ANALYSIS & PDE

Volume 8 No. 6 2015

ROBERTO MONTI AND DAVIDE VITTONE

HEIGHT ESTIMATE AND SLICING FORMULAS IN THE HEISENBERG GROUP

msp

ANALYSIS AND PDE
Vol. 8, No. 6, 2015
dx.doi.org/10.2140/apde.2015.8.1421

HEIGHT ESTIMATE AND SLICING FORMULAS IN THE HEISENBERG GROUP

ROBERTO MONTI AND DAVIDE VITTONE

We prove a height estimate (distance from the tangent hyperplane) for Λ-minimizers of the perimeter in the sub-Riemannian Heisenberg group. The estimate is in terms of a power of the excess (L^2-mean oscillation of the normal) and its proof is based on a new coarea formula for rectifiable sets in the Heisenberg group.

1. Introduction

We continue the research project started in [Monti and Vittone 2012; Monti 2014] on the regularity of H-perimeter minimizing boundaries in the Heisenberg group $\mathbb{H}^n$. Our goal is to prove the so-called *height estimate* for sets that are Λ*-minimizers* and have small *excess* inside suitable cylinders; see Theorem 1.3. The proof follows the scheme of the median choice for the measure of the boundary in certain half-cylinders together with a lower-dimensional isoperimetric inequality on slices. For minimizing currents in $\mathbb{R}^n$, the principal ideas of the argument go back to [Almgren 1968] and are carried over in [Federer 1969, Theorem 5.3.4]. The argument can be also found in the Appendix of [Schoen and Simon 1982] and, for Λ-minimizers of perimeter in $\mathbb{R}^n$, in [Maggi 2012, Section 22.2]. For minimizers of H-perimeter, the decay estimate of excess of Almgren and De Giorgi is still an open problem; see [Monti 2015].

Our main technical effort is the proof of a coarea formula (slicing formula) for intrinsic rectifiable sets; see Theorem 1.5. This formula is established in Section 2 and has a nontrivial character because the domain of integration and its slices need not be rectifiable in the standard sense. The relative isoperimetric inequalities that are used in the slices reduce to a single isoperimetric inequality in one slice that is relative to a family of varying domains with uniform isoperimetric constants. This uniformity can be established using the results on regular domains in Carnot groups of step 2 in [Monti and Morbidelli 2005] and the isoperimetric inequality in [Garofalo and Nhieu 1996]; see Section 3A.

The $(2n+1)$-dimensional Heisenberg group is the manifold $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$, $n \in \mathbb{N}$, endowed with the group product

$$(z,t) * (\zeta,\tau) = (z+\zeta, t+\tau+2\,\Im\langle z,\bar{\zeta}\rangle), \tag{1-1}$$

where $t,\tau \in \mathbb{R}$, $z,\zeta \in \mathbb{C}^n$ and $\langle z,\bar{\zeta}\rangle = z_1\bar{\zeta}_1 + \cdots + z_n\bar{\zeta}_n$. The Lie algebra of left-invariant vector fields in $\mathbb{H}^n$ is spanned by the vector fields

$$X_j = \frac{\partial}{\partial x_j} + 2y_j\frac{\partial}{\partial t}, \quad Y_j = \frac{\partial}{\partial y_j} - 2x_j\frac{\partial}{\partial t}, \quad \text{and} \quad T = \frac{\partial}{\partial t}, \tag{1-2}$$

MSC2010: 49Q05, 53C17.
Keywords: Heisenberg group, regularity of H-minimal surfaces, height estimate, slicing formula.

with $z_j = x_j + iy_j$ and $j = 1, \dots, n$. We denote by H the horizontal subbundle of $T\mathbb{H}^n$. Namely, for any $p = (z, t) \in \mathbb{H}^n$ we let

$$H_p = \operatorname{span}\{X_1(p), \dots, X_n(p), Y_1(p), \dots, Y_n(p)\}.$$

A horizontal section $\varphi \in C_c^1(\Omega; H)$, where $\Omega \subset \mathbb{H}^n$ is an open set, is a vector field of the form

$$\varphi = \sum_{j=1}^n \varphi_j X_j + \varphi_{n+j} Y_j,$$

where $\varphi_j \in C_c^1(\Omega)$, i.e., each coordinate φ_j is a continuously differentiable function with compact support contained in Ω.

Let g be the left-invariant Riemannian metric on $\mathbb{H}^n$ that makes orthonormal the vector fields $X_1, \dots X_n$, $Y_1, \dots, Y_n, T$ in (1-2). For tangent vectors $V, W \in T\mathbb{H}^n$, we let

$$\langle V, W\rangle_g = g(V, W) \quad \text{and} \quad |V|_g = g(V, V)^{1/2}.$$

The sup norm with respect to g of a horizontal section $\varphi \in C_c^1(\Omega; H)$ is

$$\|\varphi\|_g = \max_{p\in\Omega} |\varphi(p)|_g.$$

The Riemannian divergence of φ is

$$\operatorname{div}_g \varphi = \sum_{j=1}^n X_j\varphi_j + Y_j\varphi_{n+j}.$$

The metric g induces a volume form on $\mathbb{H}^n$ that is left-invariant. Also, the Lebesgue measure $\mathscr{L}^{2n+1}$ on $\mathbb{H}^n$ is left-invariant, and by the uniqueness of the Haar measure the volume induced by g is the Lebesgue measure $\mathscr{L}^{2n+1}$. In fact, the proportionality constant is 1.

The *H-perimeter* of an $\mathscr{L}^{2n+1}$-measurable set $E \subset \mathbb{H}^n$ in an open set $\Omega \subset \mathbb{H}^n$ is

$$\mu_E(\Omega) = \sup\left\{ \int_E \operatorname{div}_g \varphi \, d\mathscr{L}^{2n+1} : \varphi \in C_c^1(\Omega; H), \|\varphi\|_g \le 1 \right\}.$$

If $\mu_E(\Omega) < \infty$ we say that E has finite H-perimeter in Ω. If $\mu_E(A) < \infty$ for any open set $A \Subset \Omega$, we say that E has locally finite H-perimeter in Ω. In this case, the open sets mapping $A \mapsto \mu_E(A)$ extends to a Radon measure μ_E on Ω that is called the *H-perimeter measure* induced by E. Moreover, there exists a μ_E-measurable function $\nu_E : \Omega \to H$ such that $|\nu_E|_g = 1$ μ_E-a.e. and the Gauss–Green integration by parts formula

$$\int_\Omega \langle \varphi, \nu_E\rangle_g \, d\mu_E = -\int_\Omega \operatorname{div}_g \varphi \, d\mathscr{L}^{2n+1}$$

holds for any $\varphi \in C_c^1(\Omega; H)$. The vector ν_E is called the *horizontal inner normal* of E in Ω.

The Korànyi norm of $p = (z, t) \in \mathbb{H}^n$ is $\|p\|_K = (|z|^4 + t^2)^{1/4}$. For any $r > 0$ and $p \in \mathbb{H}^n$, we define the balls

$$B_r = \{q \in \mathbb{H}^n : \|q\|_K < r\} \quad \text{and} \quad B_r(p) = \{p * q \in \mathbb{H}^n : q \in B_r\}.$$

The *measure-theoretic boundary* of a measurable set $E \subset \mathbb{H}^n$ is the set

$$\partial E = \left\{p \in \mathbb{H}^n : \mathscr{L}^{2n+1}(E \cap B_r(p)) > 0 \text{ and } \mathscr{L}^{2n+1}(B_r(p) \setminus E) > 0 \text{ for all } r > 0\right\}.$$

For a set E with locally finite H-perimeter, the H-perimeter measure μ_E is concentrated on ∂E and, actually, on a subset $\partial^* E$ of ∂E; see below. Moreover, up to modifying E on a Lebesgue-negligible set, one can always assume that ∂E coincides with the topological boundary of E; see [Serra Cassano and Vittone 2014, Proposition 2.5].

Definition 1.1. Let $\Omega \subset \mathbb{H}^n$ be an open set, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$. We say that a set $E \subset \mathbb{H}^n$ with locally finite H-perimeter in Ω is a (Λ, r)*-minimizer of H-perimeter in* Ω if, for any measurable set $F \subset \mathbb{H}^n$, $p \in \Omega$, and $s < r$ such that $E \Delta F \Subset B_s(p) \Subset \Omega$,

$$\mu_E(B_s(p)) \leq \mu_F(B_s(p)) + \Lambda \mathscr{L}^{2n+1}(E \Delta F),$$

where $E \Delta F = E \setminus F \cup F \setminus E$.

We say that E is *locally H-perimeter minimizing in* Ω if, for any measurable set $F \subset \mathbb{H}^n$ and any open set U such that $E \Delta F \Subset U \Subset \Omega$, there holds $\mu_E(U) \leq \mu_F(U)$.

We will often use the term Λ*-minimizer*, rather than (Λ, r)-minimizer, when the role of r is not relevant. In Appendix A, we list without proof some elementary properties of Λ-minimizers.

We now introduce the notion of cylindrical excess. The *height function* $\mathfrak{h} : \mathbb{H}^n \to \mathbb{R}$ is defined by $\mathfrak{h}(p) = p_1$, where p_1 is the first coordinate of $p = (p_1, \dots, p_{2n+1}) \in \mathbb{H}^n$. The set $\mathbb{W} = \{p \in \mathbb{H}^n : \mathfrak{h}(p) = 0\}$ is the vertical hyperplane passing through $0 \in \mathbb{H}^n$ and orthogonal to the left-invariant vector field X_1. The disk in $\mathbb{W}$ of radius $r > 0$ centred at $0 \in \mathbb{W}$ induced by the Korànyi norm is the set $D_r = \{p \in \mathbb{W} : \|p\|_K < r\}$. The intrinsic cylinder with central section D_r and height $2r$ is the set

$$C_r = D_r * (-r, r) \subset \mathbb{H}^n.$$

Here and in the sequel, we use the notation $D_r * (-r, r) = \{w * (se_1) \in \mathbb{H}^n : w \in D_r,\ s \in (-r, r)\big\}$, where $se_1 = (s, 0, \dots, 0) \in \mathbb{H}^n$. The cylinder C_r is comparable with the ball $B_r = \{\|p\|_K < r\}$. Namely, there exists a constant $k = k(n) \geq 1$ such that, for any $r > 0$, we have

$$B_{r/k} \subset C_r \subset B_{kr}. \tag{1-3}$$

By a rotation of the system of coordinates, it is enough to consider excess in cylinders with basis in $\mathbb{W}$ and axis X_1.

Definition 1.2 (cylindrical excess). Let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter. The cylindrical excess of E at the point $0 \in \partial E$, at scale $r > 0$, and with respect to the direction $\nu = -X_1$ is defined as

$$\operatorname{Exc}(E, r, \nu) = \frac{1}{2r^{2n+1}} \int_{C_r} |\nu_E - \nu|_g^2 \, d\mu_E,$$

where μ_E is the H-perimeter measure of E and ν_E is its horizontal inner normal.

Theorem 1.3 (height estimate). *Let $n \geq 2$. There exist constants $\varepsilon_0 = \varepsilon_0(n) > 0$ and $c_0 = c_0(n) > 0$ with the following property: if $E \subset \mathbb{H}^n$ is a (Λ, r)-minimizer of H-perimeter in the cylinder C_{4k^2r}, $\Lambda r \leq 1$, $0 \in \partial E$, and*

$$\operatorname{Exc}(E, 4k^2r, \nu) \leq \varepsilon_0,$$

then

$$\sup\{|\mathfrak{h}(p)| \in [0,\infty) : p \in \partial E \cap C_r\} \leq c_0\, r\, \operatorname{Exc}(E, 4k^2r, \nu)^{1/(2(2n+1))}. \tag{1-4}$$

The constant $k = k(n)$ is the one in (1-3).

The estimate (1-4) does not hold when $n = 1$. In fact, there are sets $E \subset \mathbb{H}^1$ such that $\operatorname{Exc}(E, r, \nu) = 0$ but ∂E is not flat in $C_{\varepsilon r}$ for any $\varepsilon > 0$. See the conclusions of Proposition 3.7 in [Monti 2014]. Theorem 1.3 is proved in Section 3.

Besides local minimizers of H-perimeter, our interest in Λ-minimizers is also motivated by possible applications to isoperimetric sets. The height estimate is a first step in the regularity theory of Λ-minimizers of classical perimeter; we refer to [Maggi 2012, Part III] for a detailed account of the subject.

In order to state the slicing formula in its general form, we need the definition of a rectifiable set in $\mathbb{H}^n$ of codimension 1. We follow closely [Franchi et al. 2001], where this notion was first introduced.

The Riemannian and horizontal gradients of a function $f \in C^1(\mathbb{H}^n)$ are, respectively,

$$\nabla f = (X_1 f)X_1 + \cdots + (Y_n f)Y_n + (Tf)T,$$
$$\nabla_H f = (X_1 f)X_1 + \cdots + (Y_n f)Y_n.$$

We say that a continuous function $f \in C(\Omega)$, with $\Omega \subset \mathbb{H}^n$ an open set, is of class $C^1_H(\Omega)$ if the horizontal gradient $\nabla_H f$ exists in the sense of distributions and is represented by continuous functions $X_1 f, \ldots, Y_n f$ in Ω. A set $S \subset \mathbb{H}^n$ is an H-regular hypersurface if, for all $p \in S$, there exist $r > 0$ and a function $f \in C^1_H(B_r(p))$ such that $S \cap B_r(p) = \{q \in B_r(p) : f(q) = 0\}$ and $\nabla_H f(p) \neq 0$. Sets with H-regular boundary have locally finite H-perimeter.

For any $p = (z,t) \in \mathbb{H}^n$, let us define the box norm $\|p\|_\infty = \max\{|z|, |t|^{1/2}\}$ and the balls $U_r = \{q \in \mathbb{H}^n n \|q\|_\infty < r\}$ and $U_r(p) = p * U_r$ for $r > 0$. Let $E \subset \mathbb{H}^n$ be a set. For any $s \geq 0$ define the measure

$$\mathscr{S}^s(E) = \sup_{\delta > 0} \inf\Big\{c(n,s)\sum_{i \in \mathbb{N}} r_i^s : E \subset \bigcup_{i \in \mathbb{N}} U_{r_i}(p_i),\ r_i < \delta\Big\}.$$

Above, $c(n,s) > 0$ is a normalization constant that we do not need to specify here. By Carathéodory's construction, $E \mapsto \mathscr{S}^s(E)$ is a Borel measure in $\mathbb{H}^n$. When $s = 2n+2$, it turns out that $\mathscr{S}^{2n+2}$ is the Lebesgue measure $\mathscr{L}^{2n+1}$. Thus, the correct dimension to measure hypersurfaces is $s = 2n+1$. In fact, if E is a set with locally finite H-perimeter in $\mathbb{H}^n$, then we have

$$\mu_E = \mathscr{S}^{2n+1} \llcorner \partial^* E, \tag{1-5}$$

where $\llcorner$ denotes restriction and $\partial^* E$ is the H-reduced boundary of E, namely the set of points $p \in \mathbb{H}^n$ such that $\mu_E(U_r(p)) > 0$ for all $r > 0$, $\fint_{U_r(p)} \nu_E \, d\mu_E \to \nu_E(p)$ as $r \to 0$, and $|\nu_E(p)|_g = 1$. The validity of formula (1-5) depends on the geometry of the balls $U_r(p)$; see [Magnani 2014]. We refer the reader to [Franchi et al. 2001] for more details on the H-reduced boundary.

Definition 1.4. A set $R \subset \mathbb{H}^n$ is $\mathscr{S}^{2n+1}$-rectifiable if there exists a sequence of H-regular hypersurfaces $(S_j)_{j\in\mathbb{N}}$ in $\mathbb{H}^n$ such that

$$\mathscr{S}^{2n+1}\Big(R \setminus \bigcup_{j\in\mathbb{N}} S_j\Big) = 0.$$

By the results of [Franchi et al. 2001], the H-reduced boundary $\partial^* E$ is $\mathscr{S}^{2n+1}$-rectifiable. Definition 1.4 is generalized in [Mattila et al. 2010], which studies the notion of an s-rectifiable set in $\mathbb{H}^n$ for any integer $1 \le s \le 2n+1$.

