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A CHARACTERIZATION OF 1-RECTIFIABLE DOUBLING MEASURES WITH CONNECTED SUPPORTS

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Garnett, Killip, and Schul have exhibited a doubling measure μ with support equal to $\mathbb{R}^d$ that is 1-*rectifiable*, meaning there are countably many curves Γ_i of finite length for which $\mu\big(\mathbb{R}^d \setminus \bigcup \Gamma_i\big) = 0$. In this note, we characterize when a doubling measure μ with support equal to a connected metric space X has a 1-rectifiable subset of positive measure and show this set coincides up to a set of μ-measure zero with the set of $x \in X$ for which $\liminf_{r\to 0} \mu(B_X(x,r))/r > 0$.

1. Introduction

Recall that a Borel measure μ on a metric space X is *doubling* if there is $C_\mu > 0$ so that

$$\mu(B_X(x,2r)) \le C_\mu \mu(B_X(x,r)) \quad \text{for all } x \in X \text{ and } r > 0. \tag{1-1}$$

Garnett, Killip, and Schul [Garnett et al. 2010] exhibit a doubling measure μ with support equal to $\mathbb{R}^n$, $n > 1$, that is 1-rectifiable in the sense that there are countably many curves Γ_i of finite length such that $\mu\big(\mathbb{R}^n \setminus \bigcup \Gamma_i\big) = 0$. This is surprising given that such measures give zero measure to smooth or bi-Lipschitz curves in $\mathbb{R}^d$. To see this, note that, for such a curve Γ and for each $x \in \Gamma$, there are $r_x, \delta_x > 0$ so that for all $r \in (0, r_x)$ there is $B_{\mathbb{R}^d}(y_{x,r}, \delta_x r) \subseteq B_{\mathbb{R}^n}(x, r_x) \setminus \Gamma$, so by the Lebesgue differentiation theorem, $\mu(\Gamma) = 0$. If Γ is just Lipschitz and not bi-Lipschitz, however, we only know this property holds for every point in Γ outside a set of zero length. The aforementioned result shows that Lipschitz curves of finite length can in some sense be coiled up tightly enough that this zero-length set accumulates on a set of positive doubling measure.

The notion of rectifiability of a measure that we are using is not universal. In [Azzam et al. 2015], a measure μ in Euclidean space being d-rectifiable means $\mu \ll \mathscr{H}^d$ and $\operatorname{supp}\mu$ is d-rectifiable. In our setting, however, we don't require absolute continuity of our measures. To avoid ambiguity, we fix our definition below, which is the convention used in [Federer 1969, §3.2.14].

Definition 1.1. If μ is a Borel measure on a metric space X, d is an integer, and $E \subseteq X$ a Borel set, we say E is (μ,d)*-rectifiable* if $\mu\big(E \setminus \bigcup_{i=1}^{\infty} \Gamma_i\big) = 0$ where $\Gamma_i = f_i(E_i)$, $E_i \subseteq \mathbb{R}^d$, and $f_i : E_i \to X$ is Lipschitz. We say μ is d*-rectifiable* if $\operatorname{supp}\mu$ is (μ,d)-rectifiable.

A set $E \subseteq \mathbb{R}^n$ of positive and finite $\mathscr{H}^d$-measure is d*-rectifiable* if it is $(\mathscr{H}^d, d)$-rectifiable (see [Mattila 1995, Definition 15.3] and the few paragraphs preceding it). This is also equivalent to being covered up

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to set of $\mathscr{H}^d$-measure zero by Lipschitz graphs [Mattila 1995, Lemma 15.4]. The example from [Garnett et al. 2010], however, shows that being almost covered by Lipschitz graphs versus Lipschitz images are not equivalent definitions for rectifiability of a measure.

Since this example was published, it has been an open question to classify which doubling measures on $\mathbb{R}^d$ are rectifiable. Very recently, Badger and Schul have given a complete description. First, for a general Radon measure in $\mathbb{R}^d$ and A compact with $\mu(A) > 0$, define

$$\beta_2^{(1)}(\mu, A)^2 = \inf_L \int_A \left(\frac{\operatorname{dist}(x, L)}{\operatorname{diam} A}\right)^2 \frac{d\mu(x)}{\mu(A)}$$

where the infimum is taken over all lines $L \subseteq \mathbb{R}^d$.

Theorem 1.2 [Badger and Schul 2015b, Corollary 1.12]. *If μ is a Radon measure on $\mathbb{R}^d$ such that $\liminf_{r\to 0} \beta_2^{(1)}(\mu, B_{\mathbb{R}^d}(x, r)) > 0$ for μ-almost every $x \in \mathbb{R}^d$, then μ is 1-rectifiable if and only if*

$$\sum_{\substack{x\in Q \\ \ell(Q)\leq 1}} \frac{\operatorname{diam} Q}{\mu(Q)} < \infty \quad \mu\text{-a.e.} \tag{1-2}$$

where the sum is over half-open dyadic cubes Q.

It is not hard to show that, if μ is a doubling measure with $\operatorname{supp} \mu = \mathbb{R}^d$, $d \geq 2$, then there is $c > 0$ depending on the doubling constant such that $\beta_2^{(1)}(\mu, B) \geq c > 0$ for any ball $B \subseteq \mathbb{R}^d$, so the above theorem characterizes all 1-rectifiable doubling measures with support equal to all of $\mathbb{R}^d$.

