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We prove a height estimate (distance from the tangent hyperplane) for Λ -minimizers of the perimeter in the sub-Riemannian Heisenberg group. The estimate is in terms of a power of the excess (L^2 -mean oscillation of the normal) and its proof is based on a new coarea formula for rectifiable sets in the Heisenberg group.

1. Introduction

We continue the research project started in [Monti and Vittone 2012; Monti 2014] on the regularity of H -perimeter minimizing boundaries in the Heisenberg group \mathbb{H}^n . Our goal is to prove the so-called *height estimate* for sets that are Λ -minimizers and have small *excess* inside suitable cylinders; see Theorem 1.3. The proof follows the scheme of the median choice for the measure of the boundary in certain half-cylinders together with a lower-dimensional isoperimetric inequality on slices. For minimizing currents in \mathbb{R}^n , the principal ideas of the argument go back to [Almgren 1968] and are carried over in [Federer 1969, Theorem 5.3.4]. The argument can be also found in the Appendix of [Schoen and Simon 1982] and, for Λ -minimizers of perimeter in \mathbb{R}^n , in [Maggi 2012, Section 22.2]. For minimizers of H -perimeter, the decay estimate of excess of Almgren and De Giorgi is still an open problem; see [Monti 2015].

Our main technical effort is the proof of a coarea formula (slicing formula) for intrinsic rectifiable sets; see Theorem 1.5. This formula is established in Section 2 and has a nontrivial character because the domain of integration and its slices need not be rectifiable in the standard sense. The relative isoperimetric inequalities that are used in the slices reduce to a single isoperimetric inequality in one slice that is relative to a family of varying domains with uniform isoperimetric constants. This uniformity can be established using the results on regular domains in Carnot groups of step 2 in [Monti and Morbidelli 2005] and the isoperimetric inequality in [Garofalo and Nhieu 1996]; see Section 3A.

The $(2n+1)$ -dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group product

$$(z, t) * (\zeta, \tau) = (z + \zeta, t + \tau + 2\Im\langle z, \bar{\zeta} \rangle), \quad (1-1)$$

where $t, \tau \in \mathbb{R}$, $z, \zeta \in \mathbb{C}^n$ and $\langle z, \bar{\zeta} \rangle = z_1 \bar{\zeta}_1 + \cdots + z_n \bar{\zeta}_n$. The Lie algebra of left-invariant vector fields in \mathbb{H}^n is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j \frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j \frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}, \quad (1-2)$$

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with $z_j = x_j + iy_j$ and $j = 1, \dots, n$. We denote by H the horizontal subbundle of $T\mathbb{H}^n$. Namely, for any $p = (z, t) \in \mathbb{H}^n$ we let

$$H_p = \text{span}\{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\}.$$

A horizontal section $\varphi \in C_c^1(\Omega; H)$, where $\Omega \subset \mathbb{H}^n$ is an open set, is a vector field of the form

$$\varphi = \sum_{j=1}^n \varphi_j X_j + \varphi_{n+j} Y_j,$$

where $\varphi_j \in C_c^1(\Omega)$, i.e., each coordinate φ_j is a continuously differentiable function with compact support contained in Ω .

Let g be the left-invariant Riemannian metric on \mathbb{H}^n that makes orthonormal the vector fields $X_1, \dots, X_n, Y_1, \dots, Y_n, T$ in (1-2). For tangent vectors $V, W \in T\mathbb{H}^n$, we let

$$\langle V, W \rangle_g = g(V, W) \quad \text{and} \quad |V|_g = g(V, V)^{1/2}.$$

The sup norm with respect to g of a horizontal section $\varphi \in C_c^1(\Omega; H)$ is

$$\|\varphi\|_g = \max_{p \in \Omega} |\varphi(p)|_g.$$

The Riemannian divergence of φ is

$$\text{div}_g \varphi = \sum_{j=1}^n X_j \varphi_j + Y_j \varphi_{n+j}.$$

The metric g induces a volume form on \mathbb{H}^n that is left-invariant. Also, the Lebesgue measure \mathcal{L}^{2n+1} on \mathbb{H}^n is left-invariant, and by the uniqueness of the Haar measure the volume induced by g is the Lebesgue measure \mathcal{L}^{2n+1} . In fact, the proportionality constant is 1.

The H -perimeter of an \mathcal{L}^{2n+1} -measurable set $E \subset \mathbb{H}^n$ in an open set $\Omega \subset \mathbb{H}^n$ is

$$\mu_E(\Omega) = \sup \left\{ \int_E \text{div}_g \varphi \, d\mathcal{L}^{2n+1} : \varphi \in C_c^1(\Omega; H), \|\varphi\|_g \leq 1 \right\}.$$

If $\mu_E(\Omega) < \infty$ we say that E has finite H -perimeter in Ω . If $\mu_E(A) < \infty$ for any open set $A \Subset \Omega$, we say that E has locally finite H -perimeter in Ω . In this case, the open sets mapping $A \mapsto \mu_E(A)$ extends to a Radon measure μ_E on Ω that is called the H -perimeter measure induced by E . Moreover, there exists a μ_E -measurable function $\nu_E : \Omega \rightarrow H$ such that $|\nu_E|_g = 1$ μ_E -a.e. and the Gauss–Green integration by parts formula

$$\int_{\Omega} \langle \varphi, \nu_E \rangle_g \, d\mu_E = - \int_{\Omega} \text{div}_g \varphi \, d\mathcal{L}^{2n+1}$$

holds for any $\varphi \in C_c^1(\Omega; H)$. The vector ν_E is called the *horizontal inner normal* of E in Ω .

The Korányi norm of $p = (z, t) \in \mathbb{H}^n$ is $\|p\|_K = (|z|^4 + t^2)^{1/4}$. For any $r > 0$ and $p \in \mathbb{H}^n$, we define the balls

$$B_r = \{q \in \mathbb{H}^n : \|q\|_K < r\} \quad \text{and} \quad B_r(p) = \{p * q \in \mathbb{H}^n : q \in B_r\}.$$

The *measure-theoretic boundary* of a measurable set $E \subset \mathbb{H}^n$ is the set

$$\partial E = \{p \in \mathbb{H}^n : \mathcal{L}^{2n+1}(E \cap B_r(p)) > 0 \text{ and } \mathcal{L}^{2n+1}(B_r(p) \setminus E) > 0 \text{ for all } r > 0\}.$$

For a set E with locally finite H -perimeter, the H -perimeter measure μ_E is concentrated on ∂E and, actually, on a subset $\partial^* E$ of ∂E ; see below. Moreover, up to modifying E on a Lebesgue-negligible set, one can always assume that ∂E coincides with the topological boundary of E ; see [Serra Cassano and Vittone 2014, Proposition 2.5].

Definition 1.1. Let $\Omega \subset \mathbb{H}^n$ be an open set, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$. We say that a set $E \subset \mathbb{H}^n$ with locally finite H -perimeter in Ω is a (Λ, r) -*minimizer of H -perimeter in Ω* if, for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$, and $s < r$ such that $E \Delta F \Subset B_s(p) \Subset \Omega$,

$$\mu_E(B_s(p)) \leq \mu_F(B_s(p)) + \Lambda \mathcal{L}^{2n+1}(E \Delta F),$$

where $E \Delta F = E \setminus F \cup F \setminus E$.

We say that E is *locally H -perimeter minimizing in Ω* if, for any measurable set $F \subset \mathbb{H}^n$ and any open set U such that $E \Delta F \Subset U \Subset \Omega$, there holds $\mu_E(U) \leq \mu_F(U)$.

We will often use the term Λ -*minimizer*, rather than (Λ, r) -minimizer, when the role of r is not relevant. In Appendix A, we list without proof some elementary properties of Λ -minimizers.

We now introduce the notion of cylindrical excess. The *height function* $\xi : \mathbb{H}^n \rightarrow \mathbb{R}$ is defined by $\xi(p) = p_1$, where p_1 is the first coordinate of $p = (p_1, \dots, p_{2n+1}) \in \mathbb{H}^n$. The set $\mathbb{W} = \{p \in \mathbb{H}^n : \xi(p) = 0\}$ is the vertical hyperplane passing through $0 \in \mathbb{H}^n$ and orthogonal to the left-invariant vector field X_1 . The disk in \mathbb{W} of radius $r > 0$ centred at $0 \in \mathbb{W}$ induced by the Korányi norm is the set $D_r = \{p \in \mathbb{W} : \|p\|_K < r\}$. The intrinsic cylinder with central section D_r and height $2r$ is the set

$$C_r = D_r * (-r, r) \subset \mathbb{H}^n.$$

Here and in the sequel, we use the notation $D_r * (-r, r) = \{w * (se_1) \in \mathbb{H}^n : w \in D_r, s \in (-r, r)\}$, where $se_1 = (s, 0, \dots, 0) \in \mathbb{H}^n$. The cylinder C_r is comparable with the ball $B_r = \{\|p\|_K < r\}$. Namely, there exists a constant $k = k(n) \geq 1$ such that, for any $r > 0$, we have

$$B_{r/k} \subset C_r \subset B_{kr}. \tag{1-3}$$

By a rotation of the system of coordinates, it is enough to consider excess in cylinders with basis in \mathbb{W} and axis X_1 .

Definition 1.2 (cylindrical excess). Let $E \subset \mathbb{H}^n$ be a set with locally finite H -perimeter. The cylindrical excess of E at the point $0 \in \partial E$, at scale $r > 0$, and with respect to the direction $\nu = -X_1$ is defined as

$$\text{Exc}(E, r, \nu) = \frac{1}{2r^{2n+1}} \int_{C_r} |\nu_E - \nu|_g^2 d\mu_E,$$

where μ_E is the H -perimeter measure of E and ν_E is its horizontal inner normal.

Theorem 1.3 (height estimate). *Let $n \geq 2$. There exist constants $\varepsilon_0 = \varepsilon_0(n) > 0$ and $c_0 = c_0(n) > 0$ with the following property: if $E \subset \mathbb{H}^n$ is a (Λ, r) -minimizer of H -perimeter in the cylinder C_{4k^2r} , $\Lambda r \leq 1$, $0 \in \partial E$, and*

$$\text{Exc}(E, 4k^2r, \nu) \leq \varepsilon_0,$$

then

$$\sup\{|\zeta(p)| \in [0, \infty) : p \in \partial E \cap C_r\} \leq c_0 r \text{Exc}(E, 4k^2r, \nu)^{1/(2(2n+1))}. \tag{1-4}$$

The constant $k = k(n)$ is the one in (1-3).

The estimate (1-4) does not hold when $n = 1$. In fact, there are sets $E \subset \mathbb{H}^1$ such that $\text{Exc}(E, r, \nu) = 0$ but ∂E is not flat in $C_{\varepsilon r}$ for any $\varepsilon > 0$. See the conclusions of Proposition 3.7 in [Monti 2014]. Theorem 1.3 is proved in Section 3.

Besides local minimizers of H -perimeter, our interest in Λ -minimizers is also motivated by possible applications to isoperimetric sets. The height estimate is a first step in the regularity theory of Λ -minimizers of classical perimeter; we refer to [Maggi 2012, Part III] for a detailed account of the subject.

In order to state the slicing formula in its general form, we need the definition of a rectifiable set in \mathbb{H}^n of codimension 1. We follow closely [Franchi et al. 2001], where this notion was first introduced.

The Riemannian and horizontal gradients of a function $f \in C^1(\mathbb{H}^n)$ are, respectively,

$$\begin{aligned} \nabla f &= (X_1 f)X_1 + \dots + (Y_n f)Y_n + (Tf)T, \\ \nabla_H f &= (X_1 f)X_1 + \dots + (Y_n f)Y_n. \end{aligned}$$

We say that a continuous function $f \in C(\Omega)$, with $\Omega \subset \mathbb{H}^n$ an open set, is of class $C^1_H(\Omega)$ if the horizontal gradient $\nabla_H f$ exists in the sense of distributions and is represented by continuous functions $X_1 f, \dots, Y_n f$ in Ω . A set $S \subset \mathbb{H}^n$ is an H -regular hypersurface if, for all $p \in S$, there exist $r > 0$ and a function $f \in C^1_H(B_r(p))$ such that $S \cap B_r(p) = \{q \in B_r(p) : f(q) = 0\}$ and $\nabla_H f(p) \neq 0$. Sets with H -regular boundary have locally finite H -perimeter.

For any $p = (z, t) \in \mathbb{H}^n$, let us define the box norm $\|p\|_\infty = \max\{|z|, |t|^{1/2}\}$ and the balls $U_r = \{q \in \mathbb{H}^n n \|q\|_\infty < r\}$ and $U_r(p) = p * U_r$ for $r > 0$. Let $E \subset \mathbb{H}^n$ be a set. For any $s \geq 0$ define the measure

$$\mathcal{S}^s(E) = \sup_{\delta > 0} \inf \left\{ c(n, s) \sum_{i \in \mathbb{N}} r_i^s : E \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i), r_i < \delta \right\}.$$

Above, $c(n, s) > 0$ is a normalization constant that we do not need to specify here. By Carathéodory’s construction, $E \mapsto \mathcal{S}^s(E)$ is a Borel measure in \mathbb{H}^n . When $s = 2n + 2$, it turns out that \mathcal{S}^{2n+2} is the Lebesgue measure \mathcal{L}^{2n+1} . Thus, the correct dimension to measure hypersurfaces is $s = 2n + 1$. In fact, if E is a set with locally finite H -perimeter in \mathbb{H}^n , then we have

$$\mu_E = \mathcal{S}^{2n+1} \llcorner \partial^* E, \tag{1-5}$$

where \llcorner denotes restriction and $\partial^* E$ is the H -reduced boundary of E , namely the set of points $p \in \mathbb{H}^n$ such that $\mu_E(U_r(p)) > 0$ for all $r > 0$, $\int_{U_r(p)} \nu_E d\mu_E \rightarrow \nu_E(p)$ as $r \rightarrow 0$, and $|\nu_E(p)|_g = 1$. The validity

of formula (1-5) depends on the geometry of the balls $U_r(p)$; see [Magnani 2014]. We refer the reader to [Franchi et al. 2001] for more details on the H -reduced boundary.

