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CHARACTERISATION OF THE ENERGY OF GAUSSIAN BEAMS ON LORENTZIAN MANIFOLDS: WITH APPLICATIONS TO BLACK HOLE SPACETIMES

JAN SBIERSKI

It is known that, using the Gaussian beam approximation, one can show that there exist solutions of the wave equation on a general globally hyperbolic Lorentzian manifold whose energy is localised along a given null geodesic for a finite, but arbitrarily long, time. We show that the energy of such a localised solution is determined by the energy of the underlying null geodesic. This result opens the door to various applications of Gaussian beams on Lorentzian manifolds that *do not admit a globally timelike Killing vector field*. In particular, we show that trapping in the exterior of Kerr or at the horizon of an extremal Reissner–Nordström black hole necessarily leads to a "loss of derivative" in a local energy decay statement. We also demonstrate the obstruction formed by the red-shift effect at the event horizon of a Schwarzschild black hole to scattering constructions from the future (where the red-shift turns into a blue-shift): we construct solutions to the backwards problem whose energies grow exponentially for a finite, but arbitrarily long, time. Finally, we give a simple mathematical realisation of the heuristics for the blue-shift effect near the Cauchy horizon of subextremal and extremal black holes: we construct a sequence of solutions to the wave equation whose initial energies are uniformly bounded, whereas the energy near the Cauchy horizon goes to infinity.

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1. Introduction

Part I of this paper is concerned with the study of the temporal behaviour of Gaussian beams on *general* globally hyperbolic Lorentzian manifolds. Here, a Gaussian beam is a highly oscillatory wave packet of the form

$$\tilde{u}_\lambda = \frac{1}{\sqrt{E(\lambda, a, \phi)}} \cdot a \cdot e^{i\lambda\phi},$$

where $E(\lambda, a, \phi)$ is a renormalisation factor keeping the initial energy of $\tilde{u}_\lambda$ independent of $\lambda \in \mathbb{R}^+$, and the complex-valued functions a and ϕ are chosen in such a way that for $\lambda \gg 0$ the Gaussian beam $\tilde{u}_\lambda$ is an approximate solution to the wave equation on the underlying Lorentzian manifold (M, g). The failure of $\tilde{u}_\lambda$ being an actual solution to the wave equation

$$\Box_g u = 0 \tag{1.1}$$

is measured in terms of an energy norm — and this error can be made arbitrarily small up to a finite, but arbitrarily long, time, by choosing λ large enough. The construction of the functions a and ϕ allows for restricting the support of a to a small neighbourhood of a given null geodesic. Thus, one can infer from $\tilde{u}_\lambda$ being an approximate solution with respect to some energy norm that:[1]

> There exist *actual* solutions of the wave equation (1.1) whose "energy" is localised along a given null geodesic up to some finite, but arbitrarily long, time. (1.2)

This is, roughly, the state of the art knowledge of Gaussian beams (see, for instance, [Ralston 1982]).

The main new result of Part I is to provide a geometric characterisation of the temporal behaviour of the localised energy of a Gaussian beam. More precisely, given a timelike vector field N (with respect to which we measure the energy) and a Gaussian beam $\tilde{u}_\lambda$ supported in a small neighbourhood of an affinely parametrised null geodesic γ, we show in Theorem 4.1 that

$$\int_{\Sigma_\tau} J^N(\tilde{u}_\lambda) \cdot n_{\Sigma_\tau} \approx -g(N, \dot{\gamma})\big|_{\mathrm{Im}(\gamma)\cap\Sigma_\tau} \tag{1.3}$$

holds up to some finite time T. Here, we consider a foliation of the Lorentzian manifold (M, g) by spacelike slices Σ_τ, $J^N(\tilde{u}_\lambda)$ denotes the contraction of the stress–energy tensor[2] of $\tilde{u}_\lambda$ with N, and n_{Σ_τ} is the normal of Σ_τ. The left-hand side of (1.3) is called the N-*energy* of the Gaussian beam $\tilde{u}_\lambda$. The approximation in (1.3) can be made arbitrarily good and the time T arbitrarily large if we only take $\lambda > 0$ to be big enough. This characterisation of the energy allows then for a refinement of (1.2):[3]

> There exist (actual) solutions of the wave equation (1.1) whose N-energy is localised along a given null geodesic γ and behaves approximately like $-g(N, \dot{\gamma})\big|_{\mathrm{Im}(\gamma)\cap\Sigma_\tau}$ up to some finite, but arbitrarily large, time T. Here, $\dot{\gamma}$ is with respect to some affine parametrisation of γ. (1.4)

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[1] See Theorem 2.1.

[2] We refer the reader to (1.8) in Section 1E for the definition of the stress–energy tensor.

[3] See Theorem 5.1.

It is worth emphasising that the need for an understanding of the temporal behaviour of the energy only arises for Gaussian beams on Lorentzian manifolds that *do not admit a globally timelike Killing vector field*[4] — otherwise there is a canonical energy which is conserved for solutions to the wave equation (1.1). Thus, for the majority of problems which so far found applications of Gaussian beams, for example the obstacle problem or the wave equation in time-independent inhomogeneous media, the question of the temporal behaviour of the energy did not arise (since it is trivial). However, understanding this behaviour on general Lorentzian manifolds is crucial for widening the application of Gaussian beams to problems arising, in particular, from general relativity.

In Part II, by applying (1.4), we derive some new results on the study of the wave equation on the familiar Schwarzschild, Reissner–Nordström, and Kerr black hole backgrounds (see [Hawking and Ellis 1973] for an introduction to these spacetimes):

(1) It is well-known folklore that the trapping[5] at the *photon sphere* in Reissner–Nordström and in Kerr necessarily leads to a "loss of derivative" in a local energy decay (LED) statement. We give a rigorous proof of this fact.

(2) We also show that the trapping at the *horizon* of an extremal Reissner–Nordström (and Kerr) black hole necessarily leads to a loss of derivative in an LED statement.

(3) When solving the wave equation (1.1) on the exterior of a Schwarzschild black hole backwards in time, the red-shift effect at the event horizon turns into a blue-shift: we construct solutions to the backwards problem whose energies grow exponentially for a finite, but arbitrarily long, time. This demonstrates the obstruction formed by the red-shift effect at the event horizon to scattering constructions from the future.

(4) Finally, we give a simple mathematical realisation of the heuristics for the blue-shift effect near the Cauchy horizon of (sub)extremal Reissner–Nordström and Kerr black holes: we construct a sequence of solutions to the wave equation whose initial energy is uniformly bounded whereas the energy near the Cauchy horizon goes to infinity.

Outline of the paper. We start by giving a short historical review of Gaussian beams in Section 1A. Thereafter we briefly explain how the notion of "energy" arises in the study of the wave equation and why it is important. We also discuss how the results we obtain allow us to disprove certain uniform statements about the temporal behaviour of the energy of waves. Section 1C elaborates on the wide applicability of the Gaussian beam approximation and explains its advantage over the geometric optics approximation. In the physics literature a similar "characterisation of the energy of high frequency waves" is folklore — we discuss its origin in Section 1D and put it into context with the work presented in this paper. Section 1E lays down the notation we use.

[4]One could add here "*uniformly*" timelike, meaning that the timelike Killing vector field does not "degenerate" when approaching the "boundary" of the manifold. Let us just state here that one can give precise meaning to "degenerating at the boundary".

[5]We do not intend to give a precise definition in this paper of what we mean by "trapping". However, loosely speaking "trapping" refers here to the presence of null geodesics that stay for all time in a compact region of "space".

Part I discusses the theory of Gaussian beams on Lorentzian manifolds. Sections 2 and 3 recall the construction of Gaussian beams and sketch the proof of Theorem 2.1, which basically says (1.2) and is more or less well known. In Section 4 we characterise the energy of a Gaussian beam, which is the main result of Part I. This result is then incorporated into Theorem 2.1, which yields Theorem 5.1 (or (1.4)). Moreover, Section 5 contains some general theorems which are tailored to the needs of many applications.

In Part II, we prove the above mentioned new results on the behaviour of waves on various black hole backgrounds. The important ideas are first introduced in Section 6 by the example of the Schwarzschild and Reissner–Nordström family, whose simple form of the metric allows for an uncomplicated presentation. Thereafter, in Section 7, we proceed to the Kerr family.

In the Appendix we give a sufficient criterion for the formation of caustics, i.e., a breakdown criterion for solutions of the eikonal equation, which shows the limitations of the "naive" geometric optics approximation.

1A. ***A brief historical review of Gaussian beams.*** The ansatz

$$u_\lambda = e^{i\lambda\phi}\left(a_0 + \frac{1}{\lambda}a_1 + \cdots + \frac{1}{\lambda^N}a_N\right) \tag{1.5}$$

for either a highly oscillatory approximate solution to some PDE or for a highly oscillatory approximate eigenfunction to some partial differential operator is known as the *geometric optics ansatz*. Here, $N \in \mathbb{N}$, ϕ is a real function (called the eikonal), the a_k are complex-valued functions, and λ is a positive parameter determining how quickly the function u_λ oscillates. In the widest sense, we understand under a Gaussian beam a function of the form (1.5) with a *complex*-valued eikonal ϕ that is real-valued along a bicharacteristic and has growing imaginary part off this bicharacteristic. This then leads to an exponential fall off in λ away from the bicharacteristic.

The use of a complex eikonal, although in a slightly different context, appears already in work of Keller [1956]. It was, however, only in the 1960s that the method of Gaussian beams was systematically applied and explored — mainly from a physics perspective. For more on these early developments we refer the reader to [Arnaud 1973, Chapter 4] and references therein. A general, mathematical theory of Gaussian beams, or what he called the complex WKB method, was developed by Maslov; see his book [1994] for an overview and also for references. Several of the later papers on Gaussian beams have their roots in this work.

The earliest application of the Gaussian beam method was to the construction of *quasimodes*; see, for example, [Ralston 1976]. Quasimodes approximately satisfy some type of Helmholtz equation, and thus they give rise to time-harmonic, approximate solutions to a wave equation. In this way quasimodes can be interpreted as standing waves. Later, various people used the Gaussian beam method for the construction of Gaussian *wave packets* (but also called "Gaussian beams") which form approximate solutions to a hyperbolic PDE.[6] Those wave packets, in contrast to quasimodes, are not stationary waves, but they move

[6]It is this sort of "Gaussian beam" that is the subject of this paper for the case of the wave equation on Lorentzian manifolds. More appropriately, one could name them "Gaussian wave packets" or "Gaussian pulses" to distinguish them from the standing waves — which are actually *beams*. However, we stick to the standard terminology.

through space, the trajectory in spacetime being a bicharacteristic of the partial differential operator. A detailed reference for this construction is [Ralston 1982], which goes back to 1977. Another presentation of this construction scheme was given by Babich and Ulin [1981].

Since then, there have been a lot of papers applying Gaussian beams to various problems.[7] For instance, in quantum mechanics Gaussian beams correspond to semiclassical approximate solutions to the Schrödinger equation and thus help understand the classical limit; or, in geophysics, one models seismic waves using the Gaussian beam approximation for solutions to a wave equation in an inhomogeneous (time-independent) medium.

1B. *Gaussian beams and the energy method.*

1B1. *The energy method as a versatile method for studying the wave equation.* The study of the wave equation on various geometries has a long history in mathematics and physics. A very successful and widely applicable method for obtaining quantitative results on the long-time behaviour of waves is the energy method. It was pioneered by Morawetz [1961; 1962], where she proved pointwise decay results in the context of the obstacle problem. In [Morawetz 1968] she established what is now known as *integrated local energy decay* (ILED) for solutions of the Klein–Gordon equation (and thus inferring decay). In the past ten years, her methods were adapted and extended by many people in order to prove boundedness and decay of waves on various (black hole) spacetimes — a study which is mainly motivated by the black hole stability conjecture (see the introduction of [Dafermos and Rodnianski 2013]). A small selection of examples is [Klainerman 1985; Dafermos and Rodnianski 2009; 2010a; 2011a; 2011b; Andersson and Blue 2009; Tataru and Tohaneanu 2011; Luk 2010; Schlue 2013; Aretakis 2011a; Holzegel and Smulevici 2013; Civin 2014; Dyatlov 2011].

The philosophy of the energy method is first to derive estimates on a suitable energy (and higher-order energies)[8] and then to establish pointwise estimates using Sobolev embeddings. Thus, given a spacetime on which one intends to study the wave equation using the energy method, one first has to set up such a suitable energy (and higher-order energies — but in this paper we focus on the first-order energy). A general procedure is to construct an energy from a foliation of the spacetime by spacelike slices Σ_τ together with a timelike vector field N; see (1.9) in Section 1E. We refrain from discussing here what choices of foliation and timelike vector field lead to a "suitable" notion of energy.[9] Let us just mention here that, in the presence of a globally timelike Killing vector field T, one obtains a particularly well-behaved energy by choosing $N = T$ and a foliation that is invariant under the flow of T.[10] We invite the reader to convince him- or herself that the familiar notions of energy for the wave equation on the Minkowski spacetime or in time-independent inhomogeneous media arise as special cases of this more general scheme.

[7] We refer the reader to [Maslov 1994] for a list of references.

[8] A *first-order* energy controls the *first* derivatives of the wave and is referred to in the following just as "energy". *Higher-order* energies control *higher* derivatives of the wave. A special case of the energy method is the so-called *vector field method*. Higher-order energies arise there naturally by commutation with suitable vector fields; see [Klainerman 1985].

[9] However, see Section II for some examples and footnote 27 on page 1400 for some further comments.

[10] Such a choice corresponds to what we called in the introduction a "canonical energy".

1B2. *Gaussian beams as obstructions to certain uniform behaviour of the energy of waves.* The approximation with Gaussian beams allows us to construct solutions to the wave equation whose energy is localised for an arbitrarily long, but finite, time along a null geodesic. Such solutions naturally form an obstruction to certain uniform statements about the temporal behaviour of the energy of waves. A classical example is the case in which one has a null geodesic that does not leave a compact region in "space" and which has constant energy.[11] Such null geodesics form obstructions to certain formulations of local energy decay being true.[12] However, it is very important to be aware of the fact that, in general, none of the solutions from (1.4) has localised energy for *all* time. Thus, in order to contradict, for instance, an LED statement, it is in general inevitable to resort to a *sequence* of solutions of the form (1.4) which exhibit the contradictory behaviour in the limit. For this scheme to work, however, it is clearly crucial that the LED statement in question is *uniform* with respect to some energy which is left constant by the sequence of Gaussian beam solutions. Note here that (1.4) states in particular that the time T up to which one has good control over the wave can be made arbitrarily large without changing the initial energy! Higher-order initial energies, however, will blow up when T is taken bigger and bigger. In this paper we restrict our consideration to disproving statements that are uniform with respect to the first-order energy. In Sections 6A, 6F and 7A, we demonstrate this important application of Gaussian beams: we show that certain (I)LED statements derived by various people in the presence of "trapping" are sharp in the sense that *some* loss of derivative is necessary (however, one does not necessarily need to lose a whole derivative; see the discussion at the end of Section 6A).

We conclude this section with the remark that in the presence of a globally timelike Killing vector field one can already infer such obstructions from (1.2), since the (canonical) energy of solutions to the wave equation is then constant. In this way, one can easily infer from (1.2) alone that an LED statement in Schwarzschild has to lose differentiability due to the trapping at the photon sphere. But already for trapping in Kerr one needs to know how the "trapped" energy of the solutions referred to in (1.2) behaves in order to infer the analogous result. This knowledge is provided by (1.3) and/or (1.4).

1C. *Gaussian beams are parsimonious.* The approximation by Gaussian beams can be carried out on a Lorentzian manifold (M, g) under minimal assumptions:

(1) One needs a well-posed initial value problem. This is ensured by requiring that (M, g) is globally hyperbolic.[13] However, one can also replace the well-posed initial value problem by a well-posed initial–boundary value problem — and one can obtain, with small changes and some additional work in the proof, qualitatively identical results.

(2) Having fixed an N-energy to work with, one has to have an energy estimate of the form (2.8) at one's disposal, which is guaranteed by the condition (2.3). The estimate (2.8) allows us to infer that the approximation by the Gaussian beam is *global* in space. It is only under this condition that it is justified to

[11] We refer to the right-hand side of (1.3) as the N-energy of the null geodesic.

[12] A classic regarding such a result is by Ralston [1969]. However, he does not use the Gaussian beam approximation in this work, but the geometric optics approximation.

[13] The assumption of global hyperbolicity has another simplifying, but not essential, feature; see the discussion after Definition 3.13.

say in (1.2) and (1.4) that the energy of the actual solution is *localised* along a null geodesic.[14] However, as we show in Remark 2.9, one *always* has a *local* approximation, which is, together with the geometric characterisation of the energy, sufficient for obtaining control of the wave in a small neighbourhood of the underlying null geodesic *regardless of condition* (2.3). This then allows us to establish, for example, the very general Theorem 5.5, which only requires global hyperbolicity (or some other form of well-posedness for the wave equation; see (1)).

In particular, the method of Gaussian beams is not in need of any special structure on the Lorentzian manifold like Killing vector fields (as, for example, needed for the mode analysis or for the construction of quasimodes).

