cf. (2-1)),

$$\tilde{Q}^0 := \tilde{Q}(x; \alpha, \beta, 0, 0), \tag{4-1}$$

and similarly for Q^0 and $\tilde{Q}^0_t$. Recall that, by Lemma 2.2, the complex soliton Q^0 is well-defined everywhere if (2-9) is not satisfied. Finally, given any

$$\tilde{z}^0_b \in \dot{H}^1(\mathbb{R}; \mathbb{C}),$$

we define z^0_b by the identity (see (3-1), for instance)

$$z^0_b := (\tilde{z}^0_b)_x,$$

and, in term of distributions,

$$w^0_b := -((z^0_b)_{xx} + (z^0_b)^3) \in H^{-1}(\mathbb{R}; \mathbb{C}).$$

Lemma 4.1. *There exists $\nu_0 > 0$ and $C > 0$ such that, for all $0 < \nu < \nu_0$, the following holds. For all $z^0_b \in H^1(\mathbb{R}; \mathbb{C})$ satisfying*

$$\|z^0_b\|_{H^1(\mathbb{R};\mathbb{C})} < \nu, \tag{4-2}$$

there exist unique $y^0_a \in H^1(\mathbb{R}, \mathbb{C})$, $y^1_a \in H^{-1}(\mathbb{R}, \mathbb{C})$ and $m \in \mathbb{C}$ of the form

$$y^0_a(x) = y^0_a[z^0_b](x), \quad y^1_a(x) = y^1_a[z^0_b, w^0_b](x), \quad m := \beta + i\alpha + q^0 \tag{4-3}$$

such that

$$\begin{aligned} &\|y^0_a\|_{H^1(\mathbb{R};\mathbb{C})} + |q^0| \le C\nu, \quad \tilde{z}^0_a + \tilde{y}^0_a \in H^2(\mathbb{R}; \mathbb{C}), \\ &\textit{and} \quad G(Q^0 + z^0_b, y^0_a, \tilde{Q}^0_t + w^0_b, y^1_a, m) \equiv (0, 0). \end{aligned} \tag{4-4}$$

Note that both $\tilde{z}^0_a$ and $\tilde{y}^0_a$ may be unbounded functions, but the sum is bounded on $\mathbb{R}$.

Proof. Let Q^0 be the soliton profile with parameters β, α and $x_1 = x_2 = 0$ (cf. (4-1)). We apply Proposition 3.3 with

$$u^0_a := Q^0, \quad u^0_b := 0, \quad v^0_a := \tilde{Q}^0_t, \quad v^0_b := 0 \quad \text{and} \quad m^0 := \beta + i\alpha.$$

Clearly $\tilde{u}^0_a + \tilde{u}^0_b = \tilde{Q}_0$ satisfies (3-6)–(3-7). From (2-15) we have

$$(Q^0)_x - (\beta + i\alpha)\cos\left(\frac{\tilde{Q}^0}{\sqrt{2}}\right) Q^0 = 0, \quad Q^0(-\infty) = 0, \tag{4-5}$$

so that we have (cf. (3-8)–(3-9))

$$\mu^0 = Q^0.$$

Clearly Q^0 is never zero. Moreover, $|(Q^0)^{-1}Q^0_x|$ is bounded on $\mathbb{R}$. Now we prove that

$$\int_{\mathbb{R}} \sin\left(\frac{\tilde{Q}^0}{\sqrt{2}}\right) Q^0 \neq 0.$$

From (2-15) and (2-18),

$$\int_{\mathbb{R}} \sin\left(\frac{\tilde{Q}^0}{\sqrt{2}}\right) Q^0 = \frac{1}{\sqrt{2}(\beta+i\alpha)} \int_{\mathbb{R}} (Q^0)^2 = \frac{4(\beta+i\alpha)}{\sqrt{2}(\beta+i\alpha)} = 2\sqrt{2}. \qquad \square$$

Before continuing, we need some definitions. We write

$$\alpha^* := \alpha + \operatorname{Im} q^0, \quad \beta^* := \beta + \operatorname{Re} q^0, \tag{4-6}$$

so that m in (4-3) satisfies

$$m = \beta + i\alpha + q^0 = \beta^* + i\alpha^*.$$

Since q^0 is small, we have that β^* and α^* are positive quantities. Similarly, define

$$\delta^* := (\alpha^*)^2 - 3(\beta^*)^2, \quad \gamma^* := 3(\alpha^*)^2 - (\beta^*)^2, \tag{4-7}$$

and compare with (1-4).

Consider the kink profile $\tilde{Q}$ introduced in (2-1). We consider, for all $t \in \mathbb{R}$, the complex (kink) profile

$$\tilde{Q}^*(t,x) := \tilde{Q}(x; \alpha^*, \beta^*, \delta^* t + x_1, \gamma^* t + x_2), \tag{4-8}$$

with δ^* and γ^* defined in (4-7), x_1 and x_2 *possibly* depending on time, and

$$Q^*(t,x) := \partial_x \tilde{Q}^*(t,x). \tag{4-9}$$

It is not difficult to see that (see, e.g., (1-10))

$$Q^*(t,x) = Q_c(x - ct - \hat{x}), \quad \sqrt{c} = \beta^* + i\alpha^*, \quad \hat{x} \in \mathbb{C},$$

which is a *complex-valued* solution of mKdV (1-1). Technically, the complex soliton $Q^*(t)$ has velocity $-\gamma^* = (\beta^*)^2 - 3(\alpha^*)^2$, a quantity that is always smaller than the corresponding speed $(\beta^*)^2$ of the associated real-valued soliton $Q_{(\beta^*)^2}$ obtained by sending α^* to zero. Finally, as in (2-6) we define

$$\tilde{Q}^*_t(t,x) := -(\beta^* + i\alpha^*)^2 Q^*(t,x).$$

Lemma 4.2. *Fix $\alpha, \beta > 0$. Assume that x_1, x_2 are time-dependent functions such that*

$$|x_1'(t)| + |x_2'(t)| \ll |\delta^* - \gamma^*| = 2((\alpha^*)^2 + (\beta^*)^2). \tag{4-10}$$

Then there exists a sequence of times $t_k \in \mathbb{R}$, $k \in \mathbb{Z}$ such that (2-9) *is satisfied. In particular*, (t_k) *is a sequence with no limit points.*

Proof. Note that (2-9) now reads

$$(\delta^* - \gamma^*)t_k + (x_1 - x_2)(t_k) = \frac{\pi}{\alpha^*}\left(k + \tfrac{1}{2}\right).$$

By (4-7), $\delta^* - \gamma^* = -2((\alpha^*)^2 + (\beta^*)^2) \neq 0$, and using (4-10) and the mean and intermediate value theorems applied to the smooth function

$$f(t) := (\delta^* - \gamma^*)t + (x_1 - x_2)(t),$$

at each value $\frac{\pi}{\alpha^*}\big(k + \frac{1}{2}\big)$, $k \in \mathbb{Z}$, we see that f satisfies

$$f'(t) = -2((\alpha^*)^2 + (\beta^*)^2) + (x_1' - x_2')(t) \sim -2((\alpha^*)^2 + (\beta^*)^2). \qquad \square$$

We conclude that $\tilde{Q}^*$ and Q^* defined in (4-8) and (4-9) are well-defined except for an isolated sequence of times t_k. We impose now the condition

$$t \in \mathbb{R} \quad \text{satisfies} \quad t \neq t_k \ \text{ for all } \ k \in \mathbb{Z}. \tag{4-11}$$

In what follows we will solve the Cauchy problem associated to mKdV with suitable initial data. Indeed, we will assume that

$$y_a^0 \ \text{ is a real-valued function and } \ y_a^0 \in H^1(\mathbb{R}). \tag{4-12}$$

We will need the following:[10]

Theorem 4.3 ([Kenig et al. 1993]). *For any $y_a^0 \in H^1(\mathbb{R})$, there exists a unique*[11] *solution $y_a \in C(\mathbb{R}, H^1(\mathbb{R}))$ with initial data $y_a(0) = y_a^0 \in H^1(\mathbb{R})$ to mKdV, and*

$$\sup_{t\in\mathbb{R}} \|y_a(t)\|_{H^1(\mathbb{R})} \leq C\|y_a^0\|_{H^1(\mathbb{R})}$$

with $C > 0$ independent of time. Moreover, the mass

$$M[y_a](t) := \frac{1}{2}\int_{\mathbb{R}} y_a^2(t,x)\,dx = M[y_a^0] \tag{4-13}$$

and energy

$$E[y_a](t) := \frac{1}{2}\int_{\mathbb{R}} (y_a)_x^2(t,x)\,dx - \frac{1}{4}\int_{\mathbb{R}} (y_a)^4(t,x)\,dx = E[y_a^0] \tag{4-14}$$

are conserved quantities.

Let $y_a \in C(\mathbb{R}, H^1(\mathbb{R}))$ denote the corresponding solution for mKdV with initial data y_a^0. Since $\|y_a^0\|_{H^1} \leq C\eta$, we have, for a (possibly different) constant $C > 0$,

$$\sup_{t\in\mathbb{R}} \|y_a(t)\|_{H^1(\mathbb{R})} \leq C\eta. \tag{4-15}$$

In particular, we can define, for all $t \in \mathbb{R}$,

$$\tilde{y}_a(t) := \int_0^x y_a(t,s)\,ds,$$

and

$$(\tilde{y}_a)_t(t) := -((y_a)_{xx}(t) + y_a^3(t)) \in H^{-1}(\mathbb{R}) \tag{4-16}$$

[10] We recall that this result is consequence of the local Cauchy theory and the conservation of mass and energy (4-13)–(4-14).

[11] In a certain sense; see [Kenig et al. 1993].

because $y_a(t) \in L^p(\mathbb{R})$ for all $p \geq 2$.

Lemma 4.4. *Assume that a time $t \in \mathbb{R}$ and y_a^0 are such that* (4-11) *and* (4-12) *hold. Then there are unique $z_b = z_b(t) \in H^1(\mathbb{R}; \mathbb{C})$ and $w_b = w_b(t) \in H^{-1}(\mathbb{R}; \mathbb{C})$ such that, for all $t \neq t_k$,*

$$\tilde{z}_b + \tilde{y}_a \in H^2(\mathbb{R}; \mathbb{C}), \tag{4-17}$$

$$\frac{1}{\sqrt{2}}(Q^* + z_b - y_a) = (\beta + i\alpha + q^0) \sin \frac{\tilde{Q}^* + \tilde{z}_b + \tilde{y}_a}{\sqrt{2}}, \tag{4-18}$$

where $\tilde{Q}$ and Q are defined in (4-8) *and* (4-9). *Moreover, we have*

$$0 = \tilde{Q}_t^* + w_b - (\tilde{y}_a)_t + (\beta + i\alpha + q^0)\left[(Q_x^* + (z_b)_x + (y_a)_x) \cos \frac{\tilde{Q}^* + \tilde{z}_b + \tilde{y}_a}{\sqrt{2}} + \frac{(Q^* + z_b)^2 + y_a^2}{\sqrt{2}} \sin \frac{\tilde{Q}^* + \tilde{z}_b + \tilde{y}_a}{\sqrt{2}}\right], \tag{4-19}$$

and, for all $t \neq t_k$,

$$\|z_b(t)\|_{H^1(\mathbb{R};\mathbb{C})} < C\nu \tag{4-20}$$

with C uniformly bounded provided t is uniformly far from each t_k.