In this short note, we take a different approach and provide a complete classification of 1-rectifiable doubling measures not just with support equal to $\mathbb{R}^d$ but with support equal to any topologically connected metric space. It turns out that the rectifiable part of such a measure coincides up to a set of μ-measure zero with the set of points where the lower 1-density is positive, where for $s > 0$ we define the *lower s-density* as

$$\underline{D}^s(\mu, x) := \liminf_{r\to 0} \frac{\mu(B_X(x, r))}{r^s}.$$

Theorem 1.3 (main theorem). *Let μ be a doubling measure whose support is a topologically connected metric space X, and let $E \subseteq X$ be compact. Then E is $(\mu, 1)$-rectifiable if and only if $\underline{D}^1(\mu, x) > 0$ for μ-a.e. $x \in E$.*

Note that there are no other topological or geometric restrictions on X: the support of μ may have topological dimension two (like $\mathbb{R}^2$ for example), yet if $\underline{D}^1(\mu, x) > 0$ μ-a.e., then μ is supported on a countable union of Lipschitz images of $\mathbb{R}$. Also observe that the condition $\underline{D}^1(\mu, x) > 0$ is a weaker condition than (1-2). An interesting corollary of the main theorem and Theorem 1.2 is the following.

Corollary 1.4. *If μ is a doubling measure in $\mathbb{R}^d$ with connected support such that*

$$\liminf_{r\to 0} \beta_2^{(1)}(\mu, B_{\mathbb{R}^d}(x, r)) > 0$$

and $\underline{D}^1(\mu, x) > 0$ μ-a.e., then (1-2) *holds.*

2. Proof of the main theorem: sufficiency

When dealing with any metric space X, we will let $B_X(x,r)$ denote the set of points *in* X of distance less than $r>0$ from x. If $B=B_X(x,r)$ and $M>0$, we will denote $MB=B_X(x,Mr)$. For a Borel set $A\subseteq X$, we define the (spherical) 1-Hausdorff measure as

$$\mathscr{H}^1_\delta(A)=\inf\left\{\sum_{i=1}^\infty 2r_i : A\subseteq\bigcup_{i=1}^\infty B_X(x_i,r_i),\ x_i\in A,\ r_i\in(0,\delta)\right\}$$

and $\mathscr{H}^1(A)=\inf_{\delta>0}\mathscr{H}^1_\delta(A)$.

For $A,B\subseteq X$, we set

$$\operatorname{dist}(A,B)=\inf\{|x-y| : x\in A,\ y\in B\}$$

and, for $x\in X$, $\operatorname{dist}(x,A)=\operatorname{dist}(\{x\},A)$.

Remark 2.1. By the Kuratowski embedding theorem, if X is separable (which happens, for example, if $X=\operatorname{supp}\mu$ for a locally finite measure μ), X is isometrically embeddable into $C(X)$, where $C(X)$ is the Banach space of bounded continuous functions on X equipped with the supremum norm $|f|=\sup_{x\in X}|f(x)|$. Thus, we can assume without loss of generality that X is the subset of a complete Banach space, and we will abuse notation by calling this space $C(X)$ as well so that $X\subseteq C(X)$.

The forward direction of the main theorem is proven for general measures in Euclidean space by Badger and Schul [2015a, Lemma 2.7], who in fact prove a higher-dimensional version. Below we provide a proof that works for metric spaces in the one-dimensional case.

Proposition 2.2. *Let μ be a finite measure with $X:=\operatorname{supp}\mu$ a metric space, and suppose μ is 1-rectifiable. Then $\underline{D}^1(\mu,x)>0$ for μ-a.e. $x\in\operatorname{supp}\mu$.*

Proof. Let

$$F=\{x\in\operatorname{supp}\mu : \underline{D}^1(\mu,x)=0\},$$

and let $\varepsilon,\delta>0$. Since μ is rectifiable, there are Lipschitz functions $f_i : A_i\to X$, where $A_i\subseteq[0,1]$ are compact Borel sets of positive measure and $i=1,\dots,N$, so that

$$\mu\left(E\setminus\bigcup_{i=1}^N f_i(A_i)\right)<\delta.$$

We can extend each f_i affinely on the intervals in the complement of A_i to a Lipschitz function $f_i:[0,1]\to C(X)$. Let $d=\min_{i=1,\dots,N}\operatorname{diam} f_i([0,1])$ so that $r\in(0,d)$ and $x\in G:=\bigcup_{i=1}^N f_i([0,1])$ implies $\mathscr{H}^1(B_{C(X)}(x,r)\cap G)\geq r$ (simply because now the images of the f_i are connected).

For each $x\in F\cap G$, there is $r_x\in(0,d/5)$ so that $\mu(B_X(x,5r_x))<\varepsilon r_x$. By the Vitali covering theorem [Heinonen 2001, Lemma 1.2], there are countably many disjoint balls $B_i=B_X(x_i,r_i)$ with centers in F so that $\bigcup 5B_i\supseteq F$. Thus,

$$\mu(F\cap G)\leq\sum_i\mu(5B_i)\leq\varepsilon\sum_i r_i\leq\varepsilon\sum_i\mathscr{H}^1(B_{C(X)}(x_i,r_i)\cap G)\leq\varepsilon\mathscr{H}^1(G).$$

Thus,

$$\mu(F) < \delta + \varepsilon \mathscr{H}^1(G).$$

Keeping δ (and hence G) fixed and sending $\varepsilon \to 0$, we get $\mu(F) < \delta$ for all $\delta > 0$ and thus $\mu(F) = 0$. □

3. Proof of the main theorem: necessity

What remains is to prove the reverse direction of the main theorem, which we summarize in the next lemma.