We would also like to emphasise here that in order to apply (1.4) one only needs to understand the behaviour of the null geodesics of the underlying Lorentzian manifold! This knowledge is often in reach and thus Gaussian beams provide in many cases an easy and feasible way for obtaining control of highly oscillatory solutions to the wave equation. In this sense the theory presented in Part I forms a good "black box result" which can be applied to various different problems.

We conclude this section with a brief comparison of the Gaussian beam approximation with the geometric optics approximation: Let us call the geometric optics approximation, which considers approximate solutions of the form (1.5), the "naive" geometric optics approximation. Although it applies under the same general conditions as the Gaussian beam approximation, in general the time T up to which one has good control over the solution cannot be chosen arbitrarily large, since the approximate solution breaks down at caustics. In the Appendix we show that caustics necessarily form along null geodesics that possess conjugate points. A prominent example of such null geodesics are the trapped null geodesics at the photon sphere in the Schwarzschild spacetime (see Section 6A for the proof that these null geodesics have conjugate points). However, the formation of caustics is not a serious limitation of the geometric optics approximation, since one can extend the approximate solution through the caustics, making use of Maslov's canonical operator. The approximate solution obtained in this way is, however, no longer of the simple form (1.5). The advantage of the Gaussian beam approximation is that the simple ansatz (1.5) does not break down at caustics; it yields an approximation up to all finite times T.

1D. *"High-frequency" waves in the physics literature.* In physics, the notion of a *local observer's energy* arose with the emergence of Einstein's theory of relativity: Suppose an observer travels along a timelike curve $\sigma : I \to M$ with unit velocity $\dot{\sigma}$. Then, with respect to a Lorentz frame of his, he measures the local energy density of a wave u to be $\mathbb{T}(u)(\dot{\sigma}, \dot{\sigma})$, where $\mathbb{T}(u)$ is the stress–energy tensor of the wave u; see (1.8) in Section 1E. By considering the 3-parameter family of observers whose velocity vector field is given by the normal n_{Σ_τ} to a foliation of M by spacelike slices Σ_τ, the physical definition of energy is contained in the mathematical one (which is given by (1.9)).

[14]That one needs condition (2.3) for ensuring that the energy is indeed localised is in fact another minor novelty in the study of Gaussian beams on *general* Lorentzian manifolds (note that, in the case of N being a Killing vector field, condition (2.3) is trivially satisfied). For an example for a violation of condition (2.3) we refer to the discussion after (6.8) on page 1406.

The prevalent description of highly oscillatory (or "high-frequency") waves in the physics literature is that the waves (or "photons") propagate along null geodesics γ and each of these rays (or photons) carries an energy–momentum 4-vector $\dot{\gamma}$, where the dot is with respect to some affine parametrisation. In the high-frequency limit, the number of photons is preserved. Thus, the energy of the wave, as measured by a local observer with world line σ, is determined by the energy component $-g(\dot{\gamma}, \dot{\sigma})$ of the momentum 4-vector $\dot{\gamma}$. By considering a highly oscillatory wave that "gives rise to just one photon", one recovers the characterisation of the energy of a Gaussian beam, (1.3), given in this paper.

In the physics literature (see, for example, the classic [Misner et al. 1973, Chapter 22.5]), this description is justified using the naive geometric optics approximation. Here, it suffices to take $N = 0$ in (1.5); one then considers approximate solutions to the wave equation of the form $u_\lambda = a \cdot e^{i\lambda\phi}$, where a and ϕ satisfy

$$d\phi \cdot d\phi = 0 \quad \text{and} \quad 2 \operatorname{grad} \phi(a) + \Box\phi \cdot a = 0. \tag{1.6}$$

The conservation law

$$\operatorname{div}(a^2 \operatorname{grad} \phi) = 0, \tag{1.7}$$

which can be easily inferred from the second equation in (1.6), is interpreted as the conservation of the number-flux vector $S = a^2 \operatorname{grad} \phi$ of the photons. The leading component in λ of the renormalised[15] stress–energy tensor $T(u_\lambda)$ of the wave $u_\lambda = a \cdot e^{i\lambda\phi}$ in the geometric optics limit is then given by

$$\mathbb{T}(u_\lambda) = \operatorname{grad} \phi \otimes S,$$

from which it then follows that each photon carries a 4-momentum $\operatorname{grad} \phi = \dot{\gamma}$.

In particular, making use of the conservation law (1.7), it is not difficult[16] to prove a geometric characterisation of the energy of waves in the naive geometric optics limit analogous to the one we prove in this paper for Gaussian beams. However, as we have mentioned in the previous section, the naive geometric optics approximation has the undesirable feature that it breaks down at caustics.

The characterisation of the energy of Gaussian beams is more difficult, since (1.7) is replaced only by an *approximate* conservation law.[17] Moreover, it provides a rigorous justification of the temporal behaviour of the local observer's energy of photons, *which also applies to photons along whose trajectory caustics would form*.

1E. *Notation.* Given a Lorentzian manifold (M, g), we denote the canonical isomorphisms induced by the metric g between the tangent and cotangent space by $\sharp : T_x^*M \to T_xM$ and $\flat : T_xM \to T_x^*M$, where $x \in M$ and, for $\alpha \in T_x^*M$ and $X \in T_xM$, the isomorphisms $\sharp$ and $\flat$ are given by $\alpha^\sharp := g^{-1}(\alpha, \cdot)$ and $X^\flat := g(X, \cdot)$. Here g^{-1} denotes the inverse of the metric g. Moreover, we denote with $\cdot$ the inner product of two vectors as well as the inner product of two covectors, i.e., for $\alpha, \beta \in T_x^*M$ we write $\alpha \cdot \beta := g^{-1}(\alpha, \beta)$, and for $X, Y \in T_xM$ we write $X \cdot Y := g(X, Y)$. We also introduce the notation $\operatorname{grad} f := (df)^\sharp$ for the gradient of a function $f \in C^\infty(M, \mathbb{R})$. The Levi-Civita connection on the

[15] Divided by λ^2.

[16] Although, to the best of our knowledge, it is nowhere done explicitly.

[17] See the discussion below (4.6) in Section 4.

Lorentzian manifold (M, g) is denoted by ∇, and we write $\operatorname{div} Z := \nabla_\mu Z^\mu$ for the divergence of a smooth vector field Z on M. Furthermore, we define the *wave operator* $\Box_g$ by

$$\Box_g u := \nabla^\mu \nabla_\mu u.$$

From here on we will, however, omit the index g on $\Box_g$, since it is clear from the context which Lorentzian metric is referred to.

Whenever we are given a time-oriented Lorentzian manifold (M, g) that is (partly) foliated by spacelike slices $\{\Sigma_\tau\}_{\tau\in[0,\tau^*)}$, $0<\tau^*\leq\infty$, we denote the future-directed unit normal to the slice Σ_τ by n_{Σ_τ}. Moreover, the induced Riemannian metric on Σ_τ is then denoted by $\bar{g}_\tau$ and we set $R_{[0,T]} := \bigcup_{0\leq\tau\leq T} \Sigma_\tau$.

For $u \in C^\infty(M, \mathbb{C})$ we define the *stress–energy tensor* $\mathbb{T}(u)$ by

$$\mathbb{T}(u) := \tfrac{1}{2}\overline{du} \otimes du + \tfrac{1}{2} du \otimes \overline{du} - \tfrac{1}{2} g(\,\cdot\,,\cdot\,) g^{-1}(du, \overline{du}). \tag{1.8}$$

Given in addition a vector field N, we define the current $J^N(u)$ by

$$J^N(u) := [\mathbb{T}(u)(N, \cdot\,)]^\sharp.$$

Finally, if N is future-directed timelike, we call

$$E^N_\tau(u) := \int_{\Sigma_\tau} J^N(u) \cdot n_{\Sigma_\tau} \operatorname{vol}_{\bar{g}_\tau} \tag{1.9}$$

the *N-energy* of u at time τ, where $\operatorname{vol}_{\bar{g}_\tau}$ denotes the volume element corresponding to the metric $\bar{g}_\tau$.[18] If $A \subseteq \Sigma_\tau$, then $E^N_{\tau,A}(u)$ denotes the N-energy of u at time τ in the volume A, i.e., the integration in (1.9) is only over A.

The notion (1.9) of the N-energy of a function u is especially helpful whenever we have an adequate knowledge of $\Box u$, since one can then infer detailed information about the behaviour of the N-energy (see the energy estimate (2.8) in the next section), and thus also about the behaviour of u itself. Hence, the stress–energy tensor (1.8) together with the notion of the N-energy is particularly useful for solutions u of the *wave equation*

$$\Box u = 0. \tag{1.10}$$

For more on the stress–energy tensor and the notion of energy, we refer the reader to [Taylor 2011, Chapters 2.7 and 2.8].

Given a Lorentzian manifold (M, g) and $A \subseteq M$, we denote with $J^+(A)$ the causal future of A, namely, all the points $x \in M$ such that there exists a future-directed causal curve starting at some point of A and ending at x. The causal past of A, $J^-(A)$, is defined analogously.[19] Finally, C and c will always denote positive constants.

For simplicity of notation we restrict our considerations to 3+1-dimensional Lorentzian manifolds (M, g). However, all results extend in an obvious way to dimensions $n+1$, $n \geq 1$. Moreover, all given

[18] See also [Choquet-Bruhat 2009, Appendix III, Sections 2.3 and 2.4] (in particular Definition (2.27)) for a detailed discussion of the notion of N-energy.

[19] See also Chapter 14 in [O'Neill 1983].

manifolds, functions and tensor fields are assumed to be smooth, although this is only for convenience and clearly not necessary.

Part I. The theory of Gaussian beams on Lorentzian manifolds

2. Solutions of the wave equation with localised energy

This section and the next are devoted to a sketch of the proof of Theorem 2.1, which summarises the state of the art knowledge concerning the construction of solutions with localised energy using the approximation by Gaussian beams.

Theorem 2.1. *Let (M, g) be a time-oriented, globally hyperbolic Lorentzian manifold with time function t, foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface.*[20] *Furthermore, let γ be a null geodesic that intersects Σ_0 and N a timelike, future-directed vector field.*

For any neighbourhood $\mathcal{N}$ of γ, any $T > 0$ with $\Sigma_T \cap \operatorname{Im}(\gamma) \neq \varnothing$ (see Figure 1), and any $\mu > 0$, there exists a solution $v \in C^\infty(M, \mathbb{C})$ of the wave equation (1.10) with $E^N_0(v) = 1$ and $\tilde{u} \in C^\infty(M, \mathbb{C})$ with $\operatorname{supp}(\tilde{u}) \subseteq \mathcal{N}$ *such that*

$$E^N_\tau(v - \tilde{u}) < \mu \quad \textit{for all } 0 \le \tau \le T \tag{2.2}$$

provided that we have, on $R_{[0,T]} \cap J^+(\mathcal{N} \cap \Sigma_0)$,

$$\frac{1}{|dt(n_{\Sigma_\tau})|} + |g(N, n_{\Sigma_\tau})| \le C < \infty \quad \text{and} \quad 0 < c \le |g(N, N)|,$$

$$|\nabla N(n_{\Sigma_\tau}, n_{\Sigma_\tau})| + \sum_{i=1}^{3} |\nabla N(n_{\Sigma_\tau}, e_i)| + \sum_{i,j=1}^{3} |\nabla N(e_i, e_j)| \le C < \infty, \tag{2.3}$$

where c and C are positive constants and $\{n_{\Sigma_\tau}, e_1, e_2, e_3\}$ is an orthonormal frame.

Note that (2.2) together with $\operatorname{supp}(\tilde{u}) \subseteq \mathcal{N}$ make rigorous the statement that the solution v hardly disperses up to time T. The energy of the solution v stays localised for finite time.

Proof. The function $\tilde{u}$ in the theorem is the Gaussian beam, the *approximate* solution to the wave equation (1.10) which we need to construct. Recall that a Gaussian beam $u_\lambda \in C^\infty(M, \mathbb{C})$ is of the form

$$u_\lambda(x) = a_{\mathcal{N}}(x) e^{i\lambda\phi(x)}, \tag{2.4}$$

where $\lambda > 0$ is a parameter that determines how quickly the Gaussian beam oscillates, and $a_{\mathcal{N}}$ and ϕ are smooth, complex-valued functions on M that do *not* depend on λ. However, $a_{\mathcal{N}}$ depends on the neighbourhood $\mathcal{N}$ of the null geodesic γ. In Section 3 we outline how one constructs the functions $a_{\mathcal{N}}$ and ϕ in such a way that u_λ satisfies the following three conditions: The *first condition* is

$$\|\Box u_\lambda\|_{L^2(R_{[0,T]})} \le C(T), \tag{2.5}$$

[20] Bernal and Sánchez [2005] showed that every globally hyperbolic Lorentzian manifold admits a smooth time function.

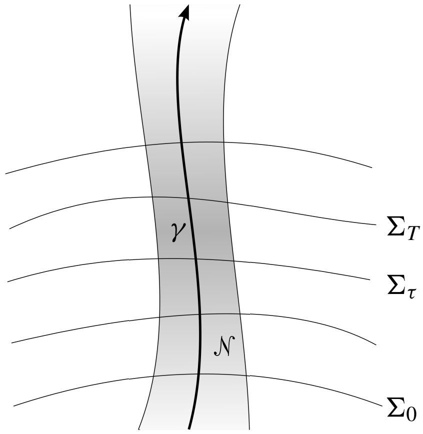

Figure 1. The setting of Theorem 2.1.

where the constant $C(T)$ depends on $a_{\mathcal{N}}$, ϕ and T, but not on λ. The *second condition* is

$$E_0^N(u_\lambda) \to \infty \quad \text{for } \lambda \to \infty, \tag{2.6}$$

where N is the timelike vector field from Theorem 2.1. Finally, the *third condition* is

$$u_\lambda \text{ is supported in } \mathcal{N}. \tag{2.7}$$

Assuming for now that we have already found functions $a_{\mathcal{N}}$ and ϕ such that the conditions (2.5), (2.6) and (2.7) are satisfied, we finish the proof of Theorem 2.1. In order to normalise the initial energy of the approximate solutions u_λ, we define

$$\tilde{u}_\lambda := \frac{u_\lambda}{\sqrt{E_0^N(u_\lambda)}},$$

which, moreover, yields

$$\|\Box\tilde{u}_\lambda\|_{L^2(R_{[0,T]})} \to 0 \quad \text{for } \lambda \to \infty.$$

This says that as the Gaussian beam becomes more and more oscillatory (i.e., for bigger and bigger λ), the closer it comes to being a proper solution to the wave equation.

We now define the actual solution v_λ of the wave equation — the one that is being approximated by the $\tilde{u}_\lambda$ — to be the solution of the following initial value problem:

$$\begin{aligned} \Box v &= 0, \\ v\big|_{\Sigma_0} &= \tilde{u}_\lambda\big|_{\Sigma_0}, \\ n_{\Sigma_0} v\big|_{\Sigma_0} &= n_{\Sigma_0}\tilde{u}_\lambda\big|_{\Sigma_0}. \end{aligned}$$

Here, we make use of the fact that the Lorentzian manifold (M, g) is globally hyperbolic and thus allows for a well-posed initial value problem for the wave equation. Moreover, the condition (2.3) ensures that we have an energy estimate of the form

$$\int_{\Sigma_\tau} J^N(u)\cdot n_{\Sigma_\tau}\,\mathrm{vol}_{\bar g_\tau} \le C(T,N,\{\Sigma_\tau\})\left(\int_{\Sigma_0} J^N(u)\cdot n_{\Sigma_0}\,\mathrm{vol}_{\bar g_0} + \|\Box u\|^2_{L^2(R_{[0,T]})}\right) \quad \text{for all } 0\le\tau\le T \tag{2.8}$$

at our disposal (see for example [Taylor 2011, Chapter 2.8]). Thus, we obtain

$$E^N_\tau(v_\lambda - \tilde u_\lambda) \le C(T,N,\Sigma_\tau)\cdot \|\Box\tilde u_\lambda\|^2_{L^2(R_{[0,T]})} \quad \text{for all } 0\le\tau\le T,$$

which goes to zero for $\lambda\to\infty$. Given now $\mu>0$, it suffices to choose $\lambda_0>0$ big enough and to set $\tilde u := \tilde u_{\lambda_0}$ and $v := v_{\lambda_0}$, which then finishes the proof under the assumption of the conditions (2.5), (2.6) and (2.7). □

We end this section with a couple of remarks about Theorem 2.1:

Remark 2.9. As already mentioned, the condition (2.3) ensures that we have the energy estimate (2.8). It is automatically satisfied if the region under consideration, $R_{[0,T]}\cap J^+(\mathcal N\cap\Sigma_0)$, is relatively compact, which will be the case in many concrete applications.

