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REFINED AND MICROLOCAL KAKEYA–NIKODYM BOUNDS FOR EIGENFUNCTIONS IN TWO DIMENSIONS

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We obtain some improved essentially sharp Kakeya–Nikodym estimates for eigenfunctions in two dimensions. We obtain these by proving stronger related microlocal estimates involving a natural decomposition of phase space that is adapted to the geodesic flow.

1. Introduction and main results

Suppose that (M, g) is a two-dimensional compact Riemannian manifold and $\{e_\lambda\}$ are the associated eigenfunctions. That is, if Δ_g is the Laplace–Beltrami operator, we have

$$-\Delta_g e_\lambda(x) = \lambda^2 e_\lambda(x),$$

and we assume throughout that the eigenfunctions are normalized to have L^2-norm one, i.e.,

$$\int_M |e_\lambda|^2 \, dV_g = 1,$$

where dV_g is the volume element.

The purpose of this paper is to obtain essentially sharp estimates that link, in two dimensions, the size of L^p-norms of eigenfunctions with $2 < p < 6$ to their L^2-concentration near geodesics. Specifically, we have the following:

Theorem 1.1. *For every* $0 < \varepsilon_0 \le \frac{1}{2}$, *we have*

$$\|e_\lambda\|_{L^4(M)} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/4} \|e_\lambda\|^{1/2}_{L^2(M)} \times |||e_\lambda|||^{1/2}_{KN(\lambda,\varepsilon_0)} \tag{1-1}$$

if

$$|||e_\lambda|||_{KN(\lambda,\varepsilon_0)} = \left(\sup_{\gamma \in \Pi} \lambda^{1/2-\varepsilon_0} \int_{\mathcal{T}_{\lambda^{-1/2+\varepsilon_0}}(\gamma)} |e_\lambda|^2 \, dV \right)^{1/2}. \tag{1-2}$$

Equivalently, *if* $\varepsilon_0 > 0$, *then there is a* $C = C(\varepsilon_0, M)$ *such that*

$$\|e_\lambda\|_{L^4} \le C\lambda^{1/8} \|e_\lambda\|^{1/2}_{L^2(M)} \times \left(\sup_{\gamma \in \Pi} \int_{\mathcal{T}_{\lambda^{-1/2+\varepsilon_0}}(\gamma)} |e_\lambda|^2 \, dV \right)^{1/4}, \tag{1-3}$$

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and therefore if $\int_M |e_\lambda|^2\,dV = 1$, for any $\varepsilon > 0$ there is a $C = C(\varepsilon, M)$ such that

$$\|e_\lambda\|_{L^4(M)} \le C\lambda^{1/8+\varepsilon} \sup_{\gamma\in\Pi} \|e_\lambda\|^{1/2}_{L^2(\mathcal{T}_{\lambda^{-1/2}}(\gamma))} \le C\lambda^{1/16+\varepsilon} \sup_{\gamma\in\Pi} \|e_\lambda\|^{1/2}_{L^4(\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma))}. \tag{1-4}$$

Here Π denotes the space of unit-length geodesics in M and the last factor in (1-2) involves averages of $|e_\lambda|^2$ over $\lambda^{-1/2+\varepsilon_0}$ tubes about $\gamma \in \Pi$. Also, for simplicity, we are only stating things here and throughout for eigenfunctions, but the results easily extend to quasimodes using results from [Sogge and Zelditch 2014].

Note that if $\varepsilon_0 = \frac{1}{2}$, then (1-1) is equivalent to the eigenfunction estimates from [Sogge 1988]

$$\|e_\lambda\|_{L^4(M)} \lesssim \lambda^{1/8}\|e_\lambda\|_{L^2(M)},$$

which are saturated by highest weight spherical harmonics on the standard two-sphere. We also remark that, up to the factor $\lambda^{\varepsilon_0/4}$, the estimate (1-1) is saturated by both the highest weight spherical harmonics and zonal functions on S^2. This is because the highest weight spherical harmonics are given by the restriction of the harmonic polynomials $\lambda^{1/4}(x_1 + ix_2)^k$, $\lambda = \sqrt{k(k+1)}$ to the unit sphere, while the L^2-normalized zonal functions centered about the north pole on S^2 behave like $\big(\lambda^{-1} + \operatorname{dist}(x, \pm(0,0,1))\big)^{-1/2}$. See, for instance, [Sogge 1986].

In [Bourgain 2009] (with a slight loss) and in [Sogge 2011], inequalities of the form (1-1) and (1-3) were proved, where the first norm on the right is raised to the $\frac{3}{4}$ power and the second to the $\frac{1}{4}$ power. The inequalities in [Sogge 2011] were not formulated in this way but easily lead to this result. The approach in [Sogge 2011] made inefficient use of the Cauchy–Schwarz inequality to handle the "easy" term (not the bilinear one), which led to the loss. The strategy for proving (1-1) will be to make an angular dyadic decomposition of a bilinear expression and pay close attention to the dependence of the bilinear estimates in terms of the angles, which we shall exploit using a multilayered microlocal decomposition of phase space.

Before turning to the details of the proof, let us record a few simple corollaries of our main estimate. If $\{a_{\lambda_{j_k}}\}_{k=0}^\infty$ is a sequence depending on a subsequence $\{\lambda_{j_k}\}$ of the eigenvalues of Δ_g, then we say that

$$a_\lambda = o_-(\lambda^\sigma)$$

if there are some $\varepsilon > 0$ and $C < \infty$ such that

$$|a_\lambda| \le C(1+\lambda)^{\sigma-\varepsilon}.$$

Then using Theorem 1.1, we get:

Corollary 1.2. *The following are equivalent*:

$$\|e_{\lambda_{j_k}}\|_{L^4(M)} = o_-(\lambda_{j_k}^{1/8}), \tag{1-5}$$

$$\sup_{\gamma\in\Pi} \|e_{\lambda_{j_k}}\|_{L^4(\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma))} = o_-(\lambda_{j_k}^{1/8}), \tag{1-6}$$

$$\sup_{\gamma\in\Pi} \|e_{\lambda_{j_k}}\|_{L^2(\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma))} = o_-(1). \tag{1-7}$$

Also, if either

$$\sup_{\gamma\in\Pi}\int_\gamma |e_\lambda|^2\,ds = O(\lambda_{j_k}^{\varepsilon}), \quad \textit{for all } \varepsilon>0 \tag{1-8}$$

or

$$\sup_{\gamma\in\Pi}\|e_{\lambda_{j_k}}\|_{L^2(\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma))} = O(\lambda_{j_k}^{-1/4+\varepsilon}), \quad \textit{for all } \varepsilon>0, \tag{1-9}$$

then

$$\|e_{\lambda_{j_k}}\|_{L^4(M)} = O(\lambda_{j_k}^{\varepsilon}), \quad \textit{for all } \varepsilon>0. \tag{1-10}$$

Here, ds denotes the arc length measure on γ.

Clearly (1-5) implies (1-6). Also, (1-7) follows from (1-6) and Hölder's inequality. Since (1-1) shows that (1-7) implies (1-5), the last part of the corollary is also an easy consequence of Theorem 1.1.

Note also that (1-4) says that if $e_{\lambda_{j_k}}$ is a sequence of eigenfunctions with

$$\|e_{\lambda_{j_k}}\|_{L^4(M)} = \Omega(\lambda_{j_k}^{1/8}),$$

then for any ε, there must be a sequence of shrinking geodesic tubes $\{\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma_k)\}$ for which, for some $c=c_\varepsilon>0$, we have

$$\|e_{\lambda_{j_k}}\|_{L^4(\mathcal{T}_{\lambda_{j_k}^{-1/2}}(\gamma_k))} \geq c\,\lambda_{j_k}^{1/8-\varepsilon}.$$

In other words, up to a factor of $\lambda^{-\varepsilon}$ for any $\varepsilon>0$, they fit the profile of the highest weight spherical harmonics by having maximal L^4-mass on a sequence of shrinking $\lambda^{-1/2}$ tubes.