Lemma 3.1. *Let μ be a doubling measure with constant $C_\mu > 0$ and support X, a topologically connected metric space. Then $\{x \in X : \underline{D}^1(\mu, x) > 0\}$ is $(\mu, 1)$-rectifiable.*

To prove Lemma 3.1, it suffices to show the following lemma.

Lemma 3.2. *Let μ be a doubling measure and support X a topologically connected complete metric space. If $E \subseteq X$ is a compact set for which $E \subseteq B_X(\xi_0, r_0/2)$ for some $\xi_0 \in X$, $r_0 > 0$, and*

$$\mu(B_X(x, r)) \geq 2r \quad \textit{for all } x \in E \textit{ and } r \in (0, r_0), \tag{3-1}$$

then $E = f(A)$ for some $A \subseteq \mathbb{R}$ and Lipschitz function $f : A \to X$.

Proof of Lemma 3.1 using Lemma 3.2. First, note that, if we define $\bar{\mu}(A) = \mu(A \cap X)$, then $\bar{\mu}$ is a doubling measure on $\overline{X}$, where the closure is in $C(X)$ (recall Remark 2.1). Moreover, the closure $\overline{X}$ is still topologically connected but now is a complete metric space since $C(X)$ is complete. Thus, for proving Lemma 3.1, we can assume without loss of generality that X is complete.

Let $F := \{x \in X : \underline{D}^1(\mu, x) > 0\}$. For $j, k \in \mathbb{N}$, let

$$F_{j,k} = \{x \in F : \mu(B_X(x, r)) \geq r/j \text{ for } 0 < r < k^{-1}\}.$$

Then $F = \bigcup_{j,k \in \mathbb{N}} F_{j,k}$. Furthermore, we can write $F_{j,k}$ as a countable union of sets $\{F_{j,k,\ell}\}_{\ell \in \mathbb{N}}$ with diameters less than $1/(3k)$. It suffices then to show that each one of these sets is 1-rectifiable. Fix $j, k, \ell \in \mathbb{N}$. Then the measure $j\mu$ and the set $F_{j,k,\ell}$ satisfy the conditions for Lemma 3.2 with $r_0 = k^{-1}$ except that $F_{j,k,\ell}$ is not necessarily compact. However, $\overline{F}_{j,k,\ell}$ is a closed set still satisfying these conditions, it is totally bounded since μ is doubling, and since X is complete, the Heine–Borel theorem implies $\overline{F}_{j,k,\ell}$ is compact. Thus, we can apply Lemma 3.2 to get that $\overline{F}_{j,k,\ell}$ is rectifiable. Since $F = \bigcup_{j,k,\ell} F_{j,k,\ell}$, we now have that F is also rectifiable. □

The rest of the paper is devoted to proving Lemma 3.2, so fix μ, E, ξ_0, and r_0 as in the lemma.

Proof of Lemma 3.2. We will require the notion of dyadic cubes on a metric space. This theorem was originally developed by David [1988] and Christ [1990], but the current formulation we take from Hytönen and Martikainen [2012].

Theorem 3.3. *Let X be a metric space equipped with a doubling measure μ. Let X_n be a nested sequence of maximal ρ^n-nets for X where $\rho < 1/1000$, and let $c_0 = 1/500$. For each $n \in \mathbb{Z}$, there is a collection $\mathscr{D}_n$ of "cubes", which are Borel subsets of X such that:*

(1) *For every n, $X = \bigcup_{\Delta \in \mathscr{D}_n} \Delta$.*

(2) *If $\Delta, \Delta' \in \mathscr{D} = \bigcup \mathscr{D}_n$ and $\Delta \cap \Delta' \neq \varnothing$, then $\Delta \subseteq \Delta'$ or $\Delta' \subseteq \Delta$.*

(3) *For $\Delta \in \mathscr{D}$, let $n(\Delta)$ be the unique integer so that $\Delta \in \mathscr{D}_n$ and set $\ell(\Delta) = 5\rho^{n(\Delta)}$. Then there is $\zeta_\Delta \in X_n$ so that*
$$B_X(\zeta_\Delta, c_0\ell(\Delta)) \subseteq \Delta \subseteq B_X(\zeta_\Delta, \ell(\Delta))$$
and
$$X_n = \{\zeta_\Delta : \Delta \in \mathscr{D}_n\}.$$

It is not necessary for there to exist a doubling measure but just that the metric space is geometrically doubling. Moreover, Hytönen and Martikainen [2012] use sequences of sets X_n slightly more general than maximal nets.

Let X_n be a nested sequence of maximal ρ^n-nets for X where $\rho < 1/1000$ and $\mathscr{D}$ the resulting cubes from Theorem 3.3. By picking our net points X_n appropriately, we may assume that $E \subseteq \Delta_0 \in \mathscr{D}$.

