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ON CHARACTERIZATION OF THE SHARP STRICHARTZ INEQUALITY FOR THE SCHRÖDINGER EQUATION

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We study the extremal problem for the Strichartz inequality for the Schrödinger equation on $\mathbb{R} \times \mathbb{R}^2$. We show that the solutions to the associated Euler–Lagrange equation are exponentially decaying in the Fourier space and thus can be extended to be complex analytic. Consequently, we provide a new proof of the characterization of the extremal functions: the only extremals are Gaussian functions, as investigated previously by Foschi, Hundertmark and Zharnitsky.

1. Introduction

We begin with some notation. For a Schwarz function f on \mathbb{R}^d , $d \geq 1$, define the Fourier transform

$$\mathcal{F}(f)(\xi) = \hat{f}(\xi) = \int_{\mathbb{R}^d} e^{-ix \cdot \xi} f(x) dx, \quad \xi \in \mathbb{R}^d.$$

The inverse of the Fourier transform,

$$\mathcal{F}^{-1}(f)(x) = f^\vee(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot \xi} f(\xi) d\xi, \quad x \in \mathbb{R}^d.$$

The linear Strichartz inequality for the Schrödinger equation [Keel and Tao 1998; Tao 2006] asserts that

$$\|e^{it\Delta} f\|_{L_{t,x}^{2+4/d}(\mathbb{R} \times \mathbb{R}^d)} \leq C_d \|f\|_{L^2(\mathbb{R}^d)}, \quad (1)$$

where $e^{it\Delta} f(x) = (1/(2\pi)^d) \int_{\mathbb{R}^d} e^{ix \cdot \xi + it|\xi|^2} \hat{f}(\xi) d\xi$. We specify $d = 2$ and consider

$$\|e^{it\Delta} f\|_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^2)} \leq \mathbf{R} \|f\|_{L^2(\mathbb{R}^2)}, \quad (2)$$

where

$$\mathbf{R} := \sup \left\{ \frac{\|e^{it\Delta} f\|_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^2)}}{\|f\|_{L^2(\mathbb{R}^2)}} : f \in L^2, f \neq 0 \right\}. \quad (3)$$

We define an extremal function or extremal to (2) to be a nonzero function $f \in L^2$ such that the inequality is optimized, in the sense that

$$\|e^{it\Delta} f\|_{L_{t,x}^4(\mathbb{R} \times \mathbb{R}^2)} = \mathbf{R} \|f\|_{L^2(\mathbb{R}^2)}. \quad (4)$$

The extremal problem of (2) concerns:

- (i) Whether there exists an extremal function?

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(ii) How to characterize the extremal functions? What are the explicit forms of extremal functions? Are they unique up to the symmetry of the inequality?

From Foschi [2007] and Hundertmark and Zharnitsky [2006], it is known that the Gaussian functions are the only extremal functions of the linear Strichartz inequality (2) for the dimensions $d = 1, 2$. Here Gaussian functions $\mathbb{R}^d \rightarrow \mathbb{C}$, $d = 1, 2$, are of the form

$$e^{A|x|^2+B \cdot x+C}$$

with $A, C \in \mathbb{C}$, $B \in \mathbb{C}^d$ and the real part of A negative. The existence of extremizers was established previously by Kunze [2003] for the Strichartz inequality (1) when $d = 1$. When $d \geq 3$, existence of extremizers is proved by the second author in [Shao 2009].

In this note, we are interested in the problem of how to characterize extremals for (2) via the study of the associated Euler–Lagrange equation. We show that the solutions of this generalized Euler–Lagrange equation enjoy fast decay in the Fourier space and thus can be extended to be complex analytic; see Theorem 1.1. Then, as an easy consequence, we give an alternative proof that all extremal functions to (2) are Gaussians, based on solving a functional equation of extremizers derived in [Foschi 2007]; see (7) and Theorem 1.2. Indeed, in the proof given below we use the information that f is twice continuously differentiable, i.e., $f \in C^2$, which can be lowered to continuity by a more refined argument. The functional inequality (7) is a key ingredient in Foschi’s proof. To prove f in (7) to be a Gaussian function, local integrability of f is assumed in [Foschi 2007], which is further reduced to measurable functions in [Charalambides 2013].

