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ON SMALL ENERGY STABILIZATION IN THE NLS WITH A TRAPPING POTENTIAL

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We describe the asymptotic behavior of small energy solutions of an NLS with a trapping potential, generalizing work of Soffer and Weinstein, and of Tsai and Yau. The novelty is that we allow generic spectra associated to the potential. This is a new application of the idea of interpreting the *nonlinear Fermi golden rule* as a consequence of the Hamiltonian structure.

1. Introduction

We consider the initial value problem

$$\mathrm{i}u_t = Hu + |u|^2u, \quad (t,x) \in \mathbb{R}^{1+3}, \qquad u(0) = u_0, \tag{1-1}$$

where $H = -\Delta + V$. For $f, g : \mathbb{R}^3 \to \mathbb{C}$, we introduce the bilinear form

$$\langle f, g\rangle = \int_{\mathbb{R}^3} f(x)g(x)\,dx. \tag{1-2}$$

We assume the following:

(H1) $V \in \mathscr{S}(\mathbb{R}^3)$, where $\mathscr{S}(\mathbb{R}^3)$ is the space of Schwartz functions.

(H2) $\sigma_p(H) = \{e_1 < e_2 < e_3 < \cdots < e_n < 0\}$. Here we assume that all the eigenvalues have multiplicity 1. Zero is neither an eigenvalue nor a resonance (that is, if $(-\Delta + V)u = 0$ with $u \in C^\infty$ and $|u(x)| \le C|x|^{-1}$ for a fixed C, then $u = 0$).

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(H3) There is an $N \in \mathbb{N}$ with $N > |e_1|(\min\{e_i - e_j : i > j\})^{-1}$ such that, if $\mu \in \mathbb{Z}^n$ satisfies $|\mu| \le 4N+8$ and $\boldsymbol{e} := (e_1, \dots, e_n)$, then we have

$$\mu \cdot \boldsymbol{e} := \mu_1 e_1 + \cdots + \mu_n e_n = 0 \iff \mu = 0 .$$

(H4) The following Fermi golden rule (FGR) holds: the expression

$$\sum_{L \in \Lambda} \langle \delta(H-L)\overline{G}_L(\zeta), G_L(\zeta)\rangle,$$

which is defined in the course of the paper (for $\Lambda \subset \mathbb{R}_+$ see (6-25) and for G_L see (6-44)) and which is always nonnegative, satisfies formula (6-47).

To each e_j we associate an eigenfunction ϕ_j. We choose them so that $\langle \phi_j, \overline{\phi}_k\rangle = \delta_{jk}$ and, since we can, we also choose the ϕ_j to be all real valued. To each ϕ_j we associate nonlinear bound states.

Proposition 1.1 (bound states). *Fix $j \in \{1, \dots, n\}$. Then there exists $a_0 > 0$ such that, for all $z \in B_{\mathbb{C}}(0, a_0)$, there is a unique $Q_{jz} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}) := \bigcap_{t\ge 0} \Sigma_t(\mathbb{R}^3, \mathbb{C})$ (for the spaces Σ_t, see Section 2) such that*

$$HQ_{jz} + |Q_{jz}|^2 Q_{jz} = E_{jz} Q_{jz}, \quad Q_{jz} = z\phi_j + q_{jz}, \quad \langle q_{jz}, \overline{\phi}_j\rangle = 0, \tag{1-3}$$

and such that we have, for any $r \in \mathbb{N}$:

(1) *$(q_{jz}, E_{jz}) \in C^\infty(B_{\mathbb{C}}(0, a_0), \Sigma_r \times \mathbb{R})$, $q_{jz} = z\hat{q}_j(|z|^2)$ with $\hat{q}_j(t^2) = t^2\tilde{q}_j(t^2)$, where $\tilde{q}_j(t)$ is in $C^\infty((-a_0{}^2, a_0{}^2), \Sigma_r(\mathbb{R}^3, \mathbb{R}))$, and $E_{jz} = E_j(|z|^2)$ with $E_j(t) \in C^\infty((-a_0{}^2, a_0{}^2), \mathbb{R})$.*

(2) *$\|q_{jz}\|_{\Sigma_r} \le C|z|^3$, $|E_{jz} - e_j| < C|z|^2$ for some $C > 0$.*

For the proof of Proposition 1.1 see Appendix A.

Definition 1.2. Let $b_0 > 0$ be sufficiently small so that, for $z_j \in B_{\mathbb{C}}(0, b_0)$, the function Q_{jz_j} exists for all $j \in \{1, \dots, n\}$. For such z_j and for D_{jI} and D_{jR}, defined in Section 2, we set

$$\mathcal{H}_c[z] = \mathcal{H}_c[z_1, \dots, z_n] := \left\{\eta \in L^2 : \mathrm{Re}\langle \mathrm{i}\overline{\eta}, D_{jR} Q_{jz_j}\rangle = \mathrm{Re}\langle \mathrm{i}\overline{\eta}, D_{jI} Q_{jz_j}\rangle = 0 \text{ for all } j\right\}. \tag{1-4}$$

In particular, as an elementary consequence of (1-4) and Proposition 1.1, we have

$$\mathcal{H}_c[0] = \{\eta \in L^2 : \langle \overline{\eta}, \phi_j\rangle = 0 \text{ for all } j\}. \tag{1-5}$$

We denote by P_c the orthogonal projection of L^2 onto $\mathcal{H}_c[0]$.

A pair (p, q) is *admissible* when

$$\frac{2}{p} + \frac{3}{q} = \frac{3}{2}, \quad 6 \ge q \ge 2, \quad p \ge 2. \tag{1-6}$$

The following theorem is our main result:

Theorem 1.3. *Assume* (H1)–(H4). *Then there exist $\epsilon_0 > 0$ and $C > 0$ such that, if $\epsilon = \|u(0)\|_{H^1} < \epsilon_0$, the solution $u(t)$ of* (1-1) *can be written uniquely for all times as*

$$u(t) = \sum_{j=1}^n Q_{jz_j(t)} + \eta(t) \quad \textit{with} \quad \eta(t) \in \mathcal{H}_c[z(t)] \tag{1-7}$$

in such a way that there exist a unique j_0, *a* $\rho_+ \in [0,\infty)^n$ *with* $\rho_{+j} = 0$ *for* $j \neq j_0$ *and* $|\rho_+| \le C\|u(0)\|_{H^1}$, *and an* $\eta_+ \in H^1$ *with* $\|\eta_+\|_{H^1} \le C\|u(0)\|_{H^1}$, *such that*

$$\lim_{t\to+\infty} \|\eta(t,x) - e^{\mathrm{i}t\Delta}\eta_+(x)\|_{H^1_x} = 0, \qquad \lim_{t\to+\infty} |z_j(t)| = \rho_{+j}. \tag{1-8}$$

Furthermore, we have $\eta = \tilde{\eta} + A(t,x)$ *such that, for all admissible pairs* (p,q),

$$\|z\|_{L^\infty_t(\mathbb{R}_+)} + \|\tilde{\eta}\|_{L^p_t(\mathbb{R}_+, W^{1,q}_x)} \le C\|u(0)\|_{H^1} \quad \text{and} \quad \|\dot{z}_j + \mathrm{i}e_j z_j\|_{L^\infty_t(\mathbb{R}_+)} \le C\|u(0)\|^2_{H^1} \tag{1-9}$$

and such that $A(t,\cdot) \in \Sigma_2$ *for all* $t \ge 0$ *and*

$$\lim_{t\to+\infty} \|A(t,\cdot)\|_{\Sigma_2} = 0. \tag{1-10}$$

As an interesting corollary to Theorem 1.3, we show rather simply that the excited states are *orbitally unstable*. We recall that $e^{-\mathrm{i}tE_{jz}}Q_{jz}$ is called *orbitally stable* in $H^1(\mathbb{R}^3)$ for (1-1) if

$$\forall\varepsilon > 0 \ \exists\delta > 0 \quad \|u_0 - Q_{jz}\|_{H^1(\mathbb{R}^3)} < \delta \implies \sup_{t\in\mathbb{R}} \inf_{\vartheta\in\mathbb{R}} \|u(t) - e^{\mathrm{i}\vartheta}e^{-\mathrm{i}tE_{jz}}Q_{jz}\|_{H^1(\mathbb{R}^3)} < \varepsilon \tag{1-11}$$

and is orbitally unstable if (1-11) does not hold. We prove:

Theorem 1.4. *Assume* (H1)–(H4). *Then there exists* $\epsilon_0 > 0$ *such that, if* $j \ge 2$, *and for* $|z| < \epsilon_0$, *the standing wave* $e^{-\mathrm{i}tE_{jz}}Q_{jz}$ *is orbitally unstable. Furthermore,* $e^{-\mathrm{i}tE_{1z}}Q_{1z}$ *is orbitally stable.*

Notice that [Tsai and Yau 2002b; 2002c; 2002d; Soffer and Weinstein 2004; Gang and Weinstein 2008; 2011; Gustafson and Phan 2011; Nakanishi et al. 2012] contain only very partial proofs of the instability of the second excited state. Theorem 1.4 will be proved in Section 7 and, until then, and in particular in the sequel of this introduction, we will focus only on Theorem 1.3.

We recall that [Gustafson et al. 2004] proved Theorem 1.3 for $|u|^2u$ replaced by more general functions in the case when H has one eigenvalue (for the NLS with an electromagnetic potential, we refer to [Koo 2011]). The case of two eigenvalues is discussed in the series [Tsai and Yau 2002a; 2002b; 2002c] and in [Soffer and Weinstein 2004] under more stringent conditions on the initial data, which are such that $\|u_0\|_{H^{k,s}}$ is small for $k > 2$ and some s large enough in [Soffer and Weinstein 2004] and $\|u_0\|_{H^1\cap L^{2,s}}$ small for $s > 3$ in [Tsai and Yau 2002a; 2002b; 2002c]. A crucial restriction in these papers is that $2e_2 > e_1$. They then prove versions of Theorem 1.3 involving also rates of decay of $|z(t)|$, of $\|\eta(t)\|_{L^\infty(\mathbb{R}^3)}$, and of $\|\eta(t)\|_{L^{2,s}(\mathbb{R}^3)}$ for appropriate $s > 0$.

The ideas used in proofs such as in [Tsai and Yau 2002a; 2002b; 2002c; Soffer and Weinstein 2004] appear to be very difficult to extend to operators with more than 2 eigenvalues, where only partial results like in [Nakanishi et al. 2012] are known, and for initial data small only in H^1. On one hand, the Poincaré–Dulac normal form argument in these papers seems not suited to discuss the higher-order FGR needed when $2e_2 < e_1$. Furthermore, in these papers there is a subdivision of the evolution into distinct phases, which the solution enters in a somewhat irreversible fashion and which are considered one by one. This division into distinct phases might become unclear in cases when $u(t)$ oscillates from one phase to the other, as is not unlikely to happen in the H^1 case, or when the passage from one phase to the other is very slow, as is certainly true in the H^1 case. Moreover, an increase in the number of eigenvalues of H increases also the number of distinct phases that need to be accounted for and the complexity of the argument. So, any hope of proving Theorem 1.3 should rely on an argument which yields the asymptotics in a single stroke and which does not distinguish distinct cases. This is what we do; see, for example, the second part of Section 6. We did not check if our method yields the decay estimates of [Tsai and Yau 2002a; 2002b; 2002c; Soffer and Weinstein 2004] under more stringent conditions on u_0.

We give a new application of the interpretation of the FGR in terms of the Hamiltonian structure of the equation. This interpretation was first introduced in [Cuccagna 2009] and was then applied in [Bambusi and Cuccagna 2011] to generalize the result of [Soffer and Weinstein 1999]. It was later applied to the problem of asymptotic stability of ground states of the NLS, first not allowing translation symmetries in [Cuccagna 2011a], and then with translation in [Cuccagna 2014]; see also [Cuccagna 2012].

The link between FGR and Hamiltonian structure rests in the fact that the latter yields algebraic identities between coefficients of different coordinates in the system (compare the right-hand side in (6-13) with the second line in (6-27)). These allow us to show that some other coefficients in the equations of the z_j have a square power structure and have a fixed sign (in the case of the NLS); see Lemma 6.8. This then yields decay of the z_j, except for at most one of the j here. We refer to pp. 287–288 in [Cuccagna 2011a] for the original intuition behind this approach to the FGR, which views the FGR as a simple consequence of Schwartz's lemma on mixed derivatives, and which has been used in [Bambusi and Cuccagna 2011; Cuccagna 2009; 2011a; 2014; 2012], among others. For other applications of this theory we refer to the references in [Cuccagna 2012; Cuccagna and Maeda 2014]. We refer also to [Cuccagna 2011b], whose treatment of the FGR is similar to the one in this paper. Earlier treatments of FGR, are in [Tsai and Yau 2002a; 2002b; 2002c; Soffer and Weinstein 2004] and, still earlier, in [Buslaev and Perel′man 1995; Soffer and Weinstein 1999], but they seem to work only in relatively simple cases, because they run into trouble if the normal form argument requires more than a very few steps. For more references and comments see [Cuccagna 2011a].

As we will see below, the FGR can be seen relatively easily after one finds an appropriate effective Hamiltonian in the right system of coordinates. This coordinate system is obtained by a normal form argument. Right from the beginning, though, it is crucial to choose the right ansatz and system of coordinates. For example, since H has eigenvalues, it would seem natural to split the NLS (1-1) into a system using the coordinates of the spectral decomposition of H; see (4-2). However, this would not be a good choice for our nonlinear system. Following [Gustafson et al. 2004], it is better to pick as coordinates the z_j of Proposition 1.1, complementing them with an appropriate continuous coordinate. There is the natural ansatz (2-1) (the same used in [Soffer and Weinstein 2004]), which, following [Gustafson et al. 2004], can be used to obtain the continuous coordinate, here denoted η and introduced in Lemma 2.4.

Once we have coordinates (z, η) with $z = (z_1, \dots, z_n)$, where z_1 is the ground state coordinate, z_j for $j > 1$ the excited states coordinates, and η the radiation coordinate, Theorem 1.3 can be loosely paraphrased as

$$\eta(t) \to 0 \quad \text{in } H^1_{\mathrm{loc}} \quad \text{and} \quad z_j(t) \to 0 \quad \text{except for at most one } j. \tag{1-12}$$

In particular, if $z(t) \to 0$ the solution $u(t)$ of (1-1) scatters like a solution of $\mathrm{i}\dot{u} = -\Delta u$ in H^1. Otherwise there is one j such that $u(t)$ scatters to $e^{\mathrm{i}\vartheta(t)} Q_{z_{+j}}$, with $\vartheta(t)$ a phase term which we do not control here. We have convergence by scattering to a ground state if $j = 1$, and to an excited state if $j > 1$. The latter presumably occurs for the $u(t)$ whose trajectory is contained in an appropriate manifold; see [Tsai and Yau 2002d; Beceanu 2012; Gustafson and Phan 2011].

It is not easy to see (1-12) in the initial coordinate system. So we need a Birkhoff normal form argument to identify an effective Hamiltonian, like in [Bambusi and Cuccagna 2011]. Unlike there, but like in [Cuccagna 2011a], the initial coordinates, while quite natural from the point of view of the NLS (1-1), are not Darboux coordinates for the natural symplectic form Ω in the problem; see (4-1). Hence, before doing normal forms, we have first to implement the Darboux theorem to diagonalize the problem (of course, the coordinates arising from the spectral decomposition of H — see (4-2) — are Darboux coordinates, but, as we wrote, they are not suited for our nonlinear asymptotic analysis). So in this paper we need to perform a number of coordinate changes: first a Darboux theorem and then normal form analysis. At the end of the process we get new coordinates $(z_1, \dots, z_n, \eta)$ where the Hamiltonian is sufficiently simple that we can prove (1-12) relatively easily using the FGR (which tells us that all the z_j, except at most one, are damped) and a semilinear NLS for η that shows scattering of η because of linear dispersion. In the context of the theory developed in [Bambusi and Cuccagna 2011; Cuccagna 2011a] and other literature, the work in the last system of coordinates, that is, all the material in Section 6, is rather routine.

Having proved (1-12) for the last system of coordinates (z, η), the obvious question is why (1-12) should hold, as Theorem 1.3 is saying, also for the initial coordinates, which we now denote by (z', η') to distinguish them from the final coordinates (z, η). Keeping in mind that all coordinate changes are small nonlinear perturbations of the identity, the only simple reason why this might happen is that different coordinates must be related in the form

$$\begin{aligned} z'_k &= z_k + O(z\eta) + O(\eta^2) + \sum_{i \neq j} O(z_i z_j) \quad \text{for } k = 1, \dots, n, \\ \eta' &= \eta + O(z\eta) + O(\eta^2) + \sum_{i \neq j} O(z_i z_j). \end{aligned} \tag{1-13}$$

This relation between any two systems of coordinates forbids relations like $z'_1 = z_1 + z_2^2$. Indeed, with the latter relations it would not be true (except for the case $z(t) \to 0$) that (1-12) for (z, η) implies (1-12) for (z', η'). So our main strategy is to prove (1-12) for the final (z, η) with some relatively standard method using FGR and linear dispersion, and to be careful to implement only coordinate changes like in (1-13). This latter point is the novel problem we need to face in this paper. It is not obvious from the outset that (1-13) should hold.

As we wrote above, [Gustafson et al. 2004] suggests a very natural choice of functions z_j, based on Proposition 1.1, which can be completed in a system of independent coordinates. Loosely speaking, the z_j have the problem that they are defined somewhat independently to each other. This shows up in the expansion of the Hamiltonian in Lemma 3.1, with a certain lack of decoupling inside the energy between distinct z_j; see (3-9) and Remark 3.2. This leads in (3-3) (see the second line) to terms whose elimination in a normal form argument would seem incompatible with coordinate changes satisfying (1-13). These bad terms of the energy can be better seen in (4-45): they are the $l = 0$ terms in the third line. Other additional bad terms arise in the course of the Darboux theorem transformation. Bad terms in the differential form Γ in (4-17) (used in the classical formula (4-40)) are those in I_1 in (4-22). Specifically, they are the first term in the right-hand side of (4-22). The right-hand side of (4-28) is also filled with bad terms, in the sense that they yield a coordinate change $\mathfrak{F}$ in Lemma 4.8 leading to more $l = 0$ terms in the third line in (4-45). Specifically, they originate from the pullback $\mathfrak{F}^* \sum_{j=1}^n E(Q_{jz_j})$ of the first term in the right-hand side of (3-3) (more bad terms seem to arise if we use Ω_0' — see (4-8) — rather than the slightly more complicated Ω_0 — see (4-13) — as the local model of Ω). In a somewhat empirical fashion, for which we don't have a simple conceptual reason, a plain and simple computation shows that all the bad terms cancel out and that there are no $l = 0$ terms in (4-45). This is proved in the cancellation lemma, Lemma 4.11, which is the main new ingredient in the paper. This lemma proves that the change of coordinates designed to diagonalize Ω is also decoupling the discrete coordinates inside the Hamiltonian. From that point on, the structure (1-13) for the coordinate changes is automatic and the various steps of the proof of Theorem 1.3 are similar to arguments such as [Cuccagna 2011b; 2012], which have been repeated in a number of papers. So they are fairly standard, even though we are able to discuss them only in a rather technical way. We have to go into the details of the proof, rather than refer to the references, because of some technical novelties required by the fact that in general $z \nrightarrow 0$, and what converges to 0 is instead the vector $\boldsymbol{Z}$ introduced in Definition 2.2, whose components are products of distinct components of z.

In the second part of Section 6, the FGR and the asymptotics of the z_j in the final coordinate system are rather simple to see in a single stroke. Furthermore, Theorem 6.1 is more or less the same as [Cuccagna 2011a; 2011b].

One limitation in our present paper is that we do not generate examples of equations which satisfy hypothesis (H4). Notice though that our result, for solutions only in H^1, is new even in the 2-eigenvalues case of [Tsai and Yau 2002a; 2002b; 2002c; Soffer and Weinstein 2004], where our FGR is the same. Still, we believe that (H4) holds for generic V. And even if it fails at one stage, this is not necessarily a problem: the strict positive sign in the FGR is only an obstruction to performing further the normal form argument, so, if there is a 0, in principle it is enough to proceed with some further coordinate change until, after a finite number of steps, there will finally be a positive sign in the FGR, and so the stabilization will occur, just at a slower rate. And if the FGR is always 0, then maybe this is because the NLS has a special structure; see [Soffer and Weinstein 1999, p. 69] for some thoughts.

Proposition 2.2 of [Bambusi and Cuccagna 2011] proves validity in general of the FGR. Transposing here that proof would require replacing the cubic nonlinearity with a more general nonlinearity $\beta(|u|^2)u$. This seems rather simple to do because the cubic power is only used to simplify the discussion in Lemma 3.1. But it is not so clear how to offset here the absence of a meaningful mass term m^2u, which in [Bambusi and Cuccagna 2011, pp. 1444–1445], by choosing m generic, is used to move some appropriate spheres in phase space. Adding to the NLS a term m^2u would not change the spheres here.

We reiterate that Proposition 1.1 is valid for small $z_j \in \mathbb{C}$. As z_j increases there are interesting symmetry-breaking bifurcation phenomena; see [Kirr et al. 2008; 2011] and references therein and see also [Fukuizumi and Sacchetti 2011; Grecchi et al. 2002; Sacchetti 2005] and references therein for the semiclassical NLS. Notice that Theorem 1.3 should allow one to prove asymptotic breakdown of the beating motion in the case $\mu_\infty = 0$ in [Grecchi et al. 2002]. Finite-dimensional approximations of the solutions at energies close to the symmetry breaking point of [Kirr et al. 2008] have been considered by [Goodman 2011; Marzuola and Weinstein 2010], who prove the long time existence of interesting patterns for the full NLS. Unfortunately, it is beyond the scope of our analysis, and it remains an interesting open problem, to understand the eventual asymptotic behavior of the solutions in [Goodman 2011; Marzuola and Weinstein 2010].

2. Notation, coordinates and resonant sets

Notation.

- We denote by $\mathbb{N} = \{1, 2, \dots\}$ the set of natural numbers and set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.
- We denote $z = (z_1, \dots, z_n)$, $|z| := \sqrt{\sum_{j=1}^n |z_j|^2}$.
- Given a Banach space X, $v \in X$ and $\delta > 0$, we set $B_X(v, \delta) := \{x \in X : \|v - x\|_X < \delta\}$.
- Let A be an operator on $L^2(\mathbb{R}^3)$. Then $\sigma_p(A) \subset \mathbb{C}$ is the set of eigenvalues of A and $\sigma_e(A) \subset \mathbb{C}$ is the essential spectrum of A.
- For $\mathbb{K} = \mathbb{R}, \mathbb{C}$, we denote by $\Sigma_r = \Sigma_r(\mathbb{R}^3, \mathbb{K})$ for $r \in \mathbb{N}_0$ the Banach spaces defined by the completion of $C_c(\mathbb{R}^3, \mathbb{K})$ by the norms
$$\|u\|_{\Sigma_r}^2 := \sum_{|\alpha| \le r} (\|x^\alpha u\|_{L^2(\mathbb{R}^3)}^2 + \|\partial_x^\alpha u\|_{L^2(\mathbb{R}^3, \mathbb{K})}^2).$$
For $m < 0$ we consider the topological dual $\Sigma_m = (\Sigma_{-m})'$. Notice — see [Cuccagna 2014] — that the spaces Σ_r can be equivalently defined using, for $r \in \mathbb{R}$, the norm $\|u\|_{\Sigma_r} := \|(1 - \Delta + |x|^2)^{r/2} u\|_{L^2}$.
- $\mathcal{S}(\mathbb{R}^3) = \bigcap_{m \ge 0} \Sigma_m$ is the space of Schwartz functions; $\mathcal{S}'(\mathbb{R}^3) = \bigcup_{m \le 0} \Sigma_m$ is the space of tempered distributions.
- We set $z_j = z_{jR} + \mathrm{i} z_{jI}$ for $z_{jR}, z_{jI} \in \mathbb{R}$.
- For $f : \mathbb{C}^n \to \mathbb{C}$, set $D_{jR} f(z) := \partial f / \partial z_{jR}(z)$ and $D_{jI} f(z) := \partial f / \partial z_{jI}(z)$.
- We set $\partial_l := \partial_{z_l}$ and $\partial_{\bar{l}} := \partial_{\bar{z}_l}$. Here, as is customary, $\partial_{z_l} = \frac{1}{2}(D_{lR} - \mathrm{i} D_{lI})$ and $\partial_{\bar{z}_l} = \frac{1}{2}(D_{lR} + \mathrm{i} D_{lI})$.
- Occasionally we use a single index $\ell = j, \bar{j}$. To define $\bar{\ell}$ we use the convention $\bar{\bar{j}} = j$. We will also write $z_{\bar{j}} = \bar{z}_j$.
- We will consider vectors $z = (z_1, \dots, z_n) \in \mathbb{C}^n$ and, for vectors $\mu, \nu \in (\mathbb{N} \cup \{0\})^n$, we set $z^\mu \bar{z}^\nu := z_1^{\mu_1} \cdots z_n^{\mu_n} \bar{z}_1^{\nu_1} \cdots \bar{z}_n^{\nu_n}$. We will set $|\mu| = \sum_j \mu_j$.
- We have $dz_j = dz_{jR} + \mathrm{i}\, dz_{jI}$, $d\bar{z}_j = dz_{jR} - \mathrm{i}\, dz_{jI}$.
- We consider the vector $\boldsymbol{e} = (e_1, \dots, e_n)$ whose entries are the eigenvalues of H.
- P_c is the orthogonal projection of L^2 onto $\mathcal{H}_c[0]$.

- Given two Banach spaces X and Y we denote by $B(X,Y)$ the space of bounded linear operators $X\to Y$ with the norm of the uniform operator topology.

Coordinates. The first thing we need is an ansatz. This is provided by the following lemma:

Lemma 2.1. *There exist $c_0>0$ and $C>0$ such that, for all $u\in H^1$ with $\|u\|_{H^1}<c_0$, there exists a unique pair $(z,\Theta)\in\mathbb{C}^n\times(H^1\cap\mathcal{H}_c[z])$ such that*

$$u=\sum_{j=1}^{n}Q_{jz_j}+\Theta \quad \textit{with} \quad |z|+\|\Theta\|_{H^1}\le C\|u\|_{H^1}. \tag{2-1}$$

Finally, the map $u\mapsto(z,\Theta)$ is $C^\infty(B_{H^1}(0,c_0),\mathbb{C}^n\times H^1)$ and satisfies the gauge property

$$z(e^{i\vartheta}u)=e^{i\vartheta}z(u) \quad \textit{and} \quad \Theta(e^{i\vartheta}u)=e^{i\vartheta}\Theta(u). \tag{2-2}$$

Proof. We consider the functions

$$F_{jA}(u,z):=\operatorname{Re}\left\langle u-\sum_{l=1}^{n}Q_{lz_l},\, \mathrm{i}\overline{D_{jA}Q_{jz_j}}\right\rangle \quad \text{for } A=R,I.$$

We have $F_{jR}(0,0)=F_{jI}(0,0)=0$. These functions are smooth in $L^2\times B_{\mathbb{C}^n}(0,b_0)$ for the b_0 in Definition 1.2. We have $F_{jR}(0,z)=\operatorname{Im}z_j+O(z^3)$ and $F_{jI}(0,z)=\operatorname{Re}z_j+O(z^3)$ by Proposition 1.1. By the implicit function theorem, there is a map $u\to z$ which is $C^\infty(B_{L^2}(0,c_0),\mathbb{C}^n)$ for $c_0>0$ sufficiently small. Set $\Theta:=u-\sum_{j=1}^{n}Q_{jz_j}$. Then $\Theta\in C^\infty(B_{H^1}(0,c_0),H^1)$. The inequalities follow from $|z(u)|\le C\|u\|_{H^1}$, which follows from $z\in C^1$ and $z(0)=0$. Formula (2-2) follows from

$$e^{i\vartheta}u=\sum_{j=1}^{n}e^{i\vartheta}Q_{jz_j}+e^{i\vartheta}\Theta=\sum_{j=1}^{n}Q_{je^{i\vartheta}z_j}+e^{i\vartheta}\Theta$$

and from the fact that $\Theta\in\mathcal{H}_c[z]$ implies $e^{i\vartheta}\Theta\in\mathcal{H}_c[z']$, where $z'=e^{i\vartheta}z$. This last fact is elementary. Indeed, setting only for this proof $z_j=x_j+\mathrm{i}y_j$ and $z'_j=x'_j+\mathrm{i}y'_j$, we have

$$\operatorname{Re}\langle \mathrm{i}\overline{e^{i\vartheta}\Theta},\partial_{x'_j}Q_{jz'_j}\rangle=\partial_{x'_j}x_j\operatorname{Re}\langle \mathrm{i}\overline{e^{i\vartheta}\Theta},e^{i\vartheta}\partial_{x_j}Q_{jz_j}\rangle+\partial_{x'_j}y_j\operatorname{Re}\langle \mathrm{i}\overline{e^{i\vartheta}\Theta},e^{i\vartheta}\partial_{y_j}Q_{jz_j}\rangle=0$$

if $\Theta\in\mathcal{H}_c[z]$. Similarly, $\operatorname{Re}\langle \mathrm{i}\overline{e^{i\vartheta}\Theta},\partial_{y'_j}Q_{jz'_j}\rangle=0$. Hence $\Theta\in\mathcal{H}_c[z]$ implies $e^{i\vartheta}\Theta\in\mathcal{H}_c[e^{i\vartheta}z]$. □

Definition 2.2. Given $z\in\mathbb{C}^n$, we denote by $\widehat{\boldsymbol{Z}}$ the vector with entries $(z_i\bar z_j)$ with $i,\ j\in[1,n]$, in lexicographic order. We denote by $\boldsymbol{Z}$ the vector with entries $(z_i\bar z_j)$ with $i,\ j\in[1,n]$, in lexicographic order but only for pairs of indexes with $i\ne j$. Here, $\boldsymbol{Z}$ is in L, the subspace of $\mathbb{C}^{n_0}=\{(a_{i,j})_{i,j=1,\dots,n}:i\ne j\}$, $n_0=n(n-1)$, with $(a_{i,j})\in L$ if and only if $a_{i,j}=\bar a_{j,i}$ for all i,j. For a multiindex $\boldsymbol{m}=\{m_{ij}\in\mathbb{N}_0:i\ne j\}$, we set $\boldsymbol{Z}^{\boldsymbol{m}}=\prod(z_i\bar z_j)^{m_{ij}}$ and $|\boldsymbol{m}|:=\sum_{i,j}m_{ij}$.

We need a system of independent coordinates, which the (z,Θ) in (2-1) are not. The following lemma is used to complete the z with a continuous coordinate.

Lemma 2.3. *There exists $d_0 > 0$ such that, for any $z \in \mathbb{C}$ with $|z| < d_0$, there exists an $\mathbb{R}$-linear operator $R[z] : \mathcal{H}[0] \to \mathcal{H}_c[z]$ such that $P_c|_{\mathcal{H}_c[z]} = R[z]^{-1}$, with P_c the orthogonal projection of L^2 onto $\mathcal{H}_c[0]$; see Definition 1.2. Furthermore, for $|z| < d_0$ and $\eta \in \mathcal{H}_c[0]$, we have the following properties*:

(1) *$R[z] \in C^\infty(B_{\mathbb{C}^n}(0, d_0), B(H^1, H^1))$ with $B(H^1, H^1)$ the Banach space of $\mathbb{R}$-linear bounded operators from H^1 into itself.*

(2) *For any $r > 0$, we have $\|(R[z] - 1)\eta\|_{\Sigma_r} \le c_r |z|^2 \|\eta\|_{\Sigma_{-r}}$ for a fixed c_r.*

(3) *We have the covariance property $R[e^{\mathrm{i}\vartheta} z] = e^{\mathrm{i}\vartheta} R[z] e^{-\mathrm{i}\vartheta}$.*

(4) *We have, summing on repeated indexes,*

$$R[z]\eta = \eta + (\alpha_j[z]\eta)\phi_j \quad \textit{with} \quad \alpha_j[z]\eta = \langle B_j(z), \eta\rangle + \langle C_j(z), \bar{\eta}\rangle, \tag{2-3}$$

where $B_j(z) = \widehat{B}_j(\widehat{\boldsymbol{Z}})$ and $C_j(z) = z_i z_\ell \widehat{C}_{i\ell j}(\widehat{\boldsymbol{Z}})$ for $\widehat{B}$ and $\widehat{C}_{i\ell j}$ smooth and the $\widehat{\boldsymbol{Z}}$ of Definition 2.2.

(5) *We have, for $r \in \mathbb{R}$ and $\boldsymbol{Z}$ as in Definition 2.2,*

$$\|B_j(z) + \partial_{\bar{z}_j} \bar{q}_{j z_j}\|_{\Sigma_r} + \|C_j(z) - \partial_{\bar{z}_j} q_{j z_j}\|_{\Sigma_r} \le c_r |\boldsymbol{Z}|^2. \tag{2-4}$$

Proof. Summing over repeated indexes, we search for a map $R[z] : L^2 \to \mathcal{H}_c[z]$ of the form

$$R[z] f = f + (\alpha_j[z] f)\phi_j \quad \text{with} \quad \alpha_j[z] f = \langle B_j'(z), f\rangle + \langle C_j(z), \bar{f}\rangle$$

such that $R[z] f \in \mathcal{H}_c[z]$ for all $f \in L^2$. The latter condition can be expressed as

$$\operatorname{Re}\langle \bar{f}, \mathrm{i} D_{lA} Q_{l z_l} + \langle \phi_j, \mathrm{i} D_{lA} Q_{l z_l}\rangle \overline{B}_j' - \langle \phi_j, \mathrm{i} D_{lA} \overline{Q}_{l z_l}\rangle C_j\rangle = 0 \quad \text{for all} \;\; f \in L^2.$$

This and the equalities

$$\begin{aligned} &\langle \phi_j, \mathrm{i} D_{lR} Q_{l z_l}\rangle = \mathrm{i}\delta_{jl} + \langle \phi_j, \mathrm{i} D_{lR} q_{l z_l}\rangle, \quad \langle \phi_j, \mathrm{i} D_{lI} Q_{l z_l}\rangle = -\delta_{jl} + \langle \phi_j, \mathrm{i} D_{lI} q_{l z_l}\rangle, \\ &\langle \phi_j, \mathrm{i} D_{lR} \overline{Q}_{l z_l}\rangle = \mathrm{i}\delta_{jl} + \langle \phi_j, \mathrm{i} D_{lR} \bar{q}_{l z_l}\rangle, \quad \langle \phi_j, \mathrm{i} D_{lI} \overline{Q}_{l z_l}\rangle = \delta_{jl} + \langle \phi_j, \mathrm{i} D_{lI} \bar{q}_{l z_l}\rangle, \end{aligned}$$

yield the equalities

$$\begin{aligned} D_{lR} Q_{l z_l} + (\delta_{jl} + \langle \phi_j, D_{lR} q_{l z_l}\rangle)\overline{B}_j' - (\delta_{jl} + \langle \phi_j, D_{lR} \bar{q}_{l z_l}\rangle) C_j &= 0, \\ \mathrm{i} D_{lI} Q_{l z_l} + (-\delta_{jl} + \mathrm{i}\langle \phi_j, D_{lI} q_{l z_l}\rangle)\overline{B}_j' - (\delta_{jl} + \mathrm{i}\langle \phi_j, D_{lI} \bar{q}_{l z_l}\rangle) C_j &= 0. \end{aligned}$$

They can be rewritten as

$$\begin{aligned} \phi_l + \partial_l q_{l z_l} + (\delta_{jl} + \mathrm{i}\langle \phi_j, \partial_l q_{l z_l}\rangle)\overline{B}_j' - \langle \phi_j, \partial_l \bar{q}_{l z_l}\rangle C_j &= 0, \\ \partial_{\bar{l}} q_{l z_l} + \langle \phi_j, \partial_{\bar{l}} q_{l z_l}\rangle \overline{B}_j' - (\delta_{jl} + \langle \phi_j, \partial_{\bar{l}} \bar{q}_{l z_l}\rangle) C_j &= 0. \end{aligned} \tag{2-5}$$

For $z^2 = \{z_j^2 \delta_{ij}\}$ and $\bar{z}^2 = \{\bar{z}_j^2 \delta_{ij}\}$ two $n \times n$ matrices, the solution of this system is of the form

$$\begin{pmatrix} \overline{B}' \\ C \end{pmatrix} = \sum_{m=0}^{\infty} (-1)^m \begin{pmatrix} \boldsymbol{A}_1 & \bar{z}^2 \boldsymbol{A}_2 \\ z^2 \boldsymbol{A}_3 & \boldsymbol{A}_4 \end{pmatrix}^m \begin{pmatrix} u_1 \\ z^2 u_2 \end{pmatrix}, \tag{2-6}$$

where $\boldsymbol{A}_l = \boldsymbol{A}_l(|z_1|^2, \dots, |z_n|^2)$ are $n \times n$ matrices and $u_l = u_l(|z_1|^2, \dots, |z_n|^2)$ are $n \times 1$ matrices for $l = 1$ (resp. $l = 2$) with entries $\phi_j + \partial_j q_{jz_j}$ (resp. $\partial_{\bar{j}} q_{jz_j}$) as $j = 1, \dots, n$. This yields the structure $\bar{B}'(z) = \widehat{B}'(\widehat{\boldsymbol{Z}})$ and $C_j(z) = z_i z_\ell \widehat{C}_{i\ell j}(\widehat{\boldsymbol{Z}})$.