An H-regular surface S has a continuous horizontal normal ν_S that is locally defined up to the sign. This normal is given by the formula

$$\nu_S = \frac{\nabla_H f}{|\nabla_H f|_g}, \tag{1-6}$$

where f is a defining function for S. When $S = \partial E$ is the boundary of a smooth set, ν_S agrees with the horizontal normal ν_E. Then, for an $\mathscr{S}^{2n+1}$-rectifiable set $R \subset \mathbb{H}^n$, there is a unit horizontal normal $\nu_R : R \to H$ that is Borel regular. This normal is uniquely defined $\mathscr{S}^{2n+1}$-a.e. on R up to the sign; see Appendix B. However, (1-8) below does not depend on the sign.

In the following theorem, $\Omega \subset \mathbb{H}^n$ is an open set and $u \in C^\infty(\Omega)$ is a smooth function. For any $s \in \mathbb{R}$, we denote by $\Sigma^s = \{p \in \Omega : u(p) = s\}$ the level sets of u.

Theorem 1.5. *Let $R \subset \Omega$ be an $\mathscr{S}^{2n+1}$-rectifiable set. Then, for a.e. $s \in \mathbb{R}$ there exists a Radon measure μ_R^s on $R \cap \Sigma^s$ such that, for any Borel function $h : \Omega \to [0, \infty)$, the function*

$$s \mapsto \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_R^s \tag{1-7}$$

is $\mathscr{L}^1$-measurable and we have the coarea formula

$$\int_{\mathbb{R}} \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_R^s\, ds = \int_R h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u\rangle_g^2}\, d\mathscr{S}^{2n+1}. \tag{1-8}$$

Theorem 1.5 is proved in Section 2. When $R \cap \Sigma^s$ is a regular subset of Σ^s, the measures μ_R^s are natural horizontal perimeters defined in Σ^s.

Coarea formulas in the Heisenberg group are known only for slicing of sets with positive Lebesgue measure; see [Magnani 2004; 2008]. Theorem 1.5 is, to our knowledge, the first example of slicing of lower-dimensional sets in a sub-Riemannian framework. Also, Theorem 1.5 is a nontrivial extension of the Riemannian coarea formula, because the set R and the slices $R \cap \Sigma^s$ need not be rectifiable in the standard sense; see [Kirchheim and Serra Cassano 2004]. We need the coarea formula (1-8) in the proof of Theorem 1.3; see Section 3C.

We conclude the introduction by stating a different but equivalent formulation of the coarea formula (1-8) that is closer to standard coarea formulas. This alternative formulation holds only when $n \ge 2$: when $n = 1$, the right-hand side in (1-9) might not be well defined; see Remark 2.11.

Theorem 1.6. *Let $\Omega \subset \mathbb{H}^n$, $n \geq 2$, be an open set, $u \in C^\infty(\Omega)$ be a smooth function, and $R \subset \Omega$ be an $\mathscr{S}^{2n+1}$-rectifiable set. Then, for any Borel function $h : \Omega \to [0, \infty)$,*

$$\int_{\mathbb{R}} \int_{\Omega} h \, d\mu_R^s \, ds = \int_R h |\nabla u|_g \sqrt{1 - \langle \nu_R, \nabla_H u / |\nabla_H u|_g \rangle_g^2} \, d\mathscr{S}^{2n+1}, \tag{1-9}$$

where μ_R^s are the measures given by Theorem 1.5.

2. Proof of the coarea formula

2A. *Horizontal perimeter on submanifolds.* Let $\Sigma \subset \mathbb{H}^n$ be a C^∞ hypersurface. We define the horizontal tangent bundle $H\Sigma$ by letting, for any $p \in \Sigma$,

$$H_p\Sigma = H_p \cap T_p\Sigma.$$

In general, the rank of $H\Sigma$ is not constant. This depends on the presence of *characteristic points* on Σ, i.e., points such that $H_p = T_p\Sigma$. For points $p \in \Sigma$ such that $H_p \neq T_p\Sigma$, we have $\dim(H_p\Sigma) = 2n - 1$.

We denote by σ_Σ the surface measure on Σ induced by the Riemannian metric g restricted to the tangent bundle $T\Sigma$.

Definition 2.1. Let $F \subset \Sigma$ be a Borel set and let $\Omega \subset \Sigma$ be an open set. We define the *H-perimeter of F in Ω*,

$$\mu_F^\Sigma(\Omega) = \sup\left\{ \int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma : \varphi \in C_c^1(\Omega; H\Sigma), \|\varphi\|_g \leq 1 \right\}. \tag{2-10}$$

We say that the set $F \subset \Sigma$ has locally finite H-perimeter in Ω if $\mu_F^\Sigma(A) < \infty$ for any open set $A \Subset \Omega$.

By the Riesz theorem, if $F \subset \Sigma$ has locally finite H-perimeter in Ω, then the open sets mapping $A \mapsto \mu_F^\Sigma(A)$ extends to a Radon measure on Ω, called the *H-perimeter measure* of F.

Remark 2.2. If $F \subset \Sigma$ is an open set with smooth boundary, then, by the divergence theorem, we have, for any $\varphi \in C_c^1(\Omega; H\Sigma)$,

$$\int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma = \int_{\partial F} \langle N_{\partial F}, \varphi \rangle_g \, d\lambda_{\partial F}, \tag{2-11}$$

where $N_{\partial F}$ is the Riemannian outer unit normal to ∂F and $d\lambda_{\partial F}$ is the Riemannian $(2n-1)$-dimensional volume form on ∂F induced by g.

From the sup definition (2-10) and from (2-11), we deduce that the H-perimeter measure of F has the representation

$$\mu_F^\Sigma = |N_{\partial F}^{H\Sigma}|_g \, \lambda_{\partial F},$$

where $N_{\partial F}^{H\Sigma} \in H\Sigma$ is the g-orthogonal projection of $N_{\partial F} \in T\Sigma$ onto $H\Sigma$.

This formula can be generalized as follows. We denote by $\mathscr{H}_g^{2n-1}$ the $(2n-1)$-dimensional Hausdorff measure in $\mathbb{H}^n$ induced by the metric g.

Lemma 2.3. *Let $F, \Omega \subset \Sigma$ be open sets and assume that there exists a compact set $N \subset \partial F$ such that $\mathscr{H}_g^{2n-1}(N) = 0$ and $(\partial F \setminus N) \cap \Omega$ is a smooth $(2n-1)$-dimensional surface. Then, we have*

$$\mu_F^{\Sigma} \llcorner \Omega = |N_{\partial F}^{H\Sigma}|_g \, \lambda_{\partial F \setminus N} \llcorner \Omega. \tag{2-12}$$

Proof. For any $\varepsilon > 0$ there exist points $p_i \in \mathbb{H}^n$ and radii $r_i \in (0, 1)$, $i = 1, \dots, M$, such that

$$N \subset \bigcup_{i=1}^{M} B_g(p_i, r_i) \quad \text{and} \quad \sum_{i=1}^{M} r_i^{2n-1} < \varepsilon,$$

where $B_g(p, r)$ denotes the ball in $\mathbb{H}^n$ with centre p and radius r with respect to the metric g. By a partition of unity argument, there exist functions f^ε, $g_i^\varepsilon \in C^\infty(\Omega; [0, 1])$, $i = 1, \dots, M$, such that:

(i) $f^\varepsilon + g_1^\varepsilon + \cdots + g_M^\varepsilon = \chi_\Omega$;

(ii) $f^\varepsilon = 0$ on $\bigcup_{i=1}^{M} B_g(p_i, r_i/2)$;

(iii) for each i, the support of g_i^ε is contained in $B_g(p_i, r_i)$;

(iv) $|\nabla g_i^\varepsilon|_g \le C r_i^{-1}$ for a constant $C > 0$ independent of ε.

Hence, for any horizontal section $\varphi \in C_c^1(\Omega; H\Sigma)$, we have

$$\begin{aligned}
\int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma &= \int_F \operatorname{div}_g(f^\varepsilon \varphi) \, d\sigma_\Sigma + \sum_{i=1}^{M} \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g(g_i^\varepsilon \varphi) \, d\sigma_\Sigma \\
&= \int_{\partial F \setminus N} \langle f^\varepsilon \varphi, N_{\partial F} \rangle_g \, d\lambda_{\partial F \setminus N} + \sum_{i=1}^{M} \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g(g_i^\varepsilon \varphi) \, d\sigma_\Sigma,
\end{aligned} \tag{2-13}$$

where, by (iv),

$$\left| \sum_{i=1}^{M} \int_{F \cap B_g(p_i, r_i)} \operatorname{div}_g(g_i^\varepsilon \varphi) \, d\sigma_\Sigma \right| \le \sum_{i=1}^{M} \int_{B_g(p_i, r_i)} (\| \operatorname{div}_g \varphi \|_{L^\infty} + C r_i^{-1}) \, d\sigma_\Sigma \le C' \sum_{i=1}^{M} r_i^{2n-1} \le C' \varepsilon \tag{2-14}$$

with a constant $C' > 0$ independent of ε.

Letting $\varepsilon \to 0$, we have $f^\varepsilon \to 1$ pointwise on $\partial F \setminus N$, by (i) and (iii). Then, from (2-13) and (2-14), we obtain

$$\int_F \operatorname{div}_g \varphi \, d\sigma_\Sigma = \int_{\partial F \setminus N} \langle \varphi, N_{\partial F} \rangle_g \, d\lambda_{\partial F \setminus N}$$

and claim (2-12) follows by standard arguments. □

2B. ***Proof of Theorem 1.5.*** Let $\Omega \subset \mathbb{H}^n$ be an open set and $u \in C^\infty(\Omega)$. By Sard's theorem, for a.e. $s \in \mathbb{R}$ the level set

$$\Sigma^s = \{p \in \Omega : u(p) = s\}$$

is a smooth hypersurface and, moreover, we have $\nabla u \neq 0$ on Σ^s.

Let $E \subset \mathbb{H}^n$ be a Borel set such that $E \cap \Sigma^s$ has (locally) finite H-perimeter in $\Omega \cap \Sigma^s$, in the sense of Definition 2.1. Then on $\Omega \cap \Sigma^s$ we have the H-perimeter measure $\mu^{\Sigma^s}_{E\cap\Sigma^s}$ induced by $E \cap \Sigma^s$. We shall use the notation

$$\mu^s_E = \mu^{\Sigma^s}_{E\cap\Sigma^s}$$

to denote a measure on Ω that is supported on $\Omega \cap \Sigma^s$.

We start with the following coarea formula in the smooth case, which is deduced from the Riemannian formula.

Lemma 2.4. *Let $\Omega \subset \mathbb{H}^n$ be an open set and $u \in C^\infty(\Omega)$. Let $E \subset \mathbb{H}^n$ be an open set with C^∞ boundary in Ω such that $\mu_E(\Omega) < \infty$. Then we have*

$$\int_{\mathbb{R}} \int_\Omega \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu^s_E\, ds = \int_\Omega \sqrt{|\nabla_H u|^2_g - \langle \nu_E, \nabla_H u\rangle^2_g}\, d\mu_E, \tag{2-15}$$

where μ_E is the H-perimeter measure of E and ν_E is its horizontal normal.

Proof. The integral in the left-hand side is well defined, because for a.e. $s \in \mathbb{R}$ there holds $\nabla u \neq 0$ on Σ^s. By the coarea formula for Riemannian manifolds — see, e.g., [Burago and Zalgaller 1988] — for any Borel function $h : \partial E \to [0, \infty]$ we have

$$\int_{\mathbb{R}} \int_{\partial E\cap\Sigma^s} h\, d\lambda_{\partial E\cap\Sigma^s}\, ds = \int_{\partial E} h |\nabla^{\partial E} u|_g\, d\sigma_{\partial E}, \tag{2-16}$$

where $\nabla^{\partial E} u$ is the tangential gradient of u on ∂E. Then we have

$$\nabla^{\partial E} u = \nabla u - \langle \nabla u, N_{\partial E}\rangle_g N_{\partial E} \quad \text{and} \quad |\nabla^{\partial E} u|_g = \sqrt{|\nabla u|^2_g - \langle \nabla u, N_{\partial E}\rangle^2_g}. \tag{2-17}$$

Step 1. Let us define the set

$$C = \left\{ p \in \partial E \cap \Omega : \nabla u(p) \neq 0 \ \text{ and } \ N_{\partial E}(p) = \pm \frac{\nabla u(p)}{|\nabla u(p)|_g} \right\}.$$

If $s \in \mathbb{R}$ is such that $\nabla u \neq 0$ on Σ^s, then $C \cap \Sigma^s$ is a closed set in Σ^s. Using the coarea formula (2-16) with the function $h = \chi_C$, we get

$$\int_{\mathbb{R}} \lambda_{\partial E\cap\Sigma^s}(C)\, ds = \int_C |\nabla^{\partial E} u|_g\, d\sigma_{\partial E} = 0,$$

because we have $\nabla^{\partial E} u = 0$ on C. In particular, we deduce that

$$C \cap \Sigma^s \ \text{ is a closed set in } \ \Sigma^s \quad \text{and} \quad \lambda_{\partial E\cap\Sigma^s}(C \cap \Sigma^s) = 0 \quad \text{for a.e. } s \in \mathbb{R}. \tag{2-18}$$

If $p \in \Sigma^s$ is a point such that $\nabla u(p) \neq 0$ and $p \notin C$, then Σ^s is a smooth hypersurface in a neighbourhood of p and $E^s = E \cap \Sigma^s$ is a domain in Σ^s with smooth boundary in a neighbourhood of p. Moreover, we have $(\partial E \cap \Sigma^s) \setminus C = \partial E^s \setminus C$. Then, from (2-18) and Lemma 2.3 we conclude that for a.e. $s \in \mathbb{R}$ we have

$$\mu^s_E = |N^{H\Sigma^s}_{\partial E^s}|_g \lambda_{\partial E^s}. \tag{2-19}$$

By (2-18) and (2-19),

$$\mu^s_E(C \cap \Sigma^s) = \int_{C\cap\Sigma^s} |N^{H\Sigma^s}_{\partial E^s}|_g \, d\lambda_{\partial E^s} = 0 \quad \text{for a.e. } s \in \mathbb{R}. \tag{2-20}$$

Step 2. We prove (2-15) by plugging into (2-16) the Borel function $h : \partial E \to [0, \infty]$,

$$h = \begin{cases} \dfrac{|N^H_{\partial E}|_g \sqrt{|\nabla_H u|^2_g - \langle \nu_E, \nabla_H u\rangle^2_g}}{|\nabla u|_g \sqrt{1 - \langle N_{\partial E}, \nabla u/|\nabla u|_g\rangle^2_g}} & \text{on } \partial E \setminus (C \cup \{\nabla u = 0\}), \\ 0 & \text{on } C \cup \{\nabla u = 0\}. \end{cases}$$

Above, $N^H_{\partial E}$ is the projection of the Riemannian normal $N_{\partial E}$ onto H and ν_E is the horizontal normal. Namely, we have

$$N^H_{\partial E} = N_{\partial E} - \langle N_{\partial E}, T\rangle_g T \quad \text{and} \quad \nu_E = \frac{N^H_{\partial E}}{|N^H_{\partial E}|_g}.$$

