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NONEXISTENCE OF SMALL DOUBLY PERIODIC SOLUTIONS FOR DISPERSIVE EQUATIONS

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NONEXISTENCE OF SMALL DOUBLY PERIODIC SOLUTIONS
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DAVID M. AMBROSE AND J. DOUGLAS WRIGHT

We study the question of existence of time-periodic, spatially periodic solutions for dispersive evolution equations, and in particular, we introduce a framework for demonstrating the nonexistence of such solutions. We formulate the problem so that doubly periodic solutions correspond to fixed points of a certain operator. We prove that this operator is locally contracting, for almost every temporal period, if the Duhamel integral associated to the evolution exhibits a weak smoothing property. This implies the nonexistence of nontrivial, small-amplitude time-periodic solutions for almost every period if the smoothing property holds. This can be viewed as a partial analogue of scattering for dispersive equations on periodic intervals, since scattering in free space implies the nonexistence of small coherent structures. We use a normal form to demonstrate the smoothing property on specific examples, so that it can be seen that there are indeed equations for which the hypotheses of the general theorem hold. The nonexistence result is thus established through the novel combination of small-divisor estimates and dispersive smoothing estimates. The examples treated include the Korteweg–de Vries equation and the Kawahara equation.

1. Introduction

In the absence of the ability to "explicitly" compute solutions of the Cauchy problem for a nonlinear dispersive system by some specialized technique particular to the equation at hand (such as complete integrability), coherent structures often form the backbone for both qualitative and quantitative descriptions of the dynamics of the system. Such structures, be they traveling waves, self-similar solutions, time-periodic solutions or some other sort of solution, give great insight into the short-time behavior of the system and often provide possible states towards which solutions trend as time goes to infinity.

For dispersive equations in free space, many authors have proved scattering results; we cannot hope to list all such results here, but a sampling is [Christ and Weinstein 1991; Ginibre and Velo 1986; Liu 1997; Ponce and Vega 1990; Strauss 1974]. For instance, Strauss [1974] showed that for a generalized Korteweg–de Vries (KdV) equation and for a nonlinear Schrödinger equation, all sufficiently small solutions decay to zero. There is no generally agreed upon meaning for scattering on periodic intervals, and one cannot expect

decay of solutions. However, decay of solutions implies the nonexistence of small-amplitude coherent struc-
tures, and the nonexistence of small coherent structures is a question which can be studied on periodic in-
tervals. In the present work, we study the nonexistence of small time-periodic solutions for dispersive equa-
tions on periodic intervals. We prove a general theorem, showing that the existence of sufficiently strong
dispersive smoothing effects implies the nonexistence of small doubly periodic solutions for almost every
temporal period; we then demonstrate the required dispersive smoothing for particular examples. We have
partially described the results of the present work briefly in the announcement [Ambrose and Wright 2014].

One method of constructing doubly periodic solutions of dispersive equations is to use Nash–Moser-
type methods. These methods typically work in a function space with periodic boundary conditions
in both space and time, so that the Fourier transform of the evolution equation can be taken in both
variables. Then, a solution is sought nearby to an equilibrium solution. The implicit function theorem
cannot be used due to the presence of small divisors. Instead, the small divisors are compensated for
by the fast convergence afforded by Newton's method. A version of such arguments is now known
as the Craig–Wayne–Bourgain method [Bourgain 1994; 1995a; Craig and Wayne 1993; Wayne 1990],
which they used to demonstrate existence of doubly periodic solutions for a number of equations, such as
nonlinear wave and nonlinear Schrödinger equations.

Such methods have since been extended to other systems, such as the irrotational gravity water wave, on
finite or infinite depth, by Plotnikov, Toland, and Iooss [Plotnikov and Toland 2001; Iooss et al. 2005], or
irrotational gravity-capillary waves by Alazard and Baldi [2015]. Also, Baldi [2013] used such methods to
demonstrate existence of doubly periodic solutions for perturbations of the Benjamin–Ono equation. The
typical result of these small-divisor methods is the existence of small-amplitude doubly periodic waves
for the system under consideration, for certain values of the relevant parameters. One such parameter is
the frequency (or equivalently, the temporal period) of the solution; other parameters may arise in specific
applications, such as the surface tension parameter in [Alazard and Baldi 2015]. With these methods, the
parameter values for which solutions are shown to exist are typically in a Cantor set.

The most classical version of small-divisor theory is due to Kolmogorov, Arnold, and Moser (KAM).
KAM theory has been used by Kuksin [1988; 1998] to show that quasiperiodic solutions of the KdV
equation persist under certain perturbations; see also the book of Kappeler and Pöschel [2003] and the
references therein. Our work is complementary to these approaches, as we offer a nonexistence theory.
That is, we give certain regions of frequency-amplitude space in which doubly periodic solutions cannot
exist, whereas KAM theory gives instead regions where such solutions do in fact exist. Furthermore, we
mention that the method of the present work requires no special structure on the equations; we only need
certain estimates to be satisfied to conclude nonexistence of small doubly periodic solutions, and these
are likely unrelated to any integrable or near-integrable structure.

For certain completely integrable equations, time-periodic waves can be shown to exist by producing
exact, closed-form solutions. This is the case, for instance, for the KdV equation [Dubrovin 1975] and the
Benjamin–Ono equation [Dobrokhotov and Krichever 1991; Matsuno 2004; Satsuma and Ishimori 1979].
The first author and Wilkening found that these time-periodic solutions of the Benjamin–Ono equation
form continuous families [Ambrose and Wilkening 2009; 2010; Wilkening 2008]; this is in sharp contrast

to results proved by small-divisor methods, then. The small-divisor methods do not address the question of whether such results are optimal; it is not possible at present to conclude whether or not time-periodic solutions exist as continuous families for such equations.

Thus, we find it of significant interest to develop further tools to answer the questions of both existence and nonexistence of doubly periodic waves for dispersive partial differential equations. In the present contribution, we develop a framework by which the nonexistence of small-amplitude time-periodic waves can be established. We do this by first using the Duhamel formula together with the time-periodic ansatz. Then we rewrite this formula, factoring out the linear operator. The resulting equation yields the notion of time-periodic solutions as fixed points of a new operator, which is the composition of the inverse of the linear operator with the Duhamel integral.

We then prove a general theorem, showing that under certain conditions, this operator is contracting in a neighborhood of the origin in a certain function space. Since the operator is the composition of two operators, we prove estimates for these individual operators separately. We are able to prove that the inverse of the linear operator acts like differentiation of order p for some $p > 1$. Of course, in order to have the contracting property, the composition must map from some function space, X, to itself. We use Sobolev spaces, so since the inverse of the linear operator acts like differentiation of order p, the Duhamel integral must satisfy an estimate with some form of a gain at least p derivatives. In Theorem 4, then, we have a general condition for the nonexistence of small-amplitude time-periodic waves for almost every possible temporal period: if the Duhamel integral possesses a weak form of smoothing (with an associated estimate), then the equation does not possess arbitrarily small doubly periodic waves for almost every period. The results described thus far are the content of Section 2. We mention that we are aware of one other result in the literature on nonexistence of small time-periodic solutions for almost every period; de la Llave [2000] uses a variational method to demonstrate nonexistence of small doubly periodic solutions for nonlinear wave equations. The methods of proof of the current work and of [de la Llave 2000] appear to be quite different, since in the present work we rely on smoothing estimates which derive from the linear part of the evolution; such smoothing estimates would not be expected to hold in the case of wave equations.

Clearly, we must address the question of whether there is an equation for which the truth of the hypotheses of Theorem 4 can be demonstrated. We demonstrate the required smoothing for the Duhamel integral associated to some dispersive evolution equations with fifth-order dispersion in Sections 3 and 4.1, and with seventh-order dispersion in Section 4.2. For these equations, the Duhamel integral satisfies a strong smoothing property: the Duhamel integral gains $p > 1$ derivatives, compensating for the loss of derivatives from the inverse of the linear operator. Theorem 4 does not, however, require so strong a smoothing property. In Section 5, we demonstrate that the Duhamel integral for the KdV equation satisfies a weaker smoothing property, allowing Theorem 4 to be applied and demonstrating the nonexistence of small doubly periodic solutions for almost every temporal period.

The estimates for the inverse of the linear operator (Lemma 1, Lemma 2, and Corollary 3 below), in which we demonstrate that the inverse of our linear operator acts like differentiation of order p, are proved by small-divisor techniques. In fact, these are versions of classical results, such as can be found, for instance, in [Ghys 2007]. As with all small-divisor results, some parameter values are discarded;

in the present case, the parameter is the temporal period of the solution. Thus, we arrive at a result about nonexistence of small solutions for almost every possible temporal period. Even though such a small-divisor argument is classical, we provide our own proof because the detailed information about the set of temporal periods in the proof is helpful.

For our particular examples of dispersive equations, we prove smoothing estimates for the Duhamel integral by following the lines of an argument by Erdoğan and Tzirakis [2013]. By using a normal form representation, they showed that the Duhamel integral for the KdV equation gains $1 - \varepsilon$ derivatives as compared to the initial data, for any $\varepsilon > 0$. (We mention that a similar result has been demonstrated on the real line by Linares and Scialom [1993].) For the equations with fifth-order dispersion which we consider in Sections 3 and 4.1, we find a gain of two derivatives. This is in line with the usual, expected gain of regularity from dispersion. In general, if the dispersion relation is of order r (say, for instance, the linearized evolution equation is, in the Fourier transform, $\hat{w}_t = i k^r \hat{w}$), then one expects to gain $(r-1)/2$ derivatives in some sense [Kenig et al. 1991]; this is known as the Kato smoothing effect [1983]. With fifth-order dispersion, this means the expected gain is two derivatives. Given the results of [Erdoğan and Tzirakis 2013] as well as the present work, it does appear that it is reasonable to expect the same order of smoothing in the spatially periodic setting, for the Duhamel integral. In fact, in Section 4.2, we show that for an evolution equation with seventh-order dispersion, the gain of regularity on the Duhamel integral is four derivatives; thus, the smoothing effect in [Erdoğan and Tzirakis 2013] and the present work is not the same as Kato smoothing, but is still due to the presence of dispersion. To demonstrate our weaker smoothing estimate for the KdV equation in Section 5, we begin with the same normal form as before, but we estimate the terms differently.

We close with some discussion in Section 6.