Using $\langle \phi_j, q_{jz_j} \rangle = 0$, we can rewrite (2-5) in the form

$$\begin{aligned} \bar{B}'_l &= -\phi_l - \partial_l q_{lz_l} - \sum_{j \neq l} (\mathrm{i} \langle \phi_j, \partial_l q_{lz_l} \rangle \bar{B}'_j - \langle \phi_j, \partial_l \bar{q}_{lz_l} \rangle C_j), \\ C_l &= \partial_{\bar{l}} q_{lz_l} + \sum_{j \neq l} (\langle \phi_j, \partial_{\bar{l}} q_{lz_l} \rangle \bar{B}'_j - \langle \phi_j, \partial_{\bar{l}} \bar{q}_{lz_l} \rangle) C_j. \end{aligned} \tag{2-7}$$

By Proposition 1.1, this implies

$$\| \bar{B}'_l + \phi_l \|_{\Sigma_r} + \| C_l \|_{\Sigma_r} \le C |z_l|^2. \tag{2-8}$$

Reiterating this estimate, from (2-7) and for B_l defined by the following formula, we get

$$\begin{aligned} \Big\| \overbrace{\bar{B}'_l + \phi_l - \sum_{j \neq l} \mathrm{i} \langle \phi_j, \partial_l q_{lz_l} \rangle \phi_j}^{\bar{B}_l} + \partial_l q_{lz_l} \Big\|_{\Sigma_r} &\le C |\boldsymbol{Z}|^2 \\ \| C_l - \partial_{\bar{l}} q_{lz_l} \|_{\Sigma_r} &\le C |\boldsymbol{Z}|^2. \end{aligned}$$

This yields (2-4). Claim (3) follows by

$$\alpha_j[e^{\mathrm{i}\vartheta} z] \eta = e^{\mathrm{i}\vartheta} \alpha_j[z] e^{-\mathrm{i}\vartheta} \eta, \tag{2-9}$$

which in turn follows by claim (4). Indeed,

$$\begin{aligned} \alpha_j[e^{\mathrm{i}\vartheta} z] \eta &= \langle \widehat{B}_j(\widehat{\boldsymbol{Z}}), \eta \rangle + \langle e^{2\mathrm{i}\vartheta} z_i z_\ell \widehat{C}_{i\ell j}(\widehat{\boldsymbol{Z}}), \bar{\eta} \rangle \\ &= e^{\mathrm{i}\vartheta} \langle \widehat{B}_j(\widehat{\boldsymbol{Z}}), e^{-\mathrm{i}\vartheta} \eta \rangle + e^{\mathrm{i}\vartheta} \langle z_i z_\ell \widehat{C}_{i\ell j}(\widehat{\boldsymbol{Z}}), \overline{e^{-\mathrm{i}\vartheta} \eta} \rangle = e^{\mathrm{i}\vartheta} \alpha_j[z] e^{-\mathrm{i}\vartheta} \eta. \end{aligned} \qquad \square$$

We are now able to define a system of coordinates near the origin in L^2.

Lemma 2.4. *For the d_0 of Lemma 2.3, the map $(z, \eta) \mapsto u$ defined by*

$$u = \sum_{j=1}^{n} Q_{jz_j} + R[z]\eta \quad \textit{for } (z, \eta) \in B_{\mathbb{C}^n}(0, d_0) \times (H^1 \cap \mathcal{H}_c[0]) \tag{2-10}$$

has values in H^1 and is C^∞. Furthermore, there is a $d_1 > 0$ such that the above map is a diffeomorphism for $(z, \eta) \in B_{\mathbb{C}^n}(0, d_1) \times (B_{H^1}(0, d_1) \cap \mathcal{H}_c[0])$ and

$$|z| + \|\eta\|_{H^1} \sim \|u\|_{H^1}. \tag{2-11}$$

Finally, we have the gauge properties $u(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta} \eta) = e^{\mathrm{i}\vartheta} u(z, \eta)$,

$$z(e^{\mathrm{i}\vartheta} u) = e^{\mathrm{i}\vartheta} z(u) \quad \textit{and} \quad \eta(e^{\mathrm{i}\vartheta} u) = e^{\mathrm{i}\vartheta} \eta(u). \tag{2-12}$$

Proof. The smoothness follows from the smoothness in z in Proposition 1.1 and Lemma 2.3. Property $u(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta}u(z,\eta)$ and its equivalent formula (2-12) follow from (2-2) and Lemma 2.3(3). Notice that $u = u(z,\eta)$ is the inverse of the smooth map $u \mapsto (z,\Theta) \mapsto (z, P_c\Theta)$. Formula (2-11) follows by the estimates in Proposition 1.1 and by Lemma 2.3(2). □

Resonant sets.

Definition 2.5. Consider the set of multiindexes $\boldsymbol{m}$ as in Definition 2.2 and, for any $k \in \{1,\dots,n\}$, the set

$$\begin{aligned} \mathcal{M}_k(r) &= \big\{\boldsymbol{m} : \textstyle\sum_{i=1}^n \sum_{j=1}^n m_{ij}(e_i - e_j) - e_k < 0 \ \text{ and } \ |\boldsymbol{m}| \le r\big\}, \\ \mathcal{M}_0(r) &= \big\{\boldsymbol{m} : \textstyle\sum_{i=1}^n \sum_{j=1}^n m_{ij}(e_i - e_j) = 0 \ \text{ and } \ |\boldsymbol{m}| \le r\big\}. \end{aligned} \tag{2-13}$$

Set now

$$\begin{aligned} M_k(r) &= \{(\mu,\nu) \in \mathbb{N}_0^n \times \mathbb{N}_0^n : z^\mu \bar{z}^\nu = \bar{z}_k \boldsymbol{Z}^{\boldsymbol{m}} \ \text{ for some } \ \boldsymbol{m} \in \mathcal{M}_k(r)\}, \\ M(r) &= \bigcup_{k=1}^n M_k(r) \quad \text{and} \quad M = M(2N+4) \end{aligned} \tag{2-14}$$

Lemma 2.6. *Assuming* (H3) *we have the following facts*:

(1) *If* $\boldsymbol{Z}^{\boldsymbol{m}} = z^\mu \bar{z}^\nu$, *then* $\boldsymbol{m} \in \mathcal{M}_0(2N+4)$ *implies* $\mu = \nu$. *In particular*, $\boldsymbol{m} \in \mathcal{M}_0(2N+4)$ *implies* $\boldsymbol{Z}^{\boldsymbol{m}} = |z_1|^{2l_1} \cdots |z_n|^{2l_n}$ *for some* $(l_1,\dots,l_n) \in \mathbb{N}_0^n$.

(2) *For* $|\boldsymbol{m}| \le 2N+3$ *and any* j, *we have* $\sum_{a,b}(e_a - e_b)m_{ab} - e_j \ne 0$.

Proof. First of all, if $\mu = \nu$ then $z^\mu \bar{z}^\nu = |z_1|^{2\mu_1} \cdots |z_n|^{2\mu_n}$. So the first sentence in claim (1) implies the second sentence in claim (1). We have

$$\boldsymbol{Z}^{\boldsymbol{m}} = \prod_{i,l=1}^n (z_i \bar{z}_l)^{m_{il}} = \prod_{i=1}^n z_i^{\sum_{l=1}^n m_{il}} \bar{z}_i^{\sum_{l=1}^n m_{li}} = z^\mu \bar{z}^\nu.$$

The pair (μ,ν) satisfies $|\mu| = |\nu| \le 2N+4$, by

$$|\mu| = \sum_l \mu_l = \sum_{i,l} m_{il} = |\nu|.$$

We have $(\mu - \nu)\cdot \boldsymbol{e} = 0$ by $\boldsymbol{m} \in \mathcal{M}_0(2N+4)$ and

$$\sum_i \mu_i e_i - \sum_l \nu_l e_l = \sum_{i,l} m_{il}(e_i - e_l) = 0.$$

We conclude, by (H3), that $\mu - \nu = 0$. This proves the first sentence of claim (1).

The proof of claim (2) is similar. Set

$$\boldsymbol{Z}^{\boldsymbol{m}} \bar{z}_j = \prod_{i,l=1}^n (z_i \bar{z}_l)^{m_{il}} \bar{z}_j = \prod_{i=1}^n z_i^{\sum_{l=1}^n m_{il}} \bar{z}_i^{\sum_{l=1}^n m_{li}} \bar{z}_j = z^\mu \bar{z}^\nu.$$

We have

$$(\mu - \nu)\cdot \boldsymbol{e} = \sum_i \mu_i e_i - \sum_l \nu_l e_l = \sum_{i,l} m_{il}(e_i - e_l) - e_j$$

and

$$|\mu| = \sum_l \mu_l = \sum_{i,l} m_{il} = |\nu| - 1. \tag{2-15}$$

If $(\mu - \nu) \cdot \boldsymbol{e} = 0$ then, by $|\mu - \nu| \le 4N + 5$ and (H3), we would have $\mu = \nu$, which is impossible by (2-15). □

Lemma 2.7. (1) *Consider* $\boldsymbol{m} = (m_{ij}) \in \mathbb{N}_0^{n_0}$ *such that* $\sum_{i<j} m_{ij} > N$ *for* $N > |e_1|(\min\{e_j - e_i : j > i\})^{-1}$; *see* (H3). *Then, for any eigenvalue* e_k, *we have*

$$\sum_{i<j} m_{ij}(e_i - e_j) - e_k < 0. \tag{2-16}$$

(2) *Consider* $\boldsymbol{m} \in \mathbb{N}_0^{n_0}$ *with* $|\boldsymbol{m}| \ge 2N + 3$ *and the monomial* $z_j \boldsymbol{Z}^{\boldsymbol{m}}$. *Then there exist* $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}_0^{n_0}$ *such that*

$$\sum_{i<j} a_{ij} = N + 1 = \sum_{i<j} b_{ij},$$
$$a_{ij} = b_{ij} = 0 \quad \text{for all } i > j \qquad \text{and} \qquad a_{ij} + b_{ij} \le m_{ij} + m_{ji} \quad \text{for all } (i, j), \tag{2-17}$$

and moreover there is a pair of indexes (k, l) *such that*

$$\sum_{i<j} a_{ij}(e_i - e_j) - e_k < 0 \quad \text{and} \quad \sum_{i<j} b_{ij}(e_i - e_j) - e_l < 0 \tag{2-18}$$

and such that, for $|z| \le 1$,

$$|z_j \boldsymbol{Z}^{\boldsymbol{m}}| \le |z_j|\,|z_k \boldsymbol{Z}^{\boldsymbol{a}}|\,|z_l \boldsymbol{Z}^{\boldsymbol{b}}|. \tag{2-19}$$

(3) *For* $\boldsymbol{m}$ *with* $|\boldsymbol{m}| \ge 2N + 3$, *there exist* (k, l), $\boldsymbol{a} \in \mathcal{M}_k$ *and* $\boldsymbol{b} \in \mathcal{M}_l$ *such that* (2-19) *holds.*

Proof. Equation (2-16) follows immediately from

$$\sum_{i<j} m_{ij}(e_i - e_j) - e_k \le -\min\{e_j - e_i : j > i\}N - e_1 < 0,$$

where the latter inequality follows by the definition of N.

Given $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{N}_0^{n_0}$ satisfying (2-17), by claim (1) they satisfy (2-18) for any pair of indexes (k, l). Consider now the monomial $z_j \boldsymbol{Z}^{\boldsymbol{m}}$. Since $|\boldsymbol{m}| \ge 2N + 3$, there are vectors $\boldsymbol{c}, \boldsymbol{d} \in \mathbb{N}_0^{n_0}$ such that $|\boldsymbol{c}| = |\boldsymbol{d}| = N + 1$ and $c_{ij} + d_{ij} \le m_{ij}$ for all (i, j). Furthermore, we have

$$z_j \boldsymbol{Z}^{\boldsymbol{m}} = z_j z^{\mu} \bar{z}^{\nu} \boldsymbol{Z}^{\boldsymbol{c}} \boldsymbol{Z}^{\boldsymbol{d}} \quad \text{with} \quad |\mu| > 0 \text{ and } |\nu| > 0. \tag{2-20}$$

So, for z_k a factor of z^{μ} and $\bar{z}_l$ a factor of $\bar{z}^{\nu}$, and for

$$a_{ij} = \begin{cases} c_{ij} + c_{ji} & \text{for } i < j, \\ 0 & \text{for } i > j, \end{cases} \qquad b_{ij} = \begin{cases} d_{ij} + d_{ji} & \text{for } i < j, \\ 0 & \text{for } i > j, \end{cases} \tag{2-21}$$

for $|z| \le 1$ we have, from (2-20),

$$|z_j \boldsymbol{Z}^{\boldsymbol{m}}| \le |z_j|\,|z_k \boldsymbol{Z}^{\boldsymbol{c}}|\,|z_l \boldsymbol{Z}^{\boldsymbol{d}}| = |z_j|\,|z_k \boldsymbol{Z}^{\boldsymbol{a}}|\,|z_l \boldsymbol{Z}^{\boldsymbol{b}}|.$$

Furthermore, (2-17) is satisfied.

Since our $(\boldsymbol{a}, \boldsymbol{b})$ satisfy $\boldsymbol{a} \in \mathcal{M}_k$ and $\boldsymbol{b} \in \mathcal{M}_l$, claim (3) is a consequence of claim (2). □

We end this section by exploiting the notation introduced in Lemma 2.3(5) to introduce two classes of functions. First of all, notice that the linear maps $\eta \mapsto \langle \eta, \phi_j \rangle$ extend to bounded linear maps $\Sigma_r \to \mathbb{R}$ for any $r \in \mathbb{R}$. We set

$$\Sigma_r^c := \{\eta \in \Sigma_r : \langle \eta, \phi_j \rangle = 0,\ j = 1, \dots, n\}. \tag{2-22}$$

The following two classes of functions will be used in the rest of the paper. Recall that in Definition 2.2 we introduced the space L with $\dim L = n(n-1)$. In Definitions 2.8–2.9, we denote by $\boldsymbol{Z}$ an auxiliary variable independent of z which takes values in L.

Definition 2.8. Let $\mathfrak{B}$ be an open subset of a Banach space. We will say that $F(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)$ in $C^M(I \times \mathfrak{B} \times \mathcal{A}, \mathbb{R})$, with I a neighborhood of 0 in $\mathbb{R}$ and $\mathcal{A}$ a neighborhood of 0 in $\mathbb{C}^n \times L \times \Sigma_{-K}^c$, is $F = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)$ if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of 0 such that

$$|F(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)| \le C(\|\eta\|_{\Sigma_{-K}} + |\boldsymbol{Z}|)^j (\|\eta\|_{\Sigma_{-K}} + |\boldsymbol{Z}| + |z|)^i \quad \text{in } I \times \mathfrak{B} \times \mathcal{A}'. \tag{2-23}$$

We will specify $F = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \boldsymbol{Z})$ if

$$|F(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)| \le C|\boldsymbol{Z}|^j |z|^i \tag{2-24}$$

and $F = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \eta)$ if

$$|F(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)| \le C\|\eta\|^j_{\Sigma_{-K}} (\|\eta\|_{\Sigma_{-K}} + |z|)^i. \tag{2-25}$$

We will omit t or $\mathfrak{b}$ if there is no dependence on such variables.

We write $F = \mathcal{R}^{i,j}_{K,\infty}$ if $F = \mathcal{R}^{i,j}_{K,m}$ for all $m \ge M$. We write $F = \mathcal{R}^{i,j}_{\infty,M}$ if, for all $k \ge K$, the above F is the restriction of an $F(t, \mathfrak{b}, z, \eta) \in C^M(I \times \mathfrak{B} \times \mathcal{A}_k, \mathbb{R})$ with $\mathcal{A}_k$ a neighborhood of 0 in $\mathbb{C}^n \times L \times \Sigma_{-k}^c$ and which is $F = \mathcal{R}^{i,j}_{k,M}$. Finally we write $F = \mathcal{R}^{i,j}_{\infty,\infty}$ if $F = \mathcal{R}^{i,j}_{k,\infty}$ for all k.

Definition 2.9. We will say that $T(t, \mathfrak{b}, z, \eta) \in C^M(I \times \mathfrak{B} \times \mathcal{A}, \Sigma_K(\mathbb{R}^3, \mathbb{C}))$, with the above notation, is $T = \boldsymbol{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)$ if there exists a $C > 0$ and a smaller neighborhood $\mathcal{A}'$ of 0 such that

$$\|T(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)\|_{\Sigma_K} \le C(\|\eta\|_{\Sigma_{-K}} + |\boldsymbol{Z}|)^j (\|\eta\|_{\Sigma_{-K}} + |\boldsymbol{Z}| + |z|)^i \quad \text{in } I \times \mathfrak{B} \times \mathcal{A}'. \tag{2-26}$$

We use notations $\boldsymbol{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \boldsymbol{Z})$, $\boldsymbol{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \eta)$, etc. as above.

Notice that we have the elementary formulas

$$\mathcal{R}^{a,b}_{K,M} \boldsymbol{S}^{i,j}_{K,M} = \boldsymbol{S}^{i+a,j+b}_{K,M} \quad \text{and} \quad \mathcal{R}^{a,b}_{K,M} \mathcal{R}^{i,j}_{K,M} = \mathcal{R}^{i+a,j+b}_{K,M}. \tag{2-27}$$

Remark 2.10. For functions $F(t, \mathfrak{b}, z, \eta)$ and $T(t, \mathfrak{b}, z, \eta)$, we write $F(t, \mathfrak{b}, z, \eta) = \mathcal{R}^{i,j}_{K,M}(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)$ and $T(t, \mathfrak{b}, z, \eta) = \boldsymbol{S}^{i,j}_{K,M}(t, \mathfrak{b}, z, \boldsymbol{Z}, \eta)$ when the equality holds restricting the variable $\boldsymbol{Z}$ to the $\boldsymbol{Z}$ of Definition 2.2 for symbols satisfying Definitions 2.8–2.9.

Furthermore, later, when we write $\mathcal{R}^{i,j}_{K,M}$ and $\boldsymbol{S}^{i,j}_{K,M}$, we will mean $\mathcal{R}^{i,j}_{K,M}(z, \boldsymbol{Z}, \eta)$ and $\boldsymbol{S}^{i,j}_{K,M}(z, \boldsymbol{Z}, \eta)$, respectively.

Notice that $F = \mathcal{R}^{i,j}_{K,M}(z, \boldsymbol{Z})$ or $T = \boldsymbol{S}^{i,j}_{K,M}(z, \boldsymbol{Z})$ do not mean independence from the variable η.

3. Invariants

Equation (1-1) admits energy and mass invariants, defined as follows:

$$E(u) := E_K(u) + E_P(u), \quad \text{where} \quad E_K(u) := \langle Hu, \bar{u}\rangle \text{ and } E_P(u) = \frac{1}{2}\int_{\mathbb{R}^3} |u(x)|^4\, dx, \qquad Q(u) := \langle u, \bar{u}\rangle. \tag{3-1}$$

We have $E \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}), \mathbb{R})$ and $Q \in C^\infty(L^2(\mathbb{R}^3, \mathbb{C}), \mathbb{R})$. We denote by dE the Fréchet derivative of E. We define $\nabla E \in C^\infty(H^1(\mathbb{R}^3, \mathbb{C}), H^{-1}(\mathbb{R}^3, \mathbb{C}))$ by $dEX = \mathrm{Re}\langle \nabla E, \overline{X}\rangle$ for any $X \in H^1$. We define also $\nabla_u E$ and $\nabla_{\bar{u}} E$ by

$$dEX = \langle \nabla_u E, X\rangle + \langle \nabla_{\bar{u}} E, \overline{X}\rangle, \quad \text{that is,} \quad \nabla_u E = 2^{-1}\overline{\nabla E} \text{ and } \nabla_{\bar{u}} E = 2^{-1}\nabla E.$$

Notice that $\nabla E = 2Hu + 2|u|^2 u$. Then (1-1) can be interpreted as

$$\mathrm{i}\dot{u} = \nabla_{\bar{u}} E(u). \tag{3-2}$$

Lemma 3.1. *Consider the coordinates $(z, \eta) \mapsto u$ in Lemma 2.4. Then there exists some functions as in Definitions 2.8–2.9 such that, for $(z, \eta) \in B_{\mathbb{C}^n}(0, d_0) \times (B_{H^1}(0, d_0) \cap \mathcal{H}_c[0])$, we have, for any preassigned $r_0 \in \mathbb{N}$, the expansion (where* c.c. *means complex conjugate)*

$$\begin{aligned} E(u) = {} & \sum_{j=1}^n E(Q_{jz_j}) + \langle H\eta, \overline{\eta}\rangle + \mathcal{R}^{1,2}_{r_0,\infty}(z, \eta) \\ & + \sum_{j\neq k}\big[E_{jz_j}(\mathrm{Re}\langle q_{jz_j}, \bar{z}_k\phi_k\rangle + \mathrm{Re}\langle q_{kz_k}, \bar{z}_j\phi_j\rangle) + \mathrm{Re}\langle |Q_{kz_k}|^2 Q_{kz_k}, \bar{z}_j\phi_j\rangle\big] \\ & + \mathcal{R}^{0,2N+5}_{r_0,\infty}(z, \boldsymbol{Z}) + \sum_{j=1}^n \sum_{l=1}^{2N+3} \sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} a_{j\boldsymbol{m}}(|z_j|^2) + \mathrm{Re}\langle \boldsymbol{S}^{0,2N+4}_{r_0,\infty}(z, \boldsymbol{Z}), \overline{\eta}\rangle \\ & + \sum_{j,k=1}^n \sum_{l=1}^{2N+3} \sum_{|\boldsymbol{m}|=l} (\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{jk\boldsymbol{m}}(|z_k|^2), \eta\rangle + \text{c.c.}) + \sum_{i+j=2} \sum_{|\boldsymbol{m}|\le 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{2\boldsymbol{m}ij}(z), \eta^i \overline{\eta}^j\rangle \\ & + \sum_{d+c=3} \sum_{i+j=d} \langle G_{dij}(z), \eta^i \overline{\eta}^j\rangle \mathcal{R}^{0,c}_{r_0,\infty}(z, \eta) + E_P(\eta), \end{aligned} \tag{3-3}$$

where:

- $(a_{j\boldsymbol{m}}, G_{jk\boldsymbol{m}}) \in C^\infty(B_{\mathbb{R}}(0, d_0), \mathbb{C} \times \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}))$;
- $(G_{2\boldsymbol{m}ij}, G_{dij}) \in C^\infty(B_{\mathbb{C}^n}(0, d_0), \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}) \times \Sigma_{r_0}(\mathbb{R}^3, \mathbb{C}))$;
- *for $|\boldsymbol{m}| = 0$, where, in particular, $G_{2\boldsymbol{0}ij}(0) = 0$, we have*

$$\sum_{i+j=2} \langle G_{2\boldsymbol{0}ij}(z), \eta^i \overline{\eta}^j\rangle = \sum_{j=1}^n \langle |Q_{jz_j}|^2 \eta, \overline{\eta}\rangle + 2\sum_{j=1}^n \mathrm{Re}\langle Q_{jz_j}\, \mathrm{Re}(Q_{jz_j}\overline{\eta}), \overline{\eta}\rangle; \tag{3-4}$$

- $\mathcal{R}^{1,2}_{r_0,\infty}(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) = \mathcal{R}^{1,2}_{r_0,\infty}(z, \eta)$ *for all $\vartheta \in \mathbb{R}$ for the third term in the right-hand side of* (3-3).

Remark 3.2. In (3-3) the terms of the second line could potentially derail our proof. They appear in (3-7)–(3-9). Similarly problematic is the first term in the right-hand side in (4-18) later. All these terms are tied up. Indeed, in Lemma 4.11 we will show that in a system of coordinates better suited to search for an effective Hamiltonian the problematic terms in the expansion of E cancel out.

In the proof of Lemma 3.1 we use the following lemma:

Lemma 3.3. *We have, for $j \neq k$ and $\delta E_{jz_j} := E_{jz_j} - e_j$,*

$$E_{jz_j}\langle q_{kz_k}, \phi_j\rangle + \langle |Q_{kz_k}|^2 Q_{kz_k}, \phi_j\rangle = E_{kz_k}\langle q_{kz_k}, \phi_j\rangle + \delta E_{jz_j}\langle q_{kz_k}, \phi_j\rangle. \tag{3-5}$$

Proof. We apply $\langle \cdot, \phi_j\rangle$ to

$$Hq_{kz_k} + |Q_{kz_k}|^2 Q_{kz_k} = z_k \delta E_{kz_k}\phi_k + E_{kz_k} q_{kz_k}$$

to get the following equality, which, from $e_j = E_{jz_j} - \delta E_{jz_j}$, yields (3-5):

$$e_j\langle q_{kz_k}, \phi_j\rangle + \langle |Q_{kz_k}|^2 Q_{kz_k}, \phi_j\rangle = E_{kz_k}\langle q_{kz_k}, \phi_j\rangle. \qquad \square$$

Proof of Lemma 3.1. First of all, we have the Taylor expansion

$$E(u) = E\Big(\sum_{j=1}^n Q_{jz_j}\Big) + \operatorname{Re}\Big\langle \nabla E\Big(\sum_{j=1}^n Q_{jz_j}\Big), \overline{R[z]\eta}\Big\rangle + 2^{-1}\operatorname{Re}\Big\langle \nabla^2 E\Big(\sum_{j=1}^n Q_{jz_j}\Big) R[z]\eta, \overline{R[z]\eta}\Big\rangle + E_3(\eta) \tag{3-6}$$

with

$$E_3(\eta) := \int_0^1 (1-t)\operatorname{Re}\Big\langle \Big[\nabla^2 E_P\Big(\sum_{j=1}^n Q_{jz_j} + tR[z]\eta\Big) - \nabla^2 E_P\Big(\sum_{j=1}^n Q_{jz_j}\Big)\Big] R[z]\eta, \overline{R[z]\eta}\Big\rangle\, dt.$$

Step 1. We consider the expansion of the first term in the right-hand side of (3-6). We have

$$\begin{aligned}\Big|\sum Q_{jz_j}\Big|^4 &= \sum |Q_{jz_j}|^4 + 4\sum_{j\neq k} |Q_{jz_j}|^2 \operatorname{Re}(Q_{jz_j}\bar{Q}_{kz_k}) \\ &\quad + 2\sum_{j<k} |Q_{jz_j}|^2 |Q_{kz_k}|^2 + \sum_{\substack{j\neq k\\ j'\neq k'}} \operatorname{Re}(Q_{jz_j}\bar{Q}_{kz_k})\operatorname{Re}(Q_{j'z_{j'}}\bar{Q}_{k'z_{k'}}) + 4\sum_{\substack{k<l\\ j\neq k,l}} |Q_{jz_j}|^2 \operatorname{Re}(Q_{kz_k}\bar{Q}_{lz_l}).\end{aligned}$$

All terms are invariant under the change of variable $z \rightsquigarrow e^{\mathrm{i}\vartheta}z$. The second line is $O(|\mathbf{Z}|^2)$. We conclude that

$$\begin{aligned}E\Big(\sum_{j=1,\dots,n} Q_{jz_j}\Big) &= \sum_{j,k}\langle HQ_{jz_j}, \bar{Q}_{kz_k}\rangle + \frac{1}{2}\int\Big|\sum_{j=1,\dots,n} Q_{jz_j}\Big|^4 \\ &= \sum_{j=1,\dots,n} E(Q_{jz_j}) + R_1 + \sum_{j\neq k}\big[\operatorname{Re}\langle HQ_{jz_j}, \bar{Q}_{kz_k}\rangle + 2\operatorname{Re}\langle |Q_{jz_j}|^2 Q_{jz_j}, \bar{Q}_{kz_k}\rangle\big], \end{aligned} \tag{3-7}$$

where

$$R_1 := \sum_{j<k} \int |Q_{jz_j}|^2 |Q_{kz_k}|^2 + \frac{1}{2} \sum_{\substack{j\neq k \\ j'\neq k'}} \int \operatorname{Re}(Q_{jz_j} \bar{Q}_{kz_k}) \operatorname{Re}(Q_{j'z_{j'}} \bar{Q}_{k'z_{k'}}) + 2 \sum_{\substack{k<l \\ j\neq k,l}} \int |Q_{jz_j}|^2 \operatorname{Re}(Q_{kz_k} \bar{Q}_{lz_l})$$
$$= O(|\mathbf{Z}|^2).$$

By Proposition 1.1 and by (3-5), the second summation in the last line of (3-7) equals

$$\sum_{j\neq k} [E_{jz_j} \operatorname{Re}\langle Q_{jz_j}, \bar{Q}_{kz_k}\rangle + \operatorname{Re}\langle |Q_{jz_j}|^2 Q_{jz_j}, \bar{Q}_{kz_k}\rangle]$$
$$= \sum_{j\neq k} \big[E_{jz_j} (\operatorname{Re}\langle q_{jz_j}, \bar{z}_k \phi_k\rangle + \operatorname{Re}\langle q_{kz_k}, \bar{z}_j \phi_j\rangle) + \operatorname{Re}\langle |Q_{kz_k}|^2 Q_{kz_k}, \bar{z}_j \phi_j\rangle \big] + R_2, \tag{3-8}$$

where

$$R_2 := \sum_{j\neq k} E_{jz_j} \operatorname{Re}\langle q_{jz_j}, \bar{q}_{kz_k}\rangle + \operatorname{Re}\langle |Q_{kz_k}|^2 Q_{kz_k}, \bar{q}_{jz_j}\rangle = O(|\mathbf{Z}|^2).$$

The summation in (3-8) is $O(|z|^2|\mathbf{Z}|)$ and not of the form $O(|\mathbf{Z}|^2)$. Indeed, in the particular case when $z_k = \rho_k$ and $z_j = \rho_j$ are real numbers, we have what follows, which is not $O(\rho_k^2 \rho_j^2)$:

$$E_{jz_j} \operatorname{Re}\langle q_{jz_j}, \bar{z}_k \phi_k\rangle + E_{kz_k} \operatorname{Re}\langle q_{kz_k}, \bar{z}_j \phi_j\rangle + \operatorname{Re}\langle |Q_{kz_k}|^2 Q_{kz_k}, \bar{z}_j \phi_j\rangle$$
$$= \rho_k \rho_j \big[E_{j\rho_j} \rho_j^2 \langle \tilde{q}_j(\rho_j^2), \phi_k\rangle + E_{k\rho_k} \rho_k^2 \langle \tilde{q}_k(\rho_k), \phi_j\rangle + \rho_k^2 \langle (\phi_k + \hat{q}_k(\rho_k^2))^3, \phi_j\rangle \big]. \tag{3-9}$$

Finally, we observe that $R_1 + R_2 = O(|\mathbf{Z}|^2)$ summed up together yield the first two terms on the third line of (3-3).

Indeed, since $R_1 + R_2$ is gauge invariant, by Lemma B.3 in Appendix B we have

$$R_1 + R_2 = \sum_{j=1}^{n} \sum_{l=1}^{2N+3} \sum_{|\boldsymbol{m}|=l+1} \mathbf{Z}^{\boldsymbol{m}} b_{j\boldsymbol{m}}(|z_j|^2) + O(|\mathbf{Z}|^{2N+5}) \tag{3-10}$$

with $O(|\mathbf{Z}|^{2N+5})$ smooth in z, independent of η and gauge invariant.

We have discussed the contribution to (3-3) of the first term in the expansion (3-6). Now we consider the other terms in (3-6).

Step 2. We consider the expansion of the second term in the right-hand side of (3-6).

By $\operatorname{Re}\langle \nabla E(Q_{jz_j}), \overline{R[z]\eta}\rangle = 2 \operatorname{Re} E_{jz_j} \langle Q_{jz_j}, \overline{R[z]\eta}\rangle = 0$, which follows from $R[z]\eta \in \mathcal{H}_c[z]$ and $\mathrm{i} Q_{jz_j} = -z_{jI} D_{jR} Q_{jz_j} + z_{jR} D_{jI} Q_{jz_j}$ — see (11) in [Gustafson et al. 2004] (and which is an immediate consequence of $Q_{jz_j} = e^{\mathrm{i}\theta} Q_{j|z_j|}$ for $z_j = e^{\mathrm{i}\theta}|z_j|$) — we have

$$\operatorname{Re}\Big\langle \nabla E\Big(\sum_{j=1}^{n} Q_{jz_j}\Big), \overline{R[z]\eta}\Big\rangle$$
$$= \overbrace{\operatorname{Re}\langle \nabla E(Q_{1z_1}), \overline{R[z]\eta}\rangle}^{0} + \int_0^1 \partial_t \operatorname{Re}\Big\langle \nabla E\Big(Q_{1z_1} + t \sum_{j>1} Q_{jz_j}\Big), \overline{R[z]\eta}\Big\rangle dt$$

$$= \mathrm{Re}\Big\langle \nabla E\Big(\sum_{j>1} Q_{jz_j}\Big), \overline{R[z]\eta}\Big\rangle + \int_{[0,1]^2} \partial_s\partial_t \,\mathrm{Re}\Big\langle \nabla E_P\Big(sQ_{1z_1} + t\sum_{l>1} Q_{lz_l}\Big), \overline{R[z]\eta}\Big\rangle\, dt\, ds$$

$$= \sum_{j=1}^{n-1} \int_{[0,1]^2} \partial_s\partial_t \,\mathrm{Re}\Big\langle \nabla E_P\Big(sQ_{jz_j} + t\sum_{l>j} Q_{lz_l}\Big), \overline{R[z]\eta}\Big\rangle\, dt\, ds, \tag{3-11}$$

where the last line is obtained by repeating the argument in the first three lines. For $\widehat{Q}_j = \sum_{l>j} Q_{lz_l}$, by $\nabla E_P(u) = 2|u|^2 u$, the last line of (3-11) is, in the notation of Lemma 2.3,

$$2\sum_{j=1}^{n-1} \mathrm{Re}\big\langle 2Q_{jz_j}|\widehat{Q}_j|^2 + 2|Q_{jz_j}|^2\widehat{Q}_j + Q_{jz_j}^2\overline{\widehat{Q}}_j + \overline{Q}_{jz_j}\widehat{Q}_j^2,\ \overline{\eta} + \phi_j\big(\langle \overline{\widehat{B}}_j(\widehat{\boldsymbol{Z}}), \overline{\eta}\rangle + \langle \bar{z}_i\bar{z}_\ell \overline{\widehat{C}}_{i\ell j}(\widehat{\boldsymbol{Z}}), \eta\rangle\big)\big\rangle.$$

Further expanding $\widehat{Q}_j = \sum_{l>j} Q_{lz_l}$ and using $Q_{lz_l} = z_l(\phi_l + \hat{q}_l(|z_l|^2))$, the above term is of the form

$$\sum_{j=1}^{n}\sum_{|\boldsymbol{m}|=1} \big(\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}}\langle G_{j\boldsymbol{m}}(\widehat{\boldsymbol{Z}}), \eta\rangle + \text{c.c.}\big).$$

As in Step 1, by Lemma B.4, this can be expanded into

$$\sum_{j=1}^{n}\sum_{1\le|\boldsymbol{m}|\le 2N+3} \big(\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}}\langle G_{jk\boldsymbol{m}}(|z_k|^2), \eta\rangle + \text{c.c.}\big) + \sum_{|\boldsymbol{m}|=2N+4}\big(\boldsymbol{Z}^{\boldsymbol{m}}\langle G_{\boldsymbol{m}}(z), \eta\rangle + \text{c.c.}\big). \tag{3-12}$$

Thus the last line in (3-11) can be absorbed in the third and fourth lines of (3-3).