The H-perimeter measure of E is

$$\mu_E = |N^H_{\partial E}|_g \sigma_{\partial E}. \tag{2-21}$$

Using (2-17) and (2-21), we find

$$\begin{aligned} \int_{\partial E} h|\nabla^{\partial E} u| \, d\sigma_{\partial E} &= \int_{\partial E\setminus(C\cup\{\nabla u=0\})} |N^H_{\partial E}|_g \sqrt{|\nabla_H u|^2_g - \langle \nu_E, \nabla_H u\rangle^2_g} \, d\sigma_{\partial E} \\ &= \int_{\partial E\setminus(C\cup\{\nabla u=0\})} \sqrt{|\nabla_H u|^2_g - \langle \nu_E, \nabla_H u\rangle^2_g} \, d\mu_E \\ &= \int_{\partial E} \sqrt{|\nabla_H u|^2_g - \langle \nu_E, \nabla_H u\rangle^2_g} \, d\mu_E, \end{aligned} \tag{2-22}$$

where the last equality is justified by the fact that if $p \in C \cup \{\nabla u = 0\}$ then

$$\sqrt{|\nabla_H u(p)|^2_g - \langle \nu_E(p), \nabla_H u(p)\rangle^2_g} = 0.$$

For a.e. $s \in \mathbb{R}$, we have $\nabla u \neq 0$ on Σ^s. Using (2-21) and the fact that $h = 0$ on $C \cup \{\nabla_H u = 0\}$, letting $\Lambda^s = (\partial E \cap \Sigma^s) \setminus (C \cup \{\nabla_H u = 0\})$, we obtain

$$\begin{aligned} \int_{\mathbb{R}} \int_{\partial E\cap\Sigma^s} h \, d\lambda_{\partial E^s} \, ds &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|N^H_{\partial E}|_g \sqrt{|\nabla_H u|^2_g - \langle \nu_E, \nabla_H u\rangle^2_g}}{|\nabla u|_g \sqrt{1 - \langle N_{\partial E}, \nabla u/|\nabla u|_g\rangle^2_g}} \, d\lambda_{\partial E^s} \, ds \\ &= \int_{\mathbb{R}} \int_{\Lambda^s} \frac{|\nabla_H u|_g}{|\nabla u|_g} \vartheta^s \, d\lambda_{\partial E^s} \, ds, \end{aligned} \tag{2-23}$$

where we let

$$\vartheta^s = \frac{\sqrt{|N^H_{\partial E}|^2_g - \langle N^H_{\partial E}, \nabla_H u/|\nabla_H u|_g\rangle^2_g}}{\sqrt{1 - \langle N_{\partial E}, \nabla u/|\nabla u|_g\rangle^2_g}}.$$

We will prove in Step 3 that, for any $s \in \mathbb{R}$ such that $\nabla u \neq 0$ on Σ^s,

$$\vartheta^s = |N^{H\Sigma^s}_{\partial E^s}|_g \quad \text{on } \Lambda^s. \tag{2-24}$$

Using (2-24), (2-19), and (2-20), formula (2-23) becomes

$$\begin{aligned}\int_{\mathbb{R}}\int_{\partial E\cap\Sigma^s} h\,d\lambda_{\partial E\cap\Sigma^s}\,ds &= \int_{\mathbb{R}}\int_{\Lambda^s}\frac{|\nabla_H u|_g}{|\nabla u|_g}|N^{H\Sigma^s}_{\partial E^s}|_g\,d\lambda_{\partial E^s}\,ds\\ &= \int_{\mathbb{R}}\int_{\Lambda^s}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu^s_E\,ds\\ &= \int_{\mathbb{R}}\int_{\partial E\cap\Sigma^s}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu^s_E\,ds.\end{aligned}\tag{2-25}$$

The proof is complete, because (2-15) follows from (2-16), (2-22), and (2-25).

Step 3. We prove claim (2-24). Let us introduce the vector field W in $\Omega\setminus\{\nabla_H u=0\}$,

$$W=\frac{Tu}{|\nabla u|_g}\frac{\nabla_H u}{|\nabla_H u|_g}-\frac{|\nabla_H u|_g}{|\nabla u|_g}T.$$

It can be checked that $|W|_g=1$ and $Wu=0$. In particular, for a.e. s we have $W\in T\Sigma^s$. Moreover, W is g-orthogonal to $H\Sigma^s$ because any vector in $H\Sigma^s$ is orthogonal both to $\nabla_H u$ and to T. It follows that

$$N^{H\Sigma^s}_{\partial E^s}=N_{\partial E^s}-\langle N_{\partial E^s},W\rangle_g$$

and, in particular,

$$|N^{H\Sigma^s}_{\partial E^s}|^2_g=1-\langle N_{\partial E^s},W\rangle^2_g.$$

Starting from the formula

$$N_{\partial E^s}=\frac{N_{\partial E}-\langle N_{\partial E},\nabla u/|\nabla u|_g\rangle_g\nabla u/|\nabla u|_g}{|N_{\partial E}-\langle N_{\partial E},\nabla u/|\nabla u|_g\rangle_g\nabla u/|\nabla u|_g|_g}=\frac{N_{\partial E}-\langle N_{\partial E},\nabla u/|\nabla u|_g\rangle_g\nabla u/|\nabla u|_g}{\sqrt{1-\langle N_{\partial E},\nabla u/|\nabla u|_g\rangle^2_g}},$$

we find

$$|N^{H\Sigma^s}_{\partial E^s}|^2_g=\frac{M}{1-\langle N_{\partial E},\nabla u/|\nabla u|_g\rangle^2_g},$$

where we let

$$M=1-\left\langle N_{\partial E},\frac{\nabla u}{|\nabla u|_g}\right\rangle^2_g-\left\langle N_{\partial E}-\left\langle N_{\partial E},\frac{\nabla u}{|\nabla u|_g}\right\rangle_g\frac{\nabla u}{|\nabla u|_g},W\right\rangle^2_g.$$

We claim that, on the open set $\{\nabla_H u\neq 0\}$,

$$M=|N^H_{\partial E}|^2_g-\left\langle N^H_{\partial E},\frac{\nabla_H u}{|\nabla_H u|_g}\right\rangle^2_g,\tag{2-26}$$

and formula (2-24) follows from (2-26). Using the identity $\nabla u=\nabla_H u+(Tu)T$ and the orthogonality

$$\left\langle N_{\partial E}-\left\langle N_{\partial E},\frac{\nabla u}{|\nabla u|_g}\right\rangle_g\frac{\nabla u}{|\nabla u|_g},\nabla u\right\rangle_g=0,$$

we find

$$
\begin{aligned}
M &= 1-\left\langle N_{\partial E}, \frac{\nabla_H u+(Tu)T}{|\nabla u|_g}\right\rangle_g^2 - \left(\frac{Tu}{|\nabla u|_g}\left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\right\rangle_g - \frac{|\nabla_H u|_g}{|\nabla u|_g}\langle N_{\partial E}, T\rangle_g\right)^2 \\
&= 1-\left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\right\rangle_g^2 \frac{|\nabla_H u|_g^2+(Tu)^2}{|\nabla u|_g^2} - \langle N_{\partial E}, T\rangle_g^2 \frac{|\nabla_H u|_g^2+(Tu)^2}{|\nabla u|_g^2} \\
&= 1-\left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\right\rangle_g^2 - \langle N_{\partial E}, T\rangle_g^2 \\
&= 1-\langle N_{\partial E}, T\rangle_g^2 - \left(\left\langle N_{\partial E}, \frac{\nabla_H u}{|\nabla_H u|_g}\right\rangle_g - \left\langle \langle N_{\partial E}, T\rangle_g T, \frac{\nabla_H u}{|\nabla_H u|_g}\right\rangle_g\right)^2 \\
&= |N_{\partial E}^H|_g^2 - \left\langle N_{\partial E}^H, \frac{\nabla_H u}{|\nabla_H u|_g}\right\rangle_g^2.
\end{aligned}
\tag{2-27}
$$

This ends the proof. ☐

We prove a coarea inequality:

Proposition 2.5. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ a set with finite H-perimeter in Ω, and let $h : \partial E \to [0, \infty]$ be a Borel function. Then we have*

$$
\int_{\mathbb{R}} \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g}\, d\mu_E^s\, ds \le \int_\Omega h \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2}\, d\mu_E. \tag{2-28}
$$

Proof. The coarea inequality (2-28) follows from the smooth case of Lemma 2.4 by an approximation and lower semicontinuity argument.

Step 1. By [Franchi et al. 1996, Theorem 2.2.2], there exists a sequence of smooth sets $(E_j)_{j\in\mathbb{N}}$ in Ω such that

$$
\chi_{E_j} \xrightarrow{L^1(\Omega)} \chi_E \quad \text{as } j\to\infty \qquad \text{and} \qquad \lim_{j\to\infty} \mu_{E_j}(\Omega) = \mu(\Omega).
$$

By a straightforward adaptation of the proof of [Ambrosio et al. 2000, Proposition 3.13], we also have that $\nu_{E_j}\mu_{E_j} \to \nu_E \mu_E$ weakly* in Ω. Namely, for any $\psi \in C_c(\Omega; H)$,

$$
\lim_{j\to\infty} \int_\Omega \langle \psi, \nu_{E_j}\rangle_g\, d\mu_{E_j} = \int_\Omega \langle \psi, \nu_E\rangle_g\, d\mu_E.
$$

Let $A \Subset \Omega$ be an open set such that $\lim_{j\to\infty} \mu_{E_j}(A) = \mu_E(A)$. By Reshetnyak's continuity theorem (see, e.g., [Ambrosio et al. 2000, Theorem 2.39]), we have

$$
\lim_{j\to\infty} \int_A f(p, \nu_{E_j}(p))\, d\mu_{E_j} = \int_A f(p, \nu_E(p))\, d\mu_E
$$

for any continuous and bounded function f. In particular,

$$
\lim_{j\to\infty} \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_{E_j}, \nabla_H u\rangle_g^2}\, d\mu_{E_j} = \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u\rangle_g^2}\, d\mu_E. \tag{2-29}
$$

Step 2. Let $(E_j)_{j\in\mathbb{N}}$ be the sequence introduced in Step 1. Then, for a.e. $s\in\mathbb{R}$, we have

$$\nabla u \neq 0 \quad \text{on } \Sigma^s \qquad \text{and} \qquad \chi_{E_j}\to\chi_E \quad \text{in } L^1(\Sigma^s,\sigma_{\Sigma^s}) \text{ as } j\to\infty.$$

In particular, for any such s and for any open set $A\subset\Sigma^s\cap\Omega$,

$$\mu_E^s(A)\le\liminf_{j\to\infty}\mu_{E_j}^s(A).$$

From Fatou's lemma and the continuity of $|\nabla_H u|_g/|\nabla u|_g$ on Σ^s, it follows that

$$\begin{aligned}\int_A\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s&=\int_0^\infty\mu_E^s\Big(\Big\{p\in A:\frac{|\nabla_H u|_g}{|\nabla u|_g}(p)>t\Big\}\Big)\,dt\\&\le\int_0^\infty\liminf_{j\to\infty}\mu_{E_j}^s\Big(\Big\{p\in A:\frac{|\nabla_H u|_g}{|\nabla u|_g}(p)>t\Big\}\Big)\,dt\\&\le\liminf_{j\to\infty}\int_0^\infty\mu_{E_j}^s\Big(\Big\{p\in A:\frac{|\nabla_H u|_g}{|\nabla u|_g}(p)>t\Big\}\Big)\,dt\\&=\liminf_{j\to\infty}\int_A\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_{E_j}^s.\end{aligned}$$

Using again Fatou's lemma and Lemma 2.4,

$$\begin{aligned}\int_{\mathbb{R}}\int_A\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds&\le\int_{\mathbb{R}}\liminf_{j\to\infty}\int_A\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_{E_j}^s\,ds\\&\le\liminf_{j\to\infty}\int_{\mathbb{R}}\int_A\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_{E_j}^s\,ds\\&=\liminf_{j\to\infty}\int_A\sqrt{|\nabla_H u|_g^2-\langle\nu_{E_j},\nabla_H u\rangle_g^2}\,d\mu_{E_j}.\end{aligned}$$

This, together with (2-29), gives

$$\int_{\mathbb{R}}\int_A\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds\le\int_A\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.$$

Step 3. Any open set $A\subset\Omega$ can be approximated by a sequence $(A_k)_{k\in\mathbb{N}}$ of open sets such that

$$A_k\Subset\Omega,\qquad A_k\subset A_{k+1},\qquad\bigcup_{k=1}^\infty A_k=A\quad\text{and}\quad\mu_E(\partial A_k)=0.$$

In particular, for each $k\in\mathbb{N}$, we have

$$\liminf_{j\to\infty}\mu_{E_j}(A_k)\le\limsup_{j\to\infty}\mu_{E_j}(\bar A_k)\le\mu_E(\bar A_k)=\mu_E(A_k)\le\liminf_{j\to\infty}\mu_{E_j}(A_k).$$

Hence, the inequalities are equalities, i.e., $\mu_E(A_k)=\lim_{j\to\infty}\mu_{E_j}(A_k)$. By Step 2, for any $k\in\mathbb{N}$,

$$\int_{\mathbb{R}}\int_{A_k}\frac{|\nabla_H u|_g}{|\nabla u|_g}\,d\mu_E^s\,ds\le\int_{A_k}\sqrt{|\nabla_H u|_g^2-\langle\nu_E,\nabla_H u\rangle_g^2}\,d\mu_E.$$

By monotone convergence, letting $k \to \infty$ we obtain, for any open set $A \subset \Omega$,

$$\int_{\mathbb{R}} \int_A \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \le \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E.$$

By a standard approximation argument, it is enough to prove (2-28) for the characteristic function $h = \chi_B$ of a Borel set $B \subset \partial E$. Since the measure $\sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, \mu_E$ is a Radon measure on ∂E, there exists a sequence of open sets B_j such that $B \subset B_j$ for each $j \in \mathbb{N}$ and

$$\lim_{j \to \infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E = \int_B \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E.$$

Therefore, we have

$$\begin{aligned} \int_{\mathbb{R}} \int_B \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds &\le \liminf_{j \to \infty} \int_{\mathbb{R}} \int_{B_j} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \\ &\le \lim_{j \to \infty} \int_{B_j} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E \\ &= \int_B \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E, \end{aligned}$$

and this concludes the proof. □

In the next step, we prove an approximate coarea formula for sets E such that the boundary ∂E is an H-regular surface.

Lemma 2.6. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $E \subset \mathbb{H}^n$ an open set such that $\partial E \cap \Omega$ is an H-regular hypersurface, and $\bar{p} \in \partial E \cap \Omega$ a point such that*

$$\nabla_H u(\bar{p}) \neq 0 \quad \text{and} \quad \nu_E(\bar{p}) \neq \pm \frac{\nabla_H u(\bar{p})}{|\nabla_H u(\bar{p})|_g}.$$

Then, for any $\varepsilon > 0$, there exists $\bar{r} = \bar{r}(\bar{p}, \varepsilon) > 0$ such that $B_{\bar{r}}(\bar{p}) \subset \Omega$ and, for any $r \in (0, \bar{r})$,

$$\begin{aligned} (1-\varepsilon) \int_{B_r(\bar{p})} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E &\le \int_{\mathbb{R}} \int_{B_r(\bar{p})} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \\ &\le (1+\varepsilon) \int_{B_r(\bar{p})} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E. \end{aligned}$$

Proof. We can, without loss of generality, assume that $\bar{p} = 0$ and $u(0) = 0$. We divide the proof into several steps.

Step 1: preliminary considerations. The horizontal vector field $V_{2n} = \nabla_H u / |\nabla_H u|_g$ is well defined in a neighbourhood $\Omega_\varepsilon \subset \mathbb{H}^n$ of 0. For any $s \in \mathbb{R}$, the hypersurface $\Sigma^s = \{p \in \Omega : u(p) = s\}$ is smooth in Ω_ε because $\nabla_H u \neq 0$ on Ω_ε.