2. Nonexistence of doubly periodic solutions

We begin with the evolution equation

$$\partial_t u = Au + Nu, \tag{1}$$

where A is a linear operator and N is a nonlinear operator. Then, the solution of (1) with initial data $u(\,\cdot\,, 0) = u_0$, if there is a solution, can be represented with the usual Duhamel formula,

$$u(\,\cdot\,, t) = e^{At} u_0 + \int_0^t e^{A(t-\tau)} N(u(\,\cdot\,, \tau)) \, d\tau. \tag{2}$$

Given a time t, we define the linear solution operator $S_L(t) = e^{At}$ and the difference of the solution operator and the linear solution operator to be $S_D(t)$; thus, S_D is exactly the Duhamel integral:

$$S_D(t) u_0 = \int_0^t e^{A(t-\tau)} N(u(\,\cdot\,, \tau)) \, d\tau.$$

We work in the spatially periodic case, so we assume that solutions u of (1) satisfy

$$u(x + 2\pi, t) = u(x, t) \quad \forall x \in \mathbb{R}.$$

We assume that (1) maintains the mean of solutions; that is, given any u in a reasonable function space, we have

$$\int_0^{2\pi} u(x,t)\,dx = \int_0^{2\pi} u_0(x)\,dx.$$

For the remainder of the present section, we will assume that u_0 (and thus $u(\cdot,t)$) has mean zero.

If u_0 is the initial data for a time-periodic solution of (1) with temporal period T, then we have

$$u_0 = S_L(T)u_0 + S_D(T)u_0.$$

We rewrite this as

$$\big(I - S_L(T) - S_D(T)\big)u_0 = 0. \tag{3}$$

Our goal is to demonstrate nonexistence of small-amplitude doubly periodic solutions of (3) for certain temporal periods. We begin now to focus only on certain values of T, and our first restriction on values of T is to ensure that $I - S_L(T)$ is invertible. For $s > 0$, define the space H_0^s to be the subset of the usual spatially periodic Sobolev space H^s such that for all $f \in H_0^s$, the mean of f is equal to zero. We assume that the operator $S_L(t)$ is bounded,

$$S_L(t) : H_0^s \to H_0^s \quad \forall t \in \mathbb{R}.$$

Then, we define the set W to be

$$W = \big\{t \in (0,\infty) : \ker(I - S_L(t)) = \{0\}\big\}.$$

For any $T \in W$, we rewrite (3) by factoring out $I - S_L(T)$:

$$(I - S_L(T))\big(I - (I - S_L(T))^{-1}S_D(T)\big)u_0 = 0.$$

We see, then, that if u_0 is the initial data for a time-periodic solution of (1) with temporal period $T \in W$, then u_0 is a fixed point of the operator

$$K(T) := (I - S_L(T))^{-1}S_D(T).$$

If we can show that this is (locally) a contraction on H_0^s, then there are no (small) nontrivial time-periodic solutions in the space H_0^s with temporal period T. To establish this, we will need estimates both for $(I - S_L(T))^{-1}$ and for $S_D(T)$. In Sections 2.1 and 2.2, we establish estimates for $(I - S_L(T))^{-1}$; the results are that the symbol can be bounded as $|k|^p$, where k is the variable in Fourier space for some $p > 1$, for certain values of T. Thus, the inverse of the linear operator acts like differentiation of order $p > 1$. In Section 2.3, then, we will state a corollary of these estimates: if the operator S_D satisfies a certain estimate related to a gain of p derivatives, then the operator $K(T)$ is locally contracting, and thus there are no small time-periodic solutions with temporal period T.

2.1. *The linear estimate: the homogeneous case.* We now prove our estimate for $(I - S_L(T))^{-1}$ in the case that the linear operator A has symbol

$$\mathcal{F}(A)(k) = ik^r \quad \forall k \in \mathbb{Z}, \tag{4}$$

with r being an odd integer. This estimate is the content of Lemma 1. We note that this is not, strictly speaking, useful to us, as we will prove a version of the lemma for a more general class of operators A in the following section. However, we include Lemma 1 because the simplicity of the form of (4) allows for clarity of exposition. The proof of Lemma 2 below will build upon the proof of Lemma 1, which we now present.

We mention that the restriction that r is an odd integer is for two reasons, neither of which is essential. First, since we are interested in equations of KdV-type, these are the natural values of r to consider. Second, the restriction is for ease of exposition, since Lemma 1 would be equally true if we allowed the symbol of A to be defined by

$$\mathcal{F}(A)(k) = i|k|^{r_1} k^{r_2} \tag{5}$$

for all $k \in \mathbb{Z}$, with $r_1 \in (0, \infty)$ and $r_2 \in \{0, 1\}$. The presentation, however, is slightly streamlined if we use (4) instead of (5).

Lemma 1. *Let the linear operator A be given by (4). Let $0 < T_1 < T_2$ be given. Let $0 < \delta < T_2 - T_1$ be given. Let $p > 1$ be given. There exists a set $W_{p,\delta} \subseteq [T_1, T_2] \cap W$ and there exists $c_1 > 0$ such that the Lebesgue measure of $W_{p,\delta}$ satisfies $\mu(W_{p,\delta}) > T_2 - T_1 - \delta$, and for all $k \in \mathbb{Z} \setminus \{0\}$, for all $T \in W_{p,\delta}$, we have*

$$\left| \mathcal{F}(I - S_L(T))^{-1}(k) \right| < c_1 |k|^p.$$

Proof. For the moment, we fix $k \in \mathbb{Z} \setminus \{0\}$. We need to estimate

$$\left| \mathcal{F}(I - S_L(T))^{-1}(k) \right| = \left| \frac{1}{1 - \exp\{ik^r T\}} \right| = \frac{1}{\sqrt{2}} (1 - \cos(k^r T))^{-1/2}. \tag{6}$$

Clearly, the symbol of the inverse operator is undefined if there exists $n \in \mathbb{N}$ such that $T = 2\pi n / |k|^r$. With the assumption that $T \in [T_1, T_2]$, the associated values of n comprise the set

$$\mathcal{N} := \left[\frac{|k|^r T_1}{2\pi}, \frac{|k|^r T_2}{2\pi} \right] \cap \mathbb{N}.$$

We remove a small set of possible periods around each of these values; that is to say, we consider

$$T \in [T_1, T_2] \setminus \bigcup_{n \in \mathcal{N}} \left[\frac{2\pi n}{|k|^r} - \varepsilon, \frac{2\pi n}{|k|^r} + \varepsilon \right] \tag{7}$$

for some $0 < \varepsilon \ll 1$ to be specified.

To start, we may notice that the collection of intervals $\left[2\pi n / |k|^r - \varepsilon, 2\pi n / |k|^r + \varepsilon \right]$ do not overlap for different values of n as long as, for all $n \in \mathcal{N}$, we have

$$\frac{2\pi n}{|k|^r} + \varepsilon < \frac{2\pi (n+1)}{|k|^r} - \varepsilon.$$

This is satisfied as long as $\varepsilon < \pi / |k|^r$. When we choose ε, this condition will be satisfied.

A simple calculation shows that, on an interval of values of θ which does not include an integer multiple of 2π, the value of $(1 - \cos \theta)$ is minimized at the endpoints of the interval. Thus, the value of (6) is

largest on our set of possible periods at the values $T = 2\pi n/|k|^r \pm \varepsilon$ for $n \in \mathcal{N}$. At such values, we find

$$\left| \mathcal{F}\left(\left(I - S_L\left(\frac{2\pi n}{|k|^r} \pm \varepsilon\right)\right)^{-1}\right)(k) \right| = \frac{1}{\sqrt{2}\left(1 - \cos\left(k^r\left(\frac{2\pi n}{|k|^r} \pm \varepsilon\right)\right)\right)^{1/2}}$$

$$= \frac{1}{\sqrt{2}(1 - \cos(\pm 2\pi n \pm \varepsilon k^r))^{1/2}} = \frac{1}{\sqrt{2}(1 - \cos(\varepsilon k^r))^{1/2}}. \quad (8)$$

We will now perform a Taylor expansion for cosine, paying attention to the error estimates. For any $\theta \in \mathbb{R}$, we have the formula

$$\cos \theta = 1 - \tfrac{1}{2}\theta^2 + \tfrac{1}{6}\sin(\xi)\theta^3$$

for some ξ between 0 and θ. We notice that $\left|\tfrac{1}{6}\sin(\xi)\theta^3\right| \leq \tfrac{1}{4}\theta^2$, as long as $|\theta| \leq \tfrac{3}{2}$. Thus, for $|\theta| \leq \tfrac{3}{2}$, we have

$$|1 - \cos \theta| = \left|\tfrac{1}{2}\theta^2 - \tfrac{1}{6}\sin(\xi)\theta^3\right| \geq \tfrac{1}{4}\theta^2. \quad (9)$$

Combining (9) with (8), we find that for any T satisfying (7), if $\varepsilon|k|^r < \tfrac{3}{2}$, then we have

$$\left| \mathcal{F}\left((I - S_L(T))^{-1}(k)\right) \right| \leq \frac{1}{(\sqrt{2})\left(\tfrac{1}{2}\varepsilon|k|^r\right)} = \frac{\sqrt{2}}{\varepsilon|k|^r}. \quad (10)$$

We choose $\varepsilon = c_0|k|^{-p-r}$. This choice of ε immediately yields the claimed estimate for the symbol. Recall that we have specified $p > 1$; the positive constant c_0 is to be specified. The conditions we have placed on ε above are (1) $\varepsilon < \pi/|k|^r$ and (2) $\varepsilon|k|^r < \tfrac{3}{2}$. These conditions are both satisfied as long as $c_0 < \tfrac{3}{2}$.

For fixed $k \in \mathbb{Z} \setminus \{0\}$ and for fixed $n \in \mathcal{N}$, we have removed a set of measure $2\varepsilon = 2c_0|k|^{-r-p}$ from the interval $[T_1, T_2]$. Since $\mathcal{N}$ is the intersection of an interval with the natural numbers, we see that the cardinality of $\mathcal{N}$ is less than $|k|^r (T_2 - T_1) + 1$, so for fixed k, we have removed a set of measure no more than $2c_0(1 + T_2 - T_1)|k|^{-p}$. Summing over k, since we have chosen $p > 1$, the measure of the set we have removed is finite. Taking c_0 sufficiently small, we can conclude that the measure of the set which is removed has Lebesgue measure smaller than δ. To be definite, we write the definition of the set $W_{p,\delta}$, which is

$$W_{p,\delta} = [T_1, T_2] \setminus \bigcup_{k \in \mathbb{Z} \setminus \{0\}} \bigcup_{n \in \mathcal{N}} \left[\frac{2\pi n}{|k|^r} - \frac{c_0}{|k|^{r+p}}, \frac{2\pi n}{|k|^r} + \frac{c_0}{|k|^{r+p}}\right],$$

where c_0 is chosen so that $0 < c_0 < \tfrac{3}{2}$, and also so that

$$c_0 < \frac{\delta}{2(1 + T_2 - T_1)\sum_{k \in \mathbb{Z} \setminus \{0\}} |k|^{-p}}.$$

Finally, the constant c_1 is given by $c_1 = \sqrt{2}/c_0$. This completes the proof. $\square$

We note that we can see clearly the dependence of the constant c_1 on the parameters T_1, T_2, p, and δ. If we want a larger set of potential periods, we could take $T_2 - T_1$ larger or δ smaller; this would result in a larger value of c_1. Choosing smaller values of $p > 1$ also leads to larger values of c_1.

