ANALYSIS & PDE

Volume 8 No. 6 2015

GRÉGOIRE NADIN AND LUCA ROSSI

TRANSITION WAVES FOR FISHER–KPP EQUATIONS WITH GENERAL TIME-HETEROGENEOUS AND SPACE-PERIODIC COEFFICIENTS

msp

ANALYSIS AND PDE
Vol. 8, No. 6, 2015
dx.doi.org/10.2140/apde.2015.8.1351

msp

TRANSITION WAVES FOR FISHER–KPP EQUATIONS WITH GENERAL TIME-HETEROGENEOUS AND SPACE-PERIODIC COEFFICIENTS

GRÉGOIRE NADIN AND LUCA ROSSI

We study existence and nonexistence results for generalized transition wave solutions of space-time heterogeneous Fisher–KPP equations. When the coefficients of the equation are periodic in space but otherwise depend in a fairly general fashion on time, we prove that such waves exist as soon as their speed is sufficiently large in a sense. When this speed is too small, transition waves do not exist anymore; this result holds without assuming periodicity in space. These necessary and sufficient conditions are proved to be optimal when the coefficients are periodic both in space and time. Our method is quite robust and extends to general nonperiodic space-time heterogeneous coefficients, showing that transition wave solutions of the nonlinear equation exist as soon as one can construct appropriate solutions of a given linearized equation.

1. Introduction

We are concerned with transition wave solutions of the space-time heterogeneous reaction-diffusion equation

$$\partial_t u - \mathrm{Tr}(A(x,t)D^2u) + q(x,t)\cdot Du = f(x,t,u), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{1}$$

Here D and D^2 denote respectively the gradient and the Hessian with respect to the space variables. We assume that the terms in the equation are periodic in x, with the same period. The matrix field A is uniformly elliptic and the nonlinearity $f(x,t,\cdot)$ vanishes at 0 and 1. The steady states 0 and 1 are respectively unstable and stable.

When the coefficients do not depend on (x,t), Equation (1) becomes a classical homogeneous monostable reaction-diffusion equation. The pioneering works on such equations are due to Kolmogorov, Petrovski and Piskunov [Kolmogorov et al. 1937] and Fisher [1937], when $f(u) = u(1-u)$. They investigated the existence of traveling wave solutions, that is, solutions of the form $u(x,t) = \phi(x\cdot e - ct)$, with $\phi(-\infty) = 1,\ \phi(+\infty) = 0,\ \phi > 0$. The quantity $c \in \mathbb{R}$ is the speed of the wave and $e \in S^{N-1}$ is its direction. Kolmogorov, Petrovski and Piskunov [Kolmogorov et al. 1937] proved that when $A = I_N$, $q \equiv 0$ and $f = u(1-u)$, there exists $c^* > 0$ such that (1) admits traveling waves of speed c if and

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP/2007-2013) / ERC Grant Agreement n.321186, ReaDi (Reaction-Diffusion Equations, Propagation and Modelling). Luca Rossi was partially supported by GNAMPA-INdAM.
MSC2010: 35B40, 35K10, 35K57, 35P05, 35B51.
Keywords: Fisher–KPP equation, reaction-diffusion, traveling waves, generalized transition waves, generalized principal eigenvalues.

only if $c \geq c^*$. This property was extended to more general monostable nonlinearities by Aronson and Weinberger [1978]. The properties (uniqueness, stability, attractivity, decay at infinity) of these waves have been extensively studied since then.

An increasing attention has been paid to heterogeneous reaction-diffusion since the 2000s. In particular, the existence of appropriate generalizations of traveling wave solutions has been proved for various classes of heterogeneities such as shear [Berestycki and Nirenberg 1992], time periodic [Alikakos et al. 1999], space-periodic [Berestycki and Hamel 2002; Berestycki et al. 2005; Xin 1992], space-time periodic [Nolen et al. 2005; Nadin 2009], time almost periodic [Shen 1999] and time uniquely ergodic [Shen 2011b], under several types of hypotheses on the nonlinearity. Now, the topical question is to understand whether reaction-diffusion equations with general heterogeneous coefficients admit wave-like solutions or not. A generalization of the notion of traveling waves has been given by Berestycki and Hamel [2007; 2012].

Definition 1.1 [Berestycki and Hamel 2007; 2012]. A *generalized transition wave* (in the direction $e \in S^{N-1}$) is a positive time-global solution u of (1) such that there exists a function $c \in L^\infty(\mathbb{R})$ satisfying[1]

$$\lim_{x\cdot e\to-\infty} u\left(x+e\int_0^t c(s)\,ds,t\right)=1, \qquad \lim_{x\cdot e\to+\infty} u\left(x+e\int_0^t c(s)\,ds,t\right)=0, \tag{2}$$

uniformly with respect to $t \in \mathbb{R}$. The function c is called the *speed* of the generalized transition wave u, and $\phi(x,t) := u\big(x + e\int_0^t c(s)\,ds, t\big)$ is the associated *profile*.

The profile of a generalized transition wave satisfies

$$\lim_{x\cdot e\to-\infty}\phi(x,t)=1, \qquad \lim_{x\cdot e\to+\infty}\phi(x,t)=0, \qquad \text{uniformly with respect to } t\in\mathbb{R}.$$

It is clear that any perturbation of c obtained by adding a function with bounded integral is still a speed of u, with a different profile. Reciprocally, if $\tilde{c}$ is another speed associated with u, then it is easy to check that $t \mapsto \int_0^t (c-\tilde{c})\,ds$ is bounded. Obviously, all the notions of waves used previously when the coefficients belong to particular classes of heterogeneities can be viewed as transition waves.

The existence of such waves has been proved for one-dimensional space heterogeneous equations with ignition-type nonlinearities (that is, $f(x,u) = 0$ if $u \in [0,\theta) \cup \{1\}$ and $f(x,u) > 0$ if $u \in (\theta,1)$) in parallel ways by Nolen and Ryzhik [2009] and Mellet, Roquejoffre and Sire [Mellet et al. 2010], and their stability was proved in [Mellet et al. 2009]. For space heterogeneous monostable nonlinearities, when $f(x,u) > 0$ if $u \in (0,1)$ and $f(x,0) = f(x,1) = 0$, transition waves might not exist [Nolen et al. 2012] in general. This justified the introduction of the alternative notion of *critical traveling wave* in [Nadin 2014] for one-dimensional equations. Some existence results have also been obtained by Zlatos for partially periodic multidimensional equations of ignition-type [Zlatoš 2013].

When the coefficients only depend on t in a general way, the existence of transition waves was first proved by Shen for bistable nonlinearities [2006] (that is, nonlinearities vanishing at $u = 0$ and $u = 1$ but negative near these two equilibria) and for monostable equations with time uniquely ergodic coefficients

[1] For a given function $g = g(x,t)$, the condition $\lim_{x\cdot e\to\pm\infty} g(x,t) = l$ uniformly with respect to $t \in \mathbb{R}$ means that

$$\lim_{r\to+\infty} \sup_{\substack{\pm x\cdot e>r\\ t\in\mathbb{R}}} |g(x,t)-l| = 0.$$

[2011b]. The case of general time heterogeneous monostable equations was investigated in [Nadin and Rossi 2012], where it was observed that the notions of least and upper mean play a crucial role in such frameworks.

Definition 1.2. The *least mean* (resp. the *upper mean*) *over* $\mathbb{R}$ of a function $g \in L^\infty(\mathbb{R})$ is given by

$$\lfloor g \rfloor := \lim_{T\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{T} \int_t^{t+T} g(s)\,ds \quad \left(\text{resp. } \lceil g \rceil := \lim_{T\to+\infty} \sup_{t\in\mathbb{R}} \frac{1}{T} \int_t^{t+T} g(s)\,ds\right).$$

As shown in Proposition 3.1 of [Nadin and Rossi 2012], the definitions of $\lfloor g \rfloor$, $\lceil g \rceil$ do not change if one replaces $\lim_{T\to+\infty}$ with $\sup_{T>0}$ and $\inf_{T>0}$ respectively in the above expressions; this shows that $\lfloor g \rfloor$, $\lceil g \rceil$ are well defined for any $g \in L^\infty(\mathbb{R})$. Notice that g admits a uniform mean $\langle g \rangle$, that is, $\langle g \rangle := \lim_{T\to+\infty} \frac{1}{T} \int_t^{t+T} g(s)\,ds$ exists uniformly with respect to $t \in \mathbb{R}$, if and only if $\lfloor g \rfloor = \lceil g \rceil = \langle g \rangle$. This is the case in particular when the coefficients are uniquely ergodic.

Note that if c and $\tilde{c}$ are two speeds associated with the same wave u, then $c - \tilde{c}$ has a bounded integral and thus $\lfloor c \rfloor = \lfloor \tilde{c} \rfloor$.

It is proved in [Nadin and Rossi 2012] that when $A \equiv I_N$, $q \equiv 0$ and f only depends on (t, u) and is concave and positive with respect to $u \in (0, 1)$, there exists a speed $c_* > 0$ such that, for all $\gamma > c_*$ and $|e| = 1$, Equation (1) admits a generalized transition wave with speed $c = c(t)$ in the direction e such that $\lfloor c \rfloor = \gamma$, while no such waves exist when $\gamma < c_*$.

When the coefficients not only depend on t in a general way but also on x periodically, some of the above results have been extended. Assuming in addition that the coefficients are uniquely ergodic and recurrent with respect to t and that $A \equiv I_N$, Shen [2011a] proved the existence of a quantity c_* such that, for all $\gamma > c_*$, there exists a generalized transition wave for monostable equations with speed c.

The case of space-periodic and time general monostable equations was first studied in [Rossi and Ryzhik 2014], under the additional assumption that the dependences in t and x are separated, in the sense that A and q only depend on x, periodically, while f only depends on (t, u). They proved both the existence of generalized transition waves of speed c such that $\lfloor c \rfloor > c_*$ and the nonexistence of such waves with $\lfloor c \rfloor < c_*$. Moreover, they provided a more general nonexistence result, without assuming that the dependence on x of A and q is periodic.

The aim of the present paper is to consider the general case of coefficients depending on both x and t. As in [Rossi and Ryzhik 2014], we assume the periodicity in x only for the existence result.

2. Hypotheses and results

2A. ***Statement of the main results.*** Throughout the paper, the terms in (1) will always be assumed to satisfy the following (classical) regularity hypotheses:

(3) A is symmetric and uniformly continuous, and there exist $0 < \underline{\alpha} \le \bar{\alpha}$ such that, for all $(x, t) \in \mathbb{R}^{N+1}$, $\underline{\alpha} I \le A(x, t) \le \bar{\alpha} I$.

(4) q is bounded and uniformly continuous on $\mathbb{R}^{N+1}$.

(5) f is a Caratheodory function on $\mathbb{R}^{N+1} \times [0,1]$, and there exists $\delta > 0$ such that $f(x,t,\cdot) \in W^{1,\infty}([0,1]) \cap C^1([0,\delta))$ uniformly in $(x,t) \in \mathbb{R}^{N+1}$.

The assumption that q is uniformly continuous is a technical hypothesis that is used in the proofs in order to pass to the limit in sequences of translations of the equation. It could be replaced by $\operatorname{div} q = 0$. We further assume that f is of monostable type, 0 being the unstable equilibrium and 1 being the stable one. Namely,

$$\forall (x,t) \in \mathbb{R}^{N+1}, \quad f(x,t,0) = 0, \tag{6}$$

$$\forall (x,t) \in \mathbb{R}^{N+1}, \quad f(x,t,1) = 0, \tag{7}$$

$$\forall u \in (0,1), \quad \inf_{(x,t)\in\mathbb{R}^{N+1}} f(x,t,u) > 0. \tag{8}$$

In order to derive the existence result, we need some additional hypotheses. The first one is the standard KPP condition,

$$\forall (x,t) \in \mathbb{R}^{N+1},\ u \in [0,1], \quad f(x,t,u) \le \mu(x,t)u, \tag{9}$$

where, here and in the sequel, μ denotes the function

$$\mu(x,t) := \partial_u f(x,t,0).$$

Conditions (8), (9) imply that $\inf \mu > 0$. The second condition is

$$\exists C > 0,\ \delta, \nu \in (0,1], \quad \forall x \in \mathbb{R}^N,\ t \in \mathbb{R},\ u \in (0,\delta), \quad f(x,t,u) \ge \mu(x,t)u - Cu^{1+\nu}. \tag{10}$$

Note that a sufficient condition for (10) to hold is $f(x,t,\cdot) \in C^{1+\nu}([0,\delta])$, uniformly with respect to x, t. The last condition for the existence result is

$$\exists l = (l_1,\dots,l_N) \in \mathbb{R}^N_+, \quad \forall t \in \mathbb{R},\ u \in (0,1), \quad A, q, f \text{ are } l\text{-periodic in } x, \tag{11}$$

where a function g is said to be l-periodic in x if it satisfies

$$\forall j \in \{1,\dots,N\}, \forall x \in \mathbb{R}^N, \quad g(x + l_j e_j) = g(x),$$

$(e_1,\dots,e_N)$ being the canonical basis of $\mathbb{R}^N$.

When we say that a function is a solution (or subsolution or supersolution) of (1) we always mean that it is between 0 and 1. We deal with strong solutions whose derivatives ∂_t, D, D^2 belong to some $L^p(\mathbb{R}^{N+1})$, $p \in (1,\infty)$. Many of our statements and equations, such as (1), are understood to hold almost everywhere, even if we omit to specify it, and inf, sup are used in place of ess inf, ess sup.

The main results of this paper consist of sufficient and necessary conditions for the existence of generalized transition waves, expressed in terms of their speeds.