Like in Bourgain's estimate, (1-1) involves a slight loss, but this is not so important in view of the above application. In a later work we hope to show that (1-1) holds without this loss (in other words with $\varepsilon_0=0$), which should mainly involve refining the $S_{1/2,1/2}$ microlocal arguments that are to follow. Note that, because of the zonal functions on S^2, this result would be sharp.

This paper is organized as follows. In Section 2 we shall introduce a microlocal Kakeya–Nikodym norm and an inequality involving it, (2-14), which implies (1-1). This norm is associated to a decomposition of phase space which is naturally associated to the geodesic flow on the cosphere bundle. In particular, each term in the decomposition will involve bump functions which are supported in tubular neighborhoods of unit geodesics in S^*M. This decomposition and the resulting square function arguments are similar to the earlier ones in the joint paper of Mockenhaupt, Seeger and the second author [Mockenhaupt et al. 1993], but there are some differences and new technical issues that must be overcome. We do this and prove our microlocal Kakeya–Nikodym estimate in Section 3. There, after some pseudodifferential arguments, we reduce matters to an oscillatory integral estimate which is a technical variation on the classical one in Hörmander [1973], which was the main step in his proof of the Carleson–Sjölin theorem [1972]. The result which we need does not directly follow from the results in [Hörmander 1973]; however, we can prove it by adapting Hörmander's argument and using Gauss's lemma. After doing this, in Section 4 we shall see how our results are in some sense related to Zygmund's theorem [1974] saying that in two dimensions, eigenfunctions on the standard torus have bounded L^4-norms. Specifically, we shall see there that if we could obtain the endpoint version of (1-1), we would be able to recover Zygmund's theorem with no loss if we also knew a conjectured result that arcs on λS^1 of length $\lambda^{1/2}$ contain a uniformly bounded number of lattice points.

In a later paper with S. Zelditch, we hope to strengthen our results and also extend them to higher dimensions, as well as to present applications in the spirit of [Sogge and Zelditch 2012] of the microlocal bounds which we obtain. The current authors would like to thank S. Zelditch for a number of stimulating discussions.

2. Microlocal Kakeya–Nikodym norms

As in [Sogge 2011; Sogge 1993, §5.1], we use the fact that we can use a reproducing operator to write $e_\lambda = \chi_\lambda f = \rho(\lambda - \sqrt{\Delta_g})e_\lambda$, for $\rho \in \mathcal{S}$ satisfying $\rho(0) = 1$, where, if $\operatorname{supp} \hat{\rho} \subset (1, 2)$, we also have modulo $O(\lambda^{-N})$ errors (see [Sogge 1993, Lemma 5.1.3])

$$\chi_\lambda f(x) = \frac{1}{2\pi} \int \hat{\rho}(t) e^{i\lambda t} (e^{-it\sqrt{\Delta_g}} f)(x)\, dt = \lambda^{1/2} \int e^{i\lambda\psi(x,y)} a_\lambda(x, y) f(y)\, dV(y), \tag{2-1}$$

where

$$\psi(x, y) = d_g(x, y) \tag{2-2}$$

is the Riemannian distance function, and if, as we may, we assume that the injectivity radius is 10 or more, a_λ belongs to a bounded subset of C^∞ and satisfies

$$a_\lambda(x, y) = 0, \quad \text{if } d_g(x, y) \notin (1, 2). \tag{2-3}$$

Thus, in order to prove (1-1), it suffices to work in a local coordinate patch and show that if a is smooth and satisfies the support assumptions in (2-3), if $0 < \delta < \frac{1}{10}$ is small but fixed, and if

$$x_0 = (0, y_0), \quad \tfrac{1}{2} < y_0 < 4$$

is also fixed, then

$$\Big\| \lambda^{1/2} \int e^{i\lambda\psi(x,y)} a(x, y) f(y)\, dy \Big\|^2_{L^4(B(0,\delta))} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/2} \|f\|_{L^2} \times |||f|||_{KN(\lambda,\varepsilon_0)}, \quad \text{if } \operatorname{supp} f \subset B(x_0, \delta). \tag{2-4}$$

Here $B(x, \delta)$ denotes the δ-ball about x in our coordinates. We may assume that in our local coordinate system the line segment $(0, y)$, $|y| < 4$ is a geodesic.

In order to prove (2-4) we also need to define a microlocal version of the above Kakeya–Nikodym norm. We first choose $0 \le \beta \in C_0^\infty(\mathbb{R}^2)$ satisfying

$$\sum_{\nu \in \mathbb{Z}^2} \beta(z + \nu) = 1 \quad \text{and} \quad \operatorname{supp} \beta \subset \{x \in \mathbb{R}^2 : |x| \le 2\}. \tag{2-5}$$

To use this bump function, let $\Phi_t(x, \xi) = (x(t), \xi(t))$ denote the geodesic flow on the cotangent bundle. Then if (x, ξ) is a unit cotangent vector with $x \in B(x_0, \delta)$ and $|\xi_1| < \delta$, with δ small enough, it follows that there is a unique $0 < t < 10$ such that $x(t) = (s, 0)$ for some $s(x, \xi)$. If $\xi(t) = (\xi_1(t), \xi_2(t))$ for this t, it follows that $\xi_2(t)$ is bounded from below. Let us then set $\varphi(x, \xi) = (s(x, \xi), \xi_1(t)/|\xi(t)|)$. Note that φ then is a smooth map from such unit cotangent vectors to $\mathbb{R}^2$. Also, φ is constant on the orbit of Φ. Therefore, $|\varphi(x,\xi)-\varphi(y,\eta)|$ can be thought of as measuring the distance from the geodesic in our coordinate patch through (x,ξ) to that of the one through (y,η).

Let $\alpha(x)$ be a nonnegative C_0^∞ function which is one in $B\big(x_0,\frac{3}{2}\delta\big)$ and zero outside of $B(x_0,2\delta)$. Given $\theta=2^{-k}$ with $\lambda^{-1/2}\le\theta\le 1$ and $\nu\in\mathbb{Z}^2$, let $\Upsilon\in C^\infty(\mathbb{R})$ satisfy

$$\Upsilon(s)=1,\ s\in[c,c^{-1}],\quad \Upsilon(s)=0,\ s\notin\left[\frac{c}{2},2c^{-1}\right], \tag{2-6}$$

for some $c>0$ to be specified later. We then put

$$Q_\theta^\nu(x,\xi)=\alpha(x)\beta\big(\theta^{-1}\varphi(x,\xi)+\nu\big)\Upsilon(|\xi|/\lambda). \tag{2-7}$$

This is a function of unit cotangent vectors, and we also denote its homogeneous of degree zero extension to the cotangent bundle with the zero section removed by $Q_\theta^\nu(x,\xi)$, $\xi\neq 0$, and the resulting pseudodifferential operator by $Q_\theta^\nu(x,D)$. Then if f is as in (2-4), we define its microlocal Kakeya–Nikodym norm corresponding to frequency λ and angle $\theta_0=\lambda^{-1/2+\varepsilon_0}$ to be

$$|||f|||_{MKN(\lambda,\varepsilon_0)}=\sup_{\theta_0\le\theta\le 1}\Big(\sup_{\nu\in\mathbb{Z}^2}\theta^{-1/2}\|Q_\theta^\nu(x,D)f\|_{L^2(\mathbb{R}^2)}\Big)+\|f\|_{L^2(\mathbb{R}^2)},\quad \theta_0=\lambda^{-1/2+\varepsilon_0}. \tag{2-8}$$

Note that

$$\sup_{\nu\in\mathbb{Z}^2}\theta^{-1/2}\|Q_\theta^\nu(x,D)f\|_{L^2(\mathbb{R}^2)}$$

measures the maximal microlocal concentration of f about all unit geodesics in the scale of θ. This is because if we consider the restriction of Q_θ^ν to unit cotangent vectors and if $Q_\theta^\nu(x,\xi)\neq 0$, then $\operatorname{supp} Q_\theta^\nu$ is contained in an $O(\theta)$ tube in the space of unit cotangent vectors about the orbit $t\to\Phi_t(x,\xi)$.