Definition 1.4. A set $R \subset \mathbb{H}^n$ is \mathcal{S}^{2n+1} -rectifiable if there exists a sequence of H -regular hypersurfaces $(S_j)_{j \in \mathbb{N}}$ in \mathbb{H}^n such that

$$\mathcal{S}^{2n+1}\left(R \setminus \bigcup_{j \in \mathbb{N}} S_j\right) = 0.$$

By the results of [Franchi et al. 2001], the H -reduced boundary $\partial^* E$ is \mathcal{S}^{2n+1} -rectifiable. Definition 1.4 is generalized in [Mattila et al. 2010], which studies the notion of an s -rectifiable set in \mathbb{H}^n for any integer $1 \leq s \leq 2n + 1$.

An H -regular surface S has a continuous horizontal normal ν_S that is locally defined up to the sign. This normal is given by the formula

$$\nu_S = \frac{\nabla_H f}{|\nabla_H f|_g}, \tag{1-6}$$

where f is a defining function for S . When $S = \partial E$ is the boundary of a smooth set, ν_S agrees with the horizontal normal ν_E . Then, for an \mathcal{S}^{2n+1} -rectifiable set $R \subset \mathbb{H}^n$, there is a unit horizontal normal $\nu_R : R \rightarrow H$ that is Borel regular. This normal is uniquely defined \mathcal{S}^{2n+1} -a.e. on R up to the sign; see Appendix B. However, (1-8) below does not depend on the sign.

In the following theorem, $\Omega \subset \mathbb{H}^n$ is an open set and $u \in C^\infty(\Omega)$ is a smooth function. For any $s \in \mathbb{R}$, we denote by $\Sigma^s = \{p \in \Omega : u(p) = s\}$ the level sets of u .

Theorem 1.5. *Let $R \subset \Omega$ be an \mathcal{S}^{2n+1} -rectifiable set. Then, for a.e. $s \in \mathbb{R}$ there exists a Radon measure μ_R^s on $R \cap \Sigma^s$ such that, for any Borel function $h : \Omega \rightarrow [0, \infty)$, the function*

$$s \mapsto \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s \tag{1-7}$$

is \mathcal{L}^1 -measurable and we have the coarea formula

$$\int_{\mathbb{R}} \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s ds = \int_R h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} d\mathcal{S}^{2n+1}. \tag{1-8}$$

Theorem 1.5 is proved in Section 2. When $R \cap \Sigma^s$ is a regular subset of Σ^s , the measures μ_R^s are natural horizontal perimeters defined in Σ^s .

Coarea formulas in the Heisenberg group are known only for slicing of sets with positive Lebesgue measure; see [Magnani 2004; 2008]. Theorem 1.5 is, to our knowledge, the first example of slicing of lower-dimensional sets in a sub-Riemannian framework. Also, Theorem 1.5 is a nontrivial extension of the Riemannian coarea formula, because the set R and the slices $R \cap \Sigma^s$ need not be rectifiable in the standard sense; see [Kirchheim and Serra Cassano 2004]. We need the coarea formula (1-8) in the proof of Theorem 1.3; see Section 3C.

We conclude the introduction by stating a different but equivalent formulation of the coarea formula (1-8) that is closer to standard coarea formulas. This alternative formulation holds only when $n \geq 2$: when $n = 1$, the right-hand side in (1-9) might not be well defined; see Remark 2.11.

Theorem 1.6. *Let $\Omega \subset \mathbb{H}^n$, $n \geq 2$, be an open set, $u \in C^\infty(\Omega)$ be a smooth function, and $R \subset \Omega$ be an \mathcal{S}^{2n+1} -rectifiable set. Then, for any Borel function $h : \Omega \rightarrow [0, \infty)$,*

$$\int_{\mathbb{R}} \int_{\Omega} h \, d\mu_R^s \, ds = \int_R h |\nabla u|_g \sqrt{1 - \langle \nu_R, \nabla_{Hu} / |\nabla_{Hu}|_g \rangle_g^2} \, d\mathcal{S}^{2n+1}, \tag{1-9}$$

where μ_R^s are the measures given by Theorem 1.5.

2. Proof of the coarea formula

2A. Horizontal perimeter on submanifolds. Let $\Sigma \subset \mathbb{H}^n$ be a C^∞ hypersurface. We define the horizontal tangent bundle $H\Sigma$ by letting, for any $p \in \Sigma$,

$$H_p \Sigma = H_p \cap T_p \Sigma.$$

In general, the rank of $H\Sigma$ is not constant. This depends on the presence of *characteristic points* on Σ , i.e., points such that $H_p = T_p \Sigma$. For points $p \in \Sigma$ such that $H_p \neq T_p \Sigma$, we have $\dim(H_p \Sigma) = 2n - 1$.

We denote by σ_Σ the surface measure on Σ induced by the Riemannian metric g restricted to the tangent bundle $T\Sigma$.

Definition 2.1. Let $F \subset \Sigma$ be a Borel set and let $\Omega \subset \Sigma$ be an open set. We define the *H-perimeter of F in Ω* ,

$$\mu_F^\Sigma(\Omega) = \sup \left\{ \int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma : \varphi \in C_c^1(\Omega; H\Sigma), \|\varphi\|_g \leq 1 \right\}. \tag{2-10}$$

We say that the set $F \subset \Sigma$ has locally finite *H-perimeter* in Ω if $\mu_F^\Sigma(A) < \infty$ for any open set $A \Subset \Omega$.

By the Riesz theorem, if $F \subset \Sigma$ has locally finite *H-perimeter* in Ω , then the open sets mapping $A \mapsto \mu_F^\Sigma(A)$ extends to a Radon measure on Ω , called the *H-perimeter measure of F* .

Remark 2.2. If $F \subset \Sigma$ is an open set with smooth boundary, then, by the divergence theorem, we have, for any $\varphi \in C_c^1(\Omega; H\Sigma)$,

$$\int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma = \int_{\partial F} \langle N_{\partial F}, \varphi \rangle_g \, d\lambda_{\partial F}, \tag{2-11}$$

where $N_{\partial F}$ is the Riemannian outer unit normal to ∂F and $d\lambda_{\partial F}$ is the Riemannian $(2n-1)$ -dimensional volume form on ∂F induced by g .

From the sup definition (2-10) and from (2-11), we deduce that the *H-perimeter measure of F* has the representation

$$\mu_F^\Sigma = |N_{\partial F}^{H\Sigma}|_g \lambda_{\partial F},$$

where $N_{\partial F}^{H\Sigma} \in H\Sigma$ is the g -orthogonal projection of $N_{\partial F} \in T\Sigma$ onto $H\Sigma$.

This formula can be generalized as follows. We denote by \mathcal{H}_g^{2n-1} the $(2n-1)$ -dimensional Hausdorff measure in \mathbb{H}^n induced by the metric g .

Lemma 2.3. *Let $F, \Omega \subset \Sigma$ be open sets and assume that there exists a compact set $N \subset \partial F$ such that $\mathcal{H}_g^{2n-1}(N) = 0$ and $(\partial F \setminus N) \cap \Omega$ is a smooth $(2n-1)$ -dimensional surface. Then, we have*

$$\mu_F^\Sigma \llcorner \Omega = |N_{\partial F}^{H\Sigma}|_g \lambda_{\partial F \setminus N} \llcorner \Omega. \tag{2-12}$$

Proof. For any $\varepsilon > 0$ there exist points $p_i \in \mathbb{H}^n$ and radii $r_i \in (0, 1), i = 1, \dots, M$, such that

$$N \subset \bigcup_{i=1}^M B_g(p_i, r_i) \quad \text{and} \quad \sum_{i=1}^M r_i^{2n-1} < \varepsilon,$$

where $B_g(p, r)$ denotes the ball in \mathbb{H}^n with centre p and radius r with respect to the metric g . By a partition of unity argument, there exist functions $f^\varepsilon, g_i^\varepsilon \in C^\infty(\Omega; [0, 1]), i = 1, \dots, M$, such that:

- (i) $f^\varepsilon + g_1^\varepsilon + \dots + g_M^\varepsilon = \chi_\Omega$;
- (ii) $f^\varepsilon = 0$ on $\bigcup_{i=1}^M B_g(p_i, r_i/2)$;
- (iii) for each i , the support of g_i^ε is contained in $B_g(p_i, r_i)$;
- (iv) $|\nabla g_i^\varepsilon|_g \leq C r_i^{-1}$ for a constant $C > 0$ independent of ε .

Hence, for any horizontal section $\varphi \in C_c^1(\Omega; H\Sigma)$, we have

$$\begin{aligned} \int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma &= \int_F \operatorname{div}_g (f^\varepsilon \varphi) \, d\sigma_\Sigma + \sum_{i=1}^M \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) \, d\sigma_\Sigma \\ &= \int_{\partial F \setminus N} \langle f^\varepsilon \varphi, N_{\partial F} \rangle_g \, d\lambda_{\partial F \setminus N} + \sum_{i=1}^M \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) \, d\sigma_\Sigma, \end{aligned} \tag{2-13}$$

where, by (iv),

$$\left| \sum_{i=1}^M \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g (g_i^\varepsilon \varphi) \, d\sigma_\Sigma \right| \leq \sum_{i=1}^M \int_{B_g(p_i, r_i)} (\|\operatorname{div}_g \varphi\|_{L^\infty} + C r_i^{-1}) \, d\sigma_\Sigma \leq C' \sum_{i=1}^M r_i^{2n-1} \leq C' \varepsilon \tag{2-14}$$

with a constant $C' > 0$ independent of ε .

Letting $\varepsilon \rightarrow 0$, we have $f^\varepsilon \rightarrow 1$ pointwise on $\partial F \setminus N$, by (i) and (iii). Then, from (2-13) and (2-14), we obtain

$$\int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma = \int_{\partial F \setminus N} \langle \varphi, N_{\partial F} \rangle_g \, d\lambda_{\partial F \setminus N}$$

and claim (2-12) follows by standard arguments. □

2B. Proof of Theorem 1.5. Let $\Omega \subset \mathbb{H}^n$ be an open set and $u \in C^\infty(\Omega)$. By Sard's theorem, for a.e. $s \in \mathbb{R}$ the level set

$$\Sigma^s = \{p \in \Omega : u(p) = s\}$$

is a smooth hypersurface and, moreover, we have $\nabla u \neq 0$ on Σ^s .

Let $E \subset \mathbb{H}^n$ be a Borel set such that $E \cap \Sigma^s$ has (locally) finite H -perimeter in $\Omega \cap \Sigma^s$, in the sense of Definition 2.1. Then on $\Omega \cap \Sigma^s$ we have the H -perimeter measure $\mu_{E \cap \Sigma^s}^{\Sigma^s}$ induced by $E \cap \Sigma^s$. We shall use the notation

$$\mu_E^s = \mu_{E \cap \Sigma^s}^{\Sigma^s}$$

to denote a measure on Ω that is supported on $\Omega \cap \Sigma^s$.

We start with the following coarea formula in the smooth case, which is deduced from the Riemannian formula.

Lemma 2.4. *Let $\Omega \subset \mathbb{H}^n$ be an open set and $u \in C^\infty(\Omega)$. Let $E \subset \mathbb{H}^n$ be an open set with C^∞ boundary in Ω such that $\mu_E(\Omega) < \infty$. Then we have*

$$\int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds = \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E, \tag{2-15}$$

where μ_E is the H -perimeter measure of E and ν_E is its horizontal normal.

Proof. The integral in the left-hand side is well defined, because for a.e. $s \in \mathbb{R}$ there holds $\nabla u \neq 0$ on Σ^s . By the coarea formula for Riemannian manifolds—see, e.g., [Burago and Zalgaller 1988]—for any Borel function $h : \partial E \rightarrow [0, \infty]$ we have

$$\int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} h d\lambda_{\partial E \cap \Sigma^s} ds = \int_{\partial E} h |\nabla^{\partial E} u|_g d\sigma_{\partial E}, \tag{2-16}$$

where $\nabla^{\partial E} u$ is the tangential gradient of u on ∂E . Then we have

$$\nabla^{\partial E} u = \nabla u - \langle \nabla u, N_{\partial E} \rangle_g N_{\partial E} \quad \text{and} \quad |\nabla^{\partial E} u|_g = \sqrt{|\nabla u|_g^2 - \langle \nabla u, N_{\partial E} \rangle_g^2}. \tag{2-17}$$

Step 1. Let us define the set

$$C = \left\{ p \in \partial E \cap \Omega : \nabla u(p) \neq 0 \text{ and } N_{\partial E}(p) = \pm \frac{\nabla u(p)}{|\nabla u(p)|_g} \right\}.$$

If $s \in \mathbb{R}$ is such that $\nabla u \neq 0$ on Σ^s , then $C \cap \Sigma^s$ is a closed set in Σ^s . Using the coarea formula (2-16) with the function $h = \chi_C$, we get

$$\int_{\mathbb{R}} \lambda_{\partial E \cap \Sigma^s}(C) ds = \int_C |\nabla^{\partial E} u|_g d\sigma_{\partial E} = 0,$$

because we have $\nabla^{\partial E} u = 0$ on C . In particular, we deduce that

$$C \cap \Sigma^s \text{ is a closed set in } \Sigma^s \quad \text{and} \quad \lambda_{\partial E \cap \Sigma^s}(C \cap \Sigma^s) = 0 \quad \text{for a.e. } s \in \mathbb{R}. \tag{2-18}$$

If $p \in \Sigma^s$ is a point such that $\nabla u(p) \neq 0$ and $p \notin C$, then Σ^s is a smooth hypersurface in a neighbourhood of p and $E^s = E \cap \Sigma^s$ is a domain in Σ^s with smooth boundary in a neighbourhood of p . Moreover, we have $(\partial E \cap \Sigma^s) \setminus C = \partial E^s \setminus C$. Then, from (2-18) and Lemma 2.3 we conclude that for a.e. $s \in \mathbb{R}$ we have

$$\mu_E^s = |N_{\partial E^s}^{H\Sigma^s}|_g \lambda_{\partial E^s}. \tag{2-19}$$