Let f be an extremal function to (2) with the constant R . Then f satisfies the generalized Euler–Lagrange equation

$$\omega \langle g, f \rangle = \mathfrak{Q}(g, f, f, f) \quad \text{for all } g \in L^2, \tag{5}$$

where $\omega = \mathfrak{Q}(f, f, f, f) / \|f\|_{L^2}^2 > 0$ and $\mathfrak{Q}(f_1, f_2, f_3, f_4)$ is the integral

$$\int_{(\mathbb{R}^2)^4} \bar{f}_1(\xi_1) \bar{f}_2(\xi_2) \hat{f}_3(\xi_3) \hat{f}_4(\xi_4) \delta(\xi_1 + \xi_2 - \xi_3 - \xi_4) \delta(|\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 - |\xi_4|^2) d\xi_1 d\xi_2 d\xi_3 d\xi_4 \tag{6}$$

for $f_i \in L^2(\mathbb{R}^2)$, $1 \leq i \leq 4$, and $\delta(\xi) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{i\xi \cdot x} dx$ in the distribution sense for $d = 1, 2$. The proof of (5) is standard; see, e.g., [Evans 2010, p. 489] or [Hundertmark and Lee 2012, Section 2] for similar derivations of Euler–Lagrange equations.

Theorem 1.1. *If f solves the generalized Euler–Lagrange equation (5) for some $\omega > 0$, then there exists $\mu > 0$ such that*

$$e^{\mu|\xi|^2} \hat{f} \in L^2(\mathbb{R}^2).$$

Furthermore, f can be extended to be complex analytic on \mathbb{C}^2 .

To prove this theorem, we follow the argument in [Hundertmark and Shao 2012]. Similar reasoning has appeared previously in [Erdoğan et al. 2011; Hundertmark and Lee 2009]. It relies on a multilinear weighted Strichartz estimate and a continuity argument. See Lemmas 2.1 and 2.2.

Next we prove that the extremals to (2) are Gaussian functions. We start with the study of the functional equation derived in [Foschi 2007], which reads

$$f(x)f(y) = f(w)f(z) \tag{7}$$

for any $x, y, w, z \in \mathbb{R}^2$ such that

$$x + y = w + z \quad \text{and} \quad |x|^2 + |y|^2 = |w|^2 + |z|^2. \tag{8}$$

Note that $x, y, w, z \in \mathbb{R}^2$ satisfy the relation (8) if and only if these four points form a rectangle in \mathbb{R}^2 with vertices x, y, w and z . Indeed, by (8), these four points x, y, w and z form a parallelogram on \mathbb{R}^2 and $x \cdot y = w \cdot z$. Secondly, $w - x$ is perpendicular to $z - x$, since $(w - x) \cdot (z - x) = w \cdot z - w \cdot x - x \cdot z + |x|^2 = w \cdot z - (x + y) \cdot x + |x|^2 = w \cdot z - y \cdot x = 0$. This proves that x, y, w and z form a rectangle on \mathbb{R}^2 . In [Foschi 2007], it is proven that $f \in L^2$ satisfies (7) if and only if f is an extremal function to (2). Basically, this comes from two aspects. One is that, in the Foschi’s proof of the sharp Strichartz inequality, only the Cauchy–Schwarz inequality is used at one place besides equality. So the equality in the Strichartz inequality (2), or equivalently the equality in Cauchy-Schwarz, yields the same functional equation as (7), where f is replaced by \hat{f} . The other one is that the Strichartz norm for the Schrödinger equation satisfies the identity

$$\|e^{it\Delta} f\|_{L^4(\mathbb{R} \times \mathbb{R}^2)} = C \|e^{it\Delta} f^\vee\|_{L^4(\mathbb{R} \times \mathbb{R}^2)} \tag{9}$$

for some $C > 0$.