Lemma 3.4 [Azzam 2014, §3]. *Let μ be a C_μ-doubling measure and $\mathscr{D}$ the cubes from Theorem 3.3 for $X = \operatorname{supp} \mu$ with admissible constants c_0 and ρ. Let $E \subseteq \Delta_0 \in \mathscr{D}$ be a Borel set, $M > 1$, and $\delta > 0$, and set*
$$\mathscr{P} = \{\Delta \subseteq \Delta_0 : \Delta \cap E \neq \varnothing, \text{ there exists } \xi \in B_X(\zeta_\Delta, M\ell(\Delta)) \text{ such that } \operatorname{dist}(\xi, E) \geq \delta\ell(\Delta)\}.$$
Then there is $C_1 = C_1(M, \delta, C_\mu) > 0$ so that, for all $\Delta' \subseteq \Delta_0$,
$$\sum_{\substack{\Delta \subseteq \Delta' \\ \Delta \in \mathscr{P}}} \mu(\Delta) \leq C_1 \mu(\Delta'). \tag{3-2}$$

The theorem is stated in [Azzam 2014] in slightly more generality. For the reader's convenience, we provide a shorter proof in the Appendix.

Let $M, \delta > 0$, to be decided later, and let $\mathscr{P}$ be the set from Lemma 3.4 applied to our set E. Our goal now is to construct a metric space Y containing X, then a curve $\Gamma \subseteq Y$ that contains E as a subset, and then show it has finite length. We will do this by adding bridges through Y between net points around cubes in $\mathscr{P}$ since these are the cubes where E has large holes and thus potentially has big gaps or disconnections. We don't need the endpoints of these bridges to be in E, but their union plus the set E will be connected. We now proceed with the details.

Let $\widetilde{X} = \bigcup X_n$, and equip $C(X) \oplus \mathbb{R}^{\widetilde{X} \times \widetilde{X}}$ (where $\mathbb{R}^{\widetilde{X} \times \widetilde{X}} = \prod_{\alpha \in \widetilde{X} \times \widetilde{X}} \mathbb{R}$; see [Munkres 1975, p. 112–117] for the notation) with norm $|a \oplus b| = \max\{|a|, |b|\}$, where the norm on $\mathbb{R}^{\widetilde{X} \times \widetilde{X}}$ is the ℓ^2 norm.

For $x, y \in \widetilde{X}$, let $[x, y]$ denote the straight line segment between them in $C(X) \oplus \mathbb{R}^{\widetilde{X} \times \widetilde{X}}$, $e_{(x,y)}$ is the unit vector corresponding to the (x, y) coordinate in $\mathbb{R}^{\widetilde{X} \times \widetilde{X}}$, and define
$$\begin{aligned}[x, y]^* := \big[x, (x, |x-y| e_{(x,y)})\big] \cup \big[y, (y, |x-y| e_{(x,y)})\big] \cup \big[(x, |x-y| e_{(x,y)}), (y, |x-y| e_{(x,y)})\big] \\ \subseteq C(X) \oplus \mathbb{R}^{\widetilde{X} \times \widetilde{X}}.\end{aligned}$$

The set $[x, y]^*$ is two segments going straight up from x and y, respectively, in the $e_{(x,y)}$ direction and a segment connecting the endpoints, thus giving a polygonal curve connecting x to y that hops out of $C(X)$. Let

$$Y = X \cup \bigcup_{x,y\in\widetilde{X}} [x, y]^*,$$

and define a metric on Y (also denoted by $|\cdot|$) by setting

$$|x - y| = \inf \sum_{i=1}^{N} |x_i - x_{i+1}|$$

where $x_1 = x$, $x_{N+1} = y$, and for each i, $\{x_i, x_{i+1}\} \subseteq X$ or $\{x_i, x_{i+1}\} \subseteq [x', y']^*$ for some $x', y' \in \widetilde{X}$. It is easy to check that the resulting metric space Y is separable and X is a metric subspace in Y. Moreover, the following lemma is immediate from the definition of Y.

Lemma 3.5. *Let $F \subseteq X$ be compact and $x, y \in \widetilde{X}$. Then*

$$\operatorname{dist}([x, y]^*, F) = \operatorname{dist}(\{x, y\}, F).$$

We will let

$$B_\Delta := B_Y(\zeta_\Delta, \ell(\Delta)) \supseteq B_X(\zeta_\Delta, \ell(\Delta)).$$

For $\Delta \in \mathscr{D}_n$, let

$$\Gamma_\Delta = \bigcup \bigl\{ [x, y]^* \subseteq C(X) \oplus \mathbb{R}^{\widetilde{X}\times\widetilde{X}} : x, y \in X_{n+n_0} \cap M B_\Delta \bigr\}$$

where n_0 is an integer we will pick later. Note that Γ_Δ is connected and contains ζ_Δ.