Theorem 2.1. *Under the assumptions (3)–(11), for all $e \in S^{N-1}$, there exists $c_* \in \mathbb{R}$ such that, for every $\gamma > c_*$, there is a generalized transition wave in the direction e with a speed c such that $\lfloor c \rfloor = \gamma$.*

The minimal speed c_* we construct is explicitly given by (29), (34) and (37). A natural question is to determine whether our construction gives an optimal speed or not; that is, do generalized transition waves with speed c such that $\lfloor c \rfloor < c_*$ exist? One naturally starts by checking if our c_* coincides with the optimal speed known to exist in some particular cases, such as space-time periodic or space independent. In Section 2C we show that this is the case. The answer in the general, non-space-periodic, case is only partial. It is contained in the next theorem, where, however, we can relax the monostability hypotheses (8)–(9) by

$$\Big\lfloor \inf_{x\in\mathbb{R}^N} \mu(x,\cdot) \Big\rfloor > 0, \tag{12}$$

and we can drop (7), (10) as well as (11). We actually need an extra regularity assumption on A:

$$A \text{ is uniformly Hölder-continuous in } x, \text{ uniformly with respect to } t. \tag{13}$$

This ensures the validity of some a priori Lipschitz estimates quoted from [Porretta and Priola 2013] that will be needed in the sequel. It is not clear to us if such estimates hold without (13).

Theorem 2.2. *Under the assumptions (3)–(6), (12)–(13), for all $e \in S^{N-1}$, there exists $c^* \in \mathbb{R}$ such that if c is the speed of a generalized transition wave in the direction e then $\lfloor c \rfloor \geq c^*$.*

We point out that no spatial-periodicity condition is assumed in the previous statement. In order to prove Theorem 2.2 we derive a characterization of the least mean — Proposition 4.4 below — that we believe to be of independent interest. The definition of c^* is given in Section 4. Of course, $c^* \leq c_*$ if the hypotheses of both Theorems 2.1 and 2.2 are fulfilled. We do not know if, in general, $c_* = c^*$, that is, if the speed c_* is minimal, in the sense that there does not exist any wave with a speed having a smaller least mean. When the coefficients are periodic in space and time or only depend on time, we could identify the speed c_* more explicitly (see Section 2C below). Indeed, we recover in these frameworks some characterizations of the speeds identified in earlier papers [Nadin 2009; Nadin and Rossi 2012; Rossi and Ryzhik 2014], which were proved to be minimal. In the general framework, we leave this question open.

Finally, we leave as an open problem the case $\lfloor c \rfloor = c_*$, for which we believe that generalized transition waves still exist.

2B. *Optimality of the monostability assumption.* The assumption (8) implies that 0 and 1 are respectively unstable and stable. Let us discuss the meaning and the optimality of this hypothesis, which might seem strong. Actually, as we do not make any additional assumption on the coefficients, we can consider much more general asymptotic states $p_- = p_-(x,t) < p_+ = p_+(x,t)$ in place of 0 and 1 and try to construct generalized transition waves v connecting p_- to p_+. Indeed, if $p_\pm$ are solutions to (1), with $p_+ - p_-$ bounded and having positive infimum, then the change of variables

$$u(x,t) := \frac{v(x,t) - p_-(x,t)}{p_+(x,t) - p_-(x,t)}$$

leads to an equation of the same form, with reaction term

$$\tilde{f}(x,t,u) := \frac{f(x,t,up_+ + (1-u)p_-) - uf(x,t,p_+) - (1-u)f(x,t,p_-)}{p_+ - p_-}.$$

The new equation admits the steady states 0 and 1. Moreover, assuming that $u \mapsto f(x,t,u)$ is strictly concave, then $\tilde{f}$ satisfies conditions (8), (9), the latter following from the inequality

$$\forall u \in (0,1), \quad u(p_+ - p_-)\partial_u f(x,t,p_-) \geq f(x,t,up_+ + (1-u)p_-) - f(x,t,p_-).$$

This shows that, somehow, the concavity hypothesis of the nonlinearity with respect to u is stronger, up to some change of variables, than the positivity hypothesis of the nonlinearity.

Let us illustrate the above procedure with an explicit example where $p_- \equiv 0$. Consider the equation

$$\partial_t v = \Delta v + \mu(x,t)v - v^2, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{14}$$

with μ periodic in x, bounded and such that $\inf \mu > 0$. The later condition implies that the solution 0 is linearly unstable (actually, it can be relaxed by (12); see the discussion below). Then one can check that there is a time-global solution $p = p(x,t)$ which is bounded, has a positive infimum and is periodic in x. Let $u := v/p$. This function satisfies

$$\partial_t u = \Delta u + 2\frac{\nabla p}{p} \cdot \nabla u + p(x,t)u(1-u),$$

which is an equation of the form (1) for which (9)–(11) hold, at least if, for instance, μ is uniformly Hölder-continuous, since then ∇p is bounded by Schauder's parabolic estimates, and $\inf p > 0$.

Following this example, one can wonder whether (8) is an optimal condition (up to some change of variables) for the existence of transition waves. It is well-known that other classes of nonlinearities, such as bistable or ignition ones, could still give rise to transition waves (see for instance [Berestycki and Hamel 2002]). Thus, this question only makes sense if one reduces to the class of nonlinearities which are monostable, in a sense. Let us assume that f satisfies (6), (7) and that 0 is linearly unstable, in the weak sense that (12) holds. Then, using the properties of the least mean derived in [Nadin and Rossi 2012], one can construct arbitrarily small subsolutions $\underline{u} = \underline{u}(t)$ and thus, as 1 is a positive solution, there exists a minimal solution p of (1) in the class of bounded solutions with positive infimum. One could then check that our proof still works and gives rise to generalized transition waves connecting 0 to p. Indeed, condition (8) only ensures that $p \equiv 1$. As a conclusion, the positivity hypothesis (8) is not optimal: one could replace it by (12) but then the generalized transition waves we construct connect 0 to the minimal time-global solution, which might not be 1.

Since for the existence of positive solutions it is sufficient to require (12) rather than $\inf \mu > 0$, one may argue that, in order to guarantee that 1 is the minimal time-global solution with positive infimum, hypothesis (8) could be relaxed by

$$\forall u \in (0,1), \quad \left\lfloor \min_{x \in \mathbb{R}^N} f(x,\cdot,u) \right\rfloor > 0. \tag{15}$$

This is not true, as shown by the following example. Let $p \in C^1(\mathbb{R})$ be a strictly decreasing function such that $p(\pm\infty) \in (0,1)$. Let f satisfy $f(t,p(t)) = p'(t)$. It is clear that f can be extended in such a way that (15) holds; however p is a time-global solution of $\partial_t u = f(t,u)$ with positive infimum which is smaller than 1.

Finally, if 0 is linearly stable, in the sense that

$$\left\lceil \sup_{x\in\mathbb{R}^N} \mu(x,\cdot)\right\rceil < 0 \tag{16}$$

holds, and (9) is satisfied, then there do not exist generalized transition waves at all, and, more generally, solutions to the Cauchy problem with bounded initial data converge uniformly to 0 as $t \to \infty$. Indeed, as an easy application of the property of the least (and upper) mean (39), one can construct a supersolution $\bar{u} = e^{\sigma(t)-\varepsilon t}$, for some $\sigma \in W^{1,\infty}(\mathbb{R})$ and $\varepsilon > 0$. The convergence to 0 of bounded solutions then follows from the comparison principle.

2C. ***Description of the method and application to particular cases.*** The starting point of the construction of generalized transition waves consists of finding an explicit expression for the speed. This is not a trivial task in the case of mixed space-time dependence considered in this paper. We achieve it by a heuristic argument that we now illustrate.

Suppose that u is a generalized transition wave in a direction $e \in S^{N-1}$. Its tail at large $x \cdot e$ is close from being a solution of the linearized equation around 0:

$$\partial_t u - \mathrm{Tr}(A(x,t)D^2u) + q(x,t)\cdot Du = \mu(x,t)u. \tag{17}$$

It is natural to expect the tail of u to decay exponentially. Thus, since the equation is spatially periodic, we look for (the tail of) u under the form

$$u(x,t) = e^{-\lambda x\cdot e}\eta_\lambda(x,t), \quad \text{with } \eta_\lambda \text{ positive and } l\text{-periodic in } x. \tag{18}$$

Rewriting this expression as

$$u(x,t) = \exp\Big(-\lambda\Big(x\cdot e - \frac{1}{\lambda}\ln \eta_\lambda(x,t)\Big)\Big)$$

shows that the speed of u, namely, a function c for which (2) holds, should satisfy

$$\left|\int_0^t c(s)\,ds - \frac{1}{\lambda}\ln\eta_\lambda(x,t)\right| \le C,$$

for some C independent of $(x,t) \in \mathbb{R}^N \times \mathbb{R}$. Clearly, this can hold true only if the ratio between maximum and minimum of $\eta_\lambda(\cdot,t)$ is bounded uniformly on t. This property follows from a Harnack-type inequality, Lemma 3.1 below, which is the keystone of our proof and actually the only step where the periodicity in x really plays a role. It would be then natural to define $c(t) := \frac{1}{\lambda}\frac{d}{dt}\ln\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R})}$. The problem is that we do not know if this function is bounded, since it is not clear whether $\partial_t\eta_\lambda \in L^\infty(\mathbb{R}^{N+1})$ or not. We overcome this difficulty by showing that there exists a Lipschitz continuous function S_λ such that

$$\exists\beta > 0, \quad \forall t \in \mathbb{R}, \quad \left|S_\lambda(t) - \frac{1}{\lambda}\ln\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}\right| \le \beta. \tag{19}$$

We deduce that the function c defined (almost everywhere) by $c := S'_\lambda$ is bounded and it is an admissible speed for the wave u. The method described above provides, for any given $\lambda > 0$, a wave with speed $c = c_\lambda$ for the *linearized* equation which decays with exponential rate λ. It is known — for instance in the case of constant coefficients — that only decaying rates which are "not too fast" are admissible for waves of the nonlinear reaction-diffusion equation. In Section 3C, we identify a threshold rate λ_*. In the following section we construct generalized transition waves for any $\lambda < \lambda_*$, recovering with the least mean of their speeds the whole interval $(\lfloor c_{\lambda_*} \rfloor, +\infty)$. We do not know if the critical speed $c_* := \lfloor c_{\lambda_*} \rfloor$ is optimal, nor if an optimal speed does exist. However, we show below that this is the case if one applies the above procedure to some particular classes of heterogeneities already investigated in the literature.

In the case where the coefficients are periodic in time too, the class of admissible speeds has been characterized in [Nadin 2009] (see also [Berestycki et al. 2008]). Following the method described above, we see that an entire solution of (17) in the form (18) is given by $\eta_\lambda(x,t) = e^{k(\lambda)t}\varphi_\lambda(x,t)$, where $(k(\lambda), \varphi_\lambda)$ are the principal eigenelements of the problem[2]

$$\begin{cases} \partial_t\varphi_\lambda - \mathrm{Tr}(AD^2\varphi_\lambda) + (q + 2\lambda Ae)D\varphi_\lambda - (\mu + \lambda^2 eAe + \lambda q\cdot e)\varphi_\lambda + k(\lambda)\varphi_\lambda = 0 \ \text{ in } \mathbb{R}^N \times \mathbb{R}, \\ \varphi_\lambda > 0, \\ \varphi_\lambda \text{ is periodic in } t \text{ and } x. \end{cases} \tag{20}$$

Actually, the uniqueness up to a multiplicative constant of solutions of (17) in the form (18), provided by Lemma 3.1 (proved without assuming the time-periodicity), implies that η_λ necessarily has this form. Thus, $S_\lambda(t) := (k(\lambda)/\lambda)t$ satisfies (19), whence the speed of the wave for the linearized equation with decaying rate λ is $c_\lambda \equiv k(\lambda)/\lambda$. Since the c_λ are constant (and therefore they have uniform mean), it turns out that the threshold λ_* we obtain for the decaying rates coincides with the minimum point of $\lambda \mapsto c_\lambda$ (see Remark 1 below). We eventually derive the existence of a generalized transition wave for any speed larger than $c_* := \min_{\lambda>0} k(\lambda)/\lambda$, which is exactly the sharp critical speed for pulsating traveling fronts obtained in [Nadin 2009]. To sum up, our construction of the minimal speed c_* is optimal in the space-time periodic framework. On the other hand, in the periodic framework, the speed c^* constructed in Section 4 is identical to c_* and thus Theorem 2.2 implies that there do not exist generalized transition waves with a speed c such that $\lfloor c \rfloor < \min_{\lambda>0} k(\lambda)/\lambda$. We therefore recover also the nonexistence result for pulsating traveling fronts. Only the existence of fronts with critical speed is not recovered.

In the case investigated in [Nadin and Rossi 2012], namely, when $A \equiv I_N$, $q \equiv 0$ and f does not depend on x, one can easily check that $\eta_\lambda(t) = e^{\int_0^t \mu(s)\,ds + \lambda^2 t}$. As a function S_λ we can simply take $\frac{1}{\lambda}\ln\|\eta_\lambda(\cdot,t)\|_{L^\infty(\mathbb{R}^N)} = \frac{1}{\lambda}\int_0^t \mu(s)\,ds + \lambda t$, which is Lipschitz continuous. Hence $c_\lambda(t) := \lambda + \mu(t)/\lambda$ is a speed of a wave with decaying rate λ. In this case the critical decaying rate λ_* is equal to $\sqrt{\lfloor \mu \rfloor}$ (see again Remark 1) and thus we have $c_* = 2\sqrt{\lfloor \mu \rfloor}$. This is the same speed c_* as in [Nadin and Rossi 2012], which was proved to be minimal.

Under the assumptions made in [Rossi and Ryzhik 2014], that is, A and q only depend on x (periodically) and f only depends on (t,u), the speeds c_* derived in the present paper and in [Rossi and Ryzhik 2014] coincide, and thus it is minimal, in the sense that there do not exist any generalized transition waves with a lower speed.

[2] The properties of these eigenelements, which are unique (up to a multiplicative constant in the case of φ_λ) are described in [Nadin 2009] for instance.