Step 3. We consider the expansion of the third term in the right-hand side of (3-6). Using $\nabla^2 E_K(u) = 2H$ and proceeding as for (3-6), we obtain

$$\begin{aligned}
&2^{-1}\,\mathrm{Re}\Big\langle \nabla^2 E\Big(\sum_{j=1}^{n} Q_{jz_j}\Big) R[z]\eta, \overline{R[z]\eta}\Big\rangle \\
&\quad = 2^{-1}\,\mathrm{Re}\Big\langle \nabla^2 E_K\Big(\sum_{j=1}^{n} Q_{jz_j}\Big) R[z]\eta, \overline{R[z]\eta}\Big\rangle + 2^{-1}\sum_{j=1}^{n}\mathrm{Re}\langle \nabla^2 E_P(Q_{jz_j})R[z]\eta, \overline{R[z]\eta}\rangle \\
&\qquad + 2^{-1}\sum_{j=1}^{n-1}\int_{[0,1]^2} \partial_s\partial_t\,\mathrm{Re}\Big\langle \nabla^2 E_P\Big(sQ_{jz_j} + t\sum_{l=j+1}^{n} Q_{lz_l}\Big)R[z]\eta, \overline{R[z]\eta}\Big\rangle\, dt\, ds.
\end{aligned}$$

The third line is absorbed in the $\boldsymbol{Z}^{\boldsymbol{m}}\langle G_{2\boldsymbol{m}ij}(z), \eta^i\,\overline{\eta}^j\rangle + \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ with $|\boldsymbol{m}| = 1$ terms in (3-3). From the second line, using (2-3)–(2-4) and, in particular, $\alpha_j[z]\eta = \mathcal{R}^{1,1}_{r_0,\infty}(z,\eta)$ for the last equality, we have

$$\begin{aligned}
2^{-1}\,\mathrm{Re}\Big\langle \nabla^2 E_K\Big(\sum_{j=1}^{n} Q_{jz_j}\Big) R[z]\eta, \overline{R[z]\eta}\Big\rangle &= \langle HR[z]\eta, \overline{R[z]\eta}\rangle \\
&= \langle H\eta, \overline{\eta}\rangle + 2\sum_{j=1}^{n}\mathrm{Re}\big[(\alpha_j[z]\eta)\langle H\phi_j, \overline{\eta}\rangle\big] + \sum_{j,k=1}^{n} e_j|\alpha_j[z]\eta|^2 \\
&= \langle H\eta, \overline{\eta}\rangle + \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta),
\end{aligned}$$

which yield the second and third terms in the right-hand side of (3-3). For

$$2^{-1}\sum_{j=1}^{n}\nabla^2 E_P(Q_{jz_j})\eta=\sum_{j=1}^{n}|Q_{jz_j}|^2\eta+2\sum_{j=1}^{n}Q_{jz_j}\,\mathrm{Re}(Q_{jz_j}\overline{\eta}),$$

we have, for $G_{\mathbf{20}ij}(z)$ as in (3-4),

$$2^{-1}\sum_{j=1}^{n}\mathrm{Re}\langle\nabla^2 E_P(Q_{jz_j})R[z]\eta,\,\overline{R[z]\eta}\rangle=\mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)+\sum_{i+j=2}\langle G_{\mathbf{20}ij}(z),\eta^i\,\overline{\eta}^j\rangle. \tag{3-13}$$

This $\mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ defines the third term in the right-hand side of (3-3). Notice that $\mathcal{R}^{1,2}_{r_0,\infty}(e^{\mathrm{i}\vartheta}z,e^{\mathrm{i}\vartheta}\eta)=\mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ because this invariance is satisfied both by the left-hand side of (3-13) (by the invariance of E, (2-2) and Lemma 2.3) and by the last summation in the right-hand side of (3-13), by formula (3-4).

Step 4. We now turn to the $E_3(\eta)$ term in (3-6). By elementary computations,

$$\begin{aligned}E_3(\eta)&=\int_{[0,1]^2}t(1-t)\,d^3E_P\Big(\sum_{j\ge1}Q_{jz_j}+stR[z]\eta\Big)\cdot(R[z]\eta)^3\,dt\,ds\\&=E_P(R[z]\eta)\\&\qquad+\int_{[0,1]^3}t(1-t)\,d^4E_P\Big(\tau\sum_{j\ge1}Q_{jz_j}+stR[z]\eta\Big)\cdot(R[z]\eta)^3\sum_{j\ge1}Q_{jz_j}\,dt\,ds\,d\tau\end{aligned} \tag{3-14}$$

with $d^3E_P(u)\cdot v^3$ the trilinear differential form applied to (v,v,v) and $d^4E_P(u)\cdot v^3w$ the 4-linear differential form applied to (v,v,v,w).

In particular, we have used the fact that, since $d^jE_P(0)=0$ for $0\le j\le 2$, we have

$$E_P(R[z]\eta)=\int_{[0,1]^2}t(1-t)\,d^3E_P(stR[z]\eta)\cdot(R[z]\eta)^3\,dt\,ds. \tag{3-15}$$

For $\beta(u)=|u|^4$, and using the fact that $d^4\beta(u)\in B^4(\mathbb{C},R)$ is constant in u, the last line of (3-14) is

$$\frac{1}{12}\int_{\mathbb{R}^3}d^4\beta\cdot((R[z]\eta)(x))^3\sum_{j\ge1}Q_{jz_j}(x)\,dx,$$

and can be absorbed in the $\langle G_{dij}(z),\eta^i\,\overline{\eta}^j\rangle\mathcal{R}^{0,c}_{r_0,\infty}(z,\eta)$ terms in (3-3). We expand $E_P(R[z]\eta)$ as a sum of similar terms and of $E_P(\eta)$. □

In order to extract from the functional in (3-3) an effective Hamiltonian well suited for the FGR and dispersive estimates, we need to implement a Birkhoff normal form argument; see Section 5. This requires an intermediate change of coordinates, which will partially normalize the symplectic form Ω defined in (4-1) below, and diagonalize the homological equations. Notice that, as a bonus, this change of coordinates erases the bad terms in the expansion of E in (3-3) discussed in Remark 3.2.

4. Darboux theorem

System (3-2) is Hamiltonian with respect to the symplectic form in $H^1(\mathbb{R}^3, \mathbb{C})$,

$$\Omega(X, Y) := \mathrm{i}\langle X, \overline{Y}\rangle - \mathrm{i}\langle \overline{X}, Y\rangle = 2\,\mathrm{Im}\langle \overline{X}, Y\rangle. \tag{4-1}$$

In terms of the spectral decomposition of H (recall $\overline{\phi}_j = \phi_j$),

$$X = \sum_{j=1}^{n} \langle X, \phi_j\rangle \phi_j + P_c X, \tag{4-2}$$

$$\Omega(X, Y) = \mathrm{i}\sum_{j=1}^{n} (\langle X, \phi_j\rangle\langle \overline{Y}, \phi_j\rangle - \langle \overline{X}, \phi_j\rangle\langle Y, \phi_j\rangle) + \mathrm{i}\langle P_c X, P_c\overline{Y}\rangle - \mathrm{i}\langle P_c\overline{X}, P_c Y\rangle. \tag{4-3}$$

However, in terms of the coordinates in Lemma 2.4, Ω admits a quite more complicated representation, as we shall see. This will require us to adjust these coordinates.

Our first observation is that, for the coordinates in Lemma 2.4, we have the following facts:

Lemma 4.1. *The Fréchet derivatives of $\eta(u)$ and z_j are given by the formulas*

$$d\eta(u) = -\sum_{j=1,\dots,n}\ \sum_{A=I,R} P_c D_{jA} q_{jz_j}\, dz_{jA} + P_c, \tag{4-4}$$

$$dz_j = \langle\,\cdot\,, \phi_j\rangle - \sum_{k:k\neq j}\ \sum_{A=I,R} \langle D_{kA} q_{kz_k}, \phi_j\rangle\, dz_{kA} - \sum_{k=1}^{n}\ \sum_{A=I,R} D_{kA}\alpha_j[z]\eta\, dz_{kA} - \alpha_j[z]\circ d\eta. \tag{4-5}$$

Analogous formulas for dz_{jR} and dz_{jI} are obtained by applying Re *and* Im *to* (4-5).

Proof. We start with (4-4). By the independence of z and η, we have

$$d\eta\,\frac{\partial}{\partial z_{jR}} = d\eta\,\frac{\partial}{\partial z_{jI}} = 0, \tag{4-6}$$

where

$$\frac{\partial}{\partial z_{jA}} = D_{jA} Q_{jz_j} + \sum_{k=1}^{n} D_{jA}(\alpha_k[z]\eta)\phi_k. \tag{4-7}$$

Next, for $\xi \in \mathcal{H}_c[0]$ we have what follows, which implies $d\eta\, R[z]P_c = 1|_{\mathcal{H}_c[0]}$:

$$d\eta\, R[z]P_c\xi = \frac{d}{dt}\eta(Q_{jz_j} + R[z](\eta + t\xi))\Big|_{t=0} = \xi.$$

So $d\eta = \sum(a_j\, dz_{jR} + b_j\, dz_{jI}) + P_c$, where we used $P_c R[z] = 1$. Then a_j and b_j can be computed applying $\sum(a_j\, dz_{jR} + b_j\, dz_{jI}) + P_c$ to the vectors (4-7) and using (4-6). Finally (4-5) follows by

$$z_j(u) = \Big\langle u - \sum_{k=1}^{n} q_{kz_k} - R[z]\eta, \phi_j\Big\rangle = \Big\langle u - \sum_{k:k\neq j} q_{kz_k}, \phi_j\Big\rangle - \alpha_j[z]\eta. \qquad \square$$

We consider the function $\overline{\eta}(u)$. Notice that $d\overline{\eta}(u)X = d\overline{\eta}(u+tX)/dt\big|_{t=0} = \overline{d\eta(u)X}$. Now we introduce a new symplectic form. Notice that our final choice of symplectic form is not the Ω_0' defined here in (4-8), but rather the Ω_0 defined in (4-13).

Lemma 4.2. *Set*

$$\begin{aligned} \Omega_0' &:= 2\sum_{j=1}^{n} dz_{jR}\wedge dz_{jI} + \mathrm{i}\langle d\eta, d\overline{\eta}\rangle - \mathrm{i}\langle d\overline{\eta}, d\eta\rangle \\ \textit{and}\quad B_0' &:= \sum_{j=1}^{n}(z_{jR}\,dz_{jI} - z_{jI}\,dz_{jR}) - \frac{\mathrm{i}}{2}(\langle\overline{\eta}, d\eta\rangle - \langle\eta, d\overline{\eta}\rangle). \end{aligned} \tag{4-8}$$

Then $dB_0' = \Omega_0'$ and $\Omega = \Omega_0'$ at $u=0$ for the Ω of (4-1). *Furthermore,*

$$\Phi^* B_0' = B_0' \quad \textit{for}\ \ \Phi(u) = e^{\mathrm{i}\vartheta}u\ \textit{for any fixed}\ \vartheta\in\mathbb{R}. \tag{4-9}$$

Proof. The equality $dB_0' = \Omega_0'$ is elementary. Indeed, $d(z_{jR}\,dz_{jI} - z_{jI}\,dz_{jR}) = 2\,dz_{jR}\wedge dz_{jI}$ and, for a pair of constant vector fields X and Y, since $d^2\eta(X,Y) = d^2\eta(Y,X)$ we have

$$d\langle\overline{\eta}, d\eta\rangle(X,Y) = X\langle\overline{\eta}, d\eta\,Y\rangle - Y\langle\overline{\eta}, d\eta\,X\rangle = \langle d\overline{\eta}\,X, d\eta\,Y\rangle - \langle d\overline{\eta}\,Y, d\eta\,X\rangle.$$

This yields $d\langle\overline{\eta}, d\eta\rangle = \langle d\overline{\eta}, d\eta\rangle - \langle d\eta, d\overline{\eta}\rangle$ and also $d\langle\eta, d\overline{\eta}\rangle = -d\langle\overline{\eta}, d\eta\rangle = \langle d\eta, d\overline{\eta}\rangle - \langle d\overline{\eta}, d\eta\rangle$.

To compute Ω_0' at $u=0$, we observe that, by Lemma 4.1, we have $d\eta = P_c$ at $u=0$, so that

$$\mathrm{i}\langle d\eta\,X, d\overline{\eta}\,Y\rangle - \mathrm{i}\langle d\overline{\eta}\,X, d\eta\,Y\rangle = \mathrm{i}\langle P_cX, P_c\overline{Y}\rangle - \mathrm{i}\langle P_c\overline{X}, P_cY\rangle \quad \text{at } u=0. \tag{4-10}$$

By Lemma 4.1 and Proposition 1.1, at $u=0$ we have $dz_{jR} = \mathrm{Re}\langle\,\cdot\,,\phi_j\rangle$ and $dz_{jI} = \mathrm{Im}\langle\,\cdot\,,\phi_j\rangle$. Summing on repeated indexes, we have

$$\begin{aligned} \mathrm{i}(\langle X,\phi_j\rangle\langle\overline{Y},\phi_j\rangle - \langle\overline{X},\phi_j\rangle\langle Y,\phi_j\rangle) &= -2\,\mathrm{Im}(\langle X,\phi_j\rangle\langle\overline{Y},\phi_j\rangle) \\ &= 2(\mathrm{Re}\langle X,\phi_j\rangle\,\mathrm{Im}\langle Y,\phi_j\rangle - \mathrm{Re}\langle Y,\phi_j\rangle\,\mathrm{Im}\langle X,\phi_j\rangle) \\ &= 2\,\mathrm{Re}\langle\,\cdot\,,\phi_j\rangle\wedge\mathrm{Im}\langle\,\cdot\,,\phi_j\rangle(X,Y) \\ &= 2\,dz_{jR}\wedge dz_{jI}|_{u=0}(X,Y). \end{aligned} \tag{4-11}$$

By (4-10)–(4-11), we get $\Omega = \Omega_0'$ at $u=0$. Finally, (4-9) follows immediately by

$$B_0' := \sum_{j=1}^{n}\mathrm{Im}(\bar{z}_j dz_j) + \mathrm{Im}\langle\overline{\eta}, d\eta\rangle. \tag{4-12}$$

This conclude the proof. □

Summing on repeated indexes and using the notation in Proposition 1.1, we introduce the differential forms

$$\begin{aligned} &\Omega_0 := \Omega_0' + \mathrm{i}\gamma_j(|z_j|^2)\,dz_j\wedge d\bar{z}_j, \\ \text{where}\quad &\gamma_j(|z_j|^2) := \langle\hat{q}_j(|z_j|^2), \hat{q}_j(|z_j|^2)\rangle + 2|z_j|^2\langle\hat{q}_j(|z_j|^2), \hat{q}_j'(|z_j|^2)\rangle, \\ \text{and}\quad &B_0 := B_0' - \mathrm{Im}\langle D_{jA}\bar{q}_{jz_j}, q_{jz_j}\rangle\,dz_{jA} \end{aligned} \tag{4-13}$$

with $\hat{q}_j'(t) = d\hat{q}_j/dt$. We have the following lemma:

Lemma 4.3. *We have $\gamma_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$. We have $dB_0 = \Omega_0$ and*

$$\Phi^* B_0 = B_0 \quad \textit{for } \Phi(u) = e^{\mathrm{i}\vartheta} u \textit{ for any fixed } \vartheta \in \mathbb{R}. \tag{4-14}$$

Proof. The identity $\gamma_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$ is elementary from Proposition 1.1 and Definition 2.8. Next, $dB_0 = \Omega_0$ follows by $dB_0' = \Omega_0'$ and

$$\begin{aligned} -d\,\mathrm{Im}\langle D_{jA}\bar{q}_{jz_j}, q_{jz_j}\rangle\, dz_{jA} &= \mathrm{Im}\langle D_{jA}\bar{q}_{jz_j}, D_{jB}q_{jz_j}\rangle\, dz_{jA} \wedge dz_{jB} \\ &= 2\,\mathrm{Im}\langle D_{jR}\bar{q}_{jz_j}, D_{jI}q_{jz_j}\rangle\, dz_{jR} \wedge dz_{jI} \\ &= 2\gamma(|z_j|^2)\, dz_{jR} \wedge dz_{jI} \\ &= \mathrm{i}\gamma_j(|z_j|^2)\, dz_j \wedge d\bar{z}_j, \end{aligned}$$

where $q_{jz_j} = z_j \hat{q}_j(|z_j|^2)$.

Turning to the proof of (4-14), we have

$$\Phi^*(\mathrm{i}\gamma_j(|z_j|^2)\, dz_j \wedge d\bar{z}_j) = \mathrm{i}\gamma_j(|z_j|^2)\, d(\Phi^* z_j) \wedge d(\Phi^* \bar{z}_j) = \mathrm{i}\gamma_j(|z_j|^2)\, dz_j \wedge d\bar{z}_j. \qquad \square$$

Lemma 4.4. *We have $dB = \Omega$ with B the differential form in the manifold H^1 defined by*

$$B(u)X := \mathrm{Im}\langle \bar{u}, X\rangle. \tag{4-15}$$

Consider, for $u \in B_{H^1}(0, d_0)$ with the d_0 of Lemma 2.3, the function $\psi \in C^\infty(B_{H^1}(0, d_0), \mathbb{R})$ and the differential form $\Gamma(u)$ defined by

$$\psi(u) := \sum_{j=1}^{n} \mathrm{Im}\langle \bar{q}_{jz_j}, u\rangle + \sum_{j=1}^{n} \mathrm{Im}(\alpha_j[z]\eta\bar{z}_j), \tag{4-16}$$

$$\Gamma(u) := B(u) - B_0(u) + d\psi(u). \tag{4-17}$$

Then the map $(z, \eta) \mapsto \Gamma(u(z, \eta))$, where $u(z, \eta)$ is the right-hand side of (2-10), *which is initially defined in $B_{\mathbb{C}^n}(0, d_0) \times (H^1 \cap \mathcal{H}_c[0])$, extends to $B_{\mathbb{C}^n}(0, d_0) \times \Sigma^c_{-r}$ for any $r \in \mathbb{N}$. In particular, we have $\Gamma = \Gamma_{jA} dz_{jA} + \langle \Gamma_\eta, d\eta\rangle + \langle \Gamma_{\bar{\eta}}, d\bar{\eta}\rangle$ with, in the sense of Remark 2.10,*

$$\Gamma_{jA} = \mathcal{R}^{1,1}_{\infty,\infty}(z, \boldsymbol{Z}, \eta) \qquad \textit{and} \qquad \Gamma_\xi = \boldsymbol{S}^{1,1}_{\infty,\infty}(z, \boldsymbol{Z}, \eta) \quad \textit{for } \xi = \eta, \bar{\eta}. \tag{4-18}$$

Furthermore, Γ satisfies an invariance property in $B_{H^1}(0, d_0)$:

$$\Phi^*\Gamma = \Gamma \quad \textit{for } \Phi(u) = e^{\mathrm{i}\vartheta} u \textit{ for any fixed } \vartheta \in \mathbb{R}. \tag{4-19}$$

Proof. By the definition of the exterior differential, and focusing on constant vector fields X and Y,

$$dB(X, Y) = XB(u)Y - YB(u)X = \mathrm{Im}\langle \overline{X}, Y\rangle - \mathrm{Im}\langle \overline{Y}, X\rangle = \Omega(X, Y).$$

This is enough to prove $dB = \Omega$. Next, using $R[z]\eta = \eta + \sum_j \alpha_j[z]\eta\phi_j$, we expand

$$B(u) = \sum_j \operatorname{Im}\langle \overline{Q}_{jz_j}, \cdot\rangle + \operatorname{Im}\langle \overline{R[z]\eta}, \cdot\rangle$$
$$= \sum_j \operatorname{Im}\langle \bar{z}_j\phi_j, \cdot\rangle + \operatorname{Im}\langle \bar{\eta}, \cdot\rangle + \sum_j \operatorname{Im}\langle \bar{q}_{jz_j}, \cdot\rangle + \sum_j \operatorname{Im}(\overline{\alpha_j[z]\eta}\langle\phi_j, \cdot\rangle). \tag{4-20}$$

By the definition of B_0 in (4-13), we have

$$B - B_0 = I_1 + I_2 + I_3 + \sum_{j,A} \operatorname{Im}\langle D_{jA}\bar{q}_{jz_j}, q_{jz_j}\rangle dz_{jA} + \sum_j \operatorname{Im}\langle \bar{q}_{jz_j}, \cdot\rangle, \tag{4-21}$$

where

$$I_1 := \sum_j \operatorname{Im}[\bar{z}_j(\langle\phi_j, \cdot\rangle - dz_j)], \quad I_2 := -\operatorname{Im}\langle\bar{\eta}, d\eta - P_c\rangle, \quad I_3 := \sum_j \operatorname{Im}[\overline{\alpha_j[z]\eta}\langle\phi_j, \cdot\rangle].$$

We replace $d\eta$ using (4-4) and $\langle\phi_j, \cdot\rangle$ using (4-5). For $\alpha_j[z] \circ d\eta$, the linear operator defined by $\alpha_j[z] \circ d\eta(X) := \alpha_j[z]\, d\eta(X)$, we then get

$$I_1 = \operatorname{Im}\langle D_{jA}q_{jz_j}, \bar{z}_k\phi_k\rangle\, dz_{jA} + \operatorname{Im}(\bar{z}_j D_{kA}\alpha_j[z]\eta)\, dz_{kA} + \operatorname{Im}(\bar{z}_j\alpha_j[z] \circ d\eta)$$
$$= \sum_{jA} \mathcal{R}^{1,1}_{\infty,\infty}\, dz_{jA} + \operatorname{Im}(\bar{z}_j\alpha_j[z] \circ d\eta), \tag{4-22}$$

where, as anticipated in Remark 2.10, here we set $\mathcal{R}^{i,j}_{K,M} = \mathcal{R}^{i,j}_{K,M}(z, \boldsymbol{Z}, \eta)$ and $\boldsymbol{S}^{i,j}_{K,M} = \boldsymbol{S}^{i,j}_{K,M}(z, \boldsymbol{Z}, \eta)$, where $\boldsymbol{Z}$ is as defined in Definition 2.2.

The second term in the last line of (4-22) is incorporated into the first sum in (4-25). We have

$$I_2 = \operatorname{Im}\langle\bar{\eta}, D_{jA}q_{jz_j}\rangle dz_{jA} = \sum_{jA} \mathcal{R}^{2,1}_{\infty,\infty} dz_{jA}. \tag{4-23}$$

Substituting with (4-5), we have

$$I_3 = \sum_{jA} \mathcal{R}^{2,1}_{\infty,\infty} dz_{jA} + \langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\eta\rangle + \langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\bar{\eta}\rangle. \tag{4-24}$$

Hence, we get

$$B - B_0 = \sum_j \operatorname{Im}(\bar{z}_j\alpha_j[z] \circ d\eta) + \sum_{jA} \mathcal{R}^{1,1}_{\infty,\infty}\, dz_{jA} + \langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\eta\rangle + \langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\bar{\eta}\rangle$$
$$+ \sum_{jA} \operatorname{Im}\langle D_{jA}\bar{q}_{jz_j}, q_{jz_j}\rangle\, dz_{jA} + \sum_j \operatorname{Im}\langle \bar{q}_{jz_j}, \cdot\rangle. \tag{4-25}$$

Set now $\widetilde{\psi}(u) := -\sum_{j=1}^n \operatorname{Im}\langle \bar{q}_{jz_j}, u\rangle$. Then it is elementary that we have

$$d\widetilde{\psi} = -\sum_{j=1}^n \operatorname{Im}\langle \bar{q}_{jz_j}, \cdot\rangle - \sum_{j,A} \operatorname{Im}\langle D_{jA}\bar{q}_{jz_j}, q_{jz_j}\rangle\, dz_{jA} + \sum_{j,A} \mathcal{R}^{1,1}_{\infty,\infty}\, dz_{jA}. \tag{4-26}$$

By the Leibniz rule we have

$$\operatorname{Im}(\bar{z}_j\alpha_j[z] \circ d\eta) = d\operatorname{Im}(\bar{z}_j\alpha_j[z]\ \eta) - \operatorname{Im}(d(\bar{z}_j\alpha_j[z])\ \eta). \tag{4-27}$$

The contribution to $\sum_j \operatorname{Im}(\bar{z}_j \alpha_j[z] \circ d\eta)$ in (4-25) of the last term in the right-hand side of (4-27) can be absorbed into the term $\sum_{jA} \mathcal{R}^{1,1}_{\infty,\infty}\, dz_{jA}$. Then

$$B - B_0 + d\psi = \sum_{jA} \mathcal{R}^{1,1}_{\infty,\infty}\, dz_{jA} + \langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\eta\rangle + \langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\bar{\eta}\rangle.$$

Here we have used that the first two terms in the right-hand side of (4-26) cancel with the last two sums in (4-25) and that there is a cancellation between the contribution to $\sum_j \operatorname{Im}(\bar{z}_j \alpha_j[z] \circ d\eta)$ of the $d\operatorname{Im}(\bar{z}_j \alpha_j[z]\ \eta)$ in (4-27) and the differential of the last term in (4-16). This yields (4-18).

Lastly we consider (4-19). We have $\Phi^* B_0 = B_0$ by (4-14), while $\Phi^* B = B$ follows immediately from the definition of B in (4-15). Finally, $\Phi^*\psi = \psi$ follows immediately from $\Phi^*\langle \bar{q}_{jz_j}, u\rangle = \langle \bar{q}_{jz_j}, u\rangle$, which follows from $q_{jz_j}(e^{\mathrm{i}\vartheta} z) = e^{\mathrm{i}\vartheta} q_{jz_j}(z)$, and from (2-9) and (2-12), which imply

$$\Phi^*(\bar{z}_j \alpha_j[z]\eta) = e^{-\mathrm{i}\vartheta} \bar{z}_j \alpha_j[e^{\mathrm{i}\vartheta} z] e^{\mathrm{i}\vartheta}\eta = \bar{z}_j \alpha_j[z]\eta. \qquad \square$$

Lemma 4.5. *Consider the differential form $\Omega - \Omega_0$, which is defined in $B_{H^1}(0, d_0)$ for the d_0 of Lemma 2.3. Then, summing on repeated indexes, we have*

$$\Omega - \Omega_0 = \widetilde{\Omega}_{ijAB}\, dz_{iA} \wedge dz_{jB} + \sum_{\xi=\eta,\bar{\eta}} dz_{iA} \wedge \langle \widetilde{\Omega}_{iA\xi}, d\xi\rangle + \sum_{\xi,\xi'=\eta,\bar{\eta}} \langle \widetilde{\Omega}_{\xi'\xi}\, d\xi, d\xi'\rangle, \tag{4-28}$$

where, expressed as functions of (z, η), the coefficients extend into functions defined in $B_{\mathbb{C}^n}(0, d_0) \times \Sigma^c_{-r}$ for any $r \in \mathbb{N}$ and, in particular, we have $\widehat{\Omega}_{iA\xi} = \boldsymbol{S}^{1,0}_{\infty,\infty}(z, \boldsymbol{Z}, \eta)$, $\widehat{\Omega}_{ijAB} = \mathcal{R}^{1,0}_{\infty,\infty}(z, \boldsymbol{Z}, \eta)$ in the sense of Remark 2.10 and $\widetilde{\Omega}_{\xi'\xi} = \partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}(z, \boldsymbol{Z}, \eta) - (\partial_{\xi'} \boldsymbol{S}^{1,1}_{\infty,\infty}(z, \boldsymbol{Z}, \eta))^*$ (with the two instances of $\boldsymbol{S}$ distinct). We furthermore have

$$\Phi^*(\Omega - \Omega_0) = \Omega - \Omega_0 \quad \textit{for } \Phi(z, \eta) = (e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) \textit{ for any fixed } \vartheta \in \mathbb{R}. \tag{4-29}$$

Proof. We have

$$\Omega - \Omega_0 = d\Gamma = d\sum_{j,A} \mathcal{R}^{1,1}_{\infty,\infty} dz_{jA} + d\sum_{\xi} \langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\xi\rangle.$$

Summing over k, B and ξ, we have

$$d(\mathcal{R}^{1,1}_{\infty,\infty}\, dz_{jA}) = \partial_{z_{kB}} \mathcal{R}^{1,1}_{\infty,\infty}\, dz_{kB} \wedge dz_{jA} + \langle \partial_\xi \mathcal{R}^{1,1}_{\infty,\infty}, d\xi\rangle \wedge dz_{jA}$$

with the $\partial_\xi \mathcal{R}^{1,1}_{\infty,\infty} \in \mathcal{H}_c[0]$ defined, summing on repeated indexes and for F with values in $\mathbb{R}$, by

$$dF X = \partial_{z_{kB}} F\, dz_{kB}\, X + \langle \partial_\xi F, d\xi\ X\rangle \quad \text{for any } X \in L^2(\mathbb{R}^3, \mathbb{C}).$$

It is easy to see that $\partial_\xi \mathcal{R}^{1,1}_{\infty,\infty} = \boldsymbol{S}^{1,0}_{\infty,\infty}$ and $\partial_{z_{kB}} \mathcal{R}^{1,1}_{\infty,\infty} = \mathcal{R}^{1,0}_{\infty,\infty}$.

Furthermore, summing on repeated indexes we have

$$\begin{aligned} d\langle \boldsymbol{S}^{1,1}_{\infty,\infty}, d\xi\rangle &= dz_{kB} \wedge \langle \partial_{z_{kB}} \boldsymbol{S}^{1,1}_{\infty,\infty}, d\xi\rangle + \langle \partial_{\xi'} \boldsymbol{S}^{1,1}_{\infty,\infty}\, d\xi', d\xi\rangle - \langle d\xi, \partial_{\xi'} \boldsymbol{S}^{1,1}_{\infty,\infty}\, d\xi'\rangle \\ &= dz_{kB} \wedge \langle \partial_{z_{kB}} \boldsymbol{S}^{1,1}_{\infty,\infty}, d\xi\rangle + \langle \partial_{\xi'} \boldsymbol{S}^{1,1}_{\infty,\infty}\, d\xi', d\xi\rangle - \langle (\partial_{\xi'} \boldsymbol{S}^{1,1}_{\infty,\infty})^*\, d\xi, d\xi'\rangle, \end{aligned} \tag{4-30}$$

where, for $T \in C^1(U_{L^2}, L^2)$ with U_{L^2} an open subset in L^2, $\partial_\xi T \in B(\mathcal{H}_c[0], L^2)$ is defined by

$$dTX = \partial_{z_{kB}} T\, dz_{kB}\, X + \partial_\xi T\, d\xi\, X \quad \text{for any } X \in L^2(\mathbb{R}^3, \mathbb{C}).$$

Summing on ξ in (4-30) we get terms which are absorbed into the last two terms of (4-28).

Formula (4-29) follows from (4-19), $\Omega_0 = dB_0$ and $\Omega = dB$. □

Lemma 4.6. *Consider the form $\Omega_t := \Omega_0 + t(\Omega - \Omega_0)$ and set $i_X \Omega_t(Y) := \Omega_t(X, Y)$. For any preassigned $r \in \mathbb{N}$ recall by, (4-8), (4-13) and Lemmas 4.4 and 4.5, that $\Omega - \Omega_0$ and Γ extend to forms defined in $B_{\mathbb{C}^n}(0, d_0) \times \Sigma^c_{-r}$. Then there is $\delta_0 \in (0, d_0)$ such that, for any $(t, z, \eta) \in (-4, 4) \times B_{\mathbb{C}^n}(0, \delta_0) \times B_{\Sigma^c_{-r}}(0, \delta_0)$, there exists exactly one solution $\mathcal{X}^t(z, \eta) \in L^2$ of the equation $i_{\mathcal{X}^t} \Omega_t = -\Gamma$. Furthermore, we have the following facts*:

(1) *$\mathcal{X}^t(z, \eta) \in \Sigma_r$ and, if we set $\mathcal{X}^t_{jA}(z, \eta) = dz_{jA}\, \mathcal{X}^t(z, \eta)$ and $\mathcal{X}^t_\eta(z, \eta) = d\eta\, \mathcal{X}^t(z, \eta)$, we have $\mathcal{X}^t_{jA}(z, \eta) = \mathcal{R}^{1,1}_{r,\infty}(t, z, \mathbf{Z}, \eta)$ and $\mathcal{X}^t_\eta(z, \eta) = \boldsymbol{S}^{1,1}_{r,\infty}(t, z, \mathbf{Z}, \eta)$ in the sense of Remark 2.10.*

(2) *For $\mathcal{X}^t_j := dz_j\, \mathcal{X}^t$ and $\mathcal{X}^t_\eta := d\eta\, \mathcal{X}^t$, we have $\mathcal{X}^t_j(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta} \eta) = e^{\mathrm{i}\vartheta} \mathcal{X}^t_j(z, \eta)$ and $\mathcal{X}^t_\eta(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta} \eta) = e^{\mathrm{i}\vartheta} \mathcal{X}^t_\eta(z, \eta)$.*

Proof. We define Y such that $i_Y \Omega'_0 = -\Gamma$, which yields $Y_{jR} = -\frac{1}{2}\Gamma_{jI}$ and $Y_{jI} = \frac{1}{2}\Gamma_{jR}$ (both $\mathcal{R}^{1,1}_{\infty,\infty}$), $Y_\eta = -\mathrm{i}\Gamma_{\bar\eta}$ and $Y_{\bar\eta} = \mathrm{i}\Gamma_\eta$ (both $\boldsymbol{S}^{1,1}_{\infty,\infty}$). We use $i_{K_t X} \Omega'_0 = i_X(\Omega_0 - \Omega'_0 + t\widehat{\Omega})$, where $\widehat{\Omega} := \Omega - \Omega_0$, to define in L^2 the operator K_t. We claim the following lemma:

Lemma 4.7. *For appropriate symbols $\mathcal{R}^{1,0}_{\infty,\infty}(t, z, \mathbf{Z}, \eta)$ and $\boldsymbol{S}^{1,0}_{\infty,\infty}(t, z, \mathbf{Z}, \eta)$, which differ from one term to the other, and for $\mathbf{Z}$ as in Definition 2.2, we have*

$$\begin{aligned} (K_t X)_{jA} &= \sum_{lB} \mathcal{R}^{1,0}_{\infty,\infty} X_{lB} + \sum_{\xi=\eta,\bar\eta} \langle \boldsymbol{S}^{1,0}_{\infty,\infty}, X_\xi \rangle, \\ (K_t X)_\xi &= \sum_{lB} \boldsymbol{S}^{1,0}_{\infty,\infty} X_{lB} + \sum_{\xi'=\eta,\bar\eta} \big(\partial_{\xi'} \boldsymbol{S}^{1,1}_{\infty,\infty}(t, z, \mathbf{Z}, \eta) - (\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}(t, z, \mathbf{Z}, \eta))^* \big) X_{\xi'}. \end{aligned} \tag{4-31}$$

We assume for a moment Lemma 4.7 and complete the proof of Lemma 4.6. The equation $i_{\mathcal{X}^t} \Omega_t = -\Gamma$ becomes $\mathcal{X}^t + K_t \mathcal{X}^t = Y$. Indeed, suppose $\mathcal{X}^t + K_t \mathcal{X}^t = Y$ holds. Then, by definition of K_t, we have

$$i_{\mathcal{X}_t}(\Omega_t - \Omega') = i_{K_t \mathcal{X}_t} \Omega'_0 \quad \text{and so} \quad i_{\mathcal{X}_t} \Omega_t = i_{\mathcal{X}_t} \Omega'_0 + i_{K_t \mathcal{X}_t} \Omega'_0 = -\Gamma.$$

By Lemma 4.7, in coordinates and for $\xi = \eta, \bar\eta$, the last equation is schematically of the form

$$\begin{aligned} \mathcal{X}^t_{jA} + \sum_{\ell, B} \mathcal{R}^{1,0}_{r,\infty} \mathcal{X}^t_{\ell B} + \sum_{\xi=\eta,\bar\eta} \langle \boldsymbol{S}^{1,1}_{r,\infty}, \mathcal{X}^t_\xi \rangle &= \mathcal{R}^{1,1}_{r,\infty}, \\ \mathcal{X}^t_\xi + \sum_{\ell B} \boldsymbol{S}^{1,0}_{r,\infty} \mathcal{X}^t_{\ell B} + \sum_{\xi'=\eta,\bar\eta} \big(\partial_{\xi'} \boldsymbol{S}^{1,1}_{\infty,\infty}(t, z, \mathbf{Z}, \eta) - (\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}(t, z, \mathbf{Z}, \eta))^* \big) \mathcal{X}^t_{\xi'} &= \boldsymbol{S}^{1,1}_{r,\infty}. \end{aligned} \tag{4-32}$$

Notice that $(\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}) \boldsymbol{S}^{1,1}_{r,\infty}$ is C^∞ in $(t, z, \mathbf{Z}, \eta)$ with values in Σ_r. We have

$$\| (\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}) \boldsymbol{S}^{1,1}_{r,\infty} \|_{\Sigma_r} \le \| \partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty} \|_{B(\Sigma_{-r}, \Sigma_r)} \| \boldsymbol{S}^{1,1}_{r,\infty} \|_{\Sigma_r}.$$

By (2-26), we have $\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}(t,0,0,0)$. This implies

$$\|\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}\|_{B(\Sigma_{-r},\Sigma_r)} \le C\|\eta\|_{\Sigma_{-K}} + |\boldsymbol{Z}| + |z| \tag{4-33}$$

and so

$$\|(\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty})\boldsymbol{S}^{1,1}_{r,\infty}\|_{\Sigma_r} \le C(\|\eta\|_{\Sigma_{-K}} + |\boldsymbol{Z}|)(\|\eta\|_{\Sigma_{-K}} + |\boldsymbol{Z}| + |z|)^2.$$

So $(\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty})\boldsymbol{S}^{1,1}_{r,\infty} = \boldsymbol{S}^{2,1}_{r,\infty}$.