Let us collect a few facts about these pseudodifferential operators. First, the Q_θ^ν belong to a bounded subset of $S^0_{1/2+\varepsilon_0,1/2-\varepsilon_0}$ (pseudodifferential operators of order zero and type $(\frac12+\varepsilon_0,\frac12-\varepsilon_0)$), if $\lambda^{-1/2+\varepsilon_0}\le\theta\le 1$, with $\varepsilon_0>0$ fixed. Therefore, there is a uniform constant C_{ε_0} such that

$$\|Q_\theta^\nu(x,D)g\|_{L^2}\le C_{\varepsilon_0}\|g\|_{L^2},\quad \lambda^{-1/2+\varepsilon_0}\le\theta\le 1. \tag{2-9}$$

Similarly, if $P_\theta^\nu=(Q_\theta^\nu)^*\circ Q_\theta^\nu$ for such θ, then by (2-5), $\sum_\nu P_\theta^\nu$ belongs to a bounded subset of $S^0_{1/2+\varepsilon_0,1/2-\varepsilon_0}$, and so we also have the uniform bounds

$$\Big\|\sum_{\nu\in\mathbb{Z}^2}P_\theta^\nu(x,D)g\Big\|_{L^2}\le C_{\varepsilon_0}\|g\|_{L^2},\quad \lambda^{-1/2+\varepsilon_0}\le\theta\le 1. \tag{2-10}$$

We can relate the microlocal Kakeya–Nikodym norm to the Kakeya–Nikodym norm if we realize that if the $\delta>0$ above is small enough, then there is a unit length geodesic γ_ν such that $Q_\theta^\nu(x,\xi)=0$ for $x\notin\mathcal{T}_{C\theta_\nu}(\gamma)$, with C a uniform constant. As a result, since $Q_\theta^\nu(x,\xi)=0$ if $|\xi|$ is not comparable to λ, we can improve (2-9) and deduce that for every $N=1,2,\dots,$ there is a uniform constant C' such that

$$\|Q_\theta^\nu(x,D)g\|_{L^2}\le C_{\varepsilon_0}\Big(\int_{\mathcal{T}_{C'\theta}(\gamma_\nu)}|g|^2\,dy\Big)^{1/2}+C_N\lambda^{-N}\|g\|_{L^2},\quad \lambda^{-1/2+\varepsilon_0}\le\theta\le 1, \tag{2-11}$$

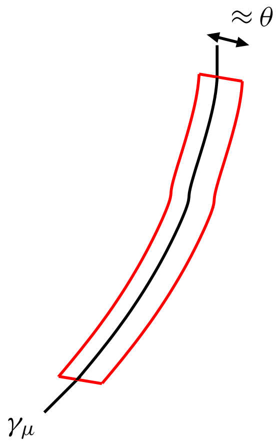

Figure 1. $\mathcal{T}_{C'\theta(\gamma_\nu)}$.

since the kernel $K^\nu_\theta(x, y)$ of $Q^\nu_\theta(x, D)$ is $O(\lambda^{-N})$ for any N if y is not in $\mathcal{T}_{C'\theta}(\gamma_\nu)$, with C' sufficiently large but fixed. (See Figure 1.) Since

$$\theta^{-1/2}\biggl(\int_{\mathcal{T}_{C'\theta}(\gamma_\nu)} |g|^2\,dy\biggr)^{1/2} \lesssim \sup_{\gamma\in\Pi}\biggl(\theta_0^{-1}\int_{\mathcal{T}_{\theta_0}(\gamma)} |g|^2\,dy\biggr)^{1/2}, \quad \lambda^{-1/2+\varepsilon_0} = \theta_0 \le \theta \le 1,$$

we have

$$\sup_{\nu\in\mathbb{Z}^2} \theta^{-1/2}\|Q^\nu_\theta(x, D)f\|_{L^2(\mathbb{R}^2)} \le C_{\varepsilon_0}|||g|||_{KN(\lambda,\varepsilon_0)}, \quad \lambda^{-1/2+\varepsilon_0} \le \theta \le 1, \tag{2-12}$$

meaning that we can dominate the microlocal Kakeya–Nikodym norm by the Kakeya–Nikodym norm.

From this, we conclude that we would have (2-4) if we could show

$$\biggl\|\int \lambda^{1/2}e^{i\lambda\psi(x,y)}a(x, y)f(y)\,dy\biggr\|^2_{L^4(B(0,\delta))} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/2}\|f\|_{L^2} \times |||f|||_{MKN(\lambda,\varepsilon_0)}, \quad \text{if } \operatorname{supp} f \subset B(x_0, \delta). \tag{2-13}$$

We note also that since $\chi_\lambda e_\lambda = e_\lambda$, this inequality of course yields the following microlocal strengthening of Theorem 1.1:

Theorem 2.1. *For every* $0 < \varepsilon_0 \le \frac{1}{2}$, *we have*

$$\|e_\lambda\|_{L^4(M)} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/4}\|e_\lambda\|^{1/2}_{L^2(M)} \times |||e_\lambda|||^{1/2}_{MKN(\lambda,\varepsilon_0)}, \tag{2-14}$$

if $|||e_\lambda|||_{MKN(\lambda,\varepsilon_0)}$ *is as in* (2-8).

3. Proof of the refined two-dimensional microlocal Kakeya–Nikodym estimates

Let us now prove the estimates in (2-13). We shall follow arguments from §6 of [Mockenhaupt et al. 1993].

We first note that if $\operatorname{supp} f \subset B(x_0, \delta)$ as in (2-4), and if

$$\theta_0 = \lambda^{-1/2+\varepsilon_0} \tag{3-1}$$

with $\varepsilon_0 > 0$ fixed,

$$\chi_\lambda f = \sum_{\nu \in \mathbb{Z}^2} \chi_\lambda \big(Q^\nu_{\theta_0}(x, D) f \big) + R_\lambda f,$$

where, if $c > 0$ in (2-6) is small enough, and $N = 1, 2, 3, \dots,$

$$\|R_\lambda f\|_{L^\infty} \lesssim \lambda^{-N} \|f\|_{L^2}.$$

Therefore, in order to prove (2-4), it suffices to show that

$$\Big\| \sum_{\nu, \nu' \in \mathbb{Z}^2} \chi_\lambda Q^\nu_{\theta_0} f \, \chi_\lambda Q^{\nu'}_{\theta_0} f \Big\|_{L^2} \lesssim_{\varepsilon_0} \lambda^{\varepsilon_0/2} \|f\|_{L^2} \times |||f|||_{MKN(\lambda,\varepsilon_0)}. \tag{3-2}$$

We split the sum on the left based on the size of $|\nu - \nu'|$. Indeed, the left side of (3-2) is dominated by

$$\Big\| \sum_\nu (\chi_\lambda Q^\nu_{\theta_0} f)^2 \Big\|_{L^2} + \sum_{\ell=1}^{\infty} \Big\| \sum_{|\nu-\nu'| \in [2^\ell, 2^{\ell+1})} \chi_\lambda Q^\nu_{\theta_0} f \, \chi_\lambda Q^{\nu'}_{\theta_0} f \Big\|_{L^2}. \tag{3-3}$$

The square of the first term in (3-3) is

$$\sum_{\nu,\nu'} \int (\chi_\lambda Q^\nu_{\theta_0} f)^2 \overline{(\chi_\lambda Q^{\nu'}_{\theta_0} f)^2} \, dx.$$

Next we need an orthogonality result, similar to Lemma 6.7 in [Mockenhaupt et al. 1993], which says that if A is large enough we have

$$\sum_{|\nu-\nu'| \geq A} \left| \int (\chi_\lambda Q^\nu_{\theta_0} f)^2 \overline{(\chi_\lambda Q^{\nu'}_{\theta_0} f)^2} \, dx \right| \lesssim_{\varepsilon_0, N} \lambda^{-N} \|f\|^4_{L^2}. \tag{3-4}$$

We shall postpone the proof of this result until the end of the section, when we will have recorded the information about the kernels of $\chi_\lambda Q^\nu_\theta$ that will be needed for the proof.