When $A \equiv I_N$ and q, f are periodic in x and uniquely ergodic in t, then one can prove that the same holds true for the function $\partial_t \eta_\lambda / \eta_\lambda$ by uniqueness, and thus $\alpha \lfloor c_\alpha \rfloor$ could be identified with the Lyapounov exponent $\lambda(\alpha, \xi)$ used by Shen in [2011a], where ξ is the direction of propagation. We thus recover in this framework the same speed c_* as in [Shen 2011a], which was not proved to be minimal since the nonexistence of transition waves with lower speed were not investigated. Note that this identification is not completely obvious. However, as the formalism of the present paper and [Shen 2011a] are very different, we leave these computations to the reader.

Lastly, let us consider the following example, where one could indeed construct directly the generalized transition waves:

$$\partial_t u - \partial_{xx} u - q(t)\partial_x u = \mu_0 u(1-u), \tag{21}$$

with q bounded and uniformly continuous and $\mu_0 > 0$. This equation satisfies assumptions (3)–(12). The change of variables $v(x,t) := u\big(x - \int_0^t q, t\big)$ leads to the classical homogeneous Fisher–KPP equation $\partial_t v - \partial_{xx} v = \mu_0 v(1-v)$. This equation admits traveling wave solutions of the form $v(x,t) = \phi_c(x-ct)$, with $\phi_c(-\infty) = 1$ and $\phi_c(+\infty) = 0$, for all $c \ge 2\sqrt{\mu_0}$. Hence, Equation (21) admits generalized transition waves $u(x,t) = \phi_c\big(x - ct + \int_0^t q, t\big)$ of speed $c - q(t)$ if and only if $c \ge 2\sqrt{\mu_0}$. That is, the set of least mean of admissible speeds is $[2\sqrt{\mu_0} - \lceil q \rceil, +\infty)$. Computing c_* in this case, one easily gets

$$\eta_\lambda = \eta_\lambda(t) = e^{\lambda^2 t - \lambda \int_0^t q(s)\,ds + \mu_0 t}, \quad c_\lambda(t) = \lambda - q(t) + \mu_0/\lambda \quad \text{and} \quad c_* = 2\sqrt{\mu_0} - \lceil q \rceil.$$

One could check that c^* coincides with this value too, meaning that Theorems 2.1 and 2.2 fully characterize the possible least means for admissible speeds, except for the critical one.

3. Existence result

Throughout this section, we fix $e \in S^{N-1}$ and we assume that conditions (3)–(11) hold. Actually, condition (8) could be weakened by (12), except for the arguments in the very last part of the proof in Section 3D. As already mentioned in Section 2B, these arguments could be easily adapted to the case where (8) is replaced by (12), leading to transition waves connecting 0 to the minimal solution with positive infimum.

3A. ***Solving the linearized equation.*** We focus on solutions with prescribed spatial exponential decay.

Lemma 3.1. *For all $\lambda > 0$, the equation (17) admits a time-global solution of the form (18). Moreover, η_λ is unique up to a multiplicative constant and satisfies, for all $t \in \mathbb{R}$, $T \ge 0$,*

$$\max_{x\in\mathbb{R}^N} \eta_\lambda(x, t+T) \le \max_{x\in\mathbb{R}^N} \eta_\lambda(x,t) \exp\bigg(\Big(\overline{\alpha}\lambda + \sup_{\mathbb{R}^{N+1}} |q|\Big)\lambda T + \int_t^{t+T} \max_{x\in\mathbb{R}^N} \mu(x,s)\,ds\bigg), \tag{22}$$

$$\min_{x\in\mathbb{R}^N} \eta_\lambda(x, t+T) \ge C \max_{x\in\mathbb{R}^N} \eta_\lambda(x,t) \exp\bigg(\Big(\underline{\alpha}\lambda - \sup_{\mathbb{R}^{N+1}} |q|\Big)\lambda T + \int_t^{t+T} \min_{x\in\mathbb{R}^N} \mu(x,s)\,ds\bigg), \tag{23}$$

with $C > 0$ only depending on a constant bounding $|\lambda|$, $|l|$, $\underline{\alpha}^{-1}$, $\overline{\alpha}$, N and the L^∞ norms of μ and q.

The function $(x,t) \mapsto e^{-\lambda x\cdot e}\eta_\lambda(x,t)$ is a solution of the linearization of (1) near the unstable equilibrium. We will show in the next section that it is somehow a transition wave solution of the linearized equation, in the sense that it moves in the direction e with a certain speed. Due to hypothesis (9), we could use it as a supersolution of the nonlinear equation. Then, in Section 3C, in order to construct an appropriate subsolution, we will need to restrict to exponents λ less than some threshold λ_*. We will eventually derive the existence of transition waves in Section 3D.

As mentioned in Section 2C, Lemma 3.1 is the only point where the spatial periodicity hypothesis (11) is used. If the coefficients depend in a general way on both x and t and if one is able to construct a solution η_λ of equation (25) for which there exists $C > 0$ such that, for all $T > 0$, $(x, t) \in \mathbb{R}^{N+1}$, one has

$$\frac{1}{C}\|\eta_\lambda(\,\cdot\,, t)\|_{L^\infty(\mathbb{R}^N)} e^{-CT} \le \eta_\lambda(x, t+T) \le C\|\eta_\lambda(\,\cdot\,, t)\|_{L^\infty(\mathbb{R}^N)} e^{CT}. \tag{24}$$

Then the forthcoming other steps of the proof still apply and it is possible to construct a generalized transition wave solution of the nonlinear equation (1). We describe this extension in Section 3E below. It would be very useful to determine optimal conditions on the coefficients enabling the derivation of a global Harnack-type inequality such as (24) for the linearized equation. We leave this question as an open problem.

Proof of Lemma 3.1. The problem for η_λ is

$$\partial_t \eta_\lambda = \operatorname{Tr}(AD^2\eta_\lambda) - (q + 2\lambda Ae)\cdot D\eta_\lambda + (\mu + \lambda^2 eAe + \lambda q\cdot e)\eta_\lambda, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}. \tag{25}$$

We find a positive, l-periodic solution to (25) as the locally uniform limit of (a subsequence of) solutions η^n of the problem in $\mathbb{R}^N \times (-n, +\infty)$, with initial datum $\eta^n(-n, \cdot\,) \equiv m_n$, where m_n is a positive constant chosen in such a way that, say, $\sup_{x\in\mathbb{R}^N} \eta^n(0, x) = 1$.

Let us show that any l-periodic solution η_λ to (25) satisfies (22) and (23). For a given $t_0 \in \mathbb{R}$, the function

$$\max_{x\in\mathbb{R}^N} \eta_\lambda(x, t_0)\ \exp\biggl(\Bigl(\bar\alpha\lambda^2 + \sup_{\mathbb{R}^{N+1}} |q|\lambda\Bigr)(t - t_0) + \int_{t_0}^{t} \max_{x\in\mathbb{R}^N} \mu(x, s)\, ds\biggr)$$

is a supersolution of (25) larger than η_λ at time t_0. Since η_λ is bounded, we can apply the parabolic comparison principle and derive (22). Let $\mathcal{C}$ denote the periodicity cell $\prod_{j=1}^{N}[0, l_j]$. By the parabolic Harnack inequality (see, e.g., Corollary 7.42 in [Lieberman 1996]), we have that

$$\forall t \in \mathbb{R}, \quad \max_{x\in\mathcal{C}} \eta_\lambda(x, t-1) \le \tilde{C} \min_{x\in\mathcal{C}} \eta_\lambda(x, t), \tag{26}$$

for some $\tilde{C} > 0$ depending on a constant bounding $|\lambda|$, $|l|$, $\underline{\alpha}^{-1}$, $\bar\alpha$, N and the L^∞ norms of μ and q, and not on t. On the other hand, the comparison principle yields, for $T \ge 0$,

$$\min_{x\in\mathbb{R}^N} \eta_\lambda(x, t+T) \ge \min_{x\in\mathbb{R}^N} \eta_\lambda(x, t)\ \exp\biggl(\Bigl(\underline{\alpha}\lambda^2 - \sup_{\mathbb{R}^{N+1}} |q|\lambda\Bigr)T + \int_{t}^{t+T} \min_{x\in\mathbb{R}^N} \mu(x, s)\, ds\biggr).$$

Combining this inequality with (26) we eventually derive

$$\min_{x\in\mathbb{R}^N} \eta_\lambda(x, t+T) \ge \tilde{C}^{-1} \max_{x\in\mathbb{R}^N} \eta_\lambda(x, t-1)\ \exp\biggl(\Bigl(\underline{\alpha}\lambda^2 - \sup_{\mathbb{R}^{N+1}} |q|\lambda\Bigr)T + \int_{t}^{t+T} \min_{x\in\mathbb{R}^N} \mu(x, s)\, ds\biggr),$$

from which (23) follows by (22).

It remains to prove the uniqueness result. Assume that (17) admits two solutions η^1, η^2 that are positive and l-periodic in x. As shown before, we know that they both satisfy (22) and (23). We first claim that there exists $K > 1$ such that

$$\forall t \in \mathbb{R},\ x \in \mathbb{R}^N, \quad K^{-1}\eta^2(x,t) \le \eta^1(x,t) \le K\eta^2(x,t). \tag{27}$$

Let $h > 0$ be such that $\eta^1 \le h\eta^2$ at $t = 0$. It follows, for $t \le 0$, that $\min_{x\in\mathbb{R}^N} \eta^1(x,t) \le h \max_{x\in\mathbb{R}^N} \eta^2(x,t)$, because otherwise the parabolic strong maximum principle would imply $\eta^1 > h\eta^2$ at $t = 0$. Hence, applying (23) with $T = 0$ to both η^1 and η^2, we find a positive constant K such that

$$\forall t < 0, \quad \max_{x\in\mathbb{R}^N} \eta^1(x,t) \le K \min_{x\in\mathbb{R}^N} \eta^2(x,t).$$

This proves the second inequality in (27), for $t < 0$, whence for all $t \in \mathbb{R}$ by the maximum principle. The first inequality, with a possibly larger K, is obtained by exchanging the roles of η^1 and η^2. Now define

$$k := \limsup_{t\to-\infty} \max_{x\in\mathbb{R}^N} \frac{\eta^1(x,t)}{\eta^2(x,t)}.$$

We know from (27) that $k \in [K^{-1}, K]$. Consider a sequence $(t_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} t_n = -\infty, \qquad \lim_{n\to\infty} \max_{x\in\mathbb{R}^N} \frac{\eta^1(x,t_n)}{\eta^2(x,t_n)} = k.$$

Define the sequences of functions $(\eta_n^1)_{n\in\mathbb{N}}$, $(\eta_n^2)_{n\in\mathbb{N}}$ as follows:

$$\forall i \in \{1,2\},\ n \in \mathbb{N}, \quad \eta_n^i(x,t) := \frac{\eta^i(x,t+t_n)}{\max_{y\in\mathbb{R}^N} \eta^1(y,t_n)}.$$

We deduce from (22) and (23) that the $(\eta_n^1)_{n\in\mathbb{N}}$ are uniformly bounded from above and uniformly bounded from below away from 0 in, say, $\mathbb{R}^N \times [-2, 2]$. The same is true for $(\eta_n^2)_{n\in\mathbb{N}}$ by (27). Thus, by parabolic estimates and periodicity in x, the sequences $(\eta_n^i)_n$, $(\partial_t \eta_n^i)_n$, $(D\eta_n^i)_n$ and $(D^2\eta_n^i)_n$ converge, up to subsequences, in $L^p_{\mathrm{loc}}(\mathbb{R}^{N+1})$. Morrey's inequality yields that the sequences $(\eta_n^1)_n$ and $(\eta_n^2)_n$ converge locally uniformly to some functions $\tilde\eta^1$ and $\tilde\eta^2$ respectively.

Define $A_n := A(\cdot,\cdot+t_n)$, $q_n := q(\cdot,\cdot+t_n)$, $\mu_n := \mu(\cdot,\cdot+t_n)$. As A and q are uniformly continuous, $(A_n)_n$ and $(q_n)_n$ converge (up to subsequences) to some functions $\tilde A$ and $\tilde q$ in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{N+1})$, whereas $(\mu_n)_n$ converges to some $\tilde\mu$ in the $L^\infty(\mathbb{R}^{N+1})$ weak-$*$ topology. Hence, taking the weak $L^p_{\mathrm{loc}}(\mathbb{R}^{N+1})$ limit as $n \to \infty$ in the equations satisfied by the $(\eta_n^i)_{n\in\mathbb{N}}$, we get

$$\partial_t \tilde\eta^i = \mathrm{Tr}(\tilde A D^2 \tilde\eta^i) - (\tilde q + 2\lambda \tilde A e) D\tilde\eta^i + (\tilde\mu + \lambda^2 e\tilde A e + \lambda \tilde q \cdot e)\tilde\eta^i, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

Clearly, these equations hold almost everywhere because all the terms are measurable functions. That is, the $\tilde\eta^i$ are strong solutions. Moreover,

$$\tilde\eta^1 \le k\tilde\eta^2, \qquad \max_{x\in\mathbb{R}^N} \frac{\tilde\eta^1(x,0)}{\tilde\eta^2(x,0)} = k.$$

The strong maximum principle then yields $\tilde\eta^1 \equiv k\tilde\eta^2$. As a consequence, for any $\varepsilon > 0$, we can find $n_\varepsilon \in \mathbb{N}$ such that, for $n \ge n_\varepsilon$, one has $(k-\varepsilon)\eta_n^2 < \eta_n^1 < (k+\varepsilon)\eta_n^2$ at $t=0$. These inequalities hold for all $t \ge 0$, again by the maximum principle. Reverting to the original functions we obtain $(k-\varepsilon)\eta^2 < \eta^1 < (k+\varepsilon)\eta^2$ for $t \ge t_n$ and $n \ge n_\varepsilon$, from which, letting $n \to \infty$ and $\varepsilon \to 0^+$, we eventually infer that $\eta^1 \equiv k\eta^2$ for all $t \in \mathbb{R}$. □

In the particular case $T = 0$, the inequality (23) reads

$$\min_{x\in\mathbb{R}^N} \eta_\lambda(x,t) \ge C \max_{x\in\mathbb{R}^N} \eta_\lambda(x,t). \tag{28}$$

Notice that, in contrast with the standard parabolic Harnack inequality, the two sides are evaluated at the same time. This particular instance of (23) will be used in the sequel.