Inequality (4-33), a Neumann expansion and formulas (2-27) yield claim (1) in Lemma 4.6.

Claim (2) in Lemma 4.6 follows from

$$i_{\Phi_*^{-1}\mathscr{X}^t}\Phi^*\Omega_t = -\Phi^*\Gamma = -\Gamma = i_{\mathscr{X}^t}\Omega_t = i_{\Phi_*^{-1}\mathscr{X}^t}\Omega_t,$$

where $\Phi^*\Gamma = \Gamma$ is (4-19) and we use (4-14) and (4-29) to conclude $\Phi^*\Omega_t = \Omega_t$. Then $\Phi_*^{-1}\mathscr{X}^t = \mathscr{X}^t$, which is equivalent to $\Phi_*\mathscr{X}^t = \mathscr{X}^t$. For the other formulas in claim (2), we have, for instance,

$$\mathscr{X}^t_j(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) = \mathscr{X}^t_j(\Phi(u)) = dz_j\,(\mathscr{X}^t(\Phi(u))) = dz_j(\Phi_*\mathscr{X}^t(u)) = d(z_j\circ\Phi)(\mathscr{X}^t(u)) = e^{\mathrm{i}\vartheta}\mathscr{X}^t_j(u).$$

This ends the proof of Lemma 4.6, assuming Lemma 4.7. □

Proof of Lemma 4.7. By (4-13) and summing over the indexes (j, A, B), we can write

$$\Omega_0 - \Omega_0' = \mathscr{R}^{4,0}_{\infty,\infty}\,dz_{jA}\wedge dz_{jB} \implies i_X(\Omega_0 - \Omega_0') = \mathscr{R}^{4,0}_{\infty,\infty}X_{jR}\,dz_{jI} + \mathscr{R}^{4,0}_{\infty,\infty}X_{jI}\,dz_{jR}. \tag{4-34}$$

So, if we define $K'X$ by setting $i_{K'X}\Omega_0' = i_X(\Omega_0 - \Omega_0')$, by comparing (4-34) with

$$i_{K'X}\Omega_0' = 2(K'X)_{jR}dz_{jI} - 2(K'X)_{jI}dz_{jR} + \mathrm{i}\langle (K'X)_\eta, X_{\bar\eta}\rangle - \mathrm{i}\langle (K'X)_{\bar\eta}, X_\eta\rangle,$$

we obtain

$$(K'X)_{jA} = \mathscr{R}^{4,0}_{\infty,\infty}X_{jA} \qquad \text{and} \qquad (K'X)_\xi = 0 \quad \text{for } \xi = \eta, \bar\eta. \tag{4-35}$$

Summing on (j, l, A, B, ξ, ξ'), we have

$$t\widehat{\Omega} = \mathscr{R}^{1,0}_{\infty,\infty}\,dz_{jA}\wedge dz_{lB} + dz_{jA}\wedge\langle \boldsymbol{S}^{1,0}_{\infty,\infty}, d\xi\rangle + t\big\langle\big(\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty}(z,\boldsymbol{Z},\eta) - (\partial_{\xi'}\boldsymbol{S}^{1,1}_{\infty,\infty}(z,\boldsymbol{Z},\eta))^*\big)\,d\xi, d\xi'\big\rangle.$$

Hence,

$$ti_X\widehat{\Omega} = \mathscr{R}^{1,0}_{\infty,\infty}X_{jA}\,dz_{lB} + \langle \boldsymbol{S}^{1,0}_{\infty,\infty}, X_\xi\rangle\,dz_{jA} + X_{jA}\langle \boldsymbol{S}^{1,0}_{\infty,\infty}, d\xi\rangle + \langle[\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty} - (\partial_{\xi'}\boldsymbol{S}^{1,1}_{\infty,\infty})^*]X_\xi, d\xi'\rangle.$$

So, if we define $K''X$ by setting $i_{K''X}\Omega_0' = ti_X\widehat{\Omega}$, we obtain

$$\begin{aligned}(K''X)_{jA} &= \sum_{\ell B}\mathscr{R}^{1,0}_{\infty,\infty}X_{\ell B} + \sum_{\xi=\eta,\bar\eta}\langle \boldsymbol{S}^{1,0}_{\infty,\infty}, X_\xi\rangle,\\ (K''X)_\xi &= \sum_{lB}\boldsymbol{S}^{1,0}_{\infty,\infty}X_{lB} + \sum_{\xi=\eta,\bar\eta}[\partial_{\xi'}\boldsymbol{S}^{1,1}_{\infty,\infty} - (\partial_\xi \boldsymbol{S}^{1,1}_{\infty,\infty})^*]X_{\xi'}.\end{aligned} \tag{4-36}$$

Since $K_t = K' + K''$, summing up (4-35) and (4-36) we get (4-31), and so Lemma 4.7. □

Having established that $\mathscr{X}^t(z,\eta)$ has components which are restrictions of symbols as in Definitions 2.8 and 2.9, we have the following result:

Lemma 4.8. *Fix $r \in \mathbb{N}$ and for the δ_0 and the $\mathcal{X}^t(z,\eta)$ of Lemma 4.6, consider the following system, which is well defined in $(t,z,\eta) \in (-4,4) \times B_{\mathbb{C}^n}(0,\delta_0) \times B_{\Sigma_k^c}(0,\delta_0)$ for all $k \in \mathbb{Z} \cap [-r,r]$:*

$$\dot{z}_j = \mathcal{X}^t_j(z,\eta) \quad \text{and} \quad \dot{\eta} = \mathcal{X}^t_\eta(z,\eta). \tag{4-37}$$

Then the following facts hold:

(1) *For $\delta_1 \in (0,\delta_0)$ sufficiently small, system* (4-37) *generates flows, for all $k \in \mathbb{Z} \cap [-r,r]$,*

$$\begin{aligned} &\mathfrak{F}^t \in C^\infty((-2,2) \times B_{\mathbb{C}^n}(0,\delta_1) \times B_{\Sigma_k^c}(0,\delta_1), B_{\mathbb{C}^n}(0,\delta_0) \times B_{\Sigma_k^c}(0,\delta_0)),\\ &\mathfrak{F}^t \in C^\infty((-2,2) \times B_{\mathbb{C}^n}(0,\delta_1) \times B_{H^1 \cap \mathcal{H}_c[0]}(0,\delta_1), B_{\mathbb{C}^n}(0,\delta_0) \times B_{H^1 \cap \mathcal{H}_c[0]}(0,\delta_0)).\end{aligned} \tag{4-38}$$

In particular, for $z_j^t := z_j \circ \mathfrak{F}^t(z,\eta)$ and $\eta^t := \eta \circ \mathfrak{F}^t(z,\eta)$, we have

$$z_j^t = z_j + S_j(t,z,\eta) \quad \text{and} \quad \eta^t = \eta + S_\eta(t,z,\eta) \tag{4-39}$$

with $S_j(t,z,\eta) = \mathcal{R}^{1,1}_{r,\infty}(t,z,\boldsymbol{Z},\eta)$ and $S_\eta(t,z,\eta) = \boldsymbol{S}^{1,1}_{r,\infty}(t,z,\boldsymbol{Z},\eta)$ in the sense of Remark 2.10.

(2) *$\mathfrak{F} = \mathfrak{F}^1$ is a local diffeomorphism of H^1 into itself near the origin such that $\mathfrak{F}^*\Omega = \Omega_0$.*

(3) *$S_j(t, e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta} S_j(t,z,\eta)$ and $S_\eta(t, e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta} S_\eta(t,z,\eta)$.*

Proof. The first sentence has been established in Lemma 4.6. Elementary theory of ODEs yields (4-38). The rest of claim (1) is a special case of a more general result; see Lemma 4.9 below. We get claim (2) by the classical formula, for L_X the Lie derivative,

$$\partial_t(\mathfrak{F}^{t*}\Omega_t) = \mathfrak{F}^{t*}(L_{\mathcal{X}^t}\Omega_t + \partial_t\Omega_t) = \mathfrak{F}^{t*}(d i_{\mathcal{X}^t}\Omega_t + d\Gamma) = 0. \tag{4-40}$$

Notice that (4-40) is well defined here, while it has no clear meaning for the NLS with translation treated in [Cuccagna 2012; 2014], where the flows $\mathfrak{F}^t$ are not differentiable (see [Cuccagna 2012] for a rigorous argument on how to get around this problem). The symmetry in claim (3) is elementary and we skip it. □

Lemma 4.9. *Consider a system*

$$\dot{z}_j = X_j(t,z,\eta) \quad \text{and} \quad \dot{\eta} = X_\eta(t,z,\eta), \tag{4-41}$$

where $X_j = \mathcal{R}^{a,b}_{r,m}(t,z,\boldsymbol{Z},\eta)$ for all j and $X_\eta = \boldsymbol{S}^{c,d}_{r,m}(t,z,\boldsymbol{Z},\eta)$ for fixed pairs (r,m), (a,b) and (c,d). Assume $m, b, d \geq 1$, with possibly $m = \infty$, and $r \geq 0$. Then, for the flow $(z^t, \eta^t) = \mathfrak{F}^t(z,\eta)$, we have

$$z_j^t = z_j + S_j(t,z,\eta) \quad \text{and} \quad \eta^t = \eta + S_\eta(t,z,\eta) \tag{4-42}$$

for appropriate functions $S_j = \mathcal{R}^{a,b}_{r,m}(t,z,\boldsymbol{Z},\eta)$ and $S_\eta = \boldsymbol{S}^{c,d}_{r,m}(t,z,\boldsymbol{Z},\eta)$ in the sense of Remark 2.10.

Proof. Consider the vectors $\boldsymbol{Z}$ of Definition 2.2. Notice that $\dot{\boldsymbol{Z}} = \mathcal{R}^{a+1,b}_{r,m}(t,z,\boldsymbol{Z},\eta)$, and this equation can be extended to a whole neighborhood of 0 in the space L. Pairing the latter equation with equations (4-42), a system remains defined which has a flow $\mathfrak{F}^t(z,\boldsymbol{Z},\eta)$ that is C^m in $(t,z,\boldsymbol{Z},\eta)$ and which reduces to the flow in (4-41) when we restrict to the vectors $\boldsymbol{Z}$ of Definition 2.2, by construction. The inequalities

(2-23) and (2-26), required to prove $S_j = \mathcal{R}^{a,b}_{r,m}$ and $S_\eta = \boldsymbol{S}^{c,d}_{r,m}$, can be obtained as follows. We have, for all $|k| \le r$,

$$\begin{aligned}
|z^t - z| &\le \int_0^t |\mathcal{R}^{a,b}_{r,m}(s, z^s, \boldsymbol{Z}^s, \eta^s)|\, ds \\
&\le C \int_0^t (\|\eta^s\|_{\Sigma_{-r}} + |\boldsymbol{Z}^s|)^b (\|\eta^s\|_{\Sigma_{-r}} + |\boldsymbol{Z}^s| + |z^s|)^a\, ds, \\
\|\eta^t - \eta\|_{\Sigma_k} &\le \int_0^t \|\boldsymbol{S}^{c,d}_{r,m}(s, z^s, \boldsymbol{Z}^s, \eta^s)\|_{\Sigma_k}\, ds \\
&\le C \int_0^t (\|\eta^s\|_{\Sigma_{-r}} + |\boldsymbol{Z}^s|)^d (\|\eta^s\|_{\Sigma_{-r}} + |\boldsymbol{Z}^s| + |z^s|)^c\, ds, \\
|\boldsymbol{Z}^t - \boldsymbol{Z}| &\le \int_0^t |\mathcal{R}^{a,b}_{r,m}(s, z^s, \boldsymbol{Z}^s, \eta^s)|\, ds \\
&\le C \int_0^t (\|\eta^s\|_{\Sigma_{-r}} + |\boldsymbol{Z}^s|)^b (\|\eta^s\|_{\Sigma_{-r}} + |\boldsymbol{Z}^s| + |z^s|)^{a+1}\, ds.
\end{aligned} \tag{4-43}$$

By Gronwall's inequality we get that $|\boldsymbol{Z}^t|$ and $\|\eta^t\|_{\Sigma_{-r}}$ are bounded by $C(|\boldsymbol{Z}| + \|\eta\|_{\Sigma_{-r}})$. Plugging this into the right-hand side of (4-43), we obtain the last part of the statement. □

We discuss the pullback of the energy E by the map $\mathfrak{F} := \mathfrak{F}^1$ in Lemma 4.8(2). We set $H_2(z, \eta) = \sum_{j=1}^n e_j |z_j|^2 + \langle H\eta, \bar{\eta}\rangle$. Our first preliminary result is the following one:

Lemma 4.10. *Consider the δ_1 of Lemma 4.8, the δ_0 of Lemma 4.6 and set $r = r_0$ with r_0 the index in Lemma 3.1. Then, for the map $\mathfrak{F}$ in Lemma 4.8(2), we have*

$$\mathfrak{F}\big(B_{\mathbb{C}^n}(0, \delta_1) \times (B_{H^1}(0, \delta_1) \cap \mathscr{H}_c[0])\big) \subset B_{\mathbb{C}^n}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap \mathscr{H}_c[0]) \tag{4-44}$$

and $\mathfrak{F}|_{B_{\mathbb{C}^n}(0,\delta_1)\times(B_{H^1}(0,\delta_1)\cap\mathscr{H}_c[0])}$ is a diffeomorphism between its domain and an open neighborhood of the origin in $\mathbb{C}^n \times (H^1 \cap \mathscr{H}_c[0])$. Furthermore, the functional $K := E \circ \mathfrak{F}$ admits an expansion

$$\begin{aligned}
&K(z, \eta) \\
&= H_2(z, \eta) + \sum_{j=1,\dots,n} \lambda_j(|z_j|^2) \\
&\quad + \sum_{l=0}^{2N+3} \sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} a^{(1)}_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) + \sum_{j=1}^{n} \sum_{l=0}^{2N+3} \sum_{|\boldsymbol{m}|=l} (\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G^{(1)}_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.}) \\
&\qquad + \mathcal{R}^{1,2}_{r_1,\infty}(z, \eta) + \mathcal{R}^{0,2N+5}_{r_1,\infty}(z, \boldsymbol{Z}, \eta) + \operatorname{Re}\langle \boldsymbol{S}^{0,2N+4}_{r_1,\infty}(z, \boldsymbol{Z}, \eta), \bar{\eta}\rangle \\
&\qquad\quad + \sum_{i+j=2} \sum_{|\boldsymbol{m}|\le 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G^{(1)}_{2\boldsymbol{m}ij}(z, \eta), \eta^i \bar{\eta}^j\rangle + \sum_{d+c=3} \sum_{i+j=d} \langle G^{(1)}_{dij}(z), \eta^i \bar{\eta}^j\rangle \mathcal{R}^{0,c}_{r,\infty}(z, \eta) + E_P(\eta),
\end{aligned} \tag{4-45}$$

where $r_1 = r_0 - 2$, $G^{(1)}_{j\boldsymbol{m}}$, $G^{(1)}_{2\boldsymbol{m}ij}$ and $G^{(1)}_{dij}$ are $\boldsymbol{S}^{0,0}_{r_1,\infty}$, $a^{(1)}_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) = \mathcal{R}^{0,0}_{\infty,\infty}(z)$, c.c. *means complex conjugate, and $\lambda_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$. For $|\boldsymbol{m}| = 0$, $G^{(1)}_{2\boldsymbol{m}ij}(z, \eta) = G_{2\boldsymbol{m}ij}(z)$ is the same as* (3-4). *Finally, we have the invariance $\mathcal{R}^{1,2}_{r_1,\infty}(e^{i\vartheta} z, e^{i\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_1,\infty}(z, \eta)$.*

Proof. Consider the expansion (3-3) for $E(u(z',\eta'))$, and substitute the formulas $z'_j = z_j + S_j(z,\eta)$ and $\eta' = \eta + S_\eta(z,\eta)$, with $S_\ell(z,\eta) = S_\ell(1,z,\eta)$ for $\ell = j, \bar{j}, \eta, \bar{\eta}$, with $S_{\bar{\ell}} = \bar{S}_\ell$. By $S_j(z,\eta) = \mathcal{R}^{1,1}_{r_0,\infty}(z,\boldsymbol{Z},\eta)$ and $S_\eta(z,\eta) = \boldsymbol{S}^{1,1}_{r_0,\infty}(z,\boldsymbol{Z},\eta)$, it is elementary to see that the last three lines of (3-3) yield terms that can be absorbed into the last three lines (4-45) (with $l \geq 1$ in the third line). Notice that the z dependence of the $a^{(1)}_{\boldsymbol{m}}$ in terms of $(|z_1|^2,\dots,|z_n|^2)$ follows by Lemmas 4.8 and B.3. The z dependence of the $G^{(1)}_{j\boldsymbol{m}}$ is obtained by Lemma B.4. Notice also that, if an $\mathcal{R}^{i,0}_{r,\infty}(z)$ depends only on z, then it is an $\mathcal{R}^{i,0}_{\infty,\infty}(z)$.

We have $\mathcal{R}^{1,2}_{r_0,\infty}(z',\eta') = \mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},\eta)$. Note that, by the invariance of $\mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ and Lemma 4.8(3), we have $\mathcal{R}^{1,2}_{r_0,\infty}(e^{\mathrm{i}\vartheta}z,\boldsymbol{Z},e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},\eta)$. By Taylor expansion (using the conventions under (3-14))

$$\mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},\eta) = \mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},0) + d_\eta \mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},0)\eta + \int_0^1 (1-t)\partial^2_\eta \mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},t\eta)\,dt\cdot\eta^2. \tag{4-46}$$

Each of the terms in the right-hand side is invariant by change of variables $(z,\eta) \rightsquigarrow (e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta)$. We have

$$\begin{aligned}\mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},\eta)|_{\eta=0} = \mathcal{R}^{1,2}_{\infty,\infty}(z,\boldsymbol{Z}) &= \sum_{k\leq 2N+4} \frac{1}{k!} d^k_{\boldsymbol{Z}} \mathcal{R}^{1,2}_{\infty,\infty}(z,0)\boldsymbol{Z}^k + \mathcal{R}^{1,2N+5}_{\infty,\infty}(z,\boldsymbol{Z})\\ &= \mathcal{R}^{1,2N+5}_{\infty,\infty}(z,\boldsymbol{Z}) + \sum_{l=2}^{2N+4}\sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} c_{\boldsymbol{m}}(z)\\ &= \mathcal{R}^{1,2N+5}_{\infty,\infty}(z,\boldsymbol{Z}) + \sum_{l=2}^{2N+4}\sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} \sum_{j=1}^{n} c_{j\boldsymbol{m}}(|z_j|^2),\end{aligned}$$

where, as in Step 1 in Lemma 3.1, the last equality is obtained by the invariance with respect to $(z,\eta) \rightsquigarrow (e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta)$ and by smoothness. We have, proceeding as above,

$$\begin{aligned}d_\eta \mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},0)\eta &= \mathrm{Re}\langle \boldsymbol{S}^{1,1}_{r_0,\infty}(z,\boldsymbol{Z}),\bar{\eta}\rangle\\ &= \sum_{k\leq 2N+3} \frac{1}{k!}\mathrm{Re}\langle d^k_{\boldsymbol{Z}} \boldsymbol{S}^{1,1}_{r_0,\infty}(z,0),\bar{\eta}\rangle \boldsymbol{Z}^k + \mathrm{Re}\langle \boldsymbol{S}^{1,2N+4}_{r_0,\infty}(z,\boldsymbol{Z},\eta),\bar{\eta}\rangle\\ &= \mathrm{Re}\langle \boldsymbol{S}^{1,2N+4}_{r_0,\infty}(z,\boldsymbol{Z},\eta),\bar{\eta}\rangle + \sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l} (\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}}\langle A_{j\boldsymbol{m}}(|z_j|^2),\eta\rangle + \text{c.c.}),\end{aligned}$$

Finally, for an $\mathcal{R}^{1,2}_{r_0,\infty}(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta)$ we have — see Definition 2.8 —

$$\int_0^1 (1-t)\partial^2_\eta \mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},t\eta)\,dt\,\eta^2 = \mathcal{R}^{1,2}_{r_0,\infty}(z,\eta).$$

By (4-46) and the subsequent formulas, we see that $\mathcal{R}^{1,2}_{r_0,\infty}(z',\eta')$ is absorbed into the last three lines of (4-45) (with $l \geq 1$ in the third line). The term $\langle H\eta',\bar{\eta}'\rangle = \langle H\eta,\bar{\eta}\rangle + \mathcal{R}^{1,2}_{r_0-2,\infty}(z,\boldsymbol{Z},\eta)$ behaves similarly, recalling that $r_1 = r_0 - 2$. Here too we have $\mathcal{R}^{1,2}_{r_0-2,\infty}(e^{\mathrm{i}\vartheta}z,\boldsymbol{Z},e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_0-2,\infty}(z,\boldsymbol{Z},\eta)$. This function can be treated like the $\mathcal{R}^{1,2}_{r_0,\infty}(z,\boldsymbol{Z},\eta)$ discussed earlier.

The terms $E(Q_{jz_j})$ and, for $j \neq k$, $\mathrm{Re}\langle q_{jz_j}, \bar{z}_k\phi_k\rangle = \mathcal{R}^{1,1}_{\infty,\infty}(z,\boldsymbol{Z})$ can be expanded similarly. But this time we need $l = 0$ in the third line. □

The expansion in Lemma 4.10 is too crude. We have the following additional and crucial fact:

Lemma 4.11 (cancellation lemma). *In the third line of* (4-45) *all the terms with* $l=0$ *are zeros.*

Proof. We first observe that the terms in the third line of (4-45) with $l=0$ can be written as

$$\sum_{k=1}^{n}\sum_{j\neq k}\sum_{A=R,I} z_{jA} b_{kjA}(z_k) + \sum_{k=1}^{n} \mathrm{Re}\langle \boldsymbol{A}_k(z_k), \bar{\eta}\rangle. \tag{4-47}$$

Indeed, they are

$$\sum_{|\boldsymbol{m}|=1} \boldsymbol{Z}^{\boldsymbol{m}} a^{(1)}_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) + \sum_{j=1}^{n} (\bar{z}_j \langle G^{(1)}_{j\boldsymbol{0}}(|z_j|^2), \eta\rangle + \text{c.c.}), \tag{4-48}$$

and it is obvious that the second term of (4-48) is the second term of (4-47). Arguing as in Lemma 3.1, the first term of (4-48) can be written as

$$\sum_{k=1}^{n}\sum_{|\boldsymbol{m}|=1} \boldsymbol{Z}^{\boldsymbol{m}} a^{(1)}_{k\boldsymbol{m}}(|z_k|^2).$$

Further, for $\boldsymbol{Z}^{\boldsymbol{m}} = z_i\bar{z}_j$, we can assume that i or j must be equal to k, because, if not, it can be absorbed into the terms with $l \geq 1$. Set $\mathcal{N}_k := \{\boldsymbol{m} : |\boldsymbol{m}| = 1,\ m_{i,j} = 0 \text{ if } i \neq k \text{ and } j \neq k\}$. We have

$$\sum_{k=1}^{n}\sum_{|\boldsymbol{m}|=1} \boldsymbol{Z}^{\boldsymbol{m}} a^{(1)}_{k\boldsymbol{m}}(|z_k|^2) = \sum_{k=1}^{n}\sum_{\boldsymbol{m}\in\mathcal{N}_k} \boldsymbol{Z}^{\boldsymbol{m}} a^{(1)}_{k\boldsymbol{m}}(|z_k|^2) = \sum_{k=1}^{n}\sum_{j\neq k} (z_j\bar{z}_k a^{(1)}_{km_{jk}}(|z_k|^2) + z_k\bar{z}_j a^{(1)}_{km_{jk}}(|z_k|^2)).$$

So, we can write the term in the form of the first term of (4-47).

Next, notice that, for $p_k = (0, \dots, 0, z_k, \dots, 0; 0)$,

$$b_{kjA}(z_k) = \partial_{z_{jA}} K(z, \eta)\big|_{p_k} \quad \text{and} \quad \boldsymbol{A}_k(z_k) = \nabla_\eta K(p_k). \tag{4-49}$$

Therefore, it suffices to show the right sides in (4-49) are both zero. Recall $u(z,\eta) = \sum_{j=1}^{n} Q_{jz_j} + R[z]\eta$. We have

$$\begin{aligned}\partial_{z_{jA}} K(z,\eta)\big|_{p_k} &= \partial_{z_{jA}} E\big(u(z'(z,\eta), \eta'(z,\eta))\big)\big|_{p_k}\\ &= \mathrm{Re}\big\langle \nabla E\big(u(z'(p_k), \eta'(p_k))\big), \overline{\partial_{z_{jA}} u\big(z'(z,\eta), \eta'(z,\eta)\big)\big|_{p_k}}\big\rangle.\end{aligned}$$

By Lemma 4.8, we have

$$(z'(p_k), \eta'(p_k)) = p_k. \tag{4-50}$$

So

$$\nabla E(u(z'(p_k), \eta'(p_k))) = \nabla E(Q_{kz_k}) = 2E_{kz_k} Q_{kz_k}.$$

By Proposition 1.1 and by (4-50), for $z_k = e^{\mathrm{i}\vartheta_k}\rho_k$ we have

$$-\mathrm{i}\mathfrak{F}_* \frac{\partial}{\partial\vartheta_k}\bigg|_{p_k} = -\mathrm{i}\frac{\partial}{\partial\vartheta_k}\bigg(\sum_{j=1}^{n} Q_{jz'_j} + R[z']\eta'\bigg)\bigg|_{p_k} = -\mathrm{i}\frac{\partial}{\partial\vartheta_k} Q_{kz_k} = -\mathrm{i}\frac{\partial}{\partial\vartheta_k} e^{\mathrm{i}\vartheta_k} Q_{k\rho_k} = Q_{kz_k},$$

where the first equality follows by definition of push forward, the second by (4-50) and the third by Proposition 1.1. Similarly, by the definition of push forward, we have

$$\partial_{z_{jA}} u(z'(z,\eta),\eta'(z,\eta))\big|_{p_k} = \mathfrak{F}_*\partial_{z_{jA}}\big|_{p_k}.$$

Therefore, $b_{kjA}(z_k)=0$ follows by

$$\partial_{z_{jA}} K(z,\eta)\big|_{p_k} = 2E_{kz_k}\operatorname{Im}\langle \mathfrak{F}_*\partial_{\vartheta_k}|_{p_k}, \overline{\mathfrak{F}_*\partial_{z_{jA}}}|_{p_k}\rangle = -E_{kz_k}\Omega_0(\partial_{\vartheta_k},\partial_{z_{j,A}})\big|_{p_k} = 0.$$

To get $\boldsymbol{A}_k(z_k)=0$, fix $\Xi\in\mathcal{H}_c[0]$ and set $p_{k,\Xi}(t):=(0,\dots,0,z_k,0,\dots,0;t\Xi)$. Then, for all Ξ,

$$\begin{aligned}\operatorname{Re}\langle\nabla K(p_k),\Xi\rangle &= \frac{d}{dt}K(p_{k,\Xi}(t))\bigg|_{t=0} = \frac{d}{dt}E\big(u\big(z'(p_{k,\Xi}(t)),\eta'(p_{k,\Xi}(t))\big)\big)\bigg|_{t=0}\\ &= \operatorname{Re}\bigg\langle\nabla E(Q_{kz_k}), \overline{\frac{d}{dt}u\big(z'(p_{k,\Xi}(t)),\eta'(p_{k,\Xi}(t))\big)\bigg|_{t=0}}\bigg\rangle\\ &= 2E_{kz_k}\operatorname{Im}\bigg\langle\mathfrak{F}_*\frac{\partial}{\partial\vartheta_k}\bigg|_{p_k},\overline{\mathfrak{F}_*\Xi}\bigg\rangle = -E_{kz_k}\Omega_0\bigg(\frac{\partial}{\partial\vartheta_k},\Xi\bigg)\bigg|_{p_k} = 0 \implies \boldsymbol{A}_k(z_k)=0. \qquad\square\end{aligned}$$

5. Birkhoff normal form

In this section, where we search for the effective Hamiltonian, the main result is Theorem 5.9.

We consider the symplectic form Ω_0 introduced in (4-13). We introduce an index $\ell=j,\bar{j}$, for $\bar{\bar{j}}=j$ with $j=1,\dots,n$. We write $\partial_j=\partial_{z_j}$ and $\partial_{\bar{j}}=\partial_{\bar{z}_j}$, $z_{\bar{j}}=\bar{z}_j$. With this notation, summing on j, by (4-8) and (4-34) for $\gamma_j(|z_j|^2)=\mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$ we have

$$\Omega_0 = \mathrm{i}(1+\gamma_j(|z_j|^2))\,dz_j\wedge d\bar{z}_j + \mathrm{i}\langle d\eta,d\bar{\eta}\rangle - \mathrm{i}\langle d\bar{\eta},d\eta\rangle. \tag{5-1}$$

Given $F\in C^1(U,\mathbb{R})$ with U an open subset of $\mathbb{C}^n\times\Sigma_r^c$, its Hamiltonian vector field X_F is defined by $i_{X_F}\Omega_0=dF$. We have, summing on j,

$$\begin{aligned} i_{X_F}\Omega_0 &= \mathrm{i}(1+\gamma_j(|z_j|^2))((X_F)_j\,d\bar{z}_j-(X_F)_{\bar{j}}\,dz_j)+\mathrm{i}\langle (X_F)_\eta,d\bar{\eta}\rangle - \mathrm{i}\langle (X_F)_{\bar{\eta}},d\eta\rangle\\ &= \partial_j F\,dz_j+\partial_{\bar{j}}F\,d\bar{z}_j+\langle\nabla_\eta F,d\eta\rangle+\langle\nabla_{\bar{\eta}}F,d\bar{\eta}\rangle.\end{aligned}$$

So, comparing the components of the two sides, we get for $1+\varpi_j(|z_j|^2)=(1+\gamma_j(|z_j|^2))^{-1}$, where $\varpi_j(|z_j|^2)=\mathcal{R}^{2,0}_{\infty,\infty}(|z_j|^2)$,

$$\begin{aligned} (X_F)_j &= -\mathrm{i}(1+\varpi_j(|z_j|^2))\partial_{\bar{j}}F, &\qquad (X_F)_\eta &= -\mathrm{i}\nabla_{\bar{\eta}}F,\\ (X_F)_{\bar{j}} &= \mathrm{i}(1+\varpi_j(|z_j|^2))\partial_j F, &\qquad (X_F)_{\bar{\eta}} &= \mathrm{i}\nabla_\eta F.\end{aligned} \tag{5-2}$$

Given $G\in C^1(U,\mathbb{R})$ and $F\in C^1(U,\boldsymbol{E})$ with $\boldsymbol{E}$ a Banach space, we set $\{F,G\}:=dF\,X_G$.

Definition 5.1 (normal form). Recall Definition 2.5 and, in particular, (2-13). Fix $r\in\mathbb{N}_0$. A real-valued function $Z(z,\eta)$ is in normal form if $Z=Z_0+Z_1$, where Z_0 and Z_1 are finite sums of the following type for $\boldsymbol{l} \geq 1$:

$$Z_1(z, \boldsymbol{Z}, \eta) = \sum_{j=1}^{n} \sum_{\substack{|\boldsymbol{m}|=\boldsymbol{l} \\ \boldsymbol{m} \in \mathcal{M}_j(\boldsymbol{l})}} (\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.}), \tag{5-3}$$

$$Z_0(z, \boldsymbol{Z}) = \sum_{\substack{|\boldsymbol{m}|=\boldsymbol{l}+1 \\ \boldsymbol{m} \in \mathcal{M}_0(\boldsymbol{l}+1)}} \boldsymbol{Z}^{\boldsymbol{m}} a_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2), \tag{5-4}$$

where $G_{j\boldsymbol{m}}(|z_j|^2) = \boldsymbol{S}^{0,0}_{r,\infty}(|z_j|^2)$, $\boldsymbol{Z}$ is as in Definition 2.2 and $a_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) = \mathcal{R}^{0,0}_{r,\infty}(|z_1|^2, \dots, |z_n|^2)$.

Remark 5.2. By Lemma 2.6, $\boldsymbol{Z}^{\boldsymbol{m}} = |z_1|^{2m_1} \cdots |z_n|^{2m_n}$ for all $\boldsymbol{m} \in \mathcal{M}_0(2N+4)$ for an $m \in \mathbb{N}_0^n$ with $2|m| = |\boldsymbol{m}|$. By Lemma 2.6 for $|\boldsymbol{m}| \leq 2N+3$, either $\sum_{a,b}(e_a - e_b)m_{ab} - e_j > 0$ or $\sum_{a,b}(e_a - e_b)m_{ab} - e_j < 0$.

For $\boldsymbol{l} \leq 2N+4$ we will consider flows associated to Hamiltonian vector fields X_χ with real-valued functions χ of the form

$$\chi = \sum_{\substack{|\boldsymbol{m}|=\boldsymbol{l}+1 \\ \boldsymbol{m} \notin \mathcal{M}_0(\boldsymbol{l}+1)}} \boldsymbol{Z}^{\boldsymbol{m}} b_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) + \sum_{j=1}^{n} \sum_{\substack{|\boldsymbol{m}|=\boldsymbol{l} \\ \boldsymbol{m} \notin \mathcal{M}_j(\boldsymbol{l})}} (\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle B_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.}) \tag{5-5}$$

with $b_{\boldsymbol{m}} = \mathcal{R}^{0,0}_{\boldsymbol{r},\infty}(|z_1|^2, \dots, |z_n|^2)$ and $B_{j\boldsymbol{m}} = \boldsymbol{S}^{0,0}_{\boldsymbol{r},\infty}(|z_j|^2)$ for some $\boldsymbol{r} \in \mathbb{N}$ defined in $B_{\mathbb{C}^n}(0, \boldsymbol{d})$ for some $\boldsymbol{d} > 0$.