Proof. We will use Proposition 3.4. For that it is enough to recall that, from (2-15) and (2-16), and for all $t \neq t_k$,[12]

$$\frac{1}{\sqrt{2}} Q^* = (\beta + i\alpha + q^0) \sin \frac{\tilde{Q}^*}{\sqrt{2}} \tag{4-21}$$

and

$$\tilde{Q}_t^* + (\beta + i\alpha + q^0)\left[Q_x^* \cos \frac{\tilde{Q}^*}{\sqrt{2}} + \frac{(Q^*)^2}{\sqrt{2}} \sin \frac{\tilde{Q}^*}{\sqrt{2}}\right] = 0,$$

so that we can apply Proposition 3.4 at $X^0 = (Q^*, 0, \tilde{Q}_t^*, 0, m)$, where, by, (4-21) we have $m = (\beta + i\alpha + q^0)$. It is not difficult to see that the function μ^1 in (3-25) is given by

$$\mu^1 = (Q^*)^{-1},$$

and (3-26) is satisfied. Note that we require the estimate (4-15) in order to obtain (4-18)–(4-19). Finally, (4-20) is a direct consequence of (3-31). ☐

Remark. Since, from (4-4), we get

$$\frac{1}{\sqrt{2}}(Q^0 + z_b^0 - y_a^0) = (\beta + i\alpha + q^0) \sin \frac{\tilde{Q}^0 + \tilde{z}_b^0 + \tilde{y}_a^0}{\sqrt{2}},$$

we have that (4-18) implies by uniqueness that

$$(Q^* + z_b - y_a)(t = 0) = Q^0 + z_b^0 - y_a^0,$$

[12] It is interesting to note that the shifts x_1, x_2 on $Q^*(t, x)$ cannot be modified, otherwise there is no continuity at $t = 0$.

i.e.,

$$(Q^* + z_b)(t = 0) = Q^0 + z_b^0.$$

We are ready to prove a detailed version of Theorem 1.3, a result on complex-valued solitons.

Theorem 4.5. *There exists $\nu_0 > 0$ such that for all $0 < \nu < \nu_0$ the following holds: Consider the initial data $u_b^0 := Q^0 + z_b^0 \in H^1(\mathbb{R}; \mathbb{C})$, where*

$$\|z_b^0\|_{H^1(\mathbb{R};\mathbb{C})} < \nu.$$

Assume in addition that the corresponding function y_a^0 given by Lemma 4.1 is real-valued and belongs to $H^1(\mathbb{R})$. Fix $\varepsilon_0 > 0$. Then, for all t such that $|t - t_k| \geq \varepsilon_0$, with t_k defined in Lemma 4.2, the function $u_b := Q^* + z_b$, with z_b introduced in Lemma 4.4, is an H^1 complex-valued solution of mKdV, it satisfies $(u_b)_t = (Q^* + z_b)_t = (\tilde{Q}_t^* + w_b)_x$ and

$$\sup_{|t-t_k|\geq\varepsilon_0} \|u_b(t) - Q^*(t)\|_{H^1(\mathbb{R};\mathbb{C})} \leq C_{\varepsilon_0}\nu. \tag{4-22}$$

Remark. The quantity $\varepsilon_0 > 0$ is just an auxiliary parameter and it can be made as small as required; however, the constant C_{ε_0} in (4-22) becomes singular as ε_0 approaches zero.

Remark. In Corollary 6.5 we will prove that there is an open set in $H^1(\mathbb{R}; \mathbb{C})$ leading to y_a^0 being real-valued. The openness of this set will be a consequence of the implicit function theorem.

Proof. We apply Lemma 4.1. Assuming (4-12) we have y_a^0 real-valued, so that there is an mKdV dynamics $y_a(t)$ constructed in Theorem 4.3. Lastly, we apply Lemma 4.4 to obtain the dynamical function $Q^*(t) + z_b(t)$. Theorem 3.5 gives the conclusion. □

Now we will prove that the mass and energy,

$$\frac{1}{2}\int_{\mathbb{R}} u_b^2(t) \quad \text{and} \quad \frac{1}{2}\int_{\mathbb{R}} (u_b)_x^2(t) - \frac{1}{4}\int_{\mathbb{R}} u_b^4(t), \tag{4-23}$$

remain conserved for all time without using the mKdV equation (1-1), only the Bäcklund transformation (4-18). The fact that $\tilde{z}_b + \tilde{y}_a$ is in $H^1(\mathbb{R}; \mathbb{C})$ will be essential for the proof.

Corollary 4.6. *Assume that $t \neq t_k$ for all $k \in \mathbb{Z}$. Then the quantity*

$$\frac{1}{2}\int_{\mathbb{R}} (Q^* + z_b)^2(t) \tag{4-24}$$

is well-defined and independent of time, and

$$\frac{1}{2}\int_{\mathbb{R}} (Q^* + z_b)^2(t) = \frac{1}{2}\int_{\mathbb{R}} (y_a^0)^2 + 2(\beta + i\alpha + q^0). \tag{4-25}$$

Moreover, (4-25) can be extended in a continuous form to every $t \in \mathbb{R}$.

Proof. Using (4-18) and multiplying each side by $(1/\sqrt{2})(Q^* + z_b + y_a)$, we obtain

$$\tfrac{1}{2}(Q^* + z_b - y_a)(Q^* + z_b + y_a) = -(\beta + i\alpha + q^0)\left[\cos\frac{\tilde{Q}^* + \tilde{z}_b + \tilde{y}_a}{\sqrt{2}}\right]_x.$$

Using (2-15) and (2-14),

$$\cos\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}} = \cos\frac{\tilde{Q}^*}{\sqrt{2}}\cos\frac{\tilde{z}_b+\tilde{y}_a}{\sqrt{2}} - \sin\frac{\tilde{Q}^*}{\sqrt{2}}\sin\frac{\tilde{z}_b+\tilde{y}_a}{\sqrt{2}} = \frac{1}{(\beta^*+i\alpha^*)}\left[\frac{Q^*_x}{Q^*}\cos\frac{\tilde{z}_b+\tilde{y}_a}{\sqrt{2}} - \frac{Q^*}{\sqrt{2}}\sin\frac{\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\right]. \tag{4-26}$$

We integrate on $\mathbb{R}$ to obtain

$$\frac{1}{2}\int_{\mathbb{R}}(Q^*+z_b-y_a)(Q^*+z_b+y_a) = -(\beta+i\alpha+q^0)\cos\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\bigg|_{-\infty}^{\infty}.$$

Since $\lim_{\pm\infty} Q^*=0$, $\lim_{\pm\infty} Q^*_x/Q^* = \mp(\beta^*+i\alpha^*)$ (see (2-4)) and $\lim_{\pm\infty}(\tilde{z}_b+\tilde{y}_a)=0$, we get (4-24)–(4-25), because the mass of $y_a(t)$ is conserved. □

Corollary 4.7. *Assume that $t\neq t_k$ for all $k\in\mathbb{Z}$. Then the quantity*

$$E[Q^*+z_b](t) := \frac{1}{2}\int_{\mathbb{R}}(Q^*+z_b)_x^2(t) - \frac{1}{4}\int_{\mathbb{R}}(Q^*+z_b)^4(t) \tag{4-27}$$

is well-defined and independent of time. Moreover, it satisfies

$$\frac{1}{2}\int_{\mathbb{R}}(Q^*+z_b)_x^2(t) - \frac{1}{4}\int_{\mathbb{R}}(Q^*+z_b)^4(t) = E[y_a^0] - \tfrac{2}{3}(\beta^*+i\alpha^*)^3.$$

Finally, this quantity can be extended in a continuous way to every $t\in\mathbb{R}$.

Proof. Let $m=(\beta+i\alpha+q^0)$. From (4-18) we have

$$(Q^*+z_b)_x-(y_a)_x = m\cos\left(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\right)(Q^*+z_b+y_a). \tag{4-28}$$

Multiplying by $(Q^*+z_b)_x+(y_a)_x$, we get

$$\begin{aligned}(Q^*+z_b)_x^2-(y_a)_x^2 &= m\cos\left(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\right)(Q^*+z_b+y_a)(Q^*+z_b+y_a)_x\\ &= m\cos\left(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\right)\big[(Q^*+z_b)(Q^*+z_b)_x+y_a(y_a)_x\\ &\qquad +y_a(Q^*+z_b)_x+(Q^*+z_b)(y_a)_x\big].\end{aligned} \tag{4-29}$$

On the other hand, we multiply (4-28) by y_a and (Q^*+z_b) to obtain

$$y_a(Q^*+z_b)_x-y_a(y_a)_x = m\cos\left(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\right)(Q^*+z_b+y_a)y_a,$$

and

$$(Q^*+z_b)(Q^*+z_b)_x-(Q^*+z_b)(y_a)_x = m\cos\left(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\right)(Q^*+z_b+y_a)(Q^*+z_b).$$

If we subtract the latter from the former we get

$$y_a(Q^*+z_b)_x+(Q^*+z_b)(y_a)_x \\ =(Q^*+z_b)(Q^*+z_b)_x+y_a(y_a)_x+m\cos\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)[y_a^2-(Q^*+z_b)^2]. \quad (4\text{-}30)$$

Substituting (4-30) into (4-29),

$$(Q^*+z_b)_x^2-(y_a)_x^2 \\ =m\cos\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)[(Q^*+z_b)^2+y_a^2]_x+m^2\cos^2\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)[y_a^2-(Q^*+z_b)^2]. \quad (4\text{-}31)$$

Finally, we use (4-18) once again. We multiply by $(Q^*+z_b+y_a)$:

$$\frac{1}{\sqrt{2}}[(Q^*+z_b)^2-y_a^2]=m\sin\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)(Q^*+z_b+y_a).$$

Substituting in (4-31) we finally arrive to the identity

$$(Q^*+z_b)_x^2-(y_a)_x^2=m\cos\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)[(Q^*+z_b)^2+y_a^2]_x \\ -m^3\sqrt{2}\cos^2\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)\sin\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)(Q^*+z_b+y_a).$$

The last term on the right-hand side above can be recognized as a total derivative. After integration and using (4-26), we obtain

$$\begin{aligned}\int_{\mathbb{R}}[(Q^*+z_b)_x^2-(y_a)_x^2]&=m\int_{\mathbb{R}}\cos\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)[(Q^*+z_b)^2+y_a^2]_x+\frac{2}{3}m^3\cos^3\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)\Big|_{-\infty}^{+\infty}\\ &=\frac{m}{\sqrt{2}}\int_{\mathbb{R}}\sin\Big(\frac{\tilde{Q}^*+\tilde{z}_b+\tilde{y}_a}{\sqrt{2}}\Big)(Q^*+z_b+y_a)[(Q^*+z_b)^2+y_a^2]-\frac{4}{3}m^3\\ &=\frac{1}{2}\int_{\mathbb{R}}[(Q^*+z_b)^2-y_a^2][(Q^*+z_b)^2+y_a^2]-\frac{4}{3}m^3\\ &=\frac{1}{2}\int_{\mathbb{R}}[(Q^*+z_b)^4-y_a^4]-\frac{4}{3}m^3.\end{aligned}$$