Since by [Sogge 1988],

$$\|\chi_\lambda\|_{L^2 \to L^4} = O(\lambda^{1/8}),$$

if we use (3-4) we conclude that the first term in (3-3) is majorized by (2-10) and (2-12):

$$\begin{aligned}\lambda^{1/2} \sum_\nu \|Q^\nu_{\theta_0} f\|^2_{L^2} \|Q^\nu_{\theta_0} f\|^2_{L^2} + \lambda^{-N} \|f\|^4_{L^2} &\lesssim \lambda^{1/2} \|f\|^2_{L^2} \times \sup_{\nu \in \mathbb{Z}^2} \|Q^\nu_{\theta_0} f\|^2_{L^2} + \lambda^{-N} \|f\|^4_{L^2} \\ &= \lambda^{\varepsilon_0} \|f\|^2_{L^2} \times \lambda^{1/2-\varepsilon_0} \sup_{\nu \in \mathbb{Z}^2} \|Q^\nu_{\theta_0} f\|^2_{L^2} + \lambda^{-N} \|f\|^4_{L^2}. \end{aligned} \tag{3-5}$$

Therefore, the first term in (3-3) satisfies the desired bounds.

Using (2-12) again, the proof of (2-13) and hence (2-4) would be complete if we could estimate the other terms in (3-3) and show that

$$\Big\| \sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} \chi_\lambda Q^\nu_{\theta_0} f\, \chi_\lambda Q^{\nu'}_{\theta_0} f \Big\|^2_{L^2} \lesssim_{\varepsilon_0} \|f\|^2_{L^2} \times (2^\ell\theta_0)^{-1} \sup_{\nu\in\mathbb{Z}^2} \|Q^\nu_{2^\ell\theta_0} f\|^2_{L^2} + \lambda^{-N}\|f\|^4_{L^2}. \tag{3-6}$$

Note that if $2^\ell\theta_0 \gg 1$, the left side of (3-6) vanishes and thus, as in (2-12), we are just considering $\ell \in \mathbb{N}$ satisfying $1 \le 2^\ell \le \lambda^{1/2-\varepsilon_0}$. In proving this, we may assume that ℓ is larger than a fixed constant, since the bound for small ℓ (with an extra factor of λ^{ε_0} on the right) follows from what we just did. We can handle the sum over ℓ in (3-3) due to the fact that the right side of (3-6) does not include a factor λ^{ε_0}.

We now turn to estimating the nondiagonal terms in (3-3). We first note that by (2-5),

$$\chi_\lambda Q^\nu_{\theta_0} f = \sum_{\mu\in\mathbb{Z}^2} \chi_\lambda Q^\mu_\theta Q^\nu_{\theta_0} f + O_N(\lambda^{-N}\|f\|_2), \quad \text{if } \operatorname{supp} f \subset B(x_0,\delta).$$

Furthermore, if, as we may, we assume that $\ell \in \mathbb{N}$ is sufficiently large, then given $N_0 \in \mathbb{N}$, there are fixed constants $c_0 > 0$ and $N_1 < \infty$ (with c_0 depending only on N_0 and the cutoff β in the definition of these pseudodifferential operators) such that if

$$\theta_\ell = \theta_0 2^\ell,$$

then

$$\sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} \chi_\lambda Q^\nu_{\theta_0} f\, \chi_\lambda Q^{\nu'}_{\theta_0} f$$
$$= \sum_{\{\mu,\mu'\in\mathbb{Z}^2: N_0\le|\mu-\mu'|\le N_1\}} \sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} \chi_\lambda Q^\mu_{c_0\theta_\ell} Q^\nu_{\theta_0} f\, \chi_\lambda Q^{\mu'}_{c_0\theta_\ell} Q^{\nu'}_{\theta_0} f + O_N(\lambda^{-N}\|f\|^2_{L^2}), \tag{3-7}$$

for each $N \in \mathbb{N}$. Also, given $\mu \in \mathbb{Z}^2$, there is a $\nu_0(\mu) \in \mathbb{Z}^2$ such that

$$\|Q^\mu_{c_0\theta_\ell} Q^\nu_{\theta_0} f\|_{L^2} \le C_N \lambda^{-N}\|f\|_{L^2}, \quad \text{if } |\nu - \nu_0(\mu)| \ge C2^\ell,$$

for some uniform constant C. If $|\mu-\mu'| \le N_1$, then $|\nu_0(\mu)-\nu_0(\mu')| \le C2^\ell$ for some uniform constant C. Since $\|(Q^{\nu'}_\theta)^* \circ Q^\nu_\theta\|_{L^2\to L^2} = O(\lambda^{-N})$ for every N if $|\nu-\nu'|$ is larger than a fixed constant, it follows that

$$\iint \Big| \sum_{|\nu_0(\mu)-\nu|,|\nu_0(\mu')-\nu'|\le C2^\ell} \sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} Q^\nu_{\theta_0} f(x) Q^{\nu'}_{\theta_0} f(y) \Big|^2 dx\,dy$$
$$\lesssim \sum_{|\nu-\nu_0(\mu)|,|\nu'-\nu_0(\mu)|\le C'2^\ell} \|Q^\nu_{\theta_0} f\|^2_{L^2} \|Q^{\nu'}_{\theta_0} f\|^2_{L^2} + O_N(\lambda^{-N}\|f\|^2_{L^2}), \quad \text{if } |\mu-\mu'| \le C_0, \tag{3-8}$$

for every N if C' is a sufficiently large but fixed constant. Also, using (2-10), we deduce that

$$\sum_{\mu\in\mathbb{Z}^2} \sum_{|\nu_0(\mu)-\nu|\le C'2^\ell} \|Q^\nu_{\theta_0} f\|^2_{L^2} \lesssim \|f\|^2_{L^2}.$$

We clearly also have

$$\sum_{|\nu(\mu)-\nu'|\le C'2^\ell} \|Q^{\nu'}_{\theta_0} f\|^2_{L^2} \lesssim \sup_{\mu\in\mathbb{Z}^2} \|Q^{\mu}_{2^\ell\theta} f\|^2_{L^2}.$$

Using these two inequalities and (3-8), we deduce that

$$\sum_{|\mu-\mu'|\le N_1} \Big\| \sum_{|\nu_0(\mu)-\nu|,|\nu_0(\mu')-\nu'|<C2^\ell} \ \sum_{|\nu-\nu'|\in[2^\ell,2^{\ell+1})} Q^{\nu}_{\theta_0} f(x) Q^{\nu'}_{\theta_0} f(y)\Big\|_{L^2(dx\,dy)} \\ \lesssim \|f\|_{L^2} \times \sup_{\mu\in\mathbb{Z}^2} \|Q^{\mu}_{2^\ell\theta} f\|_{L^2} + O_N(\lambda^{-N}\|f\|^2_{L^2}). \tag{3-9}$$

In addition to (3-4), we shall need another orthogonality result whose proof we postpone until the end of the section, which says that whenever θ is larger than a fixed positive multiple of θ_0 in (3-1) and N_1 is fixed,

$$\Big|\int \big(\chi_\lambda Q^{\mu}_{\theta} g_1\, \chi_\lambda Q^{\mu'}_{\theta} g_2\big)\overline{\big(\chi_\lambda Q^{\tilde\mu}_{\theta} g_3\, \chi_\lambda Q^{\tilde\mu'}_{\theta} g_4\big)}\, dx\Big| \lesssim_N \lambda^{-N} \prod_{j=1}^4 \|g_j\|_{L^2}, \\ \text{if } |\mu-\tilde\mu|+|\mu'-\tilde\mu'|\ge C \text{ and } |\mu-\mu'|,\ |\tilde\mu-\tilde\mu'|\le N_1, \tag{3-10}$$

for every $N = 1, 2, \dots$, with C being a sufficiently large uniform constant (depending on N_1 of course).