Moreover, by choosing $\mathcal N$ a bit smaller if necessary, we can always arrange that $\Sigma_T\cap\mathcal N$ is relatively compact and that $\mathcal N\cap R_{[0,T]}\subseteq J^-(\Sigma_T\cap\mathcal N)$. Doing, then, the energy estimate in the relatively compact region $J^-(\Sigma_T\cap\mathcal N)\cap J^+(\Sigma_0)$, we obtain

$$E^N_{\tau,\mathcal N\cap\Sigma_\tau}(v-\tilde u) < \mu \quad \text{for all } 0\le\tau\le T \tag{2.10}$$

independently of (2.3). Of course, the information given by (2.10) is not interesting here, since Theorem 2.1 does not provide more information about $\tilde u$ than its region of support. However, in Section 4 we will derive more information about the approximate solution $\tilde u$ and then (2.10) will tell us about the temporal behaviour of the localised energy of v; see Theorem 5.1.

Remark 2.11. By taking the real or the imaginary part of $\tilde u_\lambda$ and v_λ it is clear that we can choose $\tilde u$ and v in Theorem 2.1 to be real valued.

3. The construction of Gaussian beams

Before we sketch the construction of Gaussian beams, let us mention that other (and complete) presentations of this subject can be found, for example, in [Babich and Buldyrev 2009] or [Ralston 1982]. The latter reference also includes the construction of Gaussian beams for more general hyperbolic PDEs.

Given now a neighbourhood $\mathcal N$ of a null geodesic γ, we need to construct functions $a_{\mathcal N},\phi\in C^\infty(M,\mathbb C)$ such that the approximate solution $u_\lambda = a_{\mathcal N}\cdot e^{i\lambda\phi}$ satisfies the conditions (2.5), (2.6) and (2.7). This will then finish the proof of Theorem 2.1. We compute

$$\Box u_\lambda = -\lambda^2(d\phi\cdot d\phi)a_{\mathcal N}e^{i\lambda\phi} + i\lambda\Box\phi\cdot a_{\mathcal N}e^{i\lambda\phi} + 2i\lambda\ \mathrm{grad}\,\phi(a_{\mathcal N})\cdot e^{i\lambda\phi} + \Box a_{\mathcal N}\cdot e^{i\lambda\phi}. \tag{3.1}$$

Demanding $d\phi\cdot d\phi=0$ (the eikonal equation) and $2\,\mathrm{grad}\,\phi(a_{\mathcal N})+\Box\phi\cdot a_{\mathcal N}=0$ would lead us to the naive geometric optics approximation (see (1.6)), whose major drawback is that in general the solution ϕ of the eikonal equation breaks down at some point along γ due to the formation of caustics. The method of Gaussian beams takes a slightly different approach. We only require an *approximate* solution $\phi \in C^\infty(M, \mathbb{C})$ of the eikonal equation in the sense that

$$d\phi \cdot d\phi \ \text{ vanishes on } \gamma \text{ to high order.}$$[21]

Moreover, we demand that

$$\phi\big|_\gamma \ \text{ and } \ d\phi\big|_\gamma \ \text{ are real valued,} \tag{3.2}$$

$$\mathfrak{Im}\big(\nabla\nabla\phi\big|_\gamma\big) \ \text{ is positive definite on a 3-dimensional subspace transversal to } \dot{\gamma}, \tag{3.3}$$

where $\mathfrak{Im}\big(\nabla\nabla\phi\big|_x\big)$, $x \in M$, denotes the imaginary part of the bilinear map $\nabla\nabla\phi\big|_x : T_xM \times T_xM \to \mathbb{C}$. Let us assume for a moment that (3.2) and (3.3) hold. Taking slice coordinates for γ, that is, a coordinate chart (U, φ), $\varphi : U \subseteq M \to \mathbb{R}^4$, such that $\varphi(\mathrm{Im}(\gamma) \cap U) = \{x_1 = x_2 = x_3 = 0\}$, we obtain

$$\mathfrak{Im}(\phi)(x) \geq c \cdot (x_1^2 + x_2^2 + x_3^2), \tag{3.4}$$

at least if we restrict ϕ to a small enough neighbourhood of γ. Note that such slice coordinates exist, since the global hyperbolicity of (M, g) implies that γ is an embedded submanifold of M. This is easily seen by appealing to the strong causality condition.[22] Let us now denote the real part of ϕ by ϕ_1 and the imaginary part by ϕ_2. We then have

$$u_\lambda = a_{\mathcal{N}} \cdot e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2}.$$

We see that the last factor imposes the shape of a Gaussian on u_λ, centred around γ — this explains the name. Moreover, for λ large this Gaussian will become more and more narrow, i.e., less and less weight is given to the values of $a_{\mathcal{N}}$ away from γ.

We rewrite (3.1) as

$$\Box u_\lambda = -\lambda^2 \underbrace{(d\phi \cdot d\phi)} \cdot a_{\mathcal{N}} e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2} + i\lambda \underbrace{(2 \operatorname{grad}\phi\,(a_{\mathcal{N}}) + \Box\phi \cdot a_{\mathcal{N}})} \cdot e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2} + \Box a_{\mathcal{N}} \cdot e^{i\lambda\phi_1} \cdot e^{-\lambda\phi_2}. \tag{3.5}$$

Intuitively, if we can arrange for the underbraced terms to vanish on γ to some order and we choose large λ, then we will pick up only very small contributions. The next lemma makes this rigorous:

Lemma 3.6. *Let $f \in C_0^\infty([0, T] \times \mathbb{R}^3, \mathbb{C})$ vanish along $\{x_1 = x_2 = x_3 = 0\}$ to order S, that is, all partial derivatives up to and including the order S of f vanish along $\{x_1 = x_2 = x_3 = 0\}$, and let $c > 0$ be a constant. We then have*

(i) $$\int_{[0,T]\times\mathbb{R}^3} |f(x)|^2 e^{-\lambda \cdot c(x_1^2+x_2^2+x_3^2)}\, dx \leq C\lambda^{-(S+1)-3/2}$$

and

(ii) $$\int_{[0,T]\times\mathbb{R}^3} |f(x)| e^{-\lambda \cdot c(x_1^2+x_2^2+x_3^2)}\, dx \leq C\lambda^{-(S+1)/2-3/2},$$

where C depends on f (and on T).

---

[21] The exact order to which we require $d\phi \cdot d\phi$ to vanish on γ will be determined later.

[22] See, for example, [O'Neill 1983, Chapter 14] for more on the strong causality condition.

Proof. We prove (i) here, since it is used in the following. The formulation (ii) of Lemma 3.6 is appealed to in the proof of Theorem 4.1 in Section 4 — the proof is analogous.

Introduce stretched coordinates $y_0 := x_0$, $y_i := \sqrt{\lambda} x_i$ for $i = 1, 2, 3$. Since f vanishes along the x_0 axis to order S and has compact support, we get $|f(x)| \le C \cdot |\underline{x}|^{S+1}$ for all $x = (x_0, \underline{x}) \in [0, T] \times \mathbb{R}^3$; thus

$$\left| f\left(y_0, \frac{\underline{y}}{\sqrt{\lambda}}\right)\right| \le C \cdot \frac{|\underline{y}|^{S+1}}{\lambda^{(S+1)/2}}.$$

This yields

$$\int_{[0,T]\times\mathbb{R}^3} |f(x)|^2 e^{-\lambda \cdot c|\underline{x}|^2}\, dx \le \int_{[0,T]\times\mathbb{R}^3} C \cdot |\underline{y}|^{2(S+1)} e^{-c|\underline{y}|^2}\, dy \cdot \lambda^{-(S+1)-3/2}. \tag{3.7}$$

This concludes the proof. □

We summarise the approach taken by the Gaussian beam approximation in the following:

Lemma 3.8. *Within the setting of Theorem 2.1, assume we are given $a, \phi \in C^\infty(M, \mathbb{C})$ which satisfy (3.2) and (3.3). Moreover, assume*

$$d\phi \cdot d\phi \qquad \text{vanishes to second order along } \gamma, \tag{3.9}$$

$$2 \operatorname{grad}\phi(a) + \Box\phi \cdot a \qquad \text{vanishes to zeroth order along } \gamma, \tag{3.10}$$

$$a(\operatorname{Im}(\gamma) \cap \Sigma_0) \neq 0 \quad \text{and} \quad d\phi\,(\operatorname{Im}(\gamma) \cap \Sigma_0) \neq 0. \tag{3.11}$$

Given a neighbourhood $\mathcal{N}$ of γ, we can then multiply a by a suitable bump function $\chi_{\mathcal{N}}$, which is equal to one in a neighbourhood of γ and satisfies $\operatorname{supp}(\chi_{\mathcal{N}}) \subseteq \mathcal{N}$, such that

$$u_\lambda = u_{\lambda,\mathcal{N}} = a_{\mathcal{N}} e^{i\lambda\phi}$$

satisfies (2.5), (2.6) and (2.7), where $a_{\mathcal{N}} := a \cdot \chi_{\mathcal{N}}$.

Proof. Cover γ by slice coordinate patches and let $\tilde{\chi}$ be a bump function which meets the following three requirements:

(i) $\tilde{\chi}$ is equal to one in a neighbourhood of γ.

(ii) (3.4) is satisfied for all $x \in \operatorname{supp}(\tilde{\chi})$.

(iii) $R_{[0,T]} \cap \operatorname{supp}(\tilde{\chi})$ is relatively compact in M for all $T > 0$ with $\Sigma_T \cap \operatorname{Im}(\gamma) \neq \varnothing$.

Pick now a second bump function $\tilde{\chi}_{\mathcal{N}}$ which is again equal to one in a neighbourhood of γ and is supported in $\mathcal{N}$. We then define $\chi_{\mathcal{N}} := \tilde{\chi} \cdot \tilde{\chi}_{\mathcal{N}}$. Clearly, (2.7) is satisfied.

In order to see that (2.5) holds, note that the conditions (3.2), (3.3), (3.9) and (3.10) are still satisfied by the pair $(a_{\mathcal{N}}, \phi)$. Moreover note that, due to condition (iii), the integrand is supported in a compact region for each $T > 0$ with $\Sigma_T \cap \operatorname{Im}(\gamma) \neq \varnothing$. Thus, the spacetime volume of this region is finite. We thus obtain (2.5) from (3.5) and Lemma 3.6.

Finally, we have

$$E_0^N(u_\lambda) \ge C \cdot (\lambda^{1/2} - 1).$$

This follows since the highest-order term in λ in $E_0^N(u_\lambda)$ is

$$\lambda^2 \cdot \int_{\Sigma_0} |a_{\mathcal{N}}|^2 N\phi_1 \cdot n_{\Sigma_0}\phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_0},$$

and the same scaling argument used in the proof of Lemma 3.6 shows that the term $e^{-2\lambda\phi_2}$ leads to a $\lambda^{-3/2}$ damping — and only to a $\lambda^{-3/2}$ damping due to condition (3.11) (together with (3.9) and (3.2)). Thus, (2.6) is satisfied as well and the lemma is proved. □

For the actual construction of the functions ϕ and a such that (3.2), (3.3), (3.9), (3.10), and (3.11) are satisfied, we refer the reader, for example, to [Ralston 1982]. We content ourselves here with pointing out that the above conditions on the functions ϕ and a are actually only conditions on the first, second, and third derivatives of ϕ along γ and on the first derivative of a and the value of a itself along γ. Making the choice

$$d\phi(s) := \dot{\gamma}^\flat(s) \tag{3.12}$$

along γ, where s is an affine parameter for γ, the condition (3.9) turns into a quadratic ODE for the second derivatives of ϕ along γ, while the condition (3.10) turns into a linear ODE for a along γ. The important step is to show that one can find a *global* solution for the first ODE, which, moreover, also satisfies (3.3).

We conclude this section by making the following definition for future reference:

Definition 3.13. Let (M, g) be a time-oriented, globally hyperbolic Lorentzian manifold with time function t, foliated by the level sets $\Sigma_\tau = \{t = \tau\}$. Furthermore, let $\gamma : [0, S) \to M$ be an affinely parametrised future-directed null geodesic with $\gamma(0) \in \Sigma_0$, where $0 < S \le \infty$, and let N be a timelike, future-directed vector field.

Given functions $a, \phi \in C^\infty(M, \mathbb{C})$ that satisfy (3.2), (3.3), (3.9), (3.10), $a(\operatorname{Im}(\gamma) \cap \Sigma_0) \neq 0$ and (3.12), we call the function

$$u_{\lambda,\mathcal{N}} = a_{\mathcal{N}} e^{i\lambda\phi}$$

a *Gaussian beam along γ with structure functions a and ϕ and with parameters λ and $\mathcal{N}$*. Here, $a_{\mathcal{N}} = a \cdot \chi_{\mathcal{N}} = a \cdot \tilde{\chi} \cdot \tilde{\chi}_{\mathcal{N}}$ with $\tilde{\chi}$ and $\tilde{\chi}_{\mathcal{N}}$ as in the proof of Lemma 3.8. Moreover, we call the function

$$\tilde{u}_{\lambda,\mathcal{N}} = \frac{u_{\lambda,\mathcal{N}}}{\sqrt{E_0^N(u_{\lambda,\mathcal{N}})}} \cdot \sqrt{E}$$

a *Gaussian beam along γ with structure functions a and ϕ, parameters λ and $\mathcal{N}$, and initial N-energy E*, where E is a strictly positive real number. Let us emphasise that, when we say "a Gaussian beam along γ", γ encodes here not only the image of γ, but also the affine parametrisation.

We end this section with the remark that, for the sole *construction* of the Gaussian beams, the assumption of the global hyperbolicity of (M, g) can be replaced by the assumption that the null geodesic $\gamma : \mathbb{R} \supseteq I \to M$ is a smooth embedding, in particular $\gamma(I)$ is an embedded submanifold. Moreover, note that, if $\gamma : \mathbb{R} \supseteq I \to M$ is a smooth injective immersion and if $[a, b] \subseteq I$ with $a, b \in \mathbb{R}$, then $\gamma|_{(a,b)} : (a, b) \to M$ is a smooth embedding. Thus the construction of a Gaussian beam is always possible for null geodesics with no self-intersections on general Lorentzian manifolds — at least up to some finite affine time in the domain of γ.

4. Geometric characterisation of the energy of Gaussian beams

In this section we characterise the energy of a Gaussian beam in terms of the energy of the underlying null geodesic. The following theorem is the main result of Part I:

Theorem 4.1. *Let (M, g) be a time-oriented, globally hyperbolic Lorentzian manifold with time function t, foliated by the level sets $\Sigma_\tau = \{t = \tau\}$. Moreover, let N be a timelike future-directed vector field and $\gamma : [0, S) \to M$ an affinely parametrised future-directed null geodesic with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$.*

For any $T > 0$ with $\mathrm{Im}(\gamma) \cap \Sigma_T \neq \varnothing$ and any $\mu > 0$, there exists a $\lambda_0 > 0$ such that any Gaussian beam $\tilde{u}_{\lambda,\mathcal{N}}$ along γ with structure functions a and ϕ, parameters $\lambda \geq \lambda_0$ and $\mathcal{N}$, and initial N-energy equal to $-g(N, \dot{\gamma})\big|_{\gamma(0)}$, satisfies

$$\left|E^N_\tau(\tilde{u}_{\lambda,\mathcal{N}}) - \left(-g(N, \dot{\gamma})\big|_{\mathrm{Im}(\gamma)\cap\Sigma_\tau}\right)\right| < \mu \quad \textit{for all } 0 \leq \tau \leq T. \tag{4.2}$$

Before we give the proof, we make a few remarks:

(i) The only information about a Gaussian beam we made use of in Theorem 2.1, apart from it being an approximate solution, was that it is supported in a given neighbourhood $\mathcal{N}$ of the null geodesic γ. This then yielded, together with (2.2), an estimate on the energy outside of the neighbourhood $\mathcal{N}$ of the actual solution to the wave equation, so we could construct solutions to the wave equation with *localised* energy. However, Theorem 2.1 does not make any statement about the *temporal behaviour* of this localised energy. The above theorem fills this gap by investigating the temporal behaviour of the energy of the approximate solution, i.e., of the Gaussian beam. Together with (2.2) (or even with (2.10)!) this then gives an estimate on the temporal behaviour of the localised energy of the actual solution to the wave equation.

(ii) If N is a timelike Killing vector field, the N-energy $-g(N, \dot{\gamma})$ of the null geodesic γ is constant and, thus, so is approximately the N-energy of the Gaussian beam.

(iii) By our Definition 3.13 a Gaussian beam is a complex-valued function. However, by taking the real or the imaginary part, one can also define a real-valued Gaussian beam. The result of Theorem 4.1 also holds true in this case, and can be proved using exactly the same technique — only the computations become a bit longer, since we have to deal with more terms.

(iv) Although we have stated the above theorem again using the general assumptions needed for Theorem 2.1, we actually do not need more assumptions than we need for the construction of a Gaussian beam; see the final remark of the previous section.