Finally,

$$\frac{1}{2}\int_{\mathbb{R}}(Q^*+z_b)_x^2-\frac{1}{4}\int_{\mathbb{R}}(Q^*+z_b)^4=\frac{1}{2}\int_{\mathbb{R}}(y_a)_x^2-\frac{1}{4}\int_{\mathbb{R}}y_a^4-\frac{2}{3}(\beta+i\alpha+q^0)^3.$$

Since the right-hand side above is conserved for all time, we have proved (4-27). □

5. Complex solitons versus breathers

We introduce now the notion of breather profile. Given parameters $x_1, x_2\in\mathbb{R}$ and $\alpha,\beta>0$, we consider y_1 and y_2 defined in (2-2). Let $\tilde{B}$ be the localized profile

$$\tilde{B}=\tilde{B}(x;\alpha,\beta,x_1,x_2):=2\sqrt{2}\arctan\frac{\beta\sin(\alpha y_1)}{\alpha\cosh(\beta y_2)}, \quad (5\text{-}1)$$

and, with a slight abuse of notation, we redefine

$$B := \tilde{B}_x. \tag{5-2}$$

Note that

$$\tilde{B}(-\infty) = \tilde{B}(+\infty) = 0 \tag{5-3}$$

and, for $k \in \mathbb{Z}$,

$$\begin{cases} \tilde{B}(x;\alpha,\beta,x_1+k\pi/\alpha,x_2) = (-1)^k \tilde{B}(x;\alpha,\beta,x_1,x_2), \\ B(x;\alpha,\beta,x_1+k\pi/\alpha,x_2) = (-1)^k B(x;\alpha,\beta,x_1,x_2). \end{cases} \tag{5-4}$$

Now we introduce the directions associated to the shifts x_1 and x_2. Given a breather profile of parameters α, β, x_1 and x_2, we define

$$B_1 = B_1(x;\alpha,\beta,x_1,x_2) := \partial_{x_1} B, \tag{5-5}$$

$$B_2 = B_2(x;\alpha,\beta,x_1,x_2) := \partial_{x_2} B \tag{5-6}$$

and, for δ and γ defined in (1-4),

$$\tilde{B}_t := \delta B_1 + \gamma B_2. \tag{5-7}$$

We also have

$$\tilde{B}_t + B_{xx} + B^3 = 0; \tag{5-8}$$

see [Alejo and Muñoz 2013] for a proof of this identity.

If x_1 or x_2 are time-dependent variables, we assume that the associated B_j corresponds to the partial derivative with respect to the time-independent variable x_j, evaluated at $x_j(t)$.

In this section we will prove that there is a deep interplay between complex solitons and breather profiles. We start with the following identities:

Lemma 5.1. *Let (B, Q) be a pair breather-soliton profiles with scaling parameters $\alpha, \beta > 0$ and shifts $x_1, x_2 \in \mathbb{R}$. Assume that* (2-9) *is not satisfied. Then we have*

$$\frac{B-Q}{\sqrt{2}} - (\beta - i\alpha)\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} \equiv 0, \tag{5-9}$$

$$\text{and}\quad \tilde{B}_t - \tilde{Q}_t + (\beta - i\alpha)\left[(B_x+Q_x)\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} + \frac{B^2+Q^2}{\sqrt{2}}\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\right] \equiv 0. \tag{5-10}$$

Proof. Let us assume (5-9) and prove (5-10). We have from (2-6) and (2-11) that

$$\tilde{Q}_t = -(\beta+i\alpha)^2 Q = -(Q_{xx}+Q^3).$$

Using (5-9), we have

$$B_x - Q_x - (\beta - i\alpha)(B+Q)\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = 0,$$

and

$$B_{xx} - Q_{xx} - (\beta - i\alpha)(B_x+Q_x)\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} + (\beta - i\alpha)\frac{(B+Q)^2}{\sqrt{2}}\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = 0,$$

so that, using (5-9) and (5-8) once again,

$$\begin{aligned}
\tilde{B}_t - \tilde{Q}_t + (\beta - i\alpha)\Big[(B_x + Q_x)\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} + \frac{B^2+Q^2}{\sqrt{2}}\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big]& \\
= -(B_{xx}+B^3) + Q_{xx} + Q^3 + \Big[B_{xx} - Q_{xx} + (\beta - i\alpha)\frac{(B+Q)^2}{\sqrt{2}}\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big]& \\
+ (\beta - i\alpha)\frac{B^2+Q^2}{\sqrt{2}}\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}& \\
= Q^3 - B^3 + \sqrt{2}(\beta - i\alpha)(B^2+Q^2+BQ)\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}& \\
= Q^3 - B^3 + (B^2+Q^2+BQ)(B-Q) = 0.&
\end{aligned}$$

The proof of (5-9) is a tedious but straightforward computation which deeply exploits the nature of the breather and soliton profiles. For the proof of this result, see the Appendix. □

Corollary 5.2. *Under the assumptions of Lemma 5.1, for any $x \in \mathbb{R}$ one has*

$$\frac{B-\overline{Q}}{\sqrt{2}} - (\beta + i\alpha)\sin\frac{\tilde{B}+\tilde{\overline{Q}}}{\sqrt{2}} \equiv 0 \quad \textit{in } \mathbb{R},$$

where $\overline{Q}$ is the complex-valued soliton with parameters β and $-\alpha$.

In order to prove some results in the next section, we need several additional identities.

Corollary 5.3. *Under the assumptions of Lemma 5.1, for any $x \in \mathbb{R}$ one has*

$$\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = 1 - \frac{1}{2(\beta - i\alpha)}\int_{-\infty}^{x}(B^2-Q^2) \quad \textit{and} \quad \lim_{x\to\pm\infty}\cos\Big(\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big)(x) = \mp 1.$$

Remark. Note that both limits above make sense since, from (2-15) and (2-14), we have, for all x,

$$\begin{aligned}
\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} &= \cos\frac{\tilde{B}}{\sqrt{2}}\cos\frac{\tilde{Q}}{\sqrt{2}} - \sin\frac{\tilde{B}}{\sqrt{2}}\sin\frac{\tilde{Q}}{\sqrt{2}} \\
&= \frac{1}{\beta + i\alpha}\Big[\frac{Q_x}{Q}\cos\frac{\tilde{B}}{\sqrt{2}} - \frac{Q}{\sqrt{2}}\sin\frac{\tilde{B}}{\sqrt{2}}\Big].
\end{aligned}$$

In particular,

$$\lim_{\pm\infty}\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = \frac{1}{\beta + i\alpha}\times \mp(\beta + i\alpha) = \mp 1.$$

Proof. We multiply by $(1/\sqrt{2})(B+Q)$ in (5-9). We get

$$\frac{1}{2}(B^2-Q^2) - (\beta - i\alpha)\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(B+Q) = 0,$$

i.e.,

$$\frac{1}{2}(B^2-Q^2) + (\beta - i\alpha)\partial_x\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = 0.$$

From (2-1) and (5-1), one has

$$\lim_{x\to-\infty}\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}=1.$$

Therefore, after integration,

$$\frac{1}{2}\int_{-\infty}^{x}(B^2-Q^2)+(\beta-i\alpha)\left[\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}-1\right]=0,$$

as desired. □

Lemma 5.4.

$$\mathcal{M}:=2\beta\left[1+\frac{\alpha(\beta\sin(2\alpha y_1)+\alpha\sinh(2\beta y_2))}{\alpha^2+\beta^2-\beta^2\cos(2\alpha y_1)+\alpha^2\cosh(2\beta y_2)}\right].$$

Proof. See, e.g., [Alejo and Muñoz 2013]. □

The following result is not difficult to prove:

Corollary 5.5. *We have*

$$\int_0^x \mathcal{M}=2\beta x+\log(\alpha^2+\beta^2-\beta^2\cos(2\alpha y_1)+\alpha^2\cosh(2\beta y_2))-L_0, \tag{5-11}$$

where

$$L_0:=\log(\alpha^2+\beta^2-\beta^2\cos(2\alpha x_1)+\alpha^2\cosh(2\beta x_2)).$$

Corollary 5.6. *Under the assumptions of Lemma 5.1, we have*

$$\begin{aligned}-(\beta-i\alpha)\int_0^x\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\\ =(\beta+i\alpha)x+\log(\alpha^2+\beta^2-\beta^2\cos(2\alpha y_1)+\alpha^2\cosh(2\beta y_2))-\log(1+e^{2\beta y_2+2i\alpha y_1})-L_0+L_1,\end{aligned}$$

with L_0 and L_1 as defined in (5-11) *and* (2-19).

Proof. We have, from Corollaries 5.3 and 5.5 and (2-19),

$$\begin{aligned}&\int_0^x\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\\ &=x-\frac{1}{\beta-i\alpha}\int_0^x(\mathcal{M}-\mathcal{N})\\ &=x-\frac{1}{\beta-i\alpha}[2\beta x+\log(\alpha^2+\beta^2-\beta^2\cos(2\alpha y_1)+\alpha^2\cosh(2\beta y_2))-\log(1+e^{2(\beta y_2+i\alpha y_1)})-L_0+L_1]\\ &=-\frac{1}{\beta-i\alpha}\big[(\beta+i\alpha)x+\log(\alpha^2+\beta^2-\beta^2\cos(2\alpha y_1)+\alpha^2\cosh(2\beta y_2))\\ &\qquad\qquad-\log(1+e^{2(\beta y_2+i\alpha y_1)})-L_0+L_1\big],\end{aligned}$$

as desired. □

Corollary 5.7. *Assume that $x_1, x_2\in\mathbb{R}$ do not satisfy* (2-9)*. Consider the function*

$$\mu(x;\alpha,\beta,x_1,x_2):=2\sqrt{2}\alpha^2\beta^2\frac{\cosh(\beta y_2)\cos(\alpha y_1)+i\sinh(\beta y_2)\sin(\alpha y_1)}{\alpha^2\cosh^2(\beta y_2)+\beta^2\sin^2(\alpha y_1)}=\beta\tilde{B}_1-i\alpha\tilde{B}_2. \tag{5-12}$$

Then we have

$$\lim_{x\to\pm\infty}\mu(x)=0 \tag{5-13}$$

$$\textit{and}\quad \mu_x=(\beta-i\alpha)\cos\Big(\frac{\tilde B+\tilde Q}{\sqrt2}\Big)\mu. \tag{5-14}$$

Proof. Identity (5-13) is trivial. Let us prove (5-14). First of all, note that (cf. (2-7))

$$\beta\tilde Q_1-i\alpha\tilde Q_2\equiv 0. \tag{5-15}$$

On the other hand, from (5-9) we have

$$(\tilde B_1-\tilde Q_1)_x-(\beta-i\alpha)\cos\Big(\frac{\tilde B+\tilde Q}{\sqrt2}\Big)(\tilde B_1+\tilde Q_1)=0.$$

Similarly,

$$(\tilde B_2-\tilde Q_2)_x-(\beta-i\alpha)\cos\Big(\frac{\tilde B+\tilde Q}{\sqrt2}\Big)(\tilde B_2+\tilde Q_2)=0.$$