By (2-18) and (2-19),

$$\mu_E^s(C \cap \Sigma^s) = \int_{C \cap \Sigma^s} |N_{\partial E^s}^{H\Sigma^s}|_g d\lambda_{\partial E^s} = 0 \quad \text{for a.e. } s \in \mathbb{R}. \tag{2-20}$$

Step 2. We prove (2-15) by plugging into (2-16) the Borel function $h : \partial E \rightarrow [0, \infty]$,

$$h = \begin{cases} \frac{|N_{\partial E}^H|_g \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2}}{|\nabla u|_g \sqrt{1 - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g^2}} & \text{on } \partial E \setminus (C \cup \{\nabla u = 0\}), \\ 0 & \text{on } C \cup \{\nabla u = 0\}. \end{cases}$$

Above, $N_{\partial E}^H$ is the projection of the Riemannian normal $N_{\partial E}$ onto H and v_E is the horizontal normal. Namely, we have

$$N_{\partial E}^H = N_{\partial E} - \langle N_{\partial E}, T \rangle_g T \quad \text{and} \quad v_E = \frac{N_{\partial E}^H}{|N_{\partial E}^H|_g}.$$

The H -perimeter measure of E is

$$\mu_E = |N_{\partial E}^H|_g \sigma_{\partial E}. \tag{2-21}$$

Using (2-17) and (2-21), we find

$$\begin{aligned} \int_{\partial E} h |\nabla^{\partial E} u| d\sigma_{\partial E} &= \int_{\partial E \setminus (C \cup \{\nabla u = 0\})} |N_{\partial E}^H|_g \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\sigma_{\partial E} \\ &= \int_{\partial E \setminus (C \cup \{\nabla u = 0\})} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= \int_{\partial E} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E, \end{aligned} \tag{2-22}$$

where the last equality is justified by the fact that if $p \in C \cup \{\nabla u = 0\}$ then

$$\sqrt{|\nabla_H u(p)|_g^2 - \langle v_E(p), \nabla_H u(p) \rangle_g^2} = 0.$$

For a.e. $s \in \mathbb{R}$, we have $\nabla u \neq 0$ on Σ^s . Using (2-21) and the fact that $h = 0$ on $C \cup \{\nabla_H u = 0\}$, letting $\Lambda^s = (\partial E \cap \Sigma^s) \setminus (C \cup \{\nabla_H u = 0\})$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} h d\lambda_{\partial E^s} ds &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|N_{\partial E}^H|_g \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2}}{|\nabla u|_g \sqrt{1 - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g^2}} d\lambda_{\partial E^s} ds \\ &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \vartheta^s d\lambda_{\partial E^s} ds, \end{aligned} \tag{2-23}$$

where we let

$$\vartheta^s = \frac{\sqrt{|N_{\partial E}^H|_g^2 - \langle N_{\partial E}^H, \nabla_H u / |\nabla_H u|_g \rangle_g^2}}{\sqrt{1 - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g^2}}.$$

We will prove in Step 3 that, for any $s \in \mathbb{R}$ such that $\nabla u \neq 0$ on Σ^s ,

$$\vartheta^s = |N_{\partial E^s}^{H\Sigma^s}|_g \quad \text{on } \Lambda^s. \tag{2-24}$$

Using (2-24), (2-19), and (2-20), formula (2-23) becomes

$$\begin{aligned} \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} h \, d\lambda_{\partial E \cap \Sigma^s} \, ds &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} |N_{\partial E^s}^{H\Sigma^s}|_g \, d\lambda_{\partial E^s} \, ds \\ &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \\ &= \int_{\mathbb{R}} \int_{\partial E \cap \Sigma^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds. \end{aligned} \tag{2-25}$$

The proof is complete, because (2-15) follows from (2-16), (2-22), and (2-25).

Step 3. We prove claim (2-24). Let us introduce the vector field W in $\Omega \setminus \{\nabla_H u = 0\}$,

$$W = \frac{Tu}{|\nabla u|_g} \frac{\nabla_H u}{|\nabla_H u|_g} - \frac{|\nabla_H u|_g}{|\nabla u|_g} T.$$

It can be checked that $|W|_g = 1$ and $Wu = 0$. In particular, for a.e. s we have $W \in T\Sigma^s$. Moreover, W is g -orthogonal to $H\Sigma^s$ because any vector in $H\Sigma^s$ is orthogonal both to $\nabla_H u$ and to T . It follows that

$$N_{\partial E^s}^{H\Sigma^s} = N_{\partial E^s} - \langle N_{\partial E^s}, W \rangle_g$$

and, in particular,

$$|N_{\partial E^s}^{H\Sigma^s}|_g^2 = 1 - \langle N_{\partial E^s}, W \rangle_g^2.$$

Starting from the formula

$$N_{\partial E^s} = \frac{N_{\partial E} - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g \nabla u / |\nabla u|_g}{|N_{\partial E} - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g \nabla u / |\nabla u|_g|} = \frac{N_{\partial E} - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g \nabla u / |\nabla u|_g}{\sqrt{1 - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g^2}},$$

we find

$$|N_{\partial E^s}^{H\Sigma^s}|_g^2 = \frac{M}{1 - \langle N_{\partial E}, \nabla u / |\nabla u|_g \rangle_g^2},$$

where we let

$$M = 1 - \left\langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \right\rangle_g^2 - \left\langle N_{\partial E} - \left\langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \right\rangle_g \frac{\nabla u}{|\nabla u|_g}, W \right\rangle_g^2.$$

We claim that, on the open set $\{\nabla_H u \neq 0\}$,

$$M = |N_{\partial E}^H|_g^2 - \left\langle N_{\partial E}^H, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2, \tag{2-26}$$

and formula (2-24) follows from (2-26). Using the identity $\nabla u = \nabla_H u + (Tu)T$ and the orthogonality

$$\left\langle N_{\partial E} - \left\langle N_{\partial E}, \frac{\nabla u}{|\nabla u|_g} \right\rangle_g \frac{\nabla u}{|\nabla u|_g}, \nabla u \right\rangle_g = 0,$$

we find

$$\begin{aligned}
 M &= 1 - \left\langle N_{\partial E}, \frac{\nabla_H u + (Tu)T}{|\nabla u|_g} \right\rangle_g^2 - \left(\frac{Tu}{|\nabla u|_g} \left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g - \frac{|\nabla_H u|_g}{|\nabla u|_g} \langle N_{\partial E}, T \rangle_g \right)^2 \\
 &= 1 - \left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2 \frac{|\nabla_H u|_g^2 + (Tu)^2}{|\nabla u|_g^2} - \langle N_{\partial E}, T \rangle_g^2 \frac{|\nabla_H u|_g^2 + (Tu)^2}{|\nabla u|_g^2} \\
 &= 1 - \left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2 - \langle N_{\partial E}, T \rangle_g^2 \\
 &= 1 - \langle N_{\partial E}, T \rangle_g^2 - \left(\left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g - \langle N_{\partial E}, T \rangle_g T, \frac{\nabla_H u}{|\nabla_H u|_g} \right)_g^2 \\
 &= |N_{\partial E}^H|_g^2 - \left\langle N_{\partial E}^H, \frac{\nabla_H u}{|\nabla_H u|_g} \right\rangle_g^2. \tag{2-27}
 \end{aligned}$$

This ends the proof. □

We prove a coarea inequality:

Proposition 2.5. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ a set with finite H -perimeter in Ω , and let $h : \partial E \rightarrow [0, \infty]$ be a Borel function. Then we have*

$$\int_{\mathbb{R}} \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_{\Omega} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \tag{2-28}$$

Proof. The coarea inequality (2-28) follows from the smooth case of Lemma 2.4 by an approximation and lower semicontinuity argument.

Step 1. By [Franchi et al. 1996, Theorem 2.2.2], there exists a sequence of smooth sets $(E_j)_{j \in \mathbb{N}}$ in Ω such that

$$\chi_{E_j} \xrightarrow{L^1(\Omega)} \chi_E \quad \text{as } j \rightarrow \infty \quad \text{and} \quad \lim_{j \rightarrow \infty} \mu_{E_j}(\Omega) = \mu(\Omega).$$

By a straightforward adaptation of the proof of [Ambrosio et al. 2000, Proposition 3.13], we also have that $\nu_{E_j} \mu_{E_j} \rightarrow \nu_E \mu_E$ weakly* in Ω . Namely, for any $\psi \in C_c(\Omega; H)$,

$$\lim_{j \rightarrow \infty} \int_{\Omega} \langle \psi, \nu_{E_j} \rangle_g d\mu_{E_j} = \int_{\Omega} \langle \psi, \nu_E \rangle_g d\mu_E.$$

Let $A \Subset \Omega$ be an open set such that $\lim_{j \rightarrow \infty} \mu_{E_j}(A) = \mu_E(A)$. By Reshetnyak’s continuity theorem (see, e.g., [Ambrosio et al. 2000, Theorem 2.39]), we have

$$\lim_{j \rightarrow \infty} \int_A f(p, \nu_{E_j}(p)) d\mu_{E_j} = \int_A f(p, \nu_E(p)) d\mu_E$$

for any continuous and bounded function f . In particular,

$$\lim_{j \rightarrow \infty} \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_{E_j}, \nabla_H u \rangle_g^2} d\mu_{E_j} = \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \tag{2-29}$$

Step 2. Let $(E_j)_{j \in \mathbb{N}}$ be the sequence introduced in Step 1. Then, for a.e. $s \in \mathbb{R}$, we have

$$\nabla u \neq 0 \quad \text{on } \Sigma^s \quad \text{and} \quad \chi_{E_j} \rightarrow \chi_E \quad \text{in } L^1(\Sigma^s, \sigma_{\Sigma^s}) \quad \text{as } j \rightarrow \infty.$$

In particular, for any such s and for any open set $A \subset \Sigma^s \cap \Omega$,

$$\mu_E^s(A) \leq \liminf_{j \rightarrow \infty} \mu_{E_j}^s(A).$$

From Fatou’s lemma and the continuity of $|\nabla_H u|_g/|\nabla u|_g$ on Σ^s , it follows that

$$\begin{aligned} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s &= \int_0^\infty \mu_E^s \left(\left\{ p \in A : \frac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t \right\} \right) dt \\ &\leq \int_0^\infty \liminf_{j \rightarrow \infty} \mu_{E_j}^s \left(\left\{ p \in A : \frac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t \right\} \right) dt \\ &\leq \liminf_{j \rightarrow \infty} \int_0^\infty \mu_{E_j}^s \left(\left\{ p \in A : \frac{|\nabla_H u|_g}{|\nabla u|_g}(p) > t \right\} \right) dt \\ &= \liminf_{j \rightarrow \infty} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s. \end{aligned}$$

Using again Fatou’s lemma and Lemma 2.4,

$$\begin{aligned} \int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq \int_{\mathbb{R}} \liminf_{j \rightarrow \infty} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s ds \\ &\leq \liminf_{j \rightarrow \infty} \int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s ds \\ &= \liminf_{j \rightarrow \infty} \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_{E_j}, \nabla_H u \rangle_g^2} d\mu_{E_j}. \end{aligned}$$

This, together with (2-29), gives

$$\int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

Step 3. Any open set $A \subset \Omega$ can be approximated by a sequence $(A_k)_{k \in \mathbb{N}}$ of open sets such that

$$A_k \Subset \Omega, \quad A_k \subset A_{k+1}, \quad \bigcup_{k=1}^\infty A_k = A \quad \text{and} \quad \mu_E(\partial A_k) = 0.$$

In particular, for each $k \in \mathbb{N}$, we have

$$\liminf_{j \rightarrow \infty} \mu_{E_j}(A_k) \leq \limsup_{j \rightarrow \infty} \mu_{E_j}(\bar{A}_k) \leq \mu_E(\bar{A}_k) = \mu_E(A_k) \leq \liminf_{j \rightarrow \infty} \mu_{E_j}(A_k).$$

Hence, the inequalities are equalities, i.e., $\mu_E(A_k) = \lim_{j \rightarrow \infty} \mu_{E_j}(A_k)$. By Step 2, for any $k \in \mathbb{N}$,

$$\int_{\mathbb{R}} \int_{A_k} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_{A_k} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

By monotone convergence, letting $k \rightarrow \infty$ we obtain, for any open set $A \subset \Omega$,

$$\int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_A \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E.$$

By a standard approximation argument, it is enough to prove (2-28) for the characteristic function $h = \chi_B$ of a Borel set $B \subset \partial E$. Since the measure $\sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} \mu_E$ is a Radon measure on ∂E , there exists a sequence of open sets B_j such that $B \subset B_j$ for each $j \in \mathbb{N}$ and

$$\lim_{j \rightarrow \infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E = \int_B \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E.$$

Therefore, we have

$$\begin{aligned} \int_{\mathbb{R}} \int_B \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq \liminf_{j \rightarrow \infty} \int_{\mathbb{R}} \int_{B_j} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \\ &\leq \lim_{j \rightarrow \infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= \int_B \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E, \end{aligned}$$

and this concludes the proof. □

In the next step, we prove an approximate coarea formula for sets E such that the boundary ∂E is an H -regular surface.

Lemma 2.6. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ an open set such that $\partial E \cap \Omega$ is an H -regular hypersurface, and $\bar{p} \in \partial E \cap \Omega$ a point such that*

$$\nabla_H u(\bar{p}) \neq 0 \quad \text{and} \quad v_E(\bar{p}) \neq \pm \frac{\nabla_H u(\bar{p})}{|\nabla_H u(\bar{p})|_g}.$$

Then, for any $\varepsilon > 0$, there exists $\bar{r} = \bar{r}(\bar{p}, \varepsilon) > 0$ such that $B_{\bar{r}}(\bar{p}) \subset \Omega$ and, for any $r \in (0, \bar{r})$,

$$\begin{aligned} (1 - \varepsilon) \int_{B_r(\bar{p})} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E &\leq \int_{\mathbb{R}} \int_{B_r(\bar{p})} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \\ &\leq (1 + \varepsilon) \int_{B_r(\bar{p})} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E. \end{aligned}$$

Proof. We can, without loss of generality, assume that $\bar{p} = 0$ and $u(0) = 0$. We divide the proof into several steps.