Foschi [2007] is able to show that all the solutions to (7) are Gaussians under the assumption that f is a locally integrable function. This can be viewed as an investigation of the Cauchy functional equation (7) for functions supported on the paraboloids. To characterize the extremals for the Tomas–Stein inequality for the sphere in \mathbb{R}^3 , [Christ and Shao 2012] studies the same functional equation (7) for functions supported on the sphere and prove that they are exponentially affine functions. Charalambides [2013] generalizes the analysis in [Christ and Shao 2012] to some general hypersurfaces in \mathbb{R}^n that include the sphere, paraboloids and cones as special examples and proves that the solutions are exponentially affine functions. In [Charalambides 2013; Christ and Shao 2012], the functions are assumed to be measurable functions.

By the analyticity established in Theorem 1.1, equations (7) and (8) have the following easy consequence, which recovers the result in [Foschi 2007; Hundertmark and Zharnitsky 2006].

Theorem 1.2. *Suppose that f is an extremal function to (2). Then*

$$f(x) = e^{A|x|^2 + B \cdot x + C}, \tag{10}$$

where $A, C \in \mathbb{C}$, $B \in \mathbb{C}^2$ and $\Re(A) < 0$.

Let f be an extremal function to (2). Then, by Theorem 1.1, f is continuous. This, together with (7) and (8), implies that any nontrivial f is nowhere vanishing on \mathbb{R}^2 ; see, e.g., [Foschi 2007, Lemma 7.13]. For any $a \in \mathbb{R}^2$, there is a disk $D(a, r) \subset \mathbb{C}^2$, $r > 0$, such that f is C^2 by Theorem 1.1 and f is nowhere vanishing. Then $\log f$ is C^2 on $D(a, r)$; see, e.g., [Krantz 1992, Lemma 6.1.9]. Similar claims can be

made for $\log f^2$. Then, up to a multiple of 2π ,

$$\log f^2(a) = \log f(a) + \log f(a).$$

After restriction to \mathbb{R}^2 , f satisfies (7) for x, y, w and z satisfying (8). So, by taking r sufficiently small,

$$\log f(x) + \log f(y) = \log f(w) + \log f(z)$$

for $x, y, w, z \in B(a, r) \subset \mathbb{R}^2$ related as in (8). Since $\log f$ is twice differentiable, it is not hard to see that $\log f$ is a quadratic polynomial on $B(a, r)$. So $\log f$ is a quadratic polynomial on \mathbb{R}^2 . Indeed, let $a = 0$ and $\phi(x_1) = \log f(x_1, 0)$, $\psi(0, x_2) = \log f(0, x_2)$. Then, since the four points (x_1, x_2) , $(x_2, -x_1)$, $(x_1 + x_2, x_2 - x_1)$ and $(0, 0)$ satisfy (8), we see that

$$[\phi(x_1) + \psi(x_2)] + [\phi(x_2) + \psi(-x_1)] = [\phi(x_1 + x_2) + \psi(x_2 - x_1)] + \log f(0, 0).$$

By differentiating firstly in x_1 and then in x_2 , we see that $\phi'' = \psi''$ is a constant. Thus f is a quadratic polynomial. It is easy to see that this argument generalizes to any $a \in \mathbb{R}^2$.

2. Complex analyticity

In this section, we show that the solutions to the generalized Euler–Lagrange equation (5) can be extended to be complex analytic.

We define

$$\begin{aligned} \eta &:= (\eta_1, \eta_2, \eta_3, \eta_4) \in (\mathbb{R}^2)^4, \\ a(\eta) &:= \eta_1 + \eta_2 - \eta_3 - \eta_4, \\ b(\eta) &:= |\eta_1|^2 + |\eta_2|^2 - |\eta_3|^2 - |\eta_4|^2. \end{aligned}$$

Let $\varepsilon \geq 0$ and $\mu \geq 0$. For $\xi \in \mathbb{R}^2$, define

$$F(\xi) := F_{\mu, \varepsilon}(\xi) = \frac{\mu |\xi|^2}{1 + \varepsilon |\xi|^2}. \tag{11}$$