We then have

$$\begin{aligned}\mu_x&=(\beta\tilde B_1-i\alpha\tilde B_2)_x\\ &=(\beta-i\alpha)\cos\Big(\frac{\tilde B+\tilde Q}{\sqrt2}\Big)\mu+(\beta\tilde Q_1-i\alpha\tilde Q_2)_x+(\beta-i\alpha)\cos\Big(\frac{\tilde B+\tilde Q}{\sqrt2}\Big)(\beta\tilde Q_1-i\alpha\tilde Q_2)\\ &=(\beta-i\alpha)\cos\Big(\frac{\tilde B+\tilde Q}{\sqrt2}\Big)\mu.\end{aligned}$$

The proof is complete. □

Lemma 5.8. *Assume that* (2-9) *does not hold. Then μ defined in* (5-12) *has no zeroes*, *i.e.*, $|\mu(x)|>0$ *for all $x\in\mathbb{R}$.*

Proof. From (5-12) we have $\mu(x)=0$ if and only if $\cos(\beta y_1)=0$ and $\sinh(\alpha y_2)=0$, i.e., from (2-10) we have that (2-9) is satisfied. □

Now we consider the opposite case, where the sign in front of (5-14) is negative. We finish this section with the following result:

Lemma 5.9. *Assume that* (2-9) *does not hold. Then*

$$\mu^1(x;\alpha,\beta,x_1,x_2):=\frac{1}{\mu}(x;\alpha,\beta,x_1,x_2),$$

with μ as defined in (5-12), *is well-defined*, *has no zeroes and satisfies*

$$\lim_{x\to\pm\infty}|\mu^1(x)|=+\infty\quad\textit{and}\quad \mu^1_x=-(\beta-i\alpha)\cos\Big(\frac{\tilde B+\tilde Q}{\sqrt2}\Big)\mu^1.$$

Proof. This is a direct consequence of Corollary 5.7 and Lemma 5.8. □

6. Double Bäcklund transformation for mKdV

Assume that x_1 and x_2 do not satisfy (2-9). Consider the breather and soliton profiles B and Q defined in (5-2) and (2-5), which are well-defined by Lemma 2.2. From Lemma 5.1, we have the following result:

Lemma 6.1. *We have, for all $x \in \mathbb{R}$,*

$$G(B, Q, \tilde{B}_t, \tilde{Q}_t, \beta - i\alpha) = (0, 0).$$

Note that the previous identity can be extended by zero to the case where x_1 and x_2 satisfy (2-9), in such a form that now $G(B, Q, \tilde{B}_t, \tilde{Q}_t, \beta - i\alpha)$ as a function of x_1 and x_2 is well-defined and continuous everywhere in $\mathbb{R}^2$ (and identically zero).

Define (cf. (5-1)–(5-7)),

$$\begin{cases} \tilde{B}^0(x; \alpha, \beta) := \tilde{B}(x; \alpha, \beta, 0, 0), \\ \tilde{B}^0_t(x; \alpha, \beta) := \delta \tilde{B}_1(x; \alpha, \beta, 0, 0) + \gamma \tilde{B}_2(x; \alpha, \beta, 0, 0), \\ B^0(x; \alpha, \beta) := \partial_x \tilde{B}(x; \alpha, \beta, 0, 0). \end{cases} \tag{6-1}$$

Finally, for $z_a^0 \in H^1(\mathbb{R})$ we define

$$\omega_a^0 := -((z_a^0)_{xx} + (z_a^0)^3) \in H^{-1}(\mathbb{R}). \tag{6-2}$$

We will use Lemma 6.1 and apply Propositions 3.3 and 3.4 in a neighborhood of the complex soliton and the breather at time zero. Recall that, by Lemma 2.2, the complex soliton Q^0 is everywhere well-defined since (2-9) is not satisfied.

Lemma 6.2. *There exists $\eta_0 > 0$ and a constant $C > 0$ such that, for all $0 < \eta < \eta_0$, the following holds: Assume that $z_a^0 \in H^1(\mathbb{R})$ satisfies*

$$\|z_a^0\|_{H^1(\mathbb{R})} < \eta, \quad \omega_a^0 \text{ defined by } (6\text{-}2).$$

Then there exist unique $z_b^0 \in H^1(\mathbb{R}, \mathbb{C})$, $\omega_b^0 \in H^{-1}(\mathbb{R}; \mathbb{C})$ and $m_1 \in \mathbb{C}$ of the form

$$z_b^0(x) = z_b^0[z_a^0](x), \quad \omega_b^0(x) = \omega_b^0[z_a^0, \omega_a^0](x), \quad m_1 = m_1[z_a^0] := \beta - i\alpha + p^0$$

such that

$$\begin{gathered} \|z_b^0\|_{H^1(\mathbb{R};\mathbb{C})} + |p^0| \le C\eta, \\ \tilde{z}_a + \tilde{z}_b \in H^2(\mathbb{R}; \mathbb{C}), \\ \text{and} \quad G(B^0 + z_a^0, Q^0 + z_b^0, \tilde{B}^0_t + \omega_a^0, \tilde{Q}^0_t + \omega_b^0, m_1) \equiv (0, 0). \end{gathered}$$

Proof. Let Q^0 and B^0 be the soliton and breather profiles defined in (4-1) and (6-1). We will apply Proposition 3.3 with

$$u_a^0 := B^0, \quad u_b^0 := Q^0, \quad v_a^0 := \tilde{B}^0_t, \quad v_b^0 := \tilde{Q}^0_t, \quad m^0 := \beta + i\alpha.$$

Clearly $\operatorname{Re} m^0 = \beta > 0$, so that (3-4) is satisfied. On the other hand, (3-5) is a consequence of Lemma 6.1. From (5-9), condition (3-6) reads

$$\sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}} = \frac{(B^0-Q^0)}{\sqrt{2}(\beta-i\alpha)} \in H^1(\mathbb{R};\mathbb{C}).$$

Condition (3-7) is clearly satisfied (see (2-3) and (5-3)). From Corollary 5.7 we have

$$\mu^0 = \beta(\tilde{B}_1)^0 - i\alpha(\tilde{B}_2)^0.$$

Note that, from Lemmas 2.2 and 5.8, μ^0 has no zeroes in the complex plane and it is exponentially decreasing in space. Finally, let us show that

$$\int_{\mathbb{R}} \mu^0 \sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}} = \frac{4i\alpha\beta}{\beta-i\alpha}.$$

First of all, we have from (5-15) that

$$\begin{aligned}
&[\beta(\tilde{B}_1)^0 - i\alpha(\tilde{B}_2)^0]\sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}} \\
&\qquad = [\beta(\tilde{B}_1+\tilde{Q}_1)^0 - i\alpha(\tilde{B}_2+\tilde{Q}_2)^0]\sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}} + [-\beta(\tilde{Q}_1)^0 + i\alpha(\tilde{Q}_2)^0]\sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}}.
\end{aligned}$$

Consequently,

$$[\beta(\tilde{B}_1)^0 - i\alpha(\tilde{B}_2)^0]\sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}} = -\sqrt{2}\beta\partial_{x_1}\Big[\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big]\Big|^0 + i\alpha\sqrt{2}\partial_{x_2}\Big[\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big]\Big|^0.$$

Therefore, if $R_1,\ R_2 > 0$ are independent of x_1 and x_2,

$$\begin{aligned}
\int_{-R_2}^{R_1} \mu^0 \sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}} &= \sqrt{2}\int_{-R_2}^{R_1}\Big\{-\beta\partial_{x_1}\Big[\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big]\Big|^0 + i\alpha\partial_{x_2}\Big[\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big]\Big|^0\Big\} \\
&= \sqrt{2}\Big\{-\beta\partial_{x_1}\int_{-R_2}^{R_1}\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} + i\alpha\partial_{x_2}\int_{-R_2}^{R_1}\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}}\Big\}\Big|^0.
\end{aligned}$$

Now we use Corollary 5.6: we have

$$\partial_{x_1}\int_{-R_2}^{R_1}\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = -\frac{1}{\beta-i\alpha}\Big[\frac{2i\alpha e^{2\beta y_2+2i\alpha y_1}}{1+e^{2\beta y_2+2i\alpha y_1}} - \frac{2\alpha\beta^2\sin(2\alpha y_1)}{\alpha^2+\beta^2-\beta^2\cos(2\alpha y_1)+\alpha^2\cosh(2\beta y_2)}\Big]\Big|_{-R_1}^{R_2}.$$

We have that

$$\lim_{R_1,R_2\to\infty}\partial_{x_1}\int_{-R_2}^{R_1}\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = -\frac{2i\alpha}{\beta-i\alpha}.$$

Similarly,

$$\partial_{x_2}\int_{-R_2}^{R_1}\cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = -\frac{1}{\beta-i\alpha}\Big[\frac{2\beta e^{2\beta y_2+2i\alpha y_1}}{1+e^{2\beta y_2+2i\alpha y_1}} - \frac{2\alpha^2\beta\sinh(2\beta y_2)}{\alpha^2+\beta^2-\beta^2\cos(2\alpha y_1)+\alpha^2\cosh(2\beta y_2)}\Big]\Big|_{-R_1}^{R_2}$$

and

$$\lim_{R_1,R_2\to\infty} \partial_{x_2} \int_{-R_2}^{R_1} \cos\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = -\frac{2\beta-4\beta}{\beta-i\alpha} = \frac{2\beta}{\beta-i\alpha}.$$

Adding the previous identities, we finally obtain

$$\int_{\mathbb{R}} \mu^0 \sin\frac{\tilde{B}^0+\tilde{Q}^0}{\sqrt{2}} = \left[\frac{2i\alpha\beta}{\beta-i\alpha}+\frac{2i\alpha\beta}{\beta-i\alpha}\right] = \frac{4i\alpha\beta}{\beta-i\alpha} \neq 0.$$

After applying Proposition 3.3, we are done. □

Now we address the following very important question: is the y_a^0 given in Lemma 4.1 real-valued for all $x \in \mathbb{R}$? In general, it seems that the answer is negative; however, if z_a^0 in Lemma 6.2 is real-valued, and z_b^0 from Lemma 6.2 satisfies (4-2), then the corresponding function y_a^0 given in Lemma 4.1 is also real-valued. This property is a consequence of a deep result called the *permutability theorem*, which we explain below.

First of all, from Lemma 6.2 we have

$$\frac{1}{\sqrt{2}}(B^0+z_a^0-Q^0-z_b^0) = (\beta-i\alpha+p^0)\sin\frac{\tilde{B}^0+\tilde{z}_a^0+\tilde{Q}^0+\tilde{z}_b^0}{\sqrt{2}} \tag{6-3}$$

for some small $p^0 \in \mathbb{C}$, and

$$\sin\frac{\tilde{B}^0+\tilde{z}_a^0+\tilde{Q}^0+\tilde{z}_b^0}{\sqrt{2}} \in H^1(\mathbb{R};\mathbb{C}). \tag{6-4}$$

Now, by taking η_0 smaller if necessary, such that $C\eta < \nu_0$ for all $0<\eta<\eta_0$, Lemma 4.1 also applies. We get

$$\frac{1}{\sqrt{2}}(Q^0+z_b^0-y_a^0) = (\beta+i\alpha+q^0)\sin\frac{\tilde{Q}^0+\tilde{z}_b^0+\tilde{y}_a^0}{\sqrt{2}}, \tag{6-5}$$

for some small q^0.