Using (3-9) and (3-10), we conclude that we would have (3-6) (and consequently (2-4)) if we could prove the following:

Proposition 3.1. *Let*

$$(T^{\mu,\mu'}_{\lambda,\theta} F)(x) = \iint (\chi_\lambda Q^{\mu}_{\theta})(x,y)(\chi_\lambda Q^{\mu'}_{\theta})(x,y')F(y,y')\,dy\,dy', \tag{3-11}$$

where

$$(\chi_\lambda Q^{\mu}_{\theta})(x,y)$$

denotes the kernel of $\chi_\lambda Q^{\mu}_{\theta}$. Then if $\delta > 0$ is sufficiently small and if θ is larger than a fixed positive constant times θ_0 in (3-1) *and if $N_0 \in \mathbb{N}$ is sufficiently large and if $N_1 > N_0$ is fixed, we have*

$$\|T^{\mu,\mu'}_{\lambda,\theta} F\|_{L^2(B(0,\delta))} \lesssim_{\varepsilon_0} \theta^{-1/2}\|F\|_{L^2}, \quad \textit{if } N_0 \le |\mu-\mu'| \le N_1, \\ F(y,y') = 0, \qquad \textit{if } (y,y') \notin B(x_0,2\delta)\times B(x_0,2\delta). \tag{3-12}$$

To prove this we shall need some information about the kernel of $\chi_\lambda Q^{\mu}_{\theta}$. By (2-7), the kernel is highly concentrated near the geodesic in M

$$\gamma_\mu = \big\{x_\mu(t) : -2\le t\le 2,\ \Phi_t(x_\mu,\xi_\mu) = (x_\mu(t),\xi_\mu(t)),\ \theta^{-1}\varphi(x_\mu,\xi_\mu)+\mu = 0\big\}, \tag{3-13}$$

which corresponds to Q^{μ}_{θ}. We also will exploit the oscillatory behavior of the kernel near γ_μ.

Specifically, we require the following:

Lemma 3.2. *Let $\theta \in \bigl[C_0\lambda^{-1/2+\varepsilon_0}, \frac{1}{2}\bigr]$, where C_0 is a sufficiently large fixed constant, and, as above, $\varepsilon_0 > 0$. Then there is a uniform constant C such that for each $N = 1, 2, 3, \ldots,$ we have*

$$|(\chi_\lambda Q^\mu_\theta)(x, y)| \le C_N \lambda^{-N}, \quad \text{if } x \notin \mathcal{T}_{C\theta}(\gamma_\mu) \text{ or } y \notin \mathcal{T}_{C\theta}(\gamma_\mu). \tag{3-14}$$

Furthermore,

$$(\chi_\lambda Q^\mu_\theta)(x, y) = \lambda^{1/2} e^{i\lambda d_g(x,y)} a_{\mu,\theta}(x, y) + O_N(\lambda^{-N}), \tag{3-15}$$

where one has the uniform bounds

$$|\nabla^\alpha_y a_{\mu,\theta}(x, y)| \le C_\alpha \theta^{-|\alpha|}, \tag{3-16}$$

$$|\partial^j_t a_{\mu,\theta}(x, x_\mu(t))| \le C_j, \quad x \in \gamma_\mu, \tag{3-17}$$

if, as in (3-13), $\{x_\mu(t)\} = \gamma_\mu$.

Proof. To prove the lemma it is convenient to choose Fermi normal coordinates so that the geodesic becomes the segment $\{(0, s) : |s| \le 2\}$. Let us also write θ as

$$\theta = \lambda^{-1/2+\delta},$$

where, because of our assumptions, $c_1 \le \delta \le \frac{1}{2}$ for an appropriate $c_1 > 0$. Then in these coordinates, $Q^\mu_\theta(x, D)$ has symbol satisfying

$$q^\mu_\theta(x, \xi) = 0, \quad \text{if } \bigl|\xi_1/|\xi|\bigr| \ge C\lambda^{-1/2+\delta},\ |x_1| \ge C\lambda^{-1/2+\delta}, \text{ or } |\xi|/\lambda \notin [C^{-1}, C], \tag{3-18}$$

for some uniform constant C, and, additionally,

$$\bigl|\partial^j_{x_1}\partial^k_{x_2}\partial^l_{\xi_1}\partial^m_{\xi_2} q^\mu_\theta(x, \xi)\bigr| \le C_{j,k,l,m}(1 + |\xi|)^{j(1/2-\delta)-l(1/2+\delta)-m}. \tag{3-19}$$

Next we recall that $\chi_\lambda = \rho(\lambda - \sqrt{-\Delta_g})$, where $\rho \in \mathcal{S}(\mathbb{R})$ satisfies $\hat\rho \subset (1, 2)$, and that the injectivity radius of (M, g) is ten or more. Therefore, we can use Fourier integral parametrices for the wave equation to see that the kernel of χ_λ is of the form

$$\chi_\lambda(x, y) = \iint e^{iS(t,x,\xi) - iy\cdot\xi + it\lambda} \hat\rho(t) \alpha(t, x, y, \xi)\, d\xi\, dt,$$

where $\alpha \in S^1_{1,0}$, and S is homogeneous of degree one in ξ and is a generating function for the canonical relation for the half wave group $e^{-it\sqrt{-\Delta_g}}$. Thus,

$$\partial_t S(t, x, \xi) = -p(x, \nabla_x S(t, x, \xi)), \quad S(0, x, \xi) = x \cdot \xi. \tag{3-20}$$

Let $\tilde\Phi_t(x, \xi)$ denote the Hamiltonian flow generated by $p(x, \xi)$, which is homogeneous of degree one in ξ and agrees with the geodesic flow $\Phi_t(x, \xi)$ when restricted to unit cotangent vectors. The phase $S(t, x, \xi)$ also satisfies

$$\tilde\Phi_t(x, \nabla_x S) = (\nabla_\xi S, \xi). \tag{3-21}$$

Furthermore,

$$\det \frac{\partial S}{\partial x \partial \xi} \ne 0. \tag{3-22}$$

By (3-18), (3-19), and the proof of the Kohn–Nirenberg theorem, we have that

$$(\chi_\lambda Q_\theta^\mu)(x,y) = \iint e^{iS(t,x,\xi)-iy\cdot\xi+i\lambda t}\hat{\rho}(t)q(t,x,y,\xi)\,d\xi\,dt + O(\lambda^{-N}),$$
$$= \lambda^2 \iint e^{i\lambda(S(t,x,\xi)-y\cdot\xi+t)}\hat{\rho}(t)q(t,x,y,\lambda\xi)\,d\xi\,dt + O(\lambda^{-N}), \tag{3-23}$$

where for all t in the support of $\hat{\rho}$,

$$q(t,x,y,\xi)=0 \quad \text{if } \big|\xi_1/|\xi|\big| \ge C\lambda^{-1/2+\delta},\ |x_1| \ge C\lambda^{-1/2+\delta},\ \text{ or } |\xi|/\lambda \notin [C^{-1},C], \tag{3-24}$$

with C as in (3-19), and also

$$\big|\partial_{x_1}^j \partial_{x_2}^k \partial_{\xi_1}^l \partial_{\xi_2}^m q(t,x,y,\xi)\big| \le C_{j,k,l,m}(1+|\xi|)^{j(1/2-\delta)-l(1/2+\delta)-m}. \tag{3-25}$$