2.2. The linear estimate: the nonresonant case. The estimate of Section 2.1 can be generalized to allow operators which include lower-order terms, in some cases: there must not be a resonance between the different terms, in the sense that we require $\mathcal{F}A(k) \neq 0$ for any nonzero $k \in \mathbb{Z}$. To be very precise, we consider linear operators A which satisfy the conditions (**H**), which we now describe.

(**H**) Let $M \in \mathbb{N}$ be given, with $M \geq 2$. For each $m \in \{1, 2, \ldots, M\}$, let r_m be an odd integer, with these r_m satisfying $r_1 > r_2 > \cdots > r_M > 0$. For all $m \in \{1, 2, \ldots, M\}$, let $Z_m \subseteq \mathbb{R}$ be bounded. Let $Z = Z_1 \times Z_2 \times \cdots \times Z_M$. Let $\vec{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_M)$. Assume there exists $\beta_1 > 0$ and $\beta_2 > 0$ such that for all $\alpha_1 \in Z_1$,

$$|\alpha_1| > \beta_1, \tag{11}$$

and for all $\vec{\alpha} \in Z$, for all $k \in \mathbb{Z} \setminus \{0\}$,

$$\left| \sum_{m=1}^{M} \alpha_m k^{r_m} \right| \geq \beta_2. \tag{12}$$

Given $\vec{\alpha} \in Z$, let the linear operator A be defined through its symbol as

$$\mathcal{F}(A)(k) = i \sum_{m=1}^{M} \alpha_m k^{r_m} \quad \forall k \in \mathbb{Z}. \tag{13}$$

We again remark that the condition that the r_m be odd integers is for ease of exposition and because these are the relevant values for the KdV equation and for the other equations (such as Kawahara equations) which we consider in the present work; as mentioned previously, this restriction can be relaxed with no difficulty. We further remark that the condition (11) ensures that the equation is dispersive of order r_1 (i.e., the leading-order term is of the same order for all $\vec{\alpha} \in Z$). The condition (12) ensures that the symbol never vanishes.

Lemma 2. *Let the set Z and the linear operator A satisfy the hypotheses* (**H**), *so that in particular*, A *is defined by* (13). *Let* $0 < T_1 < T_2$ *be given. Let* $0 < \delta < T_2 - T_1$ *be given. Let* $p > 1$ *be given. There exists a set* $W_{p,\delta} \subseteq [T_1, T_2] \cap W$ *and there exists* $c_1 > 0$ *such that the Lebesgue measure of* $W_{p,\delta}$ *satisfies* $\mu(W_{p,\delta}) > T_2 - T_1 - \delta$, *and such that for all* $\vec{\alpha} \in Z$, *for all* $k \in \mathbb{Z} \setminus \{0\}$, *for all* $T \in W_{p,\delta}$, *we have*

$$\left| \mathcal{F}(I - S_L(T))^{-1}(k) \right| < c_1 |k|^p.$$

Proof. The proof of Lemma 1 can be repeated, with k^r replaced in every instance by $\sum_{m=1}^{M} \alpha_m k^{r_m}$, until (10); to be concise, we introduce the notation

$$\lambda_k = \sum_{m=1}^{M} \alpha_m k^{r_m}.$$

We choose $\varepsilon = c_0 |k|^{-p-r_1}$, with c_0 to be specified.

Similarly to the previous case, we have the conditions (i) $\varepsilon < \pi |\lambda_k|^{-1}$, and (ii) $\varepsilon |\lambda_k| < \frac{3}{2}$. The relevant product satisfies

$$|\varepsilon \lambda_k| = c_0 \left| \sum_{m=1}^{M} \alpha_m k^{r_m} |k|^{-r_1-p} \right| \leq c_0 \sum_{m=1}^{M} |\alpha_m| \quad \forall k \in \mathbb{Z} \setminus \{0\}.$$

Thus, recalling that the sets Z_m are bounded, we can clearly take c_0 sufficiently small to satisfy conditions (i) and (ii).

We must revisit the estimate (10) in the present setting. We have

$$\left| \mathcal{F}(I - S_L(T))^{-1}(k) \right| \le \frac{\sqrt{2}}{\varepsilon |\lambda_k|} = \frac{\sqrt{2}}{c_0} \frac{|k|^p}{|\lambda_k k^{-r_1}|}.$$

Since λ_k is never equal to zero, we conclude that $\lambda_k k^{-r_1}$ is also never equal to zero for $k \in \mathbb{Z} \setminus \{0\}$. Furthermore, there exists $K \in \mathbb{N}$ such that for all $k \in \mathbb{Z}$ satisfying $|k| > K$, for all $\vec{\alpha} \in Z$, we have

$$\left| \sum_{m=2}^{M} \alpha_m k^{r_m - r_1} \right| \le |K|^{r_2 - r_1} \sum_{m=2}^{M} |\alpha_m| < \tfrac{1}{2}\beta_1.$$

This implies that for all $k \in \mathbb{Z}$ with $|k| > K$, for all $\vec{\alpha} \in Z$,

$$|\lambda_k k^{-r_1}| = \left| \alpha_1 + \sum_{m=2}^{\infty} \alpha_m k^{r_m - r_1} \right| \ge \tfrac{1}{2}\beta_1 > 0,$$

$$|\lambda_k k^{-r_1}| = \left| \alpha_1 + \sum_{m=2}^{\infty} \alpha_m k^{r_m - r_1} \right| \le \left(\sup_{\alpha_1 \in Z_1} |\alpha_1| \right) + \tfrac{1}{2}\beta_1.$$

Furthermore, for any $k \in \mathbb{Z} \setminus \{0\}$ satisfying $|k| \le K$, we know

$$\inf_{\vec{\alpha} \in Z} \frac{|\lambda_k|}{|k^{r_1}|} = \inf_{\vec{\alpha} \in Z} |\lambda_k k^{-r_1}| \in (0, \infty),$$

$$\sup_{\vec{\alpha} \in Z} \frac{|\lambda_k|}{|k^{r_1}|} = \sup_{\vec{\alpha} \in Z} |\lambda_k k^{-r_1}| \in (0, \infty).$$

Combining this information, we see that

$$\inf_{k \in \mathbb{Z} \setminus \{0\}} \inf_{\vec{\alpha} \in Z} |\lambda_k k^{-r_1}| \in (0, \infty),$$

$$\sup_{k \in \mathbb{Z} \setminus \{0\}} \sup_{\vec{\alpha} \in Z} |\lambda_k k^{-r_1}| \in (0, \infty).$$

Our value of c_1 is therefore

$$c_1 = \frac{\sqrt{2}}{c_0} \left(\inf_{k \in \mathbb{Z} \setminus \{0\}} \inf_{\vec{\alpha} \in Z} |\lambda_k k^{-r_1}| \right)^{-1}.$$

In the current setting, for each $\vec{\alpha} \in Z$, for each $k \in \mathbb{Z} \setminus \{0\}$, the set $\mathcal{N}$ is defined by

$$\mathcal{N} = \left[\frac{T_1 |\lambda_k|}{2\pi}, \frac{T_2 |\lambda_k|}{2\pi} \right] \cap \mathbb{N}.$$

Thus, for all $\vec{\alpha} \in Z$, for all $k \in \mathbb{Z} \setminus \{0\}$, the cardinality of $\mathcal{N}$ satisfies

$$\text{card}(\mathcal{N}) \le (T_2 - T_1)|\lambda_k| + 1 \le (T_2 - T_1)|k|^{r_1} |\lambda_k k^{-r_1}| + 1$$

$$\le (T_2 - T_1)|k|^{r_1} \left(\sup_{k \in \mathbb{Z} \setminus \{0\}} \sup_{\vec{\alpha} \in Z} |\lambda_k k^{-r_1}| \right) + 1.$$

We then take the product $2\varepsilon \operatorname{card}(\mathcal{N})$, finding the estimate

$$2\varepsilon \operatorname{card}(\mathcal{N}) \leq 2c_0\big(1 + (T_2 - T_1) \sup_{k\in\mathbb{Z}\setminus\{0\}} \sup_{\vec{\alpha}\in Z} |\lambda_k k^{-r_1}|\big)|k|^{-p}.$$

As in the proof of Lemma 1, we sum over k, and we find that we can take c_0 sufficiently small to satisfy the remaining claims of the lemma. $\square$

Lemma 2 allows for two kinds of uniformity: the same set $W_{p,\delta}$ works for all $\vec{\alpha} \in Z$, and the constant c_1 is able to be used for all $T \in W_{p,\delta}$. The cost of this uniformity with respect to the constant c_1 is that the set $W_{p,\delta}$ does not have full measure. By sending δ to zero, we can achieve an estimate for almost every $T \in [T_1, T_2]$, but then the constant will depend on the choice of T. In doing this, we are able to maintain the uniformity with respect to the set Z. This is the content of the following corollary.