There are horizontal vector fields $V_1, \dots, V_{2n-1}$ on Ω_ε such that $V_1, \dots, V_{2n}$ is a g-orthonormal frame. In particular, we have $V_j u = 0$ for $j = 1, \dots, 2n-1$, i.e.,

$$H_p \Sigma^s = \operatorname{span}\{V_1(p), \dots, V_{2n-1}(p)\} \quad \text{for all } p \in \Sigma^s \cap \Omega_\varepsilon. \tag{2-30}$$

Possibly shrinking Ω_ε, reordering $\{V_j\}_{j=1,\dots,2n-1}$, and changing the sign of V_1, we can assume (see [Vittone 2012, Lemmas 4.3 and 4.4]) that there exist a function $f:\Omega_\varepsilon\to\mathbb{R}$ and a number $\delta>0$ such that:

(a) $f\in C^1_H(\Omega_\varepsilon)\cap C^\infty(\Omega_\varepsilon\setminus\partial E)$;

(b) $E\cap\Omega_\varepsilon=\{p\in\Omega_\varepsilon : f(p)>0\}$;

(c) $V_1 f\geq\delta>0$ on Ω_ε.

By [Vittone 2012, Remark 4.7], we also have $\nu_E=\nabla_H f/|\nabla_H f|_g$ on $\partial E\cap\Omega_\varepsilon$.

Step 2: change of coordinates. Let $S\subset\mathbb{H}^n$ be a $(2n-1)$-dimensional smooth submanifold such that:

(i) $0\in S$.

(ii) $S\subset\Sigma^0\cap\Omega_\varepsilon$. In particular, ∇u is g-orthogonal to S.

(iii) $V_1(0)$ is g-orthogonal to S at 0.

(iv) There exists a diffeomorphism $H:U\to\mathbb{H}^n$, where $U\subset\mathbb{R}^{2n-1}$ is an open set with $0\in U$, such that $H(0)=0$ and $H(U)=S\cap\Omega_\varepsilon$.

(v) The area element JH of H satisfies $JH(0)=1$. Namely,
$$JH(0)=\lim_{r\to0}\frac{\lambda_S(H(B^E_r))}{\mathscr{L}^{2n-1}(B^E_r)}=1,$$
where $B^E_r=\{p\in\mathbb{R}^{2n-1}:|p|<r\}$ is a Euclidean ball and λ_S is the Riemannian $(2n-1)$-volume measure on S induced by g.

For small enough $a,b>0$, and possibly shrinking U and Ω_ε, the mapping $G:(-a,a)\times(-b,b)\times U\to\mathbb{H}^n$,
$$G(v,z,w)=\exp(vV_1)\exp\left(z\frac{\nabla u}{|\nabla u|_g^2}\right)(H(w))$$
is a diffeomorphism from $\widetilde{\Omega}_\varepsilon=(-a,a)\times(-b,b)\times U$ onto Ω_ε. The differential of G satisfies
$$dG\left(\frac{\partial}{\partial v}\right)=V_1\quad\text{and}\quad dG(0)\left(\frac{\partial}{\partial z}\right)=\frac{\nabla u(0)}{|\nabla u(0)|_g^2}.$$
Moreover, the tangent space $T_0S=\operatorname{Im}dH(0)$ is g-orthogonal to $V_1(0)$ and $\nabla u(0)/|\nabla u(0)|^2_g$. We denote by G_z the restriction of G to $(-a,a)\times\{z\}\times U$, i.e., $G_z(v,w)=G(v,z,w)$. From the above considerations, we deduce that the area elements of G and G_0 satisfy
$$JG(0)=\frac{1}{|\nabla u(0)|_g}\quad\text{and}\quad JG_0(0)=1.$$
Then, possibly shrinking $\widetilde{\Omega}_\varepsilon$ further, we have
$$(1-\varepsilon)JG(v,z,w)\leq\frac{JG_z(v,w)}{|\nabla u\circ G(v,z,w)|_g}\leq(1+\varepsilon)JG(v,z,w)\tag{2-31}$$
for all $(v,z,w)\in\widetilde{\Omega}_\varepsilon$.

For $j = 1, \dots, 2n$, we define on $\widetilde{\Omega}_\varepsilon$ the vector fields $\widetilde{V}_j = (dG)^{-1}(V_j)$. By the definition of G, we have $\widetilde{V}_1 = \partial/\partial v$. We also define $\tilde{u} = u \circ G \in C^\infty(\widetilde{\Omega}_\varepsilon)$, $\tilde{f} = f \circ G : \widetilde{\Omega}_\varepsilon \to \mathbb{R}$, and $\widetilde{E} = G^{-1}(E)$. Then:

(1) $\widetilde{E} = \{q \in \widetilde{\Omega}_\varepsilon : \tilde{f}(q) > 0\}$.

(2) $\tilde{f} \in C^\infty(\widetilde{\Omega}_\varepsilon \setminus \partial \widetilde{E})$.

(3) The derivative $\widetilde{V}_j \tilde{f}$ is defined in the sense of distributions with respect to the measure $\mu = JG\mathscr{L}^{2n+1}$. Namely, for all $\psi \in C_c^\infty(\widetilde{\Omega}_\varepsilon)$, we have

$$\int_{\widetilde{\Omega}_\varepsilon} (\widetilde{V}_j \tilde{f})\, \psi \, d\mu = - \int_{\widetilde{\Omega}_\varepsilon} \tilde{f}\, \widetilde{V}_j^* \psi \, d\mu,$$

where $\widetilde{V}_j^*$ is the adjoint operator of $\widetilde{V}_j$ with respect to μ. Then $\widetilde{V}_j \tilde{f} = (V_j f) \circ G$ and so $\widetilde{V}_j \tilde{f}$ is a continuous function for any $j = 1, \dots, 2n$. In particular, $\widetilde{V}_1 \tilde{f} = \partial_v \tilde{f} \geq \delta > 0$.

Step 3: approximate coarea formula. We follow the argument of [Vittone 2012, Propositions 4.1 and 4.5]; see also Remark 4.7 therein.

Possibly shrinking $\widetilde{\Omega}_\varepsilon$ and Ω_ε, there exists a continuous function $\phi : (-b, b) \times U \to (-a, a)$ such that:

(A) $\partial \widetilde{E} \cap \widetilde{\Omega}_\varepsilon$ is the graph of ϕ. Namely, letting $\Phi : (-b, b) \times U \to \mathbb{R}^{2n+1}$, $\Phi(z, w) = (\phi(z, w), z, w)$, we have

$$\partial \widetilde{E} \cap \widetilde{\Omega}_\varepsilon = \Phi((-b, b) \times U).$$

(B) The measure μ_E is

$$\mu_E \llcorner \Omega_\varepsilon = (G \circ \Phi)_\# \left(\left(\frac{|\widetilde{V} \tilde{f}|}{\widetilde{V}_1 \tilde{f}} JG \right) \circ \Phi \, \mathscr{L}^{2n} \llcorner ((-b, b) \times U) \right), \tag{2-32}$$

where $(G \circ \Phi)_\#$ denotes the push-forward and

$$|\widetilde{V} \tilde{f}| = \left(\sum_{j=1}^{2n} (\widetilde{V}_j \tilde{f})^2 \right)^{\frac{1}{2}}.$$

Using $V_1 u = 0$ and $u \circ H = 0$ (this follows from $H(U) = S \cap \Omega_\varepsilon \subset \Sigma^0 \cap \Omega_\varepsilon$), we obtain

$$\begin{aligned} \tilde{u}(v, z, w) = u(G(v, z, w)) = u\left(\exp(v V_1) \exp\left(z \frac{\nabla u}{|\nabla u|_g^2} \right)(H(w)) \right) &= u\left(\exp\left(z \frac{\nabla u}{|\nabla u|_g^2} \right)(H(w)) \right) \\ &= z + u(H(w)) = z. \end{aligned}$$

In particular, from $\tilde{u} = u \circ G$, we deduce that

$$G^{-1}(\Sigma^s \cap \Omega_\varepsilon) = (-a, a) \times \{s\} \times U.$$

We denote by JG_s the Jacobian (area element) of G_s. We also define the restriction $\Phi_s : U \to \mathbb{R}^{2n+1}$, $\Phi_s(w) = \Phi(s, w)$, for any $s \in (-b, b)$.

By (2-30), for any $s \in \mathbb{R}$, the measure $\mu_E^s = \mu_{E \cap \Sigma_s}^{\Sigma^s}$ is the horizontal perimeter of $E \cap \Sigma^s$ with respect to the Carnot–Carathéodory structure induced by the family $V_1, \dots, V_{2n-1}$ on Σ^s. We can repeat the argument that led to (2-32) to obtain

$$\mu_E^s \llcorner \Omega_\varepsilon = (G \circ \Phi_s)_\# \bigg(\bigg(\frac{|\widetilde{V}' \tilde{f}|}{\widetilde{V}_1 \tilde{f}} \, JG_s \bigg) \circ \Phi_s \mathscr{L}^{2n-1} \llcorner U \bigg), \tag{2-33}$$

where $\widetilde{V}' \tilde{f} = (\widetilde{V}_1 \tilde{f}, \dots, \widetilde{V}_{2n-1} \tilde{f})$. We omit the details of the proof of (2-33). The proof is a line-by-line repetition of Proposition 4.5 in [Vittone 2012] with the sole difference that now the horizontal perimeter is defined in a curved manifold.

Let us fix $\bar{r} > 0$ such that $B_{\bar{r}} \subset \Omega_\varepsilon$ and, for any $r \in (0, \bar{r})$, let

$$A_{s,r} = \{ w \in U : G(0, s, w) \in B_r \} \quad \text{and} \quad A_r = \{ (s, w) \in (-b, b) \times U : w \in A_{s,r} \}.$$

By the Fubini–Tonelli theorem and (2-33), the function

$$s \mapsto \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s = \int_{A_{s,r}} \bigg(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \bigg) \bigg(\frac{|\widetilde{V}' \tilde{f}|}{\widetilde{V}_1 \tilde{f}} \, JG_s \bigg) \circ \Phi_s \, d\mathscr{L}^{2n-1} \tag{2-34}$$

is $\mathscr{L}^1$-measurable. Here and hereafter, the composition $\cdot \circ \Phi_s$ acts on the product. Thus, from the Fubini–Tonelli theorem and (2-31), we obtain

$$\begin{aligned}
&\int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \\
&\qquad = \int_{\mathbb{R}} \int_{A_{s,r}} \bigg(\frac{|\nabla_H u|_g}{|\nabla u|_g} \circ G \bigg) \bigg(\frac{|\widetilde{V}' \tilde{f}|}{\widetilde{V}_1 \tilde{f}} \, JG_s \bigg) \circ \Phi_s(w) \, d\mathscr{L}^{2n-1}(w) \, ds \\
&\qquad = \int_{A_r} (|\nabla_H u|_g \circ G) \bigg(\frac{|\widetilde{V}' \tilde{f}|}{\widetilde{V}_1 \tilde{f}} \, \frac{JG_s}{|\nabla u|_g \circ G} \bigg) \circ \Phi(s, w) \, d\mathscr{L}^{2n}(s, w) \\
&\qquad \le (1+\varepsilon) \int_{A_r} (|\nabla_H u|_g \circ G) \bigg(\frac{|\widetilde{V} \tilde{f}|}{\widetilde{V}_1 \tilde{f}} \sqrt{1 - (\widetilde{V}_{2n} \tilde{f})^2 / |\widetilde{V} \tilde{f}|^2} \, JG \bigg) \circ \Phi(s, w) \, d\mathscr{L}^{2n}(s, w).
\end{aligned} \tag{2-35}$$

From the identity

$$\frac{\widetilde{V}_{2n} \tilde{f}}{|\widetilde{V} \tilde{f}|} = \frac{V_{2n} f}{|\nabla_H f|_g} \circ G = \bigg\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \frac{\nabla_H f}{|\nabla_H f|_g} \bigg\rangle_g \circ G = \bigg\langle \frac{\nabla_H u}{|\nabla_H u|_g}, \nu_E \bigg\rangle_g \circ G \tag{2-36}$$

and from (2-32), we deduce that

$$\begin{aligned}
\int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds &\le (1+\varepsilon) \int_{B_r} |\nabla_H u|_g \sqrt{1 - \langle \nabla_H u / |\nabla_H u|_g, \nu_E \rangle_g^2} \, d\mu_E \\
&= (1+\varepsilon) \int_{B_r} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E.
\end{aligned} \tag{2-37}$$

In a similar way, we obtain

$$\int_{\mathbb{R}} \int_{B_r} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \ge (1-\varepsilon) \int_{B_r} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E.$$

This concludes the proof. □

We can now prove the coarea formula for H-regular boundaries.

Proposition 2.7. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$, and let $E \subset \mathbb{H}^n$ be an open domain such that $\partial E \cap \Omega$ is an H-regular hypersurface. Then*

$$\int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds = \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E. \tag{2-38}$$

Proof. Let us define the set

$$A = \left\{ p \in \partial E \cap \Omega : \nabla_H u(p) \neq 0 \ \text{ and } \ \nu_E(p) \neq \pm \frac{\nabla_H u(p)}{|\nabla_H u(p)|_g} \right\}.$$

The set A is relatively open in $\partial E \cap \Omega$. Let $\varepsilon > 0$ be fixed. Since the measure μ_E is locally doubling on $\partial E \cap \Omega$ (see, e.g., [Vittone 2012, Corollary 4.13]), by Lemma 2.6 and the Vitali covering theorem (see, e.g., [Heinonen 2001, Theorem 1.6]) there exists a countable (or finite) collection of balls $B_{r_i}(p_i)$, $i \in \mathbb{N}$, such that:

(i) for any $i \in \mathbb{N}$ we have $p_i \in A$ and $0 < r_i < \bar{r}(p_i, \varepsilon)$, where $\bar{r}$ is as in the statement of Lemma 2.6;

(ii) the balls $B_{r_i}(p_i)$ are contained in A and pairwise disjoint;

(iii) $\mu_E\big(A \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)\big) = 0$.

It follows that we have

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds &\leq (1+\varepsilon) \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E \\
&= (1+\varepsilon) \int_A \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E \\
&= (1+\varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E.
\end{aligned} \tag{2-39}$$

The last equality follows from the fact that $\sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} = 0$ outside A. In the same way, one also obtains

$$\int_{\mathbb{R}} \int_{\bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \geq (1-\varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E. \tag{2-40}$$

Moreover, by Proposition 2.5,

$$\int_{\mathbb{R}} \int_{\Omega \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \leq \int_{\Omega \setminus \bigcup_{i \in \mathbb{N}} B_{r_i}(p_i)} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E = 0.$$

In particular, the integral on the left-hand side of the last inequality is 0 and, by (2-39) and (2-40), we obtain

$$(1-\varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E \leq \int_{\mathbb{R}} \int_{\Omega} \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds \leq (1+\varepsilon) \int_{\Omega} \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E.$$

Since $\varepsilon > 0$ is arbitrary, this concludes the proof. $\square$

By a standard approximation argument, we also have this extension of the coarea formula (2-38):

Proposition 2.8. *Let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$, and let E be an open domain such that $\partial E \cap \Omega$ is an H-regular hypersurface. Then, for any Borel function $h : \partial E \to [0, \infty)$,*

$$\int_{\mathbb{R}} \int_\Omega h \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_E^s \, ds = \int_\Omega h \sqrt{|\nabla_H u|_g^2 - \langle \nu_E, \nabla_H u \rangle_g^2} \, d\mu_E.$$

Our next step is to prove the coarea formula for $\mathscr{S}^{2n+1}$-rectifiable sets.