Proof. Recall from Definition 3.13 that a Gaussian beam $\tilde{u}_{\lambda,\mathcal{N}}$ along γ with structure functions a and ϕ, parameters $\mathcal{N}$ and λ, and initial N-energy equal to $-g(N, \dot{\gamma})\big|_{\gamma(0)}$, is a function

$$\tilde{u}_{\lambda,\mathcal{N}} = \frac{u_{\lambda,\mathcal{N}}}{\sqrt{E^N_0(u_{\lambda,\mathcal{N}})}} \cdot \sqrt{-g(N, \dot{\gamma})|_{\gamma(0)}} = \frac{a_{\mathcal{N}} e^{i\lambda\phi}}{\sqrt{E^N_0(u_{\lambda,\mathcal{N}})}} \cdot \sqrt{-g(N, \dot{\gamma})|_{\gamma(0)}},$$

where the functions $a_{\mathcal{N}}$ and ϕ satisfy (3.2), (3.3), (3.9), (3.10), (3.11), (3.12), $\operatorname{supp}(a_{\mathcal{N}}) \subseteq \mathcal{N}$, $\mathcal{N} \cap R_{[0,T]}$ is relatively compact for all $T > 0$ with $\Sigma_T \cap \operatorname{Im}(\gamma) \neq \varnothing$, and, for a cover of γ with slice coordinate patches, (3.4) holds for all $x \in \operatorname{supp}(a_{\mathcal{N}})$.

We will show

$$E_\tau^N(\tilde{u}_{\lambda,\mathcal{N}}) = \frac{E_\tau^N(u_{\lambda,\mathcal{N}})}{E_0^N(u_{\lambda,\mathcal{N}})} \cdot \left[-g(N,\dot{\gamma})\big|_{\gamma(0)}\right] = -g(N,\dot{\gamma})\big|_{\operatorname{Im}(\gamma)\cap\Sigma_\tau} + o(\lambda), \tag{4.3}$$

where $o(\lambda)$ goes to zero uniformly in $0 \le \tau \le T$ for $\lambda \to \infty$. This would then prove the theorem.

In the following we compute the leading-order term of $E_\tau^N(u_{\lambda,\mathcal{N}})$ in λ:

$$\begin{aligned} J^N(u_{\lambda,\mathcal{N}})\cdot n_{\Sigma_\tau} &= \mathfrak{Re}(N u_{\lambda,\mathcal{N}} \cdot \overline{n_{\Sigma_\tau} u_{\lambda,\mathcal{N}}}) - \tfrac{1}{2} g(N, n_{\Sigma_\tau})\, du_{\lambda,\mathcal{N}} \cdot \overline{du_{\lambda,\mathcal{N}}} \\ &= \lambda^2 |a_{\mathcal{N}}|^2 N\phi_1 \cdot n_{\Sigma_\tau}\phi_1 \cdot e^{-2\lambda\phi_2} + \lambda^2 |a_{\mathcal{N}}|^2 N\phi_2 \cdot n_{\Sigma_\tau}\phi_2 \cdot e^{-2\lambda\phi_2} + \mathcal{O}(\lambda)\cdot e^{-2\lambda\phi_2} \\ &\qquad - \tfrac{1}{2} g(N, n_{\Sigma_\tau})[\lambda^2 |a_{\mathcal{N}}|^2\, (d\phi_1 \cdot d\phi_1)\, e^{-2\lambda\phi_2} + \lambda^2 |a_{\mathcal{N}}|^2\, (d\phi_2 \cdot d\phi_2)\, e^{-2\lambda\phi_2} + \mathcal{O}(\lambda)\cdot e^{-2\lambda\phi_2}]. \end{aligned}$$

Note that $d\phi_2\big|_{\gamma(\tau)} = 0$, so these terms are of lower order after integration over Σ_τ. The same holds for the $d\phi_1 \cdot d\phi_1$ term. Thus, we get

$$E_\tau^N(u_{\lambda,\mathcal{N}}) = \underbrace{\lambda^2 \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 N\phi_1 \cdot n_{\Sigma_\tau}\phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_\tau}}_{=\mathcal{O}(\lambda^{1/2})} + \underbrace{\text{lower-order terms}}_{=\mathcal{O}(1)}. \tag{4.4}$$

The main part of the proof is an approximate conservation law. Recall that $a_{\mathcal{N}}$ and ϕ satisfy (3.9) and (3.10). These equations yield

$$\begin{aligned} \operatorname{grad}\phi(|a_{\mathcal{N}}|^2) &= \operatorname{grad}\phi(a_{\mathcal{N}}) \cdot \overline{a_{\mathcal{N}}} + a_{\mathcal{N}} \cdot \operatorname{grad}\phi(\overline{a_{\mathcal{N}}}) \\ &= -\tfrac{1}{2}(\Box\phi \cdot a_{\mathcal{N}}\overline{a_{\mathcal{N}}} + a_{\mathcal{N}}\,\overline{\Box\phi} \cdot \overline{a_{\mathcal{N}}}) = -\mathfrak{Re}(\Box\phi)|a_{\mathcal{N}}|^2 \quad \text{along } \gamma \end{aligned} \tag{4.5}$$

and

$$d\phi \cdot d\phi = (d\phi_1 + i\,d\phi_2)\cdot(d\phi_1 + i\,d\phi_2) = d\phi_1 \cdot d\phi_1 - d\phi_2 \cdot d\phi_2 + 2i\, d\phi_1 \cdot d\phi_2$$

vanishes to second order along γ; thus, in particular,

$$d\phi_1 \cdot d\phi_2 = \operatorname{grad}\phi_1\,(\phi_2) \quad \text{vanishes along } \gamma \text{ to second order.} \tag{4.6}$$

Lemma 3.6(ii), together with (4.5) and (4.6), shows that the current

$$X_{\lambda,\mathcal{N}} = \lambda^2 \cdot |a_{\mathcal{N}}|^2 e^{-2\lambda\phi_2} \operatorname{grad}\phi_1$$

is approximately conserved in the sense that

$$\begin{aligned} &\int_{R_{[0,\tau]}} \operatorname{div} X_{\lambda,\mathcal{N}} \operatorname{vol}_g \\ &\quad = \lambda^2 \cdot \int_{R_{[0,\tau]}} \Big(\underbrace{[\operatorname{grad}\phi_1(|a_{\mathcal{N}}|^2) + \Box\phi_1 \cdot |a_{\mathcal{N}}|^2] e^{-2\lambda\phi_2}}_{=\lambda^{-1/2}\cdot\lambda^{-3/2}=\lambda^{-2} \text{ after integration}} - \underbrace{2\lambda \operatorname{grad}\phi_1(\phi_2)\cdot |a_{\mathcal{N}}|^2 e^{-2\lambda\phi_2}}_{=\lambda\cdot\lambda^{-3/2}\cdot\lambda^{-3/2}=\lambda^{-2} \text{ after integration}}\Big) \operatorname{vol}_g = \mathcal{O}(1), \end{aligned}$$

but

$$\int_{\Sigma_\tau} X_{\lambda,\mathcal{N}} \cdot n_{\Sigma_\tau} \operatorname{vol}_{\bar{g}_\tau} = \lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_\tau} = \mathcal{O}(\lambda^{1/2}).$$

In particular, we obtain[23]

$$\left|\lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_\tau} - \lambda^2 \cdot \int_{\Sigma_0} |a_{\mathcal{N}}|^2 n_{\Sigma_0} \phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_0}\right| = \left|\int_{R_{[0,\tau]}} \operatorname{div} X_{\lambda,\mathcal{N}} \operatorname{vol}_g\right| = \mathcal{O}(1). \quad (4.7)$$

We also observe that, by Lemma 3.6(ii), we have

$$\lambda^2 \cdot \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 \big(N\phi_1 - N\phi_1\big|_{\operatorname{Im}(\gamma)\cap\Sigma_\tau}\big) \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_\tau} = \mathcal{O}(1). \quad (4.8)$$

It thus follows from (4.4), (4.7), and (4.8) that

$$\begin{aligned} E^N_\tau(u_{\lambda,\mathcal{N}}) &= \lambda^2 \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 N\phi_1 \cdot n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_\tau} + \mathcal{O}(1) \\ &= \lambda^2 \cdot N\phi_1\big|_{\operatorname{Im}(\gamma)\cap\Sigma_\tau} \int_{\Sigma_\tau} |a_{\mathcal{N}}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_\tau} + \mathcal{O}(1) \\ &= \lambda^2 \cdot N\phi_1\big|_{\operatorname{Im}(\gamma)\cap\Sigma_\tau} \int_{\Sigma_0} |a_{\mathcal{N}}|^2 n_{\Sigma_\tau} \phi_1 e^{-2\lambda\phi_2} \operatorname{vol}_{\bar{g}_0} + \mathcal{O}(1) \\ &= \frac{N\phi_1\big|_{\operatorname{Im}(\gamma)\cap\Sigma_\tau}}{N\phi_1\big|_{\operatorname{Im}(\gamma)\cap\Sigma_0}} \cdot E^N_0(u_{\lambda,\mathcal{N}}) + \mathcal{O}(1) \\ &= \frac{g(N,\dot\gamma)\big|_{\operatorname{Im}(\gamma)\cap\Sigma_\tau}}{g(N,\dot\gamma)\big|_{\operatorname{Im}(\gamma)\cap\Sigma_0}} \cdot E^N_0(u_{\lambda,\mathcal{N}}) + \mathcal{O}(1). \end{aligned}$$

Substituting this into the expression for $E^N_\tau(\tilde{u}_{\lambda,\mathcal{N}})$, i.e., the first equation in (4.3), we obtain the second equation of (4.3). This finishes the proof of Theorem 4.1. □

5. Some general theorems about the Gaussian beam limit of the wave equation

We can now make a much more detailed statement about the behaviour of solutions v of the wave equation in the Gaussian beam limit than Theorem 2.1 does:

Theorem 5.1. *Let (M, g) be a time-oriented, globally hyperbolic Lorentzian manifold with time function t, foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface. Furthermore, let $\gamma : [0, S) \to M$ be an affinely parametrised future-directed null geodesic with $\gamma(0) \in \Sigma_0$, where $0 < S \le \infty$. Finally, let N be a timelike, future-directed vector field.*

For any neighbourhood $\mathcal{N}$ of γ, any $T > 0$ with $\Sigma_T \cap \operatorname{Im}(\gamma) \neq \varnothing$, and any $\mu > 0$, there exists a solution $v \in C^\infty(M, \mathbb{C})$ of the wave equation (1.10) with $E^N_0(v) = -g(N, \dot\gamma)\big|_{\gamma(0)}$ such that

$$\big|E^N_{\tau,\mathcal{N}\cap\Sigma_\tau}(v) - \big[-g(N,\dot\gamma)\big|_{\operatorname{Im}\gamma\cap\Sigma_\tau}\big]\big| < \mu \quad \textit{for all } 0 \le \tau \le T \quad (5.2)$$

[23] In the geometric optics approximation we have, indeed, a proper conservation law, which is interpreted in the physics literature as conservation of photon number; see, for example, [Misner et al. 1973, Chapter 22.5.]

and[24]

$$E^N_{\tau,\mathcal{N}^c\cap\Sigma_\tau}(v) < \mu \quad \textit{for all } 0 \le \tau \le T \tag{5.3}$$

provided that we have, on $R_{[0,T]} \cap J^+(\mathcal{N} \cap \Sigma_0)$,

$$\begin{gathered}\frac{1}{|dt(n_{\Sigma_\tau})|} + |g(N, n_{\Sigma_\tau})| \le C < \infty \quad \text{and} \quad 0 < c \le |g(N, N)|,\\ |\nabla N(n_{\Sigma_\tau}, n_{\Sigma_\tau})| + \sum_{i=1}^{3} |\nabla N(n_{\Sigma_\tau}, e_i)| + \sum_{i,j=1}^{3} |\nabla N(e_i, e_j)| \le C < \infty,\end{gathered} \tag{5.4}$$

where c and C are positive constants and $\{n_{\Sigma_\tau}, e_1, e_2, e_3\}$ is an orthonormal frame.

Moreover, by choosing $\mathcal{N}$ a bit smaller, if necessary, (5.2) holds independently of (5.4).

Proof. This follows easily from Theorem 2.1, Theorem 4.1, the second part of Remark 2.9 and the triangle inequality for the square root of the N-energy. □

Let us again remark that the solution v of the wave equation in Theorem 5.1 can also be chosen to be real valued.

The next theorem is a direct consequence of Theorem 5.1 and can be used, in particular, but not only, for proving upper bounds on the rate of the energy decay of waves on globally hyperbolic Lorentzian manifolds if we only allow the initial energy on the right-hand side of the decay statement.

Theorem 5.5. *Let (M, g) be a time-oriented globally hyperbolic Lorentzian manifold with time function t, foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface. Furthermore, let $\mathcal{T}$ be an open subset of M. Assume there is an affinely parametrised future-directed null geodesic $\gamma : [0, S) \to M$ with $\gamma(0) \in \Sigma_0$, where $0 < S \le \infty$, that is completely contained in $\mathcal{T}$. Let*

$$\tau^* := \sup\big\{\hat\tau \in [0, \infty) \;\big|\; \mathrm{Im}(\gamma) \cap \Sigma_\tau \neq \varnothing \text{ for all } 0 \le \tau < \hat\tau\big\}.$$

Moreover, let N be a timelike, future-directed vector field and $P : [0, \tau^*) \to (0, \infty)$ a function.[25]

If there is no constant $C > 0$ such that

$$-g(N, \dot\gamma)\big|_{\mathrm{Im}(\gamma)\cap\Sigma_\tau} \le P(\tau)C$$

holds for all $0 \le \tau < \tau^*$, then there exists no constant $C > 0$ such that

$$E^N_{\tau,\mathcal{T}\cap\Sigma_\tau}(u) \le P(\tau) C E^N_0(u) \tag{5.6}$$

holds for all solutions u of the wave equation (1.10) for $0 \le \tau < \tau^*$.

Proof. Assume the contrary, that is, that there exists a constant $C_0 > 0$ such that (5.6) holds. There is then a $0 \le \tau_0 < \tau^*$ with $-g(N, \dot\gamma)\big|_{\mathrm{Im}(\gamma)\cap\Sigma_{\tau_0}} > -g(N, \dot\gamma)\big|_{\mathrm{Im}(\gamma)\cap\Sigma_0} C_0 P(\tau_0)$. Choosing $\mu > 0$ small enough and a neighbourhood $\mathcal{N} \subseteq \mathcal{T}$ of γ small enough such that (5.2) of Theorem 5.1 applies without reference to (5.4), we obtain a contradiction. □

[24] We denote the complement of $\mathcal{N}$ in M by $\mathcal{N}^c$.

[25] There is no assumption on the regularity of the function P.

A very robust method for proving decay of solutions of the wave equation was given in [Dafermos and Rodnianski 2010b] (but also see [Metcalfe et al. 2012]). This method requires an integrated local energy decay (ILED) statement (possibly with loss of derivative), i.e., a statement of the form (5.8). The next theorem gives a sufficient criterion for an ILED statement having to lose regularity.

Theorem 5.7. *Let (M, g) be a time-oriented, globally hyperbolic Lorentzian manifold with time function t, foliated by the level sets $\Sigma_\tau = \{t = \tau\}$, where Σ_0 is a Cauchy hypersurface. Furthermore, let $\mathcal{T}$ be an open subset of M. Assume there is an affinely parametrised, future-directed null geodesic $\gamma : [0, S) \to M$ with $\gamma(0) \in \Sigma_0$, where $0 < S \leq \infty$, that is completely contained in $\mathcal{T}$. Let N be a timelike, future-directed vector field and set*

$$\tau^* := \sup\bigl\{\hat{\tau} \in [0, \infty) \bigm| \mathrm{Im}(\gamma) \cap \Sigma_\tau \neq \varnothing \text{ for all } 0 \leq \tau < \hat{\tau}\bigr\}.$$

If

$$\int_0^{\tau^*} -g(N, \dot{\gamma})\big|_{\mathrm{Im}(\gamma)\cap\Sigma_\tau}\, d\tau = \infty,$$

where $\dot{\gamma}$ is with respect to some affine parametrisation, then there exists no constant $C > 0$ such that

$$\int_0^{\tau^*}\int_{\Sigma_\tau\cap\mathcal{T}} J^N(u)\cdot n_{\Sigma_\tau}\, \mathrm{vol}_{\bar{g}_\tau}\, d\tau \leq C E_0^N(u) \tag{5.8}$$

holds for all solutions u of the wave equation (1.10).

The proof of this theorem goes along the same lines as the one of Theorem 5.5. The reader might have noticed that whether an ILED statement of the form (5.8) exists or not depends heavily on the choice of the time function. On the other hand, it also depends heavily on the choice of the time function whether an ILED statement is helpful or not. So, for instance, we only have an estimate of the form

$$\int_{\mathcal{T}\cap R_{[0,\tau^*]}} J^N(u)\cdot n_{\Sigma_\tau}\mathrm{vol}_g \leq C\cdot\int_0^{\tau^*}\int_{\Sigma_\tau\cap\mathcal{T}} J^N(u)\cdot n_{\Sigma_\tau}\,\mathrm{vol}_{\bar{g}_\tau}\, d\tau,$$

where $C > 0$, if the time function t is chosen such that $1/|dt(n_{\Sigma_\tau})| \leq C$ is satisfied for all $0 \leq \tau \leq \tau^*$. Such an estimate, together with an ILED statement, is very convenient whenever one needs to control spacetime integrals that are quadratic in the first derivatives of the field.

Part II. Applications to black hole spacetimes

In the following we give a selection of applications of Theorems 5.1, 5.5 and 5.7. A rich variety of behaviours of the energy is provided by black hole spacetimes arising in general relativity.[26] Although we will briefly introduce the Lorentzian manifolds that represent these black hole spacetimes, the reader completely unfamiliar with those is referred to [Hawking and Ellis 1973] for a more detailed discussion, including the concept of a so called Penrose diagram and an introduction to general relativity.