Corollary 3. *Let the set Z and the linear operator A satisfy the hypotheses* (**H**), *with A given by* (13). *Let $p > 1$ be given. Let $0 < T_1 < T_2$ be given. For almost every $T \in [T_1, T_2]$, there exists $c > 0$ such that for all $\vec{\alpha} \in Z$ and for all $k \in \mathbb{Z} \setminus \{0\}$, we have the estimate*

$$\big|\mathcal{F}\big((I - S_L(T))^{-1}\big)(k)\big| \leq c|k|^p.$$

Proof. For any δ satisfying $0 < \delta < T_2 - T_1$, let the set $W_{p,\delta}$ be as in Lemma 2. For any $T \in [T_1, T_2]$, if there exists a value of δ such that $T \in W_{p,\delta}$, then the desired estimate is satisfied. Let

$$W_p = \bigcup_{n=2}^{\infty} W_{p,(T_2-T_1)/n}.$$

Then, for all $T \in W_p$, the estimate holds. Since for all $n \geq 2$ we have $W_{p,(T_2-T_1)/n} \subseteq W_p \subseteq [T_1, T_2]$, and since $\mu(W_{p,(T_2-T_1)/n}) \geq (T_2-T_1)(1-1/n)$, we see that $\mu(W_p) = T_2-T_1$. This completes the proof. $\square$

2.3. *The general theorem.* We are now able to state a general theorem which follows from the above discussion. In Theorem 4, when we say that N is "as above", this includes the property that the evolution equation (1) preserves the mean value of the initial data. The theorem contains two statements about nonexistence of small-amplitude time-periodic solutions. The first statement is for a given $\delta > 0$; for T in the set $W_{p,\delta}$, we conclude that there is a uniform threshold for the amplitude of time-periodic solutions. For the second statement, we conclude that for almost any $T \in [T_1, T_2]$, there is a threshold for the amplitude of time-periodic solutions; this second statement is not uniform. These results are conditional on the existence of smoothing estimates. In Sections 3, 4.1, 4.2, and 5, we will demonstrate the required smoothing estimate for particular equations.

Theorem 4. *Let $0 < T_1 < T_2$ be given, and let $0 < \delta < T_2 - T_1$ be given. Let the set Z and the operator A satisfy the hypotheses* (**H**), *with A given by* (13). *Let the nonlinear operator N be as above. Assume there exists $p > 1$, $\tilde{p} \geq 0$, $q > 0$, $s \geq 0$, $c > 0$, and $\eta > 0$ such that for all $u_0 \in H_0^{s+\tilde{p}}$ with $\|u_0\|_{H^{s+\tilde{p}}} \leq \eta$, for all $\vec{\alpha} \in Z$, the following estimate is satisfied*:

$$\|S_D(T)u_0\|_{H^{s+p}} \leq c\|u_0\|_{H^s}\|u_0\|_{H^{s+\tilde{p}}}^q \tag{14}$$

for all $T \in W_{p,\delta}$. Then, there exists $r_0 > 0$ such that if u is a smooth, nontrivial, mean-zero time-periodic solution of (1) *with temporal period $T \in W_{p,\delta}$, then*

$$\inf_{t \in [0,T]} \|u\|_{H^{s+\tilde{p}}} \geq r_0. \tag{15}$$

Furthermore, if (14) *holds for every $T \in W_p$, then for every $T \in W_p$, there exists $r_0 > 0$ such that if u is a nontrivial mean-zero time-periodic solution of* (1) *with temporal period T, then* (15) *holds.*

Proof. The assumptions of the theorem, together with either Lemma 2 or Corollary 3, imply that there exists $C > 0$ such that for all $u_0 \in H_0^{s+\tilde{p}}$ satisfying $\|u_0\|_{H^{s+\tilde{p}}} < \eta$,

$$\|K(T)u_0\|_{H^s} \leq C \|u_0\|_{H^s} \|u_0\|_{H^{s+\tilde{p}}}^q.$$

If u_0 satisfies

$$0 < C \|u_0\|_{H^{s+\tilde{p}}}^q < 1,$$

then $\|K(T)u_0\|_{H^s} < \|u_0\|_{H^s}$, and thus u_0 cannot be a fixed point of $K(T)$. Thus, the only fixed point in a ball around zero is zero. □

Remark 5. In the announcement [Ambrose and Wright 2014], the version of this theorem which appeared was restricted to the case $\tilde{p} = 0$. In this case, the inverse of the linear operator acts like differentiation of order $p > 1$, and the Duhamel integral has a compensating gain of p derivatives. In Sections 3 and 4, we will give examples for which this smoothing property holds; these examples include the Kawahara equation. However, as the estimate (14) shows, what is needed is actually much weaker than the Duhamel integral gaining p derivatives; instead, it is only necessary that $s + p$ derivatives of the Duhamel integral satisfy a nonlinear estimate in which one factor involves only s derivatives. In Section 5, we will demonstrate that the estimate (14) holds with $\tilde{p} > 0$ for the KdV equation.

3. Application to a fifth-order dispersive equation

In this section, we will apply the above results to a specific dispersive equation, with sufficiently strong dispersion, with $\tilde{p} = 0$. We are using a version of the Erdoğan–Tzirakis argument [2013] for the equation

$$\partial_t \tilde{u} = \partial_x^5 \tilde{u} - 2\tilde{u}\,\partial_x \tilde{u} + \tilde{\omega}\,\partial_x \tilde{u} \tag{16}$$

(for any $\tilde{\omega} \in \mathbb{R}$) to get the desired smoothing effect. As we have discussed, we consider the spatially periodic case, with spatial period equal to 2π. We first rewrite (16) to remove the mean, and also to remove the tildes.

We consider the initial condition $\tilde{u}(x, 0) = \tilde{g}(x)$. Assume the mean of $\tilde{g}$ is equal to $\bar{g}$, which can be any real number. Let $u = \tilde{u} - \bar{g}$. Since the evolution for $\tilde{u}$ conserves the mean, the mean of u will equal zero at all times. The evolution equation satisfied by u is

$$\partial_t u = \partial_x^5 u - 2u\,\partial_x u + \omega\,\partial_x u, \tag{17}$$

where $\omega = \tilde{\omega} - 2\bar{g}$. The initial data for (17) is $g = \tilde{g} - \bar{g}$, which of course has mean zero.

We now discuss the appropriate existence theory for (17).

3.1. *Existence theory.* The well-posedness of the initial value problem for (17) (or for (16)) has been established in [Bourgain 1995b], in the space H^s for $s > 0$; a more general family of equations including these was also shown to be well-posed in H^s for $s > \frac{1}{2}$ in [Hu and Li 2015]. In the present work, we are not concerned with demonstrating results at the lowest possible regularity, but instead in finding estimates which will work with the nonexistence argument. Towards this end, we will give a simple existence theorem for the initial value problem for (17) in the space H^6, as the resulting estimates will be useful. We mention that the choice of H^6 as the function space is made so that the solutions are classical solutions.

Proposition 6. *Let $u_0 \in H^6$ be given. There exists $T > 0$ and a unique $u \in C([0, T]; H^6)$ such that u solves the initial value problem (17) with data $u(\cdot, 0) = u_0$.*

Proof. We begin by introducing a mollifier, $\mathcal{J}_\varepsilon$, for $\varepsilon > 0$. We use the mollifier to make an approximate evolution equation,

$$\partial_t u^\varepsilon = \mathcal{J}_\varepsilon^2 \, \partial_x^5 u^\varepsilon - \mathcal{J}_\varepsilon(2(\mathcal{J}_\varepsilon u^\varepsilon)(\mathcal{J}_\varepsilon u_x^\varepsilon)) + \omega \mathcal{J}_\varepsilon^2 u_x^\varepsilon. \tag{18}$$

When combined with the mollifier, all of the derivatives on the right-hand side have become bounded operators, and thus solutions for the initial value problem for u^ε, with initial data $u^\varepsilon(\cdot, 0) = g \in H^6$, exist in $C([0, T_\varepsilon]; H^6)$ by Picard's theorem (see Chapter 3 of [Majda and Bertozzi 2002]).

In order to show that the interval of existence can be taken to be independent of ε, we must make an energy estimate. We let the energy functional be an equivalent version of the square of the H^6 norm:

$$\mathcal{E}(t) = \frac{1}{2} \int_0^{2\pi} (u^\varepsilon)^2 + (\partial_x^6 u^\varepsilon)^2 \, dx.$$

Taking the time derivative, we find

$$\frac{d\mathcal{E}}{dt} = \int_0^{2\pi} (u^\varepsilon)(\partial_t u^\varepsilon) \, dx + \int_0^{2\pi} (\partial_x^6 u^\varepsilon)(\partial_t \partial_x^6 u^\varepsilon) \, dx = \mathrm{I} + \mathrm{II}.$$

For I, we plug in from the evolution equation, using the fact that $\mathcal{J}_\varepsilon$ is self-adjoint and commutes with ∂_x:

$$\mathrm{I} = \int_0^{2\pi} (\mathcal{J}_\varepsilon u^\varepsilon) \, \partial_x^5 (\mathcal{J}_\varepsilon u^\varepsilon) - 2(\mathcal{J}_\varepsilon u^\varepsilon)^2 (\mathcal{J}_\varepsilon u_x^\varepsilon) + \omega(\mathcal{J} u^\varepsilon)(\mathcal{J}_\varepsilon u_x^\varepsilon) \, dx.$$

All of these terms vanish upon integrating by parts and/or recognizing perfect derivatives; therefore, $\mathrm{I} = 0$.

To study the term II, it is helpful to first apply six spatial derivatives to (18). We use the product rule, finding

$$\partial_t \partial_x^6 u^\varepsilon = \mathcal{J}_\varepsilon^2 \partial_x^{11} u^\varepsilon - \mathcal{J}_\varepsilon(2(\mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^7 \mathcal{J}_\varepsilon u^\varepsilon)) + \omega \mathcal{J}_\varepsilon^2 \partial_x^7 u^\varepsilon - 2\mathcal{J}_\varepsilon\left(\sum_{m=1}^{6} \binom{6}{m} (\partial_x^m \mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^{7-m} \mathcal{J}_\varepsilon u^\varepsilon)\right). \tag{19}$$

We can then write

$$\mathrm{II} = \mathrm{II}_1 + \mathrm{II}_2 + \mathrm{II}_3 + \mathrm{II}_4,$$

where each of these terms corresponds to one of the four terms on the right-hand side of (19). We will now deal with these one at a time.

We again will frequently use the fact that $\mathcal{J}_\varepsilon$ is self-adjoint and commutes with ∂_x. To begin, we have

$$\text{II}_1 = \int_0^{2\pi} (\partial_x^6 \mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^{11} \mathcal{J}_\varepsilon u^\varepsilon)\, dx.$$

After integrating by parts and recognizing a perfect derivative, we see that $\text{II}_1 = 0$.

For II_2, we have

$$\text{II}_2 = -2 \int_0^{2\pi} (\mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^6 \mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^7 \mathcal{J}_\varepsilon u^\varepsilon)\, dx.$$

We recognize a perfect derivative and integrate by parts, finding

$$\text{II}_2 = \int_0^{2\pi} (\mathcal{J}_\varepsilon u_x^\varepsilon)(\partial_x^6 \mathcal{J}_\varepsilon u^\varepsilon)^2\, dx \le c\mathcal{E}^{3/2}.$$

For II_3, we have

$$\text{II}_3 = \omega \int_0^{2\pi} (\partial_x^6 \mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^7 \mathcal{J}_\varepsilon u^\varepsilon)\, dx = 0.$$

As before, this integrates to zero because it is the integral of a perfect derivative over a periodic interval.