Let us now prove (3-14). We have the assertion if $y \notin \mathcal{T}_{C\lambda^{-1/2+\delta}}(\gamma_\mu)$ by (3-24). To prove that remaining part of (3-24) which says that this is also the case when x is not in such a tube, we note that by (3-21), if $d_g(x_0,y_0)=t_0$ and $x_0, y_0 \in \gamma_\mu$, then

$$\nabla_\xi(S(t_0,x_0,\xi)-y_0\cdot\xi)=0, \quad \text{if } \xi_1=0.$$

By (3-22), we then have

$$\big|\nabla_\xi(S(t_0,x,\xi)-y_0\cdot\xi)\big| \approx d_g(x,x_0), \quad \text{if } \xi_1=0.$$

We deduce from this that if $|\xi_1|/|\xi| \le C\lambda^{-1/2+\delta}$, $|y_1| \le C\lambda^{-1/2+\delta}$, and $|\xi| \in [C^{-1},C]$, then there are a $c_0>0$ and a $C_0<\infty$ such that

$$\big|\nabla_\xi(S(t_0,x,\xi)-y\cdot\xi)\big| \ge c_0\lambda^{-1/2+\delta}, \quad \text{if } x \notin \mathcal{T}_{C_0\lambda^{-1/2+\delta}}(\gamma_\mu).$$

From this we obtain the remaining part of (3-14) via a simple integration by parts argument if we use the support properties (3-24) and size estimates (3-25) of $q(t,x,y,\xi)$. We note that every time we integrate by parts in ξ we gain by $\lambda^{-2\delta}$, which implies (3-14) since q vanishes unless $|\xi|\approx\lambda$ and δ is bounded below by a fixed positive constant.

To finish the proof of the lemma and obtain (3-15)–(3-17), we note that if we let

$$\Psi(t,x,y,\xi) = S(t,x,\xi) - y\cdot\xi + t$$

denote the phase function of the second oscillatory integral in (3-23), then at a stationary point where

$$\nabla_{\xi,t}\Psi = 0,$$

we must have $\Psi = d_g(x,y)$, due to the fact that $S(t,x,\xi)-y\cdot\xi=0$ and $t=d_g(x,y)$ at points where the ξ-gradient vanishes. Additionally, it is not difficult to check that the mixed Hessian of the phase satisfies

$$\det\left(\frac{\partial^2\Psi}{\partial(\xi,t)\partial(\xi,t)}\right) \ne 0$$

on the support of the integrand. This follows from the proof of Lemma 5.1.3 of [Sogge 1993]. Moreover, since modulo $O(\lambda^{-N})$ error terms $(\chi_\lambda Q_\theta^\mu)(x, y)$ equals

$$\lambda^2 \iint e^{i\lambda\Psi} \hat{\rho}(t) q(t, x, y, \lambda\xi)\, d\xi\, dt, \tag{3-26}$$

we obtain (3-15)–(3-16) by the proof of this result if we use the stationary phase and (3-24)–(3-25). Indeed, by (3-21), (3-26) has a stationary phase expansion (see [Hörmander 2003, Theorem 7.7.5]), where the leading term is a fixed constant times

$$\lambda^{1/2} e^{i\lambda t} q(t, x, y, \lambda\xi), \quad \text{if } t = d_g(x, y) \text{ and } \tilde{\Phi}_{-t}(y, \xi) = (x, \nabla_x S(t, x, \xi)). \tag{3-27}$$

From this, we see that the leading term in the asymptotic expansion must satisfy (3-16), and subsequent terms in the expansion will satisfy better estimates, where the right-hand side involves increasing negative powers of $\lambda^{2\delta}$ (by [Hörmander 2003, (7.7.1)] and (3-25)), from which we deduce that (3-16) must be valid. Since $\xi_1 = 0$ and $p(y, \xi) = 1$ (by (3-21)) in (3-27) when $x, y \in \gamma_\mu$, we similarly deduce from (3-25) that the leading term in the stationary phase expansion must satisfy (3-17), and since the other terms satisfy better bounds involving increasing powers of $\lambda^{-2\delta}$, we similarly obtain (3-17), which completes the proof of the lemma. □

Let us now collect some simple consequences of Lemma 3.2. First, in addition to (3-14), the kernel $(\chi_\lambda Q_\theta^\mu)(x, y)$ is also $O(\lambda^{-N})$ unless the distance between x and y is comparable to one by (2-3). From this we deduce that if $N_0 \in \mathbb{N}$ is sufficiently large,

$$(\chi_\lambda Q_\theta^\mu)(x, y)(\chi_\lambda Q_\theta^{\mu'})(x, y') = O(\lambda^{-N}),$$
$$\text{unless } \operatorname{Angle}(x; y, y') \in [\theta, C_2\theta] \text{ and } x, y, y' \in \mathcal{T}_{C_2\theta}(\gamma_\mu), \text{ if } |\mu - \mu'| \in [N_0, N_1], \tag{3-28}$$

if $\operatorname{Angle}(x, y, y')$ denotes the angle at x of the geodesic connecting x and y and the one connecting x and y', and where $C_2 = C_2(N_1)$.

This is because in this case, if $x \in \mathcal{T}_{C\theta}(\gamma_\mu) \cap \mathcal{T}_{C\theta}(\gamma_{\mu'})$, then the tubes must be disjoint at a distance bounded below by a fixed positive multiple of θ if N_0 is large enough, and in this region their separation is bounded by a fixed constant times θ if N_1 is fixed; see Figure 2.

To exploit this key fact, as above, let us choose Fermi normal coordinates (see [Gray 2004, Chapter 2]) about γ_μ so that the geodesic becomes the segment $\{(0, s) : |s| \le 2\}$. Then, as in (2-2), let

$$\psi(x; y) = d_g\big((x_1, x_2), (y_1, y_2)\big)$$

be the Riemannian distance function written in these coordinates. Then if x, y, y' are close to this segment and if the distances between x and y and x and y' are both comparable to 1 and if, as well, y is close to y', it follows from Gauss's lemma that

$$\operatorname{Angle}\big(x; (y_1, y_2), (y_1', y_2')\big) \approx \left| \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y) - \frac{\partial}{\partial y_1} \frac{\partial}{\partial x_2} \psi(x, y') \right|. \tag{3-29}$$

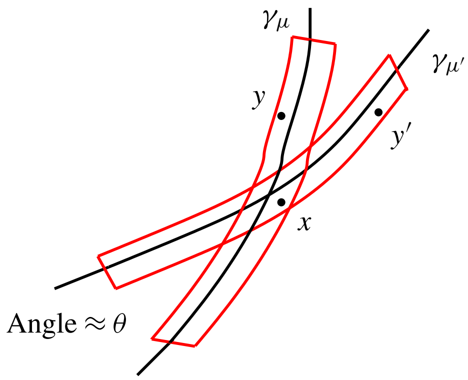

Figure 2. θ-tubes intersecting at angle $\geq N_0\theta$.