Now define

$$\Gamma = E \cup \bigcup_{\Delta\in\mathscr{P}} \Gamma_\Delta.$$

Lemma 3.6.
$$\mathscr{H}^1(\Gamma) < \infty.$$

Proof. We first claim that

$$\mathscr{H}^1(E) \le 10\mu(E). \tag{3-3}$$

Indeed, let $0 < \delta < r_0$. Take any countable collection of balls centered on E of radii less than δ that cover E. Since μ is doubling, we can use the Vitali covering theorem [Heinonen 2001, Theorem 1.2] to find a countable subcollection of disjoint balls B_i with radii $r_i < \delta$ centered on E so that $E \subseteq \bigcup 5B_i$. Then

$$\mathscr{H}^1_\delta(E) \le \sum 10 r_i \le 10 \sum \mu(B_i) \le 10\mu(\{x \in X : \operatorname{dist}(x, E) < \delta\}).$$

Since $\bigcap_{\delta>0}\{x \in X : \operatorname{dist}(x, E) < \delta\} = E$, sending $\delta \to 0$, we obtain $\mathscr{H}^1(E) \le 10\mu(E)$, which proves the claim.

With this estimate in hand, we have

$$\begin{aligned}\mathscr{H}^1(\Gamma) &\leq \mathscr{H}^1(E)+\sum_{\Delta\in\mathscr{P}}\mathscr{H}^1(\Gamma_\Delta) \overset{(3\text{-}3)}{\leq} 10\mu(E)+C\sum_{\Delta\in\mathscr{P}}\ell(\Delta)\\ &\overset{(3\text{-}1)}{\leq} 10\mu(E)+C\sum_{\Delta\in\mathscr{P}}\mu(\Delta) \overset{(3\text{-}2)}{\leq} 10\mu(E)+C\mu(\Delta_0)<\infty\end{aligned}$$

where C here stands for various constants that depend only on δ, M, n_0, ρ, and the doubling constant C_μ. □

Lemma 3.7. *Γ is compact.*

Proof. To see this, let $x_n \in \Gamma$ be any sequence. If $x_n \in \Gamma_\Delta$ infinitely many times for some $\Delta \in \mathscr{P}$ or is in E infinitely many times, then since each of these sets are compact, we can find a convergent subsequence with a limit in Γ. Otherwise, x_n visits infinitely many Γ_Δ. Let x_{n_j} be a subsequence so that $x_{n_j} \in \Gamma_{\Delta_j}$ where each $\Delta_j \in \mathscr{P}$ is distinct. Then $\ell(\Delta_j) \to 0$, and since $\Delta \cap E \neq \varnothing$ for all $\Delta \in \mathscr{P}$, $\mathrm{dist}(x_{n_j}, E) \to 0$. Pick $x'_{n_j} \in E \cap \Delta_j$. Since E is compact, there is a subsequence $x'_{n_{j_k}}$ converging to a point in E, and $x_{n_{j_k}}$ will have the same limit. We have thus shown that any sequence in Γ has a convergent subsequence, which implies Γ is compact. □

Lemma 3.8. *A compact connected metric space X of finite length can be parametrized by a Lipschitz image of an interval in $\mathbb{R}$; that is, $X = f([0, 1])$ where $f : [0, 1] \to X$ is Lipschitz.*

A proof of this fact for Hilbert spaces is given in [Schul 2007, Corollary 3.7], but the same proof works in our setting, so we omit it. Hence, to show that Γ (and hence E) is rectifiable, all that remains to show is that Γ is connected.

Lemma 3.9. *The set Γ is connected.*

Proof. Suppose for the sake of a contradiction that there exist two open and disjoint sets A and B that cover Γ, and set $\Gamma_A = \Gamma \cap A$ and $\Gamma_B = \Gamma \cap B$. Suppose without loss of generality that $\Gamma_{\Delta_0} \subseteq \Gamma_A$, which we may do since Γ_{Δ_0} is connected. We sort the proof into a series of steps.

(a) $\Gamma_B \subseteq 2B_{\Delta_0}$. To see this, suppose instead that there is $z \in \Gamma_B \setminus 2B_{\Delta_0}$. Then $z \in [x, y]^* \subseteq \Gamma_\Delta$ for some $\Delta \in \mathscr{P}$. Moreover, $\mathrm{dist}(z, \{x, y\}) \leq 2|x-y| \leq 4M\ell(\Delta)$ since $x, y \in MB_\Delta$. Since $\zeta_\Delta \in \Delta \subseteq \Delta_0$ and $x \in MB_\Delta$, we get

$$\begin{aligned}\ell(\Delta_0) &\leq \mathrm{dist}(z, B_{\Delta_0}) \leq |z-x| + \mathrm{dist}(x, B_{\Delta_0}) \leq 4M\ell(\Delta) + M\ell(\Delta)\\ &= 5M\ell(\Delta).\end{aligned}$$

For n_0 large enough so that $5M\rho^{n_0} < 1$, this implies $\zeta_\Delta \in X_{n+n_0} \cap MB_{\Delta_0}$ and so $\Gamma_\Delta \cap \Gamma_{\Delta_0} \neq \varnothing$. Hence, $\Gamma_\Delta \subseteq \Gamma_A$ since Γ_Δ is connected, contradicting that $z \in \Gamma_B$. This proves the claim.

(b) The open sets $A' = A \cup (\overline{4B_{\Delta_0}})^c$ and $B' = B \cap 2B_{\Delta_0}$ are disjoint and cover Γ. First, observe that

$$\begin{aligned}A' \cap B' &= (A \cap B \cap 2B_{\Delta_0}) \cup ((\overline{4B_{\Delta_0}})^c \cap B \cap 2B_{\Delta_0})\\ &\subseteq (A \cap B) \cup ((\overline{4B_{\Delta_0}})^c \cap 2B_{\Delta_0}) = \varnothing.\end{aligned}$$

Moreover, by part (a),

$$\Gamma \cap (A' \cup B') \supseteq \Gamma_A \cup (\Gamma_B \cap 2B_{\Delta_0}) = \Gamma_A \cup \Gamma_B = \Gamma,$$

which completes the proof of this step.

