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IMPROVEMENT OF THE ENERGY METHOD FOR STRONGLY NONRESONANT DISPERSIVE EQUATIONS AND APPLICATIONS

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We propose a new approach to prove the local well-posedness of the Cauchy problem associated with strongly nonresonant dispersive equations. As an example, we obtain unconditional well-posedness of the Cauchy problem in the energy space for a large class of one-dimensional dispersive equations with a dispersion that is greater than the one of the Benjamin–Ono equation. At the level of dispersion of the Benjamin–Ono, we also prove the well-posedness in the energy space but without unconditional uniqueness. Since we do not use a gauge transform, this enables us in all cases to prove strong convergence results in the energy space for solutions of viscous versions of these equations towards the purely dispersive solutions. Finally, it is worth noting that our method of proof works on the torus as well as on the real line.

1. Introduction

The Cauchy problem associated with dispersive equations with derivative nonlinearity has been extensively studied since the eighties. The first results were obtained by using energy methods that did not make use of the dispersive effects (see for instance [Kato 1983; Abdelouhab et al. 1989]). These methods were restricted to regular initial data ($s > d/2$, where $d \geq 1$ is the spatial dimension) and only ensured the continuity of the solution map. At the end of the eighties, Kenig, Ponce and Vega proved new dispersive estimates that enable them to lower the regularity requirement on the initial data (see for instance [Kenig et al. 1991; 1993; Ponce 1991]). They even obtained local well-posedness (LWP) for a large class of dispersive equations by a fixed point argument in a suitable Banach space related to linear dispersive estimates. Then, Bourgain [1993a; 1993b] introduced the now so-called Bourgain spaces, where one can solve by a fixed point argument a wide class of dispersive equations with very rough initial data. It is worth noting that, since the nonlinearity of these equations is in general algebraic, the fixed point argument ensures the real analyticity of the solution map. Molinet, Saut and Tzvetkov [Molinet et al. 2001] noticed that a large class of "weakly" dispersive equations, including in particular the Benjamin–Ono equation, cannot be solved by a fixed point argument for initial data in any Sobolev spaces H^s. This obstruction is due to bad interactions between high frequencies and very low frequencies. Since then, roughly speaking, two approaches have been developed to lower the regularity requirement for such equations. The first one is the so-called gauge method. This consists in introducing a nonlinear gauge transform of the solution that solved an equation with fewer bad interactions than the original one. This method proved to be very

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efficient for obtaining the lowest regularity index for solving canonical equations (see [Tao 2004; Ionescu and Kenig 2007; Burq and Planchon 2008; Molinet and Pilod 2012] for the BO equation and [Herr et al. 2010] for the dispersive generalized BO equation) but has the disadvantage of behaving very badly with respect to perturbation of the equation. The second one consists in improving the dispersive estimates by localizing it in space-frequency-depending time intervals and then mixing it with classical energy estimates. This type of method was first introduced by Koch and Tzvetkov [2003] (see also [Kenig and Koenig 2003] for some improvements) in the framework of Strichartz's spaces and then by Koch and Tataru [2007] (see also [Ionescu et al. 2008]) in the framework of Bourgain's spaces. It is less efficient for getting the best regularity index but it is surely more flexible with respect to perturbation of the equation.

In this paper we propose a new approach to derive local and global well-posedness results for dispersive equations that do not exhibit too-strong resonances. This approach combines classical energy estimates with Bourgain-type estimates on a time interval that does not depend on the space frequency. Here, we will apply this method to prove unconditional local well-posedness results on both $\mathbb{R}$ and $\mathbb{T}=\mathbb{R}/2\pi\mathbb{Z}$ without the use of a gauge transform for a large class of one-dimensional quadratic dispersive equations with a dispersion between those of the Benjamin–Ono equation and the KdV equation. This class contains, in particular, the equations with pure power dispersion that read

$$u_t+\partial_x D_x^\alpha u+uu_x=0 \tag{1-1}$$

with $\alpha\in[1,2]$.

The principle of the method is particularly simple in the regular case $s>\frac{1}{2}$. We start with the classical space-frequency-localized energy estimate

$$\|P_N u\|^2_{L^\infty_T H^s}\lesssim \|P_N u_0\|^2_{H^s}+\sup_{t\in]0,T[}\langle N\rangle^{2s}\left|\int_0^t\int_{\mathbb{R}}\partial_x P_N(u^2)P_N u\right|, \tag{1-2}$$

obtained by projecting the equation on frequencies of order N and taking the inner product with $J_x^s u$. Note that the second term in the right-hand side of (1-2) is easily controlled (after summing in N) by $\|u\|^3_{L^\infty_T H^s}$ for $s>\frac{3}{2}$. This is the main point in the standard energy method that leads to LWP in H^s, $s>\frac{3}{2}$. In order to take into account the dispersive effects of the equation, we will decompose the three factors in the integral term into dyadic pieces for the modulation variables and use the Bourgain spaces $X^{s,b}$ in a nonconventional way. Actually, it is known that standard bilinear estimates in $X^{s,b}$-spaces with $b=\frac{1}{2}+$ fail for (1-1) for any $s\in\mathbb{R}$ as soon as $\alpha<2$. On the other hand, as noticed in [Zhou 1997], it is easy to deduce from the equation that a solution $u\in L^\infty(0,T;H^s)$ to (1-3) has to belong to the space $X_T^{s-1,1}$. This means that, if we accept the loss of a few spatial derivatives on the solution, then we may gain some regularity in the modulation variable. This is particularly profitable when the equation enjoys a strong nonresonance relation such as (2-6). Actually, this formally allows us to estimate the second term in (1-2) at the desired level. However, this term involves a multiplication by $1_{]0,t[}$ and it is well known that such multiplication is not bounded in $X^{s-1,1}$. To overcome this difficulty we decompose this function into two parts: a high-frequency part that will be very small in L^1_T and a low-frequency part that will have good properties with respect to multiplication with high-modulation functions in $X^{s-1,1}$. This decomposition will depend on the space-frequency-localization of the three functions that appear in the trilinear term.

1A. *Presentation of the results.* In this paper we consider the dispersive equation

$$u_t + L_{\alpha+1}u + \tfrac{1}{2}\partial_x(u^2) = 0 \tag{1-3}$$

associated with the initial condition

$$u(0,\cdot) = u_0, \tag{1-4}$$

where $x \in \mathbb{R}$ or $\mathbb{T}$, $u = u(t,x)$ and $u_0 = u_0(x)$ are real-valued functions, $\alpha > 0$ is a real number and the linear operator $L_{\alpha+1}$ satisfies the following hypothesis:

Hypothesis 1. *$L_{\alpha+1}$ is the Fourier multiplier operator by $ip_{\alpha+1}$, where $p_{\alpha+1}$ is a real-valued odd function satisfying, for some $\lambda_0 > 0$,*

(1) *For any $|\xi| \gg 1$ and $0 < \lambda \le \lambda_0$,*

$$\lambda^{\alpha+1}|p_{\alpha+1}(\lambda^{-1}\xi)| \lesssim |\xi|^{\alpha+1}. \tag{1-5}$$

(2) *For any $(\xi_1,\xi_2) \in \mathbb{R}^2$ with $|\xi_1| \gg 1$ and any $0 < \lambda \le \lambda_0$,*

$$\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| \sim |\xi|_{\min}|\xi|^{\alpha}_{\max}, \tag{1-6}$$

where

$$\begin{aligned} \Omega(\xi_1,\xi_2) &:= p_{\alpha+1}(\xi_1+\xi_2) - p_{\alpha+1}(\xi_1) - p_{\alpha+1}(\xi_2), \\ |\xi|_{\min} &:= \min(|\xi_1|, |\xi_2|, |\xi_1+\xi_2|) \\ \textit{and} \quad |\xi|_{\max} &:= \max(|\xi_1|, |\xi_2|, |\xi_1+\xi_2|). \end{aligned}$$

Remark 1.1. We will see in Lemma 2.1 below that, for $\alpha > 0$, a very simple criterion on p ensures (1-6). With this criterion in hand, it is not too hard to check that the following linear operators satisfy Hypothesis 1:

(1) The purely dispersive operators $L := \partial_x D_x^{\alpha}$ with $\alpha > 0$.

(2) The linear intermediate long wave operator $L := \partial_x D_x \coth D_x$. Note that here $\alpha = 1$.

(3) Some perturbations of the Benjamin–Ono equation, such as the Smith operator [1972], $L := \partial_x(D_x^2+1)^{1/2}$. Here again $\alpha = 1$.

Before stating our main result, let us define what we mean by unconditional well-posedness.

Definition 1.2. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$, $T > 0$ and $s \ge 0$. We will say that $u \in L^\infty(0,T;H^s(\mathbb{K}))$ is a solution to (1-3) associated with the initial datum $u_0 \in H^s(\mathbb{K})$ if u satisfies (1-3)–(1-4) in the distributional sense, i.e., for any test function $\phi \in C_c^\infty(]-T,T[\times\mathbb{K})$,

$$\int_0^\infty \int_{\mathbb{K}} \bigl[(\phi_t + L_{\alpha+1}\phi)u + \tfrac{1}{2}\phi_x u^2\bigr]\,dx\,dt + \int_{\mathbb{K}} \phi(0,\cdot)u_0\,dx = 0 \tag{1-7}$$

Remark 1.3. For $u \in L^\infty(0,T;H^s(\mathbb{K}))$, with $s \ge 0$, u^2 is well defined and is in $L^\infty(0,T;H^{s-(1/2+)}(\mathbb{K}))$. Moreover, (1-5) forces

$$L_{\alpha+1}u \in L^\infty(0,T;H^{s-\alpha-1}(\mathbb{K})).$$

Therefore, $u_t \in L^\infty(0,T;H^{s-\alpha-1}(\mathbb{K}))$ and (1-7) ensures that (1-3) is satisfied in $L^\infty(0,T;H^{s-\alpha-1}(\mathbb{K}))$. In particular, $u \in C([0,T];H^{s-\alpha-1}(\mathbb{K}))$ and (1-7) forces the initial condition $u(0) = u_0$. Note that this actually implies that $u \in C([0,T]; H^\theta(\mathbb{K}))$ for any $\theta < s$. Finally, we note that this ensures that u satisfies the Duhamel formula associated with (1-3).

Definition 1.4. Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$ and $s \in \mathbb{R}$. We will say that the Cauchy problem associated with (1-3) is unconditionally locally well-posed in $H^s(\mathbb{K})$ if, for any initial data $u_0 \in H^s(\mathbb{K})$, there exists $T = T(\|u_0\|_{H^s}) > 0$ and a solution $u \in C([0,T]; H^s(\mathbb{K}))$ to (1-3) emanating from u_0. Moreover, u is the unique solution to (1-3) associated with u_0 that belongs to $L^\infty(]0,T[; H^s(\mathbb{K}))$. Finally, for any $R > 0$, the solution map $u_0 \mapsto u$ is continuous from the ball of $H^s(\mathbb{K})$ with radius R centered at the origin into $C([0,T(R)]; H^s(\mathbb{K}))$.

Theorem 1.5. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$, $L_{\alpha+1}$ satisfy Hypothesis 1 with $1 \le \alpha \le 2$ and let $s \ge 1 - \frac{\alpha}{2}$ with $(s,\alpha) \ne \left(\frac{1}{2}, 1\right)$. Then the Cauchy problem associated with (1-3) is unconditionally locally well-posed in $H^s(\mathbb{K})$ with a maximal time of existence $T \gtrsim (1 + \|u_0\|_{H^{1-\alpha/2}})^{-2(\alpha+1)/(2\alpha-1)}$.*

Remark 1.6. In the regular case (Cauchy problem in H^s with $s > \frac{1}{2}$), we actually need (1-6) only for $|\xi_1| \wedge |\xi_2| \gg 1$.

Remark 1.7. Our method also works in the case $\alpha > 2$. In this case we get the unconditional well-posedness in $H^s(\mathbb{K})$ for $s \ge 0$.

Remark 1.8. For $L_{\alpha+1} := \partial_x^3$, we recover the unconditional LWP results for the KdV equation in $L^2(\mathbb{R})$ and $L^2(\mathbb{T})$ obtained in [Zhou 1997; Babin et al. 2011], respectively.

For $L_{\alpha+1}$ with $\alpha \in]1,2[$ our results on unconditional uniqueness are, to our knowledge, new for both the real line case and the periodic case.

In the limit case $(s,\alpha) = \left(\frac{1}{2}, 1\right)$ we do not succeed in proving the unconditional uniqueness result. However, our method of proof enables us to prove the well-posedness without using a gauge transform. We think that this result is also of interest since $H^{1/2}$ is the energy space when $\alpha = 1$. It is worth noticing that, as far as we know, the available results without gauge transformation on the local well-posedness of the Benjamin–Ono equation in Sobolev spaces $H^s(\mathbb{R})$ were restricted to $s \ge 1$ (see [Guo et al. 2011]). Also, the well-posedness in the energy space $H^{1/2}$ seems to be new for the intermediate long waves equation, at least in the periodic setting.

Theorem 1.9. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$ and assume $L_{\alpha+1}$ satisfies Hypothesis 1 with $\alpha = 1$. Then the Cauchy problem associated with (1-3) is locally well-posed in $H^{1/2}(\mathbb{K})$ with a maximal time of existence $T \gtrsim (1 + \|u_0\|_{H^{1/2}})^{-4}$.*

Let us assume now that the symbol $p_{\alpha+1}$ satisfies, moreover,

$$|p_{\alpha+1}(\xi)| \lesssim |\xi| \quad \text{for } |\xi| \le 1 \qquad \text{and} \qquad |p_{\alpha+1}(\xi)| \sim |\xi|^{\alpha+1} \quad \text{for } |\xi| \ge 1. \tag{1-8}$$

Then it is not too hard to check that (1-3) enjoys the conservation laws

$$\frac{d}{dt}\int_{\mathbb{K}} u^2\,dx = 0 \quad \text{and} \quad \frac{d}{dt}\int_{\mathbb{K}} \left(|\Lambda^{\alpha/2}u|^2 + \tfrac{1}{3}u^3\right)dx = 0,$$

where $\Lambda^{\alpha/2}$ is the space Fourier multiplier defined by

$$\widehat{\Lambda^{\alpha/2}v}(\xi) = \left|\frac{p_{\alpha+1}(\xi)}{\xi}\right|^{\frac{1}{2}} \hat{v}(\xi).$$

Combined with the embedding $H^{\alpha/2} \hookrightarrow L^3$, we get an a priori bound of the $H^{\alpha/2}$-norm of the solution. We may then iterate Theorems 1.5 and 1.9 to obtain the following corollary:

Corollary 1.10. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$ and assume $L_{\alpha+1}$ satisfies Hypothesis 1 and (1-8). Then the Cauchy problem associated with (1-3) is unconditionally globally well-posed in $H^{\alpha/2}(\mathbb{K})$ for $1 < \alpha \leq 2$, and globally well-posed in $H^{1/2}(\mathbb{T})$ for $\alpha = 1$.*

Remark 1.11. The linear operators given in Remark 1.1 also satisfy assumption (1-8).

Remark 1.12. If one considers LWP and not unconditional LWP, then the best-known results for (1-1) with $1 < \alpha < 2$ have been obtained in [Herr et al. 2010], where the global well-posedness in $L^2(\mathbb{R})$ is proved by using a paradifferential gauge transformation. As far as we know, the best available results without gauge transformation are obtained in [Guo 2012], where the LWP in $H^s(\mathbb{R})$ with $s > 2-\alpha$ is proven. This leads to a global well-posedness result only for $\alpha > \frac{4}{3}$. Therefore, even for (1-1), our results improve the local and global available well-posedness results without the use of gauge transformation on the real line. To the best of our knowledge, they are new on the one-dimensional torus, where we are not aware of any global well-posedness result.

It is well known that, taking into account some damping or dissipative effects, dissipative versions of (1-3) can be derived (see for instance [Ott and Sudan 1970; Edwin and Roberts 1986]). One quite direct application of the fact that we do not need a gauge transform to solve (1-3) is that we can easily treat the dissipative limit of dissipative versions of (1-3). Such a dissipative limit was, for example, studied for the Benjamin–Ono equation on the real line in [Guo et al. 2011; Molinet 2013].

Let us introduce the following dissipative version of (1-3):

$$u_t + L_{\alpha+1}u + \varepsilon A_\beta u + uu_x = 0, \tag{1-9}$$

where $\varepsilon > 0$ is a small parameter, $\beta \geq 0$ and A_β is a linear operator satisfying the following hypothesis:

Hypothesis 2. *We assume that A_β is the Fourier multiplier operator by q_β, where q_β is a real-valued, even function, bounded on bounded intervals, satisfying: for all $0 < \lambda \ll 1$ and $\xi \gg 1$,*

$$\lambda^\beta q_\beta(\lambda^{-1}\xi) \sim |\xi|^\beta.$$

Remark 1.13. Both the homogeneous operators D_x^β and the nonhomogeneous operators J_x^β satisfy Hypothesis 2.

Theorem 1.14. *Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{T}$, $1 \leq \alpha \leq 2$, $0 \leq \beta \leq 1+\alpha$ and $s \geq 1 - \frac{\alpha}{2}$.*

(1) *Then the Cauchy problem associated with (1-9) is locally well-posed in $H^s(\mathbb{K})$.*

(2) *For $u_0 \in H^s(\mathbb{K})$, let u be the solution to (1-3) emanating from u_0 and let the maximal time of existence of u in H^s be $T^* \gtrsim (1+\|u_0\|_{H^{1-\alpha/2}})^{-2(\alpha+1)/(2\alpha-1)}$ (note that T^* may be infinite). Then the maximal time of existence T_ε of the solution u_ε to (1-9) emanating from u_0 satisfies $\liminf_{\varepsilon\to 0} T_\varepsilon \geq T^*$. Moreover, for any $0 < T_0 < T^*$, $u_\varepsilon \to u$ in $C([0,T_0];H^s)$ as $\varepsilon \to 0$.*

Remark 1.15. The constraint $\beta \leq 1+\alpha$ is clearly an artifact of the method we used. We think that it could be dropped by replacing, in some estimates, the dispersive Bourgain spaces by dispersive–dissipative Bourgain spaces that were first introduced in [Molinet and Ribaud 2002]. But, since the dissipative operators involved in wave motions are commonly of order less or equal to 2, we do not pursue this issue.

The rest of the paper is organized as follows: In Section 2, we introduce the notations, define the function spaces and state some preliminary lemmas. In Section 3, we develop our method in the simplest case, $s > \frac{1}{2}$, while the nonregular case is treated in Section 4. Section 5 is devoted to the proof of Theorem 1.14. We conclude the paper with an Appendix explaining how to deal with the special case $(s,\alpha) = \left(\frac{1}{2},1\right)$.

2. Notations, function spaces and preliminary lemmas

2A. *Notation.* For any positive numbers a and b, the notation $a \lesssim b$ means that there exists a positive constant c such that $a \leq cb$. We also write $a \sim b$ when $a \lesssim b$ and $b \lesssim a$. Moreover, if $\alpha \in \mathbb{R}$, then α_+ and α_- will denote a number slightly greater and less than α, respectively.

For $u = u(x,t) \in \mathcal{S}(\mathbb{R}^2)$, $\mathcal{F}u = \hat{u}$ will denote its space-time Fourier transform, whereas $\mathcal{F}_x u = (u)^{\wedge_x}$ and $\mathcal{F}_t u = (u)^{\wedge_t}$ will denote its Fourier transform in space and in time, respectively. For $s \in \mathbb{R}$, we define the Bessel and Riesz potentials of order $-s$, J_x^s and D_x^s, by

$$J_x^s u = \mathcal{F}_x^{-1}\big((1+|\xi|^2)^{\frac{s}{2}}\mathcal{F}_x u\big) \quad \text{and} \quad D_x^s u = \mathcal{F}_x^{-1}\big(|\xi|^s \mathcal{F}_x u\big).$$

Throughout the paper, we fix a smooth cutoff function η such that

$$\eta \in C_0^\infty(\mathbb{R}), \quad 0 \leq \eta \leq 1, \quad \eta_{|[-1,1]} = 1 \quad \text{and} \quad \operatorname{supp}(\eta) \subset [-2,2].$$

We set $\phi(\xi) := \eta(\xi) - \eta(2\xi)$. For $l \in \mathbb{Z}$, we define

$$\phi_{2^l}(\xi) := \phi(2^{-l}\xi),$$

and, for $l \in \mathbb{N}^*$,

$$\psi_{2^l}(\xi,\tau) = \phi_{2^l}(\tau - p_{\alpha+1}(\xi)),$$

where $i p_{\alpha+1}$ is the Fourier symbol of $L_{\alpha+1}$. By convention, we also denote

$$\psi_1(\xi,\tau) := \eta\big(2(\tau - p_{\alpha+1}(\xi))\big).$$

Any summations over capitalized variables such as N, L, K or M are presumed to be dyadic. Unless stated otherwise, we work with homogeneous dyadic decomposition for the space-frequency variables and nonhomogeneous decompositions for modulation variables, i.e., these variables range over numbers of the form $\{2^k : k \in \mathbb{Z}\}$ and $\{2^k : k \in \mathbb{N}\}$, respectively. Then we have that

$$\sum_{N>0} \phi_N(\xi) = 1 \quad \text{for all } \xi \in \mathbb{R}^* \qquad \text{and} \qquad \operatorname{supp}(\phi_N) \subset \left\{\tfrac{1}{2}N \le |\xi| \le 2N\right\} \quad \text{for } N \in \{2^k : k \in \mathbb{Z}\},$$

and

$$\sum_{L\ge 1} \psi_L(\xi,\tau) = 1 \quad \text{for all } (\xi,\tau) \in \mathbb{R}^2$$

Let us define the Littlewood–Paley multipliers by

$$P_N u = \mathcal{F}_x^{-1}(\phi_N \mathcal{F}_x u), \quad Q_L u = \mathcal{F}^{-1}(\psi_L \mathcal{F} u),$$

$P_{\ge N} := \sum_{K\ge N} P_K$, $P_{\le N} := \sum_{K\le N} P_K$, $Q_{\ge L} := \sum_{K\ge L} Q_K$ and $Q_{\le L} := \sum_{K\le L} Q_K$. For brevity we also write $u_N = P_N u$, $u_{\le N} = P_{\le N} u$, etc.