As a result, by (3-28), there must be a constant $c_0 > 0$ such that

$$(\chi_\lambda Q_\theta^\mu)(x,y)(\chi_\lambda Q_\theta^{\mu'})(x,y') = O(\lambda^{-N}),$$
$$\text{if } \left| \frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\psi(x,y) - \frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\psi(x,y')\right| \leq c_0\theta \text{ and } |\mu-\mu'| \in [N_0, N_1], \tag{3-30}$$

with, as above, $N_0 \in \mathbb{N}$ sufficiently large and N_1 fixed. Another consequence of Gauss's lemma is that if x and y as in (3-29) are close to this segment and at a distance from each other which is comparable to one, then

$$\frac{\partial}{\partial x_1}\frac{\partial}{\partial y_1}\psi(x,y) \neq 0. \tag{3-31}$$

We shall also need to make use of the fact that, in these Fermi normal coordinates, we have

$$\frac{\partial}{\partial x_2}\frac{\partial}{\partial y_1}\psi\big((0,x_2),(0,y_2)\big) = \frac{\partial}{\partial x_1}\psi\big((0,x_2),(0,y_2)\big) = 0, \quad \text{if } d_g\big((0,x_2),(0,y_2)\big) \approx 1. \tag{3-32}$$

Next, by (3-15)–(3-17), modulo terms which are $O(\lambda^{-N})$ we can write

$$(\chi_\lambda Q_\theta^\mu)(x,y)(\chi_\lambda Q_\theta^{\mu'})(x,y') = \lambda e^{i\lambda(\psi(x,y)+\psi(x,y'))} b_\mu(x;y,y'),$$

where, by (3-28) and (3-30),

$$b_\mu(x;y,y') = 0, \quad \text{if } d_g(x,y) \text{ or } d_g(x,y') \notin [1,2],$$
$$\text{or } |x_1|+|y_1|+|y_1'| \geq c_0^{-1}\theta, \text{ or } \left| \frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\psi(x,y) - \frac{\partial}{\partial y_1}\frac{\partial}{\partial x_2}\psi(x,y')\right| \leq c_0\theta, \tag{3-33}$$

and, since we are working in Fermi normal coordinates,

$$\left| \frac{\partial^j}{\partial x_1^j}\frac{\partial^k}{\partial x_2^k} b_\mu(x,y,y')\right| \leq C_0\theta^{-j}, \quad 0 \leq j,k \leq 3. \tag{3-34}$$

The constants C_0 and c_0 can be chosen to be independent of $\mu \in \mathbb{Z}^2$ and $\theta \geq \lambda^{-1/2+\varepsilon_0}$ if $\varepsilon_0 > 0$. But then, by (3-33) and (3-34) if y_2 and y_2' are fixed and close to one another, and if we set

$$\Psi(x; s, t) = \psi(x, (s+t, y_2)) + \psi(x, (s-t, y_2')) \quad \text{and} \quad b(x; s, t) = b_\mu(x; s+t, y_2, s-t, y_2'),$$

there is a fixed constant C such that

$$\begin{aligned} &b(x; s, t) = 0 \quad \text{if } |x_1| + |s| + |t| \geq C\theta, \\ \text{and} \quad &\left| \frac{\partial^j}{\partial x_1^j} \frac{\partial^k}{\partial x_2^k} b(x; s, t) \right| \leq C\theta^{-j}, \quad 0 \leq j, k \leq 3, \end{aligned} \tag{3-35}$$

while, by (3-31) and (3-32),

$$\begin{aligned} &\frac{\partial}{\partial x_2}\frac{\partial}{\partial s}\Psi(0, x_2; 0, 0) = \frac{\partial}{\partial x_2}\frac{\partial}{\partial t}\Psi(0, x_2; 0, 0) = \frac{\partial}{\partial x_1}\Psi(0, x_2; 0, 0) = 0, \\ &\qquad\text{but} \quad \frac{\partial}{\partial x_1}\frac{\partial}{\partial s}\Psi(0, x_2; 0, 0) \neq 0 \quad \text{if } b(0, x_2; 0, 0) \neq 0, \end{aligned} \tag{3-36}$$

and, moreover, by (3-33),

$$\left| \frac{\partial}{\partial x_2}\frac{\partial}{\partial t}\Psi(x; s, t) \right| \geq c\theta, \quad \text{if } b(x; s, t) \neq 0. \tag{3-37}$$

Also, if we assume that $|y_2 - y_2'| \leq \delta$, as we may because of the support assumption in (3-12), then

$$\left| \frac{\partial}{\partial x_1}\frac{\partial}{\partial t}\Psi(x; s, 0) \right| \leq C\delta, \quad \text{if } b(x; s, t) \neq 0, \tag{3-38}$$

since the quantity on the left vanishes identically when $y_2 = y_2'$.

Another consequence of Gauss's lemma is that if y, y', x are close to the second coordinate axis and if the distances between x and each of y and y' are comparable to 1, then if θ above is bounded below, the 2×2 mixed Hessian of the function $(x; y_1, y_1') \to \psi(x, y) + \psi(x, y')$ has nonvanishing determinant. Thus, in this case (3-12) just follows from Hörmander's nondegenerate L^2-oscillatory integral lemma [1973] (see [Sogge 1993, Theorem 2.1.1]). Therefore, it suffices to prove (3-12) when θ is bounded above by a fixed positive constant, and so Proposition 3.1, and hence Theorem 1.1, is a consequence of the following:

Lemma 3.3. *Suppose that $b \in C_0^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ vanishes when $|(s, t)| \geq \delta$. Then if $\Psi \in C^\infty(\mathbb{R}^2 \times \mathbb{R}^2)$ is real and* (3-35)–(3-38) *are valid, there is a uniform constant C such that if $\delta > 0$ and $\theta > 0$ are smaller than a fixed positive constant and*

$$T_\lambda F(x) = \iint e^{i\lambda\Psi(x;s,t)} b(x; s, t) F(s, t)\, ds\, dt,$$

then we have

$$\|T_\lambda F\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{-1}\theta^{-1/2}\|F\|_{L^2(\mathbb{R}^2)}. \tag{3-39}$$

We shall include the proof of this result for the sake of completeness even though it is a standard result. It is a slight variant of the main lemma in Hörmander's proof [1973] of the Carleson–Sjölin theorem (see [Sogge 1993, pp. 61–62]). Hörmander's proof gives this result in the special case where $y_2 = y_2'$, and, as above, Ψ is defined by two copies of the Riemannian distance function. The case where y_2 and y_2' are not equal to each other introduces some technicalities that, as we shall see, are straightforward to overcome.

Proof. Inequality (3-39) is equivalent to the statement that $\|T_\lambda^* T_\lambda\|_{L^2\to L^2} \le C\lambda^{-2}\theta^{-1}$. The kernel of $T_\lambda^* T_\lambda$ is

$$K(s,t;s',t') = \iint e^{i\lambda(\Psi(x;s,t)-\Psi(x;s',t'))} a(x;s,t,s',t')\,dx_1\,dx_2,$$
$$\text{if } a(x;s,t,s',t') = b(x,s,t)\overline{b(x;s',t')}.$$

Therefore, we would have this estimate if we could show that

$$|K(s,t;s',t')| \le C\theta^{1-N}\big(1+\lambda|(s-s',t-t')|\big)^{-N} + C\theta\big(1+\lambda\theta|(s-s',t-t')|\big)^{-N}, \quad N = 0,1,2,3, \tag{3-40}$$

for then by using the $N=0$ bounds for the regions where $|(s-s',t-t')| \le (\lambda\theta)^{-1}$ and the $N=3$ bounds in the complement, we see that

$$\sup_{s,t}\iint |K|\,ds'\,dt',\ \sup_{s',t'}\iint |K|\,ds\,dt \le C\lambda^{-2}\theta^{-1},$$

which means that by Young's inequality, $\|T_\lambda^* T_\lambda\|_{L^2\to L^2} \le C\lambda^{-2}\theta^{-1}$, as desired.

The bound for $N=0$ follows from the first part of (3-35). To prove the bounds for $N=1,2,3$, we need to integrate by parts.

Let us first handle the case where

$$|s-s'| \ge A^{-1}|t-t'|, \tag{3-41}$$

where $A \ge 1$ is a possibly fairly large constant which we shall specify in the next step. By the second part of (3-36) and by (3-38), we conclude that if $\delta > 0$ is sufficiently small (depending on A), we have

$$\left|\frac{\partial}{\partial x_1}\big(\Psi(x;s,t)-\Psi(x;s',t')\big)\right| \ge c|s-s'|, \quad |s-s'| \ge A^{-1}|t-t'|, \tag{3-42}$$

for some uniform constant $c > 0$.