The Hamiltonian vector field X_χ can be explicitly computed using (5-2). We have

$$(X_\chi)_j = (Y_\chi)_j + (\widetilde{Y}_\chi)_j, \quad (X_\chi)_\eta = -\mathrm{i} \sum_{j=1}^{n} \sum_{\substack{|\boldsymbol{m}|=\boldsymbol{l} \\ \boldsymbol{m} \notin \mathcal{M}_j(\boldsymbol{l})}} z_j \overline{\boldsymbol{Z}}^{\boldsymbol{m}} \overline{B}_{j\boldsymbol{m}}(|z_j|^2), \tag{5-6}$$

where

$$\begin{aligned}
(Y_\chi)_j(z, \eta) &:= -\mathrm{i}(1 + \varpi_j(|z_j|^2)) \\
&\quad \times \Bigg[\sum_{|\boldsymbol{m}|=\boldsymbol{l}+1} b_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) \partial_{\bar{j}} \boldsymbol{Z}^{\boldsymbol{m}} \\
&\qquad + \sum_{k=1}^{n} \sum_{|\boldsymbol{m}|=\boldsymbol{l}} (\langle B_{k\boldsymbol{m}}(|z_k|^2), \eta\rangle \partial_{\bar{j}}(\bar{z}_k \boldsymbol{Z}^{\boldsymbol{m}}) + \langle \overline{B}_{k\boldsymbol{m}}(|z_k|^2), \bar{\eta}\rangle \partial_{\bar{j}}(z_k \overline{\boldsymbol{Z}}^{\boldsymbol{m}})) \Bigg], \\
(\widetilde{Y}_\chi)_j(z, \eta) &:= -\mathrm{i}(1 + \varpi_j(|z_j|^2)) \Bigg[\sum_{|\boldsymbol{m}|=\boldsymbol{l}+1} \partial_{|z_j|^2} b_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) z_j \boldsymbol{Z}^{\boldsymbol{m}} \\
&\qquad + \sum_{|\boldsymbol{m}|=\boldsymbol{l}} (\langle B'_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle |z_j|^2 \boldsymbol{Z}^{\boldsymbol{m}} + \langle \overline{B}'_{j\boldsymbol{m}}(|z_j|^2), \bar{\eta}\rangle z_j^2 \overline{\boldsymbol{Z}}^{\boldsymbol{m}}) \Bigg].
\end{aligned} \tag{5-7}$$

Notice that $(Y_\chi)_j = \mathcal{R}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}$, $(\widetilde{Y}_\chi)_j = \mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}$ and $(X_\chi)_\eta = \boldsymbol{S}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}$. We now introduce a new space.

Definition 5.3. We denote by $X_{\boldsymbol{r}}(\boldsymbol{l})$ the space formed by

$$\{(b, B) = (\{b_{\boldsymbol{m}}\}_{\boldsymbol{m}\in\mathcal{A}(\boldsymbol{l})}, \{B_{j\boldsymbol{n}}\}_{j\in 1,\dots,n,\boldsymbol{n}\in\mathcal{B}_j(\boldsymbol{l})}) : b_{\boldsymbol{m}} \in \mathbb{C},\ B_{j\boldsymbol{n}} \in \Sigma^c_{\boldsymbol{r}} \text{ and } \chi(b, B) \text{ is real valued for all } z \in B_{\mathbb{C}^n(0,\boldsymbol{d})}\},$$

where

$$\mathcal{A}(\boldsymbol{l}) := \{\boldsymbol{m} : |\boldsymbol{m}| = \boldsymbol{l}+1,\ \boldsymbol{m} \notin \mathcal{M}_0(\boldsymbol{l}+1)\},$$
$$\mathcal{B}_j(\boldsymbol{l}) := \{\boldsymbol{n} : |\boldsymbol{n}| = \boldsymbol{l},\ \boldsymbol{n} \notin \mathcal{M}_j(\boldsymbol{l}+1)\},$$

where we have assigned some order in the coordinates and where

$$\chi(b, B) = \sum_{\boldsymbol{m}\in\mathcal{A}(\boldsymbol{l})} \boldsymbol{Z}^{\boldsymbol{m}} b_{\boldsymbol{m}} + \sum_{j=1}^{n} \sum_{\boldsymbol{m}\in\mathcal{B}_j(\boldsymbol{l})} (\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle B_{j\boldsymbol{m}}, \eta\rangle + \text{c.c.}).$$

We give $X_{\boldsymbol{r}}(\boldsymbol{l})$ the norm

$$\|(b, B)\|_{X_{\boldsymbol{r}}(\boldsymbol{l})} = \sum_{\boldsymbol{m}\in\mathcal{A}(\boldsymbol{l})} |b_{\boldsymbol{m}}| + \sum_{j=1}^{n} \sum_{\boldsymbol{m}\in\mathcal{B}_j(\boldsymbol{l})} \|B_{j\boldsymbol{m}}\|_{\Sigma_{\boldsymbol{r}}}.$$

Set $\varrho(z) = (\varrho_1(z), \dots, \varrho_n(z))$ with $\varrho_j(z) = |z_j|^2$.

Lemma 5.4. *Consider the χ in* (5-5) *for fixed $\boldsymbol{r} > 0$ and $\boldsymbol{l} \geq 1$, with coefficients $(b(\varrho(z)), B(\varrho(z)))$ in $C^2(B_{\mathbb{C}^n}(0, \boldsymbol{d}), X_{\boldsymbol{r}}(\boldsymbol{l}))$ and with $B_{j\boldsymbol{m}}(\varrho(z)) = B_{j\boldsymbol{m}}(\varrho_j(z))$. Consider the system*

$$\dot{z}_j = (X_\chi)_j(z, \eta) \quad \text{and} \quad \dot{\eta} = (X_\chi)_\eta(z, \eta),$$

which is defined in $(t, z) \in \mathbb{R} \times B_{\mathbb{C}^n}(0, \boldsymbol{d})$ and $\eta \in \Sigma^c_k$ for all $k \in \mathbb{Z} \cap [-\boldsymbol{r}, \boldsymbol{r}]$ (or $\eta \in H^1 \cap \mathcal{H}_c[0]$). Let $\delta \in (0, \min(\boldsymbol{d}, \delta_1))$ with δ_1 the constant of Lemma 4.8. Then the following properties hold:

(1) *If*
$$4(\boldsymbol{l}+1)\delta \big\| \big(b(\varrho(z)), B(\varrho(z))\big) \big\|_{W^{1,\infty}(B_{\mathbb{C}^n}(0,\boldsymbol{d}), X_{\boldsymbol{r}}(\boldsymbol{l}))} < 1, \tag{5-8}$$
then, for all $k \in \mathbb{Z} \cap [-\boldsymbol{r}, \boldsymbol{r}]$, for the flow $\phi^t(z, \eta)$ we have
$$\begin{aligned} &\phi^t \in C^\infty\big((-2, 2) \times B_{\mathbb{C}^n}(0, \delta/2) \times B_{\Sigma^c_k}(0, \delta/2), B_{\mathbb{C}^n}(0, \delta) \times B_{\Sigma^c_k}(0, \delta)\big) \\ \text{and} \quad &\phi^t \in C^\infty\big((-2, 2) \times B_{\mathbb{C}^n}(0, \delta/2) \times B_{H^1\cap\mathcal{H}_c[0]}(0, \delta/2), B_{\mathbb{C}^n}(0, \delta) \times B_{H^1\cap\mathcal{H}_c[0]}(0, \delta)\big). \end{aligned} \tag{5-9}$$
In particular, for $z^t_j := z_j \circ \phi^t(z, \eta)$ and $\eta^t := \eta \circ \phi^t(z, \eta)$, and in the sense of Remark 2.10,
$$\begin{aligned} z^t_j = z_j + S_j(t, z, \eta) \quad &\text{and} \quad \eta^t = \eta + S_\eta(t, z, \eta) \\ \text{with} \quad S_j(t, z, \eta) = \mathcal{R}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}(t, z, \boldsymbol{Z}, \eta) \quad &\text{and} \quad S_\eta(t, z, \eta) = \boldsymbol{S}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}(t, z, \boldsymbol{Z}, \eta). \end{aligned} \tag{5-10}$$

(2) *We have $S_j(t, e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta} S_j(t, z, \eta)$ and $S_\eta(t, e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) = e^{\mathrm{i}\vartheta} S_\eta(t, z, \eta)$.*

(3) *The flow ϕ^t is canonical, that is, $\phi^{t*}\Omega_0 = \Omega_0$ in $B_{\mathbb{C}^n}(0, \delta/2) \times B_{H^1\cap\mathcal{H}_c[0]}(0, \delta/2)$.*

Proof. Claim (2) is elementary. The same is true for (3), given that ϕ^t is a standard, sufficiently regular flow. In claim (1), (5-10) is a consequence of Lemma 4.9. The first part of claim (1) follows from elementary estimates such as

$$|(X_\chi)_j(z,\eta)| = |(1+\varpi_j(|z_j|^2))\partial_{\bar{j}}\chi(z,\eta)| \le (1+\|\varpi_j\|_{L^\infty(B_{\mathbb{C}}(0,\delta_0))})(\boldsymbol{l}+1)\|(b,B)\|_{W^{1,\infty}(B_{\mathbb{C}^n}(0,\delta_0),X_{\boldsymbol{r}}(\boldsymbol{l}))}\delta_0^{\boldsymbol{l}+1}$$

for $(z,\eta)\in B_{\mathbb{C}^n}(0,\delta)\times B_{\Sigma^c_{-\boldsymbol{r}}}(0,\delta)$. Notice that, taking δ_0 sufficiently small in Lemma 4.6, we can arrange $\|\varpi_j\|_{L^\infty(B_{\mathbb{C}}(0,\delta_0))}<1$. We also have

$$\|(X_\chi)_\eta(z,\eta)\|_{\Sigma_{\boldsymbol{r}}} \le \|(0,B)\|_{L^\infty(B_{\mathbb{C}^n}(0,\delta_0),X_{\boldsymbol{r}}(\boldsymbol{l}))}\delta_0^{\boldsymbol{l}+1}.$$

Then if (5-8) holds we obtain (5-9). ☐

The main part of ϕ^t will be given by the following lemma:

Lemma 5.5. *Consider a function χ as in* (5-5). *For a parameter $\varrho\in[0,\infty)^n$, consider the field W_χ defined as follows* (*notice that $W_\chi(z,\eta,\varrho(z)) = Y_\chi(z,\eta)$*):

$$\begin{aligned}(W_\chi)_j(z,\eta,\varrho) := -\mathrm{i}(1+\varpi_j(\varrho_j)) \\ \times\Bigg[\sum_{|\boldsymbol{m}|=\boldsymbol{l}+1} b_{\boldsymbol{m}}(\varrho)\partial_{\bar{j}}\boldsymbol{Z}^{\boldsymbol{m}} \\ +\sum_{k=1}^n\sum_{|\boldsymbol{m}|=\boldsymbol{l}}\big(\langle B_{k\boldsymbol{m}}(\varrho_k),\eta\rangle\partial_{\bar{j}}(\bar{z}_k\boldsymbol{Z}^{\boldsymbol{m}}) + \langle \bar{B}_{k\boldsymbol{m}}(\varrho_k),\bar\eta\rangle z_k\partial_{\bar{j}}\bar{\boldsymbol{Z}}^{\boldsymbol{m}}\big)\Bigg],\end{aligned} \tag{5-11}$$

$$(W_\chi)_\eta(z,\eta,\varrho) := -\mathrm{i}\sum_{k=1}^n\sum_{|\boldsymbol{m}|=\boldsymbol{l}} z_k\bar{\boldsymbol{Z}}^{\boldsymbol{m}}\bar{B}_{k\boldsymbol{m}}(\varrho_k).$$

Denote by $(w^t,\sigma^t)=\phi_0^t(z,\eta)$ the flow associated to the system

$$\begin{aligned}\dot{w}_j &= (W_\chi)_j(w,\sigma,\varrho(z)), \quad w_j(0)=z_j,\\ \dot\sigma &= (W_\chi)_\sigma(w,\sigma,\varrho(z)), \quad \sigma(0)=\eta.\end{aligned} \tag{5-12}$$

Let $\delta\in(0,\min(\boldsymbol{d},\delta_1))$, as in Lemma 5.4. Then the following facts hold:

(1) *If* (5-8) *holds*, *then*, *for $B(\varrho(z))=(B_{j\boldsymbol{m}}(\varrho_j(z)))_{j\boldsymbol{m}}$*,

$$w_j^t = z_j + T_j(t,b(\varrho(z)),B(\varrho(z)),z,\eta) \quad \text{and} \quad \sigma^t = \eta + T_\eta(t,b(\varrho(z)),B(\varrho(z)),z,\eta), \tag{5-13}$$

$$T_j \text{ and } T_\eta \text{ are } C^\infty \text{ for } (t,b,B,z,\eta)\in(-2,2)\times B_{X_{\boldsymbol{r}}}(0,c)\times B_{\mathbb{C}^n}(0,\delta)\times B_{\Sigma_{-\boldsymbol{r}}}(0,\delta) \tag{5-14}$$

with values in $\mathbb{C}$ and $\Sigma_{\boldsymbol{r}}$, respectively. Furthermore, *we have*

$$\begin{aligned}T_j(t,b,B,z,\eta) &= \mathscr{R}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}(t,b,B,z,\boldsymbol{Z},\eta),\\ T_\eta(t,b,B,z,\eta) &= \boldsymbol{S}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}(t,b,B,z,\boldsymbol{Z},\eta).\end{aligned} \tag{5-15}$$

(2) *We have the gauge covariance*, *for any fixed $\vartheta\in\mathbb{R}$*,

$$\begin{aligned}T_j(t,b,B,e^{\mathrm{i}\vartheta}z,e^{\mathrm{i}\vartheta}\eta) &= e^{\mathrm{i}\vartheta}T_j(t,b,B,z,\eta),\\ T_\eta(t,b,B,e^{\mathrm{i}\vartheta}z,e^{\mathrm{i}\vartheta}\eta) &= e^{\mathrm{i}\vartheta}T_\eta(t,b,B,z,\eta).\end{aligned} \tag{5-16}$$

(3) *Consider the Hamiltonian flow* $(z^t, \eta^t) = \phi^t(z, \eta)$ *associated to* χ; *see Lemma 5.4. Then*

$$z^t - w^t = \mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}(t, z, \boldsymbol{Z}, \eta), \quad \eta^t - \sigma^t = \boldsymbol{S}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}(t, z, \boldsymbol{Z}, \eta). \tag{5-17}$$

Proof. We have (5-13)–(5-14) by standard ODE theory. For $\boldsymbol{W} = (w_i \bar{w}_j)_{i \neq j}$ like the $\boldsymbol{Z}$ in Definition 2.2,

$$\begin{aligned} w_j^t = z_j - \mathrm{i}(1 + \varpi_j(\varrho_j(z)))\biggl[\sum_{|\boldsymbol{m}|=\boldsymbol{l}+1} b_{\boldsymbol{m}}(\varrho(z)) \int_0^t (\partial_{\bar{j}} \boldsymbol{W}^{\boldsymbol{m}})^s \, ds \\ + \sum_{k=1}^n \sum_{|\boldsymbol{m}|=\boldsymbol{l}} \biggl(\biggl\langle B_{k\boldsymbol{m}}(\varrho_k(z)), \int_0^t \sigma^s (\partial_{\bar{j}}(\bar{w}_k \boldsymbol{W}^{\boldsymbol{m}}))^s \, ds \biggr\rangle \\ + \biggl\langle \bar{B}_{k\boldsymbol{m}}(\varrho_k(z)), \int_0^t \bar{\sigma}^s w_k^s (\partial_{\bar{j}} \overline{\boldsymbol{W}^{\boldsymbol{m}}})^s \, ds \biggr\rangle \biggr) \biggr], \end{aligned} \tag{5-18}$$

where $(\partial_{\bar{j}} \overline{\boldsymbol{W}^{\boldsymbol{m}}})^s = \partial_{\bar{j}} \overline{\boldsymbol{W}^{\boldsymbol{m}}}|_{w=w^s}$. Similarly, we have

$$\sigma^t = \eta - \mathrm{i} \sum_{k=1}^n \sum_{|\boldsymbol{m}|=\boldsymbol{l}} \bar{B}_{k\boldsymbol{m}}(\varrho_k(z)) \int_0^t w_k^s (\overline{\boldsymbol{W}^{\boldsymbol{m}}})^s \, ds. \tag{5-19}$$

Like in Lemma 4.9, we have also $\boldsymbol{W}^t = \boldsymbol{Z} + \int_0^t \mathcal{R}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}\bigl(s, b(\varrho(z)), B(\varrho(z)), z, \boldsymbol{Z}, \eta\bigr) \, ds$. We can apply Gronwall's inequality like in Lemma 4.9 in these formulas to obtain (5-15). This yields claim (1).

Next,

$$\begin{aligned} (W_\chi)_j(e^{\mathrm{i}\vartheta} w, e^{\mathrm{i}\vartheta} \sigma, \varrho(z)) &= e^{\mathrm{i}\vartheta} (W_\chi)_j(w, \sigma, \varrho(z)), \\ (W_\chi)_\eta(e^{\mathrm{i}\vartheta} w, e^{\mathrm{i}\vartheta} \sigma, \varrho(z)) &= e^{\mathrm{i}\vartheta} (W_\chi)_\eta(w, \sigma, \varrho(z)) \end{aligned}$$

yield claim (2).

Consider claim (3). Observe that (5-17) holds replacing $\boldsymbol{l}+1$ by $\boldsymbol{l}$. By (5-6), we have, for a fixed C,

$$\begin{aligned} |\dot{z} - \dot{w}| &\le |(W_\chi)_j(z, \eta) - (W_\chi)_j(w, \sigma)| + |\mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}(t, z, \boldsymbol{Z}, \eta)| \\ &\le C|z - w| + C\|\eta - \sigma\|_{\Sigma_{-r}} + |\mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}(t, z, \boldsymbol{Z}, \eta)|. \end{aligned}$$

Similarly, we have

$$\|\dot{\eta} - \dot{\sigma}\|_{\Sigma_r} \le \|(W_\chi)_\eta(z, \eta, \varrho(z)) - (W_\chi)_\eta(w, \sigma, \varrho(z))\|_{\Sigma_r} \le C|z - w| + C\|\eta - \sigma\|_{\Sigma_{-\boldsymbol{r}}}.$$

We then conclude, by Gronwall's inequality,

$$|z^t - w^t| + \|\eta^t - \sigma^t\|_{\Sigma_r} \le |\mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}(t, z, \boldsymbol{Z}, \eta)|,$$

which, along with (5-17) with $\boldsymbol{l}+1$ replaced by $\boldsymbol{l}$, yields (5-17), ending Lemma 5.5. □

Using Lemma 5.5, we expand the ϕ^1 given in Lemma 5.4.

Lemma 5.6. *Let* $(z', \eta') = \phi^1(z, \eta)$, *where* ϕ^t *is the canonical flow given in Lemma 5.4. We have*:

(1) *For* $\mathcal{T}_j(b, B, z, \eta) = \mathcal{R}^{3,2\boldsymbol{l}-1}_{\boldsymbol{r},\infty}$, $\mathcal{T}_\eta(b, B, z, \eta) = \boldsymbol{S}^{3,2\boldsymbol{l}-1}_{\boldsymbol{r},\infty}$ *and* $\mathcal{T}_j$, $\mathcal{T}_\eta$ *smooth in* (b, B, z, η),

$$\begin{aligned} z_j' &= z_j + (Y_\chi)_j(z, \eta) + \mathcal{T}_j\bigl(b(\varrho(z)), B(\varrho(z)), z, \eta\bigr) + \mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}, \\ \eta' &= \eta + (X_\chi)_\eta(z, \eta) + \mathcal{T}_\eta\bigl(b(\varrho(z)), B(\varrho(z)), z, \eta\bigr) + \mathcal{S}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}. \end{aligned} \tag{5-20}$$

(2) *For* $\widetilde{\mathcal{T}}_j(b, B, z, \eta) = \mathcal{R}^{1,2\boldsymbol{l}}_{\boldsymbol{r},\infty}$ *smooth in* (b, B, z, η),

$$|z'_j|^2 = |z_j|^2 + \bar{z}_j (Y_\chi)_j(z,\eta) + z_j \overline{(Y_\chi)_j(z,\eta)} + \widetilde{\mathcal{T}}_j(b(\varrho(z)), B(\varrho(z)), z, \eta) + \mathcal{R}^{1,2\boldsymbol{l}+1}_{\boldsymbol{r},\infty}. \tag{5-21}$$

Remark 5.7. For $\boldsymbol{l} \geq 2$, $\mathcal{T}_j$ and $\mathcal{T}_\eta$ are absorbed in $\mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}$ and $\mathcal{S}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}$ and do not appear in the *homological equations* in Theorem 5.9. But, if $\boldsymbol{l} = 1$, they do, although as small perturbations.

Proof. First of all, by (5-7) and by Definition 5.3, we have $\bar{z}_j(\widetilde{Y}_\chi)_j + z_j \overline{(\widetilde{Y}_\chi)_j} = 2\,\mathrm{Re}(\bar{z}_j(\widetilde{Y}_\chi)_j) = 0$. So, using the following formula to define $\mathcal{Y}_j$, we have

$$\frac{d}{dt}|z_j|^2 = \bar{z}_j (X_\chi)_j + z_j \overline{(X_\chi)_j} = \bar{z}_j (Y_\chi)_j + z_j \overline{(Y_\chi)_j} =: \mathcal{Y}_j(z,\eta). \tag{5-22}$$

Notice that $\mathcal{Y}_j$ is $\mathcal{R}^{0,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}$. Therefore, we have

$$|z^s_j|^2 - |z_j|^2 = \mathcal{R}^{0,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}. \tag{5-23}$$

This implies

$$b(\varrho(z^s)) - b(\varrho(z)) = \mathcal{R}^{0,\boldsymbol{l}+1}_{\boldsymbol{r},\infty} \quad \text{and} \quad B(\varrho(z^s)) - B(\varrho(z)) = \mathcal{S}^{0,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}. \tag{5-24}$$

Similarly — see right before (5-2) — we have

$$\varpi_j(|z^s_j|^2) - \varpi_j(|z_j|^2) = \mathcal{R}^{2,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}. \tag{5-25}$$

Now we show (1). By (5-6) and (5-11), using (5-24) and (5-25), we have

$$(Y_\chi)_j(z^s,\eta^s) - (W_\chi)_j(z^s,\eta^s,\varrho(z)) = \mathcal{R}^{1,2\boldsymbol{l}+1}_{\boldsymbol{r},\infty}. \tag{5-26}$$

By (5-6), (5-10), (5-17) and (5-26), we have

$$\begin{aligned}
z'_j &= z_j + \int_0^1 (W_\chi)_j(z^s,\eta^s,\varrho(z))\,ds + \int_0^1 \big((Y_\chi)_j(z^s,\eta^s) - (W_\chi)_j(z^s,\eta^s,\varrho(z))\big)\,ds + \int_0^1 (\widetilde{Y}_\chi)_j(z^s,\eta^s)\,ds \\
&= z_j + \int_0^1 (W_\chi)_j(w^s + \mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}, \sigma^s + S^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}, \varrho(z))\,ds + \mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty} \\
&= z_j + \int_0^1 (W_\chi)_j(w^s,\sigma^s,\varrho(z))\,ds + \mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty} \\
&= z_j + (W_\chi)_j(z,\eta,\varrho(z)) + \mathcal{T}_j + \mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty},
\end{aligned}$$

where $\mathcal{T}_j = \int_0^1 (W_\chi)_j(w^s,\sigma^s,\varrho(z))\,ds - (W_\chi)_j(z,\eta,\varrho(z))$ and the last $\mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}$ in the second line is different from the $\mathcal{R}^{1,\boldsymbol{l}+1}_{\boldsymbol{r},\infty}$ in the third line. Finally, by Lemma 5.5(1) and the fact $(W_\chi)_j = \mathcal{R}^{1,\boldsymbol{l}}_{\boldsymbol{r},\infty}$, we have $\mathcal{T}_j = \mathcal{R}^{1,2\boldsymbol{l}-1}_{\boldsymbol{r},\infty}$ with $\mathcal{T}_j$ smooth in (t, b, B, z, η). The argument for η' is similar.

We next show (2). Set $\widetilde{\mathcal{Y}}_j(z,\eta,\varrho) := \bar{z}_j(W_\chi)_j(z,\eta,\varrho) + z_j\overline{(W_\chi)_j(z,\eta,\varrho)}$. As in (5-23)–(5-24), we have

$$\widetilde{\mathcal{Y}}_j(z^s,\eta^s,\varrho(z)) - \mathcal{Y}_j(z^s,\eta^s) = \mathcal{R}^{0,2\boldsymbol{l}+2}_{\boldsymbol{r},\infty},$$

where $\mathcal{Y}_j$ is as defined in (5-22). So we have

$$\begin{aligned}|z_j'|^2 &= |z_j|^2 + \int_0^1 \mathcal{Y}_j(z^s, \eta^s)\,ds = |z_j|^2 + \int_0^1 \widetilde{\mathcal{Y}}_j(z^s, \eta^s, \varrho(z))\,ds + \mathcal{R}^{0,2\boldsymbol{l}+2}_{\boldsymbol{r},\infty}\\ &= |z_j|^2 + \int_0^1 \widetilde{\mathcal{Y}}_j(w^s, \sigma^s, \varrho(z))\,ds + \mathcal{R}^{1,2\boldsymbol{l}+1}_{\boldsymbol{r},\infty} = |z_j|^2 + \widetilde{\mathcal{Y}}_j(z,\eta) + \widetilde{\mathcal{T}}_j + \mathcal{R}^{1,2\boldsymbol{l}+1}_{\boldsymbol{r},\infty},\end{aligned}$$

where $\widetilde{\mathcal{T}}_j = \int_0^1 \widetilde{\mathcal{Y}}_j(w^s, \sigma^s, \varrho(z))\,ds - \widetilde{\mathcal{Y}}_j(z,\eta)$. As in (1), $\widetilde{\mathcal{T}}_j = \mathcal{R}^{1,2\boldsymbol{l}}_{\boldsymbol{r},\infty}$ and $\widetilde{T}$ is C^∞ for (b, B, z, η). □

After a coordinate change $\phi = \phi^1$ as in Lemma 5.4 the Hamiltonian expands like in (4-45).

Lemma 5.8 (structure lemma). *Consider a function K which admits an expansion as in* (4-45), *defined for $(z,\eta) \in B_{\mathbb{C}^n}(0,\delta) \times (B_{H^1}(0,\delta) \cap \mathcal{H}_c[0])$ for some small $\delta > 0$ and with r_1 replaced by an r'. Suppose also that the $l=0$ terms in the third line are zero. Consider a function χ such as in* (5-5) *with $1 \le \boldsymbol{l} \le 2N+4$, with $\|(b,B)\|_{W^{1,\infty}(B_{\mathbb{C}^n}(0,\delta), X_r(\boldsymbol{l}))} \le \underline{C}$ and with $\underline{C}$ a preassigned number. Suppose also that $2c_2(2N+4)\delta\underline{C} < 1$ with c_2 the constant of Lemma 5.4. Denote by $\phi = \phi^1$ the corresponding flow. Then Lemma 5.4(1)–(3) hold and, for $(z,\eta) \in B_{\mathbb{C}^n}(0,\delta/2) \times (B_{H^1}(0,\delta/2) \cap \mathcal{H}_c[0])$, $r = r' - 2$ and $\boldsymbol{Z}$ as in Definition 2.2, we have an expansion*

$$\begin{aligned}K\circ\phi(z,\eta) &= H_2(z,\eta) + \sum_{j=1}^n \lambda_j(|z_j|^2) + \sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} a_{\boldsymbol{m}}(|z_1|^2,\dots,|z_n|^2)\\ &\quad + \sum_{j=1}^n\sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l} (\bar z_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.}) + \mathcal{R}^{1,2}_{r,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r,\infty}(z,\boldsymbol{Z},\eta)\\ &\quad + \operatorname{Re}\langle \boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z},\eta), \bar\eta\rangle + \sum_{i+j=2}\sum_{|\boldsymbol{m}|\le 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{2\boldsymbol{m}ij}(z,\eta), \eta^i\bar\eta^j\rangle\\ &\quad + \sum_{d+c=3}\sum_{i+j=d} \langle G_{dij}(z,\eta), \eta^i\bar\eta^j\rangle \mathcal{R}^{0,c}_{r,\infty}(z,\eta) + E_P(\eta),\end{aligned} \tag{5-27}$$

where $G_{j\boldsymbol{m}}$, $G_{2\boldsymbol{m}ij}$ and G_{dij} are $\boldsymbol{S}^{0,0}_{r,\infty}$ and the $a_{\boldsymbol{m}}$ are $\mathcal{R}^{0,0}_{\infty,\infty}$. For $|\boldsymbol{m}|=0$, we have $G_{2\boldsymbol{m}ij}(z,\eta) = G_{2\boldsymbol{m}ij}(z)$ are the functions in (3-4) *and the $\lambda_j(|z_j|^2)$ are the same as those of* (4-45). *Furthermore, the term $\mathcal{R}^{1,2}_{r,\infty}(z,\eta)$ in* (5-27) *satisfies $\mathcal{R}^{1,2}_{r,\infty}(e^{\mathrm{i}\vartheta}z, e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r,\infty}(z,\eta)$.*

Proof. Like in Lemma 4.10, we consider the expansion (4-45) for $K(z',\eta')$, and substitute the formulas $z_j' = z_j + S_j(z,\eta)$ and $\eta' = \eta + S_\eta(z,\eta)$. Proceeding like in Lemma 4.10, we have

$$\mathcal{R}^{1,2}_{r',\infty}(z',\eta') = \mathcal{R}^{1,2}_{r',\infty}(z,\eta) + \mathcal{R}^{1,2N+5}_{r',\infty}(z,\boldsymbol{Z},\eta) + \operatorname{Re}\langle \boldsymbol{S}^{1,2N+4}_{r',\infty}(z,\boldsymbol{Z},\eta), \bar\eta\rangle + \mathfrak{S}, \tag{5-28}$$

where $\mathfrak{S}$ consists of terms like in the second and third sums of (5-27).

Similarly, for a $\widetilde{\mathfrak{S}}$ like $\mathfrak{S}$, we have

$$\begin{aligned}\langle H\eta', \bar\eta'\rangle &= \langle H\eta, \bar\eta\rangle + \mathcal{R}^{1,\boldsymbol{l}+1}_{r'-2,\infty}(z,\boldsymbol{Z},\eta)\\ &= \langle H\eta, \bar\eta\rangle + \mathcal{R}^{1,\boldsymbol{l}+1}_{r'-2,\infty}(z,\eta) + \mathcal{R}^{1,\boldsymbol{l}+1}_{r'-2,\infty}(z,\boldsymbol{Z}) + \operatorname{Re}\langle \boldsymbol{S}^{1,\boldsymbol{l}}_{r'-2,\infty}(z,\boldsymbol{Z},\eta), \bar\eta\rangle\\ &= \langle H\eta, \bar\eta\rangle + \mathcal{R}^{1,\boldsymbol{l}+1}_{r'-2,\infty}(z,\eta) + \mathcal{R}^{1,2N+5}_{r'-2,\infty}(z,\boldsymbol{Z},\eta) + \operatorname{Re}\langle \boldsymbol{S}^{1,2N+4}_{r'-2,\infty}(z,\boldsymbol{Z},\eta), \bar\eta\rangle + \widetilde{\mathfrak{S}}.\end{aligned} \tag{5-29}$$

Consider a $\lambda_j(|z_j|^2)$ in (4-45). Then, by (5-21), we have

$$\lambda(|z_j'|^2) = \lambda(|z_j|^2 + \mathcal{R}^{0,\boldsymbol{l}+1}_{r,\infty}(z, \boldsymbol{Z}, \eta)) = \mu(|z_j|^2) + \mathcal{R}^{1,\boldsymbol{l}+1}_{r,\infty}(z, \boldsymbol{Z}, \eta). \tag{5-30}$$

The latter admits an expansion like in (4-46) and what follows it.

The term $\mathcal{R}^{1,2}_{r,\infty}(z, \eta)$ in the second line of (5-27) is either the first in the right-hand side in (5-28) for $l > 1$ in Lemma 4.8, or the sum of that with the $\mathcal{R}^{1,l+1}_{r'-2,\infty}(z, \eta)$ originating from (5-29)–(5-30) for $l = 1$ in Lemma 4.8. In either case it satisfies $\mathcal{R}^{1,2}_{r,\infty}(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r,\infty}(z, \eta)$. Other terms in (4-45) computed at (z', η') by similar elementary expansions are similarly absorbed in (5-27). □

All of the above lemmas are preparation for the following result, which will give us an effective Hamiltonian by picking $\iota = 2N + 4$.

Theorem 5.9 (Birkhoff normal form). *For any $\iota \in \mathbb{N} \cap [2, 2N + 4]$ there is a $\delta_\iota > 0$, a polynomial χ_ι as in* (5-5) *with $\boldsymbol{l} = \iota$, $\boldsymbol{d} = \delta_\iota$ and $\boldsymbol{r} = r_\iota = r_0 - 2(\iota + 1)$ such that, for all $k \in \mathbb{Z} \cap [-r(\iota), r(\iota)]$, we have for each χ_ι a flow* (*for δ_1 the constant in Lemma 4.10*)

$$\begin{aligned} &\phi^t_\iota \in C^\infty\big((-2, 2) \times B_{\mathbb{C}^n}(0, \delta_\iota) \times B_{\Sigma^c_k}(0, \delta_\iota), B_{\mathbb{C}^n}(0, \delta_{\iota-1}) \times B_{\Sigma^c_k}(0, \delta_{\iota-1})\big) \\ \text{and}\quad &\phi^t_\iota \in C^\infty\big((-2, 2) \times B_{\mathbb{C}^n}(0, \delta_\iota) \times B_{H^1\cap\mathcal{H}_c[0]}(0, \delta_\iota), B_{\mathbb{C}^n}(0, \delta_{\iota-1}) \times B_{H^1\cap\mathcal{H}_c[0]}(0, \delta_{\iota-1})\big) \end{aligned} \tag{5-31}$$

and such that, for $\mathfrak{F}^{(\iota)} := \mathfrak{F} \circ \phi_2 \circ \cdots \circ \phi_\iota$ with $\mathfrak{F}$ the transformation in Lemma 4.8 and $\phi_j = \phi^1_\iota$, for $(z, \eta) \in B_{\mathbb{C}^n}(0, \delta_\iota) \times (B_{H^1}(0, \delta_\iota) \cap \mathcal{H}_c[0])$ and for $\boldsymbol{Z}$ as in Definition 2.2, we have

$$\begin{aligned} H^{(\iota)}(z, \eta) &:= E \circ \mathfrak{F}^{(\iota)}(z, \eta) \\ &= H_2(z, \eta) + \sum_{j=1}^{n} \lambda_j(|z_j|^2) + Z^{(\iota)}(z, \boldsymbol{Z}, \eta) + \sum_{l=\iota}^{2N+3} \sum_{|\boldsymbol{m}|=l+1} \boldsymbol{Z}^{\boldsymbol{m}} a^{(\iota)}_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) \\ &\qquad + \sum_{j=1}^{n} \sum_{l=\iota}^{2N+3} \sum_{|\boldsymbol{m}|=l} (\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G^{(\iota)}_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.}) + \mathcal{R}^{1,2}_{r_\iota,\infty}(z, \eta) + \mathcal{R}^{0,2N+5}_{r_\iota,\infty}(z, \boldsymbol{Z}, \eta) \\ &\qquad + \operatorname{Re}\langle \boldsymbol{S}^{0,2N+4}_{r_\iota,\infty}(z, \boldsymbol{Z}, \eta), \bar{\eta}\rangle + \sum_{i+j=2} \sum_{|\boldsymbol{m}|\le 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G^{(\iota)}_{2\boldsymbol{m}ij}(z, \eta), \eta^i \bar{\eta}^j\rangle \\ &\qquad\quad + \sum_{d+c=3} \sum_{i+j=d} \langle G^{(\iota)}_{dij}(z, \eta), \eta^i \bar{\eta}^j\rangle \mathcal{R}^{0,c}_{r_\iota,\infty}(z, \eta) + E_P(\eta), \end{aligned} \tag{5-32}$$

where, for coefficients like in Definition 5.1 for $(r, m) = (r_\iota, \infty)$,

$$Z^{(\iota)} = \sum_{\boldsymbol{m}\in\mathcal{M}_0(\iota)} \boldsymbol{Z}^{\boldsymbol{m}} a_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2) + \sum_{j=1}^{n} \bigg(\sum_{\boldsymbol{m}\in\mathcal{M}_j(\iota-1)} \bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.} \bigg). \tag{5-33}$$

We have $\mathcal{R}^{1,2}_{r_\iota,\infty} = \mathcal{R}^{1,2}_{r_2,\infty}$ and $\mathcal{R}^{1,2}_{r_2,\infty}(e^{\mathrm{i}\vartheta} z, e^{\mathrm{i}\vartheta}\eta) \equiv \mathcal{R}^{1,2}_{r_2,\infty}(z, \eta)$.