Let χ be a (possibly complex-valued) bounded measurable function on $\mathbb{R}^2$ and define the pseudoproduct operator $\Pi = \Pi_\chi$ on $\mathcal{S}(\mathbb{R})^2$ by

$$\mathcal{F}(\Pi(f,g))(\xi) = \int_{\mathbb{R}} \hat{f}(\xi_1)\hat{g}(\xi-\xi_1)\chi(\xi,\xi_1)\,d\xi_1.$$

Throughout the paper, we write $\Pi = \Pi_\chi$, where χ may be different at each occurrence of Π. This bilinear operator behaves like a product in the sense that it satisfies the following properties:

$$\Pi(f,g) = fg \quad \text{if } \chi \equiv 1,$$
$$\int_{\mathbb{R}} \Pi_\chi(f,g)h = \int_{\mathbb{R}} f\,\Pi_{\chi_1}(g,h) = \int_{\mathbb{R}} \Pi_{\chi_2}(f,h)g \tag{2-1}$$

with $\chi_1(\theta,\theta_1) = \bar{\chi}(\theta_1,\theta)$ and $\chi_2(\theta,\theta_1) = \bar{\chi}(\theta-\theta_1,\theta)$ for any real-valued functions f, g, $h \in \mathcal{S}(\mathbb{R})$. Moreover, we easily check from the Bernstein inequality that, if $f_i \in L^2(\mathbb{R})$ has a Fourier transform localized in an annulus $\{|\xi| \sim N_i\}$, $i = 1, 2, 3$, then

$$\left|\int_{\mathbb{R}} \Pi(f_1,f_2)f_3\right| \lesssim N_{\min}^{\frac{1}{2}} \prod_{i=1}^{3} \|f_i\|_{L^2}, \tag{2-2}$$

where the implicit constant only depends on $\|\chi\|_{L^\infty(\mathbb{R}^2)}$ and $N_{\min} = \min\{N_1, N_2, N_3\}$. With this notation in hand, we will be able to systematically estimate terms of the form

$$\int_{\mathbb{R}} P_N(u^2)\,\partial_x P_N u$$

to put the derivative on the lowest frequency factor.

2B. ***Function spaces.*** For $1 \le p \le \infty$, $L^p(\mathbb{R})$ is the usual Lebesgue space with the norm $\|\cdot\|_{L^p}$ and, for $s \in \mathbb{R}$, $H^s(\mathbb{R})$ is the usual Sobolev space with its usual norm,

$$\|\phi\|_{H^s} = \|J_x^s \phi\|_{L^2}.$$

If B is one of the spaces defined above, $1 \le p \le \infty$, we will define the space-time spaces $L^p_t B$ and $\widetilde{L}^p_t B$ equipped with the norms

$$\|f\|_{L^p_t B} = \Big(\int_{\mathbb{R}} \|f(\cdot,t)\|_B^p \, dt \Big)^{\frac{1}{p}},$$

with obvious modifications for $p = \infty$, and

$$\|f\|_{\widetilde{L}^p_t B} = \Big(\sum_{N>0} \|P_N f\|^2_{L^p_t B} \Big)^{\frac{1}{2}}.$$

For $s, b \in \mathbb{R}$, we introduce the Bourgain spaces $X^{s,b}$ related to the linear part of (1-3) as the completion of the Schwartz space $\mathcal{S}(\mathbb{R}^2)$ under the norm

$$\|v\|_{X^{s,b}} := \Big(\int_{\mathbb{R}^2} \langle \tau - p_{\alpha+1}(\xi) \rangle^{2b} \langle \xi \rangle^{2s} |\hat{v}(\xi,\tau)|^2 \, d\xi \, d\tau \Big)^{\frac{1}{2}}, \tag{2-3}$$

where $\langle x \rangle := 1 + |x|$ and $i p_{\alpha+1}$ is the Fourier symbol of $L_{\alpha+1}$. Recall that

$$\|v\|_{X^{s,b}} = \|U_\alpha(-t) v\|_{H^{s,b}_{t,x}},$$

where $U_\alpha(t) = \exp(t L_{\alpha+1})$ is the generator of the free evolution associated with (1-3).

Finally, we will use restriction-in-time versions of these spaces. Let $T > 0$ be a positive time and let Y be a normed space of space-time functions. The restriction space Y_T will be the space of functions $v : \mathbb{R} \times]0,T[\to \mathbb{R}$ satisfying

$$\|v\|_{Y_T} := \inf\{ \|\tilde{v}\|_Y \mid \tilde{v} : \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \ \tilde{v}|_{\mathbb{R}\times]0,T[} = v \} < \infty.$$

2C. *Preliminary lemmas.*

Lemma 2.1. *Let $p : \mathbb{R} \to \mathbb{R}$ be an odd function belonging to $C^1(\mathbb{R}) \cap C^2(\mathbb{R}^*)$. Assume that there exist $\alpha > 0$ and $\xi_0 > 0$ such that, for all $\xi \ge \xi_0$,*

$$|p'(\xi)| \sim |\xi|^\alpha \quad \text{and} \quad |p''(\xi)| \sim |\xi|^{\alpha-1}. \tag{2-4}$$

Then the Fourier multiplier $L_{\alpha+1}$ by ip satisfies Hypothesis 1.

Proof. Let $0 < \lambda \le \xi_0^{-1}$ be a real number. First, by the mean value theorem, for $\xi \ge 1$,

$$|p(\lambda^{-1}\xi)| \lesssim |p(\xi_0)| + \lambda^{-(\alpha+1)} \xi^{\alpha+1} \lesssim \lambda^{-1}(\lambda\xi_0) \max_{\xi\in[0,\xi_0]} |p'(\xi)| + \xi^{\alpha+1}$$

and thus

$$\lambda^{\alpha+1} |p(\lambda^{-1}\xi)| \lesssim \lambda^\alpha \max_{\xi\in[0,\xi_0]} |p'(\xi)| + \xi^{\alpha+1} \lesssim \xi^{\alpha+1}$$

as soon as $\lambda \le \big(\max_{\xi\in[0,\xi_0]} |p'(\xi)|\big)^{-1/\alpha}$. This proves (1-5) for

$$\lambda_0 = \min\big(\xi_0^{-1}, \big(\max_{\xi\in[0,\xi_0]} |p'(\xi)|\big)^{-\frac{1}{\alpha}}\big). \tag{2-5}$$

Let us now prove (1-6). In the sequel, we take $0 < \lambda \le \lambda_0$ with λ_0 defined by (2-5). By symmetry, we can assume $|\xi_2| \le |\xi_1|$. We separate different cases:

Case 1: $|\xi_2| \ll |\xi_1|$. Since, by hypothesis, $|\xi_1| \gg 1$, it follows that $\lambda^{-1}|\xi_1| \gg \xi_0$ and thus there exists $\theta \in [\xi_1, \xi_1+\xi_2]$ such that

$$\lambda^{\alpha+1}\big|p(\lambda^{-1}(\xi_1+\xi_2)) - p(\lambda^{-1}\xi_1)\big| = \lambda^{\alpha}|\xi_2||p'(\lambda^{-1}\theta)| \sim \lambda^{\alpha}|\xi_2||\lambda^{-1}\theta|^{\alpha} \sim |\xi_2||\xi_1|^{\alpha}.$$

Now, if $\lambda^{-1}|\xi_2| \le \xi_0$ then

$$\lambda^{\alpha+1}|p(\lambda^{-1}\xi_2)| \le \lambda^{\alpha}|\xi_2| \max_{\xi\in[0,\xi_0]} |p'(\xi)| \ll |\xi_2||\xi_1|^{\alpha}.$$

On the other hand, if $\lambda^{-1}|\xi_2| \ge \xi_0$ then

$$\begin{aligned}\lambda^{\alpha+1}|p(\lambda^{-1}\xi_2)| &= \lambda^{\alpha+1}|p(\xi_0) + p(\lambda^{-1}\xi_2) - p(\xi_0)| \\ &\le \lambda^{\alpha+1}\big(|\xi_0| \max_{\xi\in[0,\xi_0]} |p'(\xi)| + \lambda^{-1}|\xi_2||\lambda^{-1}\xi_2|^{\alpha}\big) \\ &\le |\xi_2|^{\alpha+1} + \lambda^{\alpha}|\xi_2| \max_{\xi\in[0,\xi_0]} |p'(\xi)| \ll |\xi_2||\xi_1|^{\alpha}.\end{aligned}$$

Gathering these two estimates leads to

$$\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| \sim |\xi_2||\xi_1|^{\alpha}.$$

Case 2: $|\xi_2| \gtrsim |\xi_1|$. In this case we have $\lambda^{-1}|\xi_2| \gg \xi_0$. Since p is an odd function, by symmetry we can assume that $\xi_1 > 0$.

Case 2(a): $\xi_1\xi_2 \ge 0$. Then we have $0 < \xi_0 \ll \lambda^{-1}\xi_2 \le \lambda^{-1}\xi_1 < \lambda^{-1}\xi_1 + \xi_2$. We notice that

$$\begin{aligned}&\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| \\ &\quad = \lambda^{\alpha+1}\int_{\xi_0}^{\lambda^{-1}\xi_2} (p'(\lambda^{-1}\xi_1+\theta) - p'(\theta))\,d\theta + \lambda^{\alpha+1}(p(\lambda^{-1}\xi_1+\xi_0) - p(\lambda^{-1}\xi_1)) - \lambda^{\alpha+1}p(\xi_0)\end{aligned}$$

with

$$|p(\lambda^{-1}\xi_1+\xi_0) - p(\lambda^{-1}\xi_1)| \lesssim \xi_0\lambda^{-\alpha}\xi_1^{\alpha} \ll \lambda^{-\alpha-1}\xi_2\xi_1^{\alpha}$$

and

$$p'(\lambda^{-1}\xi_1+\theta) - p'(\theta) = \int_0^{\lambda^{-1}\xi_1} p''(\theta+\mu)\,d\mu.$$

But, for $\theta \ge \xi_0$, p'' does not change sign since $|p''(\theta)| \sim |\theta|^{\alpha-1}$ and p'' is continuous outside 0. Therefore, for $\theta \in [\xi_0, \lambda^{-1}\xi_2]$, we get

$$\int_0^{\lambda^{-1}\xi_1} p''(\theta+\mu)\,d\mu \sim \int_0^{\lambda^{-1}\xi_1} (\theta+\mu)^{\alpha-1}\,d\mu \sim ((\lambda^{-1}\xi_1+\theta)^{\alpha} - \theta^{\alpha}) \sim \lambda^{-\alpha}\xi_1^{\alpha}.$$

Gathering these estimates we obtain

$$\lambda^{\alpha+1}|\Omega(\lambda^{-1}\xi_1, \lambda^{-1}\xi_2)| \sim \xi_2\xi_1^{\alpha}.$$

Case 2(b): $\xi_1\xi_2 < 0$. For $\xi_1+\xi_2 \ll -\xi_2$, recalling that p is an odd function, we can argue exactly as in Case 1, but with $\xi_1+\xi_2$, $-\xi_2$ and ξ_1 playing the role of ξ_2, ξ_1 and $\xi_1+\xi_2$, respectively. Finally, for $\xi_1+\xi_2 \gtrsim -\xi_2$, we argue exactly as in Case 2(a) with the same exchange of roles as above. □

Lemma 2.2. *Assume that $p_{\alpha+1}$ satisfies (1-6) with $\lambda = 1$. Let $L_1, L_2, L_3 \geq 1$, $0 < N_1 \leq N_2 \leq N_3$ be dyadic numbers and $u, v, w \in \mathcal{S}'(\mathbb{R}^2)$. Then*

$$\int_{\mathbb{R}^2} \Pi(Q_{L_1}P_{N_1}u, Q_{L_2}P_{N_2}v) Q_{L_3}P_{N_3}w = 0$$

whenever the following relation is not satisfied:

$$L_{\max} \sim N_1 N_2^\alpha \qquad \textit{or} \qquad (L_{\max} \gg N_1 N_2^\alpha \quad \textit{and} \quad L_{\max} \sim L_{\mathrm{med}}), \tag{2-6}$$

where $L_{\max} = \max(L_1, L_2, L_3)$, $L_{\mathrm{med}} = \max(\{L_1, L_2, L_3\} - \{L_{\max}\})$ and where the two first implicit constants in (2-6) are related to the implicit constant in (1-6).

Proof. This is a direct consequence of the hypothesis (1-6) on the resonance function $\Omega(\xi_1, \xi_2)$, since

$$\Omega(\xi_1, \xi_2) = \sigma(\tau_1+\tau_2, \xi_1+\xi_2) - \sigma(\tau_1, \xi_1) - \sigma(\tau_2, \xi_2)$$

with $\sigma(\tau, \xi) := \tau - p_{\alpha+1}(\xi)$. □

Lemma 2.3. *Let $L \geq 1$, $1 \leq p \leq \infty$ and $s \in \mathbb{R}$. The operator $Q_{\leq L}$ is bounded in $L^p_t H^s$ uniformly in $L \geq 1$.*

Proof. Let $R_{\leq L}$ be the Fourier multiplier by $\eta(\tau/L)$, where η is as defined in Section 2A. The trick is to notice that $Q_{\leq L}u = U_\alpha(t)(R_{\leq L}U_\alpha(-t)u)$. Therefore, using the unitarity of $U_\alpha(\cdot)$ in $H^s(\mathbb{R})$, we infer that

$$\begin{aligned}\|Q_{\leq L}u\|_{L^p_t H^s} = \|U_\alpha(t)(R_{\leq L}U_\alpha(-t)u)\|_{L^p_t H^s} = \|R_{\leq L}U_\alpha(-t)u\|_{L^p_t H^s} \lesssim \|U_\alpha(-t)u\|_{L^p_t H^s} \\ = \|u\|_{L^p_t H^s}.\end{aligned}$$ □

For any $T > 0$, we consider 1_T, the characteristic function of $[0, T]$, and use the decomposition

$$1_T = 1^{\mathrm{low}}_{T,R} + 1^{\mathrm{high}}_{T,R}, \qquad \widehat{1^{\mathrm{low}}_{T,R}}(\tau) = \eta\Big(\frac{\tau}{R}\Big)\widehat{1_T}(\tau) \tag{2-7}$$

for some $R > 0$.

The properties of this decomposition we will need are listed in the following lemmas.

Lemma 2.4. *For any $R > 0$ and $T > 0$,*

$$\|1^{\mathrm{high}}_{T,R}\|_{L^1} \lesssim T \wedge R^{-1} \tag{2-8}$$

and

$$\|1^{\mathrm{low}}_{T,R}\|_{L^\infty} \lesssim 1. \tag{2-9}$$

Proof. A direct computation provides

$$\begin{aligned}\|1_{T,R}^{\text{high}}\|_{L^1} &= \int_{\mathbb{R}}\left|\int_{\mathbb{R}}\Big(1_T(t)-1_T\Big(t-\frac{s}{R}\Big)\Big)\mathcal{F}^{-1}\eta(s)\,ds\right|dt\\ &\le \int_{\mathbb{R}}\int_{[0,T]\setminus[s/R,T+s/R]\cup[s/R,T+s/R]\setminus[0,T]}|\mathcal{F}^{-1}\eta(s)|\,dt\,ds\\ &\lesssim \int_{\mathbb{R}}\Big(T\wedge\frac{|s|}{R}\Big)|\mathcal{F}^{-1}\eta(s)|\,ds\\ &\lesssim T\wedge R^{-1}.\end{aligned}$$

Finally, the proof of (2-9) follows directly from the definition of $1_{T,R}^{\text{low}}$ and Young's inequality. □

Lemma 2.5. *Let $u\in L^2(\mathbb{R}^2)$. Then, for any $T>0$, $R>0$ and $L\gg R$,*

$$\|Q_L(1_{T,R}^{\text{low}}u)\|_{L^2}\lesssim\|Q_{\sim L}u\|_{L^2}.$$

Proof. By Plancherel we get

$$\begin{aligned}I_L &= \|Q_L(1_{T,R}^{\text{low}}u)\|_{L^2}\\ &= \|\varphi_L(\tau-\omega(\xi))\widehat{1_{T,R}^{\text{low}}}*_\tau\hat{u}(\tau,\xi)\|_{L^2}\\ &= \left\|\sum_{L_1\ge 1}\varphi_L(\tau-\omega(\xi))\int_{\mathbb{R}}\varphi_{L_1}(\tau'-\omega(\xi))\hat{u}(\tau',\xi)\eta\Big(\frac{\tau-\tau'}{R}\Big)\frac{e^{-iT(\tau-\tau')}-1}{\tau-\tau'}\,d\tau'\right\|_{L^2}.\end{aligned}$$

In the region where $L_1\ll L$ or $L_1\gg L$, we have $|\tau-\tau'|\sim L\vee L_1\gg R$, thus I_L vanishes. On the other hand, for $L\sim L_1$, we get

$$I_L\lesssim\sum_{L\sim L_1}\|Q_L(1_{T,R}^{\text{low}}Q_{L_1}u)\|_{L^2}\lesssim\|Q_{\sim L}u\|_{L^2}.$$ □

3. Unconditional well-posedness in the regular case $s>\frac{1}{2}$

In this section we develop our method in the regular case $s>\frac{1}{2}$. This will emphasize the simplicity of this approach to prove unconditional well-posedness for (1-3) posed on $\mathbb{R}$ or $\mathbb{T}$.

Let $\lambda>0$ and $L_{\alpha+1}^\lambda$ be the Fourier multiplier by $i\lambda^{\alpha+1}p_{\alpha+1}(\lambda^{-1}\cdot)$. We notice that if u is a solution to (1-3) on $]0,T[$ then $u_\lambda(t,x)=\lambda^\alpha u(\lambda^{\alpha+1}t,\lambda x)$ is a solution to (1-3) on $]0,\lambda^{-(\alpha+1)}T[$ with $L_{\alpha+1}$ replaced by $L_{\alpha+1}^\lambda$. Therefore, up to this change of unknown and equation, we can always assume that the operator $L_{\alpha+1}$ verifies (1-6) with $0<\lambda\le 1$.

3A. *A priori estimates.* For $s\in\mathbb{R}$ we define the function space M^s as $M^s:=L_t^\infty H^s\cap X^{s-1,1}$, endowed with the natural norm

$$\|u\|_{M^s}=\|u\|_{L_t^\infty H^s}+\|u\|_{X^{s-1,1}}.$$

For $u_0\in H^s(\mathbb{R})$, $s>\frac{1}{2}$, we will construct a solution to (1-3) in M_T^s, whereas the difference of two solutions emanating from initial data belonging to $H^s(\mathbb{R})$ will take place in M_T^{s-1}.

Lemma 3.1. *Let* $0 < T < 2$, $s > \frac{1}{2}$ *and let* $u \in L^\infty_T H^s$ *be a solution to* (1-3) *associated with an initial datum* $u_0 \in H^s(\mathbb{R})$. *Then* $u \in M^s_T$ *and*

$$\|u\|_{M^s_T} \lesssim \|u\|_{L^\infty_T H^s} + \|u\|_{L^\infty_T H^s} \|u\|_{L^\infty_T H^{\frac{1}{2}+}}. \tag{3-1}$$

Moreover, for any pair $(u, v) \in (L^\infty_T H^s)^2$ *of solutions to* (1-3) *associated with a pair of initial data* $(u_0, v_0) \in (H^s(\mathbb{R}))^2$ *and any* $s - 1 \le r \le s$,

$$\|u - v\|_{M^r_T} \lesssim \|u - v\|_{L^\infty_T H^r} + \|u + v\|_{L^\infty_T H^s} \|u - v\|_{L^\infty_T H^r}. \tag{3-2}$$

Proof. We have to extend the function u from $(0, T)$ to $\mathbb{R}$. For this we follow [Masmoudi and Nakanishi 2005] and introduce the extension operator ρ_T defined by

$$\rho_T u(t) := \eta(t) u\Big(T\mu\Big(\frac{t}{T}\Big)\Big), \tag{3-3}$$

where η is the smooth cut-off function defined in Section 2A and $\mu(t) = \max(1 - |t - 1|, 0)$. This ρ_T is a bounded operator from $X^{\theta,b}_T$ into $X^{\theta,b}$ and from $L^p(0, T; X)$ into $L^p(\mathbb{R}; X)$ for any $b \in]-\infty, 1]$, $s \in \mathbb{R}$, $p \in [1, \infty]$ and any Banach space X. Moreover, these bounds are uniform for $0 < T < 1$.