(c) Set $\Gamma_{A'} = \Gamma \cap A'$ and $\Gamma_{B'} = \Gamma \cap B'$. These sets are disjoint by part (b), and hence, they are compact since Γ was compact. We define new open sets

$$A'' = (\overline{4B_{\Delta_0}})^c \cup \bigcup_{\xi \in \Gamma_{A'}} B_Y(\xi, \operatorname{dist}(\xi, \Gamma_{B'})/2)$$

and

$$B'' = \bigcup_{\xi \in \Gamma_{B'}} B_Y(\xi, \operatorname{dist}(\xi, \Gamma_{A'})/2).$$

We claim these sets are disjoint. Suppose there is $z \in A'' \cap B''$. Then $z \in B_Y(\xi, \operatorname{dist}(\xi, \Gamma_{A'})/2)$ for some $\xi \in \Gamma_{B'}$. If we also have $z \in B_Y(\xi', \operatorname{dist}(\xi', \Gamma_{B'})/2)$ for some $\xi' \in \Gamma_{A'}$, then

$$\max\{\operatorname{dist}(\xi, \Gamma_{B'}), \operatorname{dist}(\xi', \Gamma_{A'})\} \le |\xi - \xi'| \le |\xi - z| + |z - \xi| < \frac{\operatorname{dist}(\xi, \Gamma_{B'})}{2} + \frac{\operatorname{dist}(\xi', \Gamma_{A'})}{2},$$

which is a contradiction, so we must have $z \in (\overline{4B_{\Delta_0}})^c$. Since $\xi \in \Gamma_{B'}$, we know $\xi \in 2B_{\Delta_0}$ by part (a), and $\zeta_{\Delta_0} \in \Gamma_{\Delta_0} \subseteq \Gamma_{A'}$ implies $\operatorname{dist}(\xi, \Gamma_{A'}) \le 2\ell(\Delta_0)$. Hence,

$$B_Y(\xi, \operatorname{dist}(\xi, \Gamma_{A'})/2) \subseteq B_Y(\xi, \ell(\Delta_0)) \subseteq B_Y(\zeta_{\Delta_0}, 3\ell(\Delta_0)) = 3B_{\Delta_0},$$

which proves the claim.

(d) Note that $X \setminus (A'' \cup B'')$ is nonempty since X is connected and A'' and B'' are disjoint open sets. Moreover, $X \setminus (A'' \cup B'') \subseteq \overline{4B_{\Delta_0}}$ and hence a bounded set; since X is a doubling metric space, $X \setminus (A'' \cup B'')$ is in fact totally bounded and thus compact by the Heine–Borel theorem. This implies we can find a point

$$z \in X \setminus (A'' \cup B'') \subseteq \overline{4B_{\Delta_0}}$$

of maximal distance from the compact set Γ.

(e) Let $\xi \in E$ be the closest point to z and Δ the smallest cube containing ξ so that $z \in 5B_\Delta$; since $z \in \overline{4B_{\Delta_0}} \subseteq 5B_{\Delta_0}$, this is well defined. We claim $\Delta \in \mathscr{P}$. If Δ_1 denotes the child of Δ that contains ξ, then $z \notin 5B_{\Delta_1}$, and so

$$\begin{aligned} \operatorname{dist}(z, E) = |\xi - z| &\ge |z - \zeta_{\Delta_1}| - |\zeta_{\Delta_1} - \xi| \ge 5\ell(\Delta_1) - \ell(\Delta_1) \\ &= 4\rho\ell(\Delta). \end{aligned} \tag{3-4}$$

Thus, for $M > 10$, $B_X(z, 4\rho\ell(\Delta)) \subseteq MB_\Delta \setminus E$, so if $\delta < 4\rho$, then $\Delta \in \mathscr{P}$, which proves the claim.

(f) Since $\Delta \in \mathscr{P}$, $X_{n(\Delta)+n_0}$ is a maximal $\rho^{n(\Delta)+n_0}$-net,

$$\rho^{n(\Delta)+n_0} < \rho^{n_0}\ell(\Delta) < \ell(\Delta),$$

and $z \in 5B_\Delta$, we can find

$$\begin{aligned} \zeta &\in X_{n(\Delta)+n_0} \cap B_X(z, \rho^{n(\Delta)+n_0}) \qquad (3\text{-}5)\\ &\subseteq X_{n(\Delta)+n_0} \cap B_X(\zeta_\Delta, 5\ell(\Delta)+\rho^{n(\Delta)+n_0}) \\ &\subseteq X_{n(\Delta)+n_0} \cap B_X(\zeta_\Delta, 6\ell(\Delta)) \subseteq \Gamma_\Delta, \qquad (3\text{-}6) \end{aligned}$$

where the last containment follows if we assume $M > 6$.