Define the weighted multilinear integral for $h_i \in L^2(\mathbb{R}^2)$, $1 \leq i \leq 4$, by

$$M_F(h_1, h_2, h_3, h_4) := \int_{(\mathbb{R}^2)^4} e^{F(\eta_1) - \sum_{j=2}^4 F(\eta_j)} \prod_{j=1}^4 |h(\eta_j)| \delta(a(\eta)) \delta(b(\eta)) d\eta. \tag{12}$$

The multilinear estimate we need shows the weak interaction of Schrödinger waves between the high and low frequency. More precisely:

Lemma 2.1. *Let $h_i \in L^2(\mathbb{R}^2)$, $1 \leq i \leq 4$, and let $s > 1$ be a large number. If the Fourier transforms of h_1 and h_2 are supported in $\{\xi : |\xi| \leq s\}$ and $\{\xi : |\xi| \geq Ns\}$ with $N > 1$ a large number, respectively, then*

$$M_F(h_1, h_2, h_3, h_4) \leq CN^{-1/2} \prod_{j=1}^4 \|h_j\|_{L^2}. \tag{13}$$

Proof. The proof of this lemma needs the following two inequalities:

$$M_F(h_1, h_2, h_3, h_4) \leq \int_{(\mathbb{R}^2)^4} \prod_{j=1}^4 |h_j(\eta_j)| \delta(a(\eta)) \delta(b(\eta)) d\eta \tag{14}$$

and

$$\|e^{it\Delta} h_1 e^{it\Delta} h_2\|_{L^2_{t,x}} \leq CN^{-1/2} \|h_1\|_{L^2} \|h_2\|_{L^2}. \tag{15}$$

Together with the Cauchy–Schwarz inequality and the $L^2 \rightarrow L^4$ Strichartz inequality, the inequality (13) follows from (14) and (15). Note that (15) is established in [Bourgain 1998]. Thus it remains to establish (14), where we follow [Erdoğan et al. 2011; Hundertmark and Shao 2012].

On the support of η determined by $\delta(a(\eta))$ and $\delta(b(\eta))$, we have

$$\eta_1 + \eta_2 = \eta_3 + \eta_4 \quad \text{and} \quad |\eta_1|^2 + |\eta_2|^2 = |\eta_3|^2 + |\eta_4|^2.$$

Thus,

$$|\eta_1|^2 \leq |\eta_2|^2 + |\eta_3|^2 + |\eta_4|^2.$$

Since the function $x \mapsto x/(1 + \varepsilon x)$ is increasing on the interval $[0, \infty)$, we have

$$\frac{|\eta_1|^2}{1 + \varepsilon|\eta_1|^2} \leq \frac{\sum_{j=2}^4 |\eta_j|^2}{1 + \sum_{j=2}^4 \varepsilon|\eta_j|^2} = \sum_{j=2}^4 \frac{|\eta_j|^2}{1 + \sum_{j=2}^4 \varepsilon|\eta_j|^2} \leq \sum_{j=2}^4 \frac{|\eta_j|^2}{1 + \varepsilon|\eta_j|^2}.$$

This implies that $F(\eta_1) \leq \sum_{j=2}^4 F(\eta_j)$, since $\mu \geq 0$. Hence,

$$e^{F(\eta_1) - \sum_{j=2}^4 F(\eta_j)} \leq 1.$$

Therefore (14) follows by taking the absolute value in the integral. □

If $f \in L^2$ satisfies the generalized Euler–Lagrange equation (5), the following bootstrap lemma shows that f gains certain regularity; namely, there is a constant $\mu > 0$ depending on the function f such that $e^{\mu|\xi|^2} \hat{f} \in L^2$. This is enough to conclude that f can be extended to be complex analytic.

Lemma 2.2. *If f solves the generalized Euler–Lagrange equation (5) for some $\omega > 0$ and $\|f\|_{L^2} = 1$, then for $\hat{f}_> := \hat{f} 1_{|\xi| \geq s^2}$ with $s > 0$, there is a large constant $s \gg 1$ such that, for $\mu = s^{-4}$,*

$$\omega \|e^{F(\cdot)} \hat{f}_>\|_{L^2} \leq o_1(1) \|e^{F(\cdot)} \hat{f}_>\|_{L^2} + C \|e^{F(\cdot)} \hat{f}_>\|_{L^2}^2 + C \|e^{F(\cdot)} \hat{f}_>\|_{L^2}^3 + o_2(1), \tag{16}$$

where $\lim_{s \rightarrow \infty} o_i(1) = 0$ uniformly for all $\varepsilon > 0$, $i = 1, 2$, and the constant $C > 0$ is independent of ε and s .