Since $|K|$ is trivially bounded by the second term on the right side of (3-40) when $|s-s'| \le (\lambda\theta)^{-1}$ and (3-41) is valid, we shall assume that $|s-s'| \ge (\lambda\theta)^{-1}$.

If we then write

$$e^{i\lambda(\Psi(x;s,t)-\Psi(x;s',t'))} = L e^{i\lambda(\Psi(x;s,t)-\Psi(x;s',t'))},$$
$$\text{where } L(x,D) = \frac{1}{i\lambda\big(\Psi'_{x_1}(x;s,t)-\Psi'_{x_1}(x;s',t')\big)}\frac{\partial}{\partial x_1}, \tag{3-43}$$

then we obtain

$$|K| \le \iint \bigl|(L^*(x, D))^N a(x; s, t, s', t')\bigr|\, dx.$$

Note that

$$\begin{aligned}\bigl|\lambda\bigl(\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')\bigr)\bigr|^N |(L^*)^N a| \\ \le C_N \sum_{0\le j+k\le N} \left|\frac{\partial^j}{\partial x_1^j} a\right| \times \sum_{\alpha_1+\cdots+\alpha_k\le N} \frac{\prod_{m=1}^k \left|\frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}}\bigl(\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')\bigr)\right|}{\bigl|\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')\bigr|^k}.\end{aligned} \tag{3-44}$$

Clearly,

$$\prod_{m=1}^k \left|\frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}}\bigl(\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')\bigr)\right| \le C_k |(s-s', t-t')|^k, \tag{3-45}$$

and consequently, by (3-41) and (3-42),

$$\frac{\prod_{m=1}^k \left|\frac{\partial^{\alpha_m}}{\partial x_1^{\alpha_m}}\bigl(\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')\bigr)\right|}{\bigl|\Psi'_{x_1}(x; s, t) - \Psi'_{x_1}(x; s', t')\bigr|^k} \le C_{A,k}. \tag{3-46}$$

Since by (3-35), we have that $|\partial^j_{x_1} a| \le C\theta^{-j}$, $j = 0, 1, 2, 3$, and (3-35) also says that a vanishes when $|x_1|$ is larger than a fixed multiple of θ, we conclude from (3-42)–(3-46) that if (3-41) holds, then $|K|$ is dominated by the first term on the right side of (3-40).

We now turn to the remaining case, which is

$$|t - t'| \ge A|s - s'|, \tag{3-47}$$

and where the parameter $A \ge 1$ will be specified. By the first part of (3-36) and by (3-37) and the fact that $|s|$, $|s'|$, $|t|$, $|t'|$ are bounded by a fixed multiple of θ in the support of a, it follows that we can fix A (independent of θ small) so that if (3-47) is valid, then

$$\left|\frac{\partial}{\partial x_2}\bigl(\Psi(x; s, t) - \Psi(x; s', t')\bigr)\right| \ge c\theta|t - t'|, \quad \text{on } \operatorname{supp} a,$$

for some uniform constant $c > 0$. Then since (3-32) implies that

$$\prod_{m=1}^k \left|\frac{\partial^{\alpha_m}}{\partial x_2^{\alpha_m}}\bigl(\Psi'_{x_2}(x; s, t) - \Psi'_{x_2}(x; s', t')\bigr)\right| \le C_k \theta^k |(s-s', t-t')|^k, \quad \text{on } \operatorname{supp} a,$$

and since, by (3-35),

$$|\partial^j_{x_2} a| \le C_N, \quad 1 \le j \le N,$$

we conclude that if we repeat the argument just given but now integrate by parts with respect to x_2 instead of x_1, then $|K|$ is bounded by the second term on the right side of (3-40), which completes the proof of Lemma 3.3. □

To conclude matters, we also need to prove the orthogonality estimates (3-4) and (3-10). Since (3-4) is a special case of (3-10), we just need to establish the latter.

To see this, we note that by Lemma 3.2, if $(\chi_\lambda Q_\theta^\mu)(x, y)$ denotes the kernel of $\chi_\lambda Q_\theta^\mu$, then

$$(\chi_\lambda Q_\theta^\mu)(x, y)(\chi_\lambda Q_\theta^{\mu'})(x, y')\overline{(\chi_\lambda Q_\theta^{\tilde\mu})(x, \tilde y)(\chi_\lambda Q_\theta^{\tilde\mu'})(x, \tilde y')} = O_N(\lambda^{-N}),$$
$$\text{if } x \notin \mathcal{T}_{C\theta}(\gamma_\mu)\cap\mathcal{T}_{C\theta}(\gamma_{\mu'})\cap\mathcal{T}_{C\theta}(\gamma_{\tilde\mu})\cap\mathcal{T}_{C\theta}(\gamma_{\tilde\mu'}),$$

with C sufficiently large and the geodesics defined by (3-13). On the other hand, if x is in the above intersection of tubes, then the condition on $(\mu, \mu', \tilde\mu, \tilde\mu')$ in (3-10) ensures that if the constant C there is large enough, we have

$$\left|\nabla_x\big(d_g(x, y)+d_g(x, y')-d_g(x, \tilde y)-d_g(x, \tilde y')\big)\right| \ge c_0\theta,$$
$$\text{if } y\in\mathcal{T}_{C\theta}(\gamma_\mu),\ y'\in\mathcal{T}_{C\theta}(\gamma_{\mu'}),\ \tilde y\in\mathcal{T}_{C\theta}(\gamma_{\tilde\mu}),\ \text{and } \tilde y'\in\mathcal{T}_{C\theta}(\gamma_{\tilde\mu'}),$$

for some uniform $c_0 > 0$. Thus, (3-10) follows from Lemma 3.2 and a simple integration by parts argument since we are assuming that $\theta \ge \theta_0 = \lambda^{-1/2+\varepsilon_0}$ with $\varepsilon_0 > 0$.

4. Relationships with Zygmund's L^4-toral eigenfunction bounds

Recall that for $\mathbb{T}^2$, Zygmund [1974] showed that if e_λ is an eigenfunction on $\mathbb{T}^2$, i.e.,

$$e_\lambda(x) = \sum_{\{\varepsilon\in\mathbb{Z}^2:\,|\ell|=\lambda\}} a_\ell e^{ix\cdot\ell}, \tag{4-1}$$

then

$$\|e_\lambda\|_{L^4(\mathbb{T}^2)} \le C,$$

for some uniform constant C.

As observed in [Burq et al. 2007], using well-known pointwise estimates in two dimensions, one has

$$\sup_{\gamma\in\Pi}\int_\gamma |e_\lambda|^2\,ds = O_\varepsilon(\lambda^\varepsilon)$$

for all $\varepsilon > 0$. This of course implies that one also has

$$\sup_{\gamma\in\Pi}\int_{\mathcal{T}_{\lambda^{-1/2}}(\gamma)} |e_\lambda|^2\,dx = O_\varepsilon(\lambda^{-1/2+\varepsilon})$$

for any $\varepsilon > 0$.

Sarnak (unpublished) made an interesting observation that having $O(1)$ geodesic restriction bounds for $\mathbb{T}^2$ is equivalent to the statement that there is a uniformly bounded number of lattice points on arcs of λS^1 of aperture $\lambda^{-1/2}$. (Cilleruelo and Córdoba [1992] showed that this is the case for arcs of aperture $\lambda^{-1/2-\delta}$ for any $\delta > 0$.)

Using (1-1) we can essentially recover Zygmund's bound and obtain $\|e_\lambda\|_{L^4(\mathbb{T}^2)} = O_\varepsilon(\lambda^\varepsilon)$ for every $\varepsilon > 0$. (Of course this just follows from the pointwise estimate, but it shows how the method is natural too.)

If we could push the earlier results to include $\varepsilon_0 = 0$ and if we knew that there were uniformly bounded restriction bounds, then we would recover Zygmund's estimate.

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