Since Γ_Δ is connected and A' and B' are disjoint open sets, we may without loss of generality suppose $\Gamma_{A'} \supseteq \Gamma_\Delta$ and let $\zeta' \in \Gamma_{B'}$ be the closest point to ζ. Then

$$|z-\zeta| \geq |\zeta-\zeta'|/2 = \operatorname{dist}(\zeta, \Gamma_{B'})/2 \qquad (3\text{-}7)$$

since otherwise would imply $z \in B_Y(\zeta, \operatorname{dist}(\zeta, \Gamma_{B'})/2) \subseteq A''$, contradicting that $z \in X \setminus (A'' \cup B'')$.

We may assume $\zeta' \in \Gamma_{\Delta'}$ for some $\Delta' \in \mathscr{P}$, and we assume Δ' is the largest such cube for which this happens. Note that this implies $\Gamma_{\Delta'} \subseteq \Gamma_{B'}$ since $\zeta' \in \Gamma_{B'} \cap \Gamma_{\Delta'}$ and $\Gamma_{\Delta'}$ is connected. By Lemma 3.5 with $F = \{\zeta\}$, we can assume $\zeta' \in X$, and so $\zeta' \in X_{n(\Delta')+n_0} \cap MB_{\Delta'}$.

(g) We claim that $n(\Delta)+1 \leq n(\Delta') \leq n(\Delta)+2$. Note that, since

$$5\rho^{n(\Delta)+n_0} \leq \ell(\Delta)\rho^{n_0} \leq \rho\ell(\Delta) < \ell(\Delta), \qquad (3\text{-}8)$$

we have

$$|\zeta'-\zeta_\Delta| \;\leq\; |\zeta'-\zeta|+|\zeta-\zeta_\Delta| \overset{(3\text{-}6)}{\underset{}{\overset{(3\text{-}7)}{<}}} 2|\zeta-z|+6\ell(\Delta) \overset{(3\text{-}5)}{<} 2\rho^{n(\Delta)+n_0}+6\ell(\Delta) \overset{(3\text{-}8)}{\leq} 8\ell(\Delta). \qquad (3\text{-}9)$$

Thus, for $M > 8$, we must have $n(\Delta') > n(\Delta)$; otherwise, since $\xi \in \Delta \subseteq B_\Delta$, we would have

$$\zeta' \in X_{n(\Delta')+n_0} \cap 8B_\Delta \subseteq X_{n(\Delta)+n_0} \cap MB_\Delta \subseteq \Gamma_\Delta$$

so that $\Gamma_\Delta \cap \Gamma_{\Delta'} \neq \varnothing$, which implies $\Gamma_{A'} \cap \Gamma_{B'} \neq \varnothing$, a contradiction. Thus, $\ell(\Delta') < \ell(\Delta)$, which proves the first inequality in the claim.

Note this implies $\ell(\Delta') \leq \rho\ell(\Delta)$. Let $\xi' \in \Delta' \cap E$ (which exists since $\Delta' \in \mathscr{P}$). Since $\zeta' \in MB_{\Delta'}$,

$$\begin{aligned} 4\rho\ell(\Delta) &\overset{(3\text{-}4)}{\leq} \operatorname{dist}(z, E) \leq |\xi'-z| \leq |\xi'-\zeta_{\Delta'}|+|\zeta_{\Delta'}-\zeta'|+|\zeta'-\zeta|+|\zeta-z| \\ &\overset{(3\text{-}7)}{\leq} \ell(\Delta')+M\ell(\Delta')+2|\zeta-z|+|\zeta-z| \leq (M+1)\ell(\Delta')+3\rho^{n(\Delta)+n_0} \\ &\overset{(3\text{-}8)}{\leq} (M+1)\ell(\Delta')+\rho\ell(\Delta) \end{aligned}$$

and so

$$\frac{3\rho}{M+1}\ell(\Delta) \leq \ell(\Delta').$$

Thus, $\rho < 3/(M+1)$ implies $\rho^2\ell(\Delta) \leq \ell(\Delta')$, and so $n(\Delta') \leq n(\Delta)+2$, which finishes the claim.

(h) Now we'll show that $\Gamma_\Delta \cap \Gamma_{\Delta'} \neq \varnothing$. Observe that

$$|\zeta_\Delta-\zeta_{\Delta'}| \leq |\zeta_\Delta-\zeta'|+|\zeta'-\zeta_{\Delta'}| \overset{(3\text{-}9)}{\leq} 8\ell(\Delta)+M\ell(\Delta') \leq (8+M\rho)\ell(\Delta) < M\ell(\Delta) \qquad (3\text{-}10)$$

if $\rho^{-1} > M > 9$. Since $n(\Delta') \le n(\Delta)+2$, we have that $\zeta_{\Delta'} \in X_{n(\Delta)+n_0} \cap MB_\Delta$ for $n_0 \ge 2$ and so $\zeta_{\Delta'} \in \Gamma_\Delta$. But $\zeta_{\Delta'} \in X_{n(\Delta')+n_0} \cap MB_{\Delta'} \subseteq \Gamma_{\Delta'}$; thus, $\Gamma_\Delta \cap \Gamma_{\Delta'} \neq \varnothing$, which proves the claim.