Until the end of the proof of Theorem 2.1, for $\lambda > 0$, we let η_λ stand for the (unique up to a multiplicative constant) function given by Lemma 3.1.

3B. *The speeds of the waves.*

Lemma 3.2. *There is a uniformly Lipschitz-continuous function $S_\lambda : \mathbb{R} \to \mathbb{R}$ satisfying* (19).

Proof. Properties (22)–(23) yield the existence of a constant $\beta > 0$ such that

$$\forall t \in \mathbb{R},\ T \ge 0, \quad \bigl|\ln \|\eta_\lambda(\cdot\,,t+T)\|_{L^\infty(\mathbb{R}^N)} - \ln \|\eta_\lambda(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}\bigr| \le \beta(1+\lambda^2)T.$$

For all $n \in \mathbb{N}$, we define S_λ on $[n, n+1]$ as the affine function satisfying

$$S_\lambda(n) = \frac{1}{\lambda}\ln \|\eta_\lambda(\cdot\,,n)\|_{L^\infty(\mathbb{R}^N)}, \qquad S_\lambda(n+1) = \frac{1}{\lambda}\ln \|\eta_\lambda(\cdot\,,n+1)\|_{L^\infty(\mathbb{R}^N)}.$$

Then, for all $t \in (n, n+1)$,

$$|S'_\lambda(t)| = \left|\frac{1}{\lambda}\ln \|\eta_\lambda(\cdot\,,n+1)\|_{L^\infty(\mathbb{R}^N)} - \frac{1}{\lambda}\ln \|\eta_\lambda(\cdot\,,n)\|_{L^\infty(\mathbb{R}^N)}\right| \le \beta\frac{1+\lambda^2}{\lambda}.$$

Hence, S_λ is uniformly Lipschitz-continuous over $\mathbb{R}$. Moreover, if $t \in [n, n+1]$, one has

$$\begin{aligned}\left|S_\lambda(t) - \frac{1}{\lambda}\ln \|\eta_\lambda(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}\right| &\le |S_\lambda(t) - S_\lambda(n)| + \frac{1}{\lambda}\bigl|\ln \|\eta_\lambda(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)} - \ln \|\eta_\lambda(\cdot\,,n)\|_{L^\infty(\mathbb{R}^N)}\bigr| \\ &\le 2\beta\frac{1+\lambda^2}{\lambda}.\end{aligned}$$

Hence, $t \mapsto S_\lambda(t) - \frac{1}{\lambda}\ln \|\eta_\lambda(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}$ is uniformly bounded over $\mathbb{R}$. □

Owing to Lemma 3.2, the function c_λ, defined for (almost everywhere) $t \in \mathbb{R}$ by

$$c_\lambda(t) := S'_\lambda(t), \tag{29}$$

belongs to $L^\infty(\mathbb{R})$. We will use it as a possible speed for a transition wave to be constructed.

Let us investigate the properties of the least mean of the $(c_\lambda)_{\lambda>0}$. It follows from (19) that

$$\lfloor c_\lambda \rfloor = \frac{1}{\lambda} \lim_{T\to+\infty} \inf_{t\in\mathbb{R}} \frac{1}{T} \ln \frac{\|\eta_\lambda(\cdot\,,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_\lambda(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}}. \tag{30}$$

Hence, by (22) and (23), we derive

$$\underline{\alpha}\lambda - \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\Big\lfloor \min_{x\in\mathbb{R}^N} \mu(x,\cdot)\Big\rfloor \le \lfloor c_\lambda \rfloor \le \bar{\alpha}\lambda + \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\Big\lfloor \max_{x\in\mathbb{R}^N} \mu(x,\cdot)\Big\rfloor. \tag{31}$$

Analogous bounds hold for the upper mean:

$$\underline{\alpha}\lambda - \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\Big\lceil \min_{x\in\mathbb{R}^N} \mu(x,\cdot)\Big\rceil \le \lceil c_\lambda \rceil \le \bar{\alpha}\lambda + \sup_{\mathbb{R}^{N+1}} |q| + \frac{1}{\lambda}\Big\lceil \max_{x\in\mathbb{R}^N} \mu(x,\cdot)\Big\rceil. \tag{32}$$

We have seen in Section 2C that, when the coefficients are periodic in t, one can take $S_\lambda(t) := (k(\lambda)/\lambda)t$, whence $c_\lambda \equiv k(\lambda)/\lambda$. It follows that $\lambda c_\lambda = k(\lambda)$, and we know from the arguments in the proof of Proposition 5.7 part (iii) in [Berestycki and Hamel 2002] that the function k is convex. In the general heterogeneous framework considered in the present paper, we use the same arguments as in [Berestycki and Hamel 2002] to derive the Lipschitz continuity of the function $\lambda \mapsto \lambda\lfloor c_\lambda\rfloor$. If the functions c_λ admit a uniform mean then these arguments actually imply that $\lambda \mapsto \lambda\lfloor c_\lambda\rfloor$ is convex, but we do not know if this is true in general.

Lemma 3.3. *The functions $\lambda \mapsto \lfloor c_\lambda\rfloor$ and $\lambda \mapsto \lceil c_\lambda\rceil$ are locally uniformly Lipschitz continuous on* $(0, +\infty)$.

Proof. Fix $\Lambda > 0$ and $-\Lambda \le \lambda_0 \le \Lambda$. Let λ_1 be such that $|\lambda_1 - \lambda_0| = 2\Lambda$. For $j = 0, 1$, the function $v_j(x,t) := e^{-\lambda_j x\cdot e}\eta_{\lambda_j}(x,t)$ satisfies (17). Hence, setting $v_j = e^{w_j}$, we find that

$$\partial_t w_j - \mathrm{Tr}(AD^2 w_j) + q\cdot Dw_j = \mu + \mathrm{Tr}(ADw_j \otimes Dw_j), \quad x\in\mathbb{R}^N,\ t\in\mathbb{R}.$$

For $\tau\in(0,1)$, the function $w := (1-\tau)w_0 + \tau w_1$ satisfies, for $x\in\mathbb{R}^N$, $t\in\mathbb{R}$,

$$\begin{aligned}\partial_t w - \mathrm{Tr}(AD^2 w) + q\cdot Dw &= \mu + \mathrm{Tr}\big(A((1-\tau)Dw_0\otimes Dw_0 + \tau Dw_1\otimes Dw_1)\big)\\ &\ge \mu + \mathrm{Tr}(ADw\otimes Dw).\end{aligned}$$

As a consequence, e^w is a supersolution of (17) and then, since

$$e^{w(x,t)} = e^{-((1-\tau)\lambda_0+\tau\lambda_1)x\cdot e}\eta_{\lambda_0}^{1-\tau}(x,t)\eta_{\lambda_1}^{\tau}(x,t),$$

the function $\eta_{\lambda_0}^{1-\tau}\eta_{\lambda_1}^{\tau}$ is a supersolution of (25) with $\lambda = \lambda_\tau := (1-\tau)\lambda_0 + \tau\lambda_1$. We can therefore apply the comparison principle between this function and η_{λ_τ} and derive, for $t\in\mathbb{R}$, $T>0$,

$$\begin{aligned}\frac{\|\eta_{\lambda_\tau}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_\tau}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} &\le \frac{\|\eta_{\lambda_0}^{1-\tau}\eta_{\lambda_1}^{\tau}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\min_{x\in\mathbb{R}^N}\eta_{\lambda_0}^{1-\tau}\eta_{\lambda_1}^{\tau}(x,t)}\\ &\le \left(\frac{\|\eta_{\lambda_0}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\min_{x\in\mathbb{R}^N}\eta_{\lambda_0}(x,t)}\right)^{1-\tau}\left(\frac{\|\eta_{\lambda_1}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\min_{x\in\mathbb{R}^N}\eta_{\lambda_1}(x,t)}\right)^{\tau}.\end{aligned}$$

Hence, using the inequality (28) for η_{λ_0} and η_{λ_1} (with the same C depending on Λ), we obtain

$$\frac{\|\eta_{\lambda_\tau}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_\tau}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}} \le C^{-1}\left(\frac{\|\eta_{\lambda_0}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_0}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}\right)^{1-\tau}\left(\frac{\|\eta_{\lambda_1}(\cdot,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_1}(\cdot,t)\|_{L^\infty(\mathbb{R}^N)}}\right)^{\tau}. \tag{33}$$

Consider the function Γ defined by $\Gamma(\lambda) := \lambda\lfloor c_\lambda\rfloor$. It follows from (30) and (33) that

$$\Gamma(\lambda_\tau) \le \lim_{T\to+\infty}\inf_{t\in\mathbb{R}}\frac{1}{T}\left((1-\tau)\ln\frac{\|\eta_{\lambda_0}(\cdot\,,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_0}(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}} + \tau\ln\frac{\|\eta_{\lambda_1}(\cdot\,,t+T)\|_{L^\infty(\mathbb{R}^N)}}{\|\eta_{\lambda_1}(\cdot\,,t)\|_{L^\infty(\mathbb{R}^N)}}\right).$$

If the $(c_\lambda)_{\lambda>0}$ admit a uniform mean, the above inequality and (30) imply that Γ is convex. Otherwise, we can only infer that

$$\Gamma(\lambda_\tau) \le (1-\tau)\Gamma(\lambda_0) + \tau\lambda_1\lceil c_{\lambda_1}\rceil.$$

We have therefore shown that

$$\forall\tau\in(0,1),\quad \Gamma(\lambda_\tau)-\Gamma(\lambda_0) \le \tau(\lambda_1\lceil c_{\lambda_1}\rceil - \lambda_0\lfloor c_{\lambda_0}\rfloor).$$

Thus, by (31) and (32) there exists a constant $K>0$, depending on A, q, μ, such that

$$\forall\tau\in(0,1),\quad \Gamma(\lambda_\tau)-\Gamma(\lambda_0) \le K(\Lambda^2+1)\tau.$$

This proves the Lipschitz continuity of Γ on $[-\Lambda,\Lambda]$, because $|\lambda_\tau-\lambda_0| = 2\Lambda\tau$, concluding the proof of the lemma.

The same arguments lead to the local Lipschitz continuity of $\lambda\mapsto\lceil c_\lambda\rceil$. □

3C. *Definition of the critical speed.* In order to define the critical speed c_*, we introduce the set

$$\Lambda := \{\lambda>0 : \exists\bar{k}>0,\ \forall 0<k<\bar{k},\ \lfloor c_\lambda - c_{\lambda+k}\rfloor > 0\}. \tag{34}$$

Lemma 3.4. *There exists $\lambda_*>0$ such that $\Lambda=(0,\lambda_*)$. Moreover, the function $\lambda\mapsto\lfloor c_\lambda\rfloor$ is decreasing on Λ.*

Proof. Fix $\lambda_0,\lambda_1>0$. For $\tau\in(0,1)$, we set $\lambda_\tau := (1-\tau)\lambda_0+\tau\lambda_1$. Taking the natural log of (33) and recalling that $c_\lambda = S'_\lambda$ with S_λ satisfying (19) yields

$$\int_t^{t+T}[(1-\tau)\lambda_0 c_{\lambda_0} + \tau\lambda_1 c_{\lambda_1} - \lambda_\tau c_{\lambda_\tau}]\,ds \ge \ln C - 4\lambda_\tau\beta.$$

Hence,

$$\lambda_\tau\int_t^{t+T}(c_{\lambda_0}-c_{\lambda_\tau})\,ds \ge \tau\lambda_1\int_t^{t+T}(c_{\lambda_0}-c_{\lambda_1})\,ds + \ln C - 4\lambda_\tau\beta.$$

Dividing both sides by T, taking the infimum over $t\in\mathbb{R}$ and then taking the limit as $T\to+\infty$, we derive

$$\forall\tau\in(0,1),\quad \lfloor c_{\lambda_0}-c_{\lambda_\tau}\rfloor \ge \tau\frac{\lambda_1}{\lambda_\tau}\lfloor c_{\lambda_0}-c_{\lambda_1}\rfloor. \tag{35}$$

If instead we divide by $-T$, we get

$$\forall\tau\in(0,1),\quad \lfloor c_{\lambda_\tau}-c_{\lambda_0}\rfloor \le \tau\frac{\lambda_1}{\lambda_\tau}\lfloor c_{\lambda_1}-c_{\lambda_0}\rfloor. \tag{36}$$

Analogous estimates hold of course for the upper mean. The characterization of Λ follows from these inequalities, by suitable choices of λ_0, λ_1 and τ. We prove it in four steps.

<u>*Step 1:*</u> $\Lambda \neq \varnothing$. The first inequality in (31), together with (12), yields

$$\lim_{\lambda\to 0^+} \lfloor c_\lambda - c_1 \rfloor \geq \lim_{\lambda\to 0^+} \lfloor c_\lambda \rfloor - \lceil c_1 \rceil = +\infty.$$

Then there exists $0 < \lambda < 1$ such that $\lfloor c_\lambda - c_1 \rfloor > 0$. Applying (35) with $\lambda_0 = \lambda,\ \lambda_1 = 1$, we eventually infer that $\lfloor c_\lambda - c_{\lambda+k} \rfloor > 0$, for all $0 < k < 1-\lambda$; that is, $\lambda \in \Lambda$.