Proof. Define $h(\xi) = e^{F(\xi)} \hat{f}(\xi)$ and $h_>(\xi) = e^{F(\xi)} \hat{f}_>$, where $\hat{f}_> = \hat{f} 1_{|\xi| \geq s^2}$. Let P denote the symbol of differentiation $-i\partial_x$; under the Fourier transform, $\hat{P} = |\xi|$. Correspondingly, we write $F(P)$ with the Fourier symbol $\mu|\xi|^2/(1 + \varepsilon|\xi|^2)$.

We expand

$$\|e^{F(\cdot)} \hat{f}_>\|_{L^2}^2 = \langle e^{F(\cdot)} \hat{f}_>, e^{F(\cdot)} \hat{f}_> \rangle = \langle e^{2F(\cdot)} \hat{f}_>, \hat{f}_> \rangle = \langle e^{2F(P)} f_>, f \rangle.$$

Thus, in the generalized Euler–Lagrange equation (5), setting $g = e^{2F(P)} f_{>}$, we see that

$$\omega \|e^{F(P)} f_{>}\|_{L^2}^2 = Q(e^{2F(P)} f_{>}, f, f, f). \tag{17}$$

Since $\hat{f} = e^{-F(\xi)} h$ and $e^{2F(\xi)} \hat{f}_{>} = e^{F(\xi)} h_{>}$,

$$\begin{aligned} Q(e^{2F(P)} f_{>}, f, f, f) &= \int_{(\mathbb{R}^2)^4} e^{2F(\xi_1)} \overline{\hat{f}_{>}(\xi_1)} \overline{\hat{f}_{>}(\xi_2)} \hat{f}(\xi_3) \hat{f}_4(\xi_4) \delta(a(\xi)) \delta(b(\xi)) d\xi \\ &= \int_{(\mathbb{R}^2)^4} \overline{e^{F(\xi_1)} h_{>}(\xi_1)} \overline{e^{-F(\xi_2)} h(\xi_2)} e^{-F(\xi_3)} h(\xi_3) e^{-F(\xi_4)} h(\xi_4) \delta(a(\xi)) \delta(b(\xi)) d\xi \\ &= \int_{(\mathbb{R}^2)^4} e^{F(\xi_1) - \sum_{j=2}^4 F(\xi_j)} h_{>}(\xi_1) h(\xi_2) h(\xi_3) h(\xi_4) \delta(a(\xi)) \delta(b(\xi)) d\xi, \end{aligned}$$

where $a(\xi) = \xi_1 + \xi_2 - \xi_3 - \xi_4$ and $b(\xi) = |\xi_1|^2 + |\xi_2|^2 - |\xi_3|^2 - |\xi_4|^2$ for $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in (\mathbb{R}^2)^4$.

Thus,

$$\omega \|e^{F(P)} f_{>}\|_{L^2}^2 \leq M_F(h_{>}, h, h, h). \tag{18}$$

Define

$$h_{\sim} = h 1_{s \leq |\xi| \leq s^2}, h_{\ll} = h 1_{|\xi| < s} \quad \text{and} \quad h_{<} = h_{\ll} + h_{\sim}.$$