Step 1: preliminary considerations. The horizontal vector field $V_{2n} = \nabla_H u / |\nabla_H u|_g$ is well defined in a neighbourhood $\Omega_\varepsilon \subset \mathbb{H}^n$ of 0. For any $s \in \mathbb{R}$, the hypersurface $\Sigma^s = \{p \in \Omega : u(p) = s\}$ is smooth in Ω_ε because $\nabla_H u \neq 0$ on Ω_ε .

There are horizontal vector fields V_1, \dots, V_{2n-1} on Ω_ε such that V_1, \dots, V_{2n} is a g -orthonormal frame. In particular, we have $V_j u = 0$ for $j = 1, \dots, 2n - 1$, i.e.,

$$H_p \Sigma^s = \text{span}\{V_1(p), \dots, V_{2n-1}(p)\} \quad \text{for all } p \in \Sigma^s \cap \Omega_\varepsilon. \tag{2-30}$$

Possibly shrinking Ω_ε , reordering $\{V_j\}_{j=1,\dots,2n-1}$, and changing the sign of V_1 , we can assume (see [Vittone 2012, Lemmas 4.3 and 4.4]) that there exist a function $f : \Omega_\varepsilon \rightarrow \mathbb{R}$ and a number $\delta > 0$ such that:

- (a) $f \in C^1_H(\Omega_\varepsilon) \cap C^\infty(\Omega_\varepsilon \setminus \partial E)$;
- (b) $E \cap \Omega_\varepsilon = \{p \in \Omega_\varepsilon : f(p) > 0\}$;
- (c) $V_1 f \geq \delta > 0$ on Ω_ε .

By [Vittone 2012, Remark 4.7], we also have $v_E = \nabla_H f / |\nabla_H f|_g$ on $\partial E \cap \Omega_\varepsilon$.

Step 2: change of coordinates. Let $S \subset \mathbb{H}^n$ be a $(2n-1)$ -dimensional smooth submanifold such that:

- (i) $0 \in S$.
- (ii) $S \subset \Sigma^0 \cap \Omega_\varepsilon$. In particular, ∇u is g -orthogonal to S .
- (iii) $V_1(0)$ is g -orthogonal to S at 0 .
- (iv) There exists a diffeomorphism $H : U \rightarrow \mathbb{H}^n$, where $U \subset \mathbb{R}^{2n-1}$ is an open set with $0 \in U$, such that $H(0) = 0$ and $H(U) = S \cap \Omega_\varepsilon$.
- (v) The area element JH of H satisfies $JH(0) = 1$. Namely,

$$JH(0) = \lim_{r \rightarrow 0} \frac{\lambda_S(H(B_r^E))}{\mathcal{L}^{2n-1}(B_r^E)} = 1,$$

where $B_r^E = \{p \in \mathbb{R}^{2n-1} : |p| < r\}$ is a Euclidean ball and λ_S is the Riemannian $(2n-1)$ -volume measure on S induced by g .

For small enough $a, b > 0$, and possibly shrinking U and Ω_ε , the mapping $G : (-a, a) \times (-b, b) \times U \rightarrow \mathbb{H}^n$,

$$G(v, z, w) = \exp(vV_1) \exp\left(z \frac{\nabla u}{|\nabla u|_g^2}\right)(H(w))$$

is a diffeomorphism from $\tilde{\Omega}_\varepsilon = (-a, a) \times (-b, b) \times U$ onto Ω_ε . The differential of G satisfies

$$dG\left(\frac{\partial}{\partial v}\right) = V_1 \quad \text{and} \quad dG(0)\left(\frac{\partial}{\partial z}\right) = \frac{\nabla u(0)}{|\nabla u(0)|_g^2}.$$

Moreover, the tangent space $T_0S = \text{Im } dH(0)$ is g -orthogonal to $V_1(0)$ and $\nabla u(0)/|\nabla u(0)|_g^2$. We denote by G_z the restriction of G to $(-a, a) \times \{z\} \times U$, i.e., $G_z(v, w) = G(v, z, w)$. From the above considerations, we deduce that the area elements of G and G_0 satisfy

$$JG(0) = \frac{1}{|\nabla u(0)|_g} \quad \text{and} \quad JG_0(0) = 1.$$

Then, possibly shrinking $\tilde{\Omega}_\varepsilon$ further, we have

$$(1 - \varepsilon)JG(v, z, w) \leq \frac{JG_z(v, w)}{|\nabla u \circ G(v, z, w)|_g} \leq (1 + \varepsilon)JG(v, z, w) \tag{2-31}$$

for all $(v, z, w) \in \tilde{\Omega}_\varepsilon$.

For $j = 1, \dots, 2n$, we define on $\tilde{\Omega}_\varepsilon$ the vector fields $\tilde{V}_j = (dG)^{-1}(V_j)$. By the definition of G , we have $\tilde{V}_1 = \partial/\partial v$. We also define $\tilde{u} = u \circ G \in C^\infty(\tilde{\Omega}_\varepsilon)$, $\tilde{f} = f \circ G : \tilde{\Omega}_\varepsilon \rightarrow \mathbb{R}$, and $\tilde{E} = G^{-1}(E)$. Then:

- (1) $\tilde{E} = \{q \in \tilde{\Omega}_\varepsilon : \tilde{f}(q) > 0\}$.
- (2) $\tilde{f} \in C^\infty(\tilde{\Omega}_\varepsilon \setminus \partial\tilde{E})$.
- (3) The derivative $\tilde{V}_j \tilde{f}$ is defined in the sense of distributions with respect to the measure $\mu = JG \mathcal{L}^{2n+1}$. Namely, for all $\psi \in C_c^\infty(\tilde{\Omega}_\varepsilon)$, we have

$$\int_{\tilde{\Omega}_\varepsilon} (\tilde{V}_j \tilde{f}) \psi \, d\mu = - \int_{\tilde{\Omega}_\varepsilon} \tilde{f} \tilde{V}_j^* \psi \, d\mu,$$

where \tilde{V}_j^* is the adjoint operator of \tilde{V}_j with respect to μ . Then $\tilde{V}_j \tilde{f} = (V_j f) \circ G$ and so $\tilde{V}_j \tilde{f}$ is a continuous function for any $j = 1, \dots, 2n$. In particular, $\tilde{V}_1 \tilde{f} = \partial_v \tilde{f} \geq \delta > 0$.

Step 3: approximate coarea formula. We follow the argument of [Vittone 2012, Propositions 4.1 and 4.5]; see also Remark 4.7 therein.

Possibly shrinking $\tilde{\Omega}_\varepsilon$ and Ω_ε , there exists a continuous function $\phi : (-b, b) \times U \rightarrow (-a, a)$ such that:

- (A) $\partial\tilde{E} \cap \tilde{\Omega}_\varepsilon$ is the graph of ϕ . Namely, letting $\Phi : (-b, b) \times U \rightarrow \mathbb{R}^{2n+1}$, $\Phi(z, w) = (\phi(z, w), z, w)$, we have

$$\partial\tilde{E} \cap \tilde{\Omega}_\varepsilon = \Phi((-b, b) \times U).$$

- (B) The measure μ_E is

$$\mu_{E \llcorner \Omega_\varepsilon} = (G \circ \Phi)_\# \left(\left(\frac{|\tilde{V} \tilde{f}|}{\tilde{V}_1 \tilde{f}} JG \right) \circ \Phi \mathcal{L}^{2n} \llcorner ((-b, b) \times U) \right), \tag{2-32}$$

where $(G \circ \Phi)_\#$ denotes the push-forward and

$$|\tilde{V} \tilde{f}| = \left(\sum_{j=1}^{2n} (\tilde{V}_j \tilde{f})^2 \right)^{\frac{1}{2}}.$$

Using $V_1 u = 0$ and $u \circ H = 0$ (this follows from $H(U) = S \cap \Omega_\varepsilon \subset \Sigma^0 \cap \Omega_\varepsilon$), we obtain

$$\begin{aligned} \tilde{u}(v, z, w) &= u(G(v, z, w)) = u \left(\exp(v V_1) \exp \left(z \frac{\nabla u}{|\nabla u|_g^2} \right) (H(w)) \right) = u \left(\exp \left(z \frac{\nabla u}{|\nabla u|_g^2} \right) (H(w)) \right) \\ &= z + u(H(w)) = z. \end{aligned}$$

In particular, from $\tilde{u} = u \circ G$, we deduce that

$$G^{-1}(\Sigma^s \cap \Omega_\varepsilon) = (-a, a) \times \{s\} \times U.$$

We denote by JG_s the Jacobian (area element) of G_s . We also define the restriction $\Phi_s : U \rightarrow \mathbb{R}^{2n+1}$, $\Phi_s(w) = \Phi(s, w)$, for any $s \in (-b, b)$.

By (2-30), for any $s \in \mathbb{R}$, the measure $\mu_E^s = \mu_{E \cap \Sigma^s}^{\Sigma^s}$ is the horizontal perimeter of $E \cap \Sigma^s$ with respect to the Carnot–Carathéodory structure induced by the family V_1, \dots, V_{2n-1} on Σ^s . We can repeat the

argument that led to (2-32) to obtain

$$\mu_E^s \llcorner \Omega_\varepsilon = (G \circ \Phi_s) \# \left(\left(\frac{|\tilde{V}'\tilde{f}|}{\tilde{V}_1\tilde{f}} JG_s \right) \circ \Phi_s \mathcal{L}^{2n-1} \llcorner U \right), \tag{2-33}$$

where $\tilde{V}'\tilde{f} = (\tilde{V}_1\tilde{f}, \dots, \tilde{V}_{2n-1}\tilde{f})$. We omit the details of the proof of (2-33). The proof is a line-by-line repetition of Proposition 4.5 in [Vittone 2012] with the sole difference that now the horizontal perimeter is defined in a curved manifold.

Let us fix $\bar{r} > 0$ such that $B_{\bar{r}} \subset \Omega_\varepsilon$ and, for any $r \in (0, \bar{r})$, let

$$A_{s,r} = \{w \in U : G(0, s, w) \in B_r\} \quad \text{and} \quad A_r = \{(s, w) \in (-b, b) \times U : w \in A_{s,r}\}.$$

By the Fubini–Tonelli theorem and (2-33), the function

$$s \mapsto \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s = \int_{A_{s,r}} \left(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \right) \left(\frac{|\tilde{V}'\tilde{f}|}{\tilde{V}_1\tilde{f}} JG_s \right) \circ \Phi_s d\mathcal{L}^{2n-1} \tag{2-34}$$

is \mathcal{L}^1 -measurable. Here and hereafter, the composition $\cdot \circ \Phi_s$ acts on the product. Thus, from the Fubini–Tonelli theorem and (2-31), we obtain

$$\begin{aligned} & \int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \\ &= \int_{\mathbb{R}} \int_{A_{s,r}} \left(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \right) \left(\frac{|\tilde{V}'\tilde{f}|}{\tilde{V}_1\tilde{f}} JG_s \right) \circ \Phi_s(w) d\mathcal{L}^{2n-1}(w) ds \\ &= \int_{A_r} (|\nabla_H u|_g \circ G) \left(\frac{|\tilde{V}'\tilde{f}|}{\tilde{V}_1\tilde{f}} \frac{JG_s}{|\nabla u|_g \circ G} \right) \circ \Phi(s, w) d\mathcal{L}^{2n}(s, w) \\ &\leq (1 + \varepsilon) \int_{A_r} (|\nabla_H u|_g \circ G) \left(\frac{|\tilde{V}'\tilde{f}|}{\tilde{V}_1\tilde{f}} \sqrt{1 - (\tilde{V}_{2n}\tilde{f})^2 / |\tilde{V}'\tilde{f}|^2} JG \right) \circ \Phi(s, w) d\mathcal{L}^{2n}(s, w). \end{aligned} \tag{2-35}$$

From the identity

$$\frac{\tilde{V}_{2n}\tilde{f}}{|\tilde{V}'\tilde{f}|} = \frac{V_{2n}f}{|\nabla_H f|_g} \circ G = \left\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \frac{\nabla_H f}{|\nabla_H f|_g} \right\rangle_g \circ G = \left\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \nu_E \right\rangle_g \tag{2-36}$$

and from (2-32), we deduce that

$$\begin{aligned} \int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq (1 + \varepsilon) \int_{B_r} |\nabla_H u|_g \sqrt{1 - \langle \nabla_H u / |\nabla_H u|_g, \nu_E \rangle_g^2} d\mu_E \\ &= (1 + \varepsilon) \int_{B_r} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E. \end{aligned} \tag{2-37}$$

In a similar way, we obtain

$$\int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \geq (1 - \varepsilon) \int_{B_r} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} d\mu_E.$$

This concludes the proof. □

We can now prove the coarea formula for H -regular boundaries.

Proposition 2.7. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$, and let $E \subset \mathbb{H}^n$ be an open domain such that $\partial E \cap \Omega$ is an H -regular hypersurface. Then*

$$\int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds = \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E. \tag{2-38}$$

Proof. Let us define the set

$$A = \left\{ p \in \partial E \cap \Omega : \nabla_H u(p) \neq 0 \text{ and } v_E(p) \neq \pm \frac{\nabla_H u(p)}{|\nabla_H u(p)|_g} \right\}.$$

The set A is relatively open in $\partial E \cap \Omega$. Let $\varepsilon > 0$ be fixed. Since the measure μ_E is locally doubling on $\partial E \cap \Omega$ (see, e.g., [Vittone 2012, Corollary 4.13]), by Lemma 2.6 and the Vitali covering theorem (see, e.g., [Heinonen 2001, Theorem 1.6]) there exists a countable (or finite) collection of balls $B_{r_i}(p_i)$, $i \in \mathbb{N}$, such that:

- (i) for any $i \in \mathbb{N}$ we have $p_i \in A$ and $0 < r_i < \bar{r}(p_i, \varepsilon)$, where \bar{r} is as in the statement of Lemma 2.6;
- (ii) the balls $B_{r_i}(p_i)$ are contained in A and pairwise disjoint;
- (iii) $\mu_E(A \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)) = 0$.