In particular, we have, for $\delta_f := \delta_{2N+4}$ and for the δ_0 in Lemma 4.6,

$$\mathfrak{F}^{(2N+4)}\big(B_{\mathbb{C}^n}(0, \delta_f) \times (B_{H^1}(0, \delta_f) \cap \mathcal{H}_c[0])\big) \subset B_{\mathbb{C}^n}(0, \delta_0) \times (B_{H^1}(0, \delta_0) \cap \mathcal{H}_c[0]) \tag{5-34}$$

with $\mathfrak{F}|_{B_{\mathbb{C}^n}(0,\delta_f)\times(B_{H^1}(0,\delta_f)\cap\mathcal{H}_c[0])}$ *a diffeomorphism between its domain and an open neighborhood of the origin in* $\mathbb{C}^n\times(H^1\cap\mathcal{H}_c[0])$.

Furthermore, for $r=r_0-4N-10$, *there is a pair* $\mathcal{R}^{1,1}_{r,\infty}$ *and* $\boldsymbol{S}^{1,1}_{r,\infty}$ *such that, for* $(z',\eta')=\mathfrak{F}^{(2N+4)}(z,\eta)$,

$$z'=z+\mathcal{R}^{1,1}_{r,\infty}(z,\boldsymbol{Z},\eta)\quad\text{and}\quad \eta'=\eta+\boldsymbol{S}^{1,1}_{r,\infty}(z,\boldsymbol{Z},\eta). \tag{5-35}$$

By taking all the $\delta_\iota>0$ *sufficiently small, we can assume that all the symbols in the proof, i.e., the symbols in* (5-35) *and the symbols in the expansions* (5-32), *satisfy the estimates of Definitions 2.8 and 2.9 for* $|z|<\delta_\iota$ *and* $\|\eta\|_{\Sigma_{r(\iota)}}<\delta_\iota$ *for their respective* ι.

Proof. Notice that the functional K in Lemma 4.10 satisfies the case $\iota=1$. The proof will be by induction on ι. We assume that $H^{(\iota)}$ satisfies the statement for some $\iota\ge 1$ and prove that there is a $\phi_{\iota+1}$ such that $H^{(\iota+1)}:=H^{(\iota)}\circ\phi_{\iota+1}$ satisfies the statement for $\iota+1$. We consider the representation (5-27) for $H^{(\iota)}$, which is guaranteed by Lemma 5.8. Using (5-27), we set $\boldsymbol{h}=H^{(\iota)}(z,\boldsymbol{Z},\eta)$, interpreting $(z,\boldsymbol{Z},\eta)$ as independent variables. Then we have, for $\boldsymbol{l}=\iota$,

$$a^{(\boldsymbol{l})}_{\boldsymbol{m}}(|z_1|^2,\dots,|z_n|^2)=\frac{1}{\boldsymbol{m}!}\partial^{\boldsymbol{m}}_{\boldsymbol{Z}}\boldsymbol{h}\Big|_{(z,\eta,\boldsymbol{Z})=(z;0,0)},\qquad |\boldsymbol{m}|\le 2N+4, \tag{5-36}$$

$$\bar z_j G^{(\boldsymbol{l})}_{j\boldsymbol{m}}(|z_j|^2)=\frac{1}{\boldsymbol{m}!}\partial^{\boldsymbol{m}}_{\boldsymbol{Z}}\nabla_\eta\boldsymbol{h}\Big|_{(z,\eta,\boldsymbol{Z})=(0,\dots,z_j,0,\dots,0;0,0)},\qquad |\boldsymbol{m}|\le 2N+3. \tag{5-37}$$

The inductive hypothesis on $H^{(\iota)}$ is a statement on the Taylor coefficients in (5-36)–(5-37), that is, that, for $\boldsymbol{l}=\iota$ (see Definition 2.5 and Remark 5.2),

$$\partial^{\boldsymbol{m}}_{\boldsymbol{Z}}\boldsymbol{h}\big|_{(z,\eta,\boldsymbol{Z})=(z;0,0)}=0\quad\text{for all }\ \boldsymbol{m}\notin\mathcal{M}_0(\boldsymbol{l}), \tag{5-38}$$

$$\partial^{\boldsymbol{m}}_{\boldsymbol{Z}}\nabla_\eta\boldsymbol{h}\big|_{(z,\eta,\boldsymbol{Z})=(0,\dots,z_j,0,\dots,0;0,0)}=0\quad\text{for all }\ (j,\boldsymbol{m})\ \text{ with }\ \boldsymbol{m}\notin\mathcal{M}_j(\boldsymbol{l}-1). \tag{5-39}$$

We consider now an as yet unknown χ as in (5-5) with $\boldsymbol{l}=\iota$, $\boldsymbol{r}=r_\iota$ and a yet to be determined $\boldsymbol{d}=\delta>0$. Set $\phi:=\phi^1$, where ϕ^t is the flow of Lemma 5.4. We are seeking χ such that $H^{(\iota)}\circ\phi$ satisfies the conclusions of Theorem 5.9 for $\iota+1$, i.e., that using Lemma 5.8 again and setting this time $\boldsymbol{h}=(H^{(\iota)}\circ\phi)(z,\eta,\boldsymbol{Z})$, we will have (5-38)–(5-39) for $\boldsymbol{l}=\iota+1$. Notice that, for any χ, (5-38)–(5-39) are automatically true for $\boldsymbol{l}=\iota$. This is because $H^{(\iota)}(z,\eta,\boldsymbol{Z})$ and $(H^{(\iota)}\circ\phi)(z,\eta,\boldsymbol{Z})$ have the same derivatives in (5-36) for $|\boldsymbol{m}|\le\iota$, and in (5-37) for $|\boldsymbol{m}|\le\iota-1$. So it is enough to consider (5-38) for $|\boldsymbol{m}|=\iota+1$ and (5-39) for $|\boldsymbol{m}|=\iota$. This will be true for a specific choice of χ whose coefficients solve the *homological equations*, which we set up in the sequel.

By (5-20) and $G^{(\iota)}_{\boldsymbol{20}ij}(z,\eta)=G_{\boldsymbol{20}ij}(z)$, we have

$$\begin{aligned}H^{(\iota)}(z',\eta')=H_2(z',\eta')+\sum_{j=1}^n\lambda_j(|z'_j|^2)+Z^{(\iota)}(z',\boldsymbol{Z}',\eta')+\mathcal{R}^{1,2}_{r,\infty}(z',\eta')+\sum_{i+j=2}\langle G_{\boldsymbol{20}ij}(z'),\eta'^i\,\bar\eta'^j\rangle\\ +(*)+\sum_{|\boldsymbol{m}|=\iota+1}\boldsymbol{Z}^{\boldsymbol{m}}a^{(\iota)}_{\boldsymbol{m}}(|z|^2)+\sum_{j=1}^n\sum_{|\boldsymbol{m}|=\iota}(\bar z_j\boldsymbol{Z}^{\boldsymbol{m}}\langle G^{(\iota)}_{j\boldsymbol{m}}(|z_j|^2),\eta\rangle+\text{c.c.}),\end{aligned} \tag{5-40}$$

where $\boldsymbol{h} := (*)(z, \eta, \boldsymbol{Z})$ satisfies (5-38)–(5-39) for $\boldsymbol{l} = \iota + 1$. In the sequel, we will use $(*)$ with this meaning. Let $(z', \eta') = \phi(z, \eta)$. We have

$$\begin{aligned}\sum_{j=1}^{n} e_j(\bar{z}_j (Y_\chi)_j(z,\eta) + z_j (Y_\chi)_{\bar{j}}(z,\eta)) \\ &= \sum_{|\boldsymbol{m}|=\iota+1} \mathrm{i}\tilde{\boldsymbol{e}}\cdot(\mu(\boldsymbol{m}) - \nu(\boldsymbol{m})) b_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2)\boldsymbol{Z}^{\boldsymbol{m}} \\ &\quad + \sum_j \sum_{|\boldsymbol{m}|=\iota} \big(\mathrm{i}\tilde{\boldsymbol{e}}\cdot(\tilde{\mu}_j(\boldsymbol{m}) - \tilde{\nu}_j(\boldsymbol{m}))\langle B_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle \bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} + \text{c.c.}\big) \end{aligned} \tag{5-41}$$

for

$$\begin{aligned} \boldsymbol{Z}^{\boldsymbol{m}} &= z^{\mu(\boldsymbol{m})} \bar{z}^{\nu(\boldsymbol{m})}, \\ \bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} &= z^{\tilde{\mu}_j(\boldsymbol{m})} \bar{z}^{\tilde{\nu}_j(\boldsymbol{m})}, \\ \tilde{\boldsymbol{e}}(z) &:= \big(e_1(1+\varpi_1(|z_1|^2)), \dots, e_n(1+\varpi_n(|z_n|^2))\big), \end{aligned} \tag{5-42}$$

and, summing on repeated indexes,

$$\langle H\eta, (X_\chi)_{\bar{\eta}}(z,\eta)\rangle + \langle H(X_\chi)_\eta(z,\eta), \bar{\eta}\rangle = \mathrm{i}\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle HB_{j,\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.} \tag{5-43}$$

So, by Lemma 5.6, (5-41)–(5-43) and using the notation in (5-42), we have

$$\begin{aligned} H_2(z', \eta') &= \sum_{j=1}^{n} e_j |z'_j|^2 + \langle H\eta', \bar{\eta}'\rangle \\ &= H_2(z,\eta) + \sum_{\substack{|\boldsymbol{m}|=\boldsymbol{l}+1 \\ \boldsymbol{m}\notin\mathcal{M}_0(\boldsymbol{l}+1)}} \mathrm{i}\tilde{\boldsymbol{e}}\cdot(\mu(\boldsymbol{m}) - \nu(\boldsymbol{m})) b_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2)\boldsymbol{Z}^{\boldsymbol{m}} \\ &\quad + \sum_j \sum_{\substack{|\boldsymbol{m}|=\boldsymbol{l} \\ \boldsymbol{m}\notin\mathcal{M}_j(\boldsymbol{l})}} \big(\mathrm{i}\big\langle\big(\tilde{\boldsymbol{e}}\cdot(\tilde{\mu}_j(\boldsymbol{m}) - \tilde{\nu}_j(\boldsymbol{m})) + H\big) B_{j\boldsymbol{m}}(|z_j|^2), \eta\big\rangle \bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} + \text{c.c.}\big) \\ &\quad + \mathcal{R}^{2,2\iota}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta) + (*), \end{aligned} \tag{5-44}$$

where c.c. refers only to the third line and, in the last line,

$$\mathcal{R}^{2,2\iota}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta) = \sum_{j=1}^{n} e_j \widetilde{\mathcal{T}}_j + \langle H\eta, \overline{\mathcal{T}}_\eta\rangle + \langle H\mathcal{T}_\eta, \bar{\eta}\rangle + \langle H\mathcal{T}_\eta, \overline{\mathcal{T}}_\eta\rangle,$$

where here and in the sequel of this proof we abuse notation, denoting by (b, B) the element in $X_r(\iota)$ — see Definition 5.3 — with entries $b_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2)$ and $B_{j\boldsymbol{m}}(|z_j|^2)$. The term $\mathcal{R}^{2,2\iota}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta)$ can be absorbed in $(*)$ if $\iota \geq 2$, but if $\iota = 1$ needs to be considered explicitly. By $\lambda_j(|z_j|^2) = \mathcal{R}^{2,0}_{\infty,\infty}$ and (5-21), we have

$$\lambda_j(|z'_j|^2) = \lambda_j(|z_j|^2) + \mathcal{R}^{2,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta) + (*). \tag{5-45}$$

Next, we claim

$$Z^{(\iota)}(z', \boldsymbol{Z}', \eta') = Z^{(\iota)}(z, \boldsymbol{Z}, \eta) + \mathcal{R}^{2,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta) + (*). \tag{5-46}$$

Let us take a term $\boldsymbol{Z}^{\boldsymbol{m}} a_{\boldsymbol{m}}(\varrho(z))$ in the first sum in (5-33). Notice that, by Lemma 2.6, we have necessarily $|\boldsymbol{m}| \geq 2$. Furthermore, by (5-21) it is easy to see that we can omit the factor $a_{\boldsymbol{m}}(\varrho(z))$. For definiteness, let $\boldsymbol{Z}^{\boldsymbol{m}} = |z_1|^2|z_2|^2$ (so $|\boldsymbol{m}| = 2$; the case $|\boldsymbol{m}| > 2$ is simpler). By (5-21) we have

$$|z_1'|^2|z_2'|^2 = (|z_1|^2 + \mathcal{R}^{0,\iota+1}_{\boldsymbol{r},\infty})(|z_2|^2 + \mathcal{R}^{0,\iota+1}_{\boldsymbol{r},\infty}) = |z_1|^2|z_2|^2 + R^{2,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta),$$

where we used information, such as $\widetilde{\mathcal{T}}_j = \mathcal{R}^{1,2\iota}_{\boldsymbol{r},\infty}$, contained in Lemma 5.6 and the fact, easy to check, that $\bar{z}_j(Y_\chi)_j(z, \eta) + z_j(Y_\chi)_{\bar{j}}(z, \eta) = R^{0,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta)$.

To complete the proof of (5-46) let us take now a term of the form $\bar{z}_2\boldsymbol{Z}^{\boldsymbol{m}}\langle G(|z_2|^2), \eta\rangle$. Here we can write $G = G(|z_2|^2)$, ignoring the dependence on $|z_2|^2$ and we can focus on $|\boldsymbol{m}| = 1$. For definiteness, let $\boldsymbol{Z}^{\boldsymbol{m}} = z_1\bar{z}_2$. By Lemma 5.6,

$$z_1'(\bar{z}_2')^2\langle G, \eta'\rangle = (z_1 + \mathcal{R}^{1,\iota}_{\boldsymbol{r},\infty})(\bar{z}_2 + \mathcal{R}^{1,\iota}_{\boldsymbol{r},\infty})^2\langle G, \eta + S^{1,\iota}_{\boldsymbol{r},\infty}\rangle,$$

which for $\iota > 1$ is of the form $z_1\bar{z}_2^2\langle G, \eta\rangle + (*)$, and for $\iota = 1$, using (5-20), yields (5-46).

By Lemma 5.4(1) and $d_\eta\mathcal{R}^{1,2}_{r,\infty}(z, \eta)\cdot \boldsymbol{S}^{1,\iota}_{\boldsymbol{r},\infty}(b, B, z, \eta) = \mathcal{R}^{2,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta)$, we get

$$\begin{aligned}
\mathcal{R}^{1,2}_{r,\infty}(z', \eta') &= \mathcal{R}^{1,2}_{r,\infty}(z, \eta') + (*)\\
&= \mathcal{R}^{1,2}_{r,\infty}(z, \eta) + (*) + \int_0^1 d_\eta\mathcal{R}^{1,2}_{r,\infty}(z, \eta + \tau \boldsymbol{S}^{1,\iota}_{\boldsymbol{r},\infty}(b, B, z, \eta))\cdot \boldsymbol{S}^{1,\iota}_{\boldsymbol{r},\infty}(b, B, z, \eta)\, d\tau\\
&= \mathcal{R}^{1,2}_{r,\infty}(z, \eta) + d_\eta\mathcal{R}^{1,2}_{r,\infty}(z, \eta)\cdot \boldsymbol{S}^{1,\iota}_{\boldsymbol{r},\infty}(b, B, z, \eta) + (*).
\end{aligned} \tag{5-47}$$

Like in (5-47) and using (5-20) and $G_{\boldsymbol{2}\boldsymbol{0}ij}(z) = \mathcal{R}^{2,0}_{\infty,\infty}(z)$ — see (3-4) — we have

$$\begin{aligned}
\sum_{i+j=2}\langle G_{\boldsymbol{2}\boldsymbol{0}ij}(z'), \eta'^i\,\bar{\eta}'^j\rangle &= \sum_{i+j=2}\langle G_{\boldsymbol{2}\boldsymbol{0}ij}(z), \eta'^i\,\bar{\eta}'^j\rangle + (*)\\
&= \sum_{i+j=2}\langle G_{\boldsymbol{2}\boldsymbol{0}ij}(z), \eta^i\,\bar{\eta}^j\rangle + \mathcal{R}^{3,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta) + (*).
\end{aligned} \tag{5-48}$$

Therefore, we seek χ_ι such that the following holds, with $\varrho(z) = (|z_1|^2, \dots, |z_n|^2)$ and the notation in (5-42):

$$\begin{aligned}
(*) = &\sum_{\substack{|\boldsymbol{m}|=\iota+1\\ \boldsymbol{m}\notin\mathcal{M}_0(\iota+1)}} \mathrm{i}\tilde{\boldsymbol{e}}\cdot(\mu(\boldsymbol{m}) - \nu(\boldsymbol{m}))b_{\boldsymbol{m}}(\varrho(z))\boldsymbol{Z}^{\boldsymbol{m}}\\
&+\sum_j\sum_{\substack{|\boldsymbol{m}|=\iota\\ \boldsymbol{m}\notin\mathcal{M}_j(\iota)}}\big(\mathrm{i}\langle(\tilde{\boldsymbol{e}}\cdot(\mu_j(\boldsymbol{m}) - \nu_j(\boldsymbol{m})) + H)B_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle\bar{z}_j\boldsymbol{Z}^{\boldsymbol{m}} + \text{c.c.}\big) + \mathcal{R}^{2,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta)\\
&+\sum_{\substack{|\boldsymbol{m}|=\iota+1\\ \boldsymbol{m}\notin\mathcal{M}_0(\iota+1)}}\boldsymbol{Z}^{\boldsymbol{m}}a^{(\iota)}_{\boldsymbol{m}}(\varrho(z)) + \sum_{j=1}^n\sum_{\substack{|\boldsymbol{m}|=\iota\\ \boldsymbol{m}\notin\mathcal{M}_j(\iota)}}\big(\bar{z}_j\boldsymbol{Z}^{\boldsymbol{m}}\langle G^{(\iota)}_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.}\big).
\end{aligned} \tag{5-49}$$

By a Taylor expansion, we can write

$$\begin{aligned}
&\mathcal{R}^{2,\iota+1}_{\boldsymbol{r},\infty}(b, B, z, \boldsymbol{Z}, \eta)\\
&\quad = (*) + \sum_{\substack{|\boldsymbol{m}|=\iota+1\\ \boldsymbol{m}\notin\mathcal{M}_0(\iota+1)}} \boldsymbol{Z}^{\boldsymbol{m}} \alpha_{\boldsymbol{m}}(b, B, \varrho(z))\\
&\qquad + \sum_{j=1}^{n} \sum_{\substack{|\boldsymbol{m}|=\iota\\ \boldsymbol{m}\notin\mathcal{M}_j(\iota)}} \big(\bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle \Gamma_{j\boldsymbol{m}}(b(0,\dots,|z_j|^2,0,\dots,0), B(0,\dots,|z_j|^2,0,\dots,0), |z_j|^2), \eta\rangle + \text{c.c.}\big),
\end{aligned}$$

where $\alpha_{\boldsymbol{m}}(b, B, \varrho(z)) = \mathcal{R}^{1,0}_{\boldsymbol{r},\infty}(b, B, \varrho(z))$ and

$$\begin{aligned}
&\Gamma_{j\boldsymbol{m}}(b(0,\dots,|z_j|^2,0,\dots,0), B(0,\dots,|z_j|^2,0,\dots,0), |z_j|^2)\\
&\qquad\qquad = S^{1,0}_{\boldsymbol{r},\infty}(b(0,\dots,|z_j|^2,0,\dots,0), B(0,\dots,|z_j|^2,0,\dots,0), |z_j|^2).
\end{aligned}$$

Furthermore, by (5-42) and $\varpi_j(|z_j|^2) = \mathcal{R}^{2,0}_{r_0,\infty}(|z_j|^2)$, the sum in the second line of (5-49) has an expansion

$$\sum_j \sum_{\substack{|\boldsymbol{m}|=\iota\\ \boldsymbol{m}\notin\mathcal{M}_j(\iota)}} \big(\mathrm{i}\langle(\boldsymbol{e}\cdot(\mu_j(\boldsymbol{m})-\nu_j(\boldsymbol{m})) + \mathcal{R}^{1,0}_{r_0,\infty}(|z_j|^2) + H) B_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle \bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} + \text{c.c.}\big) + (*).$$

Then we reduce to the following system:

$$\begin{aligned}
b_{\boldsymbol{m}}(\varrho(z)) &= \frac{\mathrm{i}}{\tilde{\boldsymbol{e}}(z)\cdot(\mu(\boldsymbol{m})-\nu(\boldsymbol{m}))}\big[a^{(\iota)}_{\boldsymbol{m}}(\varrho(z)) + \alpha_{\boldsymbol{m}}\big((b_{\boldsymbol{n}}(\varrho(z)))_{\boldsymbol{n}}, (B_{j\boldsymbol{n}}(\varrho_j(z)))_{j\boldsymbol{n}}, \varrho(z)\big)\big],\\
B_{j\boldsymbol{m}}(|z_j|^2) &= \mathrm{i}R_H\big(\boldsymbol{e}\cdot(\mu_j(\boldsymbol{m})-\nu_j(\boldsymbol{m})) + \mathcal{R}^{1,0}_{r_0,\infty}(|z_j|^2)\big)\\
&\qquad \times \big[G^{(\iota)}_{j\boldsymbol{m}}(|z_j|^2) + \Gamma_{j\boldsymbol{m}}\big(b(0,\dots,|z_j|^2,0,\dots,0), B(0,\dots,|z_j|^2,0,\dots,0), |z_j|^2\big)\big].
\end{aligned} \tag{5-50}$$

The $b_{\boldsymbol{m}}(\varrho(z))$ and $B_{j\boldsymbol{m}}(|z_j|^2)$ can be found by the implicit function theorem for $|z| < \delta'_\iota$ for δ'_ι sufficiently small. This gives us the desired polynomial χ, yielding $H^{(\iota+1)}$. Formulas (5-31) for the flow ϕ^t of χ are obtained choosing $\delta_\iota > 0$ sufficiently small, by Lemma 5.4(1). For the composition $\mathcal{F}^{(2N+4)}$, we obtain (5-34) as a consequence of (5-31) and of (4-44). □

6. Dispersion

We apply Theorem 5.9, set $\mathcal{H} = H^{(2N+4)}$, so that

$$\mathcal{H}(z,\eta) = H_2(z,\eta) + \sum_{j=1}^{n} \lambda_j(|z_j|^2) + Z^{(2N+4)}(z, \boldsymbol{Z}, \eta) + \mathcal{R}, \tag{6-1}$$

where

$$\begin{aligned}
\mathcal{R} &:= \mathcal{R}^{1,2}_{r,\infty}(z,\eta) + \mathcal{R}^{0,2N+5}_{r,\infty}(z,\boldsymbol{Z},\eta) + \mathrm{Re}\langle \boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z},\eta), \bar{\eta}\rangle\\
&\quad + \sum_{i+j=2}\sum_{|\boldsymbol{m}|\le 1} \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{2\boldsymbol{m}ij}(z,\eta), \eta^i \bar{\eta}^j\rangle + \sum_{d+c=3}\sum_{i+j=d} \langle G_{dij}(z,\eta), \eta^i\bar{\eta}^j\rangle \mathcal{R}^{0,c}_{r,\infty}(z,\eta) + E_P(\eta).
\end{aligned} \tag{6-2}$$

Using formula (5-33) for $\iota = 2N+4$, we have

$$\sum_{j=1}^{n} \lambda_j(|z_j|^2) + Z^{(2N+4)}(z, \boldsymbol{Z}, \eta) = Z_0(z) + \sum_{j=1}^{n}\biggl(\sum_{\boldsymbol{m}\in\mathcal{M}_j(2N+3)} \bar{z}_j \boldsymbol{Z}^{\boldsymbol{m}} \langle G_{j\boldsymbol{m}}(|z_j|^2), \eta\rangle + \text{c.c.}\biggr) \tag{6-3}$$

$$\begin{aligned}\text{with}\quad Z_0(z) &:= \sum_{j=1}^{n} \lambda_j(|z_j|^2) + \sum_{\boldsymbol{m}\in\mathcal{M}_0(2N+4)} \boldsymbol{Z}^{\boldsymbol{m}} a_{\boldsymbol{m}}(|z_1|^2, \dots, |z_n|^2)\\ &= \mathcal{Z}_0(|z_1|^2, \dots, |z_n|^2),\end{aligned}$$

where the last equality holds for some $\mathcal{Z}_0(|z_1|^2, \dots, |z_n|^2)$ by Lemma 2.6.

Theorem 6.1 (main estimates). *There exist $\epsilon_0 > 0$ and $C_0 > 0$ such that, if the constant $0 < \epsilon$ of Theorem 1.3 satisfies $\epsilon < \epsilon_0$, then for $I = [0, \infty)$ and $C = C_0$ we have*

$$\|\eta\|_{L_t^p(I, W_x^{1,q})} \le C\epsilon \quad \textit{for all admissible pairs } (p, q), \tag{6-4}$$

$$\|z_j \boldsymbol{Z}^{\boldsymbol{m}}\|_{L_t^2(I)} \le C\epsilon \quad \textit{for all } (j, \boldsymbol{m}) \textit{ with } \boldsymbol{m} \in \mathcal{M}_j(2N+4), \tag{6-5}$$

$$\|z_j\|_{W_t^{1,\infty}(I)} \le C\epsilon \quad \textit{for all } j \in \{1, \dots, n\}. \tag{6-6}$$

Furthermore, there exists $\rho_+ \in [0, \infty)^n$ and a j_0 with $\rho_{+j} = 0$ for $j \ne j_0$, and an $\eta_+ \in H^1$ such that $\bigl|\rho_+ - |z(0)|\bigr| \le C\epsilon$ and $\|\eta_+\|_{H^1} \le C\epsilon$, such that

$$\lim_{t\to+\infty} \|\eta(t, x) - e^{\mathrm{i}t\Delta}\eta_+(x)\|_{H_x^1} = 0, \qquad \lim_{t\to+\infty} |z_j(t)| = \rho_{+j}. \tag{6-7}$$

Proof that Theorem 6.1 implies Theorem 1.3. Denote by (z', η') the initial coordinate system. By (5-35),

$$z' = z + \mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) \quad\text{and}\quad \eta' = \eta + \boldsymbol{S}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta).$$

Notice that (6-7) implies $\lim_{t\to+\infty} \boldsymbol{Z}(t) = 0$, and by standard arguments for $s > \frac{3}{2}$ we have

$$\lim_{t\to+\infty} \|e^{t\Delta}\eta_+\|_{L^{2,-s}(\mathbb{R}^3)} = 0 \quad \text{for any } \eta_+ \in L^2. \tag{6-8}$$

These two limits, Definitions 2.8–2.9 and (6-7) imply

$$\lim_{t\to+\infty} \mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) = 0 \quad \text{in } \mathbb{C}^n \qquad\text{and}\qquad \lim_{t\to+\infty} \boldsymbol{S}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) = 0 \quad \text{in } \Sigma_r.$$

This means that

$$\lim_{t\to+\infty} \|\eta'(t, x) - e^{\mathrm{i}t\Delta}\eta_+(x)\|_{H_x^1} = 0 \quad\text{and}\quad \lim_{t\to+\infty} |z'_j(t)| = \rho_{+j}, \tag{6-9}$$

so that (1-8) is true. Notice also that if we set $\tilde{\eta} = \eta$ and $A(t, x) = \boldsymbol{S}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta)$, we obtain the desired decomposition of η' satisfying (1-9) and (1-10). Finally, we have

$$\dot{z}'_j + \mathrm{i}e_j z'_j = \dot{z}_j + \mathrm{i}e_j z_j + \frac{d}{dt}\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) + \mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) = O(\epsilon^2),$$

where $\dot{z}_j + \mathrm{i}e_j z_j = O(\epsilon^2)$ by (6-27) below, $\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) = O(\epsilon^2)$ by (2-23) and $d\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta)/dt = O(\epsilon^2)$. To check the last of these, we write (it is easy that $d_w\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) = \mathcal{R}^{1,0}_{r,\infty}(z, \boldsymbol{Z}, \eta)$ for $w = z, \boldsymbol{Z}$)

$$\frac{d}{dt}\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta) = \mathcal{R}^{1,0}_{r,\infty}(z, \boldsymbol{Z}, \eta)\dot{z} + \mathcal{R}^{1,0}_{r,\infty}(z, \boldsymbol{Z}, \eta)\dot{\boldsymbol{Z}} + d_\eta\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta)\cdot\dot{\eta},$$

with $d_\eta\mathcal{R}^{1,1}_{r,\infty}$ the partial derivative in η. By a simple use of Taylor expansions and Definition 2.8,

$$\|d_\eta\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta)\|_{\Sigma^c_{-r}\to\Sigma^c_r} \le C(|z| + \|\eta\|_{\Sigma_{-r}}).$$

Then, by equations (6-12) and (6-27) below, we have $d\mathcal{R}^{1,1}_{r,\infty}(z, \boldsymbol{Z}, \eta)/dt = O(\epsilon^2)$. This yields the second inequality claimed in (1-9). □

By a standard argument, (6-4)–(6-6) for $I = [0, \infty)$ are a consequence of the following proposition:

Proposition 6.2. *There exists a constant $c_0 > 0$ such that, for any $C_0 > c_0$, there is a value $\epsilon_0 = \epsilon_0(C_0)$ such that, if the inequalities* (6-4)–(6-6) *hold for $I = [0, T]$ for some $T > 0$, for $C = C_0$ and for $0 < \epsilon < \epsilon_0$, then, in fact, for $I = [0, T]$ the inequalities* (6-4)–(6-6) *hold for $C = C_0/2$.*

Proof. We will proceed via a series of lemmas.

Lemma 6.3. *Assume the hypotheses of Proposition 6.2 and take the M of Definition 2.5. Then there is a fixed c such that*

$$\|\eta\|_{L^p_t([0,T],W^{1,q})} \le c\epsilon + c\sum_{(\mu,\nu)\in M}|z^\mu\bar{z}^\nu|_{L^2_t(0,T)} \quad \textit{for all admissible pairs } (p, q). \tag{6-10}$$

Proof. First of all, for $|z| < \delta_f$ and $\|\eta\|_{H^1\cap\mathcal{H}_c[0]} < \delta_f$ defining the domain of the Hamiltonian $\mathcal{H}(z, \eta)$ in (6-1), we will pick $\epsilon_0 \in (0, \delta_f)$ sufficiently small. Let $\epsilon \in (0, \epsilon_0)$, where $\epsilon = \|u(0)\|_{H^1}$. By (2-11), we have $|z'(0)| + \|\eta'(0)\|_X \le c_1\epsilon$, where $(z'(0), \eta'(0))$ are the coordinates in the initial system of coordinates introduced in Lemma 2.4. Let $(z(0), \eta(0))$ be the corresponding coordinates in the final system of coordinates. Then, by the relation (5-35), if ϵ_0 is sufficiently small we conclude that

$$|z(0)| + \|\eta(0)\|_{H^1} \le c_1'\epsilon \tag{6-11}$$

for some other fixed constant c_1'. We now turn to the equation of η. We have, for $\bar{G}_{j\boldsymbol{m}} = \bar{G}_{j\boldsymbol{m}}(0)$,

$$\mathrm{i}\dot{\eta} = \mathrm{i}\{\eta, \mathcal{H}\} = H\eta + \sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l} z_j\bar{\boldsymbol{Z}}^{\boldsymbol{m}}\bar{G}_{j\boldsymbol{m}} + \mathbb{A}, \tag{6-12}$$

where

$$\mathbb{A} := \sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l} z_j\bar{\boldsymbol{Z}}^{\boldsymbol{m}}[\bar{G}_{j\boldsymbol{m}}(|z_j|^2) - \bar{G}_{j\boldsymbol{m}}] + \nabla_{\bar{\eta}}\mathcal{R}.$$

We rewrite

$$\sum_{j=1}^{n}\sum_{l=1}^{2N+3}\sum_{|\boldsymbol{m}|=l} z_j\bar{\boldsymbol{Z}}^{\boldsymbol{m}}\bar{G}_{j\boldsymbol{m}} = \sum_{(\mu,\nu)\in M}\bar{z}^\mu z^\nu\bar{G}_{\mu\nu}. \tag{6-13}$$

Notice that (6-5) is the same as

$$\|z^{\mu}\bar{z}^{\nu}\|_{L^2_t(I)} \le C\epsilon \quad \text{for all } (\mu,\nu)\in M. \tag{6-14}$$

Suppose we can show that, for $I_T := [0,T]$,

$$\|\mathbb{A}\|_{L^2(I_T,H^{1,S})+L^1(I_T,H^1)} \le C(S,C_0)\epsilon^2. \tag{6-15}$$

Then, if ϵ_0 is small enough and $\epsilon\in(0,\epsilon_0)$, we obtain (6-10) by $H^{1,S}(\mathbb{R}^3)\hookrightarrow W^{1,6/5}(\mathbb{R}^3)$, by (6-11), (6-14) and (6-15) and by the Strichartz estimates, which, for P_c the orthogonal projection of L^2 onto $\mathcal{H}[0]$, are valid for P_cH by [Yajima 1995] (here notice that all the terms in (6-12) belong to $\mathcal{H}[0]$).