We need some auxiliary notation. Define

$$\beta_* := \beta + \operatorname{Re} p^0, \quad \alpha_* := \alpha - \operatorname{Im} p^0,$$

so that (compare with (4-6))

$$\beta - i\alpha + p^0 = \beta_* - i\alpha_*.$$

We also consider

$$\tilde{Q}_*^0 := \tilde{Q}(\,\cdot\,; -\alpha_*, \beta_*, 0, 0), \quad Q_*^0 := Q(\,\cdot\,; -\alpha_*, \beta_*, 0, 0).$$

Note that, since p^0 is small, we have that Q_*^0 and $\overline{Q}^0$ share the same properties, i.e., they are close enough. Indeed,

$$\|Q_*^0-\overline{Q}^0\|_{H^1(\mathbb{R};\mathbb{C})} \le C\eta. \tag{6-6}$$

Moreover, thanks to Lemma 2.4 applied to Q_*^0,

$$\frac{1}{\sqrt{2}}Q_*^0 = (\beta-i\alpha+p^0)\sin\frac{\tilde{Q}_*^0}{\sqrt{2}}.$$

Consequently, applying Proposition 3.4 starting at y_a^0 and using (6-6), we can define z_d^0 via the identity

$$\frac{1}{\sqrt{2}}(\bar{Q}^0 + z_d^0 - y_a^0) = (\beta - i\alpha + p^0)\sin\frac{\tilde{\bar{Q}}^0 + \tilde{z}_d^0 + \tilde{y}_a^0}{\sqrt{2}}. \tag{6-7}$$

Similarly, using (4-6) and (6-1) we define

$$(\tilde{B}^0)^* := \tilde{B}^0(\,\cdot\,;\alpha^*,\beta^*), \quad (B^0)^* := B(\,\cdot\,;\alpha^*,\beta^*), \tag{6-8}$$

so that from Lemma 5.1 we have

$$\frac{1}{\sqrt{2}}((B^0)^* - (Q^0)^*) = (\beta^* - i\alpha^*)\sin\frac{(\tilde{B}^0)^* + (\tilde{Q}^0)^*}{\sqrt{2}},$$

and applying Corollary 5.2 we get

$$\frac{1}{\sqrt{2}}((B^0)^* - \overline{(Q^0)^*}) = (\beta + i\alpha + q^0)\sin\frac{(\tilde{B}^0)^* + \overline{(\tilde{Q}^0)^*}}{\sqrt{2}}.$$

Using that

$$\|(B^0)^* - B^0\|_{H^1(\mathbb{R})} \le C\eta, \quad \|\overline{(Q^0)^*} - \bar{Q}^0\|_{H^1(\mathbb{R};\mathbb{C})} \le C\eta,$$

we can use Proposition 3.4 to obtain

$$\frac{1}{\sqrt{2}}(B^0 + z_c^0 - \bar{Q}^0 - z_d^0) = (\beta + i\alpha + q^0)\sin\frac{\tilde{B}^0 + \tilde{z}_c^0 + \tilde{\bar{Q}}^0 + \tilde{z}_d^0}{\sqrt{2}} \tag{6-9}$$

for some z_c^0 small. Note that the coefficients $(\beta - i\alpha + p^0)$ and $(\beta + i\alpha + q^0)$ were left fixed this time. Note additionally that z_d^0 and z_c^0 are bounded functions. Now we can state a permutability theorem [Lamb 1980, p. 246]. This is part of a more general result, standard in the mathematical physics literature; see [Wahlquist and Estabrook 1973] for a formal proof in the Korteweg–de Vries (KdV) case.

Theorem 6.3 (permutability theorem). *We have*

$$\tilde{z}_c^0 \equiv \tilde{z}_a^0. \tag{6-10}$$

In particular, z_c^0 is an H^1 real-valued function.

Proof. Define

$$u_0 := y_a^0, \quad u_1 := Q^0 + z_b^0, \quad u_2 := \bar{Q}^0 + z_d^0, \tag{6-11}$$

$$u_{12} := B^0 + z_a^0, \quad u_{21} := B^0 + z_c^0, \tag{6-12}$$

$$\text{and} \quad \kappa_1 := \beta + i\alpha + q^0, \quad \kappa_2 := \beta - i\alpha + p^0. \tag{6-13}$$

Since p^0 and q^0 are small quantities, we have $\kappa_1 \neq \kappa_2$, and both are nonzero complex numbers. Equations (6-5), (6-3), (6-7) and (6-9) now read

$$\frac{u_1 - u_0}{\sqrt{2}} = \kappa_1 \sin \frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}, \tag{6-14}$$

$$\frac{u_{12} - u_1}{\sqrt{2}} = \kappa_2 \sin \frac{\tilde{u}_{12} + \tilde{u}_1}{\sqrt{2}}, \tag{6-15}$$

$$\frac{u_2 - u_0}{\sqrt{2}} = \kappa_2 \sin \frac{\tilde{u}_2 + \tilde{u}_0}{\sqrt{2}},$$

$$\frac{u_{21} - u_2}{\sqrt{2}} = \kappa_1 \sin \frac{\tilde{u}_{21} + \tilde{u}_2}{\sqrt{2}}.$$

Note that u_1 and u_2 are obtained via the implicit function theorem and therefore there is an associated uniqueness property for solutions obtained in a small neighborhood of the breather. The idea is to prove that $\tilde{u}_{21} \equiv \tilde{u}_{12}$. Define $\tilde{u}_3$ via the identity

$$\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}} = -\arctan\left[\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \tan \frac{\tilde{u}_{12} - \tilde{u}_0}{2\sqrt{2}}\right]. \tag{6-16}$$

Whenever $u_1 = Q^0$, $u_{12} = B^0$, $u_0 = 0$, $\kappa_1 = \beta + i\alpha$ and $\kappa_2 = \beta - i\alpha$, we get from (1-2) that

$$\frac{\tilde{u}_3 - \tilde{Q}^0}{2\sqrt{2}} = -\arctan\left[i\frac{\alpha}{\beta} \tan \frac{\tilde{B}^0}{2\sqrt{2}}\right] = -\arctan\left(i\frac{\sin(\alpha x)}{\cosh(\beta x)}\right) = -\arctan \frac{e^{i\alpha x} - e^{-i\alpha x}}{e^{\beta x} + e^{-\beta x}}.$$

Therefore, using (2-1),

$$\begin{aligned}
\tilde{u}_3 &= 2\sqrt{2}\arctan(e^{(\beta+i\alpha)x}) - 2\sqrt{2}\arctan \frac{e^{i\alpha x} - e^{-i\alpha x}}{e^{\beta x} + e^{-\beta x}} \\
&= 2\sqrt{2}\arctan \frac{e^{(\beta+i\alpha)x} - (e^{i\alpha x} - e^{-i\alpha x})/(e^{\beta x} + e^{-\beta x})}{1 + e^{(\beta+i\alpha)x}(e^{i\alpha x} - e^{-i\alpha x})/(e^{\beta x} + e^{-\beta x})} \\
&= 2\sqrt{2}\arctan(e^{(\beta-i\alpha)x}) \\
&= \bar{\tilde{Q}}^0.
\end{aligned}$$

Consequently, under the smallness assumptions in (6-11)–(6-13) (the open character of these sets is essential) we have that $\tilde{u}_3$ is still well-defined on the real line with values in the complex plane, and it is close to $\bar{\tilde{Q}}^0$, as well as to $\tilde{u}_2$.

Let us find an equation for $\tilde{u}_3$. As usual, define $u_3 := (\tilde{u}_3)_x$. We claim that

$$\frac{u_3 - u_0}{\sqrt{2}} = \kappa_2 \sin \frac{\tilde{u}_3 + \tilde{u}_0}{\sqrt{2}}; \tag{6-17}$$

in other words, $\tilde{u}_3 \equiv \tilde{u}_2$. Similarly, if $\tilde{u}_4$ solves

$$\frac{\tilde{u}_2 - \tilde{u}_4}{2\sqrt{2}} = -\arctan\left[\frac{\kappa_1 - \kappa_2}{\kappa_1 + \kappa_2} \tan \frac{\tilde{u}_{21} - \tilde{u}_0}{2\sqrt{2}}\right], \tag{6-18}$$

then

$$\frac{u_4 - u_0}{\sqrt{2}} = \kappa_1 \sin\frac{\tilde{u}_4 + \tilde{u}_0}{\sqrt{2}},$$

which implies $\tilde{u}_4 \equiv \tilde{u}_1$. Finally, from (6-16) and (6-18) we have $\tilde{u}_{12} \equiv \tilde{u}_{21}$, which proves (6-10). Even better, we have[13]

$$\tan\frac{\tilde{u}_{12} - \tilde{u}_0}{2\sqrt{2}} = -\frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2}\tan\frac{\tilde{u}_2 - \tilde{u}_1}{2\sqrt{2}}. \tag{6-19}$$

Now let us prove (6-17). First of all, denote

$$\ell := \frac{\kappa_1 + \kappa_2}{\kappa_1 - \kappa_2}. \tag{6-20}$$

We have, from (6-16),

$$\frac{\tilde{u}_{12} - \tilde{u}_0}{\sqrt{2}} = -2\arctan\left[\ell\tan\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}\right],$$

so that

$$u_{12} - u_0 = \frac{-\ell(u_3 - u_1)\sec^2((\tilde{u}_3 - \tilde{u}_1)/(2\sqrt{2}))}{1 + \ell^2\tan^2((\tilde{u}_3 - \tilde{u}_1)/(2\sqrt{2}))}.$$

We also check that

$$\sin\frac{\tilde{u}_{12} - \tilde{u}_0}{\sqrt{2}} = \frac{-2\ell\tan((\tilde{u}_3 - \tilde{u}_1)/(2\sqrt{2}))}{1 + \ell^2\tan^2((\tilde{u}_3 - \tilde{u}_1)/(2\sqrt{2}))},$$

and

$$\cos\frac{\tilde{u}_{12} - \tilde{u}_0}{\sqrt{2}} = \frac{1 - \ell^2\tan^2((\tilde{u}_3 - \tilde{u}_1)/(2\sqrt{2}))}{1 + \ell^2\tan^2((\tilde{u}_3 - \tilde{u}_1)/(2\sqrt{2}))}.$$

Substituting in (6-15) and using (6-14) we obtain

$$\begin{aligned}&-\ell\frac{u_3 - u_1}{\sqrt{2}}\sec^2\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}\\ &= \kappa_1\sin\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\left[1 + \ell^2\tan^2\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}\right] + \kappa_2\sin\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\left[1 - \ell^2\tan^2\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}\right] - 2\ell\kappa_2\cos\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\tan\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}.\end{aligned}$$

Using (6-20) and (6-14), we have

$$\begin{aligned}&u_3 - u_0 - \sqrt{2}\kappa_1\sin\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\\ &\quad = -\sqrt{2}\cos^2\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}\left[(\kappa_1 - \kappa_2)\sin\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\left(1 + \ell\tan^2\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}\right) - 2\kappa_2\cos\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\tan\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}}\right],\end{aligned}$$

i.e., after some standard trigonometric simplifications,

$$u_3 - u_0 = \sqrt{2}\kappa_2\sin\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\cos\frac{\tilde{u}_3 - \tilde{u}_1}{\sqrt{2}} + \sqrt{2}\kappa_2\cos\frac{\tilde{u}_1 + \tilde{u}_0}{\sqrt{2}}\sin\frac{\tilde{u}_3 - \tilde{u}_1}{2\sqrt{2}} = \sqrt{2}\kappa_2\sin\frac{\tilde{u}_3 + \tilde{u}_0}{\sqrt{2}},$$

as desired. $\square$

Another consequence of the previous result is the following equivalent result:

[13] Note that this identity is well-defined at one particular set of functions, then extended by continuity.