It follows that we have

$$\begin{aligned} \int_{\mathbb{R}} \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds &\leq (1 + \varepsilon) \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= (1 + \varepsilon) \int_A \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E \\ &= (1 + \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E. \end{aligned} \tag{2-39}$$

The last equality follows from the fact that $\sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} = 0$ outside A . In the same way, one also obtains

$$\int_{\mathbb{R}} \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \geq (1 - \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E. \tag{2-40}$$

Moreover, by Proposition 2.5,

$$\int_{\mathbb{R}} \int_{\Omega \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq \int_{\Omega \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E = 0.$$

In particular, the integral on the left-hand side of the last inequality is 0 and, by (2-39) and (2-40), we obtain

$$(1 - \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E \leq \int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds \leq (1 + \varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E.$$

Since $\varepsilon > 0$ is arbitrary, this concludes the proof. □

By a standard approximation argument, we also have this extension of the coarea formula (2-38):

Proposition 2.8. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$, and let E be an open domain such that $\partial E \cap \Omega$ is an H -regular hypersurface. Then, for any Borel function $h : \partial E \rightarrow [0, \infty)$,*

$$\int_{\mathbb{R}} \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_E^s ds = \int_{\Omega} h \sqrt{|\nabla_H u|_g^2 - \langle v_E, \nabla_H u \rangle_g^2} d\mu_E.$$

Our next step is to prove the coarea formula for \mathcal{S}^{2n+1} -rectifiable sets.

Lemma 2.9. *Let $R \subset \mathbb{H}^n$ be an \mathcal{S}^{2n+1} -rectifiable set. Then, there exists a Borel \mathcal{S}^{2n+1} -rectifiable set $R' \subset \mathbb{H}^n$ such that $\mathcal{S}^{2n+1}(R \Delta R') = 0$.*

Proof. By assumption, there exist a \mathcal{S}^{2n+1} -negligible set N and H -regular hypersurfaces $S_j \subset \mathbb{H}^n$, $j \in \mathbb{N}$, such that

$$R \subset N \cup \bigcup_{j=1}^{\infty} S_j.$$

It is proved in [Franchi et al. 2001; Ambrosio et al. 2006] that (up to a localization argument), for any $j \in \mathbb{N}$, there exist an open set $U_j \subset \mathbb{R}^{2n}$, a homeomorphism $\Phi_j : U_j \rightarrow S_j$, and a continuous function $\rho_j : U_j \rightarrow [1, \infty)$ such that $\mathcal{S}^{2n+1} \llcorner S_j = \Phi_{j\#}(\rho_j \mathcal{L}^{2n} \llcorner U_j)$. Since the Lebesgue measure \mathcal{L}^{2n} is a complete Borel measure, for any $j \in \mathbb{N}$ there exists a Borel set $T_j \subset U_j$ such that

$$\mathcal{L}^{2n}(T_j \Delta \Phi_j^{-1}(R \cap S_j)) = 0.$$

In particular, the Borel set

$$R' = \bigcup_{j=1}^{\infty} \Phi_j(T_j)$$

is \mathcal{S}^{2n+1} -equivalent to R . □

Proof of Theorem 1.5. Step 1. We prove (1-8) when R is an H -regular hypersurface. Then, R is locally the boundary of an open set $E \subset \mathbb{H}^n$ with H -regular boundary. Moreover, we have (locally) $\mu_E = \mathcal{S}^{2n+1} \llcorner R$ and $v_E = v_R$, up to the sign.

We define the measures $\mu_R^s = \mu_E^s$ for any s such that $\nabla u \neq 0$ on Σ^s . The measurability of the function in (1-7) follows from the argument (2-34). Formula (1-8) follows from Proposition 2.8.

Step 2. We prove (1-8) when R is an \mathcal{S}^{2n+1} -rectifiable Borel set. There exist an \mathcal{S}^{2n+1} -negligible set N and H -regular hypersurfaces $S_j \subset \mathbb{H}^n$, $j \in \mathbb{N}$, such that

$$R \subset N \cup \bigcup_{j=1}^{\infty} S_j.$$

Each S_j is (locally) the boundary of an open set E_j with H -regular boundary. We denote by $\mu_{E_j}^s$ the perimeter measure on $\partial E_j \cap \Sigma^s$ induced by E_j .

We define the pairwise disjoint Borel sets $R_j = (R \cap S_j) \setminus \bigcup_{h=1}^{j-1} S_h$ and we let

$$\mu_R^s = \sum_{j=1}^{\infty} \mu_{E_j}^s \llcorner R_j.$$

The definition is well posed for any s such that $\nabla u \neq 0$ on Σ^s . We have $\nu_R = \pm \nu_{E_j}$ \mathcal{S}^{2n+1} -a.e. on R_j and the sign of ν_R does not affect (1-8). From Step 1, for each $j \in \mathbb{N}$ the function

$$s \mapsto \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s$$

is \mathcal{L}^1 -measurable; here, we were allowed to utilize Step 1 because χ_{R_j} is Borel regular. Thus also the function

$$s \mapsto \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s = \sum_{j=1}^{\infty} \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s$$

is measurable. Moreover, we have

$$\begin{aligned} \int_{\mathbb{R}} \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_R^s ds &= \sum_{j=1}^{\infty} \int_{\mathbb{R}} \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} d\mu_{E_j}^s ds \\ &= \sum_{j=1}^{\infty} \int_{R_j} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} d\mathcal{S}^{2n+1} \\ &= \int_R h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} d\mathcal{S}^{2n+1}. \end{aligned}$$

Step 3. Finally, if R is \mathcal{S}^{2n+1} -rectifiable but not Borel, we set $\mu_R^s = \mu_{R'}^s$, where R' is a Borel set as in Lemma 2.9. Again, this definition is well posed for a.e. $s \in \mathbb{R}$. This concludes the proof. \square

2C. Proof of Theorem 1.6. In this subsection we assume $n \geq 2$.

Lemma 2.10. *For $n \geq 2$, let $\Omega \subset \mathbb{H}^m$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $R \subset \Omega$ an \mathcal{S}^{2n+1} -rectifiable set. Then*

$$\mathcal{S}^{2n+1}(\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\}) = 0.$$

Proof. It is enough to prove the lemma when R is an H -regular hypersurface. Let

$$A = \{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\}.$$

We claim that $\mathcal{S}^{2n+1}(A) = 0$.

Let $p \in A$ be a fixed point and let $\nu_R(p)$ be the horizontal normal to R at p . Since $n \geq 2$, we have

$$\dim\{V(p) \in H_p : \langle V(p), \nu_R(p) \rangle_g = 0\} = 2n - 1 \geq n + 1.$$

Thus there exist left-invariant horizontal vector fields V and W such that

$$\langle V(p), \nu_R(p) \rangle_g = \langle W(p), \nu_R(p) \rangle_g = 0 \quad \text{and} \quad [V, W] = T.$$

From $\nabla_H u(p) = 0$ and $\nabla u(p) \neq 0$, we deduce that $Tu(p) \neq 0$. It follows that

$$VWu(p) - WVu(p) = Tu(p) \neq 0$$

and, in particular, we have either $VWu(p) \neq 0$ or $WVu(p) \neq 0$. Without loss of generality, we assume that $VWu(p) \neq 0$. Then the set $S = \{q \in \Omega : Wu(q) = 0\}$ is an H -regular hypersurface near the point $p \in S$. Since we have

$$\langle V(p), \nu_R(p) \rangle_g = 0 \quad \text{and} \quad \langle V(p), \nu_S(p) \rangle_g = \frac{VWu(p)}{|\nabla_H Wu(p)|_g} \neq 0,$$

we deduce that $\nu_R(p)$ and $\nu_S(p)$ are linearly independent. Then there exists $r > 0$ such that the set $R \cap S \cap B_r(p)$ is a 2-codimensional H -regular surface (see [Franchi et al. 2007]). Therefore, by [Franchi et al. 2007, Corollary 4.4], the Hausdorff dimension in the Carnot–Carathéodory metric of $A \cap B_r(p) \subset R \cap S \cap B_r(p)$ is not greater than $2n$. This is enough to conclude. \square

Remark 2.11. Lemma 2.10 is not valid if $n = 1$. Consider the smooth surface $R = \{(x, y, t) \in \mathbb{H}^1 : x = 0\}$ and the function $u(x, y, t) = t - 2xy$. We have

$$\nabla u = -4xY + T \quad \text{and} \quad \nabla_H u = -4xY.$$

Then we have

$$\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\} = R$$

and $\mathcal{S}^3(R) = \infty$.

If $n \geq 2$ and Ω, u , and R are as in Lemma 2.10, then the function

$$|\nabla u|_g \sqrt{1 - \langle \nu_E, \nabla_H u / |\nabla_H u|_g \rangle_g^2}$$

is defined \mathcal{S}^{2n+1} -a.e. on R . We agree that its value is 0 when $|\nabla u|_g = 0$. Notice that, in this case, $\nabla_H u / |\nabla_H u|_g$ is not defined.

Proof of Theorem 1.6. Let $\varepsilon > 0$ be fixed. Then (1-9) can be obtained by plugging the function $(|\nabla u|_g / (\varepsilon + |\nabla_H u|_g)) h$ into (1-8), letting $\varepsilon \rightarrow 0$ and using the monotone convergence theorem. \square

3. Height estimate

In this section, we prove Theorem 1.3. We discuss first a relative isoperimetric inequality on slices. Then we list some elementary properties of excess, and finally we proceed with the proof.

We assume throughout this section that $n \geq 2$.

3A. Relative isoperimetric inequalities. For each $s \in \mathbb{R}$, we define the level sets of the height function,

$$\mathbb{H}_s^n = \{p \in \mathbb{H}^n : \xi(p) = s\}.$$

Let H^s be the g -orthogonal projection of H onto the tangent space of \mathbb{H}_s^n . Since the vector field X_1 is orthogonal to \mathbb{H}_s^n , while the vector fields $X_2, \dots, X_n, Y_1, \dots, Y_n$ are tangent to \mathbb{H}_s^n , at any point $p \in \mathbb{H}_s^n$

we have

$$H^s = \text{span}\{X_2(p), \dots, X_n(p), Y_1^s(p), Y_2(p), \dots, Y_n(p)\},$$

where $X_2, Y_2, \dots, X_n, Y_n$ are as in (1-2) and

$$Y_1^s = \frac{\partial}{\partial y_1} - 2s \frac{\partial}{\partial t}.$$

The natural volume in \mathbb{H}_s^n is the Lebesgue measure \mathcal{L}^{2n} . For any measurable set $F \subset \mathbb{H}_s^n$ and any open set $\Omega \subset \mathbb{H}_s^n$, we define

$$\mu_F^s(\Omega) = \sup \left\{ \int_F \text{div}_g^s \varphi \, d\mathcal{L}^{2n} : \varphi \in C_c^1(\Omega; H^s), \|\varphi\|_g \leq 1 \right\},$$

where $\text{div}_g^s \varphi = X_2\varphi_2 + \dots + X_n\varphi_n + Y_1^s\varphi_{n+1} + \dots + Y_n\varphi_{2n}$. If $\mu_F^s(\Omega) < \infty$ then μ_F^s is a Radon measure in Ω .

By Theorem 1.6, for any Borel function $h : \mathbb{H}^n \rightarrow [0, \infty)$ and any set E with locally finite H -perimeter in \mathbb{H}^n , we have the coarea formula

$$\int_{\mathbb{R}} \int_{\mathbb{H}_s^n} h \, d\mu_{E^s}^s \, ds = \int_{\mathbb{H}^n} h \sqrt{1 - \langle \nu_E, X_1 \rangle_g^2} \, d\mu_E, \tag{3-41}$$

where $E^s = E \cap \mathbb{H}_s^n$ is the section of E with \mathbb{H}_s^n . Notice that $\nabla_{H\mathbb{H}} \ell = X_1$.

In the proof of Theorem 1.3, we need a relative isoperimetric inequality in each slice \mathbb{H}_s^n for $s \in (-1, 1)$. These slices are cosets of $\mathbb{W} = \mathbb{H}_0^n$ and the isoperimetric inequalities in \mathbb{H}_s^n can be reduced to an isoperimetric inequality in the central slice $\mathbb{W} = \mathbb{H}_0^n$ relative to a family of varying domains.

For any $s \in (-1, 1)$, let $\Omega_s \subset \mathbb{W}$ be the set $\Omega_s = (-se_1) * D_1 * (se_1)$. This is the left translation by $-se_1$ of the section $C_1 \cap \mathbb{H}_s^n$. See p. 1423 in the introduction for the definition of D_1 and C_1 . With the coordinates $(y_1, \hat{z}, t) \in \mathbb{W} = \mathbb{R} \times \mathbb{C}^{n-1} \times \mathbb{R}$, we have

$$\Omega_s = \{(y_1, \hat{z}, t) \in \mathbb{W} : (y_1^2 + |\hat{z}|^2)^2 + (t - 4sy_1)^2 < 1\}.$$

The sets $\Omega_s \subset \mathbb{W}$ are open and convex in the standard sense. The boundary $\partial\Omega_s$ is a $(2n-1)$ -dimensional C^∞ embedded surface with the following property: There are $4n$ open convex sets $U_1, \dots, U_{4n} \subset \mathbb{W}$ such that $\partial\Omega_s \subset \bigcup_{i=1}^{4n} U_i$ and, for each i , the portion of the boundary $\partial\Omega_s \cap U_i$ is a graph of the form $p_j = f_i^s(\hat{p}_j)$ with $j = 2, \dots, 2n+1$ and $\hat{p}_j = (p_2, \dots, p_{j-1}, p_{j+1}, \dots, p_{2n+1}) \in V_i$, where $V_i \subset \mathbb{R}^{2n-1}$ is an open convex set and $f_i^s \in C^\infty(V_i)$ is a function such that

$$|\nabla f_i^s(\hat{p}_j) - \nabla f_i^s(\hat{q}_j)| \leq K |\hat{p}_j - \hat{q}_j| \quad \text{for all } \hat{p}_j, \hat{q}_j \in V_i, \tag{3-42}$$

where $K > 0$ is a constant independent of $i = 1, \dots, 4n$ and independent of $s \in (-1, 1)$. In other words, the boundary $\partial\Omega_s$ is of class $C^{1,1}$ uniformly in $s \in (-1, 1)$.