Lemma 2.9. *Let $R \subset \mathbb{H}^n$ be an $\mathscr{S}^{2n+1}$-rectifiable set. Then, there exists a Borel $\mathscr{S}^{2n+1}$-rectifiable set $R' \subset \mathbb{H}^n$ such that $\mathscr{S}^{2n+1}(R \Delta R') = 0$.*

Proof. By assumption, there exist a $\mathscr{S}^{2n+1}$-negligible set N and H-regular hypersurfaces $S_j \subset \mathbb{H}^n$, $j \in \mathbb{N}$, such that

$$R \subset N \cup \bigcup_{j=1}^\infty S_j.$$

It is proved in [Franchi et al. 2001; Ambrosio et al. 2006] that (up to a localization argument), for any $j \in \mathbb{N}$, there exist an open set $U_j \subset \mathbb{R}^{2n}$, a homeomorphism $\Phi_j : U_j \to S_j$, and a continuous function $\rho_j : U_j \to [1, \infty)$ such that $\mathscr{S}^{2n+1} \llcorner S_j = \Phi_{j\#}(\rho_j \mathscr{L}^{2n} \llcorner U_j)$. Since the Lebesgue measure $\mathscr{L}^{2n}$ is a complete Borel measure, for any $j \in \mathbb{N}$ there exists a Borel set $T_j \subset U_j$ such that

$$\mathscr{L}^{2n}(T_j \Delta \Phi_j^{-1}(R \cap S_j)) = 0.$$

In particular, the Borel set

$$R' = \bigcup_{j=1}^\infty \Phi_j(T_j)$$

is $\mathscr{S}^{2n+1}$-equivalent to R. □

Proof of Theorem 1.5. Step 1. We prove (1-8) when R is an H-regular hypersurface. Then, R is locally the boundary of an open set $E \subset \mathbb{H}^n$ with H-regular boundary. Moreover, we have (locally) $\mu_E = \mathscr{S}^{2n+1} \llcorner R$ and $\nu_E = \nu_R$, up to the sign.

We define the measures $\mu_R^s = \mu_E^s$ for any s such that $\nabla u \neq 0$ on Σ^s. The measurability of the function in (1-7) follows from the argument (2-34). Formula (1-8) follows from Proposition 2.8.

Step 2. We prove (1-8) when R is an $\mathscr{S}^{2n+1}$-rectifiable Borel set. There exist an $\mathscr{S}^{2n+1}$-negligible set N and H-regular hypersurfaces $S_j \subset \mathbb{H}^n$, $j \in \mathbb{N}$, such that

$$R \subset N \cup \bigcup_{j=1}^\infty S_j.$$

Each S_j is (locally) the boundary of an open set E_j with H-regular boundary. We denote by $\mu_{E_j}^s$ the perimeter measure on $\partial E_j \cap \Sigma^s$ induced by E_j.

We define the pairwise disjoint Borel sets $R_j = (R \cap S_j) \setminus \bigcup_{h=1}^{j-1} S_h$ and we let

$$\mu_R^s = \sum_{j=1}^{\infty} \mu_{E_j}^s \llcorner R_j.$$

The definition is well posed for any s such that $\nabla u \neq 0$ on Σ^s. We have $\nu_R = \pm\nu_{E_j}$ $\mathscr{S}^{2n+1}$-a.e. on R_j and the sign of ν_R does not affect (1-8). From Step 1, for each $j \in \mathbb{N}$ the function

$$s \mapsto \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_{E_j}^s$$

is $\mathscr{L}^1$-measurable; here, we were allowed to utilize Step 1 because χ_{R_j} is Borel regular. Thus also the function

$$s \mapsto \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_R^s = \sum_{j=1}^{\infty} \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_{E_j}^s$$

is measurable. Moreover, we have

$$\begin{aligned}
\int_{\mathbb{R}} \int_{\Omega} h \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_R^s \, ds &= \sum_{j=1}^{\infty} \int_{\mathbb{R}} \int_{R_j} h \frac{|\nabla_H u|_g}{|\nabla u|_g} \, d\mu_{E_j}^s \, ds \\
&= \sum_{j=1}^{\infty} \int_{R_j} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} \, d\mathscr{S}^{2n+1} \\
&= \int_{R} h \sqrt{|\nabla_H u|_g^2 - \langle \nu_R, \nabla_H u \rangle_g^2} \, d\mathscr{S}^{2n+1}.
\end{aligned}$$

Step 3. Finally, if R is $\mathscr{S}^{2n+1}$-rectifiable but not Borel, we set $\mu_R^s = \mu_{R'}^s$, where R' is a Borel set as in Lemma 2.9. Again, this definition is well posed for a.e. $s \in \mathbb{R}$. This concludes the proof. $\square$

2C. *Proof of Theorem 1.6.* In this subsection we assume $n \geq 2$.

Lemma 2.10. *For $n \geq 2$, let $\Omega \subset \mathbb{H}^n$ be an open set, $u \in C^\infty(\Omega)$ a smooth function, $R \subset \Omega$ an $\mathscr{S}^{2n+1}$-rectifiable set. Then*

$$\mathscr{S}^{2n+1}\big(\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\}\big) = 0.$$

Proof. It is enough to prove the lemma when R is an H-regular hypersurface. Let

$$A = \{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\}.$$

We claim that $\mathscr{S}^{2n+1}(A) = 0$.

Let $p \in A$ be a fixed point and let $\nu_R(p)$ be the horizontal normal to R at p. Since $n \geq 2$, we have

$$\dim\{V(p) \in H_p : \langle V(p), \nu_R(p) \rangle_g = 0\} = 2n - 1 \geq n + 1.$$

Thus there exist left-invariant horizontal vector fields V and W such that

$$\langle V(p), \nu_R(p) \rangle_g = \langle W(p), \nu_R(p) \rangle_g = 0 \quad \text{and} \quad [V, W] = T.$$

From $\nabla_H u(p) = 0$ and $\nabla u(p) \neq 0$, we deduce that $Tu(p) \neq 0$. It follows that

$$VWu(p) - WVu(p) = Tu(p) \neq 0$$

and, in particular, we have either $VWu(p) \neq 0$ or $WVu(p) \neq 0$. Without loss of generality, we assume that $VWu(p) \neq 0$. Then the set $S = \{q \in \Omega : Wu(q) = 0\}$ is an H-regular hypersurface near the point $p \in S$. Since we have

$$\langle V(p), \nu_R(p) \rangle_g = 0 \quad \text{and} \quad \langle V(p), \nu_S(p) \rangle_g = \frac{VWu(p)}{|\nabla_H Wu(p)|_g} \neq 0,$$

we deduce that $\nu_R(p)$ and $\nu_S(p)$ are linearly independent. Then there exists $r > 0$ such that the set $R \cap S \cap B_r(p)$ is a 2-codimensional H-regular surface (see [Franchi et al. 2007]). Therefore, by [Franchi et al. 2007, Corollary 4.4], the Hausdorff dimension in the Carnot–Carathéodory metric of $A \cap B_r(p) \subset R \cap S \cap B_r(p)$ is not greater than $2n$. This is enough to conclude. $\square$

Remark 2.11. Lemma 2.10 is not valid if $n = 1$. Consider the smooth surface $R = \{(x, y, t) \in \mathbb{H}^1 : x = 0\}$ and the function $u(x, y, t) = t - 2xy$. We have

$$\nabla u = -4xY + T \quad \text{and} \quad \nabla_H u = -4xY.$$

Then we have

$$\{p \in R : \nabla_H u(p) = 0 \text{ and } \nabla u(p) \neq 0\} = R$$

and $\mathscr{S}^3(R) = \infty$.

If $n \geq 2$ and Ω, u, and R are as in Lemma 2.10, then the function

$$|\nabla u|_g \sqrt{1 - \langle \nu_E, \nabla_H u / |\nabla_H u|_g \rangle_g^2}$$

is defined $\mathscr{S}^{2n+1}$-a.e. on R. We agree that its value is 0 when $|\nabla u|_g = 0$. Notice that, in this case, $\nabla_H u / |\nabla_H u|_g$ is not defined.

Proof of Theorem 1.6. Let $\varepsilon > 0$ be fixed. Then (1-9) can be obtained by plugging the function $(|\nabla u|_g / (\varepsilon + |\nabla_H u|_g))\, h$ into (1-8), letting $\varepsilon \to 0$ and using the monotone convergence theorem. $\square$

3. Height estimate

In this section, we prove Theorem 1.3. We discuss first a relative isoperimetric inequality on slices. Then we list some elementary properties of excess, and finally we proceed with the proof.

We assume throughout this section that $n \geq 2$.

3A. *Relative isoperimetric inequalities.* For each $s \in \mathbb{R}$, we define the level sets of the height function,

$$\mathbb{H}^n_s = \{p \in \mathbb{H}^n : \mathscr{h}(p) = s\}.$$

Let H^s be the g-orthogonal projection of H onto the tangent space of $\mathbb{H}^n_s$. Since the vector field X_1 is orthogonal to $\mathbb{H}^n_s$, while the vector fields $X_2, \ldots, X_n, Y_1, \ldots, Y_n$ are tangent to $\mathbb{H}^n_s$, at any point $p \in \mathbb{H}^n_s$

we have

$$H_p^s = \operatorname{span}\{X_2(p), \dots, X_n(p), Y_1^s(p), Y_2(p), \dots, Y_n(p)\},$$

where $X_2, Y_2, \dots, X_n, Y_n$ are as in (1-2) and

$$Y_1^s = \frac{\partial}{\partial y_1} - 2s\frac{\partial}{\partial t}.$$

The natural volume in $\mathbb{H}_s^n$ is the Lebesgue measure $\mathscr{L}^{2n}$. For any measurable set $F \subset \mathbb{H}_s^n$ and any open set $\Omega \subset \mathbb{H}_s^n$, we define

$$\mu_F^s(\Omega) = \sup\left\{\int_F \operatorname{div}_g^s \varphi \, d\mathscr{L}^{2n} : \varphi \in C_c^1(\Omega; H^s), \|\varphi\|_g \le 1\right\},$$

where $\operatorname{div}_g^s \varphi = X_2\varphi_2 + \dots + X_n\varphi_n + Y_1^s\varphi_{n+1} + \dots + Y_n\varphi_{2n}$. If $\mu_F^s(\Omega) < \infty$ then μ_F^s is a Radon measure in Ω.

By Theorem 1.6, for any Borel function $h : \mathbb{H}^n \to [0, \infty)$ and any set E with locally finite H-perimeter in $\mathbb{H}^n$, we have the coarea formula

$$\int_{\mathbb{R}} \int_{\mathbb{H}_s^n} h \, d\mu_{E^s}^s \, ds = \int_{\mathbb{H}^n} h\sqrt{1 - \langle \nu_E, X_1\rangle_g^2} \, d\mu_E, \tag{3-41}$$

where $E^s = E \cap \mathbb{H}_s^n$ is the section of E with $\mathbb{H}_s^n$. Notice that $\nabla_H \mathscr{E} = X_1$.

In the proof of Theorem 1.3, we need a relative isoperimetric inequality in each slice $\mathbb{H}_s^n$ for $s \in (-1, 1)$. These slices are cosets of $\mathbb{W} = \mathbb{H}_0^n$ and the isoperimetric inequalities in $\mathbb{H}_s^n$ can be reduced to an isoperimetric inequality in the central slice $\mathbb{W} = \mathbb{H}_0^n$ relative to a family of varying domains.

For any $s \in (-1, 1)$, let $\Omega_s \subset \mathbb{W}$ be the set $\Omega_s = (-s\mathrm{e}_1) * D_1 * (s\mathrm{e}_1)$. This is the left translation by $-s\mathrm{e}_1$ of the section $C_1 \cap \mathbb{H}_s^n$. See p. 1423 in the introduction for the definition of D_1 and C_1. With the coordinates $(y_1, \hat{z}, t) \in \mathbb{W} = \mathbb{R} \times \mathbb{C}^{n-1} \times \mathbb{R}$, we have

$$\Omega_s = \{(y_1, \hat{z}, t) \in \mathbb{W} : (y_1^2 + |\hat{z}|^2)^2 + (t - 4sy_1)^2 < 1\}.$$

The sets $\Omega_s \subset \mathbb{W}$ are open and convex in the standard sense. The boundary $\partial\Omega_s$ is a $(2n-1)$-dimensional C^∞ embedded surface with the following property: There are $4n$ open convex sets $U_1, \dots, U_{4n} \subset \mathbb{W}$ such that $\partial\Omega_s \subset \bigcup_{i=1}^{4n} U_i$ and, for each i, the portion of the boundary $\partial\Omega_s \cap U_i$ is a graph of the form $p_j = f_i^s(\hat{p}_j)$ with $j = 2, \dots, 2n+1$ and $\hat{p}_j = (p_2, \dots, p_{j-1}, p_{j+1}, \dots, p_{2n+1}) \in V_i$, where $V_i \subset \mathbb{R}^{2n-1}$ is an open convex set and $f_i^s \in C^\infty(V_i)$ is a function such that

$$|\nabla f_i^s(\hat{p}_j) - \nabla f_i^s(\hat{q}_j)| \le K|\hat{p}_j - \hat{q}_j| \quad \text{for all } \hat{p}_j, \hat{q}_j \in V_i, \tag{3-42}$$

where $K > 0$ is a constant independent of $i = 1, \dots, 4n$ and independent of $s \in (-1, 1)$. In other words, the boundary $\partial\Omega_s$ is of class $C^{1,1}$ uniformly in $s \in (-1, 1)$.

By Theorem 3.2 in [Monti and Morbidelli 2005], the domain $\Omega_s \subset \mathbb{W}$ is a nontangentially accessible (NTA) domain in the metric space $(\mathbb{W}, d_{CC})$, where d_{CC} is the Carnot–Carathéodory metric induced by the horizontal distribution H_p^0. In particular, Ω_s is a (weak) John domain in the sense of [Hajłasz and

Koskela 2000]. Namely, there exist a point $p_0 \in \Omega_s$, e.g., $p_0 = 0$, and a constant $C_J > 0$ such that, for any point $p \in \Omega_s$, there exists a continuous curve $\gamma : [0, 1] \to \Omega_s$ such that $\gamma(1) = p_0$, $\gamma(0) = p$, and

$$d_{\mathrm{CC}}(\gamma(\sigma), \partial\Omega_s) \geq C_J d_{\mathrm{CC}}(\gamma(\sigma), p), \quad \sigma \in [0, 1]. \tag{3-43}$$

By Theorem 3.2 in [Monti and Morbidelli 2005], the John constant C_J depends only on the constant $K > 0$ in (3-42). This claim is not stated explicitly in Theorem 3.2 of [Monti and Morbidelli 2005] but it is evident from the proof. In particular, the John constant C_J is independent of $s \in (-1, 1)$. Then, by Theorem 1.22 in [Garofalo and Nhieu 1996], we have the following result:

Theorem 3.1. *Let $n \geq 2$. There exists a constant $C(n) > 0$ such that, for any $s \in (-1, 1)$ and any measurable set $F \subset \mathbb{W}$,*

$$\min\{\mathscr{L}^{2n}(F \cap \Omega_s), \mathscr{L}^{2n}(\Omega_s \setminus F)\}^{2n/(2n+1)} \leq C(n) \frac{\mathrm{diam}_{\mathrm{CC}}(\Omega_s)}{\mathscr{L}^{2n}(\Omega_s)^{1/(2n+1)}} \mu_F^0(\Omega_s). \tag{3-44}$$

An alternative proof of Theorem 3.1 can be obtained using the Sobolev–Poincaré inequalities proved in [Hajłasz and Koskela 2000] in the general setting of metric spaces.

The diameter $\mathrm{diam}_{\mathrm{CC}}(\Omega_s)$ is bounded for $s \in (-1, 1)$ and $\mathscr{L}^{2n}(\Omega_s) > 0$ is a constant independent of s. Then we obtain the following version of (3-44):

Corollary 3.2. *Let $n \geq 2$. For any $\tau \in (0, 1)$ there exists a constant $C(n, \tau) > 0$ such that, for $s \in (-1, 1)$ and any measurable set $F \subset \mathbb{W}$ satisfying*

$$\mathscr{L}^{2n}(F \cap \Omega_s) \leq \tau \mathscr{L}^{2n}(\Omega_s),$$

we have

$$\mu_F^0(\Omega_s) \geq C(n, \tau) \mathscr{L}^{2n}(F \cap \Omega_s)^{2n/(2n+1)}.$$

3B. *Elementary properties of the excess.* We list here, without proof, the most basic properties of the cylindrical excess introduced in Definition 1.2. Their proofs are easy adaptations of those for the classical excess; see, e.g., [Maggi 2012, Chapter 22]. Note that, except for property (3), they hold also in the case $n = 1$.