<u>*Step 2:*</u> Λ *is bounded from above.* By (31) we obtain

$$\lim_{\lambda\to +\infty} \lfloor c_1 - c_\lambda \rfloor \leq \lfloor c_1 \rfloor - \lim_{\lambda\to +\infty} \lfloor c_\lambda \rfloor = -\infty.$$

Then there exists $\lambda' > 1$ such that, for $\lambda > \lambda'$, we have $\lfloor c_1 - c_\lambda \rfloor < 0$. Hence, for $k > 0$, applying (36) with $\lambda_0 = \lambda + k,\ \lambda_1 = 1$ and $\tau = k/(k+\lambda-1)$, we derive

$$\lfloor c_\lambda - c_{\lambda+k} \rfloor \leq \frac{k}{(k+\lambda-1)\lambda} \lfloor c_1 - c_{\lambda+k} \rfloor < 0.$$

Namely, $\lambda \notin \Lambda$ and thus Λ is bounded from above by λ'.

<u>*Step 3:*</u> *If* $\lambda \in \Lambda$ *then* $(0, \lambda] \subset \Lambda$. Let $0 < \lambda' < \lambda$ and $k > 0$. Using first (35) and then (36) we get

$$\lfloor c_{\lambda'} - c_{\lambda'+k} \rfloor \geq \left(\frac{k}{k+\lambda-\lambda'}\right)\left(\frac{\lambda+k}{\lambda'+k}\right) \lfloor c_{\lambda'} - c_{\lambda+k} \rfloor \geq \left(\frac{\lambda+k}{\lambda'+k}\right)\frac{\lambda}{\lambda'} \lfloor c_\lambda - c_{\lambda+k} \rfloor.$$

Thus, $\lambda \in \Lambda$ implies $\lambda' \in \Lambda$.

<u>*Step 4:*</u> $\sup \Lambda \notin \Lambda$. Let $\lambda^* := \sup \Lambda$ and $k > 0$. For all $n \in \mathbb{N}$, there exists $0 < k_n < 1/n$ such that $\lfloor c_{\lambda^*+1/n} - c_{\lambda^*+1/n+k_n} \rfloor \leq 0$. For n large enough, we have that $1/n + k_n < k$ and then, by (35),

$$0 \geq \lfloor c_{\lambda^*+1/n} - c_{\lambda^*+1/n+k_n} \rfloor \geq \left(\frac{k_n}{k-1/n}\right)\left(\frac{\lambda^*+k}{\lambda^*+1/n+k_n}\right) \lfloor c_{\lambda^*+1/n} - c_{\lambda^*+k} \rfloor.$$

Hence,

$$\lfloor c_{\lambda^*} - c_{\lambda^*+k} \rfloor \leq \lfloor c_{\lambda^*+1/n} - c_{\lambda^*+k} \rfloor + \lceil c_{\lambda^*} - c_{\lambda^*+1/n} \rceil \leq \lceil c_{\lambda^*} - c_{\lambda^*+1/n} \rceil.$$

Using the analogue of (36) for the upper mean, we can control the latter term as follows:

$$\lceil c_{\lambda^*} - c_{\lambda^*+1/n} \rceil \leq \frac{1/n}{\lambda_* + 2/n} \lceil c_{\lambda^*/2} - c_{\lambda^*+1/n} \rceil \leq \frac{1/n}{\lambda_* + 2/n} (\lceil c_{\lambda^*/2} \rceil - \lfloor c_{\lambda^*+1/n} \rfloor),$$

which goes to 0 as $n \to \infty$ (recall that $\lambda \mapsto \lfloor c_\lambda \rfloor$ is continuous by Lemma 3.3). We eventually infer that $\lfloor c_{\lambda^*} - c_{\lambda^*+k} \rfloor \leq 0$; that is, $\lambda^* \notin \Lambda$.

It remains to show that $\lambda \mapsto \lfloor c_\lambda \rfloor$ is decreasing on Λ. Assume by way of contradiction that there are $0 < \lambda_1 < \lambda_2 < \lambda^*$ such that $\lfloor c_{\lambda_1} \rfloor \leq \lfloor c_{\lambda_2} \rfloor$. The function $\lambda \mapsto \lfloor c_\lambda \rfloor$, being continuous, attains its minimum on $[\lambda_1, \lambda_2]$ at some λ. Since $\lfloor c_{\lambda_1} \rfloor \leq \lfloor c_{\lambda_2} \rfloor$, we can assume that $\lambda \in [\lambda_1, \lambda_2)$. The definition of Λ implies that there exists $\lambda' \in (\lambda, \lambda_2)$ such that $\lfloor c_\lambda - c_{\lambda'} \rfloor > 0$. Hence, we obtain the following contradiction:

$$\lfloor c_{\lambda'} \rfloor \leq \lfloor c_\lambda \rfloor + \lceil c_{\lambda'} - c_\lambda \rceil = \lfloor c_\lambda \rfloor - \lfloor c_\lambda - c_{\lambda'} \rfloor < \lfloor c_\lambda \rfloor. \qquad \square$$

We are now in position to define the critical speed

$$c_* := \lfloor c_{\lambda_*} \rfloor, \tag{37}$$

where λ_* is given in Lemma 3.4.

Remark 1. When the terms in (1) are periodic in time, resuming from Section 2C, we know that the speeds $(c_\lambda)_{\lambda>0}$ are constant and satisfy $c_\lambda \equiv k(\lambda)/\lambda$, where $k(\lambda)$ is the principal eigenvalue of problem (20). Hence,

$$\lfloor c_\lambda - c_{\lambda+\kappa} \rfloor = \frac{k(\lambda)}{\lambda} - \frac{k(\lambda+\kappa)}{\lambda+\kappa}.$$

As $\lambda \mapsto k(\lambda)$ is strictly convex (see [Nadin 2009]) and, by (31),

$$\lim_{\lambda\to+\infty} \frac{k(\lambda)}{\lambda} = +\infty, \qquad \lim_{\lambda\to 0^+} k(\lambda) = \lim_{\lambda\to 0^+} \lambda c_\lambda \geq \Big\lfloor \min_{x\in\mathbb{R}^N} \mu(x,\cdot) \Big\rfloor > 0,$$

straightforward convexity arguments yield that λ_* is the unique minimizer of $\lambda \mapsto k(\lambda)/\lambda$. Therefore, $c_* = \min_{\lambda>0} k(\lambda)/\lambda$, which is known to be the minimal speed for pulsating traveling fronts (see [Nadin 2009]).

3D. *Construction of a subsolution and conclusion of the proof.* In order to prove Theorem 2.1, we introduce a family of functions $(\varphi_\lambda)_{\lambda>0}$ which play the role of the spatial-periodic principal eigenfunctions in the time-independent case. For $\lambda > 0$, let η_λ be the function given by Lemma 3.1, normalized by $\|\eta_\lambda(\cdot,0)\|_{L^\infty(\mathbb{R}^N)} = 1$. We define

$$\varphi_\lambda(x,t) := e^{-\lambda S_\lambda(t)} \eta_\lambda(x,t).$$

By (19) and (28), there exist two positive constants C_λ, β such that

$$\forall x \in \mathbb{R}^N,\ t \in \mathbb{R}, \qquad C_\lambda \leq \varphi_\lambda(x,t) \leq e^{\lambda\beta}. \tag{38}$$

We will make use of the following key property of the least mean, provided by Lemma 3.2 of [Nadin and Rossi 2012]:

$$\forall g \in L^\infty(\mathbb{R}), \qquad \lfloor g \rfloor = \sup_{\sigma\in W^{1,\infty}(\mathbb{R})} \inf_{t\in\mathbb{R}} (\sigma' + g)(t). \tag{39}$$

Proof of Theorem 2.1. Fix $\gamma > c_*$. Since the function $\lambda \mapsto \lfloor c_\lambda \rfloor$ is continuous by Lemma 3.3 and tends to $+\infty$ as $\lambda \to 0^+$ by (31), and $\Lambda = (0, \lambda_*)$ by Lemma 3.4, there exists $\lambda \in \Lambda$ such that $\lfloor c_\lambda \rfloor = \gamma$. The function w defined by

$$w(x,t) := \min(e^{-\lambda x\cdot e} \eta_\lambda(x,t), 1)$$

is a generalized supersolution of (1).

In order to construct a subsolution, consider the constant ν in (10). By the definition of Λ, there exists $\lambda < \lambda' < (1+\nu)\lambda$ such that $\lfloor c_\lambda - c_{\lambda'} \rfloor > 0$. We then set $\psi(x,t) := e^{\sigma(t)-\lambda'(x\cdot e - S_\lambda(t) + S_{\lambda'}(t))} \eta_{\lambda'}(x,t)$, where $\sigma \in W^{1,\infty}(\mathbb{R})$ will be chosen later. We have that

$$\partial_t \psi - \mathrm{Tr}(A(x,t)D^2\psi) + q(x,t)\cdot D\psi - \mu(x,t)\psi = \big(\sigma'(t) + \lambda'(c_\lambda(t) - c_{\lambda'}(t))\big)\psi.$$

Since $\lfloor \lambda'(c_\lambda - c_{\lambda'}) \rfloor = \lambda' \lfloor c_\lambda - c_{\lambda'} \rfloor > 0$, by (39) we can choose $\sigma \in W^{1,\infty}(\mathbb{R})$ in such a way that $K := \inf_{\mathbb{R}}(\sigma' + \lambda'(c_\lambda - c_{\lambda'})) > 0$. Hence,

$$\partial_t \psi - \mathrm{Tr}(A(x,t)D^2\psi) + q(x,t)\cdot D\psi \geq (\mu(x,t)+K)\psi, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

We define

$$v(x,t) := e^{-\lambda x\cdot e}\eta_\lambda(x,t) - m\psi(x,t),$$

where m is a positive constant to be chosen. By computation,

$$e^{-\lambda x\cdot e}\eta_\lambda(x,t) - m\psi(x,t) = e^{-\lambda(x\cdot e - S_\lambda(t))}\big(\varphi_\lambda(x,t) - m\varphi_{\lambda'}(x,t)e^{\sigma(t)-(\lambda'-\lambda)(x\cdot e - S_\lambda(t))}\big).$$

Since φ_λ, $\varphi_{\lambda'}$ satisfy (38) and $\sigma \in L^\infty(\mathbb{R})$, it follows that, choosing m large enough, we have $v(x,t) \leq 0$ if $x\cdot e - S_\lambda(t) \leq 0$, and that v is less than $\delta \in (0,1]$, from (10), everywhere. If $v(x,t) > 0$, and therefore $x\cdot e - S_\lambda(t) > 0$, we see that

$$\begin{aligned}
\partial_t v - \mathrm{Tr}(A(x,t)D^2 v) + q(x,t)\cdot Dv - \mu(x,t)v &\leq -mK\psi \\
&\leq -mK\psi \frac{v^{1+\nu}}{e^{-(1+\nu)\lambda x\cdot e}\eta_\lambda^{1+\nu}} \\
&= -mKv^{1+\nu}\frac{\varphi_{\lambda'}}{\varphi_\lambda^{1+\nu}} e^{\sigma(t)-(\lambda'-(1+\nu)\lambda)(x\cdot e - S_\lambda(t))} \\
&\leq -mKv^{1+\nu}C_{\lambda'}e^{-(1+\nu)\lambda\beta}\inf_{s\in\mathbb{R}} e^{\sigma(s)},
\end{aligned}$$

where, for the last inequality, we have used (38) and the fact that $\lambda' < (1+\nu)\lambda$. As a consequence, by hypothesis (10), for m sufficiently large, v is a subsolution of (1) in the set where it is positive.

Using again (38), one computes

$$\begin{aligned}
v(x+S_\lambda(t)e,t) &= e^{-\lambda x\cdot e}\big(\varphi_\lambda(x+S_\lambda(t)e,t) - m\varphi_{\lambda'}(x+S_\lambda(t)e,t)e^{\sigma(t)-(\lambda'-\lambda)x\cdot e}\big) \\
&\geq e^{-\lambda x\cdot e}(C_\lambda - mC_{\lambda'}e^{\lambda'\beta + \|\sigma\|_\infty - (\lambda'-\lambda)x\cdot e}).
\end{aligned}$$

Hence, taking R large enough, one has

$$\inf_{\substack{x\cdot e = R\\ t\in\mathbb{R}}} v(x+S_\lambda(t)e,t) \geq e^{-\lambda R}(C_\lambda - mC_{\lambda'}e^{\lambda'\beta+\|\sigma\|_\infty-(\lambda'-\lambda)R}) =: \omega \in (0,1).$$

Consequently, the function $\underline{v}$ defined by

$$\underline{v}(x,t) := \begin{cases} v(x,t) & \text{if } x\cdot e \geq S_\lambda(t)+R, \\ \max(\omega, v(x,t)) & \text{if } x\cdot e < S_\lambda(t)+R, \end{cases}$$

is continuous and, because of (8), it is a generalized subsolution of (1). Moreover, since $v \leq w$ and $w(x+S_\lambda(t)e,t) \geq e^{-\lambda R}C_\lambda > \omega$ if $x\cdot e < R$, one sees that $\underline{v} \leq w$. A solution $\underline{v} \leq u \leq w$ can therefore be obtained as the limit of (a subsequence of) the solutions $(u_n)_{n\in\mathbb{N}}$ of the problems

$$\begin{cases} \partial_t u_n - \mathrm{Tr}(A(x,t)D^2u_n) + q(x,t)\cdot Du_n = f(x,t,u_n), & x\in\mathbb{R}^N,\ t > -n \\ u_n(x,-n) = w(x,-n), & x \in \mathbb{R}^N. \end{cases}$$

The strong maximum principle yields $u > 0$. One further sees that

$$\lim_{x\cdot e\to+\infty} u\left(x+e\int_0^t c_\lambda(s)\,ds,\,t\right) \le \lim_{x\cdot e\to+\infty} w\left(x+e\int_0^t c_\lambda(s)\,ds,\,t\right) \le \lim_{x\cdot e\to+\infty} e^{-\lambda x\cdot e}\varphi_\lambda(x,t) = 0,$$

uniformly with respect to $t \in \mathbb{R}$. It remains to prove that

$$\lim_{x\cdot e\to-\infty} u\left(x+e\int_0^t c_\lambda(s)\,ds,\,t\right) = 1$$

holds uniformly with respect to $t \in \mathbb{R}$. Set

$$\vartheta := \lim_{r\to-\infty} \inf_{\substack{x\cdot e\le r\\ t\in\mathbb{R}}} u\left(x+e\int_0^t c_\lambda(s)\,ds,\,t\right).$$