Corollary 6.4. *We have*

$$z_d^0 \equiv \overline{z_b^0} \quad \text{and} \quad p^0 = \overline{q^0}.$$

In other words, $\alpha^* = \alpha_*$ *and* $\beta^* = \beta_*$.

Proof. Note that $z_a^0 \equiv z_c^0$. From (6-9) we have

$$\frac{1}{\sqrt{2}}(B^0 + z_a^0 - Q^0 - \overline{z_d^0}) = (\beta - i\alpha + \overline{q^0}) \sin \frac{\tilde{B}^0 + \tilde{z}_a^0 + \tilde{Q}^0 + \overline{\tilde{z}_d^0}}{\sqrt{2}}.$$

The result follows from (6-3) and the uniqueness of z_b^0 and p^0 as implicit functions of z_a^0. □

The key result of this paper is the following surprising property:

Corollary 6.5. *The function* y_a^0 *is real-valued. Moreover, there is a small ball of data* z_a^0 *in* $H^1(\mathbb{R})$ *for which the corresponding data* z_b^0 *lies in an open set of* $H^1(\mathbb{R}; \mathbb{C})$.

Proof. The second statement is a consequence of the implicit function theorem. On the other hand, the first one is consequence of the permutability theorem. First of all, note that

$$\overline{\beta + i\alpha + q^0} = \beta - i\alpha + p^0 = \beta^* - i\alpha^*. \tag{6-21}$$

Now, from (6-19) we get

$$\tan \frac{B^0 + z_a^0 - y_a^0}{2\sqrt{2}} = -\frac{\beta + \operatorname{Re} p^0}{i(\alpha - \operatorname{Im} p^0)} \tan \frac{\tilde{Q}^0 + \tilde{z}_b^0 - \bar{\tilde{Q}}^0 - \bar{\tilde{z}}_b{}^0}{2\sqrt{2}},$$

so

$$\tan \frac{B^0 + z_a^0 - y_a^0}{2\sqrt{2}} = -\frac{\beta + \operatorname{Re} p^0}{(\alpha - \operatorname{Im} p^0)} \tanh \frac{\operatorname{Im}(\tilde{Q}^0 + \tilde{z}_b^0)}{\sqrt{2}},$$

from which we have that $y_a^0(x)$ is real-valued for all $x \in \mathbb{R}$. □

The main advantage of the double Bäcklund transformation is that now the dynamics of y_a^0 is real-valued. We apply Theorem 4.5 with the initial data z_b^0 to get a complex solution of mKdV, $u_b(t) = Q^*(t) + z_b(t)$ defined for all $t \neq t_k$ and satisfying (4-22).

Now we reconstruct $z_a(t)$. As in (6-8), let us define, using (5-1), (4-6) and (4-7),

$$\tilde{B}^*(t,x) := \tilde{B}(x; \alpha^*, \beta^*, \delta^* t + x_1, \gamma^* t + x_2) \tag{6-22}$$

and

$$B^*(t,x) = \partial_x \tilde{B}^*(t,x), \quad \tilde{B}_j^*(t,x) := \tilde{B}_j(x; \alpha^*, \beta^*, x_1, x_2)\big|_{x_1 = \delta^* t + x_1,\, x_2 = \gamma^* t + x_2}. \tag{6-23}$$

In other words, we recover the original breather in (1-2) with scaling parameters α^* and β^* and shifts x_1, x_2, provided they do not depend on time. Finally, as in (5-7) we define

$$\tilde{B}_t^*(t,x) := \delta \tilde{B}_1^*(t,x) + \gamma \tilde{B}_2^*(t,x).$$

Lemma 6.6. *Assume that $t \in \mathbb{R}$ is such that* (4-11) *holds. Then there are unique $z_a = z_a(t) \in H^1(\mathbb{R};\mathbb{C})$ and $w_a = w_a(t) \in H^{-1}(\mathbb{R};\mathbb{C})$ such that*

$$\tilde{z}_a + \tilde{z}_b \in H^2(\mathbb{R};\mathbb{C}), \tag{6-24}$$

$$\frac{1}{\sqrt{2}}(B^* + z_a - Q^* - z_b) = (\beta - i\alpha + p^0)\sin\frac{\tilde{B}^* + \tilde{z}_a + \tilde{Q}^* + \tilde{z}_b}{\sqrt{2}}, \tag{6-25}$$

where $\tilde{B}^*$ and B^* are defined in (6-22) *and* (6-23). *Moreover*, *we have*

$$\begin{aligned} 0 = \tilde{B}^*_t + w_a - \tilde{Q}^*_t - w_b \\ + (\beta - i\alpha + p^0)\bigg[(B^*_x + (z_a)_x + Q^*_x + (z_b)_x)\cos\frac{\tilde{B}^* + \tilde{z}_a + \tilde{Q}^* + \tilde{z}_b}{\sqrt{2}} \\ + \frac{(B^* + z_a)^2 + (Q^* + z_b)^2}{\sqrt{2}} \sin\frac{\tilde{B}^* + \tilde{z}_a + \tilde{Q}^* + \tilde{z}_b}{\sqrt{2}}\bigg] \end{aligned} \tag{6-26}$$

and, *for all $t \neq t_k$*,

$$\|z_a(t)\|_{H^1(\mathbb{R};\mathbb{C})} \le C\eta.$$

Proof. We apply Proposition 3.4 at the point

$$X^0 := (B^*, Q^*, \tilde{B}^*_t, \tilde{Q}^*_t, \beta - i\alpha + p^0),$$

because a slight variation of Lemma 6.1 shows that (compare with (6-21))

$$G(B^*, Q^*, \tilde{B}^*_t, \tilde{Q}^*_t, \beta - i\alpha + p^0) = (0, 0).$$

Since p^0 is small,

$$\operatorname{Re}(\beta - i\alpha + p^0) > 0.$$

On the other hand, (3-6) is a consequence of (6-4). Similarly, from (2-3) we get that (3-7) is satisfied. Finally, in order to ensure that (3-26) is clearly satisfied, we apply Lemma 5.9: we get

$$\mu^1 = \frac{1}{\mu^*}, \quad \text{where} \quad \mu^* := \beta^* \tilde{B}^*_1 - i\alpha^* \tilde{B}^*_2;$$

see Corollary 5.7 and (6-23). Then we conclude thanks to Proposition 3.4. □

Corollary 6.7. *The function $z_a(t)$ as defined in* (6-25) *is real-valued.*

Proof. The same proof as in Corollary 6.5 works mutatis mutandis, since now $y_a(t)$ is real-valued. □

Proposition 6.8. *For all $t \neq t_k$, $u_a = B^* + z_a$ is an H^1 real-valued solution to mKdV with initial data u_0. Therefore*, *by uniqueness*,[14] *$B^* + z_a \equiv u$.*

Proof. Since $u_b = Q^* + z_b$ solves mKdV, we use (6-25)–(6-26) and Theorem 3.5 to conclude. □

[14]Technically, what we need is a result about unconditional uniqueness, however, from [Kwon and Oh 2012] one can conclude that such a result is valid for mKdV on the line if we consider data with H^1 regularity.

7. Stability of breathers

We now prove Theorem 1.2. We assume that $u_0 \in H^1(\mathbb{R})$ satisfies (1-5) for some η small. Let $u \in C(\mathbb{R}; H^1(\mathbb{R}))$ be the — unique in a certain sense — associated solution of the Cauchy problem (1-1) with initial data $u(0) = u_0$. Finally, we recall the conserved quantities of mass (4-13) and energy (4-14).

Proof of Theorem 1.2. Consider $\varepsilon_0 > 0$ small but fixed, $A_0 > 1$ and $0 < \eta < \eta_0$ small. From Lemmas 6.2 and 6.6 the proof is not difficult, and we follow standard methods; see [Martel et al. 2002] for instance. Indeed, define the tubular neighborhood

$$\mathcal{V}(A_0, \eta) := \Big\{ U \in H^1(\mathbb{R}) \;\Big|\; \inf_{\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}} \|U - B(\,\cdot\,; \alpha, \beta, \tilde{x}_1, \tilde{x}_2)\| \le A_0 \eta \Big\}. \tag{7-1}$$

Note that B represents here the breather profile defined in (5-2). The original breather $B(t)$ from (1-2) can be recovered using (6-22) as follows (there is a slight abuse of notation here, but it is easily understood):

$$B(t, x; \alpha, \beta, x_1, x_2) = B(x; \alpha, \beta, \delta t + x_1, \gamma t + x_2).$$

Clearly $u(t) \in \mathcal{V}(A_0, \eta)$ for small $t > 0$. Define the set

$$J_{\varepsilon_0} := \{ t > 0 \mid |t - t_k| > \varepsilon_0 \text{ for all } k \in \mathbb{Z} \}.$$

We will prove that $u(t)$ is in $\mathcal{V}(A_0, \eta)$ for all $t \in J_{\varepsilon_0}$ provided A_0 is chosen large enough.

We argue by reductio ad absurdum. Assume that, for some $T_0 \in J_{\varepsilon_0}$, we have

$$\inf_{\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}} \|u(T_0) - B(\,\cdot\,; \alpha, \beta, \tilde{x}_1, \tilde{x}_2)\|_{H^1(\mathbb{R})} = A_0 \eta, \tag{7-2}$$

and, for any $\delta > 0$ small, $\delta < \frac{1}{100}\varepsilon_0$, if $T_1 := T_0 + \delta$ then

$$\inf_{\tilde{x}_1, \tilde{x}_2 \in \mathbb{R}} \|u(T_1) - B(\,\cdot\,; \alpha, \beta, \tilde{x}_1, \tilde{x}_2)\|_{H^1(\mathbb{R})} > A_0 \eta. \tag{7-3}$$

We also assume that T_0 is the first positive time in J_{ε_0} with this property. We will show that, under this last assumption, after fixing $A_0 > 1$ large enough we will have

$$u(T_0) \in \mathcal{V}\big(\tfrac{1}{2} A_0, \eta\big), \tag{7-4}$$

which contradicts (7-2)–(7-3) and therefore proves the result for all positive times far from the points t_k.

First of all, by taking $\eta_0 > 0$ smaller if necessary, and $\eta \in (0, \eta_0)$, we can ensure that there are unique $x_1(t), x_2(t) \in \mathbb{R}$ defined on $[0, T_0]$ such that

$$z(t, x) := u(t, x) - B(x; \alpha, \beta, \delta t + x_1(t), \gamma t + x_2(t)) \tag{7-5}$$

satisfies

$$\int_{\mathbb{R}} z(t, x) B_1(x; \alpha, \beta, \delta t + x_1(t), \gamma t + x_2(t))\, dx = 0, \tag{7-6}$$

and

$$\int_{\mathbb{R}} z(t, x) B_2(x; \alpha, \beta, \delta t + x_1(t), \gamma t + x_2(t))\, dx = 0. \tag{7-7}$$

The directions B_1 and B_2 are defined in (5-5)–(5-6) (see [Alejo and Muñoz 2013] for a similar statement and its proof). Moreover, we have

$$\|z(0)\|_{H^1(\mathbb{R})} \lesssim \eta,$$

and similar estimates for $x_1(0)$ and $x_2(0)$, with constants not depending on large A_0. Therefore, (2-9) is not satisfied for $x_1(0)$ and $x_2(0)$. For the sake of simplicity, we can assume $x_1(0) = x_2(0) = 0$, otherwise we perform a shift in space and time on the solution to set them equal to zero.