By using this extension operator, it is clear that we only have to estimate the $X^{s-1,1}_T$-norm of u to prove (3-1). As noticed in Remark 1.3, u satisfies the Duhamel formula of (1-3) and $u \in C([0, T]; H^\theta)$ for any $\theta < s$. Hence, standard linear estimates in Bourgain's spaces lead to

$$\begin{aligned}\|u\|_{X^{s-1,1}_T} \lesssim \|u_0\|_{H^{s-1}} + \|\partial_x(u^2)\|_{X^{s-1,0}_T} &\lesssim \|u_0\|_{H^{s-1}} + \|u^2\|_{L^2_T H^s} \\ &\lesssim \|u\|_{L^\infty_T H^{s-1}} + \|u\|_{L^\infty_T H^{\frac{1}{2}+}} \|u\|_{L^\infty_T H^s}\end{aligned}$$

by standard product estimates in Sobolev spaces (see [Adams 1975]).

In the same way, for $s - 1 \le r \le s$ we have

$$\|u-v\|_{X^{r-1,1}_T} \lesssim \|u_0-v_0\|_{H^{r-1}} + \|(u+v)(u-v)\|_{L^2_T H^r} \lesssim \|u-v\|_{L^\infty_T H^{r-1}} + \|u+v\|_{L^\infty_T H^s} \|u-v\|_{L^\infty_T H^r},$$

since $s > \frac{1}{2}+$ and $r + s > 0$. This proves (3-2). ☐

Lemma 3.2. *Assume* $u_i \in M^0$, $i = 1, 2, 3$, *are functions with spatial Fourier support in* $\{|\xi| \sim N_i\}$ *with* $N_i > 0$ *dyadic satisfying* $N_1 \le N_2 \le N_3$. *For any* $t > 0$, *we set*

$$I_t(u_1, u_2, u_3) = \int_0^t \int_{\mathbb{R}} \Pi(u_1, u_2) u_3.$$

If $N_1 \lesssim 1$,

$$|I_t(u_1, u_2, u_3)| \lesssim N_1^{\frac{1}{2}} \|u_1\|_{L^\infty_t L^2_x} \|u_2\|_{L^2_{tx}} \|u_3\|_{L^2_{tx}}. \tag{3-4}$$

In the case $N_1 \gg 1$,

$$\begin{aligned}|I_t(u_1, u_2, u_3)| \lesssim{}& N_1^{-\frac{1}{2}} N_3^{1-\alpha} \|u_1\|_{L^\infty_t L^2_x} (\|u_2\|_{L^2_{tx}} \|u_3\|_{X^{-1,1}} + \|u_2\|_{X^{-1,1}} \|u_3\|_{L^2_{tx}}) \\ &+ N_1^{\frac{1}{2}} N_3^{-\alpha} \|u_1\|_{X^{-1,1}} \|u_2\|_{L^2_{tx}} \|u_3\|_{L^\infty_t L^2_x} + N_1^{-1} N_3^{-\frac{1}{8}} \|u_1\|_{L^\infty_t L^2_x} \|u_2\|_{L^\infty_t L^2_x} \|u_3\|_{L^\infty_t L^2_x}.\end{aligned}$$

Proof. Estimate (3-4) easily follows from (2-2) together with Hölder's inequality, thus it suffices to estimate $|I_t|$ for $N_1 \gg 1$. Note that I_t vanishes unless $N_2 \sim N_3$. Setting $R = N_1^{3/2} N_3^{1/8}$, we split I_t as

$$I_t(u_1, u_2, u_3) = I_\infty(1_{t,R}^{\text{high}} u_1, u_2, u_3) + I_\infty(1_{t,R}^{\text{low}} u_1, u_2, u_3) := I_t^{\text{high}} + I_t^{\text{low}}, \tag{3-5}$$

where $I_\infty(u, v, w) = \int_{\mathbb{R}^2} \Pi(u, v) w$. The contribution of I_t^{high} is estimated, thanks to Lemma 2.4 as well as (2-2) and Hölder's inequality, by

$$I_t^{\text{high}} \lesssim N_1^{\frac{1}{2}} \|1_{t,R}^{\text{high}}\|_{L^1} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|u_3\|_{L_t^\infty L_x^2} \lesssim N_1^{-1} N_3^{-\frac{1}{8}} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|u_3\|_{L_t^\infty L_x^2}. \tag{3-6}$$

To evaluate the contribution I_t^{low} we use that, according to Lemma 2.2, we have the decomposition

$$\begin{aligned} I_\infty(1_{t,R}^{\text{low}} u_1, u_2, u_3) = I_\infty&(Q_{\gtrsim N_1 N_3^\alpha}(1_{t,R}^{\text{low}} u_1), u_2, u_3) \\ &+ I_\infty(Q_{\ll N_1 N_2^\alpha}(1_{t,R}^{\text{low}} u_1), Q_{\gtrsim N_1 N_3^\alpha} u_2, u_3) \\ &\quad + I_\infty(Q_{\ll N_1 N_2^\alpha}(1_{t,R}^{\text{low}} u_1), Q_{\ll N_1 N_3^\alpha} u_2, Q_{\sim N_1 N_3^\alpha} u_3). \end{aligned} \tag{3-7}$$

It is worth noting that $R \ll N_1 N_3^\alpha$ because $N_1 \gg 1$. Therefore, the contribution $I_t^{1,\text{low}}$ of the first term of the above right-hand side to I_t^{low} is easily estimated, thanks to Lemma 2.5, by

$$I_t^{1,\text{low}} \lesssim N_1^{\frac{1}{2}} (N_1 N_3^\alpha)^{-1} \|u_1\|_{X^{0,1}} \|u_2\|_{L_{tx}^2} \|u_3\|_{L_t^\infty L_x^2} \lesssim N_1^{\frac{1}{2}} N_3^{-\alpha} \|u_1\|_{X^{-1,1}} \|u_2\|_{L_{tx}^2} \|u_3\|_{L_t^\infty L_x^2}. \tag{3-8}$$

Thanks to Lemmas 2.3 and 2.5, the contribution $I_t^{2,\text{low}}$ of the second term can be handled via

$$\begin{aligned} I_t^{2,\text{low}} &\lesssim N_1^{\frac{1}{2}} (N_1 N_3^\alpha)^{-1} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{X^{0,1}} \|u_3\|_{L_{tx}^2} \\ &\lesssim N_1^{-\frac{1}{2}} N_3^{1-\alpha} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{X^{-1,1}} \|u_3\|_{L_{tx}^2}. \end{aligned} \tag{3-9}$$

Finally, the contribution of the third term is estimated in the same way. □

Remark 3.3. From (2-1) we see that the estimates in Lemma 3.2 also hold for any other rearrangements of N_1, N_2 and N_3.

We are now in position to derive our "improved" energy estimate on smooth solutions to (1-3).

Proposition 3.4. *Let $0 < T < 2$ and let $u \in L_T^\infty H^s$ with $s > \frac{1}{2}$ be a solution to (1-3) associated with an initial datum $u_0 \in H^s(\mathbb{R})$. Then*

$$\|u\|_{L_T^\infty H^s}^2 \lesssim \|u_0\|_{H^s}^2 + (1 + \|u\|_{L_T^\infty H^{\frac{1}{2}+}}^2) \|u\|_{L_T^\infty H^{\frac{1}{2}+}} \|u\|_{L_T^\infty H^s}^2. \tag{3-10}$$

Proof. We apply the operator P_N with $N > 0$ dyadic to (1-3). On account of Remark 1.3, it is clear that $P_N u \in C([0, T]; H^\infty)$ with $\partial_t u_N \in L^\infty(0, T; H^\infty)$. Therefore, taking the L_x^2-scalar product of the resulting equation with $P_N u$, multiplying by $\langle N \rangle^{2s}$ and integrating on $]0, t[$ with $0 < t < T$, we obtain

$$\langle N \rangle^{2s} \|P_N u(t)\|_{L^2}^2 = \langle N \rangle^{2s} \|P_N u_0\|_{L^2}^2 + \langle N \rangle^{2s} \int_0^t \int_{\mathbb{R}} \partial_x P_N(u^2) P_N u.$$

Integrating by parts and applying Bernstein inequalities, this leads to

$$\|P_N u\|^2_{L^\infty_T H^s} \lesssim \|P_N u_0\|^2_{H^s} + \sup_{t\in]0,T[} \langle N\rangle^{2s} \left| \int_0^t \int_{\mathbb{R}} P_N(u^2)\,\partial_x P_N u \right|. \tag{3-11}$$

Thus it remains to estimate

$$I := \sum_{N>0} \langle N\rangle^{2s} \sup_{t\in]0,T[} \left| \int_0^t \int_{\mathbb{R}} P_N(u^2)\,\partial_x P_N u \right|. \tag{3-12}$$

According to (3-1), u belongs to M^s_T. We take an extension $\tilde{u}$ of u supported in time in $]-2,2[$ such that $\|\tilde{u}\|_{M^s} \lesssim \|u\|_{M^s_T}$. To simplify the notation we drop the tilde in the sequel.

By localization considerations, we get

$$P_N(u^2) = P_N(u_{\gtrsim N} u_{\gtrsim N}) + 2P_N(u_{\ll N} u). \tag{3-13}$$

Moreover, using a Taylor expansion of ϕ_N, we easily get

$$P_N(u_{\ll N} u) = u_{\ll N} P_N u + N^{-1}\Pi(\partial_x u_{\ll N}, u), \tag{3-14}$$

where $\Pi = \Pi_\chi$ with $\chi(\xi,\xi_1) = -i\int_0^1 \phi'(N^{-1}(\xi-\theta\xi_1))\,d\theta \in L^\infty$. Inserting (3-13)–(3-14) into (3-12) and integrating by parts, we deduce

$$\begin{aligned} I \lesssim{}& \sum_{N>0}\sum_{0<N_1\ll N} N_1\langle N\rangle^{2s} \sup_{t\in]0,T[} \left| \int_0^t\int_{\mathbb{R}} \Pi_{\chi_1}(u_{N_1}, u_N) u_N \right| \\ &+ \sum_{N>0}\sum_{0<N_1\ll N} N_1\langle N\rangle^{2s} \sup_{t\in]0,T[} \left| \int_0^t\int_{\mathbb{R}} \Pi_{\chi_2}(u_{N_1}, u_{\sim N}) u_N \right| \\ &+ \sum_{N>0}\sum_{N_1\gtrsim N} N\langle N\rangle^{2s} \sup_{t\in]0,T[} \left| \int_0^t\int_{\mathbb{R}} \Pi_{\chi_3}(u_{N_1}, u_{\sim N_1}) u_N \right|, \end{aligned}$$

where χ_i, $1\le i\le 3$, are bounded uniformly in N and N_1, and defined by

$$\chi_1(\xi,\xi_1) = \frac{\xi_1}{N_1} 1_{\operatorname{supp}\phi_{N_1}}(\xi_1), \tag{3-15}$$

$$\chi_2(\xi,\xi_1) = \chi(\xi,\xi_1)\frac{\xi_1}{N_1}\frac{\xi}{N}\frac{1_{\operatorname{supp}\phi_N}(\xi)1_{\operatorname{supp}\phi_{N_1}}(\xi_1)}{\phi_{\sim N}(\xi-\xi_1)}, \tag{3-16}$$

$$\chi_3(\xi,\xi_1) = \frac{\xi}{N}\phi_N(\xi). \tag{3-17}$$

Recalling now the definition of I_t (see Lemma 3.2), it follows from (2-1) that

$$I \lesssim \sum_{N>0}\sum_{N_1\gtrsim N} N\langle N_1\rangle^{2s} \sup_{t\in]0,T[} |I_t(u_N, u_{\sim N_1}, u_{N_1})|. \tag{3-18}$$

The contribution of the sum over $N\lesssim 1$ is easily estimated, thanks to (3-4) and Cauchy–Schwarz, by

$$\sum_{N\le 2^9}\sum_{N_1\gtrsim N} N\langle N_1\rangle^{2s} \|u_N\|_{L^\infty_t L^2_x} \|u_{N_1}\|^2_{L^2_t L^2_x} \lesssim \|u\|_{L^\infty_t L^2_x} \|u\|^2_{L^\infty_t H^s}. \tag{3-19}$$

Finally, the contribution of the sum over $N \gg 1$ is controlled with the second part of Lemma 3.2 by

$$\sum_{N>2^9}\sum_{N_1\gtrsim N} NN_1^{2s}\big[N^{-\frac{1}{2}}N_1^{1-\alpha}\|u_N\|_{L_t^\infty L_x^2}\|u_{N_1}\|_{L_{tx}^2}\|u_{N_1}\|_{X^{-1,1}} + N^{\frac{1}{2}}N_1^{-\alpha}\|u_N\|_{X^{-1,1}}\|u_{N_1}\|^2_{L_t^\infty L_x^2} + N^{-1}N_1^{-\frac{1}{8}}\|u_N\|_{L_t^\infty L_x^2}\|u_{N_1}\|^2_{L_t^\infty L_x^2}\big]$$
$$\lesssim \|u\|_{M_T^{\frac{1}{2}+}}\|u\|_{M_T^s}\|u\|_{L_T^\infty H^s}. \tag{3-20}$$

Gathering all the above estimates leads to

$$\|u\|^2_{L_T^\infty H^s} \lesssim \|u_0\|^2_{H^s} + \|u\|_{M_T^{\frac{1}{2}+}}\|u\|_{M_T^s}\|u\|_{L_T^\infty H^s}, \tag{3-21}$$

which, together with (3-1), completes the proof of the proposition. □

Let us now establish an a priori estimate at the regularity level $s-1$ on the difference of two solutions.

Proposition 3.5. *Let $0 < T < 2$ and let $u, v \in L_T^\infty H^s$ with $s > \frac{1}{2}$ be two solutions to (1-3) associated with initial data $u_0, v_0 \in H^s(\mathbb{R})$, respectively. Then*

$$\|u-v\|^2_{L_T^\infty H^{s-1}} \lesssim \|u_0-v_0\|^2_{H^{s-1}} + \|u+v\|_{M_T^s}\|u-v\|^2_{M_T^{s-1}}. \tag{3-22}$$

Proof. The difference $w = u - v$ satisfies

$$w_t + D^\alpha w_x = \partial_x(zw), \tag{3-23}$$

where $z = u + v$. Proceeding as in the proof of Proposition 3.4, we infer that, for $N > 0$,

$$\|P_N w\|^2_{L_T^\infty H^{s-1}} \lesssim \|P_N w_0\|^2_{H^{s-1}} + \sup_{t\in]0,T[}\langle N\rangle^{2(s-1)}\left|\int_0^t\int_{\mathbb{R}} P_N(zw)\,\partial_x P_N w\right| \tag{3-24}$$

Again, according to (3-1), we can take extensions $\tilde{z}$ and $\widetilde{w}$ of z and w supported in time in $]-2,2[$ such that $\|\tilde{z}\|_{M^s} \lesssim \|z\|_{M_T^s}$ and $\|\widetilde{w}\|_{M^{s-1}} \lesssim \|w\|_{M_T^{s-1}}$. To simplify the notation we drop the tilde in the sequel.

Setting

$$J := \sum_{N>0}\langle N\rangle^{2(s-1)}\sup_{t\in]0,T[}\left|\int_0^t\int_{\mathbb{R}} P_N(zw)\,\partial_x P_N w\right|, \tag{3-25}$$

it follows from (3-14) and classical dyadic decomposition that, for all $N > 0$,

$$P_N(zw) = P_N(z_{\ll N}w) + P_N(z_{\sim N}w_{\lesssim N}) + \sum_{N_1\gg N} P_N(z_{N_1}w_{\sim N_1})$$
$$= z_{\ll N}w_N + N^{-1}\Pi_\chi(\partial_x z_{\ll N}, w) + P_N(z_{\sim N}w_{\lesssim N}) + \sum_{N_1\gg N} P_N(z_{N_1}w_{\sim N_1}). \tag{3-26}$$

Inserting this into (3-25) and integrating by parts, we infer

$$J \lesssim \sum_{N>0}\sum_{N_1 \ll N} N_1 \langle N\rangle^{2(s-1)} \Big(\sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_1}(z_{N_1}, w_N) w_N \Big| + \sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_2}(z_{N_1}, w_{\sim N}) w_N \Big| \Big)$$
$$+ \sum_{N>0}\sum_{N_1 \lesssim N} N \langle N\rangle^{2(s-1)} \sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_3}(z_{\sim N}, w_{N_1}) w_N \Big|$$
$$+ \sum_{N>0}\sum_{N_1 \gg N} N \langle N\rangle^{2(s-1)} \sup_{t\in]0,T[} \Big| \int_0^t \int_{\mathbb{R}} \Pi_{\chi_3}(z_{N_1}, w_{\sim N_1}) w_N \Big|,$$

where χ_i, $1 \le i \le 3$, are as defined in (3-15)–(3-17). Therefore, it suffices to estimate

$$\begin{aligned} J &\lesssim \sum_{N>0}\sum_{N_1 \gtrsim N} N \langle N_1\rangle^{2(s-1)} \sup_{t\in]0,T[} |I_t(z_N, w_{\sim N_1}, w_{N_1})| \\ &\quad + \sum_{N>0}\sum_{N_1 \gtrsim N} N_1 \langle N_1\rangle^{2(s-1)} \sup_{t\in]0,T[} |I_t(z_{\sim N_1}, w_N, w_{N_1})| \\ &\quad + \sum_{N>0}\sum_{N_1 \gtrsim N} N \langle N\rangle^{2(s-1)} \sup_{t\in]0,T[} |I_t(z_{N_1}, w_{N_1}, w_N)| \\ &:= J_1 + J_2 + J_3. \end{aligned} \tag{3-27}$$

The contribution of the sum over $N \lesssim 1$ in (3-27) is easily estimated, thanks to (3-4), by

$$\begin{aligned} \sum_{N\lesssim 1}\sum_{N_1 \gtrsim N} N^{\frac12} \big(N \|z_N\|_{L^\infty_t L^2_x} \|w_{N_1}\|^2_{L^2_t H^{s-1}} + N_1 \langle N_1\rangle^{-1} \|z_{N_1}\|_{L^2_t H^s} \|w_N\|_{L^\infty_t L^2_x} \|w_{N_1}\|_{L^2_t H^{s-1}} \\ + N \langle N_1\rangle^{1-2s} \|z_{N_1}\|_{L^2_t H^s} \|w_{N_1}\|_{L^2_t H^{s-1}} \|w_N\|_{L^\infty_t L^2_x} \big) \\ \lesssim \|z\|_{L^\infty_t L^2_x} \|w\|^2_{L^\infty_t H^{s-1}} + \|w\|_{L^\infty_t H_x^{-\frac12}} \|z\|_{L^\infty_t H^s} \|w\|_{L^\infty_t H^{s-1}}. \end{aligned} \tag{3-28}$$

For the contribution of the sum over $N \gg 1$, it is worth noting that, since $s > \frac12$, the term J_3 is controlled by J_2. The contribution of J_1 is estimated, thanks to Lemma 3.2, by

$$\begin{aligned} \sum_{N\gg 1}\sum_{N_1 \gtrsim N} N N_1^{2(s-1)} \big[N^{-\frac12} N_1^{1-\alpha} \|z_N\|_{L^\infty_t L^2_x} \|w_{N_1}\|_{L^2_{tx}} \|w_{N_1}\|_{X^{-1,1}} \\ + N^{\frac12} N_1^{-\alpha} \|z_N\|_{X^{-1,1}} \|w_{N_1}\|^2_{L^\infty_t L^2_x} + N^{-1} N_1^{-\frac18} \|z_N\|_{L^\infty_t L^2_x} \|w_{N_1}\|^2_{L^\infty_t L^2_x} \big] \\ \lesssim \|z\|_{M^{\frac12+}} \|w\|_{M^{s-1}} \|w\|_{L^\infty_t H^{s-1}}. \end{aligned} \tag{3-29}$$

Finally, in the same way we bound J_2 by

$$\begin{aligned} \sum_{N\gg 1}\sum_{N_1 \gtrsim N} N_1^{2s-1} \big[N^{-\frac12} N_1^{1-\alpha} \|w_N\|_{L^\infty_t L^2_x} (\|z_{N_1}\|_{L^2_{tx}} \|w_{N_1}\|_{X^{-1,1}} + \|z_{N_1}\|_{X^{-1,1}} \|w_{N_1}\|_{L^2_{tx}}) \\ + N^{\frac12} N_1^{-\alpha} \|w_N\|_{X^{-1,1}} \|z_{N_1}\|_{L^\infty_t L^2_x} \|w_{N_1}\|_{L^\infty_t L^2_x} \\ + N^{-1} N_1^{-\frac18} \|w_N\|_{L^\infty_t L^2_x} \|z_{N_1}\|_{L^\infty_t L^2_x} \|w_{N_1}\|_{L^\infty_t L^2_x} \big] \\ \lesssim \|z\|_{M^s} \|w\|_{M^{-\frac12+}} \|w\|_{L^\infty_t H^{s-1}} + \|z\|_{M^s} \|w\|_{M^{s-1}} \|w\|_{L^\infty_t H^{-\frac12+}}. \end{aligned} \tag{3-30}$$