(1) For all $0 < r < s$, we have

$$\mathrm{Exc}(E, r, \nu) \leq \left(\frac{s}{r}\right)^{2n+1} \mathrm{Exc}(E, s, \nu). \tag{3-45}$$

(2) If $(E_j)_{j \in \mathbb{N}}$ is a sequence of sets with locally finite H-perimeter such that $E_j \to E$ as $j \to \infty$ in $L^1_{\mathrm{loc}}(\mathbb{H}^n)$, then we have, for any $r > 0$,

$$\mathrm{Exc}(E, r, \nu) \leq \liminf_{j \to \infty} \mathrm{Exc}(E_j, r, \nu). \tag{3-46}$$

(3) Let $n \geq 2$. If $E \subset \mathbb{H}^n$ is a set such that $\mathrm{Exc}(E, r, \nu) = 0$ and $0 \in \partial^* E$, then

$$E \cap C_r = \{p \in C_r : \mathfrak{z}(p) < 0\}. \tag{3-47}$$

In particular, we have $\nu_E = \nu$ in $C_r \cap \partial E$. See also [Monti 2014, Proposition 3.6].

(4) For any $\lambda > 0$ and $r > 0$, we have

$$\operatorname{Exc}(\lambda E, \lambda r, \nu) = \operatorname{Exc}(E, r, \nu), \tag{3-48}$$

where $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\}$.

3C. *Proof of Theorem 1.3.* The following result is a first, suboptimal version of Theorem 1.3.

Lemma 3.3. *Let $n \geq 2$. For any $s \in (0, 1)$, $\Lambda \in [0, \infty)$, and $r \in (0, \infty]$ with $\Lambda r \leq 1$, there exists a constant $\omega(n, s, \Lambda, r) > 0$ such that, if $E \subset \mathbb{H}^n$ is a (Λ, r)-minimizer of H-perimeter in the cylinder C_2, $0 \in \partial E$, and $\operatorname{Exc}(E, 2, \nu) \leq \omega(n, s, \Lambda, r)$, then*

$$\begin{gathered}|\mathscr{k}(p)| < s \quad \textit{for any } \ p \in \partial E \cap C_1, \\ \mathscr{L}^{2n+1}\big(\{p \in E \cap C_1 : \mathscr{k}(p) > s\}\big) = 0, \\ \mathscr{L}^{2n+1}\big(\{p \in C_1 \setminus E : \mathscr{k}(p) < -s\}\big) = 0.\end{gathered}$$

Proof. By contradiction, assume that there exist $s \in (0, 1)$ and a sequence of sets $(E_j)_{j \in \mathbb{N}}$ that are (Λ, r)-minimizers in C_2 and such that

$$\lim_{j \to \infty} \operatorname{Exc}(E_j, 2, \nu) = 0$$

and at least one of the following facts holds:

$$\text{there exists } \ p \in \partial E_j \cap C_1 \ \text{ such that } \ s \leq |\mathscr{k}(p)| \leq 1, \tag{3-49}$$

$$\mathscr{L}^{2n+1}\big(\{p \in E_j \cap C_1 : \mathscr{k}(p) > s\}\big) > 0, \tag{3-50}$$

$$\text{or} \quad \mathscr{L}^{2n+1}\big(\{p \in C_1 \setminus E_j : \mathscr{k}(p) < -s\}\big) > 0. \tag{3-51}$$

By Theorem A.3 in Appendix A, there exists a measurable set $F \subset C_{5/3}$ such that F is a (Λ, r)-minimizer in $C_{5/3}$, $0 \in \partial F$, and (possibly up to subsequences) $E_j \cap C_{5/3} \to F$ in $L^1(C_{5/3})$. By (3-46) and (3-45), we obtain

$$\operatorname{Exc}\big(F, \tfrac{4}{3}, \nu\big) \leq \liminf_{j \to \infty} \operatorname{Exc}\big(E_j, \tfrac{4}{3}, \nu\big) \leq \big(\tfrac{3}{2}\big)^{2n+1} \lim_{j \to \infty} \operatorname{Exc}(E_j, 2, \nu) = 0.$$

Since $0 \in \partial F$, by (3-47) the set $F \cap C_{4/3}$ is (equivalent to) a halfspace with horizontal inner normal $\nu = -X_1$, namely,

$$F \cap C_{4/3} = \{p \in C_{4/3} : \mathscr{k}(p) < 0\}.$$

Assume that (3-49) holds for infinitely many j. Then, up to a subsequence, there are points $(p_j)_{j \in \mathbb{N}}$ and p_0 such that

$$p_j \in \partial E_j \cap C_1, \quad |\mathscr{k}(p_j)| \in (s, 1] \quad \text{and} \quad p_j \to p_0 \in \partial F \cap \bar{C}_1.$$

We used again Theorem A.3 in Appendix A. This is a contradiction because $\partial F \cap \bar{C}_1 = \{p \in \bar{C}_1 : \mathscr{k}(p) = 0\}$. Here, we used $n \geq 2$. Therefore, there exists $j_0 \in \mathbb{N}$ such that

$$\{p \in \partial E_j \cap C_1 : s \leq |\mathscr{k}(p)| \leq 1\} = \varnothing \quad \text{for all } \ j \geq j_0$$

and hence

$$\mu_{E_j}\big(C_1 \setminus \{p \in \mathbb{H}^n : |\mathfrak{p}(p)| \le s\}\big) = 0.$$

This implies that, for $j \ge j_0$, χ_{E_j} is constant on the two connected components $C_1 \cap \{p : \mathfrak{p}(p) > s\}$ and $C_1 \cap \{p : \mathfrak{p}(p) < -s\}$. Since the sequence $(E_j)_{j\in\mathbb{N}}$ converges in $L^1(C_1)$ to the halfspace F, for any $j \ge j_0$ we have

$$\begin{aligned} &\chi_{E_j} = 0 \quad \mathscr{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \mathfrak{p}(p) > s\},\\ \text{and} \qquad &\chi_{E_j} = 1 \quad \mathscr{L}^{2n+1}\text{-a.e. on } C_1 \cap \{p : \mathfrak{p}(p) < -s\}. \end{aligned}$$

This contradicts both (3-50) and (3-51) and concludes the proof. □

Let $\pi : \mathbb{H}^n \to \mathbb{W}$ be the group projection defined, for any $p \in \mathbb{H}^n$, by the formula

$$p = \pi(p) * (\mathfrak{p}(p)\mathrm{e}_1).$$

For any set $E \subset \mathbb{H}^n$ and $s \in \mathbb{R}$, we let $E^s = E \cap \mathbb{H}^n_s$ and we define the projection

$$E_s = \pi(E^s) = \{w \in \mathbb{W} : w * (s\mathrm{e}_1) \in E\}.$$

Lemma 3.4. *Let $n \ge 2$, let $E \subset \mathbb{H}^n$ be a set with locally finite H-perimeter and $0 \in \partial E$, and let $s_0 \in (0, 1)$ be such that*

$$|\mathfrak{p}(p)| < s_0 \quad \textit{for any} \;\; p \in \partial E \cap C_1, \tag{3-52}$$

$$\mathscr{L}^{2n+1}\big(\{p \in E \cap C_1 : \mathfrak{p}(p) > s_0\}\big) = 0, \tag{3-53}$$

$$\mathscr{L}^{2n+1}\big(\{p \in C_1 \setminus E : \mathfrak{p}(p) < -s_0\}\big) = 0. \tag{3-54}$$

Then, for a.e. $s \in (-1, 1)$ and any continuous function $\varphi \in C_c(D_1)$, we have, with $M = \partial^* E \cap C_1$ and $M_s = M \cap \{\mathfrak{p} > s\}$,

$$\int_{E_s \cap D_1} \varphi \, d\mathscr{L}^{2n} = -\int_{M_s} \varphi \circ \pi \, \langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1}. \tag{3-55}$$

In particular, for any Borel set $G \subset D_1$, we have

$$\mathscr{L}^{2n}(G) = -\int_{M\cap\pi^{-1}(G)} \langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1}, \tag{3-56}$$

$$\mathscr{L}^{2n}(G) \le \mathscr{S}^{2n+1}(M \cap \pi^{-1}(G)). \tag{3-57}$$

Proof. It is enough to prove (3-55). Indeed, taking $s < -s_0$ in (3-55) and recalling (3-52) and (3-54), we obtain

$$\int_{D_1} \varphi \, d\mathscr{L}^{2n} = -\int_{M} \varphi \circ \pi \, \langle \nu_E, X_1\rangle_g \, d\mathscr{S}^{2n+1}. \tag{3-58}$$

Formula (3-56) follows from (3-58) by considering smooth approximations of χ_G. Formula (3-57) is immediate from (3-56) and $|\langle \nu_E, X_1\rangle_g| \le 1$.

We prove (3-55) for a.e. $s \in (-1, 1)$, namely, for those s satisfying the property (3-61) below. Up to an approximation argument, we may assume that $\varphi \in C^1_c(D_1)$. Let $r \in (0, 1)$ and $\sigma \in (\max\{s_0, s\}, 1)$ be fixed. We define

$$F = E \cap (D_r * (s, \sigma)) = E \cap \{w * (\varrho \mathrm{e}_1) \in \mathbb{H}^n : w \in D_r,\ \varrho \in (s, \sigma)\}.$$

We claim that, for a.e. $r \in (0, 1)$ and any s satisfying (3-61), we have

$$\langle \nu_F, X_1 \rangle_g \mu_F = \langle \nu_E, X_1 \rangle_g \mathscr{S}^{2n+1} \llcorner \partial^* E \cap (D_r * (s, \sigma)) + \mathscr{L}^{2n} \llcorner E \cap D_r^s. \tag{3-59}$$

Above, we let $D_r^s = \{w * (s\mathrm{e}_1) \in \mathbb{H}^n : w \in D_r\}$. We postpone the proof of (3-59). Let Z be a horizontal vector field of the form $Z = (\varphi \circ \pi) X_1$. We have $\operatorname{div}_g Z = 0$ because $X_1(\varphi \circ \pi) = 0$. Hence, we obtain

$$0 = \int_F \operatorname{div}_g Z \, d\mathscr{L}^{2n+1} = -\int_{\mathbb{H}^n} \varphi \circ \pi \, \langle \nu_F, X_1 \rangle_g \, d\mu_F,$$

i.e., by the Fubini–Tonelli theorem and (3-59),

$$-\int_{E_s \cap D_r} \varphi \, d\mathscr{L}^{2n} = -\int_{E \cap D_r^s} \varphi \circ \pi \, d\mathscr{L}^{2n} = \int_{\partial^* E \cap (D_r * (s, \sigma))} \varphi \circ \pi \, \langle \nu_E, X_1 \rangle_g \, d\mathscr{S}^{2n+1}.$$

Formula (3-55) follows on letting first $r \nearrow 1$ and then $\sigma \nearrow 1$.

We are left with the proof of (3-59). Let $\psi \in C^1_c(\mathbb{H}^n)$ be a test function. For any $w \in \mathbb{W}$, we let

$$E_w = \{\varrho \in \mathbb{R} : w * (\varrho \mathrm{e}_1) \in E\}, \qquad \psi_w(\varrho) = \psi(w * (\varrho \mathrm{e}_1)).$$

Then we have $\psi_w \in C^1_c(\mathbb{R})$ and, by the Fubini–Tonelli theorem,

$$\begin{aligned}
-\int_F X_1 \psi \, d\mathscr{L}^{2n+1} &= -\int_{D_r} \int_s^\sigma \chi_E(w * (\varrho \mathrm{e}_1)) X_1 \psi(w * (\varrho \mathrm{e}_1)) \, d\varrho \, d\mathscr{L}^{2n}(w) \\
&= -\int_{D_r} \int_s^\sigma \chi_{E_w}(\varrho) \psi_w'(\varrho) \, d\varrho \, d\mathscr{L}^{2n}(w) \\
&= \int_{D_r} \left[\int_s^\sigma \psi_w \, dD\chi_{E_w} - \psi_w(\sigma) \chi_{E_w}(\sigma^-) + \psi_w(s) \chi_{E_w}(s^+) \right] d\mathscr{L}^{2n}(w),
\end{aligned} \tag{3-60}$$

where $D\chi_{E_w}$ is the derivative of χ_{E_w} in the sense of distributions and $\chi_{E_w}(\sigma^-)$, $\chi_{E_w}(s^+)$ are the classical trace values of χ_{E_w} at the endpoints of the interval (s, σ). We used the fact that the function χ_{E_w} is of bounded variation for $\mathscr{L}^{2n}$-a.e. $w \in \mathbb{W}$, which in turn is a consequence of the fact that $X_1 \chi_E$ is a signed Radon measure. For any such w, the trace of χ_{E_w} satisfies

$$\chi_{E_w}(s^+) = \chi_{E_w}(s) = \chi_E(w * (s\mathrm{e}_1)) \quad \text{for a.e. } s,$$

so that, by Fubini's theorem, for a.e. $s \in \mathbb{R}$ we have

$$\chi_{E_w}(s^+) = \chi_E(w * (s\mathrm{e}_1)) \quad \text{for } \mathscr{L}^{2n}\text{-a.e. } w \in D_1. \tag{3-61}$$

With a similar argument, using (3-53) and the fact that $\sigma > s_0$, one can see that

$$\chi_{E_w}(\sigma^-) = \chi_E(w * (\sigma \mathrm{e}_1)) = 0 \quad \text{for } \mathscr{L}^{2n}\text{-a.e. } w \in D_1. \tag{3-62}$$

We refer the reader to [Ambrosio et al. 2000] for an extensive account on BV functions and traces. By (3-60), (3-61) and (3-62), we obtain

$$
\begin{aligned}
-\int_F X_1\psi\, d\mathscr{L}^{2n+1} &= \int_{D_r}\int_s^\sigma \psi_w\, dD\chi_{E_w}\, d\mathscr{L}^{2n}(w) + \int_{D_r} \psi_w(s)\chi_{E_w}(s)\, d\mathscr{L}^{2n}(w) \\
&= \int_{D_r*(s,\sigma)} \psi\, \langle \nu_E, X_1\rangle_g d\mu_E + \int_{E\cap D_r^s} \psi\, d\mathscr{L}^{2n} \\
&= \int_{\partial^* E\cap(D_r*(s,\sigma))} \psi\, \langle \nu_E, X_1\rangle_g\, d\mathscr{S}^{2n+1} + \int_{E\cap D_r^s} \psi\, d\mathscr{L}^{2n},
\end{aligned}
$$

and (3-59) follows. □

Corollary 3.5. *Under the same assumptions and notation as Lemma 3.4, for a.e. $s\in(-1,1)$, we have*

$$
0 \le \mathscr{S}^{2n+1}(M_s) - \mathscr{L}^{2n}(E_s\cap D_1) \le \operatorname{Exc}(E,1,\nu). \tag{3-63}
$$

Moreover,

$$
\mathscr{S}^{2n+1}(M) - \mathscr{L}^{2n}(D_1) = \operatorname{Exc}(E,1,\nu). \tag{3-64}
$$

Proof. On approximating χ_{D_1} with functions $\varphi\in C_c(D_1)$, by (3-55) we get

$$
\mathscr{L}^{2n}(E_s\cap D_1) = -\int_{M_s} \langle \nu_E, X_1\rangle_g\, d\mathscr{S}^{2n+1},
$$

and the first inequality in (3-63) follows. The second inequality follows from

$$
\begin{aligned}
\mathscr{S}^{2n+1}(M_s) - \mathscr{L}^{2n}(E_s\cap D_1) &= \int_{M_s} (1+\langle \nu_E, X_1\rangle_g)\, d\mathscr{S}^{2n+1} \\
&= \int_{M_s} \frac{|\nu_E-\nu|_g^2}{2}\, d\mathscr{S}^{2n+1} \\
&\le \operatorname{Exc}(E,1,\nu).
\end{aligned} \tag{3-65}
$$