[26] Another physically interesting application would be, for example, to the study of waves in time-dependent inhomogeneous media.

We first restrict our considerations to the 2-parameter family of Reissner–Nordström black holes, which are exact solutions to the Einstein–Maxwell equations. The spherical symmetry of these spacetimes (and the accompanying simplicity of the metric) allows for an easy presentation without hiding any crucial details. In Section 7 we then discuss the Kerr family and show that analogous results hold.

6. Applications to Schwarzschild and Reissner–Nordström black holes

The 2-parameter family of Reissner–Nordström spacetimes is given by

$$g = -\left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{e^2}{r^2}\right)^{-1} dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2, \tag{6.1}$$

initially defined on the manifold $M := \mathbb{R} \times (m + \sqrt{m^2 - e^2}, \infty) \times \mathbb{S}^2$, for which (t, r, θ, φ) are the standard coordinates. We restrict the real parameters m and e, which model the mass and the charge of the black hole, respectively, to the range $0 \le e \le m$, $m \ne 0$.

For $e = 0$ we obtain the 1-parameter Schwarzschild subfamily which solves the vacuum Einstein equations. The manifold M and the metric (6.1) can be analytically extended (so that they still solve the Einstein equations). The so-called Penrose diagram of the maximal analytic extension of the Schwarzschild family is given in Figure 2.

The diamond-shaped region to the right corresponds to the Lorentzian manifold (M, g) we started with; it represents the exterior of the black hole. The triangle to the top corresponds to the interior of the black hole, which is separated from the exterior by the so-called event horizon, the line from the centre to the top-right i^+. The remaining parts of the Penrose diagram play no role in the following discussion.

The black hole stability problem (see the introduction of [Dafermos and Rodnianski 2013]) motivates the study of the wave equation in the exterior of the black hole (the event horizon included). In accordance with our discussion in Section 1B, we consider the framework of the energy method for the study of the wave equation. A suitable notion of energy for the black hole exterior is obtained via (1.9) through the

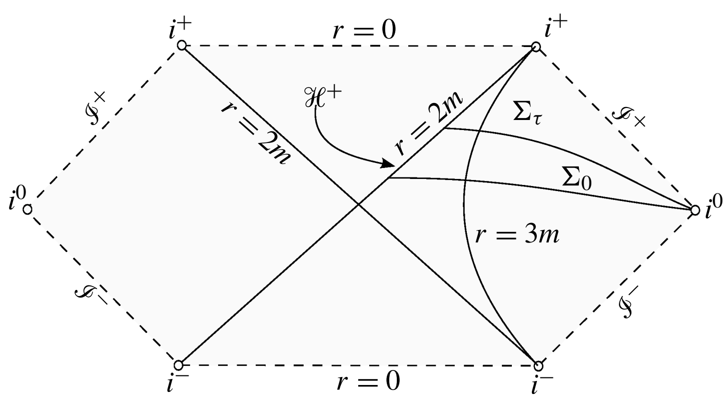

Figure 2. The Penrose diagram of the maximal analytic extension of the Schwarzschild family.

foliation given by $\Sigma_\tau = \{t^* = \tau\}$ for $t^* \geq c > -\infty$, where $t^* = t + 2m\log(r - 2m)$, together with the timelike vector field $N := -(dt^*)^\sharp$.[27]

6A. *Trapping at the photon sphere.* There are null geodesics in the Schwarzschild spacetime that stay forever on the photon sphere at $\{r = 3m\}$. Indeed, one can check that the curve γ given by

$$\gamma(s) = \bigl(s, 3m, \tfrac{1}{2}\pi, (27m^2)^{-1/2}s\bigr)$$

in (t, r, θ, φ) coordinates is an affinely parametrised null geodesic with N-energy given by $-g(N, \dot\gamma) = 1$. We now apply Theorem 5.5: The time-oriented,[28] globally hyperbolic Lorentzian manifold can be taken to be the domain of dependence $\mathcal{D}(\Sigma_0)$ of Σ_0 in (M, g). Moreover, we choose the time function to be given by the restriction of t^* to $\mathcal{D}(\Sigma_0)$, and the vector field N and null geodesic $\gamma(s)$ in Theorem 5.5 are given by N and $\gamma(s - 2m\log m)$ from above. Since $-g(N, \dot\gamma) = 1$ holds, Theorem 5.5 now states that, given any open neighbourhood $\mathcal{T}$ of $\mathrm{Im}(\gamma)$ in $\mathcal{D}(\Sigma_0)$, there is no function $P : [0, \infty) \to (0, \infty)$ with $P(\tau) \to 0$ for $\tau \to \infty$ such that

$$E^N_{\tau, \mathcal{T}\cap\Sigma_\tau}(u) \leq P(\tau) E^N_0(u)$$

holds for all solutions u of the wave equation for all $\tau \geq 0$. It follows that an LED statement for such a region can only hold if it loses differentiability. One can infer the analogous result about ILED statements from Theorem 5.7.

Let us mention here that γ has conjugate points. Indeed, the Jacobi field J with initial data $J(0) = 0$ and $D_s J(0) = \partial_\theta|_{\gamma(0)}$ vanishes in finite affine time $s > 0$: First note that the vector field

$$s \mapsto \partial_\theta\big|_{\gamma(s)}$$

along γ is parallel, i.e., $D_s\partial_\theta\big|_{\gamma(s)} = 0$. Moreover, a direct computation yields

$$R(\partial_\theta, \dot\gamma)\dot\gamma\big|_{\gamma(s)} = \frac{1}{27m^2}\partial_\theta\bigg|_{\gamma(s)},$$

where $R(\cdot\,,\cdot)\,\cdot$ is the Riemann curvature endomorphism. Thus, it follows that the vector field

$$J(s) = (27m^2)^{1/2}\sin((27m^2)^{-1/2}s)\cdot\partial_\theta\big|_{\gamma(s)}$$

---

[27] We are intentionally quite vague about what we mean by "suitable notion of energy". Instead of considering a foliation that ends at spacelike infinity ι^0, it is sometimes desirable to work with a foliation that ends at future null infinity $\mathcal{I}^+$. In a stationary spacetime, however, it is always convenient (and indeed "suitable") to work with a foliation and an energy-measuring vector field N both of which are invariant under the flow of the Killing vector field. The obvious advantage is that the constants in Sobolev embeddings do not depend on the leaf — provided, of course, that higher-energy norms are also defined accordingly. The precise choice of the timelike vector field N in a *compact* region of one leaf is completely irrelevant, since all the energy norms are equivalent in a compact region. In particular, one can deduce that the following result about trapping at the photon sphere in Schwarzschild remains unchanged if we choose a different timelike vector field N which commutes with ∂_t and a different foliation by spacelike slices. In fact, note that the behaviour of the energy of the null geodesic, $-g(N, \dot\gamma)$, does not depend at all on the choice of the foliation!

[28] The time orientation is given by the timelike vector field N.

satisfies the Jacobi equation $D_t^2 J + R(J, \dot{\gamma})\dot{\gamma} = 0$. Moreover, it clearly satisfies the above initial conditions and vanishes in finite affine time.

It now follows from Theorem A.1 that one cannot construct localised solutions to the wave equation along the trapped null geodesic γ using the naive geometric optics approximation alone. Indeed, one would need to bridge these caustics using Maslov's canonical operator.

That one can indeed prove an (I)LED statement with a loss of derivative was shown in [Dafermos and Rodnianski 2009] (see also [Blue and Sterbenz 2006]). In fact, it is sufficient to lose only an ε of a derivative; see [Blue and Soffer 2009] and also [Dafermos and Rodnianski 2013]. For a numerical study of the behaviour of a wave trapped at the photon sphere we refer the interested reader to [Zenginoglu and Galley 2012].

Other, similar, examples are trapping at the photon sphere in higher-dimensional Schwarzschild [Schlue 2013] or in Reissner–Nordström [Aretakis 2011a; Blue and Soffer 2009].

6B. ***The red-shift effect at the event horizon — and its relevance for scattering constructions from the future.*** Another kind of behaviour of the energy is exhibited by the trapping occuring at the event horizon of the Schwarzschild spacetime. Recall that the event horizon $\mathcal{H}^+$ at $\{r = 2m\}$ is a null hypersurface, spanned by null geodesics. In $(t^*, r, \theta, \varphi)$ coordinates the affinely parametrised generators are given by

$$\gamma(s) = \left(\frac{1}{\kappa}\log s, 2m, \theta_0, \varphi_0\right),$$

where $\kappa = 1/(4m)$ is the surface gravity, $s \in (0, \infty)$ and θ_0, φ_0 are constants. Thus, we have

$$-(\dot{\gamma}(s), N) = \frac{1}{\kappa s} = \frac{1}{\kappa}e^{-\kappa t^*}, \tag{6.2}$$

i.e., the energy of the corresponding Gaussian beam decays exponentially — a direct manifestation of the celebrated red-shift effect. For more on the impact of the red-shift effect on the study of the wave equation on Schwarzschild we refer the reader to the original paper by Dafermos and Rodnianski [2009], but also see [Dafermos and Rodnianski 2013].

Let us emphasise again that the null geodesics at the photon sphere as well as those at the horizon are trapped, in the sense that they never escape to null infinity — but only those at the photon sphere form an obstruction for an LED statement without loss of differentiability; the "trapped" energy at the horizon decays exponentially. This is in stark contrast to the obstacle problem, where every trapped light ray automatically leads to an obstruction for an LED statement without loss of derivatives (see [Ralston 1969]). This new variety of how the "trapped" energy behaves is due to the lack of a global timelike Killing vector field.

Let us now investigate the role played by the red-shift effect in scattering constructions from the future. While the red-shift effect is conducive to proving bounds on solutions to the wave equation in the "forward problem", it turns into a blue-shift in the "backwards problem" (see Figure 3);[29] it amplifies energy near the horizon.

[29] We call the initial value problem on Σ_0 to the future the "forward problem", while solving a mixed characteristic initial value problem on $\mathcal{H}^+(\tau) \cup \Sigma_\tau$ to the past (or indeed a scattering construction from the future with data on $\mathcal{H}^+$ and $\mathcal{I}^+$) is called the "backwards problem". Here, we have denoted the (closed) portion of the event horizon $\mathcal{H}^+$ that is cut out by Σ_0 and Σ_τ by $\mathcal{H}^+(\tau)$.

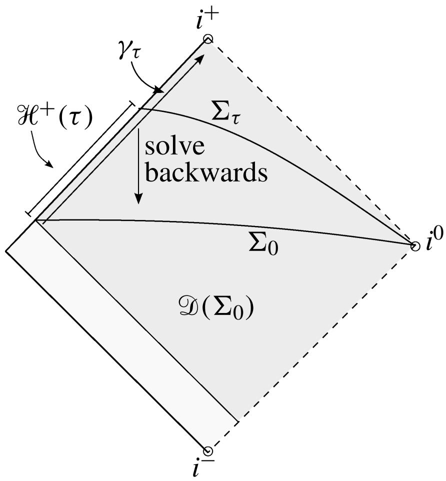

Figure 3. Illustration of the backwards problem.

Proposition 6.3. *For every $\mu > 0$ and every $\tau > 0$ there exists a smooth solution*[30] $v \in C^\infty(\overline{\mathfrak{D}(\Sigma_0)}, \mathbb{C})$ *to the wave equation* (1.10) *with* $E^N_\tau(v) = 1$ *and* $\int_{\mathcal{H}^+(\tau)} J^N(v) \lrcorner \operatorname{vol}_g < \mu$, *which satisfies* $E^N_0(v) \geq e^{\kappa\tau} - \mu$, *where* $\kappa = 1/(4m)$ *is the surface gravity of the Schwarzschild black hole.*

Here, $J^N(v) \lrcorner \operatorname{vol}_g$ denotes the 3-form obtained by inserting the vector field $J^N(v)$ into the first slot of vol_g. Let us also remark that μ should be thought of as a small positive number, while τ rather as a big one.

Proof. As in Section 6A, we consider the Lorentzian manifold $\mathfrak{D}(\Sigma_0)$ with time function t^* and timelike vector field N. Since geodesics depend smoothly on their initial data, it follows from (6.2) that we can find, for every $\tau > 0$, an affinely parametrised, radially outgoing null geodesic[31] γ_τ in $\mathfrak{D}(\Sigma_0)$ with $\big|-(\dot\gamma_\tau, N)|_{\operatorname{Im}(\gamma_\tau)\cap\Sigma_0} - e^{\kappa\tau}\big| < \mu/2$ and $-(\dot\gamma_\tau, N)|_{\operatorname{Im}(\gamma_\tau)\cap\Sigma_\tau} = 1$. We note that, for our choice of time function and vector field N, the condition (2.3) is satisfied, which does not only give us the energy estimate (2.8) but here also the refined version

$$\int_{\mathcal{H}^+(\tau)} J^N(u) \lrcorner \operatorname{vol}_g + E^N_\tau(u) \leq C(\tau)\big(E^N_0(u) + \|\Box u\|^2_{L^2(R_{[0,T]})}\big), \tag{6.4}$$

which holds in $\overline{\mathfrak{D}(\Sigma_0)}$ for all $\tau > 0$ and all $u \in C^\infty(\overline{\mathfrak{D}(\Sigma_0)}, \mathbb{R})$. The estimate (6.4) is derived in the same way as (2.8), namely by an application of Stokes' theorem to $J^N(u) \lrcorner \operatorname{vol}_g$, followed by Gronwall's inequality. The estimate (6.4) gives, in addition to (2.2) in Theorem 2.1, the estimate

$$\int_{\mathcal{H}^+(\tau)} J^N(v - \tilde u) \lrcorner \operatorname{vol}_g < \mu, \tag{6.5}$$

where $\tilde u$ is the Gaussian beam and v is the actual solution, as in Theorem 2.1. We now apply Theorem 5.1, where the Lorentzian manifold is given by $\mathfrak{D}(\Sigma_0)$, the time function by t^*, the timelike vector field

[30] We denote by $\overline{\mathfrak{D}(\Sigma_0)}$ the closure of $\mathfrak{D}(\Sigma_0)$ in the maximal analytic extension of Schwarzschild; see Figure 2 on page 1399.

[31] Radially outgoing null geodesics are the lines parallel to, and to the right of, $\mathcal{H}^+$ in the Penrose diagram. In (u, r, θ, φ) coordinates, where $u(t, r, \theta, \varphi) := t - 2m\log(r - 2m) - r$, these null geodesics are tangent to $\partial/\partial r$.

by N and, for given $\tau > 0$, the affinely parametrised null geodesic is taken to be γ_τ from above. For our purposes we can choose any neighbourhood $\mathcal{N}$ of $\mathrm{Im}(\gamma_\tau)$ in $\mathcal{D}(\Sigma_0)$. Theorem 5.1 then ensures the existence of a solution $v \in C^\infty(\mathcal{D}(\Sigma_0), \mathbb{C})$ to the wave equation with $E_0^N(v) \geq e^{\kappa\tau} - \mu$ and $E_\tau^N(v) = 1$ — possibly after renormalising the energy at time τ of v to be exactly 1. It is not difficult to show, for example by considering the Cauchy problem for a slightly larger globally hyperbolic Lorentzian manifold which contains the event horizon, that v can be chosen to extend smoothly to the event horizon. We then obtain $\int_{\mathcal{H}^+(\tau)} J^N(v) \lrcorner \mathrm{vol}_g < \mu$ from (6.5), since we recall that the Gaussian beam $\tilde{u}$ in Theorem 2.1 is supported in $\mathcal{N}$, which is disjoint from $\mathcal{H}^+$. This finishes the proof. □

The above proposition shows that for every $\tau > 0$ one can prescribe initial data for the mixed characteristic initial value problem on $\mathcal{H}^+ \cup \Sigma_\tau$ so that the total initial energy is equal to one, while the energy of the solution obtained by solving backwards grows exponentially to $\approx e^{\kappa\tau}$ on Σ_0. Dafermos, Holzegel and Rodnianski [Dafermos et al. 2013] approach the scattering problem from the future for the Einstein equations (with initial data prescribed on $\mathcal{H}^+$ and $\mathcal{I}^+$) by considering it as the limit of finite backwards problems, which — for the wave equation — are qualitatively the same as the backwards problem with initial data on $H^+(\tau)$ and Σ_τ. In order to take the limit of the finite problems, uniform control over the solutions is required: Dafermos et al. use a backwards energy estimate which bounds the energy on Σ_0 by the initial energy on $\mathcal{H}^+$ and Σ_τ, multiplied by $C \cdot \exp(c\tau)$, where c and C are constants that are independent of τ. Proposition 6.3 shows now that this estimate is sharp, in the sense that one cannot avoid exponential growth (at least not as long as one does not sacrifice regularity in the estimate). In particular, working with this estimate enforces the assumption of exponential decay on the scattering data in [Dafermos et al. 2013].

6C. ***The blue-shift near the Cauchy horizon of a subextremal Reissner–Nordström black hole.*** We now move on to the subextremal Reissner–Nordström black hole, i.e., to the parameter range $0 < e < m$ in (6.1). More precisely, we consider again its maximal analytic extension. Part of the Penrose diagram is given in Figure 4.