So now we prove (6-15). We have, for $r-1\ge S>\frac{9}{2}$,

$$\begin{aligned}\|z_j\bar{\boldsymbol{Z}}^{\boldsymbol{m}}[\bar{G}_{j\boldsymbol{m}}(|z_j|^2)-\bar{G}_{j\boldsymbol{m}}]\|_{L^2(I_T,H^{1,S})} &\le \|z_j\bar{\boldsymbol{Z}}^{\boldsymbol{m}}\|_{L^2(I_T,\mathbb{C})}\|\bar{G}_{j\boldsymbol{m}}(|z_j|^2)-\bar{G}_{j\boldsymbol{m}}\|_{L^\infty(I_T,H^{1,S})}\\ &\le C_0\epsilon\sup\{\|G'_{j\boldsymbol{m}}(|z_j|^2)\|_{\Sigma_r} : |z_j|\le\delta_0\}\|z_j^2\|_{L^\infty(I_T,\mathbb{C})}\\ &\le CC_0^3\epsilon^3 < c\epsilon.\end{aligned} \tag{6-16}$$

We have, for a fixed $c_1>0$,

$$\|\nabla_\eta E_P(\eta)\|_{L^1(I_T,H^1)} = 2\||\eta|^2\eta\|_{L^1(I_T,H^1)} \le c_1\|\eta\|_{L^\infty(I_T,H^1)}\|\eta\|^2_{L^2(I_T,L^6)} \le c_1C_0^3\epsilon^3. \tag{6-17}$$

We finally show that, for an arbitrarily preassigned $S>2$,

$$\|R_1\|_{L^2(I_T,H^{1,S})} \le C(S,C_0)\epsilon^2 \quad \text{for } R_1 = \nabla_\eta(\mathcal{R}-E_P(\eta)). \tag{6-18}$$

R_1 is a sum of various terms obtained from the expansion (6-2). Let us start by showing

$$\|\nabla_{\bar\eta}\mathcal{R}^{1,2}_{r,\infty}(z,\eta)\|_{L^2(I_T,H^{1,S})} \le C(S,C_0)\epsilon^2. \tag{6-19}$$

Recalling (2-25), it is elementary to show that $\nabla_{\bar\eta}\mathcal{R}^{1,2}_{r,\infty}(z,\eta) = \boldsymbol{S}^{1,1}_{r,\infty}(z,\eta)$ and

$$\begin{aligned}\|\boldsymbol{S}^{1,1}_{r,\infty}(z,\eta)\|_{L^2(I_T,H^{1,S})} &\le C_1\|(\|\eta\|_{\Sigma_{-r}}+|z|)\|_{L^\infty(I_T)}\|\eta\|_{L^2(I_T,\Sigma_{-r})}\\ &\le C_2\|(\|\eta\|_{H^1}+|z|)\|_{L^\infty(I_T)}\|\eta\|_{L^2(I_T,L^6)} \le C(S,C_0)\epsilon^2.\end{aligned}$$

We next show

$$\|\nabla_{\bar\eta}\mathcal{R}^{0,2N+5}_{r,\infty}(z,\boldsymbol{Z},\eta)\|_{L^2(I_T,H^{1,S})} \le C(S,C_0)\epsilon^2. \tag{6-20}$$

We have, for a remainder $\|O(\|\eta\|^2_{\Sigma_{-r}})\|_{\Sigma_r} \le C\|\eta\|^2_{\Sigma_{-r}}$,

$$\nabla_{\bar\eta}\mathcal{R}^{0,2N+5}_{r,\infty}(z,\boldsymbol{Z},\eta) = \boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z},\eta) = \boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z}) + d_\eta\boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z},0)\cdot\eta + O(\|\eta\|^2_{\Sigma_{-r}}).$$

We have, by Lemma 2.7,

$$\begin{aligned}\|\boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z})\|_{L^2(I_T,H^{1,S})} &\le C_1\sup_{|z|\le C_0\epsilon}\|\boldsymbol{S}^{0,0}_{r,\infty}(z,\boldsymbol{Z})\|_{\Sigma_{M'}}\||\boldsymbol{Z}|^{2N+4}\|_{L^2(I_T)}\\ &\le C_2\|z\|_{L^\infty(I)}\sum_j\sum_{(\mu,\nu)\in M_j(N+1)}\|z^\mu\bar{z}^\nu\|_{L^\infty(I_T)}\|z^\mu\bar{z}^\nu\|_{L^2(I_T)}\\ &\le C(S,C_0)\epsilon^3.\end{aligned}$$

We have

$$\begin{aligned}\|d_\eta \boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z},0)\cdot\eta\|_{L^2(I_T,H^{1,S})} &\le C_1(S)\|\eta\|_{L^2(I_T,\Sigma_{-r})}\sup_{|z|\le C_0\epsilon}\|d_\eta \boldsymbol{S}^{0,2N+4}_{r,\infty}(z,\boldsymbol{Z},0)\|_{\Sigma_{-r}\to\Sigma_r}\\ &\le C_2(S)\|\eta\|_{L^2(I_T,L^6)}\sup_{|z|\le C_0\epsilon}|\boldsymbol{Z}|^{2N+3}\\ &\le C(S,C_0)\epsilon^2.\end{aligned}$$

Hence (6-20) is proved. Other terms in R_1 can be bounded with similarly elementary arguments, yielding (6-18). Then (6-16), (6-17) and (6-18) imply (6-15). □

Setting $M = M(2N+4)$ — see Definition 2.5 — we now introduce a new variable g, setting

$$g=\eta+Y \quad\text{with}\quad Y:=\sum_{(\alpha,\beta)\in M}\bar z^\alpha z^\beta R^+_H(\boldsymbol{e}\cdot(\beta-\alpha))\bar G_{\alpha\beta}. \tag{6-21}$$

Lemma 6.4. *Assume the hypotheses of Proposition 6.2 and fix $S>\frac{9}{2}$. Then there is a $c_1(S)>0$ such that, for any C_0, there is an $\epsilon_0=\epsilon_0(C_0,S)>0$ such that, for $\epsilon\in(0,\epsilon_0)$ in Theorem 1.3, we have*

$$\|g\|_{L^2([0,T],L^{2,-S})}\le c_1(S)\epsilon. \tag{6-22}$$

Proof. We have

$$\mathrm{i}\dot g = Hg+\mathbb{A}+\boldsymbol{T},\quad\text{where}\quad \boldsymbol{T}:=\sum_j[\partial_{z_j}Y(\mathrm{i}\dot z_j-e_jz_j)+\partial_{\bar z_j}Y(\mathrm{i}\dot{\bar z}_j+e_j\bar z_j)]. \tag{6-23}$$

We then have

$$g(t)=e^{-\mathrm{i}Ht}\eta(0)+e^{-\mathrm{i}Ht}Y(0)-\mathrm{i}\int_0^t e^{-\mathrm{i}H(t-s)}(\mathbb{A}(s)+\boldsymbol{T}(s))\,ds. \tag{6-24}$$

We have, for fixed constants, by (6-11) and (6-15), the inequalities

$$\|e^{-\mathrm{i}Ht}\eta(0)\|_{L^2([0,T],L^{2,-S})}a\le c_2\|e^{-\mathrm{i}Ht}\eta(0)\|_{L^2([0,T],L^6)}\le c_2'\|\eta(0)\|_{L^2}\le c_3\epsilon,$$

$$\left\|\int_0^t e^{-\mathrm{i}H(t-s)}\mathbb{A}(s)\,ds\right\|_{L^2([0,T],L^{2,-S})}\le c_2\|\mathbb{A}\|_{L^2([0,T],H^{1,S})+L^1([0,T],H^1)}\le C(C_0,S)\epsilon^2.$$

For a proof of the following standard lemma see, for instance, the proof of [Cuccagna 2003, Lemma 5.4].

Lemma 6.5. *Let Λ be a compact subset of $(0,\infty)$ and let $S>\frac{9}{2}$. Then there exists a fixed $c(S,\Lambda)$ such that, for every $t\ge 0$ and $\lambda\in\Lambda$,*

$$\|e^{-\mathrm{i}Ht}R^+_H(\lambda)P_cv_0\|_{L^{2,-S}(\mathbb{R}^3)}\le c(S,\Lambda)\langle t\rangle^{-3/2}\|P_cv_0\|_{L^{2,S}(\mathbb{R}^3)}\quad\textit{for all }\ v_0\in L^{2,S}(\mathbb{R}^3).$$

By Lemma 6.5, (6-11) and $G_{\alpha\beta}=P_cG_{\alpha\beta}$, we have

$$\begin{aligned}\|e^{-\mathrm{i}Ht}Y(0)\|_{L^2([0,T],L^{2,-S})} &\le \sum_{(\alpha,\beta)\in M}|z^\alpha(0)z^\beta(0)|\,\|e^{-\mathrm{i}Ht}R^+_H(\boldsymbol{e}\cdot(\beta-\alpha))\bar G_{\alpha\beta}\|_{L^2([0,T],L^{2,-S})}\\ &\le(\sharp M)c_2\epsilon^2\|\langle t\rangle^{-3/2}\|_{L^2(0,T)}c(S,\Lambda)\|\bar G_{\alpha\beta}\|_{L^{2,S}}\le C(N,C_0,S)\epsilon^2\end{aligned}$$

with $\sharp M$ the cardinality of M and a fixed c_2, and where the set Λ is as in Lemma 6.5,

$$\Lambda := \{(\nu-\mu)\cdot \boldsymbol{e} : (\mu,\nu)\in M\}. \tag{6-25}$$

We finally consider, for definiteness (the term $\partial_{\bar z_j} Y(\mathrm{i}\dot{\bar z}_j + e_j \bar z_j)$ can be treated similarly),

$$\begin{aligned}
&\left\| \int_0^t e^{-\mathrm{i}H(t-s)} R_H^+(\boldsymbol{e}\cdot(\beta-\alpha))\overline{G}_{\alpha\beta}\partial_{z_j} Y(s)(\mathrm{i}\dot z_j - e_j z_j)(s)\,ds\right\|_{L^2([0,T],L^{2,-S})}\\
&\quad \le c(S,\Lambda)\sum_{(\alpha,\beta)\in M}\|G_{\alpha\beta}\|_{L^{2,S}}\beta_j \left\| \int_0^t \langle t-s\rangle^{-3/2}\left|\frac{\bar z^\alpha(s) z^\beta(s)}{z_j(s)}(\mathrm{i}\dot z_j - e_j z_j)(s)\right| ds\right\|_{L^2(0,T)}\\
&\quad \le c(S,\Lambda)c_2 \sum_{(\alpha,\beta)\in M}\beta_j\left\|\frac{\bar z^\alpha(s) z^\beta}{z_j}(\mathrm{i}\dot z_j - e_j z_j)\right\|_{L^2(0,T)}
\end{aligned} \tag{6-26}$$

for fixed c_2. We have

$$\begin{aligned}
\mathrm{i}\dot z_j &= (1+\varpi_j(|z_j|^2))(e_j z_j + \partial_{\bar z_j}\mathscr{Z}_0(|z_1|^2,\dots,|z_n|^2) + \partial_{\bar z_j}\mathscr{R})\\
&\quad + (1+\varpi_j(|z_j|^2))\Bigg[\sum_{(\mu,\nu)\in M}\nu_j \frac{z^\mu \bar z^\nu}{\bar z_j}\langle \eta, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M}\mu'_j \frac{z^{\nu'}\bar z^{\mu'}}{\bar z_j}\langle \bar\eta, \overline{G}_{\mu'\nu'}\rangle\Bigg]\\
&\quad + (1+\varpi_j(|z_j|^2))\Bigg[\sum_{\boldsymbol{m}\in\mathcal{M}_j(2N+3)} |z_j|^2 \boldsymbol{Z}^{\boldsymbol{m}}\langle G'_{j\boldsymbol{m}},\eta\rangle + z_j^2 \overline{\boldsymbol{Z}}^{\boldsymbol{m}}\langle \overline{G}'_{j\boldsymbol{m}},\bar\eta\rangle\Bigg].
\end{aligned} \tag{6-27}$$

To bound (6-26), we substitute $(\mathrm{i}\dot z_j - e_j z_j)$ by the other terms in (6-27) in the last line of (6-26) . So, for example, we have $\partial_{\bar z_j}\mathscr{Z}_0(|z_1|^2,\dots,|z_n|^2)\sim z_j O(\epsilon)$, which by (6-14) yields

$$\beta_j\left\|\frac{\bar z^\alpha z^\beta}{z_j}\partial_{\bar z_j}\mathscr{Z}_0(|z_1|^2,\dots,|z_n|^2)\right\|_{L^2(0,T)} \le C(C_0)\epsilon \|\bar z^\alpha z^\beta\|_{L^2(0,T)} \le C(C_0)C_0\epsilon^2.$$

For $(\mu,\nu)\in M$, we have, in $(0,T)$,

$$\beta_j\nu_j\left\|\frac{\bar z^\alpha z^\beta}{z_j}\frac{z^\mu\bar z^\nu}{\bar z_j}\langle\eta,G_{\mu\nu}\rangle\right\|_{L^2_t} \le \beta_j\nu_j\left\|\frac{\bar z^\alpha z^\beta}{z_j}\frac{z^\mu\bar z^\nu}{\bar z_j}\right\|_{L^\infty_t}\|G_{\mu\nu}\|_{L^{\frac{6}{5}}}\|\eta\|_{L^\infty_t L^6} \le C(C_0)\epsilon^2.$$

A similar argument works for the terms in the second summation in the second line of (6-27). Finally,

$$\beta_j\left\|\frac{\bar z^\alpha z^\beta}{z_j}\partial_{\bar z_j}\mathscr{R}\right\|_{L^2(0,T)} \le \beta_j\left\|\frac{\bar z^\alpha z^\beta}{z_j}\right\|_{L^\infty(0,T)}\|\partial_{\bar z_j}\mathscr{R}\|_{L^2(0,T)} \le C(C_0)\epsilon^3$$

is a consequence of the bound

$$\|\partial_{\bar z_j}\mathscr{R}\|_{L^p(0,T)} \le C(C_0)\epsilon^2 \quad \text{for any } p\in[1,\infty]. \tag{6-28}$$

Here we need to check (6-28) term by term for the sum in the right-hand side of (6-2). This is straightforward using (2-23), (2-25) and (2-26) and the fact, stated in Lemma 5.8, that $G_{2\boldsymbol{m}ij}$ and G_{dij} are $\boldsymbol{S}^{0,0}_{r,\infty}$. $\square$

We turn now to the Fermi golden rule (FGR). We substitute (6-21) into (6-27), getting

$$\begin{aligned} \mathrm{i}\dot{z}_j = (1+\varpi_j(|z_j|^2))(e_j z_j + \partial_{\bar{z}_j}\mathscr{Z}_0(|z_1|^2,\dots,|z_n|^2)) - \sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} \nu_j \frac{z^{\mu+\beta}\bar{z}^{\nu+\alpha}}{\bar{z}_j}\langle R_H^+(\boldsymbol{e}\cdot(\beta-\alpha))\bar{G}_{\alpha\beta}, G_{\mu\nu}\rangle \\ - \sum_{\substack{(\mu',\nu')\in M\\(\alpha',\beta')\in M}} \mu'_j \frac{z^{\nu'+\alpha'}\bar{z}^{\mu'+\beta'}}{\bar{z}_j}\langle R_H^-(\boldsymbol{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \bar{G}_{\mu'\nu'}\rangle + \mathscr{F}_j, \quad (6\text{-}29) \end{aligned}$$

where

$$\begin{aligned} \mathscr{F}_j := (1+\varpi_j(|z_j|^2))\partial_{\bar{z}_j}\mathscr{R} + \varpi_j(|z_j|^2)\Bigg[\sum_{(\mu,\nu)\in M}\nu_j\frac{z^\mu\bar{z}^\nu}{\bar{z}_j}\langle\eta, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M}\mu'_j\frac{z^{\nu'}\bar{z}^{\mu'}}{\bar{z}_j}\langle\bar{\eta}, \bar{G}_{\mu'\nu'}\rangle\Bigg] \\ + \sum_{(\mu,\nu)\in M}\nu_j\frac{z^\mu\bar{z}^\nu}{\bar{z}_j}\langle g, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M}\mu'_j\frac{z^{\nu'}\bar{z}^{\mu'}}{\bar{z}_j}\langle\bar{g}, \bar{G}_{\mu'\nu'}\rangle \\ + (1+\varpi_j(|z_j|^2))\Bigg[\sum_{\boldsymbol{m}\in\mathcal{M}_j(2N+3)}|z_j|^2\boldsymbol{Z}^{\boldsymbol{m}}\langle G'_{j\boldsymbol{m}}, \eta\rangle + z_j^2\bar{\boldsymbol{Z}}^{\boldsymbol{m}}\langle\bar{G}'_{j\boldsymbol{m}}, \bar{\eta}\rangle\Bigg]. \quad (6\text{-}30) \end{aligned}$$

We introduce the new variable ζ, defined by

$$\begin{aligned} z_j - \zeta_j = -\sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} \frac{\nu_j z^{\mu+\beta}\bar{z}^{\nu+\alpha}}{((\mu-\nu)\cdot\boldsymbol{e}-(\alpha-\beta)\cdot\boldsymbol{e})\bar{z}_j}\langle R_H^+(\boldsymbol{e}\cdot(\beta-\alpha))\bar{G}_{\alpha\beta}, G_{\mu\nu}\rangle \\ - \sum_{\substack{(\mu',\nu')\in M\\(\alpha',\beta')\in M}} \frac{\mu'_j z^{\nu'+\alpha'}\bar{z}^{\mu'+\beta'}}{((\alpha'-\beta')\cdot\boldsymbol{e}-(\mu'-\nu')\cdot\boldsymbol{e})\bar{z}_j}\langle R_H^-(\boldsymbol{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \bar{G}_{\mu'\nu'}\rangle, \quad (6\text{-}31) \end{aligned}$$

where we are summing only on pairs where the formula makes sense (i.e., only on pairs not in the same set M_L for an $L\in\Lambda$; see (6-33) below). It is easy to see that

$$\|\zeta - z\|_{L^2(0,T)} \le c(N, C_0)\epsilon^2 \quad\text{and}\quad \|\zeta - z\|_{L^\infty(0,T)} \le c(N, C_0)\epsilon^2. \quad (6\text{-}32)$$

Recall now the set $\Lambda = \{(\nu-\mu)\cdot\boldsymbol{e} : (\mu,\nu)\in M\}$ defined in (6-25). For any $L\in\Lambda$, set

$$M_L := \{(\mu,\nu)\in M : (\nu-\mu)\cdot\boldsymbol{e} = L\}. \quad (6\text{-}33)$$

We then get

$$\begin{aligned} \mathrm{i}\dot{\zeta}_j = (1+\varpi(|z_j|^2))(e_j\zeta_j + \partial_{\bar{j}}\mathscr{Z}_0(|\zeta_1|^2,\dots,|\zeta_n|^2)) - \sum_{L\in\Lambda}\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\nu_j\frac{\zeta^{\mu+\beta}\bar{\zeta}^{\nu+\alpha}}{\bar{\zeta}_j}\langle R_H^+(\boldsymbol{e}\cdot(\beta-\alpha))\bar{G}_{\alpha\beta}, G_{\mu\nu}\rangle \\ - \sum_{L\in\Lambda}\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}}\mu'_j\frac{\zeta^{\nu'+\alpha'}\bar{\zeta}^{\mu'+\beta'}}{\bar{\zeta}_j}\langle R_H^-(\boldsymbol{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \bar{G}_{\mu'\nu'}\rangle + \mathscr{G}_j, \quad (6\text{-}34) \end{aligned}$$

where, for some $A_{k\alpha\beta\mu\nu}$, $B_{k\alpha\beta\mu\nu}$, we have

$$\begin{aligned}\mathcal{G}_j &= \mathcal{F}_j + (1+\varpi(|z_j|^2))[\partial_{\bar{j}}\mathcal{Z}_0(|z_1|^2,\dots,|z_n|^2) - \partial_{\bar{j}}\mathcal{Z}_0(|\zeta_1|^2,\dots,|\zeta_n|^2)] \\ &\quad - e_j\varpi(|z_j|^2)\Bigg[\sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} \frac{\nu_j z^{\mu+\beta}\bar{z}^{\nu+\alpha}}{((\mu-\nu)\cdot\boldsymbol{e}-(\alpha-\beta)\cdot\boldsymbol{e})\bar{z}_j}\langle R_H^+(\boldsymbol{e}\cdot(\beta-\alpha))\bar{G}_{\alpha\beta}, G_{\mu\nu}\rangle \\ &\qquad + \sum_{\substack{(\mu',\nu')\in M\\(\alpha',\beta')\in M}} \frac{\mu'_j z^{\nu'+\alpha'}\bar{z}^{\mu'+\beta'}}{((\alpha'-\beta')\cdot\boldsymbol{e}-(\mu'-\nu')\cdot\boldsymbol{e})\bar{z}_j}\langle R_H^-(\boldsymbol{e}\cdot(\beta'-\alpha'))G_{\alpha'\beta'}, \bar{G}_{\mu'\nu'}\rangle\Bigg] \\ &\quad + \sum_k \sum_{\substack{(\mu,\nu)\in M\\(\alpha,\beta)\in M}} (\mathrm{i}\dot{z}_k - e_k z_k)\frac{z^{\mu+\beta}\bar{z}^{\nu+\alpha}}{\bar{z}_j}A_{k\alpha\beta\mu\nu} + \overline{\mathrm{i}\dot{z}_k - e_k z_k}\,\frac{z^{\mu+\beta}\bar{z}^{\nu+\alpha}}{\bar{z}_j}B_{k\alpha\beta\mu\nu}. \end{aligned} \tag{6-35}$$

Lemma 6.6. *There are fixed c_4 and $\epsilon_0 > 0$ such that, for $\epsilon \in (0,\epsilon_0)$, we have*

$$\|\mathcal{G}_j\bar{\zeta}_j\|_{L^1[0,T]} \le (1+C_0)c_4\epsilon^2. \tag{6-36}$$

Proof. We consider separately the terms in the right-hand side of (6-35) and (6-30). By (6-6), (6-28) and (6-32),

$$\|\partial_{\bar{z}_j}\mathcal{R}\bar{\zeta}_j\|_{L^1_t[0,T]} \le C(C_0)\epsilon^3.$$

For fixed constants c_2 and c_3, by (6-4) and (6-22) we have

$$\left\|\frac{z^\mu\bar{z}^\nu\bar{\zeta}_j}{\bar{z}_j}\langle g, G_{\mu\nu}\rangle\right\|_{L^1[0,T]} \le c_2\left\|\frac{z^\mu\bar{z}^\nu\bar{\zeta}_j}{\bar{z}_j}\right\|_{L^2[0,T]}\|g\|_{L^2([0,T],L^{2,-s})} \le c_3C_0\epsilon^2. \tag{6-37}$$

To get (6-37) we exploit Lemma 6.4 and the following bound:

$$\nu_j\left\|\frac{z^\mu\bar{z}^\nu\bar{\zeta}_j}{\bar{z}_j}\right\|_{L^2[0,T]} \le \nu_j\|z^\mu\bar{z}^\nu\|_{L^2[0,T]} + \nu_j\left\|\frac{z^\mu\bar{z}^\nu}{\bar{z}_j}\right\|_{L^\infty[0,T]}\|\zeta_j-\bar{z}_j\|_{L^2[0,T]} \le c_2C_0\epsilon + C(C_0)\epsilon^3 \tag{6-38}$$

for fixed c_2, where we used (6-14) and (6-32). Terms such as (6-37), that is, the terms from the second term in the right-hand side of (6-30), are the ones responsible for the $C_0c_4\epsilon^2$ in (6-36), where C_0 could be large. The other terms are $O(\epsilon^2)$ with fixed constants if ϵ_0 is small enough.

By (6-4) and (6-5), for $\boldsymbol{m} \in \mathcal{M}_j(2N+4)$ we have

$$\||z_j|^2\boldsymbol{Z}^{\boldsymbol{m}}\langle G'_{j\boldsymbol{m}}, \eta\rangle\bar{\zeta}_j\|_{L^1[0,T]} \le c_4\|z_j\zeta_j\|_{L^\infty}\|z_j\boldsymbol{Z}^{\boldsymbol{m}}\|_{L^2[0,T]}\|\eta\|_{L^2([0,T],L^{2,-s})} \le C(C_0)\epsilon^4. \tag{6-39}$$

Let $\mathfrak{S}$ be the sum of the second to fourth lines in (6-35). It is easy to see by (6-32) that

$$\|\bar{\zeta}_j(\mathfrak{S})\|_{L^2[0,T]} \le C(C_0)\epsilon^3; \tag{6-40}$$

see [Cuccagna 2011b, Lemma 4.11]. Furthermore,

$$\|[\partial_{\bar{j}}\mathcal{Z}_0(|z_1|^2,\dots,|z_n|^2) - \partial_{\bar{j}}\mathcal{Z}_0(|\zeta_1|^2,\dots,|\zeta_n|^2)]\bar{\zeta}_j\|_{L^2[0,T]} \le C(C_0)\epsilon^3; \tag{6-41}$$

see [Cuccagna 2011b, Lemma 4.10]. Finally we have, for $(\mu,\nu)\in M$,

$$\begin{aligned}\Big\|\varpi_j(|z_j|^2)\nu_j\frac{z^\mu\bar z^\nu}{\bar z_j}\langle\eta,G_{\mu\nu}\rangle\zeta_j\Big\|_{L^1_t}&\\ &\le \|\varpi_j(|z_j|^2)\nu_j z^\mu\bar z^\nu\langle\eta,G_{\mu\nu}\rangle\|_{L^1_t}+\Big\|\varpi_j(|z_j|^2)\nu_j\frac{z^\mu\bar z^\nu}{\bar z_j}\langle\eta,G_{\mu\nu}\rangle(\zeta_j-z_j)\Big\|_{L^1_t}\\ &\le C(C_0)\epsilon^3\end{aligned}$$

by $\varpi_j(|z_j|^2)=O(|z_j|^2)$, (6-4)–(6-6) and (6-32). This completes the proof of Lemma 6.6. □

We now consider

$$\begin{aligned}2^{-1}\frac{d}{dt}\sum_j|e_j|\,|\zeta_j|^2=-\sum_j e_j\,&\overbrace{\operatorname{Im}[(1+\varpi(|z_j|^2))e_j|\zeta_j|^2+\partial_{\bar\zeta_j}\mathscr{Z}_0(|\zeta_1|^2,\dots,|\zeta_n|^2)\bar\zeta_j]}^{0}\\ &-\sum_j e_j\operatorname{Im}[\mathscr{G}_j\bar\zeta_j]+\sum_{L\in\Lambda}\operatorname{Im}\Big[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\nu\cdot\boldsymbol{e}\zeta^{\mu+\beta}\bar\zeta^{\nu+\alpha}\langle R_H^+(L)\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle\\ &+\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}}\mu'\cdot\boldsymbol{e}\zeta^{\nu'+\alpha'}\bar\zeta^{\mu'+\beta'}\langle R_H^-(L)G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\rangle\Big].\end{aligned}\tag{6-42}$$

We can now substitute $R_H^{\pm}(L)=\text{P.V.}(1/(H-L))\pm\mathrm{i}\pi\delta(H-L)$.

Lemma 6.7. *The contributions to* (6-42) *from* P.V.$(1/(H-L))$ *cancel out*:

$$\begin{aligned}\operatorname{Im}\Big[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\nu\cdot\boldsymbol{e}\zeta^{\mu+\beta}\bar\zeta^{\nu+\alpha}\Big\langle\text{P.V.}\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\Big\rangle&\\ +\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}}\mu'\cdot\boldsymbol{e}\zeta^{\nu'+\alpha'}\bar\zeta^{\mu'+\beta'}\Big\langle\text{P.V.}\frac{1}{H-L}G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\Big\rangle\Big]&=0.\end{aligned}\tag{6-43}$$

Proof. We set $(\alpha',\beta')=(\mu,\nu)$ and $(\mu',\nu')=(\alpha,\beta)$ in the second sum of (6-43). With these choices,

$$\mu'\cdot\boldsymbol{e}\zeta^{\nu'+\alpha'}\bar\zeta^{\mu'+\beta'}\Big\langle\text{P.V.}\frac{1}{H-L}G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\Big\rangle=\alpha\cdot\boldsymbol{e}\zeta^{\mu+\beta}\bar\zeta^{\nu+\alpha}\Big\langle\text{P.V.}\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\Big\rangle.$$

Then 2 times the left-hand side of (6-43) becomes

$$\begin{aligned}2\operatorname{Im}\Big[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}(\alpha+\nu)\cdot\boldsymbol{e}\zeta^{\mu+\beta}\bar\zeta^{\nu+\alpha}\Big\langle\text{P.V.}\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\Big\rangle\Big]&\\ =\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\operatorname{Im}\Big[(\alpha+\nu)\cdot\boldsymbol{e}\zeta^{\mu+\beta}\bar\zeta^{\nu+\alpha}\Big\langle\text{P.V.}\frac{1}{H-L}\overline{G}_{\alpha\beta},G_{\mu\nu}\Big\rangle&\\ +(\mu+\beta)\cdot\boldsymbol{e}\bar\zeta^{\mu+\beta}\zeta^{\nu+\alpha}\Big\langle\text{P.V.}\frac{1}{H-L}\overline{G}_{\mu\nu},G_{\alpha\beta}\Big\rangle\Big]&\end{aligned}$$

$$= \operatorname{Im}\Big[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}(\alpha+\nu)\cdot\boldsymbol{e}\Big(\zeta^{\mu+\beta}\bar{\zeta}^{\nu+\alpha}\Big\langle \text{P.V.}\frac{1}{H-L}\overline{G}_{\alpha\beta}, G_{\mu\nu}\Big\rangle+\text{c.c.}\Big)\Big]=0,$$

where we exploited the fact that, if (μ,ν) and (α,β) both belong to M_L, then $(\alpha+\nu)\cdot\boldsymbol{e}=(\mu+\beta)\cdot\boldsymbol{e}$. □

Lemma 6.8. *Set, for any $L\in\Lambda$,*

$$G_L(\zeta):=\sqrt{\pi}\sum_{(\mu,\nu)\in M_L}\zeta^{\mu}\bar{\zeta}^{\nu}G_{\mu\nu}. \tag{6-44}$$

Then we have

$$\operatorname{Im}\Big[\mathrm{i}\pi\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\nu\cdot\boldsymbol{e}\zeta^{\mu+\beta}\bar{\zeta}^{\nu+\alpha}\langle\delta(H-L)\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle+\mathrm{i}\pi\sum_{\substack{(\mu',\nu')\in M_L\\(\alpha',\beta')\in M_L}}\mu'\cdot\boldsymbol{e}\zeta^{\nu'+\alpha'}\bar{\zeta}^{\mu'+\beta'}\langle\delta(H-L)G_{\alpha'\beta'},\overline{G}_{\mu'\nu'}\rangle\Big]$$
$$=L\langle\delta(H-L)\overline{G}_L(\zeta),G_L(\zeta)\rangle\geq 0. \tag{6-45}$$

Proof. First of all, the last inequality is a consequence of the formula

$$\langle F,\delta(H-L)\overline{G}\rangle=\frac{1}{2\sqrt{L}}\int_{|\xi|=\sqrt{L}}\widehat{F}(\xi)\overline{\widehat{G}}(\xi)\,d\sigma(\xi)$$

with $\widehat{F}$ and $\widehat{G}$ the Fourier transforms of F and G associated to H; see [Taylor 1997, Chapter 9, Proposition 2.2].

To prove the equality in (6-45), set $(\alpha',\beta')=(\alpha,\beta)$ and $(\mu',\nu')=(\mu,\nu)$ in the second sum of (6-45). Then the left-hand side of (6-45) equals

$$\pi\operatorname{Re}\Big[\sum_{\substack{(\mu,\nu)\in M_L\\(\alpha,\beta)\in M_L}}\overbrace{(\nu-\mu)\cdot\boldsymbol{e}}^{L}\zeta^{\mu+\beta}\bar{\zeta}^{\nu+\alpha}\langle\delta(H-L)\overline{G}_{\alpha\beta},G_{\mu\nu}\rangle\Big]=L\langle\delta(H-L)\overline{G}_L(\zeta),G_L(\zeta)\rangle. \qquad\square$$

From (6-42) and Lemmas 6.7–6.8, we obtain

$$2\sum_{L\in\Lambda}L\langle\delta(H-L)\overline{G}_L(\zeta),G_L(\zeta)\rangle=\frac{d}{dt}\sum_j|e_j||\zeta_j|^2+2\sum_j e_j\operatorname{Im}[\mathscr{G}_j\bar{\zeta}_j]. \tag{6-46}$$

We are able to restate, precisely this time, hypothesis (H4):

(H4) For some fixed constants, we have

$$\sum_{L\in\Lambda}\langle\delta(H-L)\overline{G}_L(\zeta),G_L(\zeta)\rangle\sim\sum_{(\mu,\nu)\in M}|\zeta^{\mu+\nu}|^2\quad\text{for all }\zeta\in\mathbb{C}^n\text{ with }|\zeta|\leq 1. \tag{6-47}$$

We now complete the proof of Proposition 6.2. We claim we have, for a fixed c,

$$\Big|\sum_j|e_j|\big(|\zeta_j(t)|^2-|\zeta_j(0)|^2\big)\Big|\leq c\epsilon^2. \tag{6-48}$$

Indeed, first of all we have $|\zeta_j(0)| \le c'\epsilon$ by $\epsilon := \|u_0\|_{H^1}$. Observe that, for (z', η') the initial coordinates in Lemma 2.4, by Proposition 1.1 and Lemma 2.3 it is easy to see that we have

$$\begin{aligned}\epsilon^2 > \|u_0\|_{L^2}^2 = \|u(t)\|_{L^2}^2 &= \left\| \left(\sum_{j=1}^n z'_j(t)\phi_j + \eta'(t) \right) + \left(\sum_{j=1}^n q_{j z'_j(t)} + (R[z'(t)] - 1)\eta'(t) \right) \right\|_{L^2}^2 \\ &= \sum_{j=1}^n |z'_j(t)|^2 + \|\eta'(t)\|_{L^2}^2 + O\big(|z'(t)|^6 + |z'(t)|^4 \|\eta'(t)\|_{L^2}^2\big).\end{aligned}$$

This gives the following version of (2-11):

$$\sum_{j=1}^n |z'_j(t)|^2 + \|\eta'(t)\|_{L^2}^2 \le 2\epsilon^2. \tag{6-49}$$

This yields an analogous formula for the last system of coordinates, (z, η) in (5-35). Finally, this yields the following inequality for the variables ζ introduced in (6-31):

$$\sum_{j=1}^n |\zeta_j(t)|^2 \le 3\epsilon^2. \tag{6-50}$$

Hence the claim (6-48) is proved. By Lemma 6.6, the hypothesis (6-47), (6-32) and (6-48), for $\epsilon \in (0, \epsilon_0)$ with ϵ_0 small enough we obtain, for a fixed c,

$$\sum_{(\mu,\nu)\in M} \|z^{\mu+\nu}\|_{L^2(0,t)}^2 \le c\epsilon^2 + cC_0\epsilon^2. \tag{6-51}$$

Now, (6-51) tells us that $\|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim C_0^2\epsilon^2$ implies $\|z^{\mu+\nu}\|_{L^2(0,t)}^2 \lesssim \epsilon^2 + C_0\epsilon^2$ for all $(\mu, \nu) \in M$. This means that we can take $C_0 \sim 1$. This completes the proof of Proposition 6.2. □

Proof of the asymptotics (6-9). We write (6-12) in the form $\mathrm{i}\dot{\eta} = -\Delta\eta + V\eta + \mathbb{B}$. Then $\partial_t(e^{-\mathrm{i}\Delta t}\eta) = -\mathrm{i}e^{-\mathrm{i}\Delta t}(\eta + \mathbb{B})$ and so

$$e^{-\mathrm{i}\Delta t_2}\eta(t_2) - e^{-\mathrm{i}\Delta t_1}\eta(t_1) = -\mathrm{i}\int_{t_1}^{t_2} e^{-\mathrm{i}\Delta t}(V\eta(t) + \mathbb{B}(t))\,dt \quad \text{for } t_1 < t_2.$$

Then, for a fixed c_2, by the Strichartz estimates,

$$\|e^{-\mathrm{i}\Delta t_2}\eta(t_2) - e^{-\mathrm{i}\Delta t_1}\eta(t_1)\|_{H^1} \le c_2\big(\|\eta\|_{L^2(\mathbb{R}_+, W^{1,6})} + \|\mathbb{B}(t)\|_{L^1([t_1,t_2],H^1) + L^2([t_1,t_2],W^{6/5})}\big).$$

Since we have

$$\mathbb{B} = \sum_{(\mu,\nu)\in M} \bar{z}^\mu z^\nu \bar{G}_{\mu\nu} + \mathbb{A},$$

and by (6-14) and (6-15), valid now in $[0, \infty)$, for a fixed C we have

$$\left\| \sum_{(\mu,\nu)\in M} \bar{z}^\mu z^\nu \bar{G}_{\mu\nu} \right\|_{L^2(\mathbb{R}_+, W^{1,6/5})} \le C\epsilon, \quad \|\mathbb{A}\|_{L^2(\mathbb{R}_+, W^{1,6/5}) + L^1(\mathbb{R}_+, H^1)} \le C\epsilon^2,$$

so we conclude that there exists an $\eta_+ \in H^1$ with

$$\lim_{t\to+\infty} e^{-\mathrm{i}\Delta t}\eta(t) = \eta_+ \quad \text{in } H^1 \qquad \text{and} \qquad \|\eta(t) - e^{\mathrm{i}\Delta t}\eta_+\|_{H^1} \le C\epsilon \quad \text{for all } t \ge 0.$$

So we have the first limit in (6-7) and the inequality $\|\eta_+\|_{H^1} \le C\|u(0)\|_{H^1}$ in Theorem 6.1.