Finally, we treat II_4. We have

$$\text{II}_4 = -2 \sum_{m=1}^6 \binom{6}{m} \int_0^{2\pi} (\partial_x^6 \mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^m \mathcal{J}_\varepsilon u^\varepsilon)(\partial_x^{7-m} \mathcal{J}_\varepsilon u^\varepsilon)\, dx. \tag{20}$$

Each of the integrals on the right-hand side of (20) can be bounded by $\mathcal{E}^{3/2}$.

We therefore conclude that there exists $c > 0$ such that

$$\frac{d\mathcal{E}}{dt} \le c\mathcal{E}^{3/2}.$$

This can be rephrased as

$$\frac{d\|u^\varepsilon\|_{H^6}}{dt} \le c\|u^\varepsilon\|_{H^6}^2.$$

This estimate clearly indicates that the solutions u^ε cannot blow up arbitrarily quickly, and thus exist on a common time interval. So, we have shown that there exists $T > 0$, independent of ε, such that for all $\varepsilon > 0$, we have $u^\varepsilon \in C([0, T]; H^6)$, with the norm bounded independently of ε.

Since u^ε is bounded uniformly (in both t and ε) and since u^ε solves (18), we see that $\partial_t u^\varepsilon$ is uniformly bounded in H^1, and thus in L^∞. By the Arzelà–Ascoli theorem, there exists a subsequence (which we do not relabel) and a limit, $u \in C([0, 2\pi] \times [0, T])$, such that $u_\varepsilon \to u$ in this space. Standard arguments (again, see Chapter 3 of [Majda and Bertozzi 2002], for instance) then imply that u belongs to the space $C([0, T]; H^6)$, that u obeys the same uniform bound as the u^ε, and that u is a solution of (17).

Uniqueness of solutions (and, in fact, continuous dependence on the initial data) follows from a more elementary version of the energy estimate. If we let $u \in C([0, T]; H^6)$ be a solution corresponding to initial data $u_0 \in H^6$, and if we let $v \in C([0, T]; H^6)$ be a solution corresponding to initial data $v_0 \in H^6$, then we can estimate the norm of $u - v$. If $E_d = \|u - v\|_{L^2}^2$, then a straightforward calculation, together with the uniform bounds established previously, indicates $\frac{d}{dt} E_d \le c E_d$. This implies that $E_d \le E_d(0)e^{ct}$ for all t for which the solutions are defined. If $u_0 = v_0$, then we see that $u = v$. This is the desired uniqueness result. $\square$

Remark 7. These estimates are uniform in ω, since all of the terms in the energy estimate which involve ω are equal to zero.

3.2. *Reformulation.* In this section, we use a normal form [Shatah 1985] to rewrite the evolution equation in a beneficial way.

Taking the Fourier transform, we let u_k be the Fourier coefficients of u. We get the evolution equations

$$\partial_t u_k = ik^5 u_k + i\omega k u_k - ik \sum_{j=-\infty}^{\infty}{}' u_{k-j} u_j.$$

We only consider $k \neq 0$ since we know already that $\partial_t u_0 = u_0 = 0$. The prime on the sum indicates that $j = 0$ and $j = k$ are excluded, as these modes are unnecessary since the mean of u is equal to zero. The initial condition is

$$u_k(0) = \hat{g}(k).$$

We bring the linear terms to the left-hand side:

$$\partial_t u_k - ik^5 u_k - i\omega k u_k = -ik \sum_{j=-\infty}^{\infty}{}' u_{k-j} u_j.$$

We use an integrating factor, so we define v_k through the equation

$$v_k(t) = u_k(t) e^{-ik^5 t - i\omega k t}.$$

This yields

$$\partial_t v_k = -ik \sum_{j=-\infty}^{\infty}{}' e^{-ik^5 t - i\omega k t} e^{i(k-j)^5 t + i\omega(k-j)t} e^{ij^5 t + i\omega j t} v_{k-j} v_j.$$

The exponents simplify as the terms in the exponent related to the transport speed ω all cancel. This leaves us with

$$\partial_t v_k = -ik \sum_{j=-\infty}^{\infty}{}' e^{-ik^5 t} e^{i(k-j)^5 t} e^{ij^5 t} v_{k-j} v_j.$$

It will therefore be helpful to understand the quantity $k^5 - (k-j)^5 - j^5$, as this appears in the exponent. Elementary calculations show that, upon introducing the notation $\sigma = k^2 - jk + j^2$, we have

$$k^5 - (k-j)^5 - j^5 = 5(k-j)jk\sigma.$$

Using this identity, we are able to write the evolution equation for the v_k,

$$\partial_t v_k = -ik \sum_{j=-\infty}^{\infty}{}' e^{-5i(k-j)jk\sigma t} v_{k-j} v_j. \tag{21}$$

We can then rewrite this, recognizing that the exponential is in fact the derivative of an exponential:

$$\partial_t v_k = -ik \sum_{j=-\infty}^{\infty}{}' \left(\partial_t \left(\frac{e^{-5i(k-j)jk\sigma t}}{-5i(k-j)jk\sigma} \right) \right) v_{k-j} v_j = \sum_{j=-\infty}^{\infty}{}' \left(\partial_t \left(\frac{e^{-5i(k-j)jk\sigma t}}{5(k-j)j\sigma} \right) \right) v_{k-j} v_j.$$

Next, we "differentiate by parts", moving the time derivative:

$$\partial_t v_k = \partial_t \left(\frac{1}{5} \sum_{j=-\infty}^{\infty}{}' \frac{e^{-5i(k-j)jk\sigma t}}{(k-j)j\sigma} v_{k-j} v_j \right) - \frac{1}{5} \sum_{j=-\infty}^{\infty}{}' \frac{e^{-5i(k-j)jk\sigma t}}{(k-j)j\sigma} \partial_t(v_{k-j} v_j).$$

We define B through its Fourier coefficients, $B_k(t)$, as

$$B_k(t) = -\frac{1}{5} \sum_{j=-\infty}^{\infty}{}' \frac{e^{-5i(k-j)jk\sigma t}}{(k-j)j\sigma} v_{k-j} v_j. \tag{22}$$

We then are able to write the evolution equations as

$$\partial_t[v_k + B_k] = -\frac{1}{5} \sum_{j=-\infty}^{\infty}{}' \left[\frac{e^{-5i(k-j)jk\sigma t}}{(k-j)j\sigma} \right] \left[(\partial_t v_{k-j}) v_j + v_{k-j}(\partial_t v_j) \right].$$

Next, we substitute for $\partial_t v_{k-j}$ and $\partial_t v_j$. We let $\tilde{\sigma} = j^2 - j\ell + \ell^2$; using (21), we have

$$\partial_t v_j = -ij \sum_{\ell=-\infty}^{\infty}{}' e^{-5itj\ell(j-\ell)\tilde{\sigma}} v_{j-\ell} v_\ell.$$

We are then able, using a symmetry between $k - j$ and j, to write

$$\partial_t[v_k + B_k] = \frac{2i}{5} \sum_{j=-\infty}^{\infty}{}' \sum_{\ell=-\infty}^{\infty}{}' \left[\frac{e^{-5it[(k-j)jk\sigma+(j-\ell)j\ell\tilde{\sigma}]}}{(k-j)\sigma} \right] v_{k-j} v_{j-\ell} v_\ell. \tag{23}$$

(We reiterate that $k \neq j$.) We give the name R_k to the right-hand side of (23), and we let R be the function with Fourier coefficients equal to R_k for all k. So, we have $\partial_t[v_k + B_k] = R_k$. Integrating with respect to time, we have

$$v_k(t) - v_k(0) = B_k(0) - B_k(t) + \int_0^t R_k(s)\, ds. \tag{24}$$

We transform back to u by multiplying (24) by $e^{ik^5 t + i\omega k t}$, and we note that $v_k(0) = u_k(0)$. These considerations yield

$$u_k(t) - e^{ik^5 t + i\omega k t} u_k(0) = e^{ik^5 t + i\omega k t} \left(B_k(0) - B_k(t) + \int_0^t R_k(s)\, ds \right). \tag{25}$$

Notice that (25) is the k-th Fourier coefficient of the Duhamel integral at time t, or $\mathcal{F}(S_D(t)u_0)(k)$.

3.3. *Estimates.* We now estimate B and R and associated quantities, to demonstrate the smoothing described in Theorem 4 for our equation with fifth-order dispersion.

Remark 8. It will be plain to see that all estimates made in the present section are uniform in ω.

Lemma 9. *If $s \geq 1$ and $v \in H^s$, then $B \in H^{s+3}$, with the estimate*

$$\|B\|_{H^{s+3}}^2 \leq c \|v\|_{H^s}^4.$$

(We note that for our main theorem, we only actually need $B \in H^{s+2}$, but it turns out that $B \in H^{s+3}$.)

Proof. We will show that $\partial_x^3 B$ is in H^s. We begin by taking three derivatives of B, which requires multiplying (22) by $(ik)^3$:

$$(ik)^3 B_k = \frac{i}{5} \sum_{j=-\infty}^{\infty}{}' E(k, j) \frac{k^3}{j(k-j)(k^2-kj+j^2)} v_{k-j} v_j,$$

where $E(k, j)$ represents the exponential, $E(k, j) = e^{-5itjk(k-j)\sigma}$.