Gathering the estimates (3-27)–(3-30), we obtain

$$J \lesssim (\|z\|_{M_T^{\frac{1}{2}+}}\|w\|_{M^{s-1}} + \|z\|_{M_T^s}\|w\|_{M^{-\frac{1}{2}+}})\|w\|_{L_T^\infty H^{s-1}} + \|z\|_{M_T^s}\|w\|_{M^{s-1}}\|w\|_{L_T^\infty H^{-\frac{1}{2}+}}, \quad (3\text{-}31)$$

which leads to (3-22) and completes the proof of the proposition. □

3B. ***Unconditional well-posedness.*** Fix $s > \frac{1}{2}$. First, it is worth noticing that we can always assume that we deal with data that have small H^s-norm. Indeed, if $u \in L^\infty(0, T; H^s)$ is a solution to (1-3), then, for $0 < \lambda \le 1$, $u_\lambda := \lambda^\alpha u(\lambda^{\alpha+1}\cdot, \lambda\cdot) \in L^\infty(0, \lambda^{\alpha+1}T; H^s)$ is a solution to (1-3) with $L_{\alpha+1}$ replaced by $L^\lambda_{\alpha+1}$, that is, the Fourier multiplier by $i\lambda^{\alpha+1}p_{\alpha+1}(\lambda^{-1}\cdot)$. Recall that we assumed at the beginning of this section that $L^\lambda_{\alpha+1}$ satisfies (1-6) for any $0 < \lambda \le 1$. For $0 < \varepsilon \ll 1$, let us denote by $\mathcal{B}^s(\varepsilon)$ the ball of $H^s(\mathbb{R})$ centered at the origin with radius ε. Since

$$\|u_\lambda(0)\|_{H^s} \lesssim \lambda^{\alpha-\frac{1}{2}}\|u_0\|_{H^s},$$

we see that we can force $u_{0,\lambda}$ to belong to $\mathcal{B}^s(\varepsilon)$ by choosing $\lambda = [\varepsilon(1 + \|u_0\|_{H^s})]^{-1/(\alpha-1/2)}$. Therefore, the unconditional well-posedness in $H^s(\mathbb{R})$ of (1-3) for small H^s-initial data with a time of existence $T \ge 1$ will ensure the unconditional well-posedness of (1-3) for arbitrary large H^s-initial data with a maximal time of existence

$$T \gtrsim (1 + \|u_0\|_{H^s})^{-\frac{2(\alpha+1)}{2\alpha-1}}.$$

Existence and unconditional uniqueness. It is well known (see for instance [Abdelouhab et al. 1989]) that (1-3) is locally well-posed in H^s for $s > \frac{3}{2}$ with a minimal time of existence $T = T(\|u_0\|_{H^{3/2+}}) > 0$. So, let $u \in C([0, T_0]; H^\infty(\mathbb{R}))$ be a smooth solution to (1-3) emanating from a smooth initial datum $u_0 \in H^\infty(\mathbb{R})$ with $\|u_0\|_{H^s} \ll 1$. According to (3-10),

$$\|u\|^2_{L_T^\infty H^s} \lesssim \|u(0)\|^2_{H^s} + (1 + \|u\|^2_{L_T^\infty H^{\frac{1}{2}+}})\|u\|_{L_T^\infty H^{\frac{1}{2}+}}\|u\|^2_{L_T^\infty H^s} \quad (3\text{-}32)$$

for any $0 < T \le \min(1, T_0)$ and $s > \frac{1}{2}$. Let us denote by $T^* \ge T_0$ the maximal time of existence of u in $H^\infty(\mathbb{R})$. The well-posedness result in [Abdelouhab et al. 1989] ensures that $\lim_{T\nearrow T^*}\|u\|_{L_T^\infty H^3} = +\infty$ whenever T^* is finite. Since

$$\|u(0)\|_{H^{\frac{1}{2}+}} \le \|u(0)\|_{H^s} \ll 1,$$

(3-32) together with the continuity of $T \mapsto \|u\|_{L_T^\infty H^{1/2+}}$ on $]0, T^*[$ ensure that

$$\|u\|_{L_{T'}^\infty H^{\frac{1}{2}+}} \lesssim \|u(0)\|_{H^{\frac{1}{2}+}} \ll 1$$

with $T' = \min(1, T^*)$. But then (3-32) leads, for any $s > \frac{1}{2}$, to

$$\|u\|_{L_{T'}^\infty H^s} \lesssim \|u(0)\|_{H^s}.$$

This proves that $T' < T^*$ and thus $T' = 1$ and $T^* \ge 1$.

Now, let $u_0 \in H^s(\mathbb{R})$ with $s > \frac{1}{2}$. From the above estimates, we infer that we can pass to the limit on a sequence of solutions $\{u_n\}$ emanating from smooth approximations of u_0 to obtain the existence of a solution $u \in L_T^\infty H^s$ of (1-3) with initial data u_0. Note that one can easily pass to the limit on u_n^2

by compactness arguments, since $\{u_n\}$ and $\{\partial_t u_n\}$ are bounded in $L^\infty_T H^s$ and $L^\infty_T H^{s-3}$, respectively. Estimates (3-22) and (3-1)–(3-2) then ensure that this solution is the only one in this class. Now the continuity of u with values in $H^s(\mathbb{R})$ as well as the continuity of the flow map in $H^s(\mathbb{R})$ will follow from the Bona–Smith argument [1975]. For any $\varphi \in H^s(\mathbb{R})$, dyadic integer $N \geq 1$ and $r \geq 0$, straightforward calculations in Fourier space lead to

$$\|P_{\leq N}\varphi\|_{H^{s+r}_x} \lesssim N^r \|\varphi\|_{H^s_x} \quad \text{and} \quad \|\varphi - P_{\leq N}\varphi\|_{H^{s-r}_x} \lesssim N^{-r}\|P_{>N}\varphi\|_{H^s_x}. \tag{3-33}$$

Let $u_0 \in H^s$ with $s > \frac{1}{2}$ be such that $\|u_0\|_{H^s} \ll 1$. We denote by $u^N \in L^\infty(0,1;H^s)$ the solution of (1-3) emanating from $u_0^N = P_{\leq N}u_0$ and, for $1 \leq N_1 \leq N_2$, we set

$$w := u^{N_1} - u^{N_2}.$$

Then, (3-22) and (3-2) lead to

$$\|w\|_{M_1^{s-1}} \lesssim \|w(0)\|_{H^{s-1}} \lesssim N_1^{-1}\|P_{>N_1}u_0\|_{H^s}. \tag{3-34}$$

Moreover, for any $r \geq 0$ and $s > \frac{1}{2}$, we have

$$\|u^{N_i}\|_{M_1^{s+r}} \lesssim \|u_0^{N_i}\|_{H^{s+r}} \lesssim N_i^r \|u_0\|_{H^s}. \tag{3-35}$$

Next, we observe that w solves the equation

$$w_t + L_{\alpha+1}w = \tfrac{1}{2}\partial_x(w^2) + \partial_x(u^{N_1}w). \tag{3-36}$$

Proposition 3.6. *Let $0 < T < 2$ and let $w \in M^s_T$ with $s > \frac{1}{2}$ be a solution to (3-36). Then*

$$\|w\|^2_{L^\infty_T H^s} \lesssim \|w(0)\|^2_{H^s} + \|w\|^3_{M^s_T} + \|u^{N_1}\|_{M^s_T}\|w\|^2_{M^s_T} + \|u^{N_1}\|_{M^{s+1}_T}\|w\|_{M^s_T}\|w\|_{M^{s-1}_T}. \tag{3-37}$$

Proof. Actually, this is a consequence of estimates derived in the proof of Propositions 3.4 and 3.5. We separate the contributions of $\partial_x(w^2)$ and $\partial_x(u^{N_1}w)$. Let $t \in]0,T[$. First, (3-21) leads to

$$\sum_{N>0} N^{2s}\left|\int_0^t\int_{\mathbb{R}} P_N\partial_x(w^2)P_N w\right| \lesssim \|w\|^3_{M^s_T}.$$

Second, applying (3-31) at the level s with z replaced by u^{N_1}, we obtain

$$\sum_{N>0} N^{2s}\left|\int_0^t\int_{\mathbb{R}} P_N\partial_x(u^{N_1}w)P_N w\right| \lesssim \|u^{N_1}\|_{M^s_T}\|w\|^2_{M^s_T} + \|u^{N_1}\|_{M^{s+1}_T}\|w\|_{M^s_T}\|w\|_{M_T^{-\frac{1}{2}+}},$$

which leads to (3-37) since $s > \frac{1}{2}$. □

Combining (3-2) with (3-37) and (3-35), we get

$$\|w\|^2_{M^s_1} \lesssim (1+\|u_0\|^2_{H^s})\big[\|w_0\|^2_{H^s} + \|u_0\|_{H^s}\|w\|^2_{M^s_1} + \|u_0\|_{H^s}\|w\|^2_{M^s_1} + N_1\|u_0\|_{H^s}\|w\|_{M^s_1}\|w\|_{M^{s-1}_1}\big].$$

Then, the smallness assumption on $\|u_0\|_{H^s}$ and (3-34) lead to

$$\|w\|^2_{M^s_1} \lesssim \|w_0\|^2_{H^s} + N_1^2\|w\|^2_{M^{s-1}_1} \lesssim \|P_{>N_1}u_0\|^2_{H^s}(1+\|P_{>N_1}u_0\|^2_{H^s}) \to 0 \quad \text{as } N_1 \to 0. \tag{3-38}$$

This shows that $\{u^N\}$ is a Cauchy sequence in $C([0,1];H^s)$ and thus $\{u^N\}$ converges in $C([0,1];H^s)$ to a solution of (1-3) emanating from u_0. Then, the uniqueness result ensures that $u\in C([0,1];H^s)$.

Continuity of the flow map. Now let $\{u_{0,n}\}\subset H^s(\mathbb{R})$ be such that $u_{0,n}\to u_0$ in $H^s(\mathbb{R})$. We want to prove that the emanating solution u_n tends to u in $C([0,1];H^s)$. By the triangle inequality, for n large enough,

$$\|u-u_n\|_{L_1^\infty H^s}\le \|u-u^N\|_{L_1^\infty H^s}+\|u^N-u_n^N\|_{L_1^\infty H^s}+\|u_n^N-u_n\|_{L_1^\infty H^s}.$$

Using the estimate (3-38) on the solution to (3-36) we first infer that

$$\|u-u^N\|_{M_1^s}+\|u_n-u_n^N\|_{M_1^s}\lesssim \|P_{>N}u_0\|_{H^s}+\|P_{>N}u_{0,n}\|_{H^s}$$

and thus

$$\lim_{N\to\infty}\sup_{n\in\mathbb{N}}(\|u-u^N\|_{L_1^\infty H^s}+\|u_n-u_n^N\|_{L_1^\infty H^s})=0. \tag{3-39}$$

Next, we notice that (3-22) and (3-2) ensure that

$$\|u^N-u_n^N\|_{M_1^{s-1}}\lesssim \|u_0^N-u_{0,n}^N\|_{H^{s-1}},$$

and thus (3-38) and (3-34) lead to

$$\|u^N-u_n^N\|_{M_1^s}^2\lesssim \|u_0^N-u_{0,n}^N\|_{H^s}^2+N^2\|u_0^N-u_{0,n}^N\|_{H^{s-1}}^2\lesssim \|u_0-u_{0,n}\|_{H^s}^2(1+N^2). \tag{3-40}$$

Combining (3-39) and (3-40), we obtain the continuity of the flow map. The proof of Theorem 1.5 is thus completed in the case $\mathbb{K}=\mathbb{R}$ and $s>\frac{1}{2}$.

3C. *The periodic case.* In this subsection we explain the necessary adaptations to treat the periodic case. First, we define our function spaces in the periodic setting. Since the map $u\mapsto u_\lambda$ maps $L^\infty(0,T;H^s(\mathbb{T}))$ into $L^\infty(0,\lambda^{\alpha+1}T;H^s(\lambda\mathbb{T}))$, we will have to consider space of functions on the tori $\lambda\mathbb{T}$ with $\lambda\ge 1$. We use the same notations as in [Colliander et al. 2004] to deal with Fourier transform of space-periodic functions with a large period $2\pi\lambda$. Then, $(d\xi)_\lambda$ will be the renormalized counting measure on $\lambda^{-1}\mathbb{Z}$:

$$\int a(\xi)\,(d\xi)_\lambda=\frac{1}{\lambda}\sum_{\xi\in\lambda^{-1}\mathbb{Z}}a(\xi) \quad .$$

As noticed in [Colliander et al. 2004], $(d\xi)_\lambda$ is the counting measure on the integers when $\lambda=1$ and converges weakly to the Lebesgue measure when $\lambda\to\infty$. In the definitions below, all the Lebesgue norms in ξ will be with respect to the measure $(d\xi)_\lambda$. For a $2\pi\lambda$-periodic function φ, we define its space Fourier transform on $\lambda^{-1}\mathbb{Z}$ by

$$\hat{\varphi}(\xi)=\int_{\lambda\mathbb{T}}e^{-i\xi x}f(x)\,dx \quad \text{for all } \xi\in\lambda^{-1}\mathbb{Z}.$$

The Lebesgue spaces $L^q(\lambda\mathbb{T})$, $1\le q\le\infty$, for $2\pi\lambda$-periodic functions, will be defined as usual by

$$\|\varphi\|_{L^q}=\left(\int_{\lambda\mathbb{T}}|\varphi(x)|^q\,dx\right)^{\frac{1}{q}}$$

with the obvious modification for $q=\infty$.

The Sobolev spaces $H^s(\lambda\mathbb{T})$ for $2\pi\lambda$-periodic functions are endowed with the norm

$$\|\varphi\|_{H^s} = \|\langle\xi\rangle^s\hat{\varphi}(\xi)\|_{L^2_\xi} = \|J^s_x\varphi\|_{L^2},$$

where $\langle\cdot\rangle = (1+|\cdot|^2)^{1/2}$ and $\widehat{J^s_x\varphi}(\xi) = \langle\xi\rangle^s\hat{\varphi}(\xi)$.

In the same way, for a function $u(t,x)$ on $\mathbb{R}\times\lambda\mathbb{T}$, we define its space-time Fourier transform by

$$\hat{u}(\tau,\xi) = \mathscr{F}_{t,x}(u)(\tau,\xi) = \int_{\mathbb{R}}\int_{\lambda\mathbb{T}} e^{-i(\tau t+\xi x)}u(t,x)\,dx\,dt \quad \text{for all } (\tau,\xi)\in\mathbb{R}\times\lambda^{-1}\mathbb{Z}.$$

For any $(s,b)\in\mathbb{R}^2$, we define the Bourgain space $X^{s,b}$ of $2\pi\lambda$-periodic (in x) functions as the completion of $\mathscr{S}(\lambda\mathbb{T}\times\mathbb{R})$ for the norm

$$\|u\|_{X^{s,b}} = \|\langle\tau - p_{\alpha+1}(\xi)\rangle^b\langle\xi\rangle^s\hat{u}\|_{L^2_{\tau,\xi}}.$$

Finally, we define the functions ϕ_N and ψ_L and the Fourier multipliers P_N and Q_L as in Section 2A. Since we take a homogeneous decomposition in space frequencies, in the periodic setting

$$u = P_0u + \sum_{N>0} P_N u, \tag{3-41}$$

where $\widehat{P_0u}(\xi) = \hat{u}(0)$.

Now, with these definitions, we claim that Lemma 3.1 and Propositions 3.4, 3.5 and 3.6 also hold for $2\pi\lambda$-periodic functions with an implicit constant that does not depend on $\lambda\geq 1$. Indeed, all the tools (the Sobolev and Hölder inequalities) we used in the proofs of these results work also in the periodic setting, independently of the period. However, in view of (3-41), we have to care about the contribution of the null-space frequencies, since we take an homogeneous decomposition. First, since the nonlinear term is a pure derivative, it is clear that the contribution of the null frequency of the nonlinear term vanishes in all the estimates. Now, it is also direct to check that

$$\int_{\lambda\mathbb{T}} P_N(uP_0u)\,\partial_x P_N u = 0 \tag{3-42}$$

and, in the same way,

$$\int_{\lambda\mathbb{T}} P_N(wP_0z)\,\partial_x P_N w = 0. \tag{3-43}$$

We thus just have to control the contribution of the terms $P_N(zP_0w)$ in Proposition 3.5 and $P_N(u^{N_1}P_0w)$ in Proposition 3.6. But the contribution of the first term in Proposition 3.5 can be easily estimated by

$$\begin{aligned} N^{2(s-1)}\left|\int_0^t\int_{\lambda\mathbb{T}} P_N(zP_0w)\,\partial_x P_N w\right| &\lesssim \sup_{t'\in]0,T[}|\hat{w}(t',0)|N^{2(s-1)}N\|P_N z\|_{L^2_T L^2}\|P_N w\|_{L^2_T L^2} \\ &\lesssim \delta_N\|z\|_{L^\infty_T H^s}\|w\|^2_{L^\infty_T H^{s-1}}, \end{aligned}$$

where $\|(\delta_{2^j})_{j\in\mathbb{Z}}\|_{l^1(\mathbb{Z})}\lesssim 1$. Finally, the contribution of the second term in Proposition 3.6 can be estimated in exactly the same way by

$$N^{2s}\left|\int_0^t\int_{\lambda\mathbb{T}} P_N(u^{N_1}P_0w)\,\partial_x P_N w\right| \lesssim \delta_N\|u^{N_1}\|_{L^\infty_T H^{s+1}}\|w\|_{L^\infty_T H^s}\|w\|_{L^\infty_T H^{s-1}}.$$

This completes the proof of the regular case $s > \frac{1}{2}$ in the periodic setting.

4. Estimates in the nonregular case

In this section, we provide the needed estimates at level $s \geq 1-\frac{\alpha}{2}$ for $1<\alpha\leq 2$. We introduce the space

$$F^{s,b} = F^{s,\alpha,b} = X^{s-\frac{\alpha+1}{2},b+\frac{1}{2}} + X^{s-\frac{1+\alpha}{8},b+\frac{1}{8}}, \tag{4-1}$$

endowed with the usual norm, and we define

$$Y^s = Y^{s,\alpha} = L^\infty_t H^s \cap F^{s,\alpha,\frac{1}{2}} = L^\infty_t H^s \cap (X^{s-\frac{\alpha+1}{2},1} + X^{s-\frac{1+\alpha}{8},\frac{5}{8}}).$$

For $u_0 \in H^s(\mathbb{R})$ we will construct a solution to (1-3) that belongs to Y^s_T for some $T = T(\|u_0\|_{H^{1-\alpha/2}}) > 0$. As in the regular case, by a dilation argument, we may assume that $L_{\alpha+1}$ satisfies (1-6) for $0<\lambda\leq 1$.

Remark 4.1. Except in the case $(s,\alpha) = (0,2)$, we could simply take $Y^{s,\alpha} := L^\infty_t H^s \cap X^{s-(\alpha+1)/2,1}$, since $u \in L^\infty(0,T;H^s)$ forces $\partial_x(u^2) \in L^\infty(0,T;H^{s-(\alpha+1)/2})$. To this point of view, $(s,\alpha)=(0,2)$ is a limit case since $u \in L^\infty(0,T;L^2)$ only implies $\partial_x(u^2) \in L^\infty(0,T;H^{-3/2-})$. As in [Zhou 1997], to overcome this difficulty we have to evaluate our solution in Bourgain's spaces with different conormal regularities.

Lemma 4.2. *Let $0<T<2$, $1<\alpha\leq 2$, $s\geq 1-\frac{\alpha}{2}$ and let $u \in L^\infty_T H^s$ be a solution to (1-3) associated with an initial datum $u_0 \in H^s(\mathbb{R})$. Then u belongs to $Y^{s,\alpha}_T$. Moreover, if $(s,\alpha) \neq (0,2)$,*

$$\|u\|_{Y^{s,\alpha}_T} \lesssim \|u\|_{L^\infty_T H^s}(1+\|u\|_{L^\infty_T H^{1-\frac{\alpha}{2}}}) \tag{4-2}$$

and, if $(s,\alpha)=(0,2)$,

$$\|u\|_{Y^{0,2}_T} \lesssim \|u\|_{L^\infty_T L^2_x}(1+\|u\|^2_{L^\infty_T L^2_x}). \tag{4-3}$$

Proof. As in Lemma 3.1 we will work with the extension $\tilde{u} = \rho_T u$ of u (see (3-3)). Recall that $\operatorname{supp}\tilde{u} \subset [-2,2]\times\mathbb{R}$ and that

$$\|\tilde{u}\|_{L^\infty_t H^s} \lesssim \|u\|_{L^\infty_T H^s} \quad \text{and} \quad \|\tilde{u}\|_{X^{\theta,b}} \lesssim \|u\|_{X^{\theta,b}_T}$$

for any $(\theta,b) \in \mathbb{R}\times]-\infty,1]$. It thus remains to control the $F^{s,\alpha,\frac{1}{2}}_T$-norm of u. In the case $(s,\alpha)\neq(0,2)$, we actually simply control the $X^{s-(\alpha+1)/2,1}_T$-norm of u. Using the integral formulation (see Remark 1.3), standard linear estimates in Bourgain's spaces, and standard product estimates in Sobolev spaces, we infer that

$$\begin{aligned}\|u\|_{X_T^{s-\frac{1+\alpha}{2},1}} \lesssim \|u_0\|_{H^{s-\frac{1+\alpha}{2}}} + \|\partial_x(u^2)\|_{X_T^{s-\frac{1+\alpha}{2},0}} &\lesssim \|u_0\|_{H^{s-\frac{1+\alpha}{2}}} + \|u^2\|_{L^2_T H^{s+\frac{1-\alpha}{2}}} \\ &\lesssim \|u\|_{L^\infty_T H^s} + \|u\|_{L^\infty_T H^{1-\frac{\alpha}{2}}}\|u\|_{L^\infty_T H^s},\end{aligned}$$

since, for $1<\alpha\leq 2$ and $s\geq 1-\frac{\alpha}{2}$ with $(s,\alpha)\neq(0,2)$, we have $s+1-\frac{\alpha}{2}>0$ and $s+1-\frac{\alpha}{2}-(s+\frac{1-\alpha}{2}) = \frac{1}{2}$.