Again, the diamond-shaped region I represents the black hole exterior and corresponds to the Lorentzian manifold on which the metric g from (6.1) was initially defined. The regions II, III and IV represent the black hole interior. Recall that Reissner–Nordström is a spherically symmetric spacetime. The “radius” of the spheres of symmetry is given by a globally defined function r. We write $D(r) := 1 - 2m/r + e^2/r^2$ and denote the two roots of D by $r_\pm = m \pm \sqrt{m^2 - e^2}$. The future Cauchy horizon[32] is given by $r = r_-$. The coordinate functions (θ, φ) parametrise the spheres of symmetry in the usual way and are globally defined up to one meridian. Regions I–III are covered by a (v, r, θ, φ) coordinate chart; in the region I, the function v is given by $v = t + r_I^*$, where r_I^* is a function of r satisfying $dr_I^*/dr = 1/D$. With respect to these coordinates, the Lorentzian metric takes the form

$$g = -D\, dv^2 + dv \otimes dr + dr \otimes dv + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2.$$

[32] We consider a Cauchy surface Σ_0 of the big diamond-shaped region as shown in Figure 5, i.e., a Cauchy surface of the region pictured in Figure 4 without the regions III and IV.

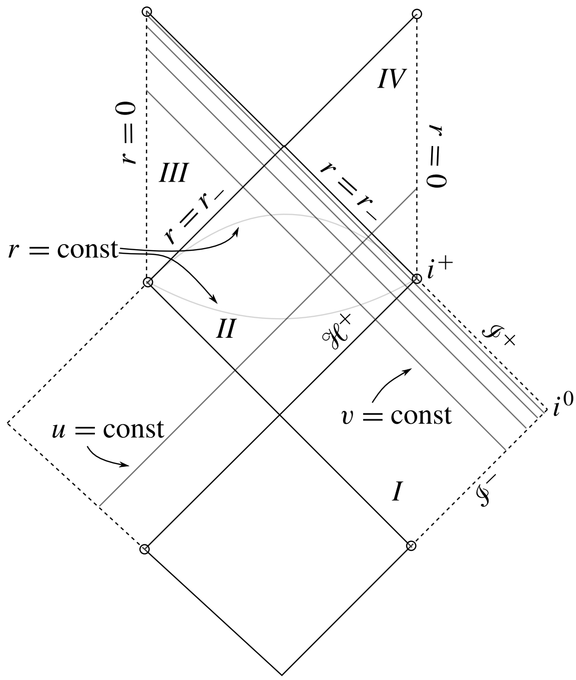

Figure 4. Part of the Penrose diagram for the subextremal Reissner–Nordström black hole.

Introducing a function r^*_{II} in region II, which satisfies $dr^*_{II}/dr = 1/D$ in this region, and defining a function[33] $t := v - r^*_{II}$, we obtain a (t, r, θ, φ) coordinate system for region II in which the metric g is again given by the algebraic expression (6.1). The regions II and IV are covered by a coordinate system (u, r, θ, φ), where the function u is given in region II by $u = t - r^*_{II}$.

Having laid down the coordinate functions we work with, we now investigate the family of affinely parametrised ingoing null geodesics, given in (v, r, θ, φ) coordinates by

$$\gamma_{v_0}(s) = (v_0, -s, \theta_0, \varphi_0),$$

where $s \in (-\infty, 0)$ and we keep θ_0, φ_0 fixed. Clearly, we have[34] $\dot\gamma_{v_0} = -\partial/\partial r\big|_v$. We are interested in the energy of these null geodesics in region II close to i^+ (in the topology of the Penrose diagram), i.e., close to the Cauchy horizon separating region II from region IV. A suitable notion of energy is given by a regular vector field that is future-directed timelike in a neighbourhood of i^+. In order to construct such a vector field, we consider (u, v, θ, φ) coordinates in region II. A straightforward computation shows that

$$N := -\frac{1}{r_+ - r}\frac{\partial}{\partial u}\bigg|_v + \frac{1}{r - r_-}\frac{\partial}{\partial v}\bigg|_u = -\frac{1}{r_+ - r}\frac{\partial}{\partial u}\bigg|_r - \frac{1}{2r^2}(r_+ - r_-)\frac{\partial}{\partial r}\bigg|_u = \frac{r_- - r_+}{2r^2}\frac{\partial}{\partial r}\bigg|_v + \frac{1}{r - r_-}\frac{\partial}{\partial v}\bigg|_r$$

[33] One could also assign the functions t an index, specifying in which region they are defined. Note that these different functions t do *not* patch together to give a globally defined smooth function!

[34] Let us denote with a subscript on the partial derivative which other coordinate (apart from θ and φ) remains fixed.

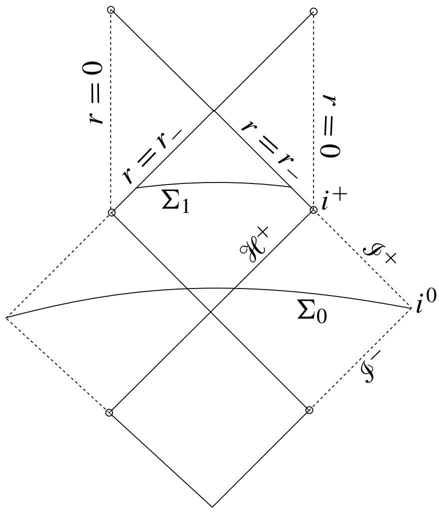

Figure 5. The spacelike slices Σ_0 and Σ_1 of Figure 4.

is future-directed timelike in a neighbourhood of i^+ intersected with region II and can be extended to a smooth timelike vector field defined on a neighbourhood of i^+. We obtain

$$-(N, \dot\gamma_{v_0}) = \frac{1}{r-r_-}; \tag{6.6}$$

the N-energy of the null geodesics γ_{v_0} gets infinitely blue-shifted near the Cauchy horizon.

For later reference let us note that the rate with which the N-energy (6.6) of γ_{v_0} blows up along a hypersurface of constant u, in advanced time v, is exponential. This is seen as follows: One has

$$r^*_{II}(r) = r + \frac{1}{2\kappa_+}\log(r_+ - r) + \frac{1}{2\kappa_-}\log(r - r_-) + \text{const},$$

where $\kappa_\pm = (r_\pm - r_\mp)/(2r_\pm^2)$ are the surface gravities of the event and the Cauchy horizon, respectively. Thus, for large r^*_{II}, one has $(1/(r-r_-))(r^*_{II}) \sim e^{-2\kappa_- r^*_{II}}$. Finally, along $\{u = u_0 = \text{const}\}$, we have $r^*_{II}(v) = \frac{1}{2}(v - u_0)$. It thus follows that the N-energy (6.6) of γ_{v_0} blows up like $e^{-\kappa_- v}$ along a hypersurface of constant u.

Let us now consider spacelike slices Σ_0 and Σ_1 as in Figure 5, where Σ_0 asymptotes to a hypersurface of constant t and Σ_1 is extendible as a smooth spacelike slice into the neighbouring regions.

Since the normal n_{Σ_1} of Σ_1 is also regular at the Cauchy horizon, it follows from (6.6) that the n_{Σ_1}-energy of the null geodesics γ_{v_0} blows up along Σ_1 when approaching the Cauchy horizon. Moreover, note that the n_{Σ_0}-energy of the geodesics γ_{v_0} along Σ_0 is uniformly bounded as $v_0 \to \infty$. We now apply Theorem 5.1 to the family of null geodesics γ_{v_0} with the following further input: the Lorentzian manifold is given by the domain of dependence $\mathscr{D}(\Sigma_0)$ of Σ_0, the time function is such that Σ_0 and Σ_1 are level sets, N is a timelike vector field that extends n_{Σ_0} and n_{Σ_1}, and finally $\mathscr{N}$ is a small enough neighbourhood of γ_{v_0}. This yields:

Theorem 6.7. *Let Σ_0 and Σ_1 be spacelike slices in the subextremal Reissner–Nordström spacetime as indicated in Figure 6. Then there exists a sequence $\{u_i\}_{i\in\mathbb{N}}$ of solutions to the wave equation with initial energy $E_0^{n_{\Sigma_0}}(u_i)=1$ on Σ_0 such that the n_{Σ_1}-energy on Σ_1 goes to infinity, i.e., $E_1^{n_{\Sigma_1}}(u_i)\to\infty$ as $i\to\infty$.*

We can infer from Theorem 6.7 that there is no uniform energy boundedness statement — that is, there is no constant $C>0$ such that

$$\int_{\Sigma_1} J^{n_{\Sigma_1}}(u)\cdot n_{\Sigma_1} \le C\int_{\Sigma_0} J^{n_{\Sigma_0}}(u)\cdot n_{\Sigma_0} \tag{6.8}$$

holds for all solutions u of the wave equation.

Let us remark here that the nonexistence of a uniform energy boundedness statement has, in particular, the following consequence: one cannot choose a time function for the region bounded by Σ_0 and Σ_1 for which these hypersurfaces are level sets and, moreover, extend the normals of Σ_0 and Σ_1 to a smooth timelike vector field N *in such a way that* an energy estimate of the form (2.8) holds. This emphasises the importance of the condition (2.3) for the *global* approximation scheme on general Lorentzian manifolds and points out the necessity of a *local* understanding of the approximate solution provided by Theorems 4.1 and 5.1.

One actually expects that there is no energy boundedness statement at all, no matter how many derivatives one loses or whether one restricts the support of the initial data:

Conjecture 6.9. *For generic compactly supported smooth initial data on Σ_0, the n_{Σ_1}-energy along Σ_1 of the corresponding solution to the wave equation is infinite.*

Let us remark here that the analysis carried out in [Dafermos 2005] shows in particular that proving the above conjecture can be reduced to proving a lower bound on the decay rate of the spherical mean of the generic solution (as in Conjecture 6.9) on the horizon.

Before we elaborate in Section 6E on the mechanism that leads to the blow-up of the energy near the Cauchy horizon in Theorem 6.7, let us investigate the situation for *extremal* Reissner–Nordström black holes.

6D. ***The blue-shift near the Cauchy horizon of an extremal Reissner–Nordström black hole.*** The extremal Reissner–Nordström black hole is given by the choice $m=e$ of the parameters in (6.1). We again consider the maximal analytic extension of the initially defined spacetime. Part of the Penrose diagram is given in Figure 6.

The region I represents again the black hole exterior and corresponds to the Lorentzian manifold on which the metric g from (6.1) was initially defined. The black hole interior extends over the regions II and III. The discussion of the functions r, θ and φ carries over from the subextremal case. However, in the extremal case, $D(r)$ has a double zero at $r=m$, the value of the radius of the spheres of symmetry on the event, as well as on the Cauchy horizon. The regions I and II can be covered by "ingoing" null coordinates (v,r,θ,φ), where the function v is given in region I by $v=t+r_I^*$, where again $r_I^*(r)$ satisfies $dr_I^*/dr=1/D$. In the same way as in the subextremal case, one introduces r_{II}^* and defines a (t,r,θ,φ) coordinate system for the region II. Finally, the regions II and III are covered by "outgoing" null coordinates (u,r,θ,φ), where we have $u=t-r_{II}^*$ in region II.

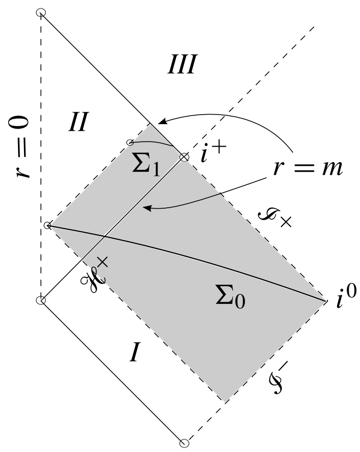

Figure 6. Part of the Penrose diagram for the extremal Reissner–Nordström black hole.

In ingoing null coordinates, the affinely parametrised, radially ingoing null geodesics are given by $\gamma_{v_0}(s) = (v_0, -s, \theta_0, \varphi_0)$, where $s \in (-\infty, 0)$. Expressing the tangent vector of γ_{v_0} in region *II* in outgoing coordinates, we obtain

$$\dot{\gamma}_{v_0} = -\frac{\partial}{\partial r}\bigg|_v = \frac{2}{D}\frac{\partial}{\partial u}\bigg|_r - \frac{\partial}{\partial r}\bigg|_u, \tag{6.10}$$

which blows up at $r = m$. Thus, we have, for any future-directed timelike vector field N in region *II* which extends to a regular timelike vector field in region *III*, that the N-energy $-g(\dot{\gamma}_{v_0}, N)$ of γ_{v_0} blows up along the hypersurface Σ_1 for $v_0 \to \infty$. Choosing now a spacelike slice Σ_0 as in the above diagram, again asymptoting to a $\{t = \text{const}\}$ hypersurface at i^0, and restricting consideration to its domain of dependence, we obtain a globally hyperbolic spacetime (the shaded region) with respect to which we can apply Theorem 5.1, inferring the analogue of Theorem 6.7 for extremal Reissner–Nordström black holes.

For the discussion in the next section, we again investigate the rate, in advanced time v, with which the N-energy $-g(\dot{\gamma}_{v_0}, N)$ blows up along a hypersurface of constant u; here, we have

$$r^*_{II}(r) = r + m\log((r-m)^2) - \frac{m^2}{(r-m)} + \text{const}.$$

It follows that for large r^*_{II} one has $(1/D)(r^*_{II}) \sim (r^*_{II})^2$. Moreover, along $\{u = u_0 = \text{const}\}$, we have $r^*_{II}(v) = \frac{1}{2}(v - u_0)$, from which it follows that the N-energy $-g(\dot{\gamma}_{v_0}, N)$ of the family of null geodesics γ_{v_0} blows up like v^2.

6E. ***The strong and the weak blue-shift — and their relevance for strong cosmic censorship.*** In the example of subextremal Reissner–Nordström as well as in the example of extremal Reissner–Nordström, the energy of the Gaussian beams is blue-shifted near the Cauchy horizon. Although not important for the proof of the *qualitative* result of Theorem 6.7 (and the analogous statement for the extremal case), the difference in the *quantitative* blow-up rate of the energy in the two cases is conspicuous.

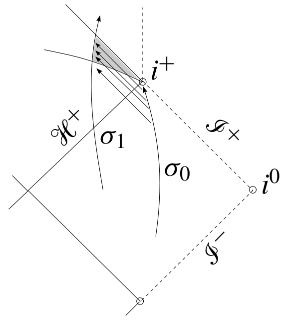

Figure 7. Illustration of the blue-shift effect in subextremal Reissner–Nordström.

Let us first recall the familiar heuristic picture that explains the basic mechanism responsible for the blue-shift effect in both cases;[35] see Figure 7. The observer σ_0 travels along a timelike curve of infinite proper time to i^+ and, in regular time intervals, sends signals of the same energy into the black hole. These signals are received by the observer σ_1, who travels into the black hole and crosses the Cauchy horizon, within *finite* proper time — which leads to an infinite blue shift. This mechanism was first pointed out by Roger Penrose [1968, page 222].[36] Although the picture, along with its heuristics, allow for inferring the *presence* of a blue-shift near the Cauchy horizon, they do not reveal the *strength* of the blue-shift. For investigating the latter, it is important to note that the region in spacetime which actually causes the blue shift is a neighbourhood of the Cauchy horizon. This neighbourhood is not well defined, however, one could think of it as being given by a neighbourhood of constant r — the shaded region in the diagram of subextremal Reissner–Nordström in Figure 7. The crucial difference between the subextremal and the extremal case is that, in the extremal case, the blue-shift degenerates *at the Cauchy horizon itself*, while, in the subextremal case, it does not: the subextremal Cauchy horizon continues to blue-shift radiation. In particular, one can prove an analogous result to Proposition 6.3 there — but for the forward problem.

This degeneration of the blue-shift towards the Cauchy horizon in the extremal case leads to the (total) blue-shift being weaker than the blue-shift in the subextremal case. Thus, the geometry of spacetime near the Cauchy horizon is crucial for understanding the strength of the blue-shift effect, and hence the blow-up rate of the energy.

We now continue with a heuristic discussion of the importance of the different blow-up rates. The reader might have noticed that we only made Conjecture 6.9 for the subextremal case; and indeed, the analogous conjecture for the extremal case is expected to be false: While in our construction we consider

[35] In Figure 7 we give the picture for the subextremal case. However, the picture and the heuristics for the extremal case are exactly the same!