We prove now the existence of z_+ and the facts about it in Theorem 6.1. First of all, from (6-27),

$$\frac{1}{2}\sum_j \frac{d}{dt}|z_j|^2 = \sum_j \mathrm{Im}\Bigl[\partial_{\bar{j}}\mathcal{R}\bar{z}_j + \sum_{(\mu,\nu)\in M} \nu_j z^\mu \bar{z}^\nu \langle \eta, G_{\mu\nu}\rangle + \sum_{(\mu',\nu')\in M} \mu'_j z^{\nu'}\bar{z}^{\mu'}\langle \bar{\eta}, \bar{G}_{\mu'\nu'}\rangle\Bigr].$$

Since the right-hand side has $L^1(0,\infty)$ norm bounded by $C\epsilon^2$ for a fixed C, we conclude that the limit

$$\lim_{t\to+\infty}(|z_1(t)|,\dots,|z_n(t)|) = (\rho_{+1},\dots,\rho_{+n}) \quad \text{exists with} \quad |\rho_+| \le C\|u(0)\|_{H^1}.$$

Since $\lim_{t\to+\infty} \boldsymbol{Z}(t) = 0$, we conclude that all but at most one of the ρ_{+j} are equal to 0. □

7. Proof of Theorem 1.4

The stability of $e^{-\mathrm{i}tE_{1z}}Q_{1z}$ is known. By [Grillakis et al. 1987, Theorem 1], the stability of $e^{-\mathrm{i}tE_{1z}}Q_{1z}$, or equivalently of $e^{-\mathrm{i}tE_{1\rho_1}}Q_{1\rho}$ for $\rho > 0$, is a consequence of the following two points:

(1) The self-adjoint operator $L_{-\rho} := H - E_{1\rho} + |Q_{1\rho}|^2$ has kernel $\ker L_{-\rho} = \{Q_{1\rho}\}$ and $L_{-\rho} > 0$ in $\{Q_{1\rho}\}^\perp$.

(2) The self-adjoint operator $L_{+\rho} = H - E_{1\rho} + 3|Q_{1\rho}|^2$ is strictly positive: $L_{+\rho} > 0$.

If $|Q_{1\rho}(x)| > 0$ for all x, then (2) is an immediate consequence of (1). The fact that $\ker L_{-\rho} = \{Q_{1\rho}\}$ follows by the facts that $Q_{1\rho} \in \ker L_{-\rho}$ and that, for $|\rho| < \epsilon_0$ with $\epsilon_0 > 0$ small, the number $E_{1\rho} \sim e_1$ is the smallest eigenvalue of $H + |Q_{1\rho}|^2$, since e_1 is the smallest eigenvalue of H.

We recall that [Tsai and Yau 2002b; 2002c; 2002d; Soffer and Weinstein 2004; Gang and Weinstein 2008; 2011; Gustafson and Phan 2011; Nakanishi et al. 2012] give partial proofs of the instability of the second excited state, and only for $2e_2 > e_1$. We now prove the instability of the excited states.

Fix $j > 1$ and assume that Q_{jr} is orbitally stable. Then Q_{jr} is asymptotically stable, by Theorem 1.3. So, if $\|u(0) - Q_{jr}\|_{H^1} \ll 1$, then $\|u(t) - Q_{jz_j} - e^{\mathrm{i}\Delta t}\eta_+\|_{H^1} \to 0$ for $t \to \infty$ and $|z_j(t)| \to \rho$ with $\rho \neq 0$ and close to r. In this case we have

$$E(u(0)) = \lim_{t\to\infty} E(u(t)) = \lim_{t\to\infty} E(Q_{jz_j(t)} + e^{\mathrm{i}\Delta t}\eta_+),$$
$$\|u(0)\|_{L^2}^2 = \lim_{t\to\infty} \|Q_{jz_j(t)} + e^{\mathrm{i}\Delta t}\eta_+\|_{L^2}^2.$$

Since $\|e^{\mathrm{i}\Delta t}\eta_+\|_{L_t^2 L_x^6} \lesssim \|\eta_+\|_{L^2}$, there exists $t_n \to \infty$ such that $\|e^{\mathrm{i}\Delta t_n}\eta\|_{L_x^6} \to 0$. So, since $\|e^{\mathrm{i}t_n\Delta}\eta_+\|_{L^4} \to 0$, $\int V|e^{\mathrm{i}t_n\Delta}\eta_+|^2\,dx \to 0$, and the cross terms in (3-3) disappear, we have

$$E(u(0)) = \lim_{n\to\infty} E(Q_{jz_j(t_n)} + e^{\mathrm{i}\Delta t_n}\eta_+) = E(Q_{j\rho}) + \|\nabla\eta_+\|_{L^2}^2,$$
$$\|u(0)\|_{L^2}^2 = \lim_{n\to\infty} \|Q_{jz_j(t_n)} + e^{\mathrm{i}\Delta t_n}\eta_+\|_{L^2}^2 = \|Q_{j\rho}\|_{L^2}^2 + \|\eta_+\|_{L^2}^2.$$

We claim that for $j \ge 2$ we can construct a curve on H^1 with the following property:

Lemma 7.1. *For sufficiently small* δ, *there exists a map* $[0, \delta) \to H^1$, $\varepsilon \mapsto \Psi(\varepsilon)$ *such that*:

- $\Psi(0) = Q_{jr}$;
- $\|\Psi(\varepsilon)\|_{L^2}^2 = \|Q_{jr}\|_{L^2}^2$;
- $E(\Psi(\varepsilon)) < E(Q_{jr})$ *if* $\varepsilon > 0$.

Before proving the lemma, we show that the assumption that Q_{jr} is asymptotically stable and the existence of Ψ lead to a contradiction.

Proof of instability. Since $\|Q_{jr}\|_{L^2}^2 = r^2 + O(r^6)$ by Proposition 1.1, $\|Q_{jr}\|_{L^2}^2$ is strictly increasing in r for r small. By Proposition 1.1, we have $E'(Q_{jr}) = (e_j + O(r^2))Q'(Q_{jr})$. This implies that $E(Q_{jr})$ is a strictly decreasing function of r. Setting $u(0) = \Psi(\varepsilon)$, we have

$$\|Q_{jr}\|_{L^2}^2 = \|\Psi(\varepsilon)\|_{L^2}^2 = \|Q_{j\rho}\|_{L^2}^2 + \|\eta_+\|_{L^2}^2.$$

Therefore we have $\|Q_{jr}\|_{L^2}^2 \geq \|Q_{j\rho}\|_{L^2}^2$. This implies $r \geq \rho$ and so $E(Q_{j\rho}) \geq E(Q_{jr})$. But, looking at the energy, we get the following contradiction, which ends the proof of Theorem 1.4:

$$E(Q_{jr}) > E(\Psi(\varepsilon)) = E(Q_{j\rho}) + \|\nabla\eta_+\|_{L^2}^2 \geq E(Q_{j\rho}) \geq E(Q_{jr}). \qquad \square$$

We now construct the curve Ψ.

Proof of Lemma 7.1. We set $\Psi(\varepsilon) = \beta(\varepsilon)Q_{j,r} + \varepsilon\phi_1$ and choose $\beta(\varepsilon)$ to make $\|\Psi(\varepsilon)\|_{L^2}^2 = \|Q_{jr}\|_{L^2}^2$:

$$\|Q_{jr}\|_{L^2}^2\beta^2 + 2\varepsilon\langle Q_{jr}, \phi_1\rangle\beta + \varepsilon^2 - \|Q_{jr}\|_{L^2}^2 = 0.$$

So, we have

$$\begin{aligned}
\beta(\varepsilon) &= \frac{-\langle Q_{jr}, \phi_1\rangle\varepsilon + \sqrt{\langle Q_{jr}, \phi_1\rangle^2\varepsilon^2 - \|Q_{jr}\|_{L^2}^2(\varepsilon^2 - \|Q_{jr}\|_{L^2}^2)}}{\|Q_{jr}\|_{L^2}^2} = \sqrt{1 - g_1(r)\varepsilon^2} + g_2(r)\varepsilon,\\
g_1(r) &:= \frac{1}{\|Q_{jr}\|_{L^2}^4}(\|Q_{jr}\|_{L^2}^2 - \langle Q_{jr}, \phi_1\rangle^2) = \frac{1}{\|Q_{jr}\|_{L^2}^4}(\|Q_{jr}\|_{L^2}^2 - \langle q_{jr}, \phi_1\rangle^2),\\
g_2(r) &:= -\frac{\langle Q_{jr}, \phi_1\rangle}{\|Q_{jr}\|_{L^2}^2} = -\frac{\langle q_{jr}, \phi_1\rangle}{\|Q_{jr}\|_{L^2}^2}.
\end{aligned}$$

We now show $E(\Psi(\varepsilon)) < E(Q_{j,r})$ for $\varepsilon > 0$. It suffices to show $S_{E_{jr}}(\Psi(\varepsilon)) < S_{E_{jr}}(Q_{jr})$, where

$$S_{E_{jr}}(u) = E(u) - E_{jr}\|u\|_{L^2}^2.$$

Notice that we have $S'_{E_{jr}}(Q_{jr}) = 0$. Therefore, setting $\gamma(\varepsilon) = \beta(\varepsilon) - 1$, we have

$$\begin{aligned}
S_{E_{jr}}(\Psi(\varepsilon)) &= S_{E_{jr}}(Q_{jr} + \gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1)\\
&= S_{E_{jr}}(Q_{jr}) + \tfrac{1}{2}\big\langle S''_{E_{jr}}(Q_{jr})(\gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1), \gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1\big\rangle + o(\|\gamma(\varepsilon)Q_{jr} + \varepsilon\phi_1\|_{H^1}^2).
\end{aligned}$$

If $g_2(r) = 0$, we have $\gamma(\varepsilon) = O(\varepsilon^2 r^{-2})$ and we conclude

$$\begin{aligned}
S_{E_{jr}}(\Psi(\varepsilon)) &= S_{E_{jr}}(Q_{jr}) + \varepsilon^2\langle S_{E_{jr}}(Q_{jr})\phi_1, \phi_1\rangle + o(\varepsilon^2)\\
&= S_{E_{jr}}(Q_{jr}) + \varepsilon^2(e_1 - e_j) + O(\varepsilon^2 r) + o(\varepsilon^2) < S_{E_{jr}}(Q_{jr}).
\end{aligned}$$

If $g_2(r) \neq 0$, we have $\gamma(\varepsilon) = O(r\varepsilon)$ and

$$S_{E_{jr}}(\Psi(\varepsilon)) = S_{E_{jr}}(Q_{jr}) + \varepsilon^2(e_1 - e_j) + O(r\varepsilon^2) < S_{E_{jr}}(Q_{jr}).$$

Therefore Lemma 7.1 is proved. This also completes the proof of Theorem 1.4. $\square$

Appendix A: A generalization of Proposition 1.1

For reference purposes, we generalize (1-1) as

$$\mathrm{i}u_t = -\Delta u + V(x)u + \beta(|u|^2)u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^3, \tag{A-1}$$

and assume that $\beta(0) = 0$, $\beta \in C^\infty(\mathbb{R}, \mathbb{R})$ and, further, there exists $p \in (1, 5)$ such that, for every $k \geq 0$, there is a fixed C_k with

$$\left|\frac{d^k}{dv^k}\beta(v^2)\right| \leq C_k|v|^{p-k-1} \quad \text{if } |v| \geq 1.$$

Proposition A.1. *Fix $j \in \{1, \dots, n\}$. Then there is $a_0 > 0$ such that, for all $z_j \in B_{\mathbb{C}}(0, a_0)$, there is a unique $Q_{jz_j} \in \mathcal{S}(\mathbb{R}^3, \mathbb{C}) := \bigcap_{t \geq 0} \Sigma_t(\mathbb{R}^3, \mathbb{C})$ such that*

$$(-\Delta + V)Q_{jz_j} + \beta(|Q_{jz_j}|^2)Q_{jz_j} = E_{jz_j}Q_{jz_j}, \quad Q_{jz_j} = z_j\phi_j + q_{jz_j}, \quad \langle q_{jz_j}, \overline{\phi}_j\rangle = 0, \tag{A-2}$$

and such that we have, for any $r \in \mathbb{N}$:

(1) $(q_{jz_j}, E_{jz_j}) \in C^\infty(B_{\mathbb{C}}(a_0), \Sigma_r \times \mathbb{R})$; *we have $q_{jz_j} = z_j\hat{q}_j(|z_j|^2)$ with $\hat{q}_j(t^2) = t^2\tilde{q}_j(t^2)$, where $\tilde{q}_j(t) \in C^\infty((-a_0{}^2, a_0{}^2), \Sigma_r(\mathbb{R}^3, \mathbb{R}))$, and $E_{jz_j} = E_j(|z_j|^2)$ with $E_j(t) \in C^\infty((-a_0{}^2, a_0{}^2), \mathbb{R})$.*

(2) *There exists $C > 0$ such that $\|q_{jz_j}\|_{\Sigma_r} \leq C|z_j|^3$ and $|E_{jz_j} - e_j| < C|z_j|^2$.*

The rest of this section is devoted to the proof of Proposition A.1.

The first step is the following lemma, which follows by a direct computation:

Lemma A.2. *Let $m \in \mathbb{N}_0$ and $k \in \{1, 2, 3\}$. Then we have*

$$\begin{aligned}
[-\Delta, |x|^{2m}] &= -2m(2m+1)|x|^{2m-2} - 4m|x|^{2m-2}x \cdot \nabla, \\
[-\Delta, |x|^{2m}x_k] &= -2m(2m+3)|x|^{2m-2}x_k - 4mx_k|x|^{2m-2}x \cdot \nabla - 2|x|^{2m}\partial_{x_k}.
\end{aligned} \tag{A-3}$$

Our second step is the following lemma:

Lemma A.3. *The eigenfunctions ϕ_j of $-\Delta + V$ satisfy $\phi_j \in \mathcal{S}(\mathbb{R}^3)$.*

Proof. First, $\phi_j \in L^2(\mathbb{R}^3)$, so we have $\phi_j \in H^2(\mathbb{R}^3)$ by

$$(-\Delta - e_j)\phi_j = -V\phi_j.$$

Furthermore, if we have $\phi_j \in H^{2m}(\mathbb{R}^3)$, then we have $\phi_j \in H^{2m+2}(\mathbb{R}^3)$. This implies $\phi_j \in \bigcap_{m=1}^\infty H^m$.

Next, by Lemma A.2, we have

$$(-\Delta - e_j)x_k\phi_j = -2\partial_{x_k}\phi_j - Vx_k\phi_j$$

for $k = 1, 2, 3$. Therefore, we have $x_k\phi_j \in H^2(\mathbb{R}^3)$. Again, by Lemma A.2, we have

$$(-\Delta - e_j)|x|^2\phi_j = -6\phi_j - 4x \cdot \nabla\phi_j - V x_k\phi_j.$$

So, by $x \cdot \nabla\phi_j = \nabla(x\phi_j) - 3\phi_j \in L^2(\mathbb{R}^3)$, we have $|x|^2\phi_j \in H^2$.

Now, suppose $|x|^{2m}\phi_j \in H^2(\mathbb{R}^3)$. By Lemma A.2, we have

$$(-\Delta - e_j)|x|^{2m}x_k\phi_j = -2m(2m+3)|x|^{2m-2}x_k\phi_j - 4mx_k|x|^{2m-2}x \cdot \nabla\phi_j - 2|x|^{2m}\partial_{x_k}\phi_j - V|x|^{2m}x_k\phi_j.$$

Since

$$|x|^{2m}\partial_{x_k}\phi_j = \partial_{x_k}(|x|^{2m}\phi_j) - 4m|x|^{2m-2}x_k\phi_j \in L^2(\mathbb{R}^3),$$

we have $|x|^{2m}x_k\phi_j \in H^2(\mathbb{R}^3)$. Finally, since

$$(-\Delta - e_j)|x|^{2m+2}\phi_j = -2(m+1)(2m+3)|x|^{2m}\phi_j - 4(m+1)|x|^{2m}x \cdot \nabla\phi_j - V|x|^{2m+2}\phi_j$$

and $|x|^{2m}x \cdot \nabla\phi_j = \nabla \cdot (|x|^{2m}x\phi_j) - (4m+3)|x|^{2m}\phi_j \in L^2(\mathbb{R}^3)$, we have $|x|^{2m+2}\phi_j \in H^2(\mathbb{R}^2)$. By induction, we have $\phi_j \in \Sigma_{2m}$ for any $m \geq 1$. $\square$

The next step is the following lemma:

Lemma A.4. *Fix $j \in \{1, \dots, n\}$ and $r \in \mathbb{N}$ with $r \geq 2$. Then there exists $\delta_r > 0$ such that, for all $z_j \in B_{\mathbb{C}}(0, \delta_r)$, there is a unique $Q_{jz_j} \in \Sigma_r(\mathbb{R}^3, \mathbb{C})$ satisfying* (1-3) *and Proposition 1.1(1)–(2).*

Proof. In this proof we write $g(u) := \beta(|u|^2)u$. Notice that it suffices to show the claim of Lemma A.4 for $z_j \in \mathbb{R}$ with real-valued Q_{j,z_j}. Indeed, if we define

$$Q_{jz_j} = e^{i\theta}Q_{j\rho} \quad \text{and} \quad E_{jz_j} = E_{j\rho} \tag{A-4}$$

for $z = e^{i\theta}\rho$, then Q_{jz} and E_{jz} satisfy (1-3) if $Q_{j\rho}$ and $E_{j\rho}$ satisfy (1-3). Further, if $B_{\mathbb{R}}(0, \delta) \to \Sigma_r \times \mathbb{R}$, $z \mapsto (Q_{jz}, E_{jz})$ is C^∞, then, by (A-4), we have $B_{\mathbb{C}}(0, \delta)\Sigma_r \times \mathbb{R}$, $z \mapsto (Q_{j,z}, E_{j,z})$ is C^∞.

Fix $j \in \{0, 1, \dots, n\}$. For simplicity we set $z_j = z$, $e_j = e$ and $\phi_j = \phi$. Set

$$Q_{j,z} = z(\phi + |z|^2\psi(z)) \quad \text{and} \quad E_{j,z} = e + |z|^2 f(z).$$

We solve (1-3) under the above ansatz. Substituting the ansatz into (1-3), we have

$$H\psi + z^{-3}g(z(\phi + z^2\psi)) = e\psi + f\phi + z^2 f\psi. \tag{A-5}$$

Set $Pu = u - \langle u, \phi\rangle\phi$. Then, we have

$$H\psi + z^{-3}Pg(z(\phi + z^2\psi)) = e\psi + z^2 f\psi, \quad \langle z^{-3}g(z(\phi + z^2\psi)), \phi\rangle = f.$$

Therefore, it suffices to solve

$$(H - e)\psi = -z^{-3}Pg(z(\phi + z^2\psi)) + z^{-1}\langle g(z(\phi + z^2\psi)), \phi\rangle\psi. \tag{A-6}$$

Now, set $\tilde{\phi}(z) := \phi + z^2\psi(z)$. Then,

$$g(z\tilde{\phi}) = \beta(z^2\tilde{\phi})z\tilde{\phi} = z^3\int_0^1 \beta'(sz^2\tilde{\phi}^2)\,ds\,\tilde{\phi}^3.$$

So, (A-6) can be rewritten as

$$(H-e)\psi = -P\left(\int_0^1 \beta'(sz^2\tilde{\phi}^2)\,ds\,\tilde{\phi}^3\right) + \langle \beta(z^2\tilde{\phi}^2)\tilde{\phi}, \phi\rangle \psi. \tag{A-7}$$

To show that $z \mapsto \psi(z) \in \Sigma_r$ exists and is C^∞, we use the inverse function theorem. Set

$$\Phi(z,\psi) := -(H-e)^{-1}P\left(\int_0^1 \beta'(sz^2\tilde{\phi}^2)\,ds\,\tilde{\phi}^3\right) + \langle \beta(z^2\tilde{\phi}^2)\tilde{\phi}, \phi\rangle (H-e)^{-1}\psi$$

and

$$F(z,\psi) := \psi - \Phi(z,\psi).$$

Then, $F : \mathbb{R}\times P\Sigma_r \to P\Sigma_r$ is C^∞. Next, since

$$F(0,\psi) = \psi + \beta'(0)(H-e)^{-1}P\phi^3,$$

we have

$$F(0, -\beta'(0)(H-e)^{-1}P\phi^3) = 0.$$

We now compute $F_\psi(z,\psi)$:

$$\begin{aligned}\Phi_\psi(z,\psi)h = -2z^4(H-e)^{-1}P\left(\int_0^1 \beta''(sz^2\tilde{\phi}^2)s\,ds\,\tilde{\phi}^4 h\right) - 3z^2(H-e)^{-1}P\left(\int_0^1 \beta'(sz^2\tilde{\phi}^2)\,ds\,\tilde{\phi}^2 h\right)\\ +2z^4\langle \beta'(z^2\tilde{\phi}^2)\tilde{\phi}^2 h, \phi\rangle (H-e)\psi + z^2\langle \beta(z^2\tilde{\phi}^2)h,\phi\rangle (H-e)\psi + \langle \beta(z^2\tilde{\phi}^2)\tilde{\phi},\phi\rangle (H-e)h.\end{aligned}$$

So, we have

$$F_\psi(0,\psi)h = h.$$

Therefore, by the inverse function theorem we have the conclusion of the lemma. □

The final step is to show that the $\delta_r > 0$ can be chosen independent of r.

Lemma A.5. *Consider the Q_{jz_j} in Lemma A.4. Then there is a $\delta > 0$ such that $Q_{jz_j} \in \mathcal{S}(\mathbb{R}^3)$ for $|z_j| < \delta$.*

Proof. We use a bootstrap argument similar to the proof of Lemma A.3. We can consider the Q_{jz} given in Lemma A.4 with $r = 4$. It is enough to consider $z = \rho \in (0,\delta)$ with $\delta < \delta_4$. For $\delta > 0$ sufficiently small, we also have $E_{j\rho} < \frac{1}{2}e_j < 0$. By (A-2) we have

$$(-\Delta - E_{j\rho})Q_{j\rho} = -VQ_{j\rho} - \int_0^1 \beta'(sQ_{j\rho}^2)\,ds\,Q_{j\rho}^3. \tag{A-8}$$

We proceed as in Lemma A.3. Since the commutator term and $-VQ_{j\rho}$ are the same as in Lemma A.3, we conclude that Lemma A.5 is a consequence of the following two simple facts for $m \ge 2$:

(i) If $Q_{j\rho} \in H^m$, then $\beta(Q_{j\rho}^2)Q_{j\rho} = \int_0^1 \beta'(sQ_{j\rho}^2)\,ds\,Q_{j\rho}^3 \in H^m$.

(ii) If $|x|^{2m}Q_{j\rho} \in L^2(\mathbb{R}^3)$, then $|x|^{2m+2}\int_0^1 \beta'(sQ_{j\rho}^2)\,ds\,Q_{j\rho}^3 \in L^2$.

Fact (i) follows from the fact that $H^m(\mathbb{R}^3)$ is a ring for $m \ge 2$. We now look at (ii). Since $Q_{j\rho}$ is a continuous function with $Q_{j\rho}(x) \to 0$ as $|x| \to \infty$, the range of $Q_{j\rho}$ (i.e., $\{Q_{j\rho}(x) \in \mathbb{R} : x \in \mathbb{R}^3\}$) is relatively compact. So, since $t \to \int_0^1 \beta'(st^2)\,ds$ is a continuous function from $\mathbb{R} \to \mathbb{R}$, the range of

$\int_0^1 \beta'(sQ_{j\rho}^2)\,ds$ is relatively compact. Therefore, we have $\int_0^1 \beta'(sQ_{j\rho}^2)\,ds \in L^\infty$. On the other hand, by $Q_{j\rho} \in \Sigma_4$ we have $|x|Q_{j\rho} \in \Sigma_3 \hookrightarrow L^\infty$. Therefore, we have

$$|x|^{2m+2} \int_0^1 \beta'(sQ_{j\rho}^2)\,ds\; Q_{j\rho}^3 = \int_0^1 \beta'(sQ_{j\rho}^2)\,ds\,(|x|Q_{j\rho})^2|x|^{2m} Q_{j\rho} \in L^2(\mathbb{R}^3).$$

This proves (ii) and completes the proof of Lemma A.5. □

Finally, Proposition A.1 is a consequence of Lemmas A.2–A.5.

Appendix B: Expansions of gauge invariant functions

We prove here (3-10) and (3-12), which are direct consequences of Lemmas B.3 and B.4.

Lemma B.1. *Let $a(z) \in C^\infty(B_{\mathbb{C}}(0,\delta), \mathbb{R})$ and $a(e^{i\theta}z) = a(z)$ for any $\theta \in \mathbb{R}$. Then there exists α in $\mathbb{C}^\infty([0,\delta^2); \mathbb{R})$ such that $\alpha(|z|^2) = a(z)$.*

Proof. For $z = re^{i\theta}$ we have $a(z) = a(r + i0)$. Since $x \mapsto a(x + i0)$ is even and smooth, we have $a(x + i0) = \alpha(x^2)$ with $\alpha(x)$ smooth; see [Whitney 1943]. So $a(z) = \alpha(|z|^2)$. □

Lemma B.2. *Let $\delta > 0$. Suppose $a \in C^\infty(B_{\mathbb{C}^n}(0,\delta); \mathbb{R})$ satisfies $a(e^{i\theta}z_1, \dots, e^{i\theta}z_n) = a(z_1, \dots, z_n)$ for all $\theta \in \mathbb{R}$ and $a(0, \dots, 0) = 0$. Then, for any $M > 0$, there exists $b_{\boldsymbol{m}}$ such that*

$$a(z_1, \dots, z_n) = \sum_{j=1}^n \alpha_j(|z_j|^2) + \sum_{|m|=1} \boldsymbol{Z}^{\boldsymbol{m}} b_{\boldsymbol{m}}(z_1, \dots, z_n) + \mathcal{R}^{0,M}(z, \boldsymbol{Z}), \tag{B-1}$$

where $\alpha_j(|z_j|^2) = a(0, \dots, 0, z_j, 0, \dots, 0)$. Furthermore, $b_{\boldsymbol{m}} \in C^\infty(B_{\mathbb{C}^n}(0,\delta); \mathbb{R})$ and satisfies

$$b_{\boldsymbol{m}}(e^{i\theta}z_1, \dots, e^{i\theta}z_n) = b_{\boldsymbol{m}}(z_1, \dots, z_n) \quad \textit{for all } \theta \in \mathbb{R}.$$

Proof. First, we expand a as

$$a(z_1, \dots, z_n) = a(z_1, 0, \dots, 0) + \int_0^1 \left(\sum_{j=2}^n \partial_j a(z_1, tz_2, \dots, tz_n) z_j + \partial_{\bar{j}} a(z_1, tz_2, \dots, tz_n) \bar{z}_j \right) dt.$$

Then, by

$$a(0, z_2, \dots, z_n) = \int_0^1 \left(\sum_{j=2}^n \partial_j a(0, tz_2, \dots, tz_n) z_j + \partial_{\bar{j}} a(0, tz_2, \dots, tz_n) \bar{z}_j \right) dt,$$

we have

$$\begin{aligned} &a(z_1, \dots, z_n) \\ &\quad = a(z_1, 0, \dots, 0) + a(0, z_2, \dots, z_n) + \int_0^1 \sum_{j=2}^n \big[(\partial_j a(z_1, tz_2, \dots, tz_n) - \partial_j a(0, tz_2, \dots, tz_n)) z_j \\ &\qquad\qquad + (\partial_{\bar{j}} a(z_1, tz_2, \dots, tz_n) - \partial_{\bar{j}} a(0, tz_2, \dots, tz_n)) \bar{z}_j \big]\, dt \end{aligned}$$

$$= a(z_1, 0, \dots, 0) + a(0, z_2, \dots, z_n) + \sum_{j\ge 2}\int_0^1\int_0^1 \big[(\partial_1\partial_j a(sz_1, tz_2, \dots, tz_n))z_1 z_j + (\partial_{\bar 1}\partial_j a(sz_1, tz_2, \dots, tz_n))\bar z_1 z_j + (\partial_1\partial_{\bar j} a(sz_1, tz_2, \dots, tz_n))\bar z_1 z_j + (\partial_{\bar 1}\partial_{\bar j} a(sz_1, tz_2, \dots, tz_n))\bar z_1 \bar z_j\big]\,ds\,dt.$$

Iterating this argument first for $a(0, z_2, \dots, z_n)$ and then for $a(0, \dots, 0, z_k, \dots, z_n)$, we have

$$\begin{aligned} a(z_1, \dots, z_n) &= a(z_1, 0, \dots, 0) + a(0, z_2, 0, \dots, 0) + \cdots + a(0, \dots, 0, z_n) \\ &\quad + \sum_{k=1}^{n-1}\sum_{j\ge k+1}\int_0^1\int_0^1 \Big[(\partial_k\partial_j a(0, \dots, 0, sz_k, tz_{k+1}, \dots, tz_n))z_k z_j \\ &\qquad + (\partial_{\bar k}\partial_j a(0, \dots, 0, sz_k, tz_{k+1}, \dots, tz_n))\bar z_k z_j \\ &\qquad + (\partial_k\partial_{\bar j} a(0, \dots, 0, sz_k, tz_{k+1}, \dots, tz_n))z_k \bar z_j \\ &\qquad + (\partial_{\bar k}\partial_{\bar j} a(0, \dots, 0, sz_k, tz_{k+1}, \dots, tz_n))\bar z_k \bar z_j\Big]\,ds\,dt. \end{aligned} \tag{B-2}$$

By Lemma B.1, there exist smooth α_j such that $\alpha_j(|z_j|^2) = a(0, \dots, 0, z_j, 0, \dots, 0)$. Furthermore, the sum of the middle two terms in the integral of (B-2) has the same form as the second term in the right-hand side of (B-1). So, it remains to handle the terms in the second and fifth lines of (B-2). Since they can be treated similarly, we focus only the second line of (B-2). Set

$$\beta_{jk}(z_k, \dots, z_n) = \int_0^1\int_0^1 (\partial_k\partial_j a(0, \dots, 0, sz_k, tz_{k+1}, \dots, tz_n))\,ds\,dt$$

with $j \ge k+1$. Notice that $\partial^\alpha\bar\partial^\beta a(0, \dots, 0) \ne 0$ by the gauge invariance of a is easily shown to imply $|\alpha| = |\beta|$. This in particular implies $\beta_{jk}(0, \dots, 0) = 0$. So, as in (B-2), we have

$$\begin{aligned} \beta_{jk}(z_k, \dots, z_n) &= \beta_{jk}(z_k, 0, \dots, 0) + \beta_{jk}(0, z_{k+1}, 0, \dots, 0) + \cdots + \beta_{jk}(0, \dots, 0, z_n) \\ &\quad + \sum_{m=k}^{n-1}\sum_{l\ge m+1}\int_0^1\int_0^1 \Big[(\partial_m\partial_l \beta_{jk}(0, \dots, 0, sz_m, tz_{m+1}, \dots, tz_n))z_m z_l \\ &\qquad + (\partial_{\bar m}\partial_l \beta_{jk}(0, \dots, 0, sz_m, tz_{m+1}, \dots, tz_n))\bar z_m z_l \\ &\qquad + (\partial_m\partial_{\bar l} \beta_{jk}(0, \dots, 0, sz_m, tz_{m+1}, \dots, tz_n))z_m \bar z_l \\ &\qquad + (\partial_{\bar m}\partial_{\bar l} \beta_{jk}(0, \dots, 0, sz_m, tz_{m+1}, \dots, tz_n))\bar z_m \bar z_l\Big]\,ds\,dt. \end{aligned} \tag{B-3}$$

Since $z_l^2\beta_{jk}(0, \dots, 0, z_l, 0, \dots, 0)$ is gauge invariant by Lemma B.1, we have

$$z_l^2\beta_{jk}(0, \dots, 0, z_l, 0, \dots, 0) = \tilde\beta_{jkl}(|z_l|^2) = \tilde\beta_{jkl}(0) + \tilde\beta'_{jkl}(0)|z_l|^2 + \gamma_{jkl}(|z_l|^2)|z_l|^4$$

for some smooth $\tilde\beta_{jkl}$ and γ_{jkl}. By the smoothness of β_{jk}, we have $\tilde\beta_{jkl}(0) = \tilde\beta'_{jkl}(0) = 0$. Therefore,

$$\beta_{jk}(0, \dots, 0, z_l, 0, \dots, 0) z_k z_j = \gamma_{jkl}(|z_l|^2) z_k z_j \bar z_l^2 \quad \text{with} \quad k < \min\{j, l\}.$$

This can be absorbed in the second term of the right-hand side of (B-1). The same is true of the contribution of the last two lines of (B-3). The term

$$\int_0^1 \int_0^1 (\partial_m \partial_l \beta_{jk}(0, \dots, 0, s z_m, t z_{m+1}, \dots, t z_n)) z_m z_l z_j z_k \, ds \, dt \tag{B-4}$$

does not have as factors components of $\boldsymbol{Z} = (z_i \bar z_j)_{i \neq j}$ but it is $O(|\boldsymbol{Z}|^2)$. Treating (B-4) the way we treated the second line of (B-2), and repeating the procedure a sufficient number of times, we can express (B-4) as a sum of a summation like the second in the right-hand side of (B-1) and of a term that is $O(|\boldsymbol{Z}|^M)$ for an arbitrary M. Furthermore, notice that, since we can think of the dependence on $\boldsymbol{Z} = (z_i \bar z_j)_{i \neq j}$ to be polynomial, and so the remainder term $R^{0,M}(z, \boldsymbol{Z})$ in (B-1) can be thought to depend polynomially on $\boldsymbol{Z} = (z_i \bar z_j)_{i \neq j}$, it can be thought as the restriction of a function in $\boldsymbol{Z} \in L$. □

Lemma B.3. *Take $a(z_1, \dots, z_n)$ like in Lemma B.2. Then, for any $M > 0$, there exist smooth a_j and $b_{j\boldsymbol{m}}$ such that, for $\alpha_j(|z_j|^2) = a(0, \dots, 0, z_j, 0, \dots, 0)$, we have*

$$a(z_1, \dots, z_n) = \sum_{j=1}^n \alpha_j(|z_j|^2) + \sum_{1 \le |\boldsymbol{m}| \le M-1} \boldsymbol{Z}^{\boldsymbol{m}} b_{j\boldsymbol{m}}(|z_j|^2) + \mathcal{R}^{0,M}(z, \boldsymbol{Z}). \tag{B-5}$$

Proof. To prove (B-5), one only has to repeatedly use Lemma B.2. □

Lemma B.4. *Suppose that $a : \mathbb{C}^n \to \mathcal{S}$ is smooth from $B_{\mathbb{R}^{2n}}(0, \delta_r)$ to Σ_r for arbitrary $r \in \mathbb{R}$ and satisfies $a(e^{i\theta} z_1, \dots, e^{i\theta} z_n) = a(z_1, \dots, z_n)$, $a(0, \dots, 0) = 0$. Then, for any $M > 0$, there exist smooth a_j and $G_{j\boldsymbol{m}}$ such that, for $\alpha_j(|z_j|^2) = a(0, \dots, 0, z_j, 0, \dots, 0)$, we have*

$$a(z_1, \dots, z_n) = \sum_{j=1}^n \alpha_j(|z_j|^2) + \sum_{1 \le |\boldsymbol{m}| \le M-1} \boldsymbol{Z}^{\boldsymbol{m}} G_{j\boldsymbol{m}}(|z_j|^2) + \boldsymbol{S}^{0,M}(z, \boldsymbol{Z}). \tag{B-6}$$

Proof. The proof is same as the proof of Lemmas B.1–B.3 □

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