We will demonstrate now that

$$\frac{k^3}{j(k-j)(k^2-kj+j^2)}$$

is bounded by a constant. To begin, we consider

$$\left| \frac{k}{j(k-j)} \right| = \left| \frac{k-j+j}{j(k-j)} \right| \le \left| \frac{1}{j} \right| + \left| \frac{1}{k-j} \right| \le 2.$$

Next, we consider $\sigma = k^2 - kj + j^2$. We observe that

$$\sigma = \tfrac{1}{2}k^2 + \tfrac{1}{2}(k-j)^2 + \tfrac{1}{2}j^2, \tag{26}$$

so clearly $\sigma \ge \tfrac{1}{2}k^2$. Thus,

$$\left| \frac{k^2}{\sigma} \right| \le 2. \tag{27}$$

Thus, for any k and any j, we have

$$\left| E(k, j) \frac{k^3}{j(k-j)(k^2-kj+j^2)} \right| \le 4. \tag{28}$$

We give the name

$$\Phi(k, j) = E(k, j) \frac{k^3}{j(k-j)(k^2-kj+j^2)}.$$

Of course, we have

$$\|\partial_x^{s+3} B\|_{L^2}^2 = \left\| |k|^s (ik^3 B_k) \right\|_{\ell^2}^2.$$

We then have

$$\left\| |k|^s (ik^3 B_k) \right\|_{\ell^2}^2 = \frac{1}{25} \sum_{k=-\infty}^{\infty} \sum_{j=-\infty}^{\infty}{}' \sum_{\ell=-\infty}^{\infty}{}' \Phi(k, j)\bar{\Phi}(k, \ell) k^{2s} v_{k-j} v_j \bar{v}_{k-\ell} \bar{v}_\ell.$$

In light of (28), we have

$$\left\| |k|^s (ik^3 B_k) \right\|_{\ell^2}^2 \le \sum_{k,j,\ell} k^{2s} |v_{k-j}||v_j||v_{k-\ell}||v_\ell|. \tag{29}$$

Let V be the function defined through its Fourier coefficients as $\mathcal{F}V(k) = |v_k|$ for all k. Note that since $v \in H^s$, we have $V \in H^s$, with $\|v\|_{H^s} = \|V\|_{H^s}$. Since $V \in H^s$ with $s \ge 1$, we can see that $V^2 \in H^s$, with $\|V^2\|_{H^s} \le c\|v\|_{H^s}^2$. Notice that the right-hand side of (29) is equal to $\|\partial_x^s (V^2)\|_{L^2}^2$. This completes the proof. $\qquad\square$

The particular estimate we need for B follows from Lemma 9:

Corollary 10. *If $s \geq 1$ and $u \in C([0, T]; H^s)$, and if $t \in [0, T]$, then*

$$\mathcal{F}^{-1}\big(e^{ik^5 t + i\omega k t}(B_k(0) - B_k(t))\big) \in H^{s+3},$$

with the bound

$$\big\|\mathcal{F}^{-1}\big(e^{ik^5 t + i\omega k t}(B_k(0) - B_k(t))\big)\big\|_{H^{s+3}} \leq c\|u\|^2_{C([0,T];H^s)}.$$

Proof. This follows immediately from Lemma 9, and from the fact that $|e^{i\theta}| = 1$ for any real θ. □

Having proved a satisfactory estimate for B, we turn to R.

Lemma 11. *If $s \geq 1$ and $v \in H^s$, then $R \in H^{s+2}$, with the estimate*

$$\|R\|^2_{H^{s+2}} \leq c\|v\|^6_{H^s}.$$

Proof. Recall the formula for R_k,

$$R_k = \frac{2i}{5} \sideset{}{'}\sum_{j=-\infty}^{\infty} \sideset{}{'}\sum_{\ell=-\infty}^{\infty} \frac{1}{(k-j)\sigma}(v_{k-j}v_{j-\ell}v_\ell) \exp\big\{(-5it)(kj(k-j)\sigma + j\ell(j-\ell)\tilde{\sigma})\big\},$$

with $\sigma = k^2 - kj + j^2$ and $\tilde{\sigma} = j^2 - j\ell + \ell^2$. As we noted in the proof of Lemma 9, we have $k^2/\sigma \leq 2$. Following the lines of the proof of Lemma 9, we arrive at

$$\|\partial_x^{s+2} R\|^2_{L^2} \leq c \sum_{k,j,\ell,m,n} k^{2s}|v_{k-j}||v_{j-\ell}||v_\ell||v_{k-m}||v_{m-n}||v_n|.$$

Letting V be as in the proof of Lemma 9, we see that the right-hand side is a constant times the square of the L^2 norm of $\partial_x^s(V^3)$. Thus, this is bounded by $\|v\|^6_{H^s}$, as claimed. □

Lemma 11 implies the following, which is the estimate we need for R:

Corollary 12. *If $s \geq 1$ and $u \in C([0, T]; H^s)$, and if $t \in [0, T]$, then*

$$\mathcal{F}^{-1}\left(e^{ik^5 t + i\omega k t} \int_0^t R_k(s)\, ds\right) \in H^{s+2},$$

with

$$\left\|\mathcal{F}^{-1}\left(e^{ik^5 t + i\omega k t} \int_0^t R_k(s)\, ds\right)\right\|_{H^{s+2}} \leq c\|u\|^3_{C([0,T];H^s)}.$$

Proof. We begin by noting that, of course,

$$\left\|\partial_x^{s+2}\mathcal{F}^{-1}\left(e^{ik^5 t + i\omega k t} \int_0^t R_k(s)\, ds\right)\right\|^2_{L^2} = \left\|(ik)^{s+2} e^{ik^5 t + i\omega k t} \int_0^t R_k(s)\, ds\right\|^2_{\ell^2}$$

$$= \left\|\int_0^t k^{s+2} R_k(s)\, ds\right\|^2_{\ell^2}$$

$$= \sum_{k=-\infty}^{\infty} \left(\int_0^t k^{s+2} R_k(s)\, ds\right)\left(\int_0^t k^{s+2} \overline{R}_k(\tau)\, d\tau\right).$$

We use the triangle inequality:

$$\left\| \partial_x^{s+2} \mathcal{F}^{-1} \left(e^{ik^5 t + i\omega k t} \int_0^t R_k(s)\, ds \right) \right\|_{L^2}^2 \leq \sum_{k=-\infty}^{\infty} \left(\int_0^t |k^{s+2} R_k(s)|\, ds \right) \left(\int_0^t |k^{s+2} R_k(\tau)|\, d\tau \right).$$

By Tonelli's theorem, we can exchange the sum and the integrals:

$$\left\| \partial_x^{s+2} \mathcal{F}^{-1} \left(e^{ik^5 t + i\omega k t} \int_0^t R_k(s)\, ds \right) \right\|_{L^2}^2 \leq \int_0^t \int_0^t \sum_{k=-\infty}^{\infty} |k^{s+2} R_k(s)| |k^{s+2} R_k(\tau)|\, ds\, d\tau.$$

We then use the Cauchy–Schwarz inequality:

$$\left\| \partial_x^{s+2} \mathcal{F}^{-1} \left(e^{ik^5 t + i\omega k t} \int_0^t R_k(s)\, ds \right) \right\|_{L^2}^2 \leq c \int_0^t \int_0^t \| R(\cdot, s) \|_{H^{s+2}} \| R(\cdot, \tau) \|_{H^{s+2}}\, ds\, d\tau.$$

We then use Lemma 11, and the proof is complete. □

We are now in a position to show that the necessary estimate for the Duhamel integral holds for (17).

Theorem 13. *Let $0 < T_1 < T_2$ be given. There exists $\gamma > 0$ such that for any $u_0 \in H^6$ such that $\|u_0\|_{H^6} < \gamma$, there is a unique solution of the initial value problem* (17) *with initial data u_0, with the solution $u \in C([0, T_2]; H^6)$. There exists $c > 0$ and $\tilde{\gamma} \in (0, \gamma)$ such that for any $T \in [T_1, T_2]$, and for any $u_0 \in H_0^6$ such that $\|u_0\|_{H^6} < \tilde{\gamma}$, we have*

$$\| S_D(T) u_0 \|_{H^8} \leq c \|u_0\|_{H^6}^2.$$

Proof. The formula (25) and the estimates of Corollaries 10 and 12 for B and R immediately imply

$$\| S_D(T) u_0 \|_{H^8} \leq c \|u\|_{C([0,T]; H^6)}^2.$$

However, we are not yet finished because we need the bound to be in terms of the initial data, and not in terms of the solution at positive times.

As discussed in Section 3.1 above, we have

$$\frac{d}{dt} \|u^\varepsilon\|_{H^6} \leq c \|u^\varepsilon\|_{H^6}^2.$$

Let $\|u_0\|_{H^6} = \delta/2$. Then, as long as $\|u^\varepsilon(\cdot, t)\|_{H^6} \leq \delta = 2\|u_0\|_{H^6}$, we have

$$\frac{d}{dt} \|u^\varepsilon\|_{H^6} \leq c\delta \|u^\varepsilon\|_{H^6},$$

and thus

$$\|u^\varepsilon\|_{H^6} \leq \|u_0\|_{H^6} e^{c\delta t} = \tfrac{1}{2}\delta e^{c\delta t}.$$

This implies that $\|u^\varepsilon\|_{H^6} \leq 2\|u_0\|_{H^6}$ as long as $e^{c\delta t} < 2$. This is valid as long as $t < \ln(2)/c\delta$; notice that this bound goes to infinity as δ vanishes (that is, the "doubling time" for solutions goes to infinity as the initial size of the solutions goes to zero). Taking the limit as ε vanishes, then (along our subsequence), we find

$$\|u\|_{H^6} \leq 2\|u_0\|_{H^6},$$

as long as $t < \ln(2)/c\delta$.

Given the set of potential temporal periods of interest, $[T_1, T_2]$, we may choose δ sufficiently small so that as long as $\|u_0\|_{H^6} < \delta/2$, for all $t \in [0, T_2]$, we have $\|u(\,\cdot\,, t)\|_{H^6} < 2\|u_0\|_{H^6}$. We have shown above that $\|S_D(T)u_0\|_{H^8} \leq c\|u\|^2_{C([0, T_2]; H^6)}$. We now can bound this in terms of $\|u_0\|_{H^6}$, so that

$$\|S_D(T)u_0\|_{H^8} \leq 4c\|u_0\|^2_{H^6}$$

for any $T \in [T_1, T_2]$ and for any u_0 satisfying our smallness assumption. This is the desired bound. $\square$

3.4. *Completion of the example.* In this section, we state a specific theorem on nonexistence of time-periodic solutions, making use of the above. Consider the equation

$$\partial_t \tilde{u} = \partial_x^5 \tilde{u} - 2\tilde{u}\,\partial_x \tilde{u}. \tag{30}$$

Consider the initial data, $\tilde{u}_0 \in H^6$, with the mean of $\tilde{u}_0$ equal to α. As above, we define $u_0 = \tilde{u}_0 - \alpha$, and we let $u = \tilde{u} - \alpha$. The evolution equation satisfied by u, as discussed above, is

$$\partial_t u = \partial_x^5 u - 2\alpha\,\partial_x u - 2u\,\partial_x u. \tag{31}$$

If we let $A = \partial_x^5 - 2\alpha\partial_x$, then we see that the symbol of A is

$$\hat{A}(k) = i(k^5 - 2\alpha k) = ik(k^4 - 2\alpha). \tag{32}$$

Thus, if $2\alpha < 1$, then there are no zeros of the symbol in $\mathbb{Z} \setminus \{0\}$. Recalling the hypotheses **(H)**, we have $M = 2$, and we let $r_1 = 5$, $r_2 = 1$, $Z_1 = \{1\}$, and $Z_2 = \left(-\frac{1}{2}, \frac{1}{2}\right)$. Letting $\alpha_1 \in Z_1$ and $\alpha_2 \in Z_2$, we see that we may take $\beta_1 = \frac{1}{2}$ and $\beta_2 = \frac{1}{2}$. Letting $\alpha_2 = -2\alpha$, we see that the hypotheses **(H)** are satisfied, and Lemma 2 and Corollary 3 hold, with uniform estimates for $\alpha \in \left(-\frac{1}{4}, \frac{1}{4}\right)$. Therefore, Theorem 4 applies for $\alpha \in \left(-\frac{1}{4}, \frac{1}{4}\right)$. Theorem 13 also applies, and in light of Remarks 7 and 8, we see that the constants in Theorem 13 can be taken to be uniform with respect to α. This implies that (31) does not possess small, nonzero time-periodic solutions, uniformly in $\alpha \in \left(-\frac{1}{4}, \frac{1}{4}\right)$.