Notice that $\nu=-X_1$. Finally, (3-64) follows on choosing a suitable $s<-s_0$ and recalling (3-52) and (3-54). In this case, the inequality in (3-65) becomes an equality and the proof is concluded. □

Proof of Theorem 1.3. Step 1. Up to replacing E with the rescaled set $\lambda E=\{(\lambda z,\lambda^2 t)\in\mathbb{H}^n : (z,t)\in E\}$ with $\lambda=1/2k^2 r$ and recalling (3-48), we can without loss of generality assume that E is a $(\Lambda', 1/(2k^2))$-minimizer of H-perimeter in C_2 with

$$
\frac{\Lambda'}{2k^2}\le 1, \qquad 0\in\partial E, \qquad \operatorname{Exc}(E,2,\nu)\le \varepsilon_0(n). \tag{3-66}
$$

Our goal is to find $\varepsilon_0(n)$ and $c_1(n)>0$ such that, if (3-66) holds, then

$$
\sup\{|\mathfrak{k}(p)| : p\in\partial E\cap C_{1/2k^2}\} \le c_1(n)\operatorname{Exc}(E,2,\nu)^{1/(2(2n+1))}. \tag{3-67}
$$

We require

$$
\varepsilon_0(n) \le \omega\left(n, \frac{1}{4k}, 2k^2, \frac{1}{2k^2}\right), \tag{3-68}
$$

where ω is as given by Lemma 3.3. Two further assumptions on $\varepsilon_0(n)$ will be made later, in (3-80) and (3-85). By (3-66), E is a $(2k^2, 1/(2k^2))$-minimizer in C_2. Letting $M = \partial E \cap C_1$, by Lemma 3.3 and (3-68) we have

$$|\mathfrak{h}(p)| < \frac{1}{4k} \quad \text{for any} \;\; p \in M, \tag{3-69}$$

$$\mathscr{L}^{2n+1}\left(\left\{p \in E \cap C_1 : \mathfrak{h}(p) > \frac{1}{4k}\right\}\right) = 0, \tag{3-70}$$

$$\mathscr{L}^{2n+1}\left(\left\{p \in C_1 \setminus E : \mathfrak{h}(p) < -\frac{1}{4k}\right\}\right) = 0. \tag{3-71}$$

By (3-64) and (3-45), we get

$$0 \le \mathscr{S}^{2n+1}(M) - \mathscr{L}^{2n}(D_1) \le \operatorname{Exc}(E, 1, \nu) \le 2^{2n+1} \operatorname{Exc}(E, 2, \nu). \tag{3-72}$$

Corollary 3.5 implies that, for a.e. $s \in (-1, 1)$,

$$0 \le \mathscr{S}^{2n+1}(M_s) - \mathscr{L}^{2n}(E_s \cap D_1) \le \operatorname{Exc}(E, 1, \nu) \le 2^{2n+1} \operatorname{Exc}(E, 2, \nu), \tag{3-73}$$

where, as before, $M_s = M \cap \{\mathfrak{h} > s\}$.

Step 2. Consider $f : (-1, 1) \to [0, \mathscr{S}^{2n+1}(M)]$ defined by

$$f(s) = \mathscr{S}^{2n+1}(M_s), \quad s \in (-1, 1).$$

The function f is nonincreasing, right-continuous and, by (3-69), it satisfies

$$\begin{aligned} f(s) &= \mathscr{S}^{2n+1}(M) && \text{for any } s \in \left(-1, -\frac{1}{4k}\right], \\ f(s) &= 0 && \text{for any } s \in \left(\frac{1}{4k}, 1\right]. \end{aligned}$$

In particular, there exists $s_0 \in (-1/(4k), 1/(4k))$ such that

$$\begin{aligned} f(s) &\ge \tfrac{1}{2}\mathscr{S}^{2n+1}(M) && \text{for any } s < s_0, \\ f(s) &\le \tfrac{1}{2}\mathscr{S}^{2n+1}(M) && \text{for any } s \ge s_0. \end{aligned} \tag{3-74}$$

Let $s_1 \in (s_0, 1/(4k))$ be such that

$$\begin{aligned} f(s) &\ge \sqrt{\operatorname{Exc}(E, 2, \nu)} && \text{for any } s < s_1, \\ f(s) &= \mathscr{S}^{2n+1}(M_s) \le \sqrt{\operatorname{Exc}(E, 2, \nu)} && \text{for any } s \ge s_1. \end{aligned} \tag{3-75}$$

We claim that there exists $c_2(n) > 0$ such that

$$\mathfrak{h}(p) \le s_1 + c_2(n) \operatorname{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any} \;\; p \in \partial E \cap C_{1/2k^2}. \tag{3-76}$$

The inequality (3-76) is trivial for any $p \in \partial E \cap C_{1/2k^2}$ with $\mathfrak{h}(p) \le s_1$. If $p \in \partial E \cap C_{1/2k^2}$ is such that $\mathfrak{h}(p) > s_1$, then

$$B_{\mathfrak{h}(p)-s_1}(p) \subset B_{1/2k}(p) \subset B_{1/k} \subset C_1.$$

We used the fact that $\|p\|_K \le 1/(2k)$ whenever $p \in C_{1/2k^2}$; see (1-3). Therefore,

$$B_{\not{g}(p)-s_1}(p) \subset C_1 \cap \{\not{g} > s_1\}$$

and, by the density estimate (A-91) of Theorem A.1 in Appendix A,

$$k_3(n)(\not{g}(p)-s_1)^{2n+1} \le \mu_E(B_{\not{g}(p)-s_1}(p)) \le \mu_E(C_1 \cap \{\not{g} > s_1\}) = \mathscr{S}^{2n+1}(M_{s_1}) = f(s_1) \le \sqrt{\operatorname{Exc}(E, 2, \nu)}.$$

This proves (3-76).

Step 3. We claim that there exists $c_3(n) > 0$ such that

$$s_1 - s_0 \le c_3(n) \operatorname{Exc}(E, 2, \nu)^{1/(2(2n+1))}. \tag{3-77}$$

By the coarea formula (3-41) with $h = \chi_{C_1}$, $D_1^s = \{p \in C_1 : \not{g}(p) = s\}$, and $E^s = \{p \in E : \not{g}(p) = s\}$, we have

$$\int_{-1}^{1} \int_{D_1^s} d\mu^s_{E^s}\, ds = \int_{C_1} \sqrt{1 - \langle \nu_E, X_1\rangle_g^2}\, d\mu_E \le \sqrt{2} \int_M \sqrt{1 + \langle \nu_E, X_1\rangle_g}\, d\mathscr{S}^{2n+1}.$$

By Hölder's inequality, (A-91), (3-56), and (3-72), we deduce that

$$\begin{aligned}\int_{-1}^{1} \int_{D_1^s} d\mu^s_{E^s}\, ds &\le \sqrt{2\mathscr{S}^{2n+1}(M)} \left(\int_M (1 + \langle \nu_E, X_1\rangle_g)\, d\mathscr{S}^{2n+1} \right)^{\frac{1}{2}} \\ &\le c_4(n)(\mathscr{S}^{2n+1}(M) - \mathscr{L}^{2n}(D_1))^{1/2} \\ &\le c_5(n) \sqrt{\operatorname{Exc}(E, 2, \nu)}.\end{aligned} \tag{3-78}$$

By Corollary 3.5 and (3-72), we obtain, for a.e. $s \in [s_0, s_1)$,

$$\begin{aligned}\mathscr{L}^{2n}(E_s \cap D_1) \le \mathscr{S}^{2n+1}(M_s) = f(s) \le f(s_0) &\le \tfrac{1}{2}\mathscr{S}^{2n+1}(M) \\ &\le \tfrac{1}{2}(\mathscr{L}^{2n}(D_1) + 2^{2n+1} \operatorname{Exc}(E, 2, \nu)) \\ &\le \tfrac{3}{4}\mathscr{L}^{2n}(D_1).\end{aligned} \tag{3-79}$$

The last inequality holds provided that

$$2^{2n+1} \varepsilon_0(n) \le \tfrac{1}{4}\mathscr{L}^{2n}(D_1). \tag{3-80}$$

Let $\Omega_s = (-se_1) * D_1^s = (-se_1) * D_1 * (se_1)$ and $F_s = (-se_1) * E^s$. We have

$$\mathscr{L}^{2n}(\Omega_s) = \mathscr{L}^{2n}(D_1^s) = \mathscr{L}^{2n}(D_1) \tag{3-81}$$

and, by (3-79),

$$\mathscr{L}^{2n}(F_s \cap \Omega_s) = \mathscr{L}^{2n}(E^s \cap D_1^s) = \mathscr{L}^{2n}(E_s \cap D_1) \le \tfrac{3}{4}\mathscr{L}^{2n}(D_1). \tag{3-82}$$

Moreover, by left invariance we have

$$\mu^s_{E^s}(D_1^s) = \mu^0_{F_s}(\Omega_s). \tag{3-83}$$

By (3-81)–(3-83) and Corollary 3.2, there exists a constant $k(n) > 0$ independent of $s \in (-1, 1)$ such that

$$\mu_{E^s}(D_1^s) = \mu^0_{F_s}(\Omega_s) \geq k(n)\mathscr{L}^{2n}(F_s \cap \Omega_s)^{2n/(2n+1)} = k(n)\mathscr{L}^{2n}(E^s \cap D_1^s)^{2n/(2n+1)}. \tag{3-84}$$

This, together with (3-78), gives

$$\begin{aligned} c_6(n)\sqrt{\operatorname{Exc}(E, 2, \nu)} &\geq \int_{s_0}^{s_1} \mathscr{L}^{2n}(E^s \cap D_1^s)^{2n/(2n+1)}\, ds \\ &\overset{(3\text{-}73)}{\geq} \int_{s_0}^{s_1} (\mathscr{S}^{2n+1}(M_s) - 2^{2n+1}\operatorname{Exc}(E, 2, \nu))^{2n/(2n+1)}\, ds \\ &\overset{(3\text{-}75)}{\geq} \int_{s_0}^{s_1} \bigl(\sqrt{\operatorname{Exc}(E, 2, \nu)} - 2^{2n+1}\operatorname{Exc}(E, 2, \nu)\bigr)^{2n/(2n+1)}\, ds \\ &\geq \frac{1}{2}\int_{s_0}^{s_1} \operatorname{Exc}(E, 2, \nu)^{n/(2n+1)}\, ds. \end{aligned}$$

In the last inequality, we require that $\varepsilon_0(n)$ satisfies

$$\sqrt{z} - 2^{2n+1}z \geq \tfrac{1}{2}\sqrt{z} \quad \text{for all } z \in [0, \varepsilon_0(n)]. \tag{3-85}$$

It follows that

$$c_6(n)\sqrt{\operatorname{Exc}(E, 2, \nu)} \geq \tfrac{1}{2}\operatorname{Exc}(E, 2, \nu)^{n/(2n+1)}(s_1 - s_0),$$

giving (3-77).

Step 4. Recalling (3-76) and (3-77), we proved that there exist $\varepsilon_0(n)$ and $c_6(n)$ such that the following holds: if E is a $(2k^2, 1/(2k^2))$-minimizer of H-perimeter in C_2 such that

$$0 \in \partial E, \quad \operatorname{Exc}(E, 2, \nu) \leq \varepsilon_0(n)$$

and $s_0 = s_0(E)$ satisfies (3-74), then

$$\mathfrak{h}(p) - s_0 \leq c_7(n)\operatorname{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3-86}$$

Let us introduce the mapping $\Psi : \mathbb{H}^n \to \mathbb{H}^n$

$$\Psi(x_1, x_2 \ldots, x_n, y_1, \ldots, y_n, t) = (-x_1, -x_2, \ldots, -x_n, y_1, \ldots, y_n, -t)\,.$$

Then we have $\Psi^{-1} = \Psi$, $\Psi(C_2) = C_2$, $\langle X_j, \nu_{\Psi(F)}\rangle_g = -\langle X_j, \nu_F\rangle_g \circ \Psi$, $\langle Y_j, \nu_{\Psi(F)}\rangle_g = \langle Y_j, \nu_F\rangle_g \circ \Psi$, and $\mu_{\Psi(F)} = \Psi_\# \mu_F$, for any set F with locally finite H-perimeter; here, $\Psi_\#$ denotes the standard push-forward of measures. Therefore, the set $\widetilde{E} = \Psi(\mathbb{H}^n \setminus E)$ satisfies the following properties:

(i) $\widetilde{E}$ is a $(2k^2, 1/(2k^2))$-minimizer of H-perimeter in C_2;

(ii) $0 \in \partial\widetilde{E}$ and

$$\operatorname{Exc}(\widetilde{E}, 2, \nu) = \frac{1}{2^Q}\int_{\partial^*\widetilde{E} \cap C_2} |\nu_{\widetilde{E}} - \nu|_g^2\, d\mathscr{S}^{2n+1} = \operatorname{Exc}(E, 2, \nu) \leq \varepsilon_0(n);$$

(iii) setting $\widetilde{M} = \partial^*\widetilde{E} \cap C_1 = \Psi(M)$ and $\tilde{f}(s) = \mathscr{S}^{2n+1}(\widetilde{M} \cap \{\ell > s\})$, we have

$$\begin{aligned} \tilde{f}(s) &\geq \tfrac{1}{2}\mathscr{S}^{2n+1}(\widetilde{M}) = \tfrac{1}{2}\mathscr{S}^{2n+1}(M) && \text{for any } s < -s_0, \\ \tilde{f}(s) &\leq \tfrac{1}{2}\mathscr{S}^{2n+1}(M) && \text{for any } s \geq -s_0. \end{aligned}$$

Formula (3-86) for the set $\widetilde{E}$ gives

$$\ell(p) + s_0 \leq c_7(n)\,\mathrm{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial\widetilde{E} \cap C_{1/2k^2}.$$

Notice that we have $p \in \partial\widetilde{E}$ if and only if $\Psi(p) \in \partial E$ and, moreover, $\ell(\Psi(p)) = -\ell(p)$. Hence, we have

$$-\ell(p) + s_0 \leq c_7(n)\,\mathrm{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}. \tag{3-87}$$

By (3-86) and (3-87), we obtain

$$|\ell(p) - s_0| \leq c_7(n)\,\mathrm{Exc}(E, 2, \nu)^{1/(2(2n+1))} \quad \text{for any } p \in \partial E \cap C_{1/2k^2}, \tag{3-88}$$

and, in particular,

$$|s_0| \leq c_7(n)\,\mathrm{Exc}(E, 2, \nu)^{1/(2(2n+1))}, \tag{3-89}$$

because $0 \in \partial E \cap C_{1/2k^2}$. Inequalities (3-88) and (3-89) give (3-67). This completes the proof. □

Appendix A

We list some basic properties of Λ-minimizers of H-perimeter in $\mathbb{H}^n$. The proofs are straightforward adaptations of the proofs for Λ-minimizers of perimeter in $\mathbb{R}^n$.