This gives us a grand contradiction since $\Gamma_\Delta \subseteq \Gamma_{A'}$ and $\Gamma_{\Delta'} \subseteq \Gamma_{B'}$, and we assumed these sets to be disjoint. □

Combining Lemmas 3.6, 3.7, 3.8, and 3.9, we have now shown that E is contained in the Lipschitz image of an interval in $\mathbb{R}$. This completes the proof of Lemma 3.2. □

Appendix: Proof of Lemma 3.4

For $\Delta \in \mathscr{D}$, define $B_\Delta = B_X(\zeta_\Delta, \ell(\Delta))$. For $\Delta \in \mathscr{P}$, let $\xi_\Delta \in MB_\Delta$ be such that $\operatorname{dist}(\xi, E) \ge \delta\ell(\Delta)$. Let $\mathscr{M}$ be the collection of maximal cubes for which $2B_\Delta \subseteq E^c$ and $\tilde\Delta \in \mathscr{M}$ be the largest cube containing ξ_Δ. Then if $\tilde\Delta^1$ denotes the parent cube of $\tilde\Delta$, $2B_{\tilde\Delta^1} \cap E \neq \varnothing$, and so

$$\delta\ell(\Delta) \le \operatorname{dist}(\xi_\Delta, E) \le \operatorname{diam} 2B_{\tilde\Delta^1} \le 4\ell(\tilde\Delta^1) = \frac{4}{\rho}\ell(\tilde\Delta). \tag{A-1}$$

Moreover,

$$\ell(\tilde\Delta) \le \frac{2M}{c_0}\ell(\Delta), \tag{A-2}$$

for otherwise $\tilde\Delta \supseteq c_0 B_{\tilde\Delta} \supseteq MB_\Delta \supseteq \Delta$, and since $\Delta \cap E \neq \varnothing$, this means $2B_{\tilde\Delta} \cap E \neq \varnothing$, contradicting our definition of $\tilde\Delta$.

Let N_Δ be such that

$$2^{N_\Delta} c_0 \ell(\tilde\Delta) > 2M\ell(\Delta) > 2^{N_\Delta - 1} c_0 \ell(\tilde\Delta). \tag{A-3}$$

Then $2^{N_\Delta} c_0 B_{\tilde\Delta} \supseteq MB_\Delta$, and $2^{N_\Delta} < \dfrac{4M\ell(\Delta)}{c_0\ell(\tilde\Delta)}$, so

$$N_\Delta < \log_2\left(\frac{4M\ell(\Delta)}{c_0\ell(\tilde\Delta)}\right). \tag{A-4}$$

Thus,

$$\begin{aligned}\frac{\mu(\tilde\Delta)}{\mu(\Delta)} &\ge \frac{\mu(c_0 B_{\tilde\Delta})}{\mu(\Delta)} \overset{\text{(1-1)}}{\ge} \frac{\mu(2^{N_\Delta} c_0 B_{\tilde\Delta})}{C_\mu^{N_\Delta}\mu(\Delta)} \overset{\text{(A-3)}}{\ge} \frac{\mu(MB_\Delta)}{C_\mu^{N_\Delta}\mu(\Delta)} \\ &\overset{\text{(A-4)}}{\ge} C_\mu^{\log_2 c_0/(4M)}\left(\frac{\ell(\tilde\Delta)}{\ell(\Delta)}\right)^{\log_2 C_\mu} \overset{\text{(A-1)}}{\ge} C_\mu^{\log_2 c_0/(4M)}\left(\frac{4}{\rho}\right)^{\log_2 C_\mu} =: a.\end{aligned} \tag{A-5}$$

Since μ is doubling and Δ and Δ' are always of comparable sizes by (A-1) and (A-2), there is b depending on M, δ, ρ, c_0, and C_μ such that at most b many cubes $\Delta \in \mathscr{M}$ with $\tilde\Delta = \Delta'$ for some fixed Δ'. Hence, for $\Delta' \subseteq \Delta_0$ with $\Delta \cap E \neq \varnothing$,

$$\begin{aligned}\sum_{\substack{\Delta \subseteq \Delta' \\ \Delta \in \mathscr{P}}} \mu(\Delta) &\overset{\text{(A-5)}}{\le} \sum_{\substack{\Delta \subseteq \Delta' \\ \Delta \in \mathscr{P}}} a\mu(\tilde\Delta) = \sum_{\substack{\Delta' \in \mathscr{M} \\ \Delta \subseteq MB_{\Delta_0}}} \sum_{\substack{\Delta \subseteq \Delta' \\ \Delta \in \mathscr{P} \\ \tilde\Delta = \Delta'}} a\mu(\tilde\Delta) \le \sum_{\substack{\Delta' \in \mathscr{M} \\ \Delta \subseteq MB_{\Delta_0}}} ab\mu(\Delta') \\ &\le ab\mu(MB_{\Delta_0} \setminus E) \le ab\mu(MB_{\Delta_0}) \overset{\text{(1-1)}}{\le} abC_\mu^{\log_2 M/c_0 + 1}\mu(c_0 B_{\Delta_0}) \le abC_\mu^{\log_2 M/c_0+1}\mu(\Delta_0).\end{aligned}$$

This finishes the proof of Lemma 3.4.

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JONAS AZZAM: jazzam@mat.uab.cat
Departament de Matemàtiques, Universitat Autònoma de Barcelona, Edifici C Facultat de Ciències, 08193 Bellaterra, Spain

MIHALIS MOURGOGLOU: mmourgoglou@crm.cat
*Departament de Matemàtiques, Universitat Autònoma de Barcelona and Centre de Recerca Matemàtica,
Edifici C Facultat de Ciències, 08193 Bellaterra, Spain*

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