Our aim is to show that $\vartheta = 1$. We know that $\vartheta \ge \omega > 0$, because $u(x,t) \ge \underline{v}(x,t) \ge \omega$ if $x\cdot e < S_\lambda(t)+R$. Let $(x_n)_{n\in\mathbb{N}}$ in $\mathbb{R}^N$ and $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ be such that

$$\lim_{n\to\infty} x_n\cdot e = -\infty, \qquad \lim_{n\to\infty} u\left(x_n+e\int_0^{t_n} c_\lambda(s)\,ds,\,t_n\right) = \vartheta.$$

For $n \in \mathbb{N}$, let $k_n \in \prod_{j=1}^N l_j\mathbb{Z}$ be such that $y_n := x_n + e\int_0^{t_n} c_\lambda(s)\,ds - k_n \in \prod_{j=1}^N [0,l_j)$ and define $v_n(x,t) := u(x+k_n, t+t_n)$. The functions $(v_n)_{n\in\mathbb{N}}$ are solutions of

$$\partial_t v_n - \mathrm{Tr}(A(x,t+t_n)D^2 v_n) + q(x,t+t_n)\cdot Dv_n = f(x,t+t_n,v_n), \quad x\in\mathbb{R}^N,\ t\in\mathbb{R}.$$

By parabolic estimates, one can show using the same types of arguments as in the proof of Lemma 3.1 that $(v_n)_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly to some function v satisfying

$$\partial_t v - \mathrm{Tr}(\tilde{A}(x,t)D^2 v) + \tilde{q}(x,t)\cdot Dv = g(x,t) \ge 0, \quad x\in\mathbb{R}^N,\ t\in\mathbb{R},$$

where $\tilde{A}$ and $\tilde{q}$ are the strong limits in $L^\infty_{\mathrm{loc}}(\mathbb{R}^{N+1})$ and g is the weak limit in $L^p_{\mathrm{loc}}(\mathbb{R}^N\times\mathbb{R})$ of (a subsequence of) $A(x,t+t_n)$, $q(x,t+t_n)$ and $f(x,t+t_n,v_n(x,t))$ respectively, the inequality $g\ge 0$ coming from hypothesis (8). Furthermore, letting y be the limit of (a converging subsequence of) $(y_n)_{n\in\mathbb{N}}$, we find that $v(y,0)=\vartheta$ and

$$\forall x\in\mathbb{R}^N,\ t\in\mathbb{R}, \quad v(x,t) = \lim_{n\to\infty} u\left(x+x_n+e\int_0^{t_n} c_\lambda(s)\,ds - y_n,\, t+t_n\right) \ge \vartheta.$$

As a consequence, the strong maximum principle yields $v=\vartheta$ in $\mathbb{R}^N\times(-\infty,0]$. In particular, $g=0$ in $\mathbb{R}^N\times(-\infty,0)$. Using the Lipschitz continuity of $f(x,t,\cdot)$, we then derive for all $(x,t)\in\mathbb{R}^N\times(-\infty,0)$,

$$\forall T>0, \quad 0 = \lim_{n\to+\infty} f(x,t+t_n,v_n(x,t)) = \lim_{n\to+\infty} f(x,t+t_n,\vartheta) \ge \inf_{(x,t)\in\mathbb{R}^{N+1}} f(x,t,\vartheta).$$

This, by (8), implies that either $\vartheta=0$ or $\vartheta=1$, whence $\vartheta=1$ because $\vartheta\ge\omega>0$. □

3E. ***A criterion for the existence of generalized transition waves in space-time general heterogeneous media.*** As already emphasized above, our proof holds in more general media, without assuming that the coefficients satisfy (11), that is, without the space periodicity assumption. We then need to assume that the linearized equation admits a family of solutions satisfying some global Harnack inequality. We conclude the existence part of the paper by stating such a result. We omit its proof since one only needs to check that the previous arguments still work.

Theorem 3.5. *In addition to (3)–(10), assume that there exists $\bar{\lambda} > 0$ such that, for all $\lambda \in (0, \bar{\lambda})$, there exists a Lipschitz-continuous time-global solution η_λ of*

$$\partial_t \eta_\lambda = \mathrm{Tr}(AD^2\eta_\lambda) - (q + 2\lambda Ae)D\eta_\lambda + (\mu + \lambda^2 eAe + \lambda q \cdot e)\eta_\lambda, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}$$

satisfying

$$\frac{1}{C}\|\eta_\lambda(\,\cdot\,, t)\|_{L^\infty(\mathbb{R}^N)} e^{-CT} \le \eta_\lambda(x, t+T) \le C\|\eta_\lambda(\,\cdot\,, t)\|_{L^\infty(\mathbb{R}^N)} e^{CT},$$

for some $C = C(\lambda) > 0$ and for all $T > 0$, $(x, t) \in \mathbb{R}^{N+1}$.

Then there exists $\lambda_* \in (0, \bar{\lambda})$ such that, for all $\gamma > c_* := \lfloor S'_{\lambda_*} \rfloor$, where S_λ is a Lipschitz continuous function satisfying (19), there exists a generalized transition wave with speed $c_\lambda = S'_\lambda$ such that $\lfloor c_\lambda \rfloor = \gamma$.

4. Nonexistence result

Our aim is to find bounded subsolutions to the linearized problem

$$\partial_t u - \mathrm{Tr}(A(x, t)D^2 u) + q(x, t) \cdot Du = \mu(x, t)u, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{40}$$

in order to get a lower bound for the speed of traveling wave solutions. We recall that no spatial-periodic condition is now assumed. Looking for solutions of (40) in the form $u(x, t) = e^{-\lambda(x\cdot e - ct)}\phi(x, t)$, with λ and c constant, leads to the equation

$$(\mathcal{P}_\lambda + c\lambda)\phi = 0, \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}, \tag{41}$$

where $\mathcal{P}_\lambda$ is the linear parabolic operator defined by

$$\mathcal{P}_\lambda w := \partial_t w - \mathrm{Tr}(A(x, t)D^2 w) + (q(x, t) - 2\lambda A(x, t)e) \cdot Dw - (\lambda^2 eA(x, t)e + \lambda q(x, t) \cdot e + \mu(x, t))w.$$

We consider the generalized principal eigenvalue introduced in [Berestycki and Nadin 2015]:

$$\kappa(\lambda) := \inf\big\{k \in \mathbb{R} : \exists\varphi,\ \inf\varphi > 0,\ \sup\varphi < \infty,\ \sup|D\varphi| < \infty,\ \mathcal{P}_\lambda\varphi \le k\varphi,\ \text{in } \mathbb{R}^N \times \mathbb{R}\big\}, \tag{42}$$

where the functions φ belong to $L^{N+1}_{\mathrm{loc}}(\mathbb{R}^{N+1})$, together with their derivatives ∂_t, D, D^2 (and therefore the differential inequalities are understood to hold almost everywhere). This is the minimal regularity required for the maximum principle to apply. See, e.g., [Lieberman 1996].

Taking $\varphi \equiv 1$ in the above definition we get, for $\lambda \in \mathbb{R}$,

$$\kappa(\lambda) \le -\underline{\alpha}\lambda^2 + \sup_{\mathbb{R}^{N+1}} |q|\,|\lambda| - \inf_{\mathbb{R}^{N+1}} \mu. \tag{43}$$

We now derive a lower bound for $\kappa(\lambda)$. Assume by way of contradiction that there exists a function φ as in the definition of $\kappa(\lambda)$, associated with some k satisfying

$$k < -\bar{\alpha}\lambda^2 - \sup_{\mathbb{R}^{N+1}} |q||\lambda| - \sup_{\mathbb{R}^{N+1}} \mu.$$

For $\beta > 0$, the function $\psi(x,t) := e^{-\beta t}$ satisfies

$$\frac{\mathcal{P}_\lambda \psi}{\psi} \geq -\beta - \bar{\alpha}\lambda^2 - \sup_{\mathbb{R}^{N+1}} |q||\lambda| - \sup_{\mathbb{R}^{N+1}} \mu.$$

Hence, β can be chosen small enough in such a way that the latter term is larger than k; that is, $\mathcal{P}_\lambda\psi \geq k\psi$. The function ψ is larger than φ for t less than some t_0, whence $\psi \geq \varphi$ for all t by the comparison principle. It follows that $\varphi \to 0$ as $t \to +\infty$, which is impossible since φ is bounded from below away from 0. This shows that $\kappa(\lambda) > -\infty$.

We can now define c^* by setting

$$c^* := -\max_{\lambda>0} \frac{\kappa(\lambda)}{\lambda}. \tag{44}$$

This definition is well posed if $\kappa(0) < 0$ because $\kappa(\lambda)/\lambda \to -\infty$ as $\lambda \to +\infty$ by (43), and we know from [Berestycki and Nadin 2015] that $\lambda \mapsto \kappa(\lambda)$ is Lipschitz-continuous.[3] Let us show that (12) implies that $\kappa(0) < 0$ and then that c^* is well defined and finite. Writing a positive function φ in the form $\varphi(t) := e^{-\sigma(t)}$, we see that

$$\mathcal{P}_0\varphi = -(\sigma'(t) + \mu(x,t))\varphi \leq -\Big(\sigma'(t) + \inf_{x\in\mathbb{R}} \mu(x,t)\Big)\varphi.$$

Thus, (39) implies that, for given $\varepsilon > 0$, there exists $\sigma \in W^{1,\infty}(\mathbb{R})$ such that

$$\mathcal{P}_0\varphi \leq -\left(\Big\lfloor \inf_{x\in\mathbb{R}^N} \mu(x,\cdot)\Big\rfloor - \varepsilon\right)\varphi.$$

Therefore, if (12) holds, taking $\varepsilon < \lfloor \min_{x\in\mathbb{R}^N} \mu(x,\cdot)\rfloor$ we derive $\kappa(0) < 0$.

The proof of Theorem 2.2 proceeds in two steps. In the following section we show that the average on $(0,+\infty)$ of the speed of a wave cannot be smaller than c^*. More precisely, we derive the following estimate.

Proposition 4.1. *Assume that (3)–(6) hold and that $\kappa(0) < 0$. Then, for any nonnegative supersolution u of (1) such that there is $c \in L^\infty(\mathbb{R})$ satisfying (2), it holds that*

$$\liminf_{t\to+\infty} \frac{1}{t}\int_0^t c(s)\,ds \geq c^* := -\max_{\lambda>0}\frac{\kappa(\lambda)}{\lambda}.$$

In this statement, the notion of solution (including subsolution and supersolution) is understood as in the definition of $\kappa(\lambda)$: namely, u, $\partial_t u$, Du, $D^2u \in L^{N+1}_{\text{loc}}(\mathbb{R}^{N+1})$. Notice that the least mean of a function is in general smaller than the average on $(0,+\infty)$. In the last section, we establish a general property

[3] The coefficients are assumed to be Hölder continuous in [Berestycki and Nadin 2015], but one can check that it does not matter in the proof of continuity.

of the least mean that allows us to deduce Theorem 2.2 by applying Proposition 4.1 to suitable time translations of the original problem.

4A. ***Lower bound on the mean speed for positive times.*** We start by constructing subsolutions with a slightly varying exponential behavior as $x \cdot e \to \pm\infty$. These will then be used to build a generalized subsolution with an arbitrary modulation of the exponential behavior. The term "generalized subsolution" refers to a function that, in a neighborhood of each point, is obtained as the supremum of some family of subsolutions. Then, using the fact that the generalized subsolutions satisfy the maximum principle, we will be able to prove Proposition 4.1.

Lemma 4.2. *Let $c, \lambda \in \mathbb{R}$ be such that $\kappa(\lambda) + c\lambda < 0$. Then there exists $\varepsilon > 0$ and $M > 1$ such that, for any $z \in \mathbb{R}$, (40) admits a subsolution $\underline{v}$ satisfying*

$$\text{if } x \cdot e - ct \geq z, \quad \frac{1}{M} e^{-(\lambda+\varepsilon)(x\cdot e - ct)} \leq \underline{v}(x,t) \leq M e^{-(\lambda+\varepsilon)(x\cdot e-ct)}, \qquad \inf_{z-1<x\cdot e-ct<z} \underline{v}(x,t) > 0,$$

$$\text{if } x \cdot e - ct \leq z-1, \quad \frac{1}{M} e^{-(\lambda-\varepsilon)(x\cdot e - ct)} \leq \underline{v}(x,t) \leq M e^{-(\lambda-\varepsilon)(x\cdot e-ct)}.$$

Proof. By the definition of $\kappa(\lambda)$, there is a bounded function φ with positive infimum satisfying

$$\mathcal{P}_\lambda \varphi \leq k\varphi, \quad x \in \mathbb{R}^N,\ t > T,$$

for some $k < -c\lambda$. It follows that $v(x,t) := e^{-\lambda(x\cdot e-ct)}\varphi(x,t)$ is a subsolution of (40). Fix $z \in \mathbb{R}$ and consider a smooth function $\zeta : \mathbb{R} \to \mathbb{R}$ satisfying

$$\zeta = \lambda - \varepsilon \text{ in } (-\infty, z-1], \quad \zeta = \lambda + \varepsilon \text{ in } [z, +\infty), \quad 0 \leq \zeta' \leq 3\varepsilon, \quad |\zeta''| \leq h\varepsilon,$$

where $\varepsilon > 0$ has to be chosen and h is a universal constant. We define the function $\underline{v}$ by setting $\underline{v}(x,t) := e^{-(x\cdot e-ct)\zeta(x\cdot e-ct)}\varphi(x,t)$. Calling $\rho := x \cdot e - ct$, we find that

$$[\partial_t \underline{v} - a_{ij}(x,t)\partial_{ij}\underline{v} + q_i(x,t)\partial_i \underline{v} - \mu(x,t)\underline{v}]e^\rho \leq (\mathcal{P}_\zeta + c\zeta)\varphi + C[(1+\rho+\rho|\zeta| + \rho^2|\zeta'|)|\zeta'| + \rho|\zeta''|],$$

where ζ, ζ', ζ'' are evaluated at ρ, and C is a constant depending on N, c and the L^∞ norms of a_{ij}, q, μ, φ, $D\varphi$. The second term of the above right-hand side is bounded by $H(\varepsilon)$, for some continuous function H vanishing at 0. The first term satisfies

$$(\mathcal{P}_\zeta + c\zeta)\varphi \leq (\mathcal{P}_\lambda + c\lambda)\varphi + C((\zeta-\lambda) + |\zeta^2 - \lambda^2|) \leq (k + c\lambda)\varphi + C(\varepsilon + 2|\lambda|\varepsilon + \varepsilon^2).$$