[36] There, he describes the above scenario in the following, more dramatic language (he considers the scenario of gravitational collapse, where the Einstein equations are coupled to some matter model and denotes the Cauchy horizon with $H_+(\mathscr{H})$):

> There is a further difficulty confronting our observer who tries to cross $H_+(\mathscr{H})$. As he looks out at the universe that he is "leaving behind", he sees, in one final flash, as he crosses $H_+(\mathscr{H})$, the entire later history of the rest of his "old universe". [...] If, for example, an unlimited amount of matter eventually falls into the star then presumably he will be confronted with an infinite density of matter along "$H_+(\mathscr{H})$". Even if only a finite amount of matter falls in, it may not be possible, in generic situations to avoid a curvature singularity in place of $H_+(\mathscr{H})$.

a family of ingoing wave packets whose energy along a fixed outgoing null ray to $\mathscr{I}^+$ does *not* decay in advanced time v, the scattered "ingoing energy" of a wave with initial data as in Conjecture 6.9 will decay in advanced time v along such an outgoing null ray. Thus, the blow-up of the energy near the Cauchy horizon can be counteracted by the decay of the energy of the wave towards null infinity. In the extremal case, the blow-up rate is v^2, which does not dominate the decay rate of the energy towards null infinity; the exponential blow-up rate $e^{-\kappa_- v}$, however, does. These are the heuristic reasons for only formulating Conjecture 6.9 for the subextremal case. We conclude with a couple of remarks: Firstly, one should actually compare the decay rate of the ingoing energy not along an outgoing null ray to $\mathscr{I}^+$, but along the event horizon — or even better, along a spacelike slice in the interior of the black hole approaching i^+ in the topology of the Penrose diagram. Secondly, we would like to repeat and stress the point made, namely that the heuristics given in the very beginning of this section, which solely ensure the *presence* of a blue-shift, are not sufficient to cause a C^1 instability of the wave at the Cauchy horizon. For this to happen, the local geometry of the Cauchy horizon is crucial. Finally, let us conjecture, based on the fact that in the extremal case one gains powers of v in the blow-up rate at the Cauchy horizon when considering higher-order energies, that there is some natural number $k > 1$ such that waves with initial data as in Conjecture 6.9 exhibit a C^k instability at the Cauchy horizon.

We conclude this section by recalling that the study of the wave equation on black hole backgrounds serves as a source of intuition for the behaviour of gravitational perturbations of these spacetimes. Thus, the following expected picture emerges: Consider a *generic* dynamical spacetime which at late times approaches a subextremal Reissner–Nordström black hole. Then the Cauchy horizon is replaced by a weak null curvature singularity (for this notion see [Dafermos 2005]).

If we restrict consideration to the class of dynamical spacetimes which at late times approach an extremal Reissner–Nordström black hole, then the *generic* spacetime *within this class* has a more regular Cauchy horizon, which in particular is not seen as a singularity from the point of view of the low regularity well-posedness theory for the Einstein equations; see the resolution [Klainerman et al. 2013] of the L^2-curvature conjecture. This picture is also supported by the recent numerical work [Murata et al. 2013].

6F. ***Trapping at the horizon of an extremal Reissner–Nordström black hole.*** We again consider the extremal Reissner–Nordström black hole. With v defined as in Section 6D, we introduce the function $t^* := v - r$. In the coordinates $(t^*, r, \theta, \varphi)$ the metric takes the form

$$g = -D(dt^*)^2 + (1-D)(dt^* \otimes dr + dr \otimes dt^*) + (2-D)\, dr^2 + r^2\, d\theta^2 + r^2 \sin^2\theta\, d\varphi^2.$$

We see that the foliation of the exterior given by $\Sigma_\tau = \{t^* = \tau\}$ is a foliation by spacelike slices, which is invariant under the flow of the stationary Killing vector field ∂_{t^*} and is regular at the event horizon $\mathscr{H}^+$ in the sense that it extends smoothly as a spacelike foliation across the event horizon; see Figure 8.

An appropriate choice of timelike vector field for measuring the energy of waves in the black hole exterior is thus given by $N = -(dt^*)^\sharp$, since it is also invariant under the flow of the Killing vector field ∂_{t^*} and extends smoothly as a timelike vector field across the event horizon. Hence, the corresponding N-energy is nondegenerate at the event horizon. These choices of foliation and timelike vector field N correspond qualitatively to the choices made in the Schwarzschild spacetime in Sections 6A and 6B.

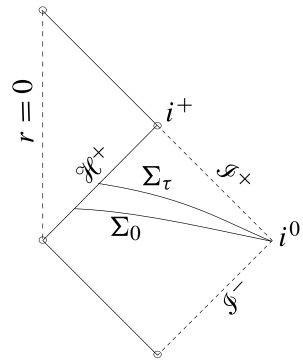

Figure 8. The extremal Reissner–Nordström black hole.

Aretakis [2011a; 2011b] investigated the behaviour of waves on this spacetime and obtained stability (i.e., boundedness and decay results) as well as instability results (blow-up of certain higher-order derivatives along the horizon); for further developments see also [Lucietti and Reall 2012]. The instability results originate from a conservation law on the extremal horizon once decay results for the wave are established. In order to obtain these stability results, Aretakis followed the new method introduced by Dafermos and Rodnianski [2010b].[37] The first important step is to prove an ILED statement. As in the Schwarzschild spacetime we have trapping at the photon sphere (here at $\{r = 2m\}$), and as shown before, an ILED statement has to degenerate there in order to hold. The fundamentally new difficulty in the extremal setting arises from the degeneration of the red-shift effect at the horizon $\mathcal{H}^+$, which was needed for proving an ILED statement that holds up to the horizon (see for example [Dafermos and Rodnianski 2013]). And indeed, the energy of the generators of the horizon is no longer decaying: In $(t^*, r, \theta, \varphi)$ coordinates, the affinely parametrised generators are given by

$$\gamma(s) = (s, m, \theta_0, \varphi_0),$$

where $s \in (-\infty, \infty)$ and again θ_0, φ_0 are fixed. Hence, we see that the N-energy of the generators of the horizon is constant: $-(N, \dot{\gamma}) = 1$.

If we consider a globally hyperbolic subset of the depicted part of extremal Reissner–Nordström that contains the horizon $\mathcal{H}^+$, for example by extending Σ_0 a bit through the event horizon and then considering its domain of dependence, we can directly infer from Theorems 5.5 and 5.7, by applying them to the null geodesic γ from above, that every (I)LED statement which concerns a neighbourhood of the horizon necessarily has to lose differentiability. However, we can also infer the same result for the wave equation on the Lorentzian manifold $\mathcal{D}(\Sigma_0)$, where "a neighbourhood of the horizon" is "a neighbourhood of the horizon in the previous, bigger spacetime, intersected with $\mathcal{D}(\Sigma_0)$": Analogous to the proof of Proposition 6.3, we consider a sequence of radially outgoing null geodesics in $\mathcal{D}(\Sigma_0)$ whose initial data on Σ_0 converges to the data of γ from above. For every "neighbourhood of the horizon", every $\tau_0 > 0$ and every (small) $\mu > 0$, there is then an element γ_0 of the sequence such that

[37] Though in addition he had to work with a degenerate energy, which makes things more complicated.

$-(N, \dot{\gamma}_0)|_{\mathrm{Im}(\gamma_0)\cap\Sigma_\tau} \in (1-\mu, 1+\mu)$ for all $0 \le \tau \le \tau_0$. This follows again from the smooth dependence of geodesics on their initial data. We now apply Theorem 5.1 to this sequence of null geodesics to infer that, for every "neighbourhood of the horizon" and for every $\tau_0 > 0$, we can construct a solution to the wave equation whose energy in this neighbourhood is, say, bigger than $\frac{1}{2}$ for all times τ with $0 \le \tau \le \tau_0$. This proves again that there is no nondegenerate (I)LED statement concerning "a neighbourhood of the horizon" in $\mathscr{D}(\Sigma_0)$; the trapping at the event horizon obstructs local energy decay — which is in stark contrast to subextremal black holes.

One should ask now whether an ILED statement with loss of derivative can actually hold. To answer this question, at least partially, it is helpful to decompose the angular part of the wave into spherical harmonics. Aretakis [2011a] proved indeed an (I)LED statement with loss of one derivative for waves that are supported on the angular frequencies $l \ge 1$. By constructing a localised solution with vanishing spherical mean we can show that this result is optimal in the sense that *some* loss of derivative is again necessary. This can be done for instance by considering the superposition of two Gaussian beams that follow the generators $\gamma_1(s) = \big(s, m, \frac{1}{2}\pi, \frac{1}{2}\pi\big)$ and $\gamma_2(s) = \big(s, m, \frac{1}{2}\pi, \frac{3}{2}\pi\big)$, where the initial value of beam one is exactly the negative of the initial value of beam two if translated in the φ variable by π.[38] The question of whether one can prove an ILED statement with loss of derivative in the case $l = 0$ is still open, though it is expected that the answer is negative. In order to obtain stability results for waves supported on all angular frequencies, Aretakis had to use the degenerate energy (of course these results are weaker than results one would obtain if an ILED statement for the case $l = 0$ actually held).

7. Applications to Kerr black holes

The Kerr family is a 2-parameter family of solutions to the vacuum Einstein equations. Let us fix the manifold $M := \mathbb{R} \times (m + \sqrt{m^2 - a^2}, \infty) \times \mathbb{S}^2$, where m and a are real parameters that will model the mass and the angular momentum per unit mass of the black hole, respectively, and which are restricted to the range $0 \le a \le m$, $0 \ne m$. Let (t, r, θ, φ) denote the standard coordinates on the manifold M and define functions

$$\rho^2 := r^2 + a^2\cos^2\theta, \qquad g_{tt} := -1 + \frac{2mr}{\rho^2},$$

$$\Delta := r^2 - 2mr + a^2, \qquad g_{t\varphi} := -\frac{2mra\sin^2\theta}{\rho^2},$$

$$g_{\varphi\varphi} := \left(r^2 + a^2 + \frac{2mra^2\sin^2\theta}{\rho^2}\right)\sin^2\theta.$$

[38] Let us mention here that in this particular situation the approximation using geometric optics is easier. Indeed, one can easily write down a solution of the eikonal equation such that the characteristics are the outgoing null geodesics. First one has to prove then the analogue of Theorem 4.1, which is easier since the approximate conservation law we used in the case of Gaussian beams is replaced by an exact conservation law for the geometric optics approximation; see footnote 23 on page 1396. But then we can easily contradict the validity of (I)LED statements for any angular frequency: working in $(t^*, r, \theta, \varphi)$ coordinates, we choose the initial value of the function a in the transport equation (i.e., the second equation in (1.6)) to have the angular dependence of a certain spherical harmonic and the radial dependence corresponds to a smooth cut-off, i.e., a initially is only nonvanishing for $r \in [m, m + \varepsilon)$.

The metric on M is then defined by

$$g = g_{tt}\,dt^2 - g_{t\varphi}(d\varphi \otimes dt + dt \otimes d\varphi) + g_{\varphi\varphi}\,d\varphi^2 + \frac{\rho^2}{\Delta}\,dr^2 + \rho^2\,d\theta^2.$$

The roots of $\Delta(r)$ are denoted by r_- and r_+, where $r_\pm = m \pm \sqrt{m^2 - a^2}$. As for the Reissner–Nordström family, one can (and should) extend these spacetimes in order to understand their physical interpretation as a black hole. For details, we refer the reader again to [Hawking and Ellis 1973]. Fixing the θ coordinate to be $\frac{1}{2}\pi$ and modding out the $\mathbb{S}^1$ corresponding to the φ coordinate, we again obtain pictorial representations of these spacetimes. For the subextremal case $0 < a < m$, the diagram is the same as the one depicted in Section 6C, while, in the extremal case $a = m$, one obtains the same diagram as in Section 6F.

7A. ***Trapping in (sub)extremal Kerr.*** As in the case of the Schwarzschild spacetime there are trapped null geodesics in the domain of outer communications of the Kerr spacetime whose energy stays bounded away from zero and infinity if the energy-measuring vector field N is sensibly chosen. In the case of $a > 0$, however, the set that accommodates trapped null geodesics is the closure of an *open* set in spacetime, which is in contrast to the 3-dimensional photonsphere in Schwarzschild and Reissner–Nordström. Before we explain in some more detail how to find the trapped geodesics, we set up a suitable choice of foliation and energy-measuring vector field:

For (sub)extremal Kerr we foliate the domain of outer communication (which is covered by the above (t, r, θ, φ) coordinates) in the same way as we did before for the Schwarzschild and the extremal Reissner–Nordström spacetimes, namely by first introducing an ingoing "null" coordinate v and then subtracting off r to get a good time coordinate t^*. Slightly more general than is needed at this point, let us define

$$v_+ := t + r^* \quad \text{and} \quad \varphi_+ := \varphi + \bar{r},$$

where r^* is defined up to a constant by $dr^*/dr = (r^2 + a^2)/\Delta$ and $\bar{r}$ is defined up to a constant by $d\bar{r}/dr = a/\Delta$. The set of functions $(v_+, r, \theta, \varphi_+)$ form ingoing "null" coordinates (v_+ is here the "null" coordinate, however, it does *not* satisfy the eikonal equation $d\phi \cdot d\phi = 0$), they cover the regions *I*, *II* and *III* in the spacetime diagram for subextremal Kerr,[39] and the metric takes the form

$$\begin{aligned} g = g_{tt}\,dv_+^2 + g_{t\varphi}(dv_+ \otimes d\varphi_+ + d\varphi_+ \otimes dv_+) + (dv_+ \otimes dr + dr \otimes dv_+) \\ - a \sin^2\theta (dr \otimes d\varphi_+ + d\varphi_+ \otimes dr) + g_{\varphi\varphi}\,d\varphi_+^2 + \rho^2\,d\theta^2. \end{aligned}$$

Finally, we define $t^* := v_+ - r$. That this is indeed a good time coordinate is easily seen from writing the metric in $(t^*, r, \theta, \varphi_+)$ coordinates and restricting it to $\{t^* = \text{const}\}$ slices: One obtains

$$\bar{g} = (g_{tt} + 2)\,dr^2 + (g_{t\varphi} - a \sin^2\theta)(d\varphi_+ \otimes dr + dr \otimes d\varphi_+) + \rho^2\,d\theta^2 + g_{\varphi\varphi}\,d\varphi_+^2,$$

and the (θ, θ) minor of this matrix is found to be $2mr \sin^2\theta + (r^2 + a^2)\sin^2\theta - a^2 \sin^4\theta$, which is positive away from the well-understood coordinate singularity $\theta = \left\{0, \frac{1}{2}\pi\right\}$. Hence, the slices $\Sigma_\tau := \{t^* = \tau\}$ are spacelike and it is easily seen that they asymptote to $\{t = \text{const}\}$ slices near spacelike infinity and end

[39] In the extremal case they cover all of the spacetime diagram depicted in Figure 8 in Section 6F.

on the *future* event horizon. A suitable timelike vector field N for measuring the energy is again given by $N := -(dt^*)^\sharp$.

To be more precise about what we mean by a null geodesic being trapped, let us call a future, complete, affinely parametrised null geodesic $\gamma : [0, \infty) \to M$ (which is in particular contained in the black hole exterior M) *trapped* if, and only if, it does not escape to infinity, i.e., for $s \to \infty$ we do *not* have $(r \circ \gamma)(s) \to \infty$. In the following we give a brief sketch of how one finds the trapped null geodesics. For a detailed discussion of the geodesic flow we refer the reader to [O'Neill 1995] or [Chandrasekhar 1998].

The starting point for the investigation of the behaviour of the geodesics in the Kerr spacetime is the observation that the geodesic flow separates. An affinely parametrised *null* geodesic $\gamma(s) = (t(s), r(s), \theta(s), \varphi(s))$ satisfies the following first-order equations:

$$\rho^2 \dot{t} = a\mathbb{D} + (r^2 + a^2)\frac{\mathbb{P}}{\Delta}, \tag{7.1}$$

$$\rho^4 (\dot{r})^2 = R(r) := -\mathcal{K}\Delta + \mathbb{P}^2, \tag{7.2}$$

$$\rho^4 (\dot{\theta})^2 = \Theta(\theta) := \mathcal{K} - \frac{\mathbb{D}^2}{\sin^2\theta}, \tag{7.3}$$

$$\rho^2 \dot{\varphi} = \frac{\mathbb{D}}{\sin^2\theta} + \frac{a\mathbb{P}}{\Delta},$$

where $\mathcal{K}$ is the Carter constant of the geodesic, $\mathbb{P}(r) = (r^2 + a^2)E - La$ and $\mathbb{D}(\theta) = L - Ea\sin^2\theta$. Here, $E = -g(\partial_t, \dot{\gamma})$ is the energy of the geodesic[40] and $L = g(\partial_\varphi, \dot{\gamma})$ is the angular momentum. Note that since the left-hand side of (7.3) is positive, it follows that the Carter constant $\mathcal{K}$ is nonnegative.

In order to find all trapped null geodesics, the investigation naturally starts with (7.2). The crucial observation is that a simple zero of $R(r)$ corresponds to a turning point (in the r-coordinate) of the geodesic, while a double zero corresponds to an orbit of constant r (or to asymptotic approach).[41] It follows that a necessary condition for a null geodesic being trapped is that the constants of motion $\mathcal{K}$, L, and E can be chosen in such a way that either $R(r)$ has a double zero in (r_+, ∞) or $R(r)$ has two simple zeros in (r_+, ∞) and is nonnegative in between. In the following we show that the latter case cannot occur.

We distinguish the two cases $E = 0$ and $E \neq 0$. In the first case, $R(r)$ is a polynomial of order two with $R(r) \to -\infty$ for $r \to \infty$ (recall that $\mathcal{K} \geq 0$). Moreover, $R(r)$ is nonnegative in $[r_-, r_+]$. This shows that $R(r)$ can have at most one real root in (r_+, ∞).