By Theorem 3.2 in [Monti and Morbidelli 2005], the domain $\Omega_s \subset \mathbb{W}$ is a nontangentially accessible (NTA) domain in the metric space (\mathbb{W}, d_{CC}) , where d_{CC} is the Carnot–Carathéodory metric induced by the horizontal distribution H_p^0 . In particular, Ω_s is a (weak) John domain in the sense of [Hajlasz and

Koskela 2000]. Namely, there exist a point $p_0 \in \Omega_s$, e.g., $p_0 = 0$, and a constant $C_J > 0$ such that, for any point $p \in \Omega_s$, there exists a continuous curve $\gamma : [0, 1] \rightarrow \Omega_s$ such that $\gamma(1) = p_0$, $\gamma(0) = p$, and

$$d_{CC}(\gamma(\sigma), \partial\Omega_s) \geq C_J d_{CC}(\gamma(\sigma), p), \quad \sigma \in [0, 1]. \tag{3-43}$$

By Theorem 3.2 in [Monti and Morbidelli 2005], the John constant C_J depends only on the constant $K > 0$ in (3-42). This claim is not stated explicitly in Theorem 3.2 of [Monti and Morbidelli 2005] but it is evident from the proof. In particular, the John constant C_J is independent of $s \in (-1, 1)$. Then, by Theorem 1.22 in [Garofalo and Nhieu 1996], we have the following result:

Theorem 3.1. *Let $n \geq 2$. There exists a constant $C(n) > 0$ such that, for any $s \in (-1, 1)$ and any measurable set $F \subset \mathbb{W}$,*

$$\min\{\mathcal{L}^{2n}(F \cap \Omega_s), \mathcal{L}^{2n}(\Omega_s \setminus F)\}^{2n/(2n+1)} \leq C(n) \frac{\text{diam}_{CC}(\Omega_s)}{\mathcal{L}^{2n}(\Omega_s)^{1/(2n+1)}} \mu_F^0(\Omega_s). \tag{3-44}$$

An alternative proof of Theorem 3.1 can be obtained using the Sobolev–Poincaré inequalities proved in [Hajlasz and Koskela 2000] in the general setting of metric spaces.

The diameter $\text{diam}_{CC}(\Omega_s)$ is bounded for $s \in (-1, 1)$ and $\mathcal{L}^{2n}(\Omega_s) > 0$ is a constant independent of s . Then we obtain the following version of (3-44):

Corollary 3.2. *Let $n \geq 2$. For any $\tau \in (0, 1)$ there exists a constant $C(n, \tau) > 0$ such that, for $s \in (-1, 1)$ and any measurable set $F \subset \mathbb{W}$ satisfying*

$$\mathcal{L}^{2n}(F \cap \Omega_s) \leq \tau \mathcal{L}^{2n}(\Omega_s),$$

we have

$$\mu_F^0(\Omega_s) \geq C(n, \tau) \mathcal{L}^{2n}(F \cap \Omega_s)^{2n/(2n+1)}.$$

3B. Elementary properties of the excess. We list here, without proof, the most basic properties of the cylindrical excess introduced in Definition 1.2. Their proofs are easy adaptations of those for the classical excess; see, e.g., [Maggi 2012, Chapter 22]. Note that, except for property (3), they hold also in the case $n = 1$.

(1) For all $0 < r < s$, we have

$$\text{Exc}(E, r, \nu) \leq \left(\frac{s}{r}\right)^{2n+1} \text{Exc}(E, s, \nu). \tag{3-45}$$

(2) If $(E_j)_{j \in \mathbb{N}}$ is a sequence of sets with locally finite H -perimeter such that $E_j \rightarrow E$ as $j \rightarrow \infty$ in $L^1_{\text{loc}}(\mathbb{H}^n)$, then we have, for any $r > 0$,

$$\text{Exc}(E, r, \nu) \leq \liminf_{j \rightarrow \infty} \text{Exc}(E_j, r, \nu). \tag{3-46}$$

(3) Let $n \geq 2$. If $E \subset \mathbb{H}^n$ is a set such that $\text{Exc}(E, r, \nu) = 0$ and $0 \in \partial^*E$, then

$$E \cap C_r = \{p \in C_r : \xi(p) < 0\}. \tag{3-47}$$

In particular, we have $\nu_E = \nu$ in $C_r \cap \partial E$. See also [Monti 2014, Proposition 3.6].

(4) For any $\lambda > 0$ and $r > 0$, we have

$$\text{Exc}(\lambda E, \lambda r, \nu) = \text{Exc}(E, r, \nu), \tag{3-48}$$

where $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$.

3C. Proof of Theorem 1.3. The following result is a first, suboptimal version of Theorem 1.3.

Lemma 3.3. *Let $n \geq 2$. For any $s \in (0, 1)$, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$ with $\Lambda r \leq 1$, there exists a constant $\omega(n, s, \Lambda, r) > 0$ such that, if $E \subset \mathbb{H}^n$ is a (Λ, r) -minimizer of H -perimeter in the cylinder C_2 , $0 \in \partial E$, and $\text{Exc}(E, 2, \nu) \leq \omega(n, s, \Lambda, r)$, then*

$$\begin{aligned} |\xi(p)| &< s \quad \text{for any } p \in \partial E \cap C_1, \\ \mathcal{L}^{2n+1}(\{p \in E \cap C_1 : \xi(p) > s\}) &= 0, \\ \mathcal{L}^{2n+1}(\{p \in C_1 \setminus E : \xi(p) < -s\}) &= 0. \end{aligned}$$

Proof. By contradiction, assume that there exist $s \in (0, 1)$ and a sequence of sets $(E_j)_{j \in \mathbb{N}}$ that are (Λ, r) -minimizers in C_2 and such that

$$\lim_{j \rightarrow \infty} \text{Exc}(E_j, 2, \nu) = 0$$

and at least one of the following facts holds:

$$\text{there exists } p \in \partial E_j \cap C_1 \text{ such that } s \leq |\xi(p)| \leq 1, \tag{3-49}$$

$$\mathcal{L}^{2n+1}(\{p \in E_j \cap C_1 : \xi(p) > s\}) > 0, \tag{3-50}$$

$$\text{or } \mathcal{L}^{2n+1}(\{p \in C_1 \setminus E_j : \xi(p) < -s\}) > 0. \tag{3-51}$$

By Theorem A.3 in Appendix A, there exists a measurable set $F \subset C_{5/3}$ such that F is a (Λ, r) -minimizer in $C_{5/3}$, $0 \in \partial F$, and (possibly up to subsequences) $E_j \cap C_{5/3} \rightarrow F$ in $L^1(C_{5/3})$. By (3-46) and (3-45), we obtain

$$\text{Exc}(F, \frac{4}{3}, \nu) \leq \liminf_{j \rightarrow \infty} \text{Exc}(E_j, \frac{4}{3}, \nu) \leq \left(\frac{3}{2}\right)^{2n+1} \lim_{j \rightarrow \infty} \text{Exc}(E_j, 2, \nu) = 0.$$

Since $0 \in \partial F$, by (3-47) the set $F \cap C_{4/3}$ is (equivalent to) a halfspace with horizontal inner normal $\nu = -X_1$, namely,

$$F \cap C_{4/3} = \{p \in C_{4/3} : \xi(p) < 0\}.$$

Assume that (3-49) holds for infinitely many j . Then, up to a subsequence, there are points $(p_j)_{j \in \mathbb{N}}$ and p_0 such that

$$p_j \in \partial E_j \cap C_1, \quad |\xi(p_j)| \in (s, 1] \quad \text{and} \quad p_j \rightarrow p_0 \in \partial F \cap \bar{C}_1.$$

We used again Theorem A.3 in Appendix A. This is a contradiction because $\partial F \cap \bar{C}_1 = \{p \in \bar{C}_1 : \xi(p) = 0\}$. Here, we used $n \geq 2$. Therefore, there exists $j_0 \in \mathbb{N}$ such that

$$\{p \in \partial E_j \cap C_1 : s \leq |\xi(p)| \leq 1\} = \emptyset \quad \text{for all } j \geq j_0$$

and hence

$$\mu_{E_j}(C_1 \setminus \{p \in \mathbb{H}^n : |\xi(p)| \leq s\}) = 0.$$

This implies that, for $j \geq j_0$, χ_{E_j} is constant on the two connected components $C_1 \cap \{p : \xi(p) > s\}$ and $C_1 \cap \{p : \xi(p) < -s\}$. Since the sequence $(E_j)_{j \in \mathbb{N}}$ converges in $L^1(C_1)$ to the halfspace F , for any $j \geq j_0$ we have

$$\begin{aligned} \chi_{E_j} &= 0 \quad \mathcal{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \xi(p) > s\}, \\ \text{and } \chi_{E_j} &= 1 \quad \mathcal{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \xi(p) < -s\}. \end{aligned}$$

This contradicts both (3-50) and (3-51) and concludes the proof. □

Let $\pi : \mathbb{H}^n \rightarrow \mathbb{W}$ be the group projection defined, for any $p \in \mathbb{H}^n$, by the formula

$$p = \pi(p) * (\xi(p)e_1).$$

For any set $E \subset \mathbb{H}^n$ and $s \in \mathbb{R}$, we let $E^s = E \cap \mathbb{H}_s^n$ and we define the projection

$$E_s = \pi(E^s) = \{w \in \mathbb{W} : w * (se_1) \in E\}.$$

Lemma 3.4. *Let $n \geq 2$, let $E \subset \mathbb{H}^n$ be a set with locally finite H -perimeter and $0 \in \partial E$, and let $s_0 \in (0, 1)$ be such that*

$$|\xi(p)| < s_0 \quad \text{for any } p \in \partial E \cap C_1, \tag{3-52}$$

$$\mathcal{L}^{2n+1}(\{p \in E \cap C_1 : \xi(p) > s_0\}) = 0, \tag{3-53}$$

$$\mathcal{L}^{2n+1}(\{p \in C_1 \setminus E : \xi(p) < -s_0\}) = 0. \tag{3-54}$$

Then, for a.e. $s \in (-1, 1)$ and any continuous function $\varphi \in C_c(D_1)$, we have, with $M = \partial^*E \cap C_1$ and $M_s = M \cap \{\xi > s\}$,

$$\int_{E_s \cap D_1} \varphi \, d\mathcal{L}^{2n} = - \int_{M_s} \varphi \circ \pi \langle \nu_E, X_1 \rangle_g \, d\mathcal{S}^{2n+1}. \tag{3-55}$$

In particular, for any Borel set $G \subset D_1$, we have

$$\mathcal{L}^{2n}(G) = - \int_{M \cap \pi^{-1}(G)} \langle \nu_E, X_1 \rangle_g \, d\mathcal{S}^{2n+1}, \tag{3-56}$$

$$\mathcal{L}^{2n}(G) \leq \mathcal{S}^{2n+1}(M \cap \pi^{-1}(G)). \tag{3-57}$$

Proof. It is enough to prove (3-55). Indeed, taking $s < -s_0$ in (3-55) and recalling (3-52) and (3-54), we obtain

$$\int_{D_1} \varphi \, d\mathcal{L}^{2n} = - \int_M \varphi \circ \pi \langle \nu_E, X_1 \rangle_g \, d\mathcal{S}^{2n+1}. \tag{3-58}$$

Formula (3-56) follows from (3-58) by considering smooth approximations of χ_G . Formula (3-57) is immediate from (3-56) and $|\langle \nu_E, X_1 \rangle_g| \leq 1$.

We prove (3-55) for a.e. $s \in (-1, 1)$, namely, for those s satisfying the property (3-61) below. Up to an approximation argument, we may assume that $\varphi \in C_c^1(D_1)$. Let $r \in (0, 1)$ and $\sigma \in (\max\{s_0, s\}, 1)$ be fixed. We define

$$F = E \cap (D_r * (s, \sigma)) = E \cap \{w * (\varrho e_1) \in \mathbb{H}^n : w \in D_r, \varrho \in (s, \sigma)\}.$$

We claim that, for a.e. $r \in (0, 1)$ and any s satisfying (3-61), we have

$$\langle \nu_F, X_1 \rangle_g \mu_F = \langle \nu_E, X_1 \rangle_g \mathcal{L}^{2n+1} \llcorner \partial^* E \cap (D_r * (s, \sigma)) + \mathcal{L}^{2n} \llcorner E \cap D_r^s. \tag{3-59}$$

Above, we let $D_r^s = \{w * (s e_1) \in \mathbb{H}^n : w \in D_r\}$. We postpone the proof of (3-59). Let Z be a horizontal vector field of the form $Z = (\varphi \circ \pi) X_1$. We have $\operatorname{div}_g Z = 0$ because $X_1(\varphi \circ \pi) = 0$. Hence, we obtain

$$0 = \int_F \operatorname{div}_g Z \, d\mathcal{L}^{2n+1} = - \int_{\mathbb{H}^n} \varphi \circ \pi \langle \nu_F, X_1 \rangle_g \, d\mu_F,$$

i.e., by the Fubini–Tonelli theorem and (3-59),

$$- \int_{E_s \cap D_r} \varphi \, d\mathcal{L}^{2n} = - \int_{E \cap D_r^s} \varphi \circ \pi \, d\mathcal{L}^{2n} = \int_{\partial^* E \cap (D_r * (s, \sigma))} \varphi \circ \pi \langle \nu_E, X_1 \rangle_g \, d\mathcal{L}^{2n+1}.$$

Formula (3-55) follows on letting first $r \nearrow 1$ and then $\sigma \nearrow 1$.