Adding the mean α back to u, we get $\tilde{u} = (\tilde{u} - \alpha) + \alpha$, with $u = \tilde{u} - \alpha$. We know that for $\alpha \in \left(-\frac{1}{4}, \frac{1}{4}\right)$, $\tilde{u} - \alpha$ does not possess small, nonzero time-periodic solutions with the associated periods ($T \in W_{p,\delta}$ or $T \in W_p$, as appropriate). Furthermore, we know that $\|\tilde{u}\|_{H^6} \geq \|\tilde{u} - \alpha\|_{H^6}$. This implies that the only small time-periodic solutions of (30) with the given temporal periods are $\tilde{u} = \alpha$. Thus (30) does not possess small, nonconstant time-periodic solutions with the given temporal periods. This proves the following corollary:

Corollary 14. *Let $0 < T_1 < T_2$ be given. Let $p \in (1, 2]$ be given. Let $0 < \delta < T_2 - T_1$ be given. Let $W_{p,\delta} \subseteq [T_1, T_2]$ be as in Lemma 2, with A given by (32) for $|\alpha| \leq \frac{1}{4}$. There exists $r_1 > 0$ such that for all $T \in W_{p,\delta}$, if u is a smooth, nonconstant time-periodic solution of (30) with temporal period T, then*

$$\inf_{t \in [0, T]} \|u\|_{H^6} > r_1.$$

Let $W_p \subseteq [T_1, T_2]$ be as in Corollary 3. Let $T \in W_p$ be given. Then there exists r_2 such that if u is a smooth, nonconstant time-periodic solution of (30) with temporal period T, then

$$\inf_{t \in [0, T]} \|u\|_{H^6} > r_2.$$

4. Further examples with $\tilde{p} = 0$

In this section, we provide a few other equations which can be treated similarly to the above. We do not provide full proofs in this section, but instead point out the differences with the prior proof.

4.1. *Nonresonant Kawahara equations.* The Kawahara equation has been justified as a model for water waves with surface tension [Düll 2012; Schneider and Wayne 2002]. It can be written as

$$\partial_t \tilde{u} = \partial_x^5 \tilde{u} - \theta \, \partial_x^3 \tilde{u} - 2\tilde{u} \, \partial_x \tilde{u}, \tag{33}$$

with $\theta > 0$. As before, we take this with initial data $\tilde{u}(\cdot, 0) = \tilde{u}_0$, and we assume that the mean of $\tilde{u}_0$ is equal to α. We again let $u = \tilde{u} - \alpha$, and we find that the equation satisfied by u is

$$\partial_t u = \partial_x^5 u - \theta \, \partial_x^3 u - 2\alpha \, \partial_x u - 2u \, \partial_x u. \tag{34}$$

Our prior results extend to the Kawahara equation as long as the constant θ is chosen to avoid resonance. In particular, we must require $k^5 - \theta k^3 \neq 0$ for all $k \in \mathbb{Z} \setminus \{0\}$. Notice that this is the same as requiring

$$\min_{k \in \mathbb{Z} \setminus \{0\}} |k^5 - \theta k^3| > 0. \tag{35}$$

This implies that there exists a constant $\bar{\alpha} > 0$ and a constant $\beta_2 > 0$ such that for all $\alpha \in (-\bar{\alpha}, \bar{\alpha})$, for all $k \in \mathbb{Z} \setminus \{0\}$,

$$|k^5 - \theta k^3 - 2\alpha k| \geq \beta_2.$$

We now verify **(H)**, taking $M = 3$. We let $r_1 = 5$, $r_2 = 3$, and $r_3 = 1$. We let $Z_1 = \{1\}$ and $Z_2 = \{\theta\}$. We take $\alpha_1 = 1$, $\alpha_2 = \theta$, and $\alpha_3 = -2\alpha$, with $Z_3 = (-2\bar{\alpha}, 2\bar{\alpha})$. Then, **(H)** is satisfied, with $\beta_1 = \frac{1}{2}$.

This means that Theorem 4 applies, and all that must be done to conclude the nonexistence of small doubly periodic waves for the nonresonant Kawahara equation is that the smoothing property must be demonstrated. We are able to demonstrate the smoothing property with $\tilde{p} = 0$.

We take the Fourier transform of (34), finding

$$\partial_t u_k = i k^5 u_k + i \theta k^3 u_k - 2i\alpha k u_k - ik \sum_{j=-\infty}^{\infty} u_{k-j} u_j.$$

As before, we use an integrating factor, defining

$$v_k = u_k \exp\{i t (-k^5 - \theta k^3 + 2\alpha k)\}.$$

We have the following evolution equation for v_k:

$$\partial_t v_k = -ik \sum_{j=-\infty}^{\infty}{}' (v_{k-j} v_j) \exp\{i t \Phi(j, k)\},$$

where the phase function, Φ, is given by

$$\Phi(j, k) = -k^5 - \theta k^3 + 2\alpha k + (k-j)^5 + \theta(k-j)^3 - 2\alpha(k-j) + j^5 + \theta j^3 - 2\alpha j.$$

This simplifies, as all the terms with α cancel, and also since we previously computed $k^5 - (k-j)^5 - j^5$. We have not previously computed $k^3 - (k-j)^3 - j^3$, but this is straightforward:

$$k^3 - (k-j)^3 - j^3 = 3jk(k-j).$$

These considerations imply

$$\Phi(j,k) = 5jk(k-j)\sigma + 3\theta jk(k-j) = 5jk(k-j)\left(\sigma + \tfrac{3}{5}\theta\right).$$

The critical step in the proof of smoothing in Section 3 was inequality (27). We see that the corresponding inequality in the present case is

$$\left| \frac{k^2}{\sigma + \tfrac{3}{5}\theta} \right| \le 2,$$

which follows immediately from (26) and the condition $\theta > 0$. The rest of the proof of Section 3 can be repeated, establishing the following:

Corollary 15. *Let $0 < T_1 < T_2$ be given. Let $p \in (1,2]$ be given. Let $0 < \delta < T_2 - T_1$ be given. Let $\theta > 0$ satisfy (35), and let $\bar{\alpha} > 0$ be as above. Let $W_{p,\delta} \subseteq [T_1, T_2]$ be as in Lemma 2, with A given by $\mathcal{F}(A) = i(-k^5 - \theta k^3 + 2\alpha k)$ for $|\alpha| \le \bar{\alpha}$. There exists $r_1 > 0$ such that for all $T \in W_{p,\delta}$, if u is a smooth, nonconstant time-periodic solution of (33) with temporal period T, then*

$$\inf_{t\in[0,T]} \|u\|_{H^6} > r_1.$$

Let $W_p \subseteq [T_1, T_2]$ be as in Corollary 3. Let $T \in W_p$ be given. Then there exists r_2 such that if u is a smooth, nonconstant time-periodic solution of (33) with temporal period T, then

$$\inf_{t\in[0,T]} \|u\|_{H^6} > r_2.$$

Remark 16. Above, we appeared to use in a fundamental way the property $\theta > 0$. We assume $\theta > 0$ only because this appears to be a feature of the Kawahara equation as it exists in the prior literature. If we instead had $\theta < 0$, our argument would still work. For a particular value of θ, if there exist $(j,k) \in (\mathbb{Z} \setminus \{0\})^2$ such that $\sigma + \tfrac{3}{5}\theta = 0$, then we treat such values of (j,k) differently. The arguments of Section 3 continue to apply whenever this quantity does not vanish. All that remains is to observe that, as can be seen from (26), the set of (j,k) for which $\sigma + \tfrac{3}{5}\theta$ does vanish is bounded for a fixed value of θ (or for values of θ in a bounded set), and that regularity is determined by behavior for large k.

4.2. *Seventh-order equations.* The proof of smoothing for the fifth-order equation above depended on the factorization of $k^5 - (k-j)^5 - j^5$. With dispersion of seventh order, we must understand $k^7 - (k-j)^7 - j^7$. By making elementary calculations, one may show that

$$k^7 - (k-j)^7 - j^7 = 7(k-j)jk\tau, \tag{36}$$

where

$$\tau = \tfrac{1}{2}k^4 + \tfrac{1}{2}(k-j)^4 + \tfrac{1}{2}j^4;$$

note the similarity to (26). Clearly, we have the inequality

$$\left|\frac{k^4}{\tau}\right| = \frac{k^4}{\tau} \leq 2.$$ (37)

We can then perform all of the calculations of Section 3 for the equation

$$\partial_t u = \partial_x^7 u - 2u\,\partial_x u.$$ (38)

Recall that B and R, defined in Section 3.2, were shown in Section 3.3 to gain two derivatives. This property hinged on the inequality (27). In the present setting, the analogues of B and R would now gain four derivatives because of (37). (As noted in the introduction, this allows one to see that the smoothing mechanism we are using is different than Kato smoothing, as Kato smoothing would provide a gain of three derivatives with seventh-order dispersion.) Following the argument of Section 3, but using (37), we arrive at the following:

Corollary 17. *Let $0 < T_1 < T_2$ be given. Let $p \in (1, 4]$ be given. Let $0 < \delta < T_2 - T_1$ be given. Let $W_{p,\delta} \subseteq [T_1, T_2]$ be as in Lemma 2, with A given by $\mathcal{F}(A) = i(-k^7 + \alpha k)$ for $|\alpha| \leq \frac{1}{2}$. There exists $r_1 > 0$ such that for all $T \in W_{p,\delta}$, if u is a smooth, nonconstant time-periodic solution of (38) with temporal period T, then*

$$\inf_{t \in [0,T]} \|u\|_{H^8} > r_1.$$

Let $W_p \subseteq [T_1, T_2]$ be as in Corollary 3. Let $T \in W_p$ be given. Then there exists r_2 such that if u is a smooth, nonconstant time-periodic solution of (38) with temporal period T, then

$$\inf_{t \in [0,T]} \|u\|_{H^8} > r_2.$$

We note that a change from Corollary 17 as compared to Corollary 14 is that we are now using H^8 instead of H^6. The function space is chosen so that the solutions under consideration are classical solutions; using the space H^8, the seventh derivative of u appearing on the right-hand side of (38) is classically defined.