Let us now tackle the case $(s,\alpha)=(0,2)$. First we notice that, since $L^1(\mathbb{R}) \hookrightarrow H^{-1/2-}(\mathbb{R})$, we have

$$\|u\|_{X_T^{-\frac{7}{4},1}} \lesssim \|u_0\|_{H^{-\frac{7}{4}}} + \|u^2\|_{L^2_t H^{-\frac{3}{4}}} \lesssim \|u\|_{L^\infty_T L^2_x}(1+\|u\|_{L^\infty_T L^2_x}). \tag{4-4}$$

To bound the $F^{0,2,1/2}$-norm of u, we first notice that linear estimates in Bourgain's spaces lead to

$$\|u\|_{F_T^{0,2,\frac{1}{2}}} \lesssim \|u_0\|_{H^{-\frac{3}{2}}} + \|u^2\|_{F_T^{0,2,-\frac{1}{2}}}$$

and then decompose u^2 as

$$u^2 = P_{\lesssim 1}u^2 + \sum_{N\gg 1}\Big(P_N(P_{\ll N}uu_{\sim N}) + \sum_{N_1'\sim N_1\gtrsim N} P_N(u_{N_1}u_{N_1'})\Big). \tag{4-5}$$

The contribution of the first term in the right-hand side is easily controlled by $\|u\|^2_{L_T^\infty L_x^2}$. The contribution of the second term is easily estimated by

$$\begin{aligned}\Big\|\sum_{N\gg 1}\partial_x P_N(P_{\ll N}uu_{\sim N})\Big\|_{F_T^{0,2,-\frac{1}{2}}} &\lesssim \Big\|\sum_{N\gg 1}P_N\partial_x(P_{\ll N}uu_{\sim N})\Big\|_{X_T^{-\frac{3}{2},0}}\\ &\lesssim \Big(\sum_{N\gg 1}\|P_N(P_{\ll N}uu_{\sim N})\|^2_{L_T^2L_x^1}\Big)^{\frac{1}{2}}\\ &\lesssim \Big(\sum_{N\geq 1}\|u_N\|^2_{L_T^2L_x^2}\|P_{\ll N}u\|^2_{L_T^\infty L_x^2}\Big)^{\frac{1}{2}}\\ &\lesssim \|u\|_{L_T^\infty L^2}\|u\|_{L_T^\infty L_x^2}.\end{aligned} \tag{4-6}$$

To estimate the third term, we take advantage of the $X^{-3/8,-3/8}$-part of $F^{0,2,-1/2}$. For $N\gg 1$, we have

$$\begin{aligned}\sum_{N_1'\sim N_1\gtrsim N}&\|\partial_x P_N(P_{N_1}uP_{N_1'}u)\|_{F_T^{0,2,-\frac{1}{2}}}\\ &\lesssim \sum_{N_1'\sim N_1\gtrsim N} N\Big\|\sum_{\substack{(L,L_1,L_2)\\ \text{satisfying (2-6)}}}\partial_x P_N Q_L(Q_{L_1}\tilde u_{N_1}Q_{L_2}\tilde u_{N_1'})\Big\|_{X^{-\frac{3}{8},-\frac{3}{8}}}.\end{aligned} \tag{4-7}$$

For the contribution of the sum over $L\gtrsim NN_1^2$ in (4-7), we obtain

$$\begin{aligned}\sum_{N_1\sim N_1'\gtrsim N}\|\partial_x P_N Q_{\gtrsim NN_1^2}(\tilde u_{N_1}\tilde u_{N_1'})\|_{X^{-\frac{3}{8},-\frac{3}{8}}} &\lesssim \sum_{N_1\sim N_1'\gtrsim N} N^{\frac{5}{8}}N^{\frac{1}{2}}(NN_1^2)^{-\frac{3}{8}}\|\tilde u_{N_1}\|_{L^2_{tx}}\|\tilde u_{N_1'}\|_{L_t^\infty L_x^2}\\ &\lesssim \|\tilde u\|_{L_t^\infty L_x^2}\sum_{N_1\gtrsim N}\Big(\frac{N}{N_1}\Big)^{\frac{3}{4}}\|\tilde u_{N_1}\|_{L^2_{tx}}\\ &\lesssim \gamma_N\|\tilde u\|^2_{L_t^\infty L_x^2}\end{aligned} \tag{4-8}$$

with $\|(\gamma_{2^j})\|_{l^2(\mathbb{N})}\leq 1$. The contribution of the region where $L\ll NN_1^2$ and $L_1\gtrsim NN_1^2$ in (4-7) is controlled by

$$\begin{aligned}\sum_{N_1\sim N_1'\gtrsim N}&\|\partial_x P_N Q_{\ll NN_1^2}(Q_{\gtrsim NN_1^2}\tilde u_{N_1}\tilde u_{N_1'})\|_{X^{-\frac{3}{8},-\frac{3}{8}}}\\ &\lesssim \sum_{N_1\sim N_1'\gtrsim N} N^{\frac{5}{8}}N^{\frac{1}{2}}(NN_1^2)^{-1}N_1^{\frac{7}{4}}\|\tilde u_{N_1}\|_{X^{-\frac{7}{4},1}}\|\tilde u_{N_1'}\|_{L_t^\infty L_x^2} \lesssim N^{-\frac{1}{8}}\|\tilde u\|_{L_t^\infty L_x^2}\|\tilde u\|_{X^{-\frac{7}{4},1}}.\end{aligned} \tag{4-9}$$

Finally, the contribution of the last region, where $L, L_1 \ll NN_1^2$ and $L_2 \sim NN_1^2$, in (4-7) is controlled in the same way. Gathering (4-4) and (4-7)–(4-9), we obtain the desired result for the case $(s,\alpha) = (0,2)$. □

In the sequel we will need the following straightforward estimates.

Lemma 4.3. *Let $\alpha \geq 0$ and $w \in F^{0,1/2}$. For $1 \leq B \lesssim N^{\alpha+1}$, we have*

$$\|Q_{\gtrsim B} w_N\|_{L^2} \lesssim B^{-1} N^{\frac{1+\alpha}{2}} \|Q_{\gtrsim B} w_N\|_{F^{0,\frac{1}{2}}} \tag{4-10}$$

and, for $B \gtrsim \langle N\rangle^{\alpha+1}$, we have

$$\|Q_{\gtrsim B} w_N\|_{L^2} \lesssim B^{-\frac{5}{8}} \langle N\rangle^{\frac{1+\alpha}{8}} \|Q_{\gtrsim B} w_N\|_{F^{0,\frac{1}{2}}}. \tag{4-11}$$

Proof. Noticing that $F^{0,1/2} = F^{0,\alpha,1/2} = X^{-(1+\alpha)/2,1} + X^{-(1+\alpha)/8,5/8}$, it is easy to check that

$$\begin{aligned} \|Q_{\gtrsim B} w_N\|_{L^2} &\lesssim \max\big(B^{-1}\langle N\rangle^{\frac{1+\alpha}{2}}, B^{-\frac{5}{8}}\langle N\rangle^{\frac{1+\alpha}{8}}\big) \|Q_{\gtrsim B} w_N\|_{F^{0,\frac{1}{2}}} \\ &\lesssim B^{-\frac{5}{8}}\langle N\rangle^{\frac{1+\alpha}{8}} \max\Big(\Big(\frac{\langle N\rangle^{1+\alpha}}{B}\Big)^{\frac{3}{8}}, 1\Big) \|Q_{\gtrsim B} w_N\|_{F^{0,\frac{1}{2}}}, \end{aligned}$$

which leads to the desired result. □

Now we rewrite Lemma 3.2 in the context of the $F^{s,b}$ spaces.

Lemma 4.4. *Assume $u_i \in Y^0$, $i = 1, 2, 3$, are functions with spatial Fourier support in $\{|\xi| \sim N_i\}$ with $N_i > 0$ dyadic satisfying $N_1 \leq N_2 \leq N_3$.*

If $N_3 \gg 1$ and $N_1 \gtrsim N_3^{2(1-\alpha)/3}$, for $(p,q) \in \{(2,\infty), (\infty,2)\}$,

$$\begin{aligned} |I_t(u_1,u_2,u_3)| \lesssim \sum_{L>1} L^{-1} N_1^{-\frac{1}{2}} N_3^{\frac{1-\alpha}{2}} \|u_1\|_{L_t^p L_x^2} \|Q_{\sim LN_1N_3^\alpha} u_2\|_{F^{0,\frac{1}{2}}} \|u_3\|_{L_t^q L_x^2} \\ + N_1^{-\frac{1}{2}} N_3^{\frac{1-\alpha}{2}} \|u_1\|_{L_t^p L_x^2} \|u_2\|_{L_t^q L_x^2} \|Q_{\sim N_1N_3^\alpha} u_3\|_{F^{0,\frac{1}{2}}} \\ + N_1^{-\frac{1}{8}} \langle N_1\rangle^{\frac{1+\alpha}{8}} N_3^{\frac{-5\alpha}{8}} \|u_1\|_{F^{0,\frac{1}{2}}} \|u_2\|_{L^2_{tx}} \|u_3\|_{L_t^\infty L_x^2} \\ + N_1^{-\frac{1}{4}} N_3^{\frac{1}{8}-\frac{\alpha}{2}} \|u_1\|_{L_t^\infty L_x^2} \|u_2\|_{L_t^\infty L_x^2} \|u_3\|_{L_t^\infty L_x^2}. \end{aligned}$$

Proof. For $R = N_1^{3/4} N_3^{\alpha/2-1/8}$, we decompose I_t as in (3-5) and obtain from (3-6) that

$$|I_t^{\text{high}}| \lesssim N_1^{-\frac{1}{4}} N_3^{\frac{1}{8}-\frac{\alpha}{2}} \prod_{i=1}^3 \|u_i\|_{L_t^\infty L_x^2}.$$

To evaluate I_t^{low} we use the decomposition (3-7) and notice that

$$R = N_1^{\frac{3}{4}} N_3^{\frac{\alpha}{2}-\frac{1}{8}} \leq N_1 N_3^{\frac{2\alpha}{3}-\frac{7}{24}} \ll N_1 N_3^\alpha \quad \text{and} \quad N_1 N_3^\alpha \gtrsim N_3^{\frac{2+\alpha}{3}} \gg 1.$$

Therefore, the contribution $I_t^{1,\text{low}}$ of the first term of the right-hand side of (3-7) to I_t^{low} is easily estimated, thanks to Lemmas 2.5 and 4.3, by

$$|I_t^{1,\text{low}}| \lesssim N_1^{\frac{1}{2}} (N_1 N_3^\alpha)^{-\frac{5}{8}} \langle N_1\rangle^{\frac{\alpha+1}{8}} \|u_1\|_{F^{0,\frac{1}{2}}} \|u_2\|_{L^2_{tx}} \|u_3\|_{L_t^\infty L_x^2},$$

which is acceptable. Thanks to Lemmas 2.3, 2.5 and 4.3, the contribution $I_t^{2,\mathrm{low}}$ of the second term can be handled in the following way:

$$
\begin{aligned}
|I_t^{2,\mathrm{low}}| &\lesssim \sum_{L>1} N_1^{\frac12}(LN_1N_3^\alpha)^{-1}N_3^{\frac{\alpha+1}{2}}\|u_1\|_{L_t^pL_x^2}\|Q_{\sim LN_1N_3^\alpha}u_2\|_{F^{0,\frac12}}\|u_3\|_{L_t^qL_x^2} \\
&\lesssim \sum_{L>1} L^{-1}N_1^{-\frac12}N_3^{\frac{1-\alpha}{2}}\|u_1\|_{L_t^pL_x^2}\|Q_{\sim LN_1N_3^\alpha}u_2\|_{F^{0,\frac12}}\|u_3\|_{L_t^qL_x^2}. \qquad (4\text{-}12)
\end{aligned}
$$

In the same way, we get that the contribution $I_t^{3,\mathrm{low}}$ of the third term in I_t^{low} is bounded by

$$
\begin{aligned}
|I_t^{3,\mathrm{low}}| &\lesssim N_1^{\frac12}(N_1N_3^\alpha)^{-1}N_3^{\frac{\alpha+1}{2}}\|u_1\|_{L_t^pL_x^2}\|u_2\|_{L_t^qL_x^2}\|Q_{\sim N_1N_3^\alpha}u_3\|_{F^{0,\frac12}} \\
&\lesssim N_1^{-\frac12}N_3^{\frac{1-\alpha}{2}}\|u_1\|_{L_t^pL_x^2}\|u_2\|_{L_t^qL_x^2}\|Q_{\sim N_1N_3^\alpha}u_3\|_{F^{0,\frac12}}. \qquad (4\text{-}13)
\end{aligned}
$$

Gathering all these estimates, we obtain the desired bound. □

Proposition 4.5. *Let $0<T<2$, $1<\alpha\le 2$, $s\ge 1-\frac{\alpha}{2}$ and let $u\in L_T^\infty H^s$ be a solution to (1-3) associated with an initial datum $u_0\in H^s(\mathbb{R})$. Then u belongs to $\widetilde{L}_T^\infty H^s$ and*

$$
\|u\|_{\widetilde{L}_T^\infty H^s}^2 \lesssim \|u_0\|_{H^s}^2 + \|u\|_{L_T^\infty H^s}\big(\|u\|_{L_T^\infty H^{1-\frac{\alpha}{2}}}\|u\|_{Y_T^s} + \|u\|_{L_T^\infty H^s}\|u\|_{Y_T^{1-\frac{\alpha}{2}}}\big). \qquad (4\text{-}14)
$$

Proof. First, we notice that Lemma 4.2 ensures that $u\in Y_T^s$. Applying the operator P_N with $N>0$ dyadic to (1-3), arguing as in (3-11), we obtain

$$
\|P_Nu\|_{L_T^\infty H^s}^2 \lesssim \|P_Nu_0\|_{H^s}^2 + \sup_{t\in]0,T[}\langle N\rangle^{2s}\left|\int_0^t\int_{\mathbb{R}} P_N(u^2)\,\partial_x P_Nu\right|. \qquad (4\text{-}15)
$$

We take an extension $\tilde u$ of u supported in time in $]-4,4[$ such that $\|\tilde u\|_{Y^s}\lesssim\|u\|_{Y_T^s}$. To simplify the notation we drop the tilde in the sequel. We infer from (3-18) that it suffices to estimate

$$
I=\sum_{N>0}\sum_{N_1\gtrsim N} N\langle N_1\rangle^{2s}\sup_{t\in]0,T[}|I_t(u_N,u_{\sim N_1},u_{N_1})|.
$$

The low frequencies part, $N\lesssim 1$, is estimated exactly as in (3-19) by

$$
\|u\|_{L_t^\infty L_x^2}\|u\|_{L_t^\infty H^s}^2.
$$

On the other hand, the contribution of the sum over $N\gg 1$ is controlled, thanks to Lemma 4.4, by

$$
\begin{aligned}
\sum_{N\gg1}\sum_{N_1\gtrsim N}\Big[\Big(\frac{N}{N_1}\Big)^{\frac{\alpha-1}{2}}&\|u_N\|_{L_t^2H^{1-\frac{\alpha}{2}}}\|u_{N_1}\|_{L_t^\infty H^s}\|u_{N_1}\|_{F^{s,\frac12}} \\
&+\Big(\frac{N}{N_1}\Big)^{\frac{5\alpha}{8}}\|u_N\|_{F^{1-\frac{\alpha}{2},\frac12}}\|u_{N_1}\|_{L_t^2H^s}\|u_{N_1}\|_{L_t^\infty H^s} \\
&+N^{\frac{\alpha}{2}-\frac14}N_1^{\frac18-\frac{\alpha}{2}}\|u_N\|_{L_t^\infty H^{1-\frac{\alpha}{2}}}\|u_{N_1}\|_{L_t^\infty H^s}^2\Big] \\
&\lesssim \|u\|_{Y^{1-\frac{\alpha}{2}}}\|u\|_{L_t^\infty H^s}^2 + \|u\|_{L_t^\infty H^{1-\frac{\alpha}{2}}}\|u\|_{L_t^\infty H^s}\|u\|_{Y^s}, \qquad (4\text{-}16)
\end{aligned}
$$

where we use the discrete Young's inequality in N_1 and then Cauchy–Schwarz in N to bound the first two terms.

Gathering the above estimates we eventually obtain

$$I \lesssim \|u\|_{Y_T^{1-\frac{\alpha}{2}}} \|u\|_{L_T^\infty H^s}^2 + \|u\|_{L_T^\infty H^{1-\frac{\alpha}{2}}} \|u\|_{L_T^\infty H^s} \|u\|_{Y_T^s}, \tag{4-17}$$

which completes the proof of the proposition. □

4A. ***Estimates on the difference of two solutions.*** First we introduce the function spaces where we will estimate the difference of two solutions of (1-3). Contrary to the regular case, we will have to work in a function space that puts a weight on the very low frequencies. This kind of weighted space for the difference of two solutions was, for instance, used in [Ionescu et al. 2008] in the context of short-time Bourgain spaces.

For $\theta \in \mathbb{R}$ we define the Banach space

$$\overline{H}^\theta(\mathbb{R}) = \{\varphi \in H^\theta(\mathbb{R}) \mid \|\varphi\|_{\overline{H}^\theta} < \infty\}$$

with

$$\|\varphi\|_{\overline{H}^\theta} := \|\langle |\xi|^{-\frac{1}{2}}\rangle \langle \xi \rangle^\theta \hat{\varphi}\|_{L^2},$$

equipped with the norm $\|\cdot\|_{\overline{H}^\theta}$. Then we define the space $\widetilde{L}_t^\infty \overline{H}^\theta$ by

$$\|w\|_{\widetilde{L}_t^\infty \overline{H}^\theta} := \Big(\sum_{N>0} \|w_N\|_{L_t^\infty \overline{H}^\theta}^2\Big)^{\frac{1}{2}}. \tag{4-18}$$

Finally, we define the function spaces $\widetilde{Y}^\theta$ and Z^θ, $\theta \in \mathbb{R}$, by

$$\widetilde{Y}^\theta = \widetilde{L}_t^\infty H^\theta \cap F^{\theta,\frac{1}{2}} \quad \text{and} \quad Z^\theta = \widetilde{L}_t^\infty \overline{H}^\theta \cap F^{\theta,\frac{1}{2}},$$

with $F^{\theta,b}$ as defined in (4-1).

If $u, v \in L_T^\infty H^s$ are two solutions of (1-3) with $s \geq 1 - \frac{\alpha}{2}$, then, by Lemma 4.2 and Proposition 4.5, we know that u and v belong to $Y_T^s \cap \widetilde{L}_T^\infty H^s$. Moreover, again using the extension operator ρ_T, it is easy to check that

$$Y_T^s \cap \widetilde{L}_T^\infty H^s \hookrightarrow \widetilde{Y}_T^s \tag{4-19}$$

with an embedding constant that does not depend on $0 < T \leq 2$. Hence, u and v belong to $\widetilde{Y}_T^s$. Assuming that $u_0 - v_0 \in \overline{H}^s$, we claim that the difference $u - v$ belongs to Z_T^s. Indeed, according to the above definitions of $\widetilde{Y}^s$ and Z^s, it suffices to check that $P_1(u-v)$ belongs to $\widetilde{L}_T^\infty \overline{H}^s$. But this is straightforward, since, by the Duhamel formula, for any dyadic integer $0 < N < 1$ we have

$$\|P_N(u-v)\|_{L_T^\infty \overline{H}^s} \lesssim \|u_0 - v_0\|_{\overline{H}^s} + N^{\frac{1}{2}}(\|u\|_{L_T^\infty L_x^2}^2 + \|v\|_{L_T^\infty L_x^2}^2).$$

We are thus allowed to estimate the difference $w = u - v$ in the space $Z_T^{s-3/2+\alpha/2}$.

Remark 4.6. For $\alpha > 1$, we have $s - \frac{3}{2} + \frac{\alpha}{2} > s - 1$ and thus, contrary to the preceding section, the derivative of a solution does not belong to the space where we estimate the difference $w = u - v$ of two solutions. This fact is crucial in the preceding section to recover the derivative in terms as J_2 in (3-27) that contains small space frequencies of w. In this section, we will instead combine the weight on the low space frequencies of w with the resonance relation to control such contributions.