We thus derive

$$\partial_t \underline{v} - a_{ij}(x,t)\partial_{ij}\underline{v} + q_i(x,t)\partial_i \underline{v} - \mu(x,t)\underline{v} \leq e^\rho[(k+c\lambda)\varphi + C\varepsilon(1+2|\lambda|+\varepsilon^2) + H(\varepsilon)].$$

Since $k < -c\lambda$ and $\inf\varphi > 0$, we can choose ε small enough in such a way that $\underline{v}$ is a subsolution of (40). □

Lemma 4.3. *Let $\underline{\lambda}, \bar{\lambda}, c \in \mathbb{R}$ satisfy $\underline{\lambda} < \bar{\lambda}$ and*

$$\max_{\lambda\in[\underline{\lambda},\bar{\lambda}]} (\kappa(\lambda) + c\lambda) < 0.$$

Then there exists a generalized, bounded subsolution $\underline{v}$ of (40) satisfying

$$\lim_{r\to-\infty}\sup_{x\cdot e-ct<r}\underline{v}(x,t)e^{\underline{\lambda}(x\cdot e-ct)}=0,\qquad \lim_{r\to+\infty}\sup_{x\cdot e-ct>r}\underline{v}(x,t)e^{\bar{\lambda}(x\cdot e-ct)}=0,$$

and

$$\forall r_1<r_2,\qquad \inf_{r_1<x\cdot e-ct<r_2}\underline{v}(x,t)>0. \tag{45}$$

Proof. For $\lambda\in[\underline{\lambda},\bar{\lambda}]$, let ε_λ, M_λ be the constants given by Lemma 4.2 associated with c and λ. Call I_λ the interval $(\lambda-\varepsilon_\lambda,\lambda+\varepsilon_\lambda)$. The family $(I_\lambda)_{\underline{\lambda}\le\lambda\le\bar{\lambda}}$ is an open covering of $[\underline{\lambda},\bar{\lambda}]$. Let $(I_{\lambda_i})_{i=1,\dots,n}$ be a finite subcovering and set for short $\varepsilon_i:=\varepsilon_{\lambda_i}$, $M_i:=M_{\lambda_i}$. Up to rearranging the indices and extracting another subcovering if need be, we can assume that

$$\forall i=1,\dots,n-1,\qquad \lambda_{i+1}-\varepsilon_{i+1}<\lambda_i-\varepsilon_i<\lambda_{i+1}+\varepsilon_{i+1}<\lambda_i+\varepsilon_i.$$

Let v_1 be the subsolution of (40) given by Lemma 4.2 associated with $\lambda=\lambda_1$ and $z=0$. Set $z_1:=0$, $k_1:=1$ and

$$k_2:=\frac{e^{(\lambda_2+\varepsilon_2-(\lambda_1-\varepsilon_1))(z_1-1)}}{M_1M_2}.$$

Consider then the subsolution v_2 associated with $\lambda=\lambda_2$ and z equal to some value $z_2<z_1-1$ to be chosen. We have that

$$\text{if } x\cdot e-ct=z_1-1,\qquad \frac{v_1(x,t)}{v_2(x,t)}\ge\frac{k_2}{k_1}$$

$$\text{if } x\cdot e-ct=z_2,\qquad \frac{k_1v_1(x,t)}{k_2v_2(x,t)}\le(M_1M_2)^2e^{(\lambda_2+\varepsilon_2-(\lambda_1-\varepsilon_1))(z_2-z_1+1)}.$$

Since $\lambda_2+\varepsilon_2>\lambda_1-\varepsilon_1$, we can choose z_2 in such a way that the latter term is less than 1. By a recursive argument, we find some constants $(z_i)_{i=1,\dots,n}$ satisfying $z_n<z_{n-1}-1<\dots<z_1-1=-1$, such that the family of subsolutions $(v_i)_{i=1,\dots,n}$ given by Lemma 4.2 associated with the $(\lambda_i)_{i=1,\dots,n}$ and $(z_i)_{i=1,\dots,n}$ satisfies, for some positive $(k_i)_{i=1,\dots,n}$,

$$\forall i=1,\dots,n-1,\quad k_{i+1}v_{i+1}\le k_iv_i\ \text{ if } x\cdot e-ct=z_i-1,\quad k_{i+1}v_{i+1}\ge k_iv_i\ \text{ if } x\cdot e-ct=z_{i+1}.$$

The function $\underline{v}$, defined by

$$\underline{v}(x,t):=\begin{cases} v_1(x,t) & \text{if } x\cdot e-ct\ge z_1,\\ \max(k_iv_i(x,t),k_{i+1}v_{i+1}(x,t)) & \text{if } z_{i+1}\le x\cdot e-ct<z_i,\\ k_nv_n(x,t) & \text{if } x\cdot e-ct<z_n,\end{cases}$$

is a generalized subsolution of (40) satisfying the desired properties. □

Proof of Proposition 4.1. Let u, c be as in the statement of the proposition, and define $\phi(x,t):=u(x+e\int_0^t c(s)\,ds,t)$. Since $\phi(x,t)\to1$ as $x\cdot e\to-\infty$, uniformly with respect to $t\in\mathbb{R}$, one can find $\rho\in\mathbb{R}$ such that

$$\inf_{\substack{x\cdot e<\rho\\ t\in\mathbb{R}}}\phi(x,t)>0.$$

We now make use of Lemma 3.1 in [Rossi and Ryzhik 2014], which, under the above condition, establishes a lower bound for the exponential decay of an entire supersolution ϕ of a linear parabolic equation (notice that the differential inequality for ϕ can be written in linear form with a bounded zero order term: $f(x,t,\phi)=[f(x,t,\phi)/\phi]\phi$). The result of [Rossi and Ryzhik 2014] implies the existence of a positive constant λ_0 such that

$$\inf_{\substack{x\cdot e>\rho-1\\ t\in\mathbb{R}}} \phi(x,t)e^{\lambda_0 x\cdot e}>0.$$

By the definition of c^*, the hypotheses of Lemma 4.3 are fulfilled with $\underline{\lambda}=0$, $\bar{\lambda}=\lambda_0$ and $c=c^*-\varepsilon$, for any given $\varepsilon>0$. This is also true if one penalizes the nonlinear term $f(x,t,u)$ by subtracting δu, with δ small enough, since this just raises the principal eigenvalues $\kappa(\lambda)$ by δ. Therefore, Lemma 4.3 provides a function $\underline{v}$ such that, for $h>0$ small enough, $h\underline{v}$ is a subsolution of (1). We choose h in such a way that, together with the above property, $h\underline{v}(x,0)<u(x,0)$. This can be done, due to the lower bounds of $u(x,0)=\phi(x,0)$, because $\underline{v}$ is bounded and decays faster than $e^{-\lambda_0 x\cdot e}$ as $x\cdot e\to+\infty$. Applying the parabolic comparison principle we eventually infer that $h\underline{v}<u$ for all $x\in\mathbb{R}^N$, $t\geq 0$. It follows that u satisfies (45) with $c=c^*-\varepsilon$ for $t>0$. We derive, in particular,

$$0<\inf_{t>0} u((c^*-\varepsilon)te,t)=\inf_{t>0} u\biggl(\biggl((c^*-\varepsilon)t-\int_0^t c(s)\,ds\biggr)e+e\int_0^t c(s)\,ds,t\biggr),$$

which, in virtue of the second condition in (2), implies that

$$\limsup_{t\to+\infty}\biggl((c^*-\varepsilon)t-\int_0^t c(s)\,ds\biggr)<+\infty.$$

This concludes the proof due to the arbitrariness of ε. □

4B. *Property of the least mean and proof of Theorem 2.2.* Roughly speaking, the least mean of a function is the infimum of its averages in sufficiently large intervals. We show that, in some sense, this infimum is achieved up to replacing the function with an element of its *ω-limit* set. The ω-limit (in the L^∞ weak-$*$ topology) of a function g, denoted by ω_g, is the set of functions obtained as L^∞ weak-$*$ limits of translations of g.

Proposition 4.4. *Let $g\in L^\infty(\mathbb{R})$ and let ω_g denote its ω-limit set (in the L^∞ weak-$*$ topology). Then*

$$\lfloor g\rfloor=\min_{\tilde{g}\in\omega_g}\biggl(\lim_{t\to+\infty}\frac{1}{t}\int_0^t \tilde{g}(s)\,ds\biggr).$$

Proof. We can assume without loss of generality that $\lfloor g\rfloor=0$. Clearly, any $\tilde{g}\in\omega_g$ satisfies $\lfloor\tilde{g}\rfloor\geq\lfloor g\rfloor$, whence

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t \tilde{g}(s)\,ds\geq\lfloor\tilde{g}\rfloor\geq\lfloor g\rfloor=0.$$

Our aim is to find a function $\tilde{g}\in\omega_g$ satisfying

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t \tilde{g}(s)\,ds\leq 0. \tag{46}$$

We claim that, for any $n \in \mathbb{N}$, there exists $t_n \in \mathbb{N}$ such that

$$\forall j = 1, \dots, n, \quad n \int_{t_n}^{t_n+j} g(s)\,ds \leq j.$$

Assume by way of contradiction that this property fails for some $n \in \mathbb{N}$. By the definition of least mean, for $K \in \mathbb{N}$ large enough, there is $\tau \in \mathbb{R}$ such that

$$\frac{1}{K} \int_{\tau}^{\tau+Kn} g(s)\,ds < \frac{1}{2}.$$

On the other hand, there is $j \in \{1, \dots, n\}$ such that $n \int_{\tau}^{\tau+j} g(s)\,ds > j$. Then, there is $h \in \{1, \dots, n\}$ such that $n \int_{\tau+j}^{\tau+j+h} g(s)\,ds > h$, and hence $n \int_{\tau}^{\tau+j+h} g(s)\,ds > j+h$. We repeat this argument until we find $k \in \{1, \dots, n\}$ such that $n \int_{\tau}^{\tau+Kn+k} g(s)\,ds > Kn+k$. From this we deduce that

$$\int_{\tau}^{\tau+Kn} g(s)\,ds > K + \frac{k}{n} - \int_{\tau+Kn}^{\tau+Kn+k} g(s)\,ds > K - n\|g\|_{L^\infty(\mathbb{R})}.$$

A contradiction follows taking $K > 2n\|g\|_{L^\infty(\mathbb{R})}$, and the claim is proved. The L^∞ weak-$*$ limit $\tilde{g}$ as $n \to \infty$ of (a subsequence of) $g(\cdot + t_n)$ satisfies the desired property. Indeed,

$$\forall j \in \mathbb{N}, \quad \int_0^j \tilde{g}(s)\,ds = \lim_{n\to\infty} \int_{t_n}^{t_n+j} g(s)\,ds = 0,$$

from which (46) follows since $\tilde{g}$ is bounded. $\square$

Proof of Theorem 2.2. Let u be a generalized transition wave with speed c. Proposition 4.4 yields that there exists $\tilde{c} \in \omega_c$ such that

$$\lfloor c \rfloor = \lim_{T\to+\infty} \frac{1}{T} \int_0^T \tilde{c}(s)\,ds. \tag{47}$$

The definition of ω_c gives a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ such that $c(\cdot + t_n) \rightharpoonup \tilde{c}$ as $n \to +\infty$ for the L^∞ weak-$*$ topology. For $n \in \mathbb{N}$, consider the functions

$$A_n(x,t) := A\Big(x + e\int_0^{t_n} c(s)\,ds, t + t_n\Big), \quad q_n(x,t) := q\Big(x + e\int_0^{t_n} c(s)\,ds, t + t_n\Big),$$
$$\mu_n(x,t) := \mu\Big(x + e\int_0^{t_n} c(s)\,ds, t + t_n\Big), \quad u_n(x,t) := u\Big(x + e\int_0^{t_n} c(s)\,ds, t + t_n\Big).$$

For any $\varepsilon \in (0,1)$ there exists $m \in (0,1)$ such that

$$\forall (x,t) \in \mathbb{R}^{N+1},\ u \in [0,1], \quad f(x,t,u) \geq (\mu(x,t) - \varepsilon)u(m-u).$$

It follows that the u_n satisfy

$$\partial_t u_n - \mathrm{Tr}(A_n(x,t)D^2 u_n) + q_n(x,t)Du_n \geq (\mu_n(x,t) - \varepsilon)u_n(m - u_n), \quad x \in \mathbb{R}^N,\ t \in \mathbb{R}.$$