We are left with the proof of (3-59). Let $\psi \in C_c^1(\mathbb{H}^n)$ be a test function. For any $w \in \mathbb{W}$, we let

$$E_w = \{\varrho \in \mathbb{R} : w * (\varrho e_1) \in E\}, \quad \psi_w(\varrho) = \psi(w * (\varrho e_1)).$$

Then we have $\psi_w \in C_c^1(\mathbb{R})$ and, by the Fubini–Tonelli theorem,

$$\begin{aligned} - \int_F X_1 \psi \, d\mathcal{L}^{2n+1} &= - \int_{D_r} \int_s^\sigma \chi_E(w * (\varrho e_1)) X_1 \psi(w * (\varrho e_1)) \, d\varrho \, d\mathcal{L}^{2n}(w) \\ &= - \int_{D_r} \int_s^\sigma \chi_{E_w}(\varrho) \psi'_w(\varrho) \, d\varrho \, d\mathcal{L}^{2n}(w) \\ &= \int_{D_r} \left[\int_s^\sigma \psi_w \, dD\chi_{E_w} - \psi_w(\sigma) \chi_{E_w}(\sigma^-) + \psi_w(s) \chi_{E_w}(s^+) \right] d\mathcal{L}^{2n}(w), \end{aligned} \tag{3-60}$$

where $D\chi_{E_w}$ is the derivative of χ_{E_w} in the sense of distributions and $\chi_{E_w}(\sigma^-)$, $\chi_{E_w}(s^+)$ are the classical trace values of χ_{E_w} at the endpoints of the interval (s, σ) . We used the fact that the function χ_{E_w} is of bounded variation for \mathcal{L}^{2n} -a.e. $w \in \mathbb{W}$, which in turn is a consequence of the fact that $X_1 \chi_E$ is a signed Radon measure. For any such w , the trace of χ_{E_w} satisfies

$$\chi_{E_w}(s^+) = \chi_{E_w}(s) = \chi_E(w * (s e_1)) \quad \text{for a.e. } s,$$

so that, by Fubini’s theorem, for a.e. $s \in \mathbb{R}$ we have

$$\chi_{E_w}(s^+) = \chi_E(w * (s e_1)) \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } w \in D_1. \tag{3-61}$$

With a similar argument, using (3-53) and the fact that $\sigma > s_0$, one can see that

$$\chi_{E_w}(\sigma^-) = \chi_E(w * (\sigma e_1)) = 0 \quad \text{for } \mathcal{L}^{2n}\text{-a.e. } w \in D_1. \tag{3-62}$$

We refer the reader to [Ambrosio et al. 2000] for an extensive account on BV functions and traces. By (3-60), (3-61) and (3-62), we obtain

$$\begin{aligned} - \int_F X_1 \psi \, d\mathcal{L}^{2n+1} &= \int_{D_r} \int_s^\sigma \psi_w \, dD\chi_{E_w} \, d\mathcal{L}^{2n}(w) + \int_{D_r} \psi_w(s) \chi_{E_w}(s) \, d\mathcal{L}^{2n}(w) \\ &= \int_{D_r * (s, \sigma)} \psi \langle v_E, X_1 \rangle_g \, d\mu_E + \int_{E \cap D_r^s} \psi \, d\mathcal{L}^{2n} \\ &= \int_{\partial^* E \cap (D_r * (s, \sigma))} \psi \langle v_E, X_1 \rangle_g \, d\mathcal{S}^{2n+1} + \int_{E \cap D_r^s} \psi \, d\mathcal{L}^{2n}, \end{aligned}$$

and (3-59) follows. □

Corollary 3.5. *Under the same assumptions and notation as Lemma 3.4, for a.e. $s \in (-1, 1)$, we have*

$$0 \leq \mathcal{S}^{2n+1}(M_s) - \mathcal{L}^{2n}(E_s \cap D_1) \leq \text{Exc}(E, 1, \nu). \tag{3-63}$$

Moreover,

$$\mathcal{S}^{2n+1}(M) - \mathcal{L}^{2n}(D_1) = \text{Exc}(E, 1, \nu). \tag{3-64}$$

Proof. On approximating χ_{D_1} with functions $\varphi \in C_c(D_1)$, by (3-55) we get

$$\mathcal{L}^{2n}(E_s \cap D_1) = - \int_{M_s} \langle v_E, X_1 \rangle_g \, d\mathcal{S}^{2n+1},$$

and the first inequality in (3-63) follows. The second inequality follows from

$$\begin{aligned} \mathcal{S}^{2n+1}(M_s) - \mathcal{L}^{2n}(E_s \cap D_1) &= \int_{M_s} (1 + \langle v_E, X_1 \rangle_g) \, d\mathcal{S}^{2n+1} \\ &= \int_{M_s} \frac{|v_E - \nu|_g^2}{2} \, d\mathcal{S}^{2n+1} \\ &\leq \text{Exc}(E, 1, \nu). \end{aligned} \tag{3-65}$$

Notice that $\nu = -X_1$. Finally, (3-64) follows on choosing a suitable $s < -s_0$ and recalling (3-52) and (3-54). In this case, the inequality in (3-65) becomes an equality and the proof is concluded. □

Proof of Theorem 1.3. Step 1. Up to replacing E with the rescaled set $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$ with $\lambda = 1/2k^2r$ and recalling (3-48), we can without loss of generality assume that E is a $(\Lambda', 1/(2k^2))$ -minimizer of H -perimeter in C_2 with

$$\frac{\Lambda'}{2k^2} \leq 1, \quad 0 \in \partial E, \quad \text{Exc}(E, 2, \nu) \leq \varepsilon_0(n). \tag{3-66}$$

Our goal is to find $\varepsilon_0(n)$ and $c_1(n) > 0$ such that, if (3-66) holds, then

$$\sup\{|\xi(p)| : p \in \partial E \cap C_{1/2k^2}\} \leq c_1(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))}. \tag{3-67}$$

We require

$$\varepsilon_0(n) \leq \omega\left(n, \frac{1}{4k}, 2k^2, \frac{1}{2k^2}\right), \tag{3-68}$$

where ω is as given by Lemma 3.3. Two further assumptions on $\varepsilon_0(n)$ will be made later, in (3-80) and (3-85). By (3-66), E is a $(2k^2, 1/(2k^2))$ -minimizer in C_2 . Letting $M = \partial E \cap C_1$, by Lemma 3.3 and (3-68) we have

$$|\xi(p)| < \frac{1}{4k} \quad \text{for any } p \in M, \tag{3-69}$$

$$\mathcal{L}^{2n+1} \left(\left\{ p \in E \cap C_1 : \xi(p) > \frac{1}{4k} \right\} \right) = 0, \tag{3-70}$$

$$\mathcal{L}^{2n+1} \left(\left\{ p \in C_1 \setminus E : \xi(p) < -\frac{1}{4k} \right\} \right) = 0. \tag{3-71}$$

By (3-64) and (3-45), we get

$$0 \leq \mathcal{I}^{2n+1}(M) - \mathcal{L}^{2n}(D_1) \leq \text{Exc}(E, 1, \nu) \leq 2^{2n+1} \text{Exc}(E, 2, \nu). \tag{3-72}$$

Corollary 3.5 implies that, for a.e. $s \in (-1, 1)$,

$$0 \leq \mathcal{I}^{2n+1}(M_s) - \mathcal{L}^{2n}(E_s \cap D_1) \leq \text{Exc}(E, 1, \nu) \leq 2^{2n+1} \text{Exc}(E, 2, \nu), \tag{3-73}$$

where, as before, $M_s = M \cap \{\xi > s\}$.

Step 2. Consider $f : (-1, 1) \rightarrow [0, \mathcal{I}^{2n+1}(M)]$ defined by

$$f(s) = \mathcal{I}^{2n+1}(M_s), \quad s \in (-1, 1).$$

The function f is nonincreasing, right-continuous and, by (3-69), it satisfies

$$\begin{aligned} f(s) &= \mathcal{I}^{2n+1}(M) && \text{for any } s \in \left(-1, -\frac{1}{4k}\right], \\ f(s) &= 0 && \text{for any } s \in \left(\frac{1}{4k}, 1\right]. \end{aligned}$$

In particular, there exists $s_0 \in (-1/(4k), 1/(4k))$ such that

$$\begin{aligned} f(s) &\geq \frac{1}{2} \mathcal{I}^{2n+1}(M) && \text{for any } s < s_0, \\ f(s) &\leq \frac{1}{2} \mathcal{I}^{2n+1}(M) && \text{for any } s \geq s_0. \end{aligned} \tag{3-74}$$

Let $s_1 \in (s_0, 1/(4k))$ be such that

$$\begin{aligned} f(s) &\geq \sqrt{\text{Exc}(E, 2, \nu)} && \text{for any } s < s_1, \\ f(s) &= \mathcal{I}^{2n+1}(M_s) \leq \sqrt{\text{Exc}(E, 2, \nu)} && \text{for any } s \geq s_1. \end{aligned} \tag{3-75}$$

We claim that there exists $c_2(n) > 0$ such that

$$\xi(p) \leq s_1 + c_2(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3-76}$$

The inequality (3-76) is trivial for any $p \in \partial E \cap C_{1/2k^2}$ with $\xi(p) \leq s_1$. If $p \in \partial E \cap C_{1/2k^2}$ is such that $\xi(p) > s_1$, then

$$B_{\xi(p)-s_1}(p) \subset B_{1/2k}(p) \subset B_{1/k} \subset C_1.$$

We used the fact that $\|p\|_K \leq 1/(2k)$ whenever $p \in C_{1/2k^2}$; see (1-3). Therefore,

$$B_{\xi(p)-s_1}(p) \subset C_1 \cap \{\xi > s_1\}$$

and, by the density estimate (A-91) of Theorem A.1 in Appendix A,

$$k_3(n)(\xi(p) - s_1)^{2n+1} \leq \mu_E(B_{\xi(p)-s_1}(p)) \leq \mu_E(C_1 \cap \{\xi > s_1\}) = \mathcal{I}^{2n+1}(M_{s_1}) = f(s_1) \leq \sqrt{\text{Exc}(E, 2, \nu)}.$$

This proves (3-76).

Step 3. We claim that there exists $c_3(n) > 0$ such that

$$s_1 - s_0 \leq c_3(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))}. \tag{3-77}$$

By the coarea formula (3-41) with $h = \chi_{C_1}$, $D_1^s = \{p \in C_1 : \xi(p) = s\}$, and $E^s = \{p \in E : \xi(p) = s\}$, we have

$$\int_{-1}^1 \int_{D_1^s} d\mu_{E^s}^s ds = \int_{C_1} \sqrt{1 - \langle v_E, X_1 \rangle_g^2} d\mu_E \leq \sqrt{2} \int_M \sqrt{1 + \langle v_E, X_1 \rangle_g} d\mathcal{I}^{2n+1}.$$

By Hölder’s inequality, (A-91), (3-56), and (3-72), we deduce that

$$\begin{aligned} \int_{-1}^1 \int_{D_1^s} d\mu_{E^s}^s ds &\leq \sqrt{2\mathcal{I}^{2n+1}(M)} \left(\int_M (1 + \langle v_E, X_1 \rangle_g) d\mathcal{I}^{2n+1} \right)^{\frac{1}{2}} \\ &\leq c_4(n)(\mathcal{I}^{2n+1}(M) - \mathcal{L}^{2n}(D_1))^{1/2} \\ &\leq c_5(n)\sqrt{\text{Exc}(E, 2, \nu)}. \end{aligned} \tag{3-78}$$

By Corollary 3.5 and (3-72), we obtain, for a.e. $s \in [s_0, s_1)$,

$$\begin{aligned} \mathcal{L}^{2n}(E_s \cap D_1) &\leq \mathcal{I}^{2n+1}(M_s) = f(s) \leq f(s_0) \leq \frac{1}{2}\mathcal{I}^{2n+1}(M) \\ &\leq \frac{1}{2}(\mathcal{L}^{2n}(D_1) + 2^{2n+1} \text{Exc}(E, 2, \nu)) \\ &\leq \frac{3}{4}\mathcal{L}^{2n}(D_1). \end{aligned} \tag{3-79}$$

The last inequality holds provided that

$$2^{2n+1} \varepsilon_0(n) \leq \frac{1}{4}\mathcal{L}^{2n}(D_1). \tag{3-80}$$

Let $\Omega_s = (-se_1) * D_1^s = (-se_1) * D_1 * (se_1)$ and $F_s = (-se_1) * E^s$. We have

$$\mathcal{L}^{2n}(\Omega_s) = \mathcal{L}^{2n}(D_1^s) = \mathcal{L}^{2n}(D_1) \tag{3-81}$$

and, by (3-79),

$$\mathcal{L}^{2n}(F_s \cap \Omega_s) = \mathcal{L}^{2n}(E^s \cap D_1^s) = \mathcal{L}^{2n}(E_s \cap D_1) \leq \frac{3}{4}\mathcal{L}^{2n}(D_1). \tag{3-82}$$

Moreover, by left invariance we have

$$\mu_{E^s}^s(D_1^s) = \mu_{F_s}^0(\Omega_s). \tag{3-83}$$

By (3-81)–(3-83) and Corollary 3.2, there exists a constant $k(n) > 0$ independent of $s \in (-1, 1)$ such that

$$\mu_{E^s}(D_1^s) = \mu_{F_s}^0(\Omega_s) \geq k(n)\mathcal{L}^{2n}(F_s \cap \Omega_s)^{2n/(2n+1)} = k(n)\mathcal{L}^{2n}(E^s \cap D_1^s)^{2n/(2n+1)}. \tag{3-84}$$

This, together with (3-78), gives

$$\begin{aligned} c_6(n)\sqrt{\text{Exc}(E, 2, \nu)} &\geq \int_{s_0}^{s_1} \mathcal{L}^{2n}(E^s \cap D_1^s)^{2n/(2n+1)} ds \\ &\stackrel{(3-73)}{\geq} \int_{s_0}^{s_1} (\mathcal{L}^{2n+1}(M_s) - 2^{2n+1} \text{Exc}(E, 2, \nu))^{2n/(2n+1)} ds \\ &\stackrel{(3-75)}{\geq} \int_{s_0}^{s_1} (\sqrt{\text{Exc}(E, 2, \nu)} - 2^{2n+1} \text{Exc}(E, 2, \nu))^{2n/(2n+1)} ds \\ &\geq \frac{1}{2} \int_{s_0}^{s_1} \text{Exc}(E, 2, \nu)^{n/(2n+1)} ds. \end{aligned}$$

In the last inequality, we require that $\varepsilon_0(n)$ satisfies

$$\sqrt{z} - 2^{2n+1}z \geq \frac{1}{2}\sqrt{z} \quad \text{for all } z \in [0, \varepsilon_0(n)]. \tag{3-85}$$

It follows that

$$c_6(n)\sqrt{\text{Exc}(E, 2, \nu)} \geq \frac{1}{2} \text{Exc}(E, 2, \nu)^{n/(2n+1)}(s_1 - s_0),$$

giving (3-77).