Proposition 4.7. *Let $0 < T < 2$, $1 < \alpha \le 2$, $s \ge 1 - \frac{\alpha}{2}$ and $u, v \in L^\infty_T H^s$ be two solutions to (1-3) on $]0, T[$ associated with initial data $u_0, v_0 \in H^s$ such that $u_0 - v_0 \in \overline{H}^s$. Then $u - v \in Z_T^{s-3/2+\alpha/2}$ and we have*

$$\|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|u-v\|_{L^\infty_T \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u+v\|_{\widetilde{Y}^s_T} \|u-v\|_{Z_T^{-\frac{1}{2}}} + \|u+v\|_{\widetilde{Y}^{1-\frac{\alpha}{2}}} \|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}}. \tag{4-20}$$

Proof. The fact that $u - v \in Z_T^{s-3/2+\alpha/2}$ follows from the discussion above. Now, recall that $w = u - v$ satisfies (3-23) with $z = u + v$. We extend w from $(0, T)$ to $\mathbb{R}$ by using the extension operator ρ_T defined in (3-3). On account of the uniform bounds on ρ_T (see the paragraph just after (3-3)), it remains to estimate the $F_T^{s-3/2+\alpha/2,\alpha,1/2}$-norm of w. From classical linear estimates in the framework of Bourgain's spaces, the Duhamel formulation associated with (3-23) leads to

$$\|w\|_{F_T^{s-\frac{3}{2}+\frac{\alpha}{2},\frac{1}{2}}} \lesssim \|w_0\|_{H^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|\partial_x(zw)\|_{F_T^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}}. \tag{4-21}$$

Let $\tilde{z}$ and $\widetilde{w}$ be time extensions of z and w satisfying $\|\tilde{z}\|_{\widetilde{Y}^s} \lesssim \|z\|_{\widetilde{Y}^s_T}$ and $\|\widetilde{w}\|_{Z^{s-3/2+\alpha/2}} \lesssim \|w\|_{Z_T^{s-3/2+\alpha/2}}$. To simplify the notation we drop the tilde in the sequel. From (4-21) we see that it suffices to estimate

$$\|\partial_x(zw)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}} \lesssim \Big(\sum_{N>0} \|P_N \partial_x(zw)\|^2_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}} \Big)^{\frac{1}{2}}.$$

We first estimate the low-high contribution $P_N(P_{\lesssim N} z P_{\sim N} w)$:

$$\begin{aligned}
\|\partial_x P_N(P_{\lesssim N} z P_{\sim N} w)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}} &\lesssim \sum_{N_1 \lesssim N} N \|P_N(P_{N_1} z P_{\sim N} w)\|_{X^{s-2,0}} \\
&\lesssim \sum_{N_1 \lesssim N} N_1^{\frac{1}{2}} N \langle N\rangle^{s-2} \|P_{N_1} z\|_{L^\infty_t L^2_x} \|P_{\sim N} w\|_{L^2_t L^2_x} \\
&\lesssim \|P_{\sim N} w\|_{L^2_t H^{s-\frac{3}{2}+\frac{\alpha}{2}}} \sum_{N_1 \lesssim N} \Big(\frac{N_1}{\langle N\rangle} \Big)^{\frac{\alpha-1}{2}} \|P_{N_1} z\|_{L^\infty_t H^{1-\frac{\alpha}{2}}} \\
&\lesssim \|z\|_{L^\infty_t H^{1-\frac{\alpha}{2}}} \|P_{\sim N} w\|_{L^\infty_t H^{s-\frac{3}{2}+\frac{\alpha}{2}}}.
\end{aligned}$$

Similarly, the high-low interactions are estimated as follows:

$$\begin{aligned}
\|\partial_x P_N(P_{\sim N} z P_{\lesssim N} w)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}} &\lesssim N \|P_N(P_{\sim N} z P_{\lesssim N} w)\|_{X^{s-2,0}} \\
&\lesssim \|P_{\sim N} z\|_{L^2_t H^s} \sum_{N_1 \lesssim N} \Big(\frac{N_1}{\langle N\rangle} \Big)^{\frac{1}{2}} \|P_{N_1} w\|_{L^\infty_t H^{-\frac{1}{2}}} \\
&\lesssim \|P_{\sim N} z\|_{L^2_t H^s} \|w\|_{L^\infty_t H^{-\frac{1}{2}}}.
\end{aligned}$$

Now we deal with the high-high interactions term:

$$\|\partial_x P_N(P_{\gg N} z P_{\gg N} w)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}} \lesssim \sum_{N_1 \gg N} N \Big\| \sum_{\substack{(L,L_1,L_2)\\ \text{satisfying (2-6)}}} \partial_x P_N Q_L(Q_{L_1} z_{N_1} Q_{L_2} w_{N_1}) \Big\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}}.$$

We may assume that $N_1 \gg 1$ since, otherwise, $N \ll N_1 \lesssim 1$ and we have

$$\|P_{\lesssim 1}\partial_x(P_{\lesssim 1} z P_{\lesssim 1} w)\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}} \lesssim \|P_{\lesssim 1} z\|_{L_t^\infty L^2} \|P_{\lesssim 1} w\|_{L_t^\infty H^{-\frac{1}{2}}}.$$

For $N_1 \gg 1$, we will take advantage of the fact that $X^{s-13/8+3\alpha/8,-3/8} \hookrightarrow F^{s-3/2+\alpha/2,-1/2}$. The contribution of the sum over $L \gtrsim N N_1^\alpha$ can be thus controlled by

$$\begin{aligned}
&\sum_{N_1 \gg N} \|\partial_x P_N Q_{\gtrsim N N_1^\alpha}(z_{N_1} w_{N_1})\|_{F^{s-\frac{3}{2}+\frac{\alpha}{2},-\frac{1}{2}}} \\
&\qquad \lesssim \sum_{N_1 \gg N} N \|P_N Q_{\gtrsim N N_1^\alpha}(z_{N_1} w_{N_1})\|_{X^{s-\frac{13}{8}+\frac{3\alpha}{8},-\frac{3}{8}}} \\
&\qquad \lesssim \sum_{N_1 \gg N} \sum_{L \gtrsim N N_1^\alpha} N \langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}} L^{-\frac{3}{8}} \|P_N Q_L(z_{N_1} w_{N_1})\|_{L^2} \\
&\qquad \lesssim \sum_{N_1 \gg N} N^{\frac{3}{2}} \langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}} (N N_1^\alpha)^{-\frac{3}{8}} N_1^{\frac{1}{2}-s} \|z_{N_1}\|_{L_t^2 H^s} \|w_{N_1}\|_{L_t^\infty H^{-\frac{1}{2}}} \\
&\qquad \lesssim \sum_{N_1 \gg N} \left(\frac{N}{N_1}\right)^{\frac{1}{2}-\frac{\alpha}{8}} \left(\frac{\langle N\rangle}{\langle N_1\rangle}\right)^{s-1+\frac{\alpha}{2}} \|z_{N_1}\|_{L_t^2 H^s} \|w_{N_1}\|_{L_t^\infty H^{-\frac{1}{2}}} \\
&\qquad \lesssim \delta_N \|z\|_{L_t^2 H^s} \|w\|_{L_t^\infty H^{-\frac{1}{2}}},
\end{aligned}$$

where $\|(\delta_{2^j})_j\|_{l^2(\mathbb{Z})} \lesssim 1$. The contribution of the region where $L \ll N N_1^\alpha$ and $L_1 \gtrsim N N_1^\alpha$ is estimated, thanks to (4-10), by

$$\begin{aligned}
&\sum_{N_1 \gg N} \|\partial_x P_N Q_{\ll N N_1^\alpha}(Q_{\gtrsim N N_1^\alpha} z_{N_1} w_{N_1})\|_{X^{s-\frac{13}{8}+\frac{3\alpha}{8},-\frac{3}{8}}} \\
&\qquad \lesssim \sum_{N_1 \gg N} N \langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}} \|P_N(Q_{\gtrsim N N_1^\alpha} z_{N_1} w_{N_1})\|_{L^2} \\
&\qquad \lesssim \sum_{N_1 \gg N} N^{\frac{3}{2}} \langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}} (N N_1^\alpha)^{-1} N_1^{1-s+\frac{\alpha}{2}} \|Q_{\gtrsim N N_1^\alpha} z_{N_1}\|_{F^{s,\frac{1}{2}}} \|w_{N_1}\|_{L_t^\infty H^{-\frac{1}{2}}} \\
&\qquad \lesssim \sum_{N_1 \gg N} \left(\frac{N}{\langle N\rangle}\right)^{\frac{1}{2}} \langle N\rangle^{-\frac{1+\alpha}{8}} \left(\frac{\langle N\rangle}{\langle N_1\rangle}\right)^{s-1+\frac{\alpha}{2}} \|Q_{\gtrsim N N_1^\alpha} z_{N_1}\|_{F^{s,\frac{1}{2}}} \|w_{N_1}\|_{L_t^\infty H^{-\frac{1}{2}}} \\
&\qquad \lesssim \delta_N \|z\|_{Y^s} \|w\|_{\widetilde{L}_t^\infty H^{-\frac{1}{2}}},
\end{aligned}$$

where $\|(\delta_{2^j})_j\|_{l^2(\mathbb{Z})} \lesssim 1$. Finally the contribution of the last region can be bounded, thanks to (4-10), by

$$\begin{aligned}
\sum_{N_1 \gg N} & \|\partial_x P_N Q_{\ll NN_1^\alpha}(Q_{\ll NN_1^\alpha} z N_1 Q_{\sim NN_1^\alpha} w_{N_1})\|_{X^{s-\frac{13}{8}+\frac{3\alpha}{8},-\frac{3}{8}}} \\
&\lesssim \sum_{N_1 \gg N} N\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}} \|P_N Q_{\ll NN_1^\alpha}(Q_{\ll NN_1^\alpha} z_{N_1} Q_{\sim NN_1^\alpha} w_{N_1})\|_{L^2} \\
&\lesssim \sum_{N_1 \gg N} N^{\frac{3}{2}}\langle N\rangle^{s-\frac{13}{8}+\frac{3\alpha}{8}} N_1^{-s}(NN_1^\alpha)^{-1} N_1^{1+\frac{\alpha}{2}} \|Q_{\ll NN_1^\alpha} z_{N_1}\|_{L_t^\infty H^s} \|Q_{\sim NN_1^\alpha} w_{N_1}\|_{F^{-\frac{1}{2},\frac{1}{2}}} \\
&\lesssim \sum_{N_1 \gg N} \left(\frac{N}{\langle N\rangle}\right)^{\frac{1}{2}} \langle N\rangle^{-\frac{1+\alpha}{8}} \left(\frac{\langle N\rangle}{\langle N_1\rangle}\right)^{s-1+\frac{\alpha}{2}} \|z_{N_1}\|_{L_t^\infty H^s} \|w_{N_1}\|_{F^{-\frac{1}{2},\frac{1}{2}}} \\
&\lesssim \delta_N \|z\|_{\widetilde{L}_t^\infty H^s} \|w\|_{Z^{-\frac{1}{2}}},
\end{aligned}$$

which is acceptable. This concludes the proof of Proposition 4.7. □

Proposition 4.8. *Let $1 \le \alpha \le 2$, $0 < T < 2$ and let $u, v \in L_T^\infty H^s$ with $s \ge 1 - \frac{\alpha}{2}$ be two solutions to (1-3) associated with initial data $u_0, v_0 \in H^s$ such that $u_0 - v_0 \in \overline{H}^s$. Then*[1]

$$\|u-v\|^2_{\widetilde{L}_T^\infty \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|u_0-v_0\|^2_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u+v\|_{Y_T^s} \|u-v\|_{\widetilde{L}_T^\infty \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} \|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}}. \tag{4-22}$$

Proof. Recall that the difference $w = u - v$ satisfies (3-23) with $z = u + v$. Applying the operator P_N with $N > 0$ dyadic to (3-23), taking the L^2 scalar product with $P_N w$ and integrating on $]0,t[$, we obtain

$$\|w_N\|^2_{L_T^\infty \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|P_N w_0\|_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3}{2}+\frac{\alpha}{2})} \sup_{t\in[0,T]} \left|\int_0^t\int_{\mathbb{R}} P_N(zw)\partial_x w_N\right|.$$

Therefore, we have to estimate

$$J := \sum_{N>0} \langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in[0,T]} \left|\int_0^t\int_{\mathbb{R}} P_N(zw)\partial_x w_N\right|.$$

We take extensions $\tilde{z}$ and $\widetilde{w}$ of z and w supported in time in $]-4,4[$ such that $\|\tilde{z}\|_{Y^s} \lesssim \|u\|_{Y_T^s}$ and $\|\widetilde{w}\|_{Z^s} \lesssim \|w\|_{Z_T^s}$. To simplify the notation we drop the tilde in the sequel.

Proceeding as in (3-27), we get

$$\begin{aligned}
J \lesssim & \sum_{N>0}\sum_{N_1 \gtrsim N} N\langle N_1^{-1}\rangle\langle N_1\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in]0,T[} |I_t(z_N, w_{\sim N_1}, w_{N_1})| \\
&+ \sum_{N>0}\sum_{N_1 \gtrsim N} N_1\langle N_1^{-1}\rangle\langle N_1\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in]0,T[} |I_t(z_{\sim N_1}, w_N, w_{N_1})| \\
&\quad + \sum_{N>0}\sum_{N_1 \gtrsim N} N\langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3-\alpha}{2})} \sup_{t\in]0,T[} |I_t(z_{N_1}, w_{N_1}, w_N)| \\
:= & J_1 + J_2 + J_3.
\end{aligned} \tag{4-23}$$

[1] We include the case $\alpha = 1$ here since it does not lead to additional difficulties and will be useful in the Appendix to prove LWP for $(\alpha, s) = \left(1, \frac{1}{2}\right)$.

Estimates for J_1. The contribution of the sum over $N \lesssim 1$ in J_1 is estimated, thanks to (3-4), by

$$\sum_{N\lesssim 1}\sum_{N_1\gtrsim N} N^{\frac{3}{2}}\|z_N\|_{L^\infty_t L^2_x}\|w_{N_1}\|^2_{L^\infty_t \overline{H}^{s-\frac{3-\alpha}{2}}} \lesssim \|z\|_{L^\infty_t L^2_x}\|w\|^2_{\widetilde{L}^\infty_T \overline{H}^{s-\frac{3-\alpha}{2}}}.$$

The contribution $N \gg 1$ in J_1 can be controlled with Lemma 4.4 by

$$\begin{aligned}\sum_{N\gg 1}\sum_{N_1\gtrsim N}\Big[\sum_{L>1} L^{-1}\Big(\frac{N}{N_1}\Big)^{\frac{\alpha-1}{2}}\|z_N\|_{L^2_t H^{1-\frac{\alpha}{2}}}\|Q_{\sim LNN_1^\alpha} w_{N_1}\|_{F^{s-\frac{3-\alpha}{2},\frac12}}\|w_{N_1}\|_{L^\infty_t H^{s-\frac{3-\alpha}{2}}}\\ +\Big(\frac{N}{N_1}\Big)^{\frac{5\alpha}{8}}\|z_N\|_{F^{1-\frac{\alpha}{2},\frac12}}\|w_{N_1}\|_{L^2_t H^{s-\frac{3-\alpha}{2}}}\|w_{N_1}\|_{L^\infty_t H^{s-\frac{3-\alpha}{2}}}\\ + N^{\frac{\alpha}{2}-\frac14}N_1^{\frac18-\frac{\alpha}{2}}\|z_N\|_{L^\infty_t H^{1-\frac{\alpha}{2}}}\|w_{N_1}\|^2_{L^\infty_t H^{s-\frac{3-\alpha}{2}}}\Big]\\ \lesssim \|z\|_{Y^{1-\frac{\alpha}{2}}}\|w\|_{\widetilde{L}^\infty_t \overline{H}^{s-\frac{3-\alpha}{2}}}\|w\|_{Z^{s-\frac{3-\alpha}{2}}},\end{aligned}$$

where for the first term we used Cauchy–Schwarz in (N, N_1) and then summed in L. Note that for $\alpha > 1$ we could replace the $\widetilde{L}^\infty_t H^{s-3/2+\alpha/2}$-norm by a standard $L^\infty_t H^{s-3/2+\alpha/2}$-norm by invoking the discrete Young inequality.

Estimates for J_2. We separate different contributions. First, the contribution of the sum over $N_1 \lesssim 1$ is directly estimated by $\|z\|_{L^\infty_T L^2}\|w\|^2_{L^\infty_T H^{-1/2}}$. The contribution of the sum over $N \le N_1^{2(1-\alpha)/3}$ and $N_1 \gg 1$ is then easily estimated by

$$\begin{aligned}\sum_{N_1\gg 1}\sum_{N\le N_1^{\frac23(1-\alpha)}} NN_1^{\frac{\alpha-1}{2}}\|z_{N_1}\|_{L^2_T H^s}\|w_N\|_{L^\infty_T \overline{H}^{-\frac12}}\|w_{N_1}\|_{L^2_T H^{s-\frac32+\frac{\alpha}{2}}}\\ \lesssim \sum_{N_1\gg 1} N_1^{\frac{1-\alpha}{6}}\|z_{N_1}\|_{L^2_T H^s}\|w\|_{L^\infty_T \overline{H}^{-\frac12}}\|w_{N_1}\|_{L^2_T H^{s-\frac32+\frac{\alpha}{2}}}\\ \lesssim \|z\|_{L^\infty_T H^s}\|w\|_{L^\infty_T \overline{H}^{-\frac12}}\|w\|_{L^\infty_T H^{s-\frac32+\frac{\alpha}{2}}}.\end{aligned}\tag{4-24}$$

Finally the contribution of the sum over $N_1 \gg 1$ and $N \gg N_1^{2(1-\alpha)/3}$ is bounded, thanks to Lemma 4.4, by

$$\begin{aligned}\sum_{N_1\gg 1}\sum_{N\gg N_1^{\frac23(1-\alpha)}}\Big[\sum_{L>1}\|w_N\|_{L^\infty_t \overline{H}^{-\frac12}}\|Q_{\sim LNN_1^\alpha} w_{N_1}\|_{F^{s-\frac{3-\alpha}{2},\frac12}}\|z_{N_1}\|_{L^2_t H^s}\\ +\|w_N\|_{L^\infty_t \overline{H}^{-\frac12}}\|w_{N_1}\|_{L^2_t H^{s-\frac{3-\alpha}{2}}}\|Q_{\sim NN_1^\alpha} z_{N_1}\|_{F^{s,\frac12}}\\ + N^{-\frac18}\langle N\rangle^{\frac{5+\alpha}{8}}N_1^{-\frac{\alpha}{8}-\frac12}\|w_N\|_{F^{-\frac12,\frac12}}\|w_{N_1}\|_{L^\infty_t H^{s-\frac{3-\alpha}{2}}}\|z_{N_1}\|_{L^2_t H^s}\\ + N^{\frac14}N_1^{-\frac38}\|w_N\|_{L^\infty_t \overline{H}^{-\frac12}}\|w_{N_1}\|_{L^\infty_t H^{s-\frac{3-\alpha}{2}}}\|z_{N_1}\|_{L^\infty_t H^s}\Big]\\ \lesssim \|z\|_{Y^s}\big(\|w\|_{\widetilde{L}^\infty_t \overline{H}^{-\frac12}}\|w\|_{Z^{s-\frac{3-\alpha}{2}}}+\|w\|_{Z^{-\frac12}}\|w\|_{\widetilde{L}^\infty_t \overline{H}^{-\frac12}}\big),\end{aligned}$$

where again we used Cauchy–Schwarz in (N, N_1) and then summed over L.

Estimates for J_3. We first notice that for $N \lesssim N_1$ and $N_1 \gg 1$, since $1+2\big(s-\frac{3-\alpha}{2}\big) \geq 0$,

$$N\langle N^{-1}\rangle\langle N\rangle^{2(s-\frac{3-\alpha}{2})} \lesssim N_1\langle N_1^{-1}\rangle\langle N_1\rangle^{2(s-\frac{3-\alpha}{2})}.$$

Therefore, the contribution of this region to J_3 is controlled by J_2. Finally the contribution of $N \lesssim N_1 \lesssim 1$ is easily bounded by $\|z\|_{L^\infty_t L^2_x}\|w\|^2_{L^\infty_t \overline{H}^{-1/2}}$.

Gathering all the estimates, we eventually obtain

$$J \lesssim \|z\|_{Y^s}\|w\|_{L^\infty_T \overline{H}^{-\frac{1}{2}}}\|w\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|z\|_{Y_T^{1-\frac{\alpha}{2}}}\|w\|_{\widetilde{L}^\infty_T H^{s-\frac{3}{2}+\frac{\alpha}{2}}}\|w\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}}, \tag{4-25}$$

which completes the proof of (4-22). □

4B. *Unconditional well-posedness.* Let us fix $s \geq 1-\frac{\alpha}{2}$. We notice that $1-\frac{\alpha}{2} \geq 0 > s_c = \frac{1}{2}-\alpha$, which is the critical Sobolev exponent associated with (1-3) for dilation symmetry. Therefore, as in Section 3B, the unconditional well-posedness in $H^s(\mathbb{R})$ of (1-3) for small H^s-initial data with a maximal time of existence $T \geq 1$ will ensure the unconditional well-posedness of (1-3) for arbitrary large H^s-initial data with a maximal time of existence

$$T \geq (1+\|u_0\|_{H^s})^{-\frac{2(\alpha+1)}{2\alpha-1}}.$$

Moreover, as in Section 3B, estimates (4-2), (4-3), (4-14), and a continuity argument ensure that a smooth solution with small H^s-initial datum has got a time of existence T in $H^\infty(\mathbb{R})$ that is greater than 1. Now, to prove the existence of a solution with initial data $u_0 \in H^{1-\alpha/2}$, we cannot argue exactly as in Section 3B since, for $s=0$, we miss compactness to pass to the limit on the nonlinear term. Instead, we construct below a sequence of smooth solutions to (1-3) that converges strongly to a solution of (1-3) emanating from u_0. This will be done by using the Bona–Smith argument.