We split the integral $M_F(h_{>}, h, h, h)$ into the following pieces:

$$M_F(h_{>}, h_{<}, h_{<}, h_{<}) + \sum_{j_2, j_3, j_4} M_F(h_{>}, h_{j_2}, h_{j_3}, h_{j_4}) =: A + B,$$

where h_{j_k} is either $h_{>}$ or $h_{<}$, but at least one is $h_{>}$. We further split A into two terms,

$$M_F(h_{>}, h_{\ll}, h_{<}, h_{<}) + M_F(h_{>}, h_{\sim}, h_{<}, h_{<});$$

we estimate this term by using Lemma 2.1:

$$A \lesssim s^{-1/2} \|h_{>}\|_{L^2} \|h_{\ll}\|_{L^2} \|h_{<}\|_{L^2}^2 + \|h_{>}\|_{L^2} \|h_{\sim}\|_{L^2} \|h_{<}\|_{L^2}^2 \lesssim \|h_{>}\|_{L^2} (s^{-1/2} \|h_{\ll}\|_{L^2} + \|h_{\sim}\|_{L^2}) \|h_{<}\|_{L^2}^2.$$

Since $\|f\|_{L^2} = 1$,

$$\begin{aligned} \|h_{<}\|_{L^2} &\leq e^{\mu s^4} \|f\|_{L^2} = e^{\mu s^4}, \\ \|h_{\ll}\|_{L^2} &\leq e^{\mu s^2}, \\ \|h_{\sim}\|_{L^2} &\leq e^{\mu s^4} \|f_{\sim}\|_{L^2}, \end{aligned}$$

where f_{\sim} is defined by $\hat{f}_{\sim} = \hat{f} 1_{s \leq |\xi| \leq s^2}$. Thus we have

$$A \lesssim e^{3\mu s^4} \|h_{>}\|_{L^2} (s^{-1/2} e^{\mu s^2 - \mu s^4} + \|f_{\sim}\|_{L^2}). \tag{19}$$

Similarly we estimate the term B . We split B as $B_1 + B_2$, where $B_1 = \sum_{j_2, j_3, j_4} M_F(h_{>}, h_{j_2}, h_{j_3}, h_{j_4})$ contains exactly one $h_{>}$ in $\{h_{j_2}, h_{j_3}, h_{j_4}\}$, while $B_2 = \sum_{j_2, j_3, j_4} M_F(h_{>}, h_{j_2}, h_{j_3}, h_{j_4})$ contains two or more $h_{>}$.

To estimate B_1 ,

$$B_1 \lesssim e^{\mu s^4} \|h_{>}\|_{L^2}^2 \|h_{<}\|_{L^2} (s^{-1/2} e^{\mu s^2 - \mu s^4} + \|f_{\sim}\|_{L^2}) \lesssim e^{2\mu s^4} \|h_{>}\|_{L^2}^2 (s^{-1/2} e^{\mu s^2 - \mu s^4} + \|f_{\sim}\|_{L^2}). \quad (20)$$

To estimate B_2 ,

$$B_2 \lesssim \|h_{>}\|_{L^2}^3 \|h_{<}\|_{L^2} + \|h_{>}\|_{L^2}^4 \lesssim e^{\mu s^4} \|h_{>}\|_{L^2}^3 + \|h_{>}\|_{L^2}^4. \quad (21)$$

Thus, from (19), (20) and (21), we obtain

$$\begin{aligned} & \|e^{F(\cdot)} \hat{f}_{>}\|_{L^2}^2 \\ & \lesssim e^{3\mu s^4} \|h_{>}\|_{L^2} (s^{-1/2} e^{\mu s^2 - \mu s^4} + \|f_{\sim}\|_{L^2}) + e^{2\mu s^4} \|h_{>}\|_{L^2}^2 (s^{-1/2} e^{\mu s^2 - \mu s^4} + \|f_{\sim}\|_{L^2}) + e^{\mu s^4} \|h_{>}\|_{L^2}^3 + \|h_{>}\|_{L^2}^4. \end{aligned}$$

Since $\lim_{s \rightarrow \infty} \|f_{\sim}\|_{L^2} = 0$, we take s sufficiently large and set $\mu = s^{-4}$:

$$\omega \|e^{F(\cdot)} \hat{f}_{>}\|_{L^2} \leq o_1(1) \|e^{F(\cdot)} \hat{f}_{>}\|_{L^2} + C \|e^{F(\cdot)} \hat{f}_{>}\|_{L^2}^2 + C \|e^{F(\cdot)} \hat{f}_{>}\|_{L^2}^3 + o_2(1), \quad (22)$$

which completes the proof of Lemma 2.2. □

Remark 2.3. Clearly the choice of μ in the preceding lemma depends on the function f itself.

Now we conclude that f in Lemma 2.2 gains certain regularity.

Proof of Theorem 1.1. Let $f \in L^2$ and $f \neq 0$. We normalize f so that $\|f\|_{L^2} = 1$. In Lemma 2.2, we choose s sufficiently large such that $o_1(1) \leq \frac{1}{2}\omega$ and $o_2(1) \leq \frac{1}{2}M$, where $M = \sup\{G(x) : x \in [0, \infty)\}$, and

$$G(x) := \frac{1}{2}\omega x - Cx^2 - Cx^3, \quad x \in [0, \infty), \quad (23)$$

and C is the same constant as in (16). It is easy to see that $0 \leq M < \infty$. Then $G(x) \leq M$ for all $x \in [0, \infty)$ by Lemma 2.2. Also the function G is continuous on $[0, \infty)$. On the other hand, $G''(x) < 0$ for all $x \in (0, \infty)$; thus G is concave. The line $G = \frac{1}{2}M$ intersects at two points of the positive x axis, $x = x_0$ and $x = x_1 > 0$.

We define $H : (0, \infty) \rightarrow [0, \infty)$ via

$$H(\varepsilon) = \left(\int_{|\xi| \geq s^2} |e^{F_{s^{-4}, \varepsilon}(\xi)} \hat{f}|^2 d\xi \right)^{\frac{1}{2}}.$$

The function H is continuous on $(0, \infty)$ by the dominated convergence theorem and $H(0, \infty)$ is connected. Hence $G^{-1}([0, \frac{1}{2}M])$ is either contained in $[0, x_0]$ or $[x_1, \infty)$; only one alternative holds. For $\varepsilon = 1$ and s sufficiently large, $H(1) \geq x_1$ is impossible. Hence the first alternative holds.

Therefore $G^{-1}([0, \frac{1}{2}M]) \subset [0, x_0]$, which yields that

$$\|e^{F(\cdot)} \hat{f}_{>}\|_{L^2} \leq C_0, \quad \text{that is,} \quad \|e^{s^{-4}|\xi|^2/(1+\varepsilon|\xi|^2)} \hat{f}_{>}\|_{L^2} \leq C_0, \quad (24)$$

uniformly in all $\varepsilon > 0$. By the monotone convergence theorem,

$$\|e^{s^{-4}|\xi|^2} \hat{f}_{>}\|_{L^2} \leq C_0 < \infty.$$

It is clear that $e^{s^{-4}|\xi|^2} \hat{f} 1_{|\xi| \leq s^2} \in L^2$. Therefore,

$$e^{s^{-4}|\xi|^2} \hat{f} \in L^2.$$

Let $\mu = s^{-4}$. This proves the first half of Theorem 1.1.

To prove that f can be extended to be complex analytic on \mathbb{C}^2 , we observe that, by the Cauchy–Schwarz inequality, for any $\lambda \in \mathbb{R}$,

$$e^{\lambda|\xi|} \hat{f}(\xi) = e^{\lambda|\xi| - \mu|\xi|^2} e^{\mu|\xi|^2} \hat{f}(\xi) \in L^2(\mathbb{R}^2). \quad (25)$$

So it is not hard to see that f can be extended to be complex analytic on \mathbb{C}^2 ; see, e.g., [Reed and Simon 1975, Theorem IX.13]. Alternatively, analyticity can be obtained in the following way. Similarly to in (25) for $k \in \mathbb{N} \cup \{0\}$, $|\xi|^k e^{\lambda|\xi|} \hat{f} \in L^1(\mathbb{R}^2)$. For $z \in \mathbb{C}^2$, we choose $\lambda > |z|$, then

$$f(z) = (2\pi)^{-2} \int_{\mathbb{R}^2} e^{iz \cdot \xi - \lambda|\xi|} e^{\lambda|\xi|} \hat{f}(\xi) d\xi.$$

Then, by taking differentiation under the integral sign, complex analyticity follows. \square

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References

- [Bourgain 1998] J. Bourgain, “Refinements of Strichartz’ inequality and applications to 2D-NLS with critical nonlinearity”, *Internat. Math. Res. Notices* **1998**:5 (1998), 253–283. MR 99f:35184 Zbl 0917.35126
- [Charalambides 2013] M. Charalambides, “On restricting Cauchy–Pexider functional equations to submanifolds”, *Aequationes Math.* **86**:3 (2013), 231–253. MR 3127007 Zbl 1281.39021
- [Christ and Shao 2012] M. Christ and S. Shao, “On the extremizers of an adjoint Fourier restriction inequality”, *Adv. Math.* **230**:3 (2012), 957–977. MR 2921167 Zbl 1258.35007
- [Erdoğan et al. 2011] M. B. Erdoğan, D. Hundertmark, and Y.-R. Lee, “Exponential decay of dispersion managed solitons for vanishing average dispersion”, *Math. Res. Lett.* **18**:1 (2011), 11–24. MR 2012f:35501 Zbl 1239.35021
- [Evans 2010] L. C. Evans, *Partial differential equations*, 2nd ed., Graduate Studies in Mathematics **19**, American Mathematical Society, Providence, RI, 2010. MR 2011c:35002 Zbl 1194.35001
- [Foschi 2007] D. Foschi, “Maximizers for the Strichartz inequality”, *J. Eur. Math. Soc.* **9**:4 (2007), 739–774. MR 2008k:35389 Zbl 1231.35028
- [Hundertmark and Lee 2009] D. Hundertmark and Y.-R. Lee, “Decay estimates and smoothness for solutions of the dispersion managed non-linear Schrödinger equation”, *Comm. Math. Phys.* **286**:3 (2009), 851–873. MR 2009k:35299 Zbl 1173.35686
- [Hundertmark and Lee 2012] D. Hundertmark and Y.-R. Lee, “On non-local variational problems with lack of compactness related to non-linear optics”, *J. Nonlinear Sci.* **22**:1 (2012), 1–38. MR 2012m:49003 Zbl 1244.49008
- [Hundertmark and Shao 2012] D. Hundertmark and S. Shao, “Analyticity of extremizers to the Airy–Strichartz inequality”, *Bull. Lond. Math. Soc.* **44**:2 (2012), 336–352. MR 2914611 Zbl 1280.35126
- [Hundertmark and Zharnitsky 2006] D. Hundertmark and V. Zharnitsky, “On sharp Strichartz inequalities in low dimensions”, *Int. Math. Res. Not.* **2006** (2006), Art. ID 34080, 18. MR 2007b:35277 Zbl 1131.35308

- [Keel and Tao 1998] M. Keel and T. Tao, “Endpoint Strichartz estimates”, *Amer. J. Math.* **120**:5 (1998), 955–980. MR 2000d:35018 Zbl 0922.35028
- [Krantz 1992] S. G. Krantz, *Function theory of several complex variables*, 2nd ed., Wadsworth & Brooks/Cole Advanced Books & Software, Pacific Grove, CA, 1992. MR 93c:32001 Zbl 0776.32001
- [Kunze 2003] M. Kunze, “On the existence of a maximizer for the Strichartz inequality”, *Comm. Math. Phys.* **243**:1 (2003), 137–162. MR 2004i:35006 Zbl 1060.35133
- [Reed and Simon 1975] M. Reed and B. Simon, *Methods of modern mathematical physics, II: Fourier analysis, self-adjointness*, Academic Press, New York, 1975. MR 58 #12429b Zbl 0308.47002
- [Shao 2009] S. Shao, “Maximizers for the Strichartz and the Sobolev–Strichartz inequalities for the Schrödinger equation”, *Electron. J. Differential Equations* **2009** (2009), art. ID #3. MR 2010a:35032 Zbl 1173.35692
- [Tao 2006] T. Tao, *Nonlinear dispersive equations: local and global analysis*, CBMS Regional Conference Series in Mathematics **106**, American Mathematical Society, Providence, RI, 2006. MR 2008i:35211 Zbl 1106.35001

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