Step 4. Recalling (3-76) and (3-77), we proved that there exist $\varepsilon_0(n)$ and $c_6(n)$ such that the following holds: if E is a $(2k^2, 1/(2k^2))$ -minimizer of H -perimeter in C_2 such that

$$0 \in \partial E, \quad \text{Exc}(E, 2, \nu) \leq \varepsilon_0(n)$$

and $s_0 = s_0(E)$ satisfies (3-74), then

$$\frac{p}{2} - s_0 \leq c_7(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3-86}$$

Let us introduce the mapping $\Psi : \mathbb{H}^n \rightarrow \mathbb{H}^n$

$$\Psi(x_1, x_2, \dots, x_n, y_1, \dots, y_n, t) = (-x_1, -x_2, \dots, -x_n, y_1, \dots, y_n, -t).$$

Then we have $\Psi^{-1} = \Psi$, $\Psi(C_2) = C_2$, $\langle X_j, \nu_{\Psi(F)} \rangle_g = -\langle X_j, \nu_F \rangle_g \circ \Psi$, $\langle Y_j, \nu_{\Psi(F)} \rangle_g = \langle Y_j, \nu_F \rangle_g \circ \Psi$, and $\mu_{\Psi(F)} = \Psi_\# \mu_F$, for any set F with locally finite H -perimeter; here, $\Psi_\#$ denotes the standard push-forward of measures. Therefore, the set $\tilde{E} = \Psi(\mathbb{H}^n \setminus E)$ satisfies the following properties:

- (i) \tilde{E} is a $(2k^2, 1/(2k^2))$ -minimizer of H -perimeter in C_2 ;
- (ii) $0 \in \partial \tilde{E}$ and

$$\text{Exc}(\tilde{E}, 2, \nu) = \frac{1}{2Q} \int_{\partial^* \tilde{E} \cap C_2} |\nu_{\tilde{E}} - \nu|_g^2 d\mathcal{S}^{2n+1} = \text{Exc}(E, 2, \nu) \leq \varepsilon_0(n);$$

(iii) setting $\tilde{M} = \partial^* \tilde{E} \cap C_1 = \Psi(M)$ and $\tilde{f}(s) = \mathcal{I}^{2n+1}(\tilde{M} \cap \{\xi > s\})$, we have

$$\begin{aligned} \tilde{f}(s) &\geq \frac{1}{2} \mathcal{I}^{2n+1}(\tilde{M}) = \frac{1}{2} \mathcal{I}^{2n+1}(M) && \text{for any } s < -s_0, \\ \tilde{f}(s) &\leq \frac{1}{2} \mathcal{I}^{2n+1}(M) && \text{for any } s \geq -s_0. \end{aligned}$$

Formula (3-86) for the set \tilde{E} gives

$$\xi(p) + s_0 \leq c_7(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial \tilde{E} \cap C_{1/2k^2}.$$

Notice that we have $p \in \partial \tilde{E}$ if and only if $\Psi(p) \in \partial E$ and, moreover, $\xi(\Psi(p)) = -\xi(p)$. Hence, we have

$$-\xi(p) + s_0 \leq c_7(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3-87}$$

By (3-86) and (3-87), we obtain

$$|\xi(p) - s_0| \leq c_7(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}, \tag{3-88}$$

and, in particular,

$$|s_0| \leq c_7(n) \text{Exc}(E, 2, \nu)^{1/(2(2n+1))}, \tag{3-89}$$

because $0 \in \partial E \cap C_{1/2k^2}$. Inequalities (3-88) and (3-89) give (3-67). This completes the proof. \square

Appendix A

We list some basic properties of Λ -minimizers of H -perimeter in \mathbb{H}^n . The proofs are straightforward adaptations of the proofs for Λ -minimizers of perimeter in \mathbb{R}^n .

Theorem A.1 (density estimates). *There exist positive constants $k_1(n), k_2(n), k_3(n)$ and $k_4(n)$ with the following property: if E is a (Λ, r) -minimizer of H -perimeter in $\Omega \subset \mathbb{H}^n$, $p \in \partial E \cap \Omega$, $B_r(p) \subset \Omega$ and $s < r$, then*

$$k_1(n) \leq \frac{\mathcal{I}^{2n+1}(E \cap B_s(p))}{s^{2n+2}} \leq k_2(n), \tag{A-90}$$

$$k_3(n) \leq \frac{\mu_E(B_s(p))}{s^{2n+1}} \leq k_4(n). \tag{A-91}$$

For a proof, see [Maggi 2012, Theorem 21.11]. By standard arguments, Theorem A.1 implies the following corollary:

Corollary A.2. *If E is a (Λ, r) -minimizer of H -perimeter in $\Omega \subset \mathbb{H}^n$, then*

$$\mathcal{I}^{2n+1}((\partial E \setminus \partial^* E) \cap \Omega) = 0.$$

Theorem A.3. *Let $(E_j)_{j \in \mathbb{N}}$ be a sequence of (Λ, r) -minimizers of H -perimeter in an open set $\Omega \subset \mathbb{H}^n$, $\Lambda r \leq 1$. Then there exists a (Λ, r) -minimizer E of H -perimeter in Ω and a subsequence $(E_{j_k})_{k \in \mathbb{N}}$ such that*

$$E_{j_k} \longrightarrow E \quad \text{in } L^1_{\text{loc}}(\Omega) \quad \text{and} \quad \nu_{E_{j_k}} \mu_{E_{j_k}} \xrightarrow{*} \nu_E \mu_E$$

as $k \rightarrow \infty$. Moreover, the measure-theoretic boundaries ∂E_{j_k} converge to ∂E in the sense of Kuratowski, i.e.,

- (i) if $p_{j_k} \in \partial E_{j_k} \cap \Omega$ and $p_{j_k} \rightarrow p \in \Omega$, then $p \in \partial E$;
- (ii) if $p \in \partial E \cap \Omega$, then there exists a sequence $(p_{j_k})_{k \in \mathbb{N}}$ such that $p_{j_k} \in \partial E_{j_k} \cap \Omega$ and $p_{j_k} \rightarrow p$.

For a proof in the case of the perimeter in \mathbb{R}^n , see [Maggi 2012, Chapter 21].

Appendix B

We define a Borel unit normal ν_R to an \mathcal{S}^{2n+1} -rectifiable set $R \subset \mathbb{H}^n$ and we show that the definition is well posed \mathcal{S}^{2n+1} -a.e., up to the sign. The normal ν_S to an H -regular hypersurface $S \subset \mathbb{H}^n$ is defined in (1-6).

Definition B.1. Let $R \subset \mathbb{H}^n$ be an \mathcal{S}^{2n+1} -rectifiable set such that

$$\mathcal{S}^{2n+1} \left(R \setminus \bigcup_{j \in \mathbb{N}} S_j \right) = 0 \tag{B-92}$$

for a sequence of H -regular hypersurfaces $(S_j)_{j \in \mathbb{N}}$ in \mathbb{H}^n . For any $p \in R \cap \bigcup_{j \in \mathbb{N}} S_j$, we define

$$\nu_R(p) = \nu_{S_{\bar{j}}}(p),$$

where \bar{j} is the unique integer such that $p \in S_{\bar{j}} \setminus \bigcup_{j < \bar{j}} S_j$.

We show that Definition B.1 is well posed, up to a sign, for \mathcal{S}^{2n+1} -a.e. p . Namely, let $(S_j^1)_{j \in \mathbb{N}}$ and $(S_j^2)_{j \in \mathbb{N}}$ be two sequences of H -regular hypersurfaces in \mathbb{H}^n for which (B-92) holds and denote by ν_R^1 and ν_R^2 , respectively, the associated normals to R according to Definition B.1. We show that $\nu_R^1 = \nu_R^2$ \mathcal{S}^{2n+1} -a.e. on R , up to the sign.

Let $A \subset R$ be the set of points such that either $\nu_R^1(p)$ is not defined, or $\nu_R^2(p)$ is not defined, or they are both defined and $\nu_R^1(p) \neq \pm \nu_R^2(p)$. It is enough to show that $\mathcal{S}^{2n+1}(A) = 0$. This is a consequence of the following lemma:

Lemma B.2. Let S_1, S_2 be two H -regular hypersurfaces in \mathbb{H}^n and let

$$A = \{p \in S_1 \cap S_2 : \nu_{S_1}(p) \neq \pm \nu_{S_2}(p)\}.$$

Then, the Hausdorff dimension of A in the Carnot–Carathéodory metric is at most $2n$, $\dim_{\text{CC}}(A) \leq 2n$, and, in particular, $\mathcal{S}^{2n+1}(A) = 0$.

Proof. The blow-up of S_i , $i = 1, 2$, at a point $p \in A$ is a vertical hyperplane $\Pi_i \times \mathbb{R} \subset \mathbb{R}^{2n} \times \mathbb{R} \equiv \mathbb{H}^n$ — see, e.g., [Franchi et al. 2001] — where:

- (i) By blow-up of S_i at p , we mean the limit

$$\lim_{\lambda \rightarrow \infty} \lambda(p^{-1} * S_i)$$

in the Gromov–Hausdorff sense. Recall that, for $E \subset \mathbb{H}^n$, we define $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$.

(ii) For $i = 1, 2$, $\Pi_i \subset \mathbb{R}^{2n}$ is the normal hyperplane to $v_{S_i}(p) \in H_p \equiv \mathbb{R}^{2n}$.

It follows that the blow-up of A at p is contained in the blow-up of $S_1 \cap S_2$ at p , i.e., in $(\Pi_1 \cap \Pi_2) \times \mathbb{R}$. Since $v_{S_1}(p) \neq \pm v_{S_2}(p)$, $\Pi_1 \cap \Pi_2$ is a $(2n-2)$ -dimensional plane in \mathbb{R}^{2n} , and we conclude thanks to the following lemma. \square

Lemma B.3. *Let $k = 0, 1, \dots, 2n$ and $A \subset \mathbb{H}^n$ be such that, for any $p \in A$, the blow-up of A at p is contained in $\Pi_p \times \mathbb{R}$ for some plane $\Pi_p \subset \mathbb{R}^{2n}$ of dimension k . Then we have $\dim_{\text{CC}}(A) \leq k + 2$.*

Proof. We claim that, for any $\eta > 0$, we have

$$\mathcal{S}^{k+2+\eta}(A) = 0. \tag{B-93}$$

Let $\varepsilon \in (0, \frac{1}{2})$ be such that $C\varepsilon^\eta \leq \frac{1}{2}$, where $C = C(n)$ is a constant that will be fixed later in the proof. By the definition of blow-up, for any $p \in A$ there exists $r_p > 0$ such that, for all $r \in (0, r_p)$, we have

$$(p^{-1} * A) \cap U_r \subset (\Pi_p)_{\varepsilon r} \times \mathbb{R},$$

where $(\Pi_p)_{\varepsilon r}$ denotes the (εr) -neighbourhood of Π_p in \mathbb{R}^{2n} . For any $j \in \mathbb{N}$, set

$$A_j = \{p \in A \cap B_j : r_p > 1/j\}.$$

To prove (B-93), it is enough to prove that

$$\mathcal{S}^{k+2+\eta}(A_j) = 0$$

for any fixed $j \geq 1$. This, in turn, will follow if we show that, for any fixed $\delta \in (0, 1/(2j))$, one has

$$\inf \left\{ \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i), r_i < 2\varepsilon\delta \right\} \leq \frac{1}{2} \inf \left\{ \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i), r_i < \delta \right\}. \tag{B-94}$$

Let $(U_{r_i}(p_i))_{i \in \mathbb{N}}$ be a covering of A_j with balls of radius smaller than δ . There exist points $\bar{p}_i \in A_j$ such that $(U_{2r_i}(\bar{p}_i))_{i \in \mathbb{N}}$ is a covering of A_j with balls of radius smaller than $2\delta < 1/j$. By definition of A_j , we have

$$(\bar{p}_i^{-1} * A_j) \cap U_{2r_i} \subset ((\Pi_{\bar{p}_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}.$$

The set $((\Pi_{\bar{p}_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}$ can be covered by a family of balls $(U_{\varepsilon r_i}(p_h^i))_{h \in H_i}$ of radius $\varepsilon r_i < 2\varepsilon\delta$ in such a way that the cardinality of H_i is bounded by $C\varepsilon^{-k-2}$, where the constant C depends only on n and not on ε . In particular, the family of balls $(U_{\varepsilon r_i}(\bar{p}_i * p_h^i))_{i \in \mathbb{N}, h \in H_i}$ is a covering of A_j and

$$\begin{aligned} \sum_{i \in \mathbb{N}} \sum_{h \in H_i} (\text{radius } U_{\varepsilon r_i}(\bar{p}_i * p_h^i))^{k+2+\eta} &= \sum_{i \in \mathbb{N}} \sum_{h \in H_i} (\varepsilon r_i)^{k+2+\eta} \leq C\varepsilon^{-k-2} \sum_{i \in \mathbb{N}} (\varepsilon r_i)^{k+2+\eta} \\ &= C\varepsilon^\eta \sum_{i \in \mathbb{N}} r_i^{k+2+\eta} \leq \frac{1}{2} \sum_i r_i^{k+2+\eta}. \end{aligned}$$

This proves (B-94) and concludes the proof. \square

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