Let $u_0 \in H^s$ with $s \geq 1-\frac{\alpha}{2}$ and $\|u_0\|_{H^s} \ll 1$. We denote by u^N the solution of (1-3) emanating from $P_{\leq N}u_0$. From the discussion above, $u_N \in C([0,1];H^\infty(\mathbb{R}))$ and, for $1 \leq N_1 \leq N_2$, we set

$$w := u^{N_1}-u^{N_2}.$$

Let us note that $P_{\leq 1}w_0 = P_{\leq 1}(u^{N_1}-u^{N_2}) = 0$ and thus $w_0 \in \overline{H}^s(\mathbb{R})$ with $\|w_0\|_{\overline{H}^s} \sim \|w_0\|_{H^s}$. It then follows from (4-20)–(4-22) that

$$\|w\|_{Z_1^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|w(0)\|_{H^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim N_1^{\frac{\alpha-3}{2}}\|P_{>N_1}u_0\|_{H^s}. \tag{4-26}$$

Moreover, on account of Lemma 4.2, Proposition 4.5 and (4-19), for any $r \geq 0$ we have

$$\|u^{N_i}\|_{Y_T^{s+r}} \lesssim \|u^{N_i}\|_{\widetilde{Y}_T^{s+r}} \lesssim \|u_0^{N_i}\|_{H^{s+r}} \lesssim N_i^r\|u_0\|_{H^s}. \tag{4-27}$$

Next, since w satisfies (3-36), the Duhamel formula leads, for any $0<N<1$, to

$$\|P_N w\|_{L^\infty_1 \overline{H}^s} \lesssim \|P_N w_0\|_{\overline{H}^s} + N^{\frac{1}{2}}(\|u^{N_1}\|^2_{L^\infty_1 L^2_x} + \|w\|^2_{L^\infty_1 L^2_x})$$

and thus

$$\|P_{\leq 1}w\|_{\widetilde{L}^\infty_1 \overline{H}^s} \lesssim \|w_0\|_{H^s} + (\|u^{N_1}\|^2_{L^\infty_1 L^2_x} + \|w\|^2_{L^\infty_1 L^2_x}). \tag{4-28}$$

This proves that $w \in Z^s_T$. We will also need the following estimates on w:

Proposition 4.9. *Let $1 < \alpha \le 2$, $0 < T < 2$ and $w \in Z^s_T$ with $s \ge 1 - \frac{\alpha}{2}$ be a solution to (3-36). Then*

$$\|w\|_{Y^s_T} \lesssim \|w\|_{L^\infty_T H^s}(1 + \|u^{N_1}\|^2_{L^\infty_T H^s} + \|w\|^2_{L^\infty_T H^s}) \tag{4-29}$$

and

$$\|w\|^2_{\tilde{L}^\infty_T H^s} \lesssim \|w_0\|^2_{H^s} + \|w\|^3_{Y^s_T} + \|u^{N_1}\|_{Y^s_T}\|w\|^2_{Z^s_T} + \|u^{N_1}\|_{Y_T^{s+\frac{3}{2}-\frac{\alpha}{2}}}\|w\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}}\|w\|_{Z^s_T}. \tag{4-30}$$

Proof. First, (4-29) can be derived exactly as (4-2)–(4-3) of Lemma 4.2. Now, to prove (4-30), we separate the contribution of $\partial_x(w^2)$ and $\partial_x(u^{N_1}w)$. First, (4-17) leads to

$$\sum_{N>0} N^{2s}\left|\int_0^t \int_{\mathbb{R}} P_N\partial_x(w^2)P_N w\right| \lesssim \|w\|^3_{Y^s_T}.$$

Second, applying (4-25) at the level s with z replaced by u^{N_1}, we obtain

$$\sum_{N>0} N^{2s}\left|\int_0^t \int_{\mathbb{R}} P_N\partial_x(u^{N_1}w)P_N w\right| \lesssim \|u^{N_1}\|_{Y_T^{s+\frac{3}{2}-\frac{\alpha}{2}}}\|w\|_{Z_T^{-\frac{1}{2}}}\|w\|_{Z^s_T} + \|u^{N_1}\|_{Y_T^{1-\frac{\alpha}{2}}}\|w\|^2_{Z^s_T},$$

which leads to (4-30) since $s - \frac{3}{2} + \frac{\alpha}{2} \ge -\frac{1}{2}$ for $s \ge 1 - \frac{\alpha}{2}$ and $Z^s_T \hookrightarrow Y^s_T$. □

Combining (4-28), (4-29), (4-30) and (4-19), we infer that

$$\|w\|^2_{Z^s_1} \lesssim (1 + \|u_0\|^2_{H^s})^2\big[\|w_0\|^2_{H^s} + \|u_0\|_{H^s}\|w\|^2_{Y^s_1} + \|u_0\|_{H^s}\|w\|^2_{Z^s_1} + N_1^{\frac{3-\alpha}{2}}\|u_0\|_{H^s}\|w\|_{Z_1^{s-\frac{3}{2}+\frac{\alpha}{2}}}\|w\|_{Z^s_1}\big].$$

Then, the smallness assumption on $\|u_0\|_{H^s}$, (4-26) and the continuous injection $Z^s_T \hookrightarrow Y^s_T$, lead to

$$\begin{aligned}\|w\|^2_{Z^s_1} &\lesssim \|w_0\|^2_{H^s} + N_1^{3-\alpha}\|w\|^2_{Z_1^{s-\frac{3}{2}+\frac{\alpha}{2}}} \\ &\lesssim \|P_{>N_1}u_0\|^2_{H^s}(1 + \|P_{>N_1}u_0\|^2_{H^s}) \to 0 \quad \text{as } N_1 \to 0.\end{aligned} \tag{4-31}$$

This shows that $\{u^N\}$ is a Cauchy sequence in $C([0,1]; H^s)$ and thus $\{u^N\}$ converges in $C([0,1]; H^s)$ to a solution of (1-3) emanating from u_0. Note that there is no problem passing to the limit on the nonlinear term here, since we have strong convergence.

Now, Lemma 4.2, Proposition 4.5 and (4-19) ensure that any $L^\infty_1 H^s$-solution to (1-3) on $]0,1[$ belongs to $\tilde{Y}^s_T$. Therefore, according to Propositions 4.7 and 4.8, u is the only solution to (1-3) associated with the initial datum u_0 that belongs to $L^\infty_{\text{loc}} H^s$.

To prove the continuity of the solution map in $H^s(\mathbb{R})$, we proceed as in Section 3B. Let $\{u_{0,n}\} \subset H^s(\mathbb{R})$ be such that $u_{0,n} \to u_0$ in $H^s(\mathbb{R})$ and let $\{u_n\} \subset C([0,1]; H^s(\mathbb{R}))$ be the associated sequence of solutions to (1-3). Taking the same notations as above, we observe that, by construction,

$$P_{\le 1}(u_0 - u_0^N) = P_{\le 1}(u_{0,n} - u_{0,n}^N) = 0 \quad \text{for all } N \ge 1.$$

This ensures that $u-u^N$ and $u_n-u_n^N$ belong to Z^s_T. Estimate (4-31) on solutions to (3-36) then leads to

$$\|u-u^N\|_{Z^s_1}+\|u_n-u_n^N\|_{Z^s_1}\lesssim \|P_{>N}u_0\|_{H^s}+\|P_{>N}u_{0,n}\|_{H^s},$$

which yields

$$\lim_{N\to\infty}\sup_{n\in\mathbb{N}}(\|u-u^N\|_{L^\infty_1H^s}+\|u_n-u_n^N\|_{L^\infty_1H^s})=0. \tag{4-32}$$

It remains to estimate $\|u_n^N-u^N\|_{H^s}$. Note that we cannot use Propositions 4.8 and 4.9 here, since $u_{0,n}^N-u_0^N$ does not belong a priori to $\overline{H}^s(\mathbb{R})$. However, since u_0^N and $u_{0,n}^N$ belong to $H^\infty(\mathbb{R})$, we know, from the beginning of this section, that u^N and u_n^N belong to $C([0,1];H^\infty(\mathbb{R}))$. We now fix $N\gg 1$. Setting $s'=\max(1,s)$, we have

$$\|u_0^N-u_{0,n}^N\|_{H^{s'}}\to 0 \quad \text{as } n\to\infty.$$

Therefore, on account of Section 3B,

$$\|u^N-u_n^N\|_{L^\infty_TH^{s'}}\to 0 \quad \text{as } n\to\infty \quad \text{with} \quad T\sim(1+\|u_0^N\|_{H^{s'}})^{-\frac{2(\alpha+1)}{2\alpha-1}}.$$

Since $u^N\in C([0,1];H^\infty(\mathbb{R}))$ we can iterate this argument a finite number of times to obtain that the convergence of u_n^N to u^N holds actually in $C([0,1];H^{s'}(\mathbb{R}))$. The continuity of the flow map in $H^s(\mathbb{R})$ follows by combining this last result with (4-32).

4C. ***The periodic case.*** We use the notations of Section 3C. Let $H^s_0(\lambda\mathbb{T})$ be the closed subspace of zero-mean functions of $H^s_0(\lambda\mathbb{T})$. We define the Banach space $\overline{H}^s(\lambda\mathbb{T})$ as the space $H^s_0(\lambda\mathbb{T})$ endowed with the norm

$$\|u\|_{\overline{H}^s}=\|\langle|\xi|^{-\frac12}\rangle\langle\xi\rangle^s\hat{\varphi}\|_{L^2_\xi}.$$

Let $(u,v)\in(L^\infty(0,T;H^s(\lambda\mathbb{T})))^2$ be a pair of solutions to (1-3) associated with initial data (u_0,v_0) in $(H^s(\lambda\mathbb{T}))^2$ such that $u_0-v_0\in\overline{H}^s(\lambda\mathbb{T})$. As noticed in Remark 1.3, $(u,v)\in C([0,T];H^{s-\alpha-1}(\lambda\mathbb{T}))^2$ and it is not too hard to check that the mean value is a constant of the motion for such solutions. Therefore, $u(t)-v(t)$ has mean value zero for all $t\in[0,T]$.

As explained in Section 3C, to extend our result on the torus $\lambda\mathbb{T}$, uniformly for $\lambda\geq 1$, we only have to care about the contributions of the null frequencies each time we used the homogeneous decomposition in space frequencies. First we notice that in the proof of Lemma 4.2 we do not use any homogeneous decomposition in space frequencies and thus this lemma still holds in the periodic setting. Note that this is also true for (4-29), since the proof of this estimate is exactly the same. Moreover, on account of (3-42), the contributions of the null frequencies vanish in the proof of Lemma 4.2. Now, for Propositions 4.7, 4.8 and 4.9, we only have to care about the contributions of $\partial_x P_N(wP_0z)$, since, according to the discussion above, $P_0w=P_0(u-v)=0$ on $[0,T]$. On account of (3-43), these contributions vanish in (4-22) and (4-30). Finally, these contributions can be estimated in Proposition 4.7 by

$$\|\partial_xP_N(P_NwP_0z)\|_{F^{s-\frac32+\frac{\alpha}{2},-\frac12}}\lesssim N\|P_N(P_NwP_0z)\|_{X^{s-2,0}}\lesssim\delta_N\|z\|_{L^\infty_tL^2_x}\|w\|_{L^2_tH^{s-1}_x}$$

with $\|(\delta_{2^j})\|_{l^1(\mathbb{Z})} \lesssim 1$. This is acceptable, since $1-\frac{\alpha}{2} \geq 0$ and $s-\frac{3}{2}+\frac{\alpha}{2} \geq s-1$. The proof of Theorem 1.5 is now complete.

5. Dissipative limits

First, we notice that, if u is a solution to (1-9), then u_λ defined by $u_\lambda(t,x) = \lambda^\alpha u(\lambda^{1+\alpha}t, \lambda x)$ is a solution to

$$\partial_t u_\lambda + L^\lambda_{\alpha+1} u_\lambda + \varepsilon \lambda^{\alpha+1-\beta} A^\lambda_\beta u_\lambda + \tfrac{1}{2}\partial_x (u_\lambda)^2 = 0 \tag{5-1}$$

with

$$\widehat{L^\lambda_{\alpha+1} v}(\xi) = i\lambda^{\alpha+1} p_{\alpha+1}(\lambda^{-1}\xi)\hat{v}(\xi)$$

and

$$\widehat{A^\lambda_\beta v}(\xi) = \lambda^\beta q_\beta(\lambda^{-1}\xi)\hat{v}(\xi) \quad \text{for all } \xi \in \mathbb{R}.$$

Therefore, as in the preceding section, up to this change of unknown, of parameter ε and of operators, we may assume that u satisfies (1-9) with $L_{\alpha+1}$ and A_β that verify Hypotheses 1 and 2 for all $0 < \lambda \leq 1$.

Second, we notice that Hypothesis 2 now ensures that, for $0 < \lambda \leq 1$ and $N \gg 1$ dyadic,

$$(A^\lambda_\beta P_N v, P_N v)_{L^2} \gtrsim N^{\frac{\beta}{2}} \|P_N v\|^2_{L^2} \tag{5-2}$$

and

$$\|A^\lambda_\beta P_N v\|_{L^2} \lesssim N^\beta \|P_N v\|_{L^2}. \tag{5-3}$$

The main point is now to prove that the Cauchy problem (1-9) is locally well-posed in H^s uniformly in $\varepsilon > 0$.

Proposition 5.1. *Let $1 \leq \alpha \leq 2$, $0 \leq \beta \leq 1+\alpha$ and $s \geq 1-\frac{\alpha}{2}$. For any $\varphi \in H^s(\mathbb{R})$ there exists $T \sim (1+\|u_0\|_{H^{1-\alpha/2}})^{-2(\alpha+1)/(2\alpha-1)}$ and a solution $u_\varepsilon \in C([0,T];H^s)$ to (1-9) that is unique in some function space*[2] *embedded in $L^\infty_T(0,T;H^s)$. Moreover, there exists $C>0$ such that, for any $\varepsilon \in]0,1[$,*

$$\sup_{t\in[0,T]} \|u_\varepsilon(t)\|_{H^s} \leq C\|\varphi\|_{H^s}. \tag{5-4}$$

Finally, for any $R>0$, the family of solution maps $S_\varepsilon : \varphi \mapsto u_\varepsilon$, $\varepsilon \in]0,1[$, from $B(0,R)_{H^s}$ into $C([0,T(R)];H^s(\mathbb{R}))$ is equicontinuous, i.e., for any sequence $\{\varphi_n\} \subset B(0,R)_{H^s}$ converging to φ in $H^s(\mathbb{R})$,

$$\lim_{n\to+\infty} \sup_{\varepsilon\in]0,1[} \|S_\varepsilon\varphi - S_\varepsilon\varphi_n\|_{L^\infty(0,T(R);H^s(\mathbb{R}))} = 0. \tag{5-5}$$

Proof. We treat the cases $(\alpha,s) \neq \left(1,\frac{1}{2}\right)$. This last case can be treated in the same way by using the estimates derived in the Appendix. First we notice that, for (1-9), in view of (5-2), the energy estimate (4-14) becomes

$$\|u\|_{\tilde{L}^\infty_T H^s} + \sqrt{\varepsilon}\|u\|_{L^2_T H^{s+\frac{\beta}{2}}} \lesssim \|u_0\|_{H^s} + \|u\|_{L^\infty_T H^{1-\frac{\alpha}{2}}} \|u\|_{Y^s_T} + \|u\|_{L^\infty_T H^s} \|u\|_{Y^{1-\frac{\alpha}{2}}_T}. \tag{5-6}$$

[2]For $(\alpha,s) \neq \left(1,\frac{1}{2}\right)$, this space is simply the space $L^\infty_T H^s \cap L^2_T H^{s+\beta/2}$.

On the other hand, viewing $\varepsilon A_\beta u$ as a forcing term, (4-2)–(4-3) together with (5-3) lead to

$$\|u\|_{Y_T^s} \lesssim \|u\|_{L_T^\infty H^s}(1+\|u\|^2_{L_T^\infty H^{1-\frac{\alpha}{2}}}) + \varepsilon\|u\|_{L_T^2 H^{s-\frac{1+\alpha}{2}+\beta}}. \tag{5-7}$$

To derive an a priori bound from the above estimates, as in the previous section, we have to use the dilation argument that is described in the beginning of this section. So the dilation function u_λ defined by $u_\lambda(t,x)=\lambda^\alpha u(\lambda^{1+\alpha}t,\lambda x)$ satisfies (5-1) and we set

$$\|v\|_{N^s} := \|v\|_{L_T^\infty H^s} + \sqrt{\varepsilon\lambda^{\alpha+1-\beta}}\|v\|_{L_T^2 H^{s+\frac{\beta}{2}}}.$$

Since $\beta \le \alpha+1$, this ensures that, for $\lambda \lesssim (1+\|\varphi\|_{H^s})^{-2(\alpha+1)/(2\alpha-1)}$ and $0<T\le 2$,

$$\|u_\lambda\|_{N_T^s} \lesssim \|\varphi_\lambda\|_{H^s} + (1+\|u_\lambda\|^2_{N_T^{1-\frac{\alpha}{2}}})\|u_\lambda\|_{N_T^{1-\frac{\alpha}{2}}}\|u_\lambda\|_{N_T^s}$$

with $\|\varphi_\lambda\|_{H^s} \lesssim \lambda^{\alpha-1/2}\|\varphi\|_{H^s} \ll 1$. This leads to the uniform bound (5-4) for smooth solutions to (1-9) by a classical continuity argument.

Now, proceeding in the same way for the difference of two solutions, it is not too hard to check that (4-20) becomes

$$\|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} \lesssim \|u-v\|_{\tilde{L}_T^\infty \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u-v\|_{L_T^2 H^{s-\frac{3}{2}+\frac{\alpha}{2}+\beta}} + \|u+v\|_{\tilde{Y}_T^s}\|u-v\|_{Z_T^{-\frac{1}{2}}} + \|u+v\|_{\tilde{Y}^{1-\frac{\alpha}{2}}}\|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}},$$

whereas (4-22) becomes

$$\|u-v\|_{\tilde{L}_T^\infty \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \sqrt{\varepsilon}\|u-v\|_{L_T^2 \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}+\frac{\beta}{2}}} \lesssim \|u_0-v_0\|_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u+v\|_{\tilde{Y}_T^s}\|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}}.$$

By the same dilation arguments as above, this leads to

$$\|u-v\|_{Z_T^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \sqrt{\varepsilon}\|u-v\|_{L_T^2 H^{s-\frac{3}{2}+\frac{\alpha}{2}+\frac{\beta}{2}}} \lesssim \|u_0-v_0\|_{\overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}}. \tag{5-8}$$

Combining the above estimates and the Bona–Smith argument, we can proceed as in Section 4B and construct a sequence of smooth solutions that converges strongly in $C([0,T];H^s)$ towards a solution u_ε to (1-9). We thus obtain the existence of a solution $u_\varepsilon \in C([0,T];H^s)\cap L_T^2 H^{s+\beta/2}$ to (1-9) with $T \gtrsim (1+\|u_0\|_{H^{1-\alpha/2}})^{-2(\alpha+1)/(2\alpha-1)}$ and $\varphi \in H^s$ as initial data. Moreover, (5-8) ensures that this is the only solution emanating from φ in the class $L^\infty_{\text{loc}} H^s \cap L^2_{\text{loc}} H^{s+\beta/2}$. Obviously, this solution satisfies (5-4). Finally, the equicontinuity of the solution map in $C(0,T;H^s)$ follows from Bona–Smith arguments as in Section 3B. □

It is clear that the above proposition implies part (1) of Theorem 1.14. Now, part (2) will follow from general arguments (see for instance [Guo and Wang 2009]). Let us denote by S_ε and S the nonlinear group associated with, respectively, (1-9) and (1-3). Let $\varphi \in H^s(\mathbb{R})$, $s \ge 1-\frac{\alpha}{2}$ and let $T = T(\|\varphi\|_{H^{1-\alpha/2}}) > 0$

be as given by Proposition 5.1. For any $N>0$ we can rewrite $S_\varepsilon(\varphi)-S(\varphi)$ as

$$\begin{aligned} S_\varepsilon(\varphi)-S(\varphi) &= (S_\varepsilon(\varphi)-S_\varepsilon(P_{\le N}\varphi)) + (S_\varepsilon(P_{\le N}\varphi)-S(P_{\le N}\varphi)) + (S(P_{\le N}\varphi)-S(\varphi)) \\ &= I_{\varepsilon,N}+J_{\varepsilon,N}+K_N. \end{aligned}$$

By continuity with respect to initial data in $H^s(\mathbb{R})$ of the solution map associated with (1-3), we have $\lim_{N\to\infty}\|K_N\|_{L^\infty(0,T;H^s)}=0$. Moreover, (5-5) ensures that

$$\lim_{N\to\infty}\sup_{\varepsilon\in]0,1[}\|I_{\varepsilon,N}\|_{L^\infty(0,T;H^s)}=0.$$

It thus remains to check that, for any fixed $N>0$, $\lim_{\varepsilon\to 0}\|J_{\varepsilon,N}\|_{L^\infty(0,T;H^s_x)}=0$. Since $P_{\le N}\varphi\in H^\infty(\mathbb{R})$, it is worth noticing that $S_\varepsilon(P_{\le N}\varphi)$ and $S(P_{\le N}\varphi)$ belong to $C^\infty(\mathbb{R};H^\infty(\mathbb{R}))$. Moreover, according to Theorem 1.14 and Proposition 5.1, for all $\theta\in\mathbb{R}$ and $\varepsilon\in]0,1[$,

$$\|S_\varepsilon(P_{\le N}\varphi)\|_{L^\infty_T H^\theta_x}+\|S(P_{\le N}\varphi)\|_{L^\infty_T H^\theta_x}\le C(N,\theta,\|\varphi\|_{L^2_x}).$$