In the case $E \neq 0$, $R(r)$ is a polynomial of order four. Over the complex numbers, we can write $R(r)$ as

$$R(r) = E^2 \cdot (r - \lambda_1)(r - \lambda_2)(r - \lambda_3)(r - \lambda_4) = E^2 \cdot r^4 - E^2(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) \cdot r^3 + \cdots,$$

where $\lambda_i \in \mathbb{C}$, $i \in \{1, 2, 3, 4\}$, are the complex roots of $R(r)$. Since $R(r)$ does not have a term of order three, we see that the sum of the complex roots of $R(r)$ has to yield zero. This directly excludes $R(r)$

[40] Note that ∂_t is not timelike everywhere! However, one still calls this quantity the "energy" of the null geodesic.

[41] See Proposition 4.3.7 and Corollary 4.3.8 in Chapter 4 of [O'Neill 1995]

having four positive zeros. We also note that $R(r)$ tends to ∞ for $r \to \infty$; hence, for $R(r)$ to have two simple zeros in (r_+, ∞) and to be nonnegative in between, we see that $R(r)$ has to have at least three zeros in (r_+, ∞). But since $\mathcal{K} \geq 0$, we see that $R(r)$ is nonnegative in $[r_-, r_+]$; i.e., if $R(r)$ has three zeros in (r_+, ∞), then it needs to have a fourth positive zero, which we have already ruled out. This shows that trapping can only occur due to a double zero of $R(r)$.

We now sketch how one finds the values of r that accommodate trapped null geodesics (along with the constants of motion $\mathcal{K}$, L and E). A detailed discussion is found in Section 63(c) of [Chandrasekhar 1998].

Without loss of generality we can assume that $E = 1$. We then need to solve

$$R(r) = -\mathcal{K}(r^2 - 2mr + a^2) + (r^2 + a^2 - La)^2 = 0,$$
$$\frac{d}{dr}R(r) = 2\mathcal{K}(m - r) + 4r(r^2 + a^2 - La) = 0.$$

Eliminating $\mathcal{K}$, we obtain the two solutions

$$L_1(r) = \frac{r^2 + a^2}{a} \quad \text{and} \quad L_2(r) = \frac{r^3 + ra^2 - 3mr^2 + ma^2}{a(m - r)}.$$

In the first case we obtain $\mathcal{K}_1(r) = 0$, which characterises the principal null geodesics (see Corollary 4.2.8 in [O'Neill 1995]) and is thus not compatible with orbits of constant r. We are thus left with the second solution $L_2(r)$, which implies $\mathcal{K}_2(r) = (4r^2/(m - r)^2)\Delta$. For the further analysis it is helpful to introduce the quantity $\mathcal{Q} = \mathcal{K} - (L - a)^2$, since it simplifies the analysis of the θ-motion of the geodesic. We obtain

$$\mathcal{Q}_2(r) = \frac{r^3}{a^2(m - r)^2}(4a^2m - r(r - 3m)^2).$$

It can now be shown (see Section 63(c) of [Chandrasekhar 1998]) that if we evaluate the right-hand side of (7.3) at $L_2(r)$ and $\mathcal{K}_2(r)$, where r is such that $\mathcal{Q}_2(r) < 0$, then we see that it is negative for all values of θ. Hence, these values of r do not accommodate trapped null geodesics. However, one can show that the values of r where $\mathcal{Q}_2(r) \geq 0$ indeed allow the presence of trapped null geodesics. This region is bounded by the roots r_δ and r_ρ of $\mathcal{Q}_2(r)$, which are bigger than r_+.

We now show that the N-energy of a trapped null geodesic γ_{r_0}, trapped on the hypersurface $\{r = r_0\}$ with $r_0 \in [r_\delta, r_\rho]$, is bounded away from zero and infinity. One way to do this is to compute the N-energy directly:

$$-(N, \dot{\gamma}) = (dt + dr^* - dr)(\dot{\gamma}) = \dot{t} = \frac{1}{\rho^2}\left[a\mathbb{D}(\theta) + (r_0^2 + a^2)\frac{\mathbb{P}(r_0)}{\Delta(r_0)}\right]$$

where we have used (7.1). A further analysis of the behaviour of the θ component of γ_{r_0} yields that its image is a closed subset of $[0, \pi]$; thus $-(N, \dot{\gamma})(\theta)$ takes on its minimum and maximum. Since $-(N, \dot{\gamma})$ is always strictly positive, this immediately yields that it is bounded away from zero and infinity.

Invoking Theorem 5.5 we thus obtain:

Theorem 7.4 (trapping in (sub)extremal Kerr). *Let (M, g) be the domain of outer communications of a (sub)extremal Kerr spacetime, foliated by the level sets of a time function t^* as above. Moreover, let N*

be the timelike vector field from above and $\mathcal{T}$ *an open set with the property that for all* $\tau \geq 0$ *we have* $\mathcal{T} \cap \Sigma_\tau \cap [r_\delta, r_\rho] \neq \varnothing$. *Then there is no function* $P : [0, \infty) \to (0, \infty)$ *with* $P(\tau) \to 0$ *for* $\tau \to \infty$ *such that*

$$E^N_{\tau, \mathcal{T}\cap\Sigma_\tau}(u) \leq P(\tau) E^N_0(u)$$

holds for all solutions u of the wave equation.

Note that the same remark as made in footnote 27 on page 1400 applies: the theorem remains true if we choose a different timelike vector field N which commutes with the Killing vector field ∂_t and also if we choose a different foliation by timelike slices, i.e., a different time function.[42]

Another way to show that the energy of the trapped null geodesic γ_{r_0} is bounded away from zero and infinity is to choose a different suitable vector field N. Recall that the vector fields ∂_t and ∂_φ are Killing, and that at each point in the domain of outer communications they also span a timelike direction. We can thus find a timelike vector field $\tilde{N}$ that commutes with ∂_t and such that in a small r-neighbourhood of r_0 the vector field $\tilde{N}$ is given by $\partial_t + k\partial_\varphi$ with $k \in \mathbb{R}$ a constant. Thus, $\tilde{N}$ is Killing in this small r-neighbourhood and, hence, the $\tilde{N}$-energy of γ_{r_0} is constant.

7B. ***Blue-shift near the Cauchy horizon of (sub)extremal Kerr.*** In this section we show that the results of Section 6C and 6D also hold for (sub)extremal Kerr. The proof is completely analogous: In the above defined $(v_+, r, \theta, \varphi_+)$ coordinates a family of ingoing null geodesics with uniformly bounded energy on Σ_0 near spacelike infinity ι^0 is given by

$$\gamma_{v_+^0}(s) = (v_+^0, -s, \theta_0, \varphi_0),$$

where $s \in (-\infty, 0)$. The same pictures as in Sections 6C and 6D apply, along with the same spacelike hypersurfaces Σ_0 and Σ_1. In order to obtain regular coordinates in a neighbourhood of the Cauchy horizon, we define, starting with (t, r, θ, φ) coordinates in region *II*, outgoing "null" coordinates $(v_-, r, \theta, \varphi_-)$ by $v_- = t - r^*$ and $\varphi_- = \varphi - \bar{r}$. These coordinates cover the regions *II* and *IV* in the subextremal case and regions *II* and *III* in the extremal case. In these coordinates, the tangent vector of the null geodesic $\gamma_{v_+^0}$ takes the form

$$\dot{\gamma}_{v_+^0} = -\frac{\partial}{\partial r}\bigg|_+ = 2\frac{r^2+a^2}{\Delta}\frac{\partial}{\partial v_-}\bigg|_- - \frac{\partial}{\partial r}\bigg|_- + 2\frac{a}{\Delta}\frac{\partial}{\partial \varphi_-}\bigg|_-, \tag{7.5}$$

which blows up at the Cauchy horizon. It is again easy to see that the inner product with a timelike vector field, which extends smoothly to a timelike vector field over the Cauchy horizon, necessarily blows up along Σ_1 for $v_+^0 \to \infty$. Thus, we obtain, after invoking Theorem 5.1:

Theorem 7.6 (blue-shift near the Cauchy horizon in subextremal Kerr). *Let* Σ_0 *and* Σ_1 *be spacelike slices in the subextremal Kerr spacetime as indicated in Figure 5 in Section 6C. Then there exists a sequence* $\{u_i\}_{i\in\mathbb{N}}$ *of solutions to the wave equation with initial energy* $E_0^{n_{\Sigma_0}}(u_i) = 1$ *on* Σ_0 *such that the* n_{Σ_1}*-energy on* Σ_1 *goes to infinity, i.e.,* $E_1^{n_{\Sigma_1}}(u_i) \to \infty$ *for* $i \to \infty$.

In particular, there is no energy boundedness statement of the form (6.8).

[42] In the latter case one may have to alter the decay statement for the function P, i.e., replace it with $P(\tau) \to 0$ for $\tau \to \tau^*$.

As before, let us state the following:

Conjecture 7.7. *For generic compactly supported smooth initial data on Σ_0, the n_{Σ_1}-energy along Σ_1 of the corresponding solution to the wave equation is infinite.*

Let us conclude this section with a few remarks:

(i) Obviously, an analogous statement to Theorem 7.6 is true for extremal Kerr, however, one has to introduce again a suitable globally hyperbolic subset in order to be able to apply Theorem 5.1.

(ii) The discussion in Section 6E carries over to the Kerr case. In particular let us stress that Conjecture 7.7 only concerns *sub*extremal Kerr black holes — the same statement for extremal Kerr black holes is expected to be false. However, as for Reissner–Nordström black holes, we conjecture a C^k instability (for some finite k) at the Cauchy horizon of extremal Kerr black holes.

(iii) We leave it as an exercise for the reader to convince him- or herself that analogous versions of Theorems 7.4 and 7.6 also hold true for the Kerr–Newman family.

Appendix: A breakdown criterion for solutions of the eikonal equation

We give a breakdown criterion for solutions of the eikonal equation for which a given null geodesic is a characteristic.

Theorem A.1. *Let (M, g) be a Lorentzian manifold and $\gamma : [0, a) \to M$ an affinely parametrised null geodesic, $a \in (0, \infty]$. If γ has conjugate points then there exists no solution $\phi : U \to \mathbb{R}$ of the eikonal equation $d\phi \cdot d\phi = 0$ with $\operatorname{grad} \phi\big|_{\operatorname{Im}\gamma} = \dot{\gamma}$, where U is a neighbourhood of $\operatorname{Im}\gamma$.*

The theorem is motivated by the construction of localised solutions to the wave equation using the naive geometric optics approximation, where we need to find a solution of the eikonal equation for which a given null geodesic is a characteristic; see (1.6). It is well known that solutions of the eikonal equation break down whenever characteristics cross. However, by choosing the initial data (and thus the neighbouring characteristics) suitably one can try to avoid crossing characteristics. This is for example possible in the Minkowski spacetime. The theorem gives a sufficient condition for when no such choice is possible.

Our proof is a minor adaptation of Riemannian methods to the Lorentzian null case; see, for example, [Eschenburg and O'Sullivan 1976], in particular their Proposition 3.

First we need some groundwork. We pull back the tangent bundle TM via γ and denote the subbundle of vectors that are orthogonal to $\dot{\gamma}$ by $N(\gamma)$. The vectors that are proportional to $\dot{\gamma}$ give rise to a subbundle of $N(\gamma)$, which we quotient out to obtain the quotient bundle $\overline{N}(\gamma)$. It is easy to see that the metric g induces a positive-definite metric $\bar{g}$ on $\overline{N}(\gamma)$ and that the bundle map $R_\gamma : N(\gamma) \to N(\gamma)$, where $R_\gamma(X) = R(X, \dot{\gamma})\dot{\gamma}$ and R is the Riemann curvature tensor, induces a bundle map $\bar{R}_\gamma$ on $\overline{N}(\gamma)$, and finally that the Levi-Civita connection ∇ induces a connection $\overline{\nabla}$ for $\overline{N}(\gamma)$.

Definition A.2. $\bar{J} \in \operatorname{End}(\overline{N}(\gamma))$ is a *Jacobi tensor class* if and only if $\overline{D}_t^2 \bar{J} + \bar{R}_\gamma \bar{J} = 0$.[43]

[43] Here and in what follows we write $\overline{D}_t$ for $\overline{\nabla}_{\partial_t}$.

A Jacobi tensor class should be thought of as a variation field of γ that arises from a many-parameter variation by geodesics. It generalises the notion of a Jacobi field (class), an infinitesimal 1-parameter variation. Indeed, a solution ϕ of the eikonal equation for which γ is a characteristic gives rise to a Jacobi tensor class $\bar{J}$:

We denote the flow of $\operatorname{grad}\phi$ by Ψ_t and define $J \in \operatorname{End}(N(\gamma))$ by

$$J_t(X_t) := (\Psi_t)_*(X_0),$$

where we extend $X_t \in N(\gamma)_t$ by parallel propagation to a vector field X along γ whose value at 0 is X_0. Note that J is well defined, that is, we have $J_t(X_t) \in N(\gamma)$: Given $X_0 \in T_{\gamma(0)}M$, extend it to a vector field $\tilde{X}$ on M with $[\tilde{X}, \operatorname{grad}\phi] = 0$, i.e., along γ we have $\tilde{X}\big|_{\gamma(t)} = (\Psi_t)_*(X_0)$. Then

$$0 = \nabla_{\tilde{X}}(\operatorname{grad}\phi, \operatorname{grad}\phi) = 2(\nabla_{\tilde{X}} \operatorname{grad}\phi, \operatorname{grad}\phi) = 2\nabla_{\operatorname{grad}\phi}(\tilde{X}, \operatorname{grad}\phi),$$

from which it follows that $\tilde{X}\big|_{\gamma(t)}$ is orthogonal to $\operatorname{grad}\phi\big|_{\gamma(t)}$. Moreover, J is a Jacobi tensor:[44] Let X be a parallel section along γ and $\tilde{X}$ an extension of X_0 as above. Then

$$(D_t J)(X) = D_t(JX) = D_t(\Psi_{t*}X_0) = \nabla_{\operatorname{grad}\phi}\tilde{X} = \nabla_{\tilde{X}} \operatorname{grad}\phi = \nabla_{JX} \operatorname{grad}\phi.$$

Thus,

$$D_t J = (\nabla \operatorname{grad}\phi) \circ J. \tag{A.3}$$

Differentiating once more gives

$$(D_t^2 J)(X) = \nabla_{\operatorname{grad}\phi}(\nabla_{JX} \operatorname{grad}\phi) = R(\operatorname{grad}\phi, JX)\operatorname{grad}\phi = -R_\gamma \circ J(X).$$

Using that $(\Psi_t)_*(\operatorname{grad}\phi|_{\gamma(0)}) = \operatorname{grad}\phi|_{\gamma(t)}$, it is now clear that J descends to a Jacobi tensor class $\bar{J}$. Moreover, $\bar{J}$ is nonsingular, i.e., $\bar{J}^{-1}$ exists. Since the metric $\bar{g}$ is nondegenerate, we can form adjoints of sections of $\operatorname{End}(\overline{N}(\gamma))$, which we will denote by *. Note also that $(\overline{D}_t\bar{J})\bar{J}^{-1}$ is self-adjoint. This follows from (A.3) and the fact that $\nabla\nabla\phi$ is symmetric. We now prove the theorem.

Proof of Theorem A.1. Assume there exists such a solution ϕ of the eikonal equation. Say the points $\gamma(t_0)$ and $\gamma(t_1)$ are conjugate, $0 \le t_0 < t_1 < a$, and $\bar{J}$ is the Jacobi tensor class induced by ϕ, as discussed above. Using the identification of $\operatorname{End}(\overline{N}(\gamma)_t)$ with $\operatorname{End}(\overline{N}(\gamma)_{t_0})$ via parallel translation, we write

$$\overline{K}(t) := \bar{J}(t)C\int_{t_0}^{t}(\bar{J}^*\bar{J})^{-1}(\tau)\,d\tau,$$

where $C = \bar{J}^{-1}(t_0)\bar{J}^*(t_0)\bar{J}(t_0)$. A straightforward computation shows that $\overline{K}$ is a Jacobi tensor class with $\overline{K}(t_0) = 0$ and $\overline{D}_t\overline{K}(t_0) = \mathrm{id}$. Moreover, $\overline{K}(t)$ is nonsingular for $t > t_0$.

On the other hand, there exists a Jacobi field Y with $Y(t_0) = 0$ and $Y(t_1) = 0$. This implies that Y is a section of $N(\gamma)$. The Jacobi field Y induces a nontrivial Jacobi field class $\bar{Y}$ that vanishes at t_0 and t_1. However, a Jacobi field class is uniquely determined by its value and velocity at a point. Parallelly propagating $\overline{D}_t\bar{Y}\big|_{t_0}$ gives rise to a vector field class $\bar{Z}$. Then $\overline{K}\bar{Z}$ is a Jacobi field class that has the same value and velocity as $\bar{Y}$ at $t = t_0$, thus $\overline{K}\bar{Z} = \bar{Y}$. This, however, contradicts $\overline{K}$ being nonsingular for $t > t_0$. $\square$

[44] This notion is analogous to Definition A.2, without taking the quotient.

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