Define $z_a^0 := z(0)$ and apply Lemma 6.2, and then Lemma 4.1 to the corresponding z_b^0 obtained from Lemma 6.2. We will obtain a *real-valued* seed y_a^0, small in $H^1(\mathbb{R})$. Note that the constants involved in each inversion do not depend on A_0. In particular, the differences between α and α^*, and β and β^*, are not dependent on A_0:

$$|\alpha-\alpha^*| + |\beta-\beta^*| \lesssim \eta. \tag{7-8}$$

Next, we let the mKdV equation evolve with initial data y_a^0. From Theorem 4.3 we have the bound (4-15) for the dynamics $y_a(t)$. On the other hand, the decomposition (7-6)–(7-7) implies that

$$|x_1'(t)| + |x_2'(t)| \lesssim A_0\eta, \tag{7-9}$$

from which the set of points where condition (4-11) is not satisfied is still a countable set of isolated points (see Lemma 4.2).

Now we are ready to apply Lemmas 4.4 and 6.6 with parameters α^*, β^* and shifts $x_1(t)$ and $x_2(t)$ in (4-8), (4-9) and (6-22)–(6-23). In that sense, we have chosen a unique set of parameters for each fixed time t, and the mKdV solution that we choose is the same as the original $u(t)$. Indeed, just notice that, at $t=0$, we have, from (4-18) at $t=0$ and (6-5),

$$\begin{aligned}\frac{1}{\sqrt{2}}(Q^*(0)+z_b(0)-y_a^0) &= (\beta+i\alpha+q^0)\sin\frac{\tilde{Q}^*(0)+\tilde{z}_b(0)+\tilde{y}_a^0}{\sqrt{2}},\\ \frac{1}{\sqrt{2}}(Q^0+z_b^0-y_a^0) &= (\beta+i\alpha+q^0)\sin\frac{\tilde{Q}^0+\tilde{z}_b^0+\tilde{y}_a^0}{\sqrt{2}}.\end{aligned}$$

Using the uniqueness of the solution obtained by the implicit function theorem in a neighborhood of the base point, we have

$$z_b(0) = Q^0 - Q^*(0) + z_b^0 \sim z_b^0. \tag{7-10}$$

Now we use (6-25) at $t=0$ and (6-3):

$$\frac{1}{\sqrt{2}}(B^*(0)+z_a(0)-Q^*(0)-z_b(0)) = (\beta-i\alpha+p^0)\sin\frac{\tilde{B}^*(0)+\tilde{z}_a(0)+\tilde{Q}^*(0)+\tilde{z}_b(0)}{\sqrt{2}},$$

and

$$\frac{1}{\sqrt{2}}(B^0+z_a^0-Q^0-z_b^0) = (\beta-i\alpha+p^0)\sin\frac{\tilde{B}^0+\tilde{z}_a^0+\tilde{Q}^0+\tilde{z}_b^0}{\sqrt{2}}.$$

From (7-10), we have

$$\frac{1}{\sqrt{2}}(B^*(0)+z_a(0)-Q^0-z_b^0)=(\beta-i\alpha+p^0)\sin\frac{\tilde{B}^*(0)+\tilde{z}_a(0)+\tilde{Q}^0+\tilde{z}_b^0}{\sqrt{2}}.$$

Once again, since B^0 and $B^*(0)$ are close, using the uniqueness of the solution obtained via the implicit function Theorem, we conclude that

$$B^*(0)+z_a(0)=B^0+z_a^0.$$

Since both initial data are the same, we conclude that the solution obtained via the Bäcklund transformation is $u(t)$.

Note that the constants involved in the inversions are not dependent on A_0. We finally get

$$\sup_{|t-t_k|\geq\varepsilon_0}\|u(t)-B^*(t)\|_{H^1(\mathbb{R})}\leq C_0\eta, \tag{7-11}$$

where

$$B^*(t,x):=B(x;\alpha^*,\beta^*,\delta^*t+x_1(t),\gamma^*t+x_2(t)).$$

Finally, from (7-8) and after redefining the shift parameters and choosing $t=T_0$, we get the desired conclusion since, for A_0 large enough, we have $C_0\leq\frac{1}{2}A_0$ and (7-4) is proved.

Now we deal with the remaining case, $t\sim t_k$. Fix $k\in\mathbb{Z}$. Note that $z_a=u-B^*$ satisfies the equation

$$(z_a)_t+[(z_a)_{xx}+3(B^*)^2z_a+3B^*z_a^2+z_a^3]_x+x_1'(t)B_1^*+x_2'(t)B_2^*=0 \tag{7-12}$$

in the H^1-sense. In what follows, we will prove that, maybe taking ε_0 smaller but independent of k, we have

$$\sup_{|t-t_k|\leq\varepsilon_0}\|u(t)-B^*(t)\|_{H^1(\mathbb{R})}\leq 4A_0\eta. \tag{7-13}$$

Since A_0 grows with ε_0 small, this implies that, after choosing η_0 smaller if necessary, such an operation can be performed without any risk.

In what follows, we assume that there is $T^*\in(t_k-\varepsilon_0,t_k+\varepsilon_0]$ such that, for all $t\in[t_k-\varepsilon_0,T^*]$,

$$\|z_a(t)\|_{H^1(\mathbb{R})}\leq 4A_0\eta, \tag{7-14}$$

and T^* is maximal in the sense of the above definition (i.e., there is no $T^{**}>T^*$ satisfying the previous property). If $T^*=t_k+\varepsilon_0$, there is nothing to prove and (7-13) holds.

Assume $T^*<t_k+\varepsilon_0$. Now we consider the quantity

$$\frac{1}{2}\int_{\mathbb{R}}z_a^2(t),\qquad t\in[t_0-\varepsilon_0,T^*].$$

We have, from (7-12),

$$\partial_t\frac{1}{2}\int_{\mathbb{R}}z_a^2(t)=\int_{\mathbb{R}}(z_a)_x[3(B^*)^2z_a+3B^*z_a^2+z_a^3](t)+x_1'(t)\int_{\mathbb{R}}z_a(t)B_1^*+x_2'(t)\int_{\mathbb{R}}z_a(t)B_2^*.$$

Using (7-14) and (7-9), we have, for some — explicit — fixed constant $C > 0$ depending only on α, β, and η_0 even smaller if necessary,

$$\left|\partial_t \frac{1}{2}\int_{\mathbb{R}} z_a^2(t)\right| \le C A_0^2 \eta^2.$$

After integration in time and using (7-11), we have

$$\int_{\mathbb{R}} z_a^2(T^*) \le \int_{\mathbb{R}} z_a^2(t_0 - \varepsilon_0) + C\varepsilon_0 A_0^2 \eta^2 \le 1.9 A_0^2 \eta^2,$$

if ε_0 is small but fixed. A similar estimate can be obtained for $\|(z_a)_x(t)\|_{H^1(\mathbb{R})}$ by proving an estimate of the form

$$\left|\partial_t \frac{1}{2}\int_{\mathbb{R}} (z_a)_x^2(t)\right| \le C A_0^2 \eta^2.$$

Therefore, estimate (7-14) has been bootstrapped, which implies that $T^* = t_0 + \varepsilon_0$. Note that the estimates do not depend on k, but only on the length of the intervals, which is about ε_0.[15]

We conclude that there is $\tilde{A}_0 > 0$ fixed such that

$$\sup_{t\in\mathbb{R}} \|u(t) - B^*(t)\|_{H^1(\mathbb{R})} \le \tilde{A}_0 \eta.$$

Finally, estimates (1-6) and (1-7) are obtained from (7-9), using the fact that α^* and β^* are close to α and β in terms of $C\eta$. The proof is complete. □

Remark. From the proof and the results in [Colliander et al. 2003], it is easy to show that the evolution of breathers can be estimated in a polynomial form in time for any $s > \frac{1}{4}$, however, in order to make things simpler, we will not address this issue.

Corollary 7.1. *We have, for all $t \neq t_k$,*

$$\frac{1}{2}\int_{\mathbb{R}} (B^* + z_a)^2(t) = \frac{1}{2}\int_{\mathbb{R}} (Q^* + z_b)^2(t) + 2(\beta^* - i\alpha^*) = M[y_a^0] + 4\beta^*.$$

Moreover, this identity can be extended to any $t \in \mathbb{R}$.

Proof. In the same way as Corollary 4.6. □

Finally, we recall that $\gamma^* = 3(\alpha^*)^2 - (\beta^*)^2$ and $E[u] = \frac{1}{2}\int_{\mathbb{R}} u_x^2 - \frac{1}{4}\int_{\mathbb{R}} u^4$.

Corollary 7.2. *Assume that $t \neq t_k$ for all $k \in \mathbb{Z}$. Then we have*

$$E[B^* + z_a](t) = E[Q^* + z_b](t) - \tfrac{4}{3}(\beta^* - i\alpha^*)^3 = E[y_a^0] + \tfrac{4}{3}\beta^*\gamma^*.$$

Finally, this quantity can be extended in a continuous form to every $t \in \mathbb{R}$.

Proof. In the same way as Corollary 4.7. □

[15] Note that an argument involving the uniform continuity of the mKdV flow will not work in this particular case since the sequence of times (t_k) is unbounded.

8. Asymptotic stability

We finally prove Theorem 1.4. Note that, for some $c_0 > 0$ depending on $\eta > 0$,

$$\lim_{t\to+\infty} \|y_a(t)\|_{H^1(x\geq c_0 t)} = 0. \tag{8-1}$$

This result can be obtained by adapting the proof for the soliton case in [Martel and Merle 2005]. Indeed, consider

$$\phi(x) := \frac{K}{\pi}\arctan(e^{x/K}), \quad K > 0,$$

so that

$$\lim_{-\infty}\phi = 0, \quad \lim_{+\infty}\phi = 1, \quad \phi''' \leq \frac{1}{K^2}\phi', \quad \phi' > 0 \text{ on } \mathbb{R}. \tag{8-2}$$

Fix c_0, $t_0 > 0$. Consider the quantities

$$I(t) := \frac{1}{2}\int_{\mathbb{R}} y_a^2(t)\phi\big(x - c_0 t_0 + \tfrac{1}{2}c_0(t_0 - t)\big),$$
$$J(t) := \int_{\mathbb{R}} \big[\tfrac{1}{2}(y_a)_x^2(t) - \tfrac{1}{4}y_a^4(t) + \tfrac{1}{2}y_a^2(t)\big]\phi\big(x - c_0 t_0 + \tfrac{1}{2}c_0(t_0 - t)\big).$$

It is not difficult to see that

$$I'(t) = -\frac{1}{4}c_0\int_{\mathbb{R}} y_a^2\phi'(t) + \frac{1}{2}\int_{\mathbb{R}} y_a^2\phi'''(t) - \frac{3}{2}\int_{\mathbb{R}}(y_a)_x^2\phi'(t) + \frac{3}{4}\int_{\mathbb{R}} y_a^4\phi'(t),$$

so that, using (8-2), and if $c_0 > 0$ is small (and, depending on η, even smaller if necessary),

$$I'(t) \leq 0.$$

We then have

$$I(t_0) \leq I(0) = \frac{1}{2}\int_{\mathbb{R}} y_a^2(0)\phi(x - c_0 t_0)$$

and

$$\lim_{t\to+\infty} I(t) = 0.$$

A similar result holds for $J(t)$, which proves (8-1).