5. The KdV equation

We study the equation

$$\partial_t \tilde{u} = -\partial_x^3 \tilde{u} - \tilde{u}\,\partial_x \tilde{u}.$$ (39)

(This is not the most traditional choice of coefficients for the KdV equation, but the coefficients are changeable by scaling, so it makes no difference.) The evolution equation (39) is taken with initial data $u(\cdot, 0) = \tilde{g}$. We let the mean of $\tilde{g}$ be denoted as α, and we define $g = \tilde{g} - \alpha$. Then, noticing that (39) preserves the mean of the solution, we define $u = \tilde{u} - \alpha$, so that the mean of u is equal to zero, as long as the solution $\tilde{u}$ of (39) exists. The evolution equation satisfied by u is

$$\partial_t u = -\partial_x^3 u - \alpha\,\partial_x u - u\,\partial_x u.$$ (40)

We note that there are very many papers in the literature treating the well-posedness of the KdV equation. For example, in [Bourgain 1993], global well-posedness for the periodic KdV equation with initial data in H^s for $s \geq 0$ is established, and in [Kappeler and Topalov 2006], global well-posedness is established in H^s for $s \geq -1$. Nevertheless, we remark that the simple well-posedness proof for (16) given in Section 3.1 can be suitably and straightforwardly modified to the KdV equation, yielding the same results: that classical solutions exist for arbitrarily long intervals of time if the initial data is sufficiently small, and that the doubling time of such solutions goes to infinity as the size of the data vanishes.

We take the Fourier transform of (40):

$$\partial_t u_k = i(k^3 - \alpha k)u_k - i \sum_{j=-\infty}^{\infty}{}' u_{k-j}(ju_j).$$

Since the mean of u is equal to zero, the values $j = 0$ and $j = k$ are unnecessary; as before, the prime indicates that these indices are excluded from the summation. We use an integrating factor, defining $v_k = u_k \exp\{-it(k^3 - \alpha k)\}$:

$$\partial_t v_k = -i \sum_{j=-\infty}^{\infty}{}' \exp\{-it(k^3 - (k-j)^3 - j^3)\}v_{k-j}(jv_j).$$

(Notice that the terms in the exponential involving α all canceled.) From Pascal's triangle, we see that $k^3 - (k-j)^3 - j^3 = 3kj(k-j)$. Thus, we may write

$$\partial_t v_k = -i \sum_{j=-\infty}^{\infty}{}' \exp\{-3itkj(k-j)\}v_{k-j}(jv_j). \tag{41}$$

We now manipulate the exponential:

$$\partial_t v_k = -i \sum_{j=-\infty}^{\infty}{}' \partial_t \left(\frac{e^{-3itkj(k-j)}}{-3ikj(k-j)} \right) v_{k-j}(jv_j).$$

We cancel the factor of $-i$, and we also "differentiate by parts":

$$\partial_t v_k = \partial_t \left(\sum_{j=-\infty}^{\infty}{}' \frac{e^{-3itkj(k-j)}}{3kj(k-j)} v_{k-j}(jv_j) \right) - \sum_{j=-\infty}^{\infty} \frac{e^{-3itkj(k-j)}}{3kj(k-j)} \partial_t (v_{k-j}(jv_j)).$$

We can write this as

$$\partial_t(v_k + B_k) = R_k,$$

with

$$B_k = -\frac{1}{3} \sum_{j=-\infty}^{\infty} \frac{e^{-3itkj(k-j)}}{kj(k-j)} v_{k-j}(jv_j),$$

$$R_k = -\frac{1}{3} \sum_{j=-\infty}^{\infty}{}' \frac{e^{-3itkj(k-j)}}{kj(k-j)} \partial_t (v_{k-j}(jv_j)). \tag{42}$$

We continue to rewrite this, by applying the time derivative on the right-hand side of (42). Using (41), we have

$$\partial_t v_{k-j} = -i \sum_{m=-\infty}^{\infty}{}' \exp\{-3it(k-j)m(k-j-m)\}v_{k-j-m}(mv_m),$$

$$\partial_t v_j = -i \sum_{\ell=-\infty}^{\infty}{}' \exp\{-3itj\ell(j-\ell)\}v_{j-\ell}(\ell v_\ell).$$

Using these, we write $R_k = R_k^1 + R_k^2$, with

$$R_k^1 = \frac{i}{3} \sum_{j=-\infty}^{\infty}{}' \sum_{m=-\infty}^{\infty}{}' \frac{\exp\{-3it(k-j)(kj+m(k-j-m))\}}{kj(k-j)} v_{k-j-m}(jv_j)(mv_m),$$

$$R_k^2 = \frac{i}{3} \sum_{j=-\infty}^{\infty}{}' \sum_{\ell=-\infty}^{\infty}{}' \frac{\exp\{-3itj(k(k-j)+\ell(j-\ell))\}}{kj(k-j)} v_{k-j}(j(v_{j-\ell})(\ell v_\ell)).$$

We further rewrite R_k^2 by writing $j = j - \ell + \ell$:

$$R_2^k = \frac{i}{3} \sum_{j=-\infty}^{\infty}{}' \sum_{\ell=-\infty}^{\infty}{}' \frac{\exp\{-3itj(k(k-j)+\ell(j-\ell))\}}{kj(k-j)} v_{k-j}((j-\ell)v_{j-\ell})(\ell v_\ell)$$

$$+ \frac{i}{3} \sum_{j=-\infty}^{\infty}{}' \sum_{\ell=-\infty}^{\infty}{}' \frac{\exp\{-3itj(k(k-j)+\ell(j-\ell))\}}{kj(k-j)} v_{k-j}v_{j-\ell}(\ell^2 v_\ell).$$

Using the inequality

$$\left| \frac{k^2}{kj(k-j)} \right| \leq 2,$$

and our previous arguments, we find the bounds

$$\|B\|_{H^{s+2}} \leq c\|v\|_{H^s}\|v\|_{H^{s+1}},$$

$$\|R^1\|_{H^{s+2}} \leq c\|v\|_{H^s}\|v\|_{H^{s+1}}^2,$$

$$\|R^2\|_{H^{s+2}} \leq c\|v\|_{H^s}(\|v\|_{H^{s+1}}^2 + \|v\|_{H^s}\|v\|_{H^{s+2}}).$$

If $\|v\|_{H^{s+2}} < 1$, then the right-hand sides can all be bounded by

$$c\|v\|_{H^s}\|v\|_{H^{s+2}}.$$

Following the arguments of the previous cases, we are able to conclude, for sufficiently small $u_0 \in H_0^{s+2}$,

$$\|S_D(T)u_0\|_{H^{s+2}} \leq c\|u_0\|_{H^s}\|u_0\|_{H^{s+2}}.$$

This is estimate (14) with $p = \tilde{p} = 2$ and $q = 1$, so we see that the nonexistence result holds for the KdV equation:

Corollary 18. *Let $0 < T_1 < T_2$ be given. Let $p \in (1, 2]$ be given. Let $0 < \delta < T_2 - T_1$ be given. Let $W_{p,\delta} \subseteq [T_1, T_2]$ be as in Lemma 2, with A given by*

$$A = -\partial_x^3 - \alpha \, \partial_x,$$

with $|\alpha| \leq \frac{1}{2}$. There exists $r_1 > 0$ such that for all $T \in W_{p,\delta}$, if u is a smooth, nonconstant time-periodic solution of (39) with temporal period T, then

$$\inf_{t \in [0,T]} \|u\|_{H^6} > r_1.$$

Let $W_p \subseteq [T_1, T_2]$ be as in Corollary 3. Let $T \in W_p$ be given. Then there exists r_2 such that if u is a smooth, nonconstant time-periodic solution of (39) with temporal period T, then

$$\inf_{t \in [0,T]} \|u\|_{H^6} > r_2.$$

We note that the result of Corollary 18 is given for solutions in H^6. This is because we take $s = 4$ so that we work with classical solutions, and we have $\tilde{p} = 2$. An application of Theorem 4 then gives a restriction on the $H^{s+\tilde{p}} = H^6$ norm.

6. Discussion

We have developed a theoretical framework for the demonstration of nonexistence of small doubly periodic solutions for dispersive evolution equations. The abstract theorem indicates that nonexistence follows from the demonstration of dispersive smoothing estimates. In particular cases, we have demonstrated that the required dispersive smoothing estimates hold. These results are an analogue of scattering results for dispersive equations on the real line, since scattering implies the nonexistence of small-amplitude coherent structures.

Other work to be done includes treating additional specific examples, and possibly proving a general theorem about when the weak smoothing property (14) holds. For example, it should be investigated whether the necessary properties can be shown to hold for other dispersive equations in one space dimension (like the Benjamin–Ono equation) and in higher dimensions (such as Schrödinger equations). For equations with sufficiently strong dispersion, the stronger smoothing estimate (corresponding to $\tilde{p} = 0$, in which the Duhamel integral gains more than one derivative) will likely hold. A class of equations with strong dispersion are fourth-order Schrödinger equations (see, e.g., [Fibich et al. 2002; 2003; Karpman and Shagalov 2000]). Such equations are of the form

$$i \psi_t + \Delta \psi + |\psi|^{2\sigma} \psi + \varepsilon \Delta^2 \psi = 0,$$

with $\sigma > 0$ and $\varepsilon > 0$, and can arise by including higher-order corrections when deriving a Schrödinger equation from the Maxwell equations. The above linear estimates, such as Lemma 2 and Corollary 3, are valid in one spatial dimension. As pointed out to the authors by the referee of [Ambrose and Wright 2014], in n spatial dimensions, the result requires $p > n$ rather than $p > 1$. Thus, in higher dimensions, one would expect to need to use the weaker smoothing property (corresponding to $\tilde{p} > 0$) in order to follow the strategy of the present paper.

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