Now, setting $v_\varepsilon:=S_\varepsilon(P_{\le N}\varphi)$ and $v:=S(P_{\le N}\varphi)$, we observe that $w_\varepsilon:=v_\varepsilon-v$ satisfies

$$\partial_t w_\varepsilon+L_{\alpha+1}w_\varepsilon=-\tfrac{1}{2}\partial_x(w_\varepsilon(v+v_\varepsilon))-\varepsilon A_\beta v_\varepsilon$$

with initial data $w_\varepsilon(0)=0$. For $s\ge 0$, taking the H^s-scalar product of this last equation with w_ε and integrating by parts, we get

$$\frac{d}{dt}\|w_\varepsilon\|_{H^s}\lesssim(1+\|\partial_x(v+v_\varepsilon)\|_{L^\infty_x})\|w_\varepsilon\|_{H^s}^2+\|[J^s\partial_x,(v+v_\varepsilon)]w_\varepsilon\|_{L^2_x}\|w_\varepsilon\|_{H^s}+\varepsilon^2\|D^\beta_x v_\varepsilon\|_{H^s}^2.$$

Applying the mean value theorem to the Fourier transform of the commutator term, it is not too hard to check that

$$\|[J^s_x\partial_x,f]g\|_{L^2_x}\lesssim\|f_x\|_{H^{s+1}}\|g\|_{H^s_x}, \tag{5-9}$$

which leads to

$$\frac{d}{dt}\|w_\varepsilon(t)\|_{H^s}^2\lesssim C(N,s+2,\|\varphi\|_{L^2_x})\|w_\varepsilon(t)\|_{H^s_x}^2+\varepsilon^2 C(N,s+\beta,\|\varphi\|_{L^2_x})^2.$$

Integrating this differential inequality on $[0,T]$, this ensures that $\lim_{\varepsilon\to 0}\|w_\varepsilon\|_{L^\infty(0,T;H^s)}=0$ and proves that

$$u_\varepsilon\to u \quad \text{in } C([0,T];H^s) \tag{5-10}$$

with $T\sim(1+\|u_0\|_{H^{1-\alpha/2}})^{-2(\alpha+1)/(2\alpha-1)}$. Now fix $\varphi\in H^s$ and let $T^*>0$ be the maximal time of existence of $S(\varphi)$. It remains to prove that the time of existence T_ε of $S_\varepsilon(\varphi)$ in H^s satisfies $\liminf_{\varepsilon\to 0}T_\varepsilon\ge T^*$. Actually, this follows by a classical contradiction argument. Indeed, assuming that this is not true, there exist $\varepsilon_n\searrow 0$ such that $\lim T_{\varepsilon_n}=T_1<T^*$. We set

$$\delta(T_1)=(1+\|S(\varphi)\|_{L^\infty(0,T_1;H^{1-\frac{\alpha}{2}})})^{-\frac{2(\alpha+1)}{2\alpha-1}},$$

which is well defined since $T_1 < T^*$. Applying (5-10) about $T_1/\delta(T_1)$ times, we eventually obtain that, for n large enough,

$$\big\| S_{\varepsilon_n}(\varphi)\big(T_1 - \tfrac{1}{100}\delta(T_1)\big)\big\|_{H^{1-\frac{\alpha}{2}}} \le 2\|S(\varphi)\|_{L^\infty(0,T_1;H^{1-\frac{\alpha}{2}})}.$$

But then the uniform bound from below on the existence time ensures that $T_{\varepsilon_n} \ge T_1 + \frac{1}{2}\delta(T_1)$, which contradicts $\lim T_{\varepsilon_n} = T_1$ and proves the desired result. This ensures that, fixing $0 < T_0 < T^*$, we have $T_\varepsilon \ge T_0$ for $\varepsilon > 0$ small enough. Finally, applying (5-10) about $T_0/\delta(T_0)$ times, we get (5-10) with $T = T_0$. This completes the proof of Theorem 1.14.

Appendix: The case $\alpha = 1$ and $s = \frac{1}{2}$

This case is important since $H^{1/2}$ is the energy space for the Benjamin–Ono equation and also the intermediate long waves equation. Unfortunately, we are not able to prove the unconditional well-posedness in this case. However, we are able to prove the well-posedness without using a gauge transform. This is useful for treating perturbations of these equations, as we explained in the preceding section. In this section, we indicate the modifications of the proofs in this case. In the sequel we set

$$\widetilde{M}^{\frac{1}{2}} := \widetilde{L}^\infty_t H^{\frac{1}{2}} \cap X^{-\frac{1}{2},1}.$$

Lemma A.1. *Let $\alpha = 1$, $0 < T < 2$, and let $u \in \widetilde{M}^{1/2}_T$ be a solution to (1-3). Then*

$$\|u\|_{\widetilde{M}^{\frac{1}{2}}_T} \lesssim \|u\|_{\widetilde{L}^\infty_T H^{\frac{1}{2}}} + \|u\|^2_{\widetilde{M}^{\frac{1}{2}}_T}. \tag{A-1}$$

Proof. Working with the extension $\tilde{u} = \rho_T u$ (see (3-3)), still denoted u, if suffices to estimate the $X^{-1/2,1}$-norm of u. First we notice that the low frequency part can be easily controlled by

$$\|P_{\lesssim 1} u\|_{X_T^{-\frac{1}{2},1}} \lesssim \|u\|^2_{L^\infty_T L^2_x}.$$

Now, for $N \gg 1$, we have

$$\begin{aligned}
\|u_N\|_{X_T^{-\frac{1}{2},1}} &\lesssim \|P_N u_0\|_{H^{-\frac{1}{2}}} + N^{\frac{1}{2}} \Big\| \sum_{N_2' \sim N_2 \gtrsim N} u_{N_2} u_{N_2'} \Big\|_{L^2_T L^2_x} \\
&\qquad + N^{\frac{1}{2}} \Big\| \sum_{1 \le N_2 \ll N} P_N(u_{\sim N} u_{N_2}) \Big\|_{L^2_T L^2_x} + N^{\frac{1}{2}} \Big\| \sum_{N_2 < 1} P_N(u_{\sim N} u_{N_2}) \Big\|_{L^2_T L^2_x} \\
&= \|P_N u_0\|_{H^{-\frac{1}{2}}} + I_N + II_N + III_N.
\end{aligned}$$

Clearly,

$$\begin{aligned}
I_N &\lesssim N^{\frac{1}{2}} \sum_{N_2' \sim N_2 \gtrsim N} \|u_{N_2}\|_{L^2_{tx}} \|u_{N_2'}\|_{L^\infty_t H^{\frac{1}{2}}} \\
&\lesssim \|u\|_{L^\infty_t H^{\frac{1}{2}}} \sum_{N_2 \gtrsim N} \Big(\frac{N}{N_2}\Big)^{\frac{1}{2}} \|u_{N_2}\|_{L^2_t H^{\frac{1}{2}}} \\
&\lesssim \delta_N \|u\|^2_{L^\infty_t H^{\frac{1}{2}}}
\end{aligned}$$

with $\|(\delta_{2^j})\|_{l^2(\mathbb{N}^*)} \lesssim 1$. Moreover, we easily get from Bernstein estimates that

$$III_N \lesssim N^{\frac{1}{2}} \sum_{N_2<1} \|u_{\sim N}\|_{L^2_{tx}} \|u_{N_2}\|_{L^\infty_{tx}} \lesssim \|u_{\sim N}\|_{L^2_t H^{\frac{1}{2}}} \|u\|_{L^\infty_t H^{\frac{1}{2}}} \lesssim \delta_N \|u\|_{L^\infty_t H^{\frac{1}{2}}} \|u\|_{L^\infty_t H^{\frac{1}{2}}}$$

with $\|(\delta_{2^j})\|_{l^2(\mathbb{N}^*)} \lesssim 1$. On the other hand,

$$II_N \lesssim N^{\frac{1}{2}} \Big\| \sum_{1\le N_2 \ll N} Q_{\sim NN_2} P_N(u_{\sim N} u_{N_2}) \Big\|_{L^2_{tx}} + N^{\frac{1}{2}} \Big\| \sum_{1\le N_2 \ll N} Q_{\nsim NN_2} P_N(u_{\sim N} u_{N_2}) \Big\|_{L^2_{tx}} \lesssim II^1_N + II^2_N .$$

By almost orthogonality, we have

$$\begin{aligned} II^1_N &\lesssim N^{\frac{1}{2}} \Big(\sum_{1\le N_2\ll N} \Big\| Q_{\sim NN_2} P_N(u_{\sim N} u_{N_2}) \Big\|^2_{L^2_{tx}} \Big)^{\frac{1}{2}} \\ &\lesssim N^{\frac{1}{2}} \Big(\sum_{N_2\ll N} \|u_{\sim N}\|^2_{L^2_{tx}} \|u_{N_2}\|^2_{L^\infty_t H^{\frac{1}{2}}_x} \Big)^{\frac{1}{2}} \\ &\lesssim \|u_{\sim N}\|_{L^2_t H^{\frac{1}{2}}} \|u\|_{\widetilde{L}^\infty_t H^{\frac{1}{2}}} \\ &\lesssim \delta_N \|u\|_{L^\infty_t H^{\frac{1}{2}}} \|u\|_{\widetilde{L}^\infty_t H^{\frac{1}{2}}} \end{aligned}$$

with $\|(\delta_{2^j})\|_{l^2(\mathbb{N}^*)} \lesssim 1$. It remains to control II^2_N. Since the Fourier projectors ensure $\langle \tau - p_2(\xi)\rangle \nsim NN_2$, the resonance relation (1-6) leads to $|\tau_1 - p_2(\xi_1)| \vee |\tau - \tau_1 - p_2(\xi-\xi_1)|) \gtrsim NN_2$ for II^2_N. We separate the contributions of $Q_{\gtrsim NN_2} u_{\sim N}$ and $Q_{\gtrsim NN_2} u_{N_2}$. For the first contribution, we have

$$\begin{aligned} II^2_N &\lesssim N^{\frac{1}{2}} \sum_{1\le N_2\ll N} (NN_2)^{-\frac{1}{4}} N^{\frac{1}{4}} \|Q_{\gtrsim NN_2} u_{\sim N}\|_{X^{\frac{1}{4},\frac{1}{4}}} \|u_{N_2}\|_{L^\infty_t H^{\frac{1}{2}}} \\ &\lesssim \|u_{\sim N}\|_{X^{\frac{1}{4},\frac{1}{4}}} \|u\|_{L^\infty_t H^{\frac{1}{2}}} \\ &\lesssim \delta_N \|u\|^{\frac{1}{4}}_{X^{-\frac{1}{2},1}} \|u\|^{\frac{3}{4}}_{L^\infty_t H^{\frac{1}{2}}} \|u\|_{L^\infty_t H^{\frac{1}{2}}} \end{aligned}$$

with $\|(\delta_{2^j})\|_{l^2(\mathbb{N}^*)} \lesssim 1$ and where we used interpolation at the last step. For the second contribution, we write

$$\begin{aligned} II^2_N &\lesssim N^{\frac{1}{2}} \sum_{1\le N_2\ll N} \|Q_{\ll NN_2} u_{\sim N}\|_{L^\infty_t L^4_x} \|Q_{\gtrsim NN_2} u_{N_2}\|_{L^2_t L^4_x} \\ &\lesssim N^{\frac{1}{2}} \sum_{1\le N_2\ll N} N^{-\frac{1}{4}} \|Q_{\ll NN_2} u_{\sim N}\|_{L^\infty_t H^{\frac{1}{2}}} \|Q_{\gtrsim NN_2} u_{N_2}\|_{L^2_t H^{\frac{1}{4}}} \\ &\lesssim N^{\frac{1}{2}} \sum_{1\le N_2\ll N} N^{-\frac{1}{4}} (NN_2)^{-\frac{1}{4}} \|u_{\sim N}\|_{L^\infty_t H^{\frac{1}{2}}} \|u_{N_2}\|_{X^{\frac{1}{4},\frac{1}{4}}} \\ &\lesssim \delta_N \|u\|_{\widetilde{L}^\infty_t H^{\frac{1}{2}}} \|u\|^{\frac{1}{4}}_{X^{-\frac{1}{2},1}} \|u\|^{\frac{3}{4}}_{L^\infty_t H^{\frac{1}{2}}} \end{aligned}$$

with $\|(\delta_{2^j})\|_{l^2(\mathbb{N}^*)} \lesssim 1$. Gathering the above estimates, (5-2) follows. □

Lemma A.2. *Let $\alpha = 1$, $0 < T < 2$ and let $u \in \widetilde{M}_T^{1/2}$ be a solution to (1-3). Then*

$$\|u\|^2_{\widetilde{L}^\infty_T H^{\frac{1}{2}}} \lesssim \|u_0\|^2_{H^{\frac{1}{2}}} + \|u\|^2_{\widetilde{L}^\infty_t H^{\frac{1}{2}}} \|u\|_{\widetilde{M}_T^{\frac{1}{2}}}. \tag{A-2}$$

Proof. We follow the proof of Proposition 4.5. Note that $\widetilde{M}^{1/2} \hookrightarrow \widetilde{Y}^{1/2}$. According to (4-15), it suffices to control

$$I = \sum_{N>0} \sum_{N_1 \gtrsim N} N\langle N_1\rangle \sup_{t\in]0,T[} |I_t(u_N, u_{\sim N_1}, u_{N_1})|.$$

It is easy to check that the only term of the left-hand side of (4-16) that causes trouble in the case $\alpha = 1$ is the first one. This term corresponds to the contribution of $Q_{\sim LNN_1^\alpha} u_{N_1}$ and $Q_{\sim NN_1^\alpha} u_{\sim N_1}$. For $\alpha = 1$, we control these contributions by applying Cauchy–Schwarz in (N, N_1). For instance, the contribution of $Q_{\sim LNN_1^\alpha} u_{N_1}$ is estimated, thanks to Lemma 4.4, by

$$\begin{aligned}\sum_{N\gg 1} \sum_{N_1 \gtrsim N} N\langle N_1\rangle \sum_{L>1} L^{-1} N^{-\frac{1}{2}} \|u_N\|_{L^2_{tx}} \|Q_{\sim LNN_1^\alpha} u_{N_1}\|_{F^{0,\frac{1}{2}}} \|u_{\sim N_1}\|_{L^\infty_t L^2_x} \\ \lesssim \sum_{L>1} L^{-1} \Bigl(\sum_{N_1 \gtrsim N \gg 1} \|u_N\|^2_{L^2_t H^{\frac{1}{2}}} \|u_{\sim N_1}\|^2_{L^\infty_t H^{\frac{1}{2}}} \Bigr)^{\frac{1}{2}} \Bigl(\sum_{N_1 \gtrsim N \gg 1} \|Q_{\sim LNN_1^\alpha} u_{N_1}\|^2_{F^{\frac{1}{2},\frac{1}{2}}} \Bigr)^{\frac{1}{2}} \\ \lesssim \|u\|_{L^2_t H^{\frac{1}{2}}} \|u\|_{\widetilde{L}^\infty_t H^{\frac{1}{2}}} \|u\|_{X^{-\frac{1}{2},1}}.\end{aligned}$$

□

Lemma A.3. *Let $0 < T < 2$ and let $u, v \in \widetilde{M}_T^{1/2}$ be two solutions to (1-3) on $]0, T[$. Then we have*

$$\|u - v\|_{Z_T^{-\frac{1}{2}}} \lesssim \|u - v\|_{L^\infty_T \overline{H}^{s-\frac{3}{2}+\frac{\alpha}{2}}} + \|u + v\|_{\widetilde{M}_T^{\frac{1}{2}}} \|u - v\|_{Z_T^{-\frac{1}{2}}} \tag{A-3}$$

and

$$\|u - v\|^2_{\widetilde{L}^\infty_T \overline{H}^{-\frac{1}{2}}} \lesssim \|u_0 - v_0\|^2_{\overline{H}^{-\frac{1}{2}}} + \|u + v\|_{\widetilde{M}_T^{\frac{1}{2}}} \|u - v\|_{\widetilde{L}^\infty_T \overline{H}^{-\frac{1}{2}}} \|u - v\|_{Z_T^{-\frac{1}{2}}}. \tag{A-4}$$

Proof. First we notice that (A-4) is already proven in Proposition 4.8, since $\widetilde{M}_T^{1/2} \hookrightarrow \widetilde{Y}_T^{1/2} \hookrightarrow Y_T^{1/2}$. It remains to prove (A-3). We follow the proof of Proposition 4.5. It is not too hard to check that the only contribution that causes troubles in the right-hand side of (4-21), in the case $\alpha = 1$, is the contribution of the low-high interaction term, $P_N(P_{\lesssim N} z w_N)$. We proceed as in Lemma A.1 . We take extensions $\tilde{z}$ and $\widetilde{w}$, supported in $]-4, 4[$, of z and w such that $\|\tilde{z}\|_{\widetilde{M}^{1/2}} \lesssim \|z\|_{\widetilde{M}_T^{1/2}}$ and $\|\widetilde{w}\|_{Z^{-1/2}} \lesssim \|w\|_{Z_T^{-1/2}}$. For simplicity we drop the tilde. We first notice that the contribution of $P_{\lesssim 1} z$ is easily estimated by

$$\|\partial_x P_N(P_{\lesssim 1} z w_{\sim N})\|_{F^{-\frac{1}{2},1,-\frac{1}{2}}} \lesssim \langle N\rangle^{-\frac{1}{2}} \|P_N(P_{\lesssim 1} z w_{\sim N})\|_{L^2_{tx}} \lesssim \|z\|_{L^\infty_t L^2_x} \|w_{\sim N}\|_{L^2_t H^{-\frac{1}{2}}},$$

which is acceptable. Now we decompose the remaining contribution as

$$\begin{aligned}
&\|\partial_x P_N(P_{\gg 1}P_{\lesssim N} z w_{\sim N})\|_{F^{-\frac12,1,-\frac12}} \\
&\quad\lesssim N\Big\|\sum_{1\ll N_1\lesssim N} P_N(P_{N_1} z w_{\sim N})\Big\|_{X^{-\frac32,0}} \\
&\quad\lesssim \langle N\rangle^{-\frac12}\Big\|\sum_{1\ll N_1\lesssim N} Q_{\sim NN_1}P_N(P_{N_1} z w_{\sim N})\Big\|_{L^2_{tx}} + \langle N\rangle^{-\frac12}\Big\|\sum_{1\ll N_1\lesssim N} Q_{\nsim NN_1}P_N(P_{N_1} z w_{\sim N})\Big\|_{L^2_{tx}} \\
&\quad= J_{1,N}+J_{2,N}.
\end{aligned}$$

By almost-orthogonality,

$$\begin{aligned}
J_{1,N} &\lesssim \langle N\rangle^{-\frac12}\Big(\sum_{1\ll N_1\lesssim N}\|Q_{\sim NN_1}P_N(P_{N_1} z w_{\sim N})\|^2_{L^2_{tx}}\Big)^{\frac12} \\
&\lesssim \langle N\rangle^{-\frac12}\Big(\sum_{1\ll N_1\lesssim N}\|P_{N_1} z\|^2_{L^2_t H^{\frac12}}\|w_{\sim N}\|^2_{L^\infty_{tx}}\Big)^{\frac12} \\
&\lesssim \|w_{\sim N}\|_{L^\infty_t H^{-\frac12}}\|z\|_{L^2_t H^{\frac12}},
\end{aligned}$$

which is acceptable. To treat J_2, we notice that, since the Fourier projectors ensure that $\langle \tau - p_2(\xi)\rangle \nsim NN_1$, the resonance relation (1-6) leads to $|\tau_1 - p_2(\xi_1)| \vee |\tau-\tau_1-p_2(\xi-\xi_1)| \gtrsim NN_1$ for $J_{2,N}$. We separate the contributions of $Q_{\gtrsim NN_1} z_{N_1}$ and $Q_{\gtrsim NN_1} w_{\sim N}$. For the first contribution, we write

$$\begin{aligned}
J_{2,N} &\lesssim \langle N\rangle^{-\frac12}\sum_{1\ll N_1\lesssim N} N_1^{\frac12}\|Q_{\gtrsim NN_1}P_{N_1} z\|_{L^2_{tx}}\|w_{\sim N}\|_{L^\infty_t L^2_x} \\
&\lesssim \langle N\rangle^{-\frac12}\sum_{1\ll N_1\lesssim N} (NN_1)^{-\frac14}N_1^{\frac14}\|Q_{\gtrsim NN_1}P_{N_1} z\|_{X^{\frac14,\frac14}}\|w_{\sim N}\|_{L^\infty_t L^2_x} \\
&\lesssim \|z\|^{\frac14}_{X^{-\frac12,1}}\|z\|^{\frac34}_{L^\infty_t H^{\frac12}}\|w_{\sim N}\|_{L^\infty_t H^{-\frac12}},
\end{aligned}$$

which is acceptable. For the second contribution, according to (4-10), we have

$$\begin{aligned}
J_2 &\lesssim \langle N\rangle^{-\frac12}\sum_{1\ll N_1\lesssim N}\|z_{N_1}\|_{L^\infty_t H^{\frac12}}\|Q_{\gtrsim NN_1} w_{\sim N}\|_{L^2_{tx}} \\
&\lesssim \langle N\rangle^{-\frac12}\sum_{1\ll N_1\lesssim N}(NN_1)^{-1}N^{\frac32}\|z_{N_1}\|_{L^\infty_t H^{\frac12}}\|w_{\sim N}\|_{F^{-\frac12,\frac12}} \\
&\lesssim \|w_{\sim N}\|_{F^{-\frac12,\frac12}}\|z\|_{L^\infty_t H^{\frac12}},
\end{aligned}$$

which is acceptable. Gathering the above estimates we obtain (A-3). □

Gathering Lemmas A.1–A.3 and proceeding as in Section 4B we obtain the local well-posedness in $H^{1/2}$ of (1-3) for $\alpha = 1$. Note that the uniqueness holds in the space $\widetilde{M}^{1/2}_T$.

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