Theorem A.1 (density estimates). *There exist positive constants $k_1(n)$, $k_2(n)$, $k_3(n)$ and $k_4(n)$ with the following property: if E is a (Λ, r)-minimizer of H-perimeter in $\Omega \subset \mathbb{H}^n$, $p \in \partial E \cap \Omega$, $B_r(p) \subset \Omega$ and $s < r$, then*

$$k_1(n) \leq \frac{\mathscr{L}^{2n+1}(E \cap B_s(p))}{s^{2n+2}} \leq k_2(n), \tag{A-90}$$

$$k_3(n) \leq \frac{\mu_E(B_s(p))}{s^{2n+1}} \leq k_4(n). \tag{A-91}$$

For a proof, see [Maggi 2012, Theorem 21.11]. By standard arguments, Theorem A.1 implies the following corollary:

Corollary A.2. *If E is a (Λ, r)-minimizer of H-perimeter in $\Omega \subset \mathbb{H}^n$, then*

$$\mathscr{S}^{2n+1}((\partial E \setminus \partial^* E) \cap \Omega) = 0.$$

Theorem A.3. *Let $(E_j)_{j\in\mathbb{N}}$ be a sequence of (Λ, r)-minimizers of H-perimeter in an open set $\Omega \subset \mathbb{H}^n$, $\Lambda r \leq 1$. Then there exists a (Λ, r)-minimizer E of H-perimeter in Ω and a subsequence $(E_{j_k})_{k\in\mathbb{N}}$ such that*

$$E_{j_k} \longrightarrow E \quad \text{in } L^1_{\mathrm{loc}}(\Omega) \qquad \text{and} \qquad \nu_{E_{j_k}}\mu_{E_{j_k}} \xrightarrow{\;*\;} \nu_E \mu_E$$

as $k \to \infty$. Moreover, the measure-theoretic boundaries ∂E_{j_k} converge to ∂E in the sense of Kuratowski, i.e.,

(i) *if $p_{j_k} \in \partial E_j \cap \Omega$ and $p_{j_k} \to p \in \Omega$, then $p \in \partial E$;*

(ii) *if $p \in \partial E \cap \Omega$, then there exists a sequence $(p_{j_k})_{k\in\mathbb{N}}$ such that $p_{j_k} \in \partial E_{j_k} \cap \Omega$ and $p_{j_k} \to p$.*

For a proof in the case of the perimeter in $\mathbb{R}^n$, see [Maggi 2012, Chapter 21].

Appendix B

We define a Borel unit normal ν_R to an $\mathscr{S}^{2n+1}$-rectifiable set $R \subset \mathbb{H}^n$ and we show that the definition is well posed $\mathscr{S}^{2n+1}$-a.e., up to the sign. The normal ν_S to an H-regular hypersurface $S \subset \mathbb{H}^n$ is defined in (1-6).

Definition B.1. Let $R \subset \mathbb{H}^n$ be an $\mathscr{S}^{2n+1}$-rectifiable set such that

$$\mathscr{S}^{2n+1}\Big(R \setminus \bigcup_{j\in\mathbb{N}} S_j\Big) = 0 \tag{B-92}$$

for a sequence of H-regular hypersurfaces $(S_j)_{j\in\mathbb{N}}$ in $\mathbb{H}^n$. For any $p \in R \cap \bigcup_{j\in\mathbb{N}} S_j$, we define

$$\nu_R(p) = \nu_{S_{\bar{j}}}(p),$$

where $\bar{j}$ is the unique integer such that $p \in S_{\bar{j}} \setminus \bigcup_{j<\bar{j}} S_j$.

We show that Definition B.1 is well posed, up to a sign, for $\mathscr{S}^{2n+1}$-a.e. p. Namely, let $(S^1_j)_{j\in\mathbb{N}}$ and $(S^2_j)_{j\in\mathbb{N}}$ be two sequences of H-regular hypersurfaces in $\mathbb{H}^n$ for which (B-92) holds and denote by ν^1_R and ν^2_R, respectively, the associated normals to R according to Definition B.1. We show that $\nu^1_R = \nu^2_R$ $\mathscr{S}^{2n+1}$-a.e. on R, up to the sign.

Let $A \subset R$ be the set of points such that either $\nu^1_R(p)$ is not defined, or $\nu^2_R(p)$ is not defined, or they are both defined and $\nu^1_R(p) \neq \pm\nu^2_R(p)$. It is enough to show that $\mathscr{S}^{2n+1}(A) = 0$. This is a consequence of the following lemma:

Lemma B.2. *Let S_1, S_2 be two H-regular hypersurfaces in $\mathbb{H}^n$ and let*

$$A = \{p \in S_1 \cap S_2 : \nu_{S_1}(p) \neq \pm\nu_{S_2}(p)\}.$$

Then, the Hausdorff dimension of A in the Carnot–Carathéodory metric is at most $2n$, $\dim_{CC}(A) \le 2n$, and, in particular, $\mathscr{S}^{2n+1}(A) = 0$.

Proof. The blow-up of S_i, $i = 1, 2$, at a point $p \in A$ is a vertical hyperplane $\Pi_i \times \mathbb{R} \subset \mathbb{R}^{2n} \times \mathbb{R} \equiv \mathbb{H}^n$ — see, e.g., [Franchi et al. 2001] — where:

(i) By blow-up of S_i at p, we mean the limit

$$\lim_{\lambda\to\infty} \lambda(p^{-1} * S_i)$$

in the Gromov–Hausdorff sense. Recall that, for $E \subset \mathbb{H}^n$, we define $\lambda E = \{(\lambda z, \lambda^2 t) \in \mathbb{H}^n : (z, t) \in E\})$.

(ii) For $i = 1, 2$, $\Pi_i \subset \mathbb{R}^{2n}$ is the normal hyperplane to $\nu_{S_i}(p) \in H_p \equiv \mathbb{R}^{2n}$.

It follows that the blow-up of A at p is contained in the blow-up of $S_1 \cap S_2$ at p, i.e., in $(\Pi_1 \cap \Pi_2) \times \mathbb{R}$. Since $\nu_{S_1}(p) \neq \pm\nu_{S_2}(p)$, $\Pi_1 \cap \Pi_2$ is a $(2n-2)$-dimensional plane in $\mathbb{R}^{2n}$, and we conclude thanks to the following lemma. □

Lemma B.3. *Let $k = 0, 1, \dots, 2n$ and $A \subset \mathbb{H}^n$ be such that, for any $p \in A$, the blow-up of A at p is contained in $\Pi_p \times \mathbb{R}$ for some plane $\Pi_p \subset \mathbb{R}^{2n}$ of dimension k. Then we have* $\dim_{\mathrm{CC}}(A) \le k+2$.

Proof. We claim that, for any $\eta > 0$, we have

$$\mathscr{S}^{k+2+\eta}(A) = 0. \tag{B-93}$$

Let $\varepsilon \in \left(0, \frac{1}{2}\right)$ be such that $C\varepsilon^\eta \le \frac{1}{2}$, where $C = C(n)$ is a constant that will be fixed later in the proof. By the definition of blow-up, for any $p \in A$ there exists $r_p > 0$ such that, for all $r \in (0, r_p)$, we have

$$(p^{-1} * A) \cap U_r \subset (\Pi_p)_{\varepsilon r} \times \mathbb{R},$$

where $(\Pi_p)_{\varepsilon r}$ denotes the (εr)-neighbourhood of Π_p in $\mathbb{R}^{2n}$. For any $j \in \mathbb{N}$, set

$$A_j = \{p \in A \cap B_j : r_p > 1/j\}.$$

To prove (B-93), it is enough to prove that

$$\mathscr{S}^{k+2+\eta}(A_j) = 0$$

for any fixed $j \ge 1$. This, in turn, will follow if we show that, for any fixed $\delta \in (0, 1/(2j))$, one has

$$\inf\left\{\sum_{i\in\mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i\in\mathbb{N}} U_{r_i}(p_i),\, r_i < 2\varepsilon\delta\right\} \le \frac{1}{2}\inf\left\{\sum_{i\in\mathbb{N}} r_i^{k+2+\eta} : A_j \subset \bigcup_{i\in\mathbb{N}} U_{r_i}(p_i),\, r_i < \delta\right\}. \tag{B-94}$$

Let $(U_{r_i}(p_i))_{i\in\mathbb{N}}$ be a covering of A_j with balls of radius smaller than δ. There exist points $\bar p_i \in A_j$ such that $(U_{2r_i}(\bar p_i))_{i\in\mathbb{N}}$ is a covering of A_j with balls of radius smaller than $2\delta < 1/j$. By definition of A_j, we have

$$(\bar p_i^{-1} * A_j) \cap U_{2r_i} \subset ((\Pi_{\bar p_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}.$$

The set $((\Pi_{\bar p_i})_{\varepsilon r_i} \times \mathbb{R}) \cap U_{2r_i}$ can be covered by a family of balls $(U_{\varepsilon r_i}(p_h^i))_{h\in H_i}$ of radius $\varepsilon r_i < 2\varepsilon\delta$ in such a way that the cardinality of H_i is bounded by $C\varepsilon^{-k-2}$, where the constant C depends only on n and not on ε. In particular, the family of balls $(U_{\varepsilon r_i}(\bar p_i * p_h^i))_{i\in\mathbb{N}, h\in H_i}$ is a covering of A_j and

$$\begin{aligned}\sum_{i\in\mathbb{N}}\sum_{h\in H_i}(\text{radius } U_{\varepsilon r_i}(\bar p_i * p_h^i))^{k+2+\eta} &= \sum_{i\in\mathbb{N}}\sum_{h\in H_i}(\varepsilon r_i)^{k+2+\eta} \le C\varepsilon^{-k-2}\sum_{i\in\mathbb{N}}(\varepsilon r_i)^{k+2+\eta}\\ &= C\varepsilon^\eta \sum_{i\in\mathbb{N}} r_i^{k+2+\eta} \le \frac{1}{2}\sum_i r_i^{k+2+\eta}.\end{aligned}$$

This proves (B-94) and concludes the proof. □

References

[Almgren 1968] F. J. Almgren, Jr., "Existence and regularity almost everywhere of solutions to elliptic variational problems among surfaces of varying topological type and singularity structure", *Ann. of Math.* (2) **87** (1968), 321–391. MR 37 #837 Zbl 0162.24703

[Ambrosio et al. 2000] L. Ambrosio, N. Fusco, and D. Pallara, *Functions of bounded variation and free discontinuity problems*, Clarendon/Oxford University Press, New York, 2000. MR 2003a:49002 Zbl 0957.49001

[Ambrosio et al. 2006] L. Ambrosio, F. Serra Cassano, and D. Vittone, "Intrinsic regular hypersurfaces in Heisenberg groups", *J. Geom. Anal.* **16**:2 (2006), 187–232. MR 2007g:49072 Zbl 1085.49045

[Burago and Zalgaller 1988] Y. D. Burago and V. A. Zalgaller, *Geometric inequalities*, Grundlehren der Math. Wissenschaften **285**, Springer, Berlin, 1988. MR 89b:52020 Zbl 0633.53002

[Federer 1969] H. Federer, *Geometric measure theory*, Grundlehren der Math. Wissenschaften **153**, Springer, New York, 1969. MR 41 #1976 Zbl 0176.00801

[Franchi et al. 1996] B. Franchi, R. Serapioni, and F. Serra Cassano, "Meyers–Serrin type theorems and relaxation of variational integrals depending on vector fields", *Houston J. Math.* **22**:4 (1996), 859–890. MR 98c:49037 Zbl 0876.49014

[Franchi et al. 2001] B. Franchi, R. Serapioni, and F. Serra Cassano, "Rectifiability and perimeter in the Heisenberg group", *Math. Ann.* **321**:3 (2001), 479–531. MR 2003g:49062 Zbl 1057.49032

[Franchi et al. 2007] B. Franchi, R. Serapioni, and F. Serra Cassano, "Regular submanifolds, graphs and area formula in Heisenberg groups", *Adv. Math.* **211**:1 (2007), 152–203. MR 2008h:49030 Zbl 1125.28002

[Garofalo and Nhieu 1996] N. Garofalo and D.-M. Nhieu, "Isoperimetric and Sobolev inequalities for Carnot–Carathéodory spaces and the existence of minimal surfaces", *Comm. Pure Appl. Math.* **49**:10 (1996), 1081–1144. MR 97i:58032 Zbl 0880.35032

[Hajłasz and Koskela 2000] P. Hajłasz and P. Koskela, *Sobolev met Poincaré*, vol. 145, Mem. Amer. Math. Soc. **688**, American Math. Society, Providence, RI, 2000. MR 2000j:46063 Zbl 0954.46022

[Heinonen 2001] J. Heinonen, *Lectures on analysis on metric spaces*, Springer, New York, 2001. MR 2002c:30028 Zbl 0985.46008

[Kirchheim and Serra Cassano 2004] B. Kirchheim and F. Serra Cassano, "Rectifiability and parameterization of intrinsic regular surfaces in the Heisenberg group", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **3**:4 (2004), 871–896. MR 2005m:28013 Zbl 1170.28300

[Maggi 2012] F. Maggi, *Sets of finite perimeter and geometric variational problems: an introduction to geometric measure theory*, Cambridge Studies in Advanced Mathematics **135**, Cambridge University Press, Cambridge, 2012. MR 2976521 Zbl 1255.49074

[Magnani 2004] V. Magnani, "Note on coarea formulae in the Heisenberg group", *Publ. Mat.* **48**:2 (2004), 409–422. MR 2005d:28014 Zbl 1062.22020

[Magnani 2008] V. Magnani, "Non-horizontal submanifolds and coarea formula", *J. Anal. Math.* **106** (2008), 95–127. MR 2009k:53067 Zbl 1158.22006

[Magnani 2014] V. Magnani, "A new differentiation, shape of the unit ball and perimeter measure", preprint, 2014. arXiv 1408.4726

[Mattila et al. 2010] P. Mattila, R. Serapioni, and F. Serra Cassano, "Characterizations of intrinsic rectifiability in Heisenberg groups", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **9**:4 (2010), 687–723. MR 2012c:28003 Zbl 1229.28004

[Monti 2014] R. Monti, "Lipschitz approximation of $\mathbb{H}$-perimeter minimizing boundaries", *Calc. Var. Partial Differential Equations* **50**:1–2 (2014), 171–198. MR 3194680 Zbl 1290.49088

[Monti 2015] R. Monti, "Minimal surfaces and harmonic functions in the Heisenberg group", *Nonlinear Analysis* (online publication April 2015).

[Monti and Morbidelli 2005] R. Monti and D. Morbidelli, "Regular domains in homogeneous groups", *Trans. Amer. Math. Soc.* **357**:8 (2005), 2975–3011. MR 2006b:43016 Zbl 1067.43003

[Monti and Vittone 2012] R. Monti and D. Vittone, "Sets with finite $\mathbb{H}$-perimeter and controlled normal", *Math. Z.* **270**:1–2 (2012), 351–367. MR 2875838 Zbl 1257.49047

[Schoen and Simon 1982] R. Schoen and L. Simon, "A new proof of the regularity theorem for rectifiable currents which minimize parametric elliptic functionals", *Indiana Univ. Math. J.* **31**:3 (1982), 415–434. MR 84j:49039 Zbl 0516.49026

[Serra Cassano and Vittone 2014] F. Serra Cassano and D. Vittone, "Graphs of bounded variation, existence and local boundedness of non-parametric minimal surfaces in Heisenberg groups", *Adv. Calc. Var.* **7**:4 (2014), 409–492. MR 3276118 Zbl 1304.49084

[Vittone 2012] D. Vittone, "Lipschitz surfaces, perimeter and trace theorems for BV functions in Carnot–Carathéodory spaces", *Ann. Sc. Norm. Super. Pisa Cl. Sci.* (5) **11**:4 (2012), 939–998. MR 3060706 Zbl 1270.53068

Received 7 Oct 2014. Accepted 11 May 2015.

ROBERTO MONTI: monti@math.unipd.it
Dipartimento di Matematica, Universitá di Padova, via Trieste 63, I-35121 Padova, Italy

DAVIDE VITTONE: vittone@math.unipd.it
Dipartimento di Matematica, Universitá di Padova, via Trieste 63, I-35121 Padova, Italy

and

Institut für Mathematik, Universität Zürich, Winterthurerstrasse 190, CH-8057 Zürich, Switzerland

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2015 is US $205/year for the electronic version, and $390/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFLOW® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS & PDE

Volume 8 No. 6 2015