Note that $\tilde{z}_b + \tilde{y}_a \in H^2(\mathbb{R};\mathbb{C})$ (see (4-17)). In what follows, we will prove that this function satisfies better estimates than y_a and z_b if x is large.

Fix $t \neq t_k$ large with $|t - t_k| \geq \varepsilon_0$. We use the notation

$$\tilde{z}_c := \tilde{y}_a + \tilde{z}_b. \tag{8-3}$$

From (3-29) we have

$$\|\tilde{z}_c(t)\|_{H^2(\mathbb{R};\mathbb{C})} \leq C\nu$$

with $C = C(\varepsilon_0)$ independent of time. From the Bäcklund transformation (4-18) we obtain

$$\begin{aligned}(\tilde z_c)_x - 2y_a &= \sqrt{2}(\beta + i\alpha + q^0)\left[\sin\frac{\tilde Q^* + \tilde z_c}{\sqrt{2}} - \sin\frac{\tilde Q^*}{\sqrt{2}}\right]\\ &= \sqrt{2}(\beta + i\alpha + q^0)\left[\sin\frac{\tilde Q^*}{\sqrt{2}}\left\{\cos\frac{\tilde z_c}{\sqrt{2}} - 1\right\} + \sin\frac{\tilde z_c}{\sqrt{2}}\,\cos\frac{\tilde Q^*}{\sqrt{2}}\right]\\ &= Q^*\left\{\cos\frac{\tilde z_c}{\sqrt{2}} - 1\right\} + \sqrt{2}\sin\left(\frac{\tilde z_c}{\sqrt{2}}\right)\frac{Q^*_x}{Q^*}.\end{aligned}$$

Assume now that $x > c_0 t/2$. Then we have, for some fixed constant $c > 0$,

$$\left|\frac{Q^*_x}{Q^*} + m\right| \le e^{-cx}, \quad m = \beta + i\alpha + q^0 = \beta^* + i\alpha^*,$$

and

$$(\tilde z_c)_x + m\tilde z_c = g,$$

where

$$g := Q^*\left\{\cos\frac{\tilde z_c}{\sqrt{2}} - 1\right\} + \sqrt{2}\left\{\sin\frac{\tilde z_c}{\sqrt{2}} - \frac{\tilde z_c}{\sqrt{2}}\right\}\frac{Q^*_x}{Q^*} + \tilde z_c\left\{\frac{Q^*_x}{Q^*} + m\right\} + 2y_a.$$

Solving the previous ODE, we get

$$\tilde z_c(t,x) = \tilde z_c\big(t, \tfrac12 c_0 t\big)e^{-m(x - c_0 t/2)} + \int_{c_0 t/2}^{x} g(t,s)e^{-m(x-s)}\,ds,$$

so that

$$|\tilde z_c(t,x)| \lesssim \big|\tilde z_c\big(t, \tfrac12 c_0 t\big)\big|e^{-\beta^*(x - c_0 t/2)} + \int_{c_0 t/2}^{x} |g(t,s)|e^{-\beta^*(x-s)}\,ds.$$

From Young's inequality we get

$$\|\tilde z_c(t)\|_{L^2(x \ge c_0 t)} \lesssim \big|\tilde z_c\big(t, \tfrac12 c_0 t\big)\big|e^{-\beta^* c_0 t/2} + \|g(t)\|_{L^2(x \ge c_0 t)}e^{-\beta^* c_0 t}.$$

Clearly,

$$\big|\tilde z_c\big(t, \tfrac12 c_0 t\big)\big| \lesssim \|\tilde z_c(t)\|_{H^1(\mathbb{R};\mathbb{C})} \le C\nu, \quad \|g(t)\|_{L^2(x \ge c_0 t)} \le C\nu^2 + C\nu e^{-ct} + o(1).$$

Passing to the limit, we obtain that, for all $T_n \to +\infty$ with $|T_n - t_k| \ge \varepsilon_0$ for all n and k,

$$\lim_{n\to+\infty}\|\tilde z_c(T_n)\|_{L^2(x \ge c_0 T_n)} = 0.$$

A similar result can be obtained for z_c and $(z_c)_x$. From (8-3), we get

$$\lim_{n\to+\infty}\|z_b(T_n)\|_{H^1(x \ge c_0 T_n)} = 0. \tag{8-4}$$

Finally, we repeat the same strategy with (6-25) and (6-24) to obtain

$$\lim_{t\to+\infty}\|z_a(T_n)\|_{H^1(x \ge c_0 T_n)} = 0.$$

Note that, since the flow map is continuous in time with values in H^1, we can extend the result to any sequence $T_n \to +\infty$ by choosing an $\varepsilon_0 > 0$ smaller but still independent of k.

Appendix A: Proof of Lemma 5.1

We will use the specific character of the breather and soliton profiles. Since (2-9) does not hold, both $\tilde{Q}$ and Q are well-defined everywhere. We have

$$\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = \sin(2(\arctan\Theta_1+\arctan\Theta_2)),$$

where, from (2-1) and (5-1), $\Theta_2 := e^{\beta y_2+i\alpha y_1}$ and $\Theta_1 := \dfrac{\beta\sin(\alpha y_1)}{\alpha\cosh(\beta y_2)}$. The expression in the previous display equals

$$\begin{aligned}
&2\big[\sin(\arctan\Theta_1)\cos(\arctan\Theta_2)+\sin(\arctan\Theta_2)\cos(\arctan\Theta_1)]\\
&\qquad\qquad\times[\cos(\arctan\Theta_1)\cos(\arctan\Theta_2)-\sin(\arctan\Theta_1)\sin(\arctan\Theta_2)\big]\\
&=2\big[\tan(\arctan\Theta_1)\cos^2(\arctan\Theta_1)\cos^2(\arctan\Theta_2)-\sin^2(\arctan\Theta_1)\tan(\arctan\Theta_2)\cos^2(\arctan\Theta_2)\\
&\quad+\cos^2(\arctan\Theta_1)\tan(\arctan\Theta_2)\cos^2(\arctan\Theta_2)-\sin^2(\arctan\Theta_2)\tan(\arctan\Theta_1)\cos^2(\arctan\Theta_1)\big].
\end{aligned}$$

Since $\sin^2(\arctan z)=\dfrac{z^2}{1+z^2}$ and $\cos^2(\arctan z)=\dfrac{1}{1+z^2}$, we have

$$\sin\frac{\tilde{B}+\tilde{Q}}{\sqrt{2}} = \frac{2(\Theta_1-\Theta_1^2\Theta_2+\Theta_2-\Theta_2^2\Theta_1)}{(1+\Theta_1^2)(1+\Theta_2^2)}. \tag{A-1}$$

On the other hand,

$$\frac{1}{\sqrt{2}}(B-Q) = 2\partial_x(\arctan\Theta_1-\arctan\Theta_2) = 2\left(\frac{\Theta_{1,x}}{1+\Theta_1^2}-\frac{\Theta_{2,x}}{1+\Theta_2^2}\right) = 2\frac{(1+\Theta_2^2)\Theta_{1,x}-(1+\Theta_1^2)\Theta_{2,x}}{(1+\Theta_1^2)(1+\Theta_2^2)}.$$

Hence, collecting terms and factoring, from (5-9) we are led to prove that

$$(1+\Theta_2^2)\Theta_{1,x}-(1+\Theta_1^2)\Theta_{2,x}-(\beta-i\alpha)(\Theta_1-\Theta_1^2\Theta_2+\Theta_2-\Theta_2^2\Theta_1)=0. \tag{A-2}$$

Now we perform some computations. We have, from (2-1),

$$\Theta_{2,x} = (\beta+i\alpha)\Theta_2, \tag{A-3}$$

$$\alpha(\beta+i\alpha\Theta_1^2)\cosh^2(\beta y_2) = \beta(\alpha\cosh^2(\beta y_2)+i\beta\sin^2(\alpha y_1)) \tag{A-4}$$

and

$$\Theta_{1,x} = \left(\frac{\beta\sin(\alpha y_1)}{\alpha\cosh(\beta y_2)}\right)_x = \frac{\alpha\beta\cos(\alpha y_1)\cosh(\beta y_2)-\beta^2\sin(\alpha y_1)\sinh(\beta y_2)}{\alpha\cosh^2(\beta y_2)},$$

so that

$$\Theta_{1,x}-(\beta-i\alpha)\Theta_1 = \beta\left[\frac{\alpha e^{i\alpha y_1}\cosh(\beta y_2)-\beta e^{\beta y_2}\sin(\alpha y_1)}{\alpha\cosh^2(\beta y_2)}\right] \tag{A-5}$$

and

$$\begin{aligned}
[\Theta_{1,x}+(\beta-i\alpha)\Theta_1]\Theta_2^2 &= \beta\left[\frac{\alpha e^{-i\alpha y_1}\cosh(\beta y_2)+\beta e^{-\beta y_2}\sin(\alpha y_1)}{\alpha\cosh^2(\beta y_2)}\right]e^{2(\beta y_2+i\alpha y_1)}\\
&= \beta\Theta_2\left[\frac{\alpha e^{\beta y_2}\cosh(\beta y_2)+\beta e^{i\alpha y_1}\sin(\alpha y_1)}{\alpha\cosh^2(\beta y_2)}\right].
\end{aligned} \tag{A-6}$$

Using (A-3), (A-4), (A-5) and (A-6) we have that the left-hand side of (A-2) is

$$
\begin{aligned}
&(1+\Theta_2^2)\Theta_{1,x} - 2(\beta+i\alpha\Theta_1^2)\Theta_2 - (\beta-i\alpha)(1-\Theta_2^2)\Theta_1 \\
&\quad = [\Theta_{1,x} - (\beta-i\alpha)\Theta_1] + [\Theta_{1,x} + (\beta-i\alpha)\Theta_1]\Theta_2^2 - 2(\beta+i\alpha\Theta_1^2)\Theta_2 \\
&\quad = \beta\left[\frac{\alpha e^{i\alpha y_1}\cosh(\beta y_2) - \beta e^{\beta y_2}\sin(\alpha y_1)}{\alpha\cosh^2(\beta y_2)}\right] \\
&\qquad\qquad + \beta\Theta_2\left[\frac{\alpha e^{\beta y_2}\cosh(\beta y_2) + \beta e^{i\alpha y_1}\sin(\alpha y_1) - 2\alpha\cosh^2(\beta y_2) - 2i\beta\sin^2(\alpha y_1)}{\alpha\cosh^2(\beta y_2)}\right] \\
&\quad = \beta\left[\frac{\alpha e^{i\alpha y_1}\cosh(\beta y_2) - \beta e^{\beta y_2}\sin(\alpha y_1)}{\alpha\cosh^2(\beta y_2)}\right] + \beta\Theta_2\left[\frac{-\alpha e^{-\beta y_2}\cosh(\beta y_2) + \beta e^{-i\alpha y_1}\sin(\alpha y_1)}{\alpha\cosh^2(\beta y_2)}\right] \\
&\quad = 0,
\end{aligned}
$$

which proves (A-2).

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