On the other hand, the L^p parabolic interior estimates ensure that the sequences $(\partial_t u_n)_{n\in\mathbb{N}}$, $(Du_n)_{n\in\mathbb{N}}$, $(D^2 u_n)_{n\in\mathbb{N}}$ are bounded in $L^p(Q)$ for all $p \in (1,\infty)$ and $Q \Subset \mathbb{R}^{N+1}$. Hence, by the embedding theorem, $(u_n)_{n\in\mathbb{N}}$ converges (up to subsequences) locally uniformly in $\mathbb{R}^{N+1}$ to some function $\tilde{u}$, and the derivatives ∂_t, D, D^2 of the $(u_n)_{n\in\mathbb{N}}$ weakly converge to those of $\tilde{u}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^{N+1})$. Therefore, letting $\tilde{A}$, $\tilde{q}$ be the locally uniform limits of (subsequences of) $(A_n)_{n\in\mathbb{N}}$, $(q_n)_{n\in\mathbb{N}}$ and $\tilde{\mu}$ be the L^∞ weak-$*$ limit of (a subsequence of) $(\mu_n)_{n\in\mathbb{N}}$, we infer that

$$\partial_t\tilde{u}-\mathrm{Tr}(\tilde{A}(x,t)D^2\tilde{u})+\tilde{q}(x,t)D\tilde{u}\geq(\tilde{\mu}(x,t)-\varepsilon)\tilde{u}(m-\tilde{u}),\qquad x\in\mathbb{R}^N,\ t\in\mathbb{R}.$$

Hence, $\tilde{u}$ is a supersolution of an equation of the type (1) whose terms satisfy (3)–(5) and (6). Moreover, it is easily derived from the definition of the speed c and the L^∞ weak-$*$ convergence to $\tilde{c}$, that $\tilde{u}$ satisfies (2) with c replaced by $\tilde{c}$, uniformly with respect to $t\in\mathbb{R}$. In order to apply Proposition 4.1 to the function $\tilde{u}$, we need to show that $\tilde{\kappa}(0)<0$, where $\lambda\mapsto\tilde{\kappa}(\lambda)$ is defined like $\lambda\mapsto\kappa(\lambda)$, but with $\tilde{A}$, $\tilde{q}$, $\tilde{\mu}-\varepsilon$ in place of A, q, μ respectively. Namely, the $\kappa(\lambda)$ are the principal eigenvalues in the sense of (42) for the operators $\tilde{\mathcal{P}}_\lambda$ defined as follows:

$$\tilde{\mathcal{P}}_\lambda w:=\partial_t w-\mathrm{Tr}(\tilde{A}(x,t)D^2w)+(\tilde{q}(x,t)-2\lambda\tilde{A}(x,t)e)\cdot Dw-(\lambda^2e\tilde{A}(x,t)e+\lambda\tilde{q}(x,t)\cdot e+\tilde{\mu}(x,t)-\varepsilon)w.$$

This will be achieved by showing that

$$\forall\lambda>0,\qquad\tilde{\kappa}(\lambda)\leq\kappa(\lambda)+\varepsilon,\tag{48}$$

whence $\tilde{\kappa}(0)<0$ as soon as $\varepsilon<-\kappa(0)$ (recall that $\kappa(0)<0$ by (12)). Let us postpone for a moment the proof of (48). Applying Proposition 4.1 to $\tilde{u}$ yields

$$\liminf_{t\to+\infty}\frac{1}{t}\int_0^t\tilde{c}(s)\,ds\geq-\max_{\lambda>0}\frac{\tilde{\kappa}(\lambda)}{\lambda}=-\frac{\tilde{\kappa}(\hat{\lambda})}{\hat{\lambda}},$$

for some $\hat{\lambda}>0$. In virtue of (47) and (48), from this inequality we deduce

$$\lfloor c\rfloor\geq-\frac{\kappa(\hat{\lambda})+\varepsilon}{\hat{\lambda}},$$

from which $\lfloor c\rfloor\geq c^*$ follows by the arbitrariness of ε.

It remains to prove (48). Let $k>\kappa(\lambda)$. By definition (43) there exists φ such that $\inf\varphi>0$ and φ, $D\varphi\in L^\infty(\mathbb{R}^N\times\mathbb{R})$ and $\mathcal{P}_\lambda\varphi\leq k\varphi$ in $\mathbb{R}^N\times\mathbb{R}$. We would like to perform on φ the same limit of translations as done before to obtain $\tilde{u}$ from u. This would yield a function $\tilde{\varphi}$ satisfying $\tilde{\mathcal{P}}_\lambda\tilde{\varphi}\leq(k+\varepsilon)\tilde{\varphi}$. But this argument requires the L^p_{loc} estimates of the derivatives ∂_t, D, D^2 of the translated of φ, which are not available because φ is a subsolution and not a solution of an equation. However, it is possible to replace φ with a solution of a semilinear equation of the type $\mathcal{P}_\lambda w=g(w)$ in $\mathbb{R}^N\times\mathbb{R}$, with g smooth and such that $g(w)\leq(k+\varepsilon)w$, which satisfies the same properties as φ, as well as the desired additional regularity properties. This is done in the proof of Theorem A.1 of [Rossi and Ryzhik 2014], whose arguments can be exactly repeated here. We can therefore apply the translation argument that provides a function $\tilde{\varphi}$ such that $\tilde{\mathcal{P}}_\lambda\tilde{\varphi}\leq(k+\varepsilon)\tilde{\varphi}$. Moreover, $\inf\tilde{\varphi}>0$ and $\sup\tilde{\varphi}<\infty$. In order to be able to use $\tilde{\varphi}$ in the definition of $\tilde{\kappa}(\lambda)$ and derive $\tilde{\kappa}(\lambda)\leq k+\varepsilon$, we only need to have that $\sup|D\tilde{\varphi}|<\infty$. This property does not follow automatically from the L^p estimates and the embedding theorem as in the elliptic case treated in [Rossi and Ryzhik 2014]. This is the reason why we need the extra assumption (13) on A. Indeed, we use Theorem 1.4 of [Porretta and Priola 2013] with, using the same notations as in [Porretta and Priola 2013], F the nonlinear operator associated with equation $\mathcal{P}_\lambda w = g(w)$. Hypothesis 1.2 of [Porretta and Priola 2013] is satisfied since A satisfies (13), q is bounded and $f = f(x,t,u)$ is bounded with respect to $(x,t,u) \in \mathbb{R}^N \times \mathbb{R} \times [0,1]$, and Hypothesis 1.3 is satisfied with $\varphi(x,t) := e^{Mt}(1+|x|^2)$ and M large enough. Hence, we get a uniform L^∞ bound on Dw, where w is the solution of $\mathcal{P}_\lambda w = g(w)$. Using w instead of φ, we get that this bound is inherited by $\tilde{\varphi}$ and we therefore deduce $\tilde{\kappa}(\lambda) \le k + \varepsilon$. As $k > \kappa(\lambda)$ is arbitrary, we eventually get (48). □

References

[Alikakos et al. 1999] N. D. Alikakos, P. W. Bates, and X. Chen, "Periodic traveling waves and locating oscillating patterns in multidimensional domains", *Trans. Amer. Math. Soc.* **351**:7 (1999), 2777–2805. MR 99j:35101 Zbl 0929.35067

[Aronson and Weinberger 1978] D. G. Aronson and H. F. Weinberger, "Multidimensional nonlinear diffusion arising in population genetics", *Adv. in Math.* **30**:1 (1978), 33–76. MR 80a:35013 Zbl 0407.92014

[Berestycki and Hamel 2002] H. Berestycki and F. Hamel, "Front propagation in periodic excitable media", *Comm. Pure Appl. Math.* **55**:8 (2002), 949–1032. MR 2003d:35139 Zbl 1024.37054

[Berestycki and Hamel 2007] H. Berestycki and F. Hamel, "Generalized travelling waves for reaction-diffusion equations", pp. 101–123 in *Perspectives in nonlinear partial differential equations*, edited by H. Berestycki et al., Contemp. Math. **446**, Amer. Math. Soc., Providence, RI, 2007. MR 2009k:35135 Zbl 1200.35169

[Berestycki and Hamel 2012] H. Berestycki and F. Hamel, "Generalized transition waves and their properties", *Comm. Pure Appl. Math.* **65**:5 (2012), 592–648. MR 2898886 Zbl 1248.35039

[Berestycki and Nadin 2015] H. Berestycki and G. Nadin, "Asymptotic spreading for general heterogeneous Fisher-KPP type equations", preprint, 2015, available at https://edutice.archives-ouvertes.fr/hal-01171334/document.

[Berestycki and Nirenberg 1992] H. Berestycki and L. Nirenberg, "Travelling fronts in cylinders", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **9**:5 (1992), 497–572. MR 93k:35019 Zbl 0799.35073

[Berestycki et al. 2005] H. Berestycki, F. Hamel, and L. Roques, "Analysis of the periodically fragmented environment model, II: Biological invasions and pulsating travelling fronts", *J. Math. Pures Appl.* (9) **84**:8 (2005), 1101–1146. MR 2006d:35123 Zbl 1083.92036

[Berestycki et al. 2008] H. Berestycki, F. Hamel, and G. Nadin, "Asymptotic spreading in heterogeneous diffusive excitable media", *J. Funct. Anal.* **255**:9 (2008), 2146–2189. MR 2009k:35136 Zbl 1234.35033

[Fisher 1937] R. A. Fisher, "The wave of advance of advantageous genes", *Ann. Eugenics* **7**:4 (1937), 355–369. Zbl 63.1111.04

[Kolmogorov et al. 1937] A. Kolmogorov, I. Petrovsky, and N. Piskunov, "Etude de l'équation de la diffusion avec croissance de la quantité de matière et son application à un problème biologique", *Bulletin Université d'Etat à Moscou* (*Bjul. Moskowskogo Gos. Univ*) **1**:6 (1937), 1–26. Zbl 0018.32106

[Lieberman 1996] G. M. Lieberman, *Second order parabolic differential equations*, World Scientific Publishing Co., River Edge, NJ, 1996. MR 98k:35003 Zbl 0884.35001

[Mellet et al. 2009] A. Mellet, J. Nolen, J.-M. Roquejoffre, and L. Ryzhik, "Stability of generalized transition fronts", *Comm. Partial Differential Equations* **34**:4-6 (2009), 521–552. MR 2010d:35185 Zbl 1173.35021

[Mellet et al. 2010] A. Mellet, J.-M. Roquejoffre, and Y. Sire, "Generalized fronts for one-dimensional reaction-diffusion equations", *Discrete Contin. Dyn. Syst.* **26**:1 (2010), 303–312. MR 2011b:35238 Zbl 1180.35294

[Nadin 2009] G. Nadin, "Traveling fronts in space-time periodic media", *J. Math. Pures Appl.* (9) **92**:3 (2009), 232–262. MR 2010i:35193 Zbl 1182.35074

[Nadin 2014] G. Nadin, "Critical travelling waves for general heterogeneous reaction-diffusion", preprint, 2014, available at https://www.ljll.math.upmc.fr/ nadin/criticalTWrevised.pdf.

[Nadin and Rossi 2012] G. Nadin and L. Rossi, "Propagation phenomena for time heterogeneous KPP reaction-diffusion equations", *J. Math. Pures Appl.* (9) **98**:6 (2012), 633–653. MR 2994696 Zbl 06115427

[Nolen and Ryzhik 2009] J. Nolen and L. Ryzhik, "Traveling waves in a one-dimensional heterogeneous medium", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **26**:3 (2009), 1021–1047. MR 2010d:35201 Zbl 1178.35205

[Nolen et al. 2005] J. Nolen, M. Rudd, and J. Xin, "Existence of KPP fronts in spatially-temporally periodic advection and variational principle for propagation speeds", *Dyn. Partial Differ. Equ.* **2**:1 (2005), 1–24. MR 2006h:35129 Zbl 1179.35166

[Nolen et al. 2012] J. Nolen, J.-M. Roquejoffre, L. Ryzhik, and A. Zlatoš, "Existence and non-existence of Fisher-KPP transition fronts", *Arch. Ration. Mech. Anal.* **203**:1 (2012), 217–246. MR 2864411 Zbl 1267.35108

[Porretta and Priola 2013] A. Porretta and E. Priola, "Global Lipschitz regularizing effects for linear and nonlinear parabolic equations", *J. Math. Pures Appl.* (9) **100**:5 (2013), 633–686. MR 3115828 Zbl 06448859

[Rossi and Ryzhik 2014] L. Rossi and L. Ryzhik, "Transition waves for a class of space-time dependent monostable equations", *Commun. Math. Sci.* **12**:5 (2014), 879–900. MR 3187783 Zbl 1305.35092

[Shen 1999] W. Shen, "Travelling waves in time almost periodic structures governed by bistable nonlinearities, II: Existence", *J. Differential Equations* **159**:1 (1999), 55–101. MR 2001k:35158 Zbl 0939.35015

[Shen 2006] W. Shen, "Traveling waves in time dependent bistable equations", *Differential Integral Equations* **19**:3 (2006), 241–278. MR 2007a:35072 Zbl 1212.35220

[Shen 2011a] W. Shen, "Existence of generalized traveling waves in time recurrent and space periodic monostable equations", *J. Appl. Anal. Comput.* **1**:1 (2011), 69–93. MR 2881848 Zbl 1304.35351

[Shen 2011b] W. Shen, "Existence, uniqueness, and stability of generalized traveling waves in time dependent monostable equations", *J. Dynam. Differential Equations* **23**:1 (2011), 1–44. MR 2012d:35177 Zbl 1223.35103

[Xin 1992] J. X. Xin, "Existence of planar flame fronts in convective-diffusive periodic media", *Arch. Rational Mech. Anal.* **121**:3 (1992), 205–233. MR 93m:35110 Zbl 0764.76074

[Zlatoš 2013] A. Zlatoš, "Generalized traveling waves in disordered media: existence, uniqueness, and stability", *Arch. Ration. Mech. Anal.* **208**:2 (2013), 447–480. MR 3035984 Zbl 1270.35173

Received 24 Mar 2014. Revised 15 Mar 2015. Accepted 11 May 2015.

GRÉGOIRE NADIN: gregoire.nadin@upmc.fr
Sorbonne Universités, UPMC Univ Paris 06, UMR 7598, Laboratoire Jacques-Louis Lions, 75005 Paris, France

LUCA ROSSI: lucar@math.unipd.it
Dipartimento di Matematica, Università di Padova, via Trieste 63, I-35121 Padova, Italy

Analysis & PDE

msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2015 is US $205/year for the electronic version, and $390/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 8 No. 6 2015