

ANALYSIS & PDE

Volume 9

No. 2

2016

DARYA E. APUSHKINSKAYA AND ALEXANDER I. NAZAROV

**A COUNTEREXAMPLE TO THE HOPF-OLEINIK LEMMA
(ELLIPTIC CASE)**

A COUNTEREXAMPLE TO THE HOPF–OLEINIK LEMMA (ELLIPTIC CASE)

DARYA E. APUSHKINSKAYA AND ALEXANDER I. NAZAROV

Dedicated to Professor M.V. Safonov

We construct a new counterexample to the Hopf–Oleinik boundary point lemma. It shows that for convex domains, the $C^{1,Dini}$ assumption on $\partial\Omega$ is the necessary and sufficient condition providing the estimates of Hopf–Oleinik type.

1. Introduction

The influence of the properties of a domain on the behavior of a solution is one of the most important topics in the qualitative analysis of partial differential equations.

The significant result in this field is the Hopf–Oleinik lemma, known also as the “boundary point principle”. This celebrated lemma states:

Lemma. *Let u be a nonconstant solution to a second-order homogeneous uniformly elliptic nondivergence equation with bounded measurable coefficients, and let u attain its extremum at a point x^0 located on the boundary of a domain $\Omega \subset \mathbb{R}^n$. Then $(\partial u / \partial \mathbf{n})(x^0)$ is necessarily nonzero provided that $\partial\Omega$ satisfies the proper assumptions at x^0 .*

This result was established in a pioneering paper of S. Zaremba [1910] for the Laplace equation in a 3-dimensional domain Ω having an interior touching ball at x^0 and generalized by G. Giraud [1932; 1933] to equations with Hölder-continuous leading coefficients and continuous lower-order coefficients in domains Ω belonging to the class $C^{1,\alpha}$ with $\alpha \in (0, 1)$.

Notice that a related assertion about the negativity on $\partial\Omega$ of the normal derivative of the Green’s function corresponding to the Dirichlet problem for the Laplace operator was proved much earlier for 2-dimensional smooth domains by C. Neumann [1888] (see also [Korn 1901]). The result of [Neumann 1888] was extended for operators with lower-order coefficients by L. Lichtenstein [1924]. The same version of the boundary point principle for the Laplacian and 3-dimensional domains satisfying a more flexible interior paraboloid condition was obtained by M. V. Keldysh and M. A. Lavrent’ev [1937].

A crucial step in studying the boundary point principle was made by E. Hopf [1952] and O. A. Oleĭnik [1952], who simultaneously and independently proved the statement for the general elliptic equations with bounded coefficients and domains satisfying an interior ball condition at x^0 .

Later the efforts of many mathematicians were focused on the generalization of the boundary point principle in several directions (for the details, we refer the reader to [Alvarado et al. 2011; Alvarado

2011] and references therein). Among these directions are the extension of the class of operators and the class of solutions, as well as the weakening of assumptions on the boundary.

The widening of the class of operators to singular/degenerate ones was made in the papers [Kamynin and Himčenko 1975; 1977; Alvarado et al. 2011], while the uniform elliptic operators with unbounded lower-order coefficients were studied in [Safonov 2010; Nazarov 2012] (see also [Nazarov and Uraltseva 2009]). We mention also the publications [Tolksdorf 1983; Mikayelyan and Shahgholian 2015], where the boundary point principle was established for a class of degenerate quasilinear operators including the p -Laplacian.

We note that before 2010, all the results were formulated for classical solutions, i.e., $u \in C^2(\Omega)$. The class of solutions was expanded in [Safonov 2010] to strong generalized solutions with Sobolev’s second-order derivatives. The latter requirement seems to be natural in the study of nondivergent elliptic equations.

The reduction of the assumptions on the boundary of Ω up to $C^{1,\text{Dini}}$ -regularity was realized for various elliptic operators in the papers [Widman 1967; Himčenko 1970; Lieberman 1985] (see also [Safonov 2008]). A weakened form of the Hopf–Oleinik lemma (the existence of a boundary point x^1 in any neighborhood of x^0 and a direction ℓ such that $(\partial u / \partial \ell)(x^1) \neq 0$) was proved in [Nadirashvili 1983] for a much wider class of domains including all Lipschitz ones. We mention also the paper [Sweers 1997], where the behavior of superharmonic functions near the boundary of a 2-dimensional domain with corners is described in terms of the main eigenfunction of the Dirichlet Laplacian.

The sharpness of some requirements was confirmed by corresponding counterexamples constructed in [Widman 1967; Himčenko 1970; Kamynin and Himčenko 1975; Safonov 2008; Alvarado et al. 2011; Nazarov 2012]. In particular, the counterexamples from [Widman 1967; Himčenko 1970; Safonov 2008] show that the Hopf–Oleinik result fails for domains lying entirely in non-Dini paraboloids.

The main result of our paper is a new counterexample (see Theorem 4.2) showing the sharpness of the Dini condition for the boundary of Ω . The simplest version of this counterexample can be formulated as follows:

Counterexample. *Let Ω be a convex domain in \mathbb{R}^n , let $\partial\Omega$ in a neighborhood of the origin be described by the equation $x_n = F(x')$ with $F \geq 0$ and $F(0) = 0$, and let $u \in W_{n,\text{loc}}^2(\Omega) \cap C(\bar{\Omega})$ be a solution of the uniformly elliptic equation*

$$-a^{ij}(x)D_i D_j u = 0 \quad \text{in } \Omega.$$

Suppose also that $u|_{\partial\Omega}$ vanishes at a neighborhood of the origin. If, in addition, the function

$$\delta(r) = \sup_{|x'| \leq r} \frac{F(x')}{|x'|}$$

is not Dini-continuous at zero, then $(\partial u / \partial \mathbf{n})(0) = 0$.

Thus, it turns out that for convex domains, the Dini-continuity assumption on $\delta(r)$ is necessary and sufficient for the validity of the boundary point principle. We emphasize that in our counterexample the Dini condition fails for the supremum of $F(x')/|x'|$, while in all the previous results of this kind, it fails for the infimum of $F(x')/|x'|$. In other words, we show that violating the Dini condition just in one direction causes the failure of the Hopf–Oleinik lemma.

Notation and conventions. Throughout the paper we use the following notation:

- $x = (x', x_n) = (x_1, \dots, x_{n-1}, x_n)$ is a point in \mathbb{R}^n .
- $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$.
- $|x|, |x'|$ are the Euclidean norms in the corresponding spaces.
- $\chi_{\mathcal{E}}$ denotes the characteristic function of the set $\mathcal{E} \subset \mathbb{R}^n$.
- Ω is a bounded domain in \mathbb{R}^n with boundary $\partial\Omega$.
- $\mathcal{P}_{r,h}(\bar{x}') = \{x \in \mathbb{R}^n : |x' - \bar{x}'| < r, 0 < x_n < h\}$ and $\mathcal{P}_r(\bar{x}') = \mathcal{P}_{r,r}(\bar{x}')$.
- $\mathcal{P}_{r,h} = \mathcal{P}_{r,h}(0)$ and $\mathcal{P}_r = \mathcal{P}_r(0)$.
- $B_r(x^0)$ is the open ball in \mathbb{R}^n with center x^0 and radius r ; $B_r = B_r(0)$.
- For $r_1 < r_2$, we define the annulus $\mathcal{B}(x^0, r_1, r_2) = B_{r_2}(x^0) \setminus \overline{B_{r_1}(x^0)}$.
- $v_+ = \max\{v, 0\}$ and $v_- = \max\{-v, 0\}$.
- $\|\cdot\|_{\infty, \Omega}$ denotes the norm in $L_{\infty}(\Omega)$.
- We adopt the convention that the indices i and j run from 1 to n . We also adopt the convention regarding summation with respect to repeated indices.
- D_i denotes the operator of (weak) differentiation with respect to x_i .
- $D = (D', D_n) = (D_1, \dots, D_{n-1}, D_n)$.
- \mathcal{L} is a linear uniformly elliptic operator with measurable coefficients

$$\mathcal{L}u \equiv -a^{ij}(x)D_iD_ju + b^i(x)D_iu, \quad v\mathcal{I}_n \leq (a^{ij}(x)) \leq v^{-1}\mathcal{I}_n, \tag{1}$$

where \mathcal{I}_n is the $n \times n$ identity matrix. We define $\mathbf{b}(x) = (b^1(x), \dots, b^n(x))$.

- We use the letters C and N (with or without indices) to denote various constants. To indicate that, say, C depends on some parameters, we list them in parentheses: $C(\dots)$.

Definition 1.1. We say that a function $\sigma : [0, 1] \rightarrow \mathbb{R}_+$ belongs to the class \mathcal{D}_1 if

- σ is increasing, $\sigma(0) = 0$, and $\sigma(1) = 1$;
- $\sigma(t)/t$ is summable and decreasing.

Remark 1.2. Our assumption about the decay of $\sigma(t)/t$ is not restrictive. Indeed, for any increasing function $\sigma : [0, 1] \rightarrow \mathbb{R}_+$ satisfying $\sigma(0) = 0$ and $\sigma(1) = 1$ and having summable $\sigma(t)/t$, we can define

$$\tilde{\sigma}(t) = t \sup_{\tau \in [t, 1]} \frac{\sigma(\tau)}{\tau}, \quad t \in (0, 1).$$

It is easy to see that $\tilde{\sigma} \in \mathcal{D}_1$, $\tilde{\sigma}(t)/t$ decreases and $\sigma(t) \leq \tilde{\sigma}(t)$ for all $t \in (0, 1]$.

Definition 1.3. Let a function σ belong to the class \mathcal{D}_1 . We define the function \mathcal{J}_{σ} as

$$\mathcal{J}_{\sigma}(s) := \int_0^s \frac{\sigma(\tau)}{\tau} d\tau. \tag{2}$$

Remark 1.4. The decreasing of $\sigma(t)/t$ implies

$$\sigma(t) \leq \mathcal{J}_\sigma(t) \quad \forall t \in [0, 1]. \tag{3}$$

In addition, for $t \leq t_0 \leq 1$, we have

$$\sigma(t/t_0) = \frac{\sigma(t/t_0)}{t/t_0} \cdot t/t_0 \leq \frac{\sigma(t)}{t} \cdot t/t_0 = \frac{\sigma(t)}{t_0}, \tag{4}$$

and, similarly,

$$\mathcal{J}_\sigma(t/t_0) \leq \frac{\mathcal{J}_\sigma(t)}{t_0}. \tag{5}$$

Definition 1.5. We say that a function ζ satisfies the Dini condition at zero if

$$|\zeta(r)| \leq C\sigma(r),$$

and σ belongs to the class \mathcal{D}_1 .

2. Preliminaries

Properties of Ω . Let Ω be a bounded domain in \mathbb{R}^n . Without loss of generality, we may assume $0 \in \partial\Omega$.

Suppose that Ω is locally convex in a neighborhood of the origin. Without restriction, the latter means that for some $0 < \mathcal{R}_0 \leq 1$, we have

$$\mathcal{P}_{\mathcal{R}_0} \cap \Omega = \{(x', x_n) \in \mathbb{R}^n : |x'| \leq \mathcal{R}_0, F(x') < x_n < \mathcal{R}_0\},$$

where F is a convex nonnegative function satisfying $F(0) = 0$.

For $r \in (0, \mathcal{R}_0)$, we define the functions $\delta = \delta(r)$ and $\delta_1 = \delta_1(r)$ by the formulas

$$\delta(r) := \max_{|x'| \leq r} \frac{F(x')}{|x'|}, \quad \delta_1(r) := \max_{|x'| \leq r} |\nabla F(x')|. \tag{6}$$

Lemma 2.1. *The following statements hold:*

- (a) $\delta_1(r) \rightarrow 0$ as $r \rightarrow 0$ if and only if $\delta(r) \rightarrow 0$ as $r \rightarrow 0$.
- (b) $\delta_1(r)$ satisfies the Dini condition at zero if and only if $\delta(r)$ satisfies the Dini condition at zero.

Proof. By the convexity of F , we have for any x' and z' , the estimate

$$F(z') \geq F(x') + \nabla F(x') \cdot (z' - x'). \tag{7}$$

Therefore,

$$|\nabla F(x')| \geq \nabla F(x') \cdot \frac{x'}{|x'|} \geq \frac{F(x')}{|x'|},$$

and, consequently,

$$\delta_1(r) \geq \delta(r). \tag{8}$$

On the other hand, for any $r < \frac{1}{2}\mathcal{R}_0$, we can find a point x'_* such that

$$|\nabla F(x'_*)| = \delta_1(r).$$

Choosing

$$z' = x'_* + r \frac{\nabla F(x'_*)}{|\nabla F(x'_*)|},$$

we easily deduce from (7) the inequalities

$$|z'| \leq 2r \quad \text{and} \quad F(z') \geq r\delta_1(r),$$

which provide

$$\delta(2r) \geq \delta(|z'|) \geq \frac{1}{2}\delta_1(r). \tag{9}$$

Combining (8) and (9), we conclude that statement (a) is obvious and the integrals

$$\int_0^{\mathcal{R}_0} \frac{\delta(r)}{r} dr \quad \text{and} \quad \int_0^{\mathcal{R}_0} \frac{\delta_1(r)}{r} dr$$

converge simultaneously. □

If $\delta(r)$ does not converge to zero as $r \rightarrow 0$, we can easily see that the domain Ω is contained in a dihedral wedge with the angle less than π and the edge going through the origin. For this case, the statement of [Theorem 4.2](#) is proved already in [[Apushkinskaya and Nazarov 2000](#), Theorem 4.3]. For this reason, we will assume throughout this paper that

$$\delta(r) \rightarrow 0 \quad \text{as} \quad r \rightarrow 0. \tag{10}$$

In view of (10), it is evident that δ and δ_1 are moduli of continuity at the origin of the functions $F(x')/|x'|$ and $|\nabla F(x')|$, respectively.

Properties of $\mathcal{X}(\Omega)$. Let $\mathcal{X}(\Omega)$ be a function space with the norm $\|\cdot\|_{\mathcal{X},\Omega}$. For $\Omega_1 \subset \Omega$, we will assume

$$\|f\|_{\mathcal{X},\Omega_1} = \|f \cdot \chi_{\Omega_1}\|_{\mathcal{X},\Omega}.$$

We suppose that $\mathcal{X}(\Omega)$ has the following properties:

- (i) For an arbitrary measurable function g defined in Ω and any function $f \in \mathcal{X}(\Omega)$, the inequality $|g(x)| \leq |f(x)|$ implies $g \in \mathcal{X}(\Omega)$ and $\|g\|_{\mathcal{X},\Omega} \leq \|f\|_{\mathcal{X},\Omega}$.
- (ii) For $f_k \in \mathcal{X}(\Omega)$, the convergence $f_k \searrow 0$ a.e. in Ω implies $\|f_k\|_{\mathcal{X},\Omega} \rightarrow 0$.

Using the terminology of the classic monograph of Kantorovich and Akilov [[1982](#)], we may say that $\mathcal{X}(\Omega)$ is the ideal functional space with order continuous monotone norm (see [[Kantorovich and Akilov 1982](#), §3, Chapter IV, Part I] for more details).

We will also assume that

- (iii) $\mathcal{X}_{\text{loc}}(\Omega)$ contains the Orlicz space $L_{\Phi,\text{loc}}(\Omega)$ with $\Phi(\xi) = e^\xi - \xi - 1$.

Finally, the basic assumption about $\mathcal{X}(\Omega)$ is the Aleksandrov-type maximum principle. Namely, we denote by $\mathcal{W}_{\mathcal{X},\text{loc}}^2(\Omega)$ the set of the functions u satisfying $D(Du) \in \mathcal{X}_{\text{loc}}(\Omega)$, and suppose that if $u \in \mathcal{W}_{\mathcal{X},\text{loc}}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$, $u|_{\partial\Omega} \leq 0$, and $|b| \in \mathcal{X}(\Omega)$ then

$$u \leq N_0(n, \nu, \|b\|_{\mathcal{X},\Omega}) \cdot \text{diam}(\Omega) \cdot \|(\mathcal{L}u)_+\|_{\mathcal{X},\{u>0\}}. \tag{11}$$

Remark 2.2. It is well known from [Aleksandrov 1960; 1963; Bakel'man 1961] (see also the survey [Nazarov 2005] for further references) that $L_n(\Omega)$ has property (11). It is also evident that properties (i)–(iii) are satisfied in $L_n(\Omega)$. Therefore, $L_n(\Omega)$ can be treated as a “basic” example of $\mathcal{X}(\Omega)$. As other examples of the space $\mathcal{X}(\Omega)$, we mention some Lebesgue weighted spaces with power weights (see [Nazarov 2001]).

Remark 2.3. Unlike the natural properties (i)–(ii), assumption (iii) is a rather “technical” one. Without (iii), our arguments from the proof of Step 3 in Theorem 4.1 are not applicable to the approximating operator \mathcal{L}_ε . So, we cannot withdraw (iii) in abstract setting. However, in all known examples of $\mathcal{X}(\Omega)$, property (iii) is satisfied.

Remark 2.4. Some of the statements that will be referred to in the sequel were proved earlier just for the case $\mathcal{X}(\Omega) = L_n(\Omega)$. However, if all the arguments are based only on the Aleksandrov-type maximum principle, these statements remain valid for an arbitrary considered space $\mathcal{X}(\Omega)$. In such cases, we will refer to this remark without any further explanation.

We also need the following convergence lemmas.

Lemma 2.5. *Let $\{f_j\}$ be a sequence of measurable functions on Ω , and let $f \in \mathcal{X}(\Omega)$. Suppose also that $f_j \rightarrow 0$ in measure on Ω , and $|f_j(x)| \leq |f(x)|$.*

Then

$$\|f_j\|_{\mathcal{X},\Omega} \rightarrow 0 \quad \text{as } j \rightarrow \infty. \tag{12}$$

Proof. We argue by a contradiction. Suppose (12) fails. Then there exists a subsequence $\{f_{j_k}\}$ satisfying

$$\|f_{j_k}\|_{\mathcal{X},\Omega} \geq \varepsilon > 0 \quad \forall k \in \mathbb{N}. \tag{13}$$

Due to the Riesz theorem, there exists also a subsubsequence $\{f_{j_{k_l}}\}$ such that

$$f_{j_{k_l}} \rightarrow 0 \quad \text{a.e. in } \Omega.$$

For simplicity of notation, we renumber the latter subsequence $\{f_{j_{k_l}}\}$ and denote its elements again by f_j .

Setting $\tilde{f}_k := \sup_{j \geq k} |f_j|$, we can easily see that $\tilde{f}_k \searrow 0$ a.e. in Ω . Now, taking into account properties (i) and (ii) of the space $\mathcal{X}(\Omega)$, we immediately get a contradiction with inequalities (13). \square

Lemma 2.6. *Let $f \in \mathcal{X}(\Omega)$, and let $\mu(\rho) := \sup_{x \in \Omega} \|f\|_{\mathcal{X}, B_\rho(x) \cap \Omega}$.*

Then

$$\mu(\rho) \rightarrow 0 \quad \text{as } \rho \rightarrow 0.$$

Proof. For every $\rho > 0$, there exists a point $x^* = x^*(\rho) \in \Omega$ such that

$$\|f\|_{\mathcal{X}, B_\rho(x^*) \cap \Omega} \geq \frac{1}{2} \mu(\rho).$$

Next, for the sequence $f_\rho := f \cdot \chi_{B_\rho(x^*)}$, it is evident that $|f_\rho| \rightarrow 0$ in measure on Ω . An application of Lemma 2.5 finishes the proof. \square

Remark 2.7. We call $\mu(\rho) := \sup_{x \in \Omega} \|f\|_{\mathcal{X}, B_\rho(x) \cap \Omega}$ the modulus of continuity of the function f in $\mathcal{X}(\Omega)$.

Lemma 2.8. *Let $D(Du) \in \mathcal{X}(\Omega)$, let \mathcal{L} be defined by (1), and let $\mathcal{L}u \in \mathcal{X}(\Omega)$. There exists the family of operators*

$$\mathcal{L}_\varepsilon = -a_\varepsilon^{ij}(x)D_iD_j + b_\varepsilon^i(x)D_i$$

with smooth coefficients a_ε^{ij} and bounded coefficients b_ε^i satisfying

$$v\mathcal{I}_n \leq (a_\varepsilon^{ij}(x)) \leq v^{-1}\mathcal{I}_n, \quad x \in \Omega, \tag{14}$$

$$|b_\varepsilon^i(x)| \leq |b^i(x)|, \quad x \in \Omega, \tag{15}$$

$$\|(\mathcal{L} - \mathcal{L}_\varepsilon)u\|_{\mathcal{X},\Omega} \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \tag{16}$$

Proof. We start with the extension of a^{ij} on the whole \mathbb{R}^n by the identity matrix and denote by a_ε^{ij} the standard mollification of extended functions a^{ij} . By construction, the coefficients a_ε^{ij} are smooth functions converging as $\varepsilon \rightarrow 0$ to a^{ij} a.e. in Ω . Moreover, it is clear that inequalities (14) are true.

Further, we set

$$\tilde{b}_\varepsilon^i(x) := \min\{|b^i(x)|, \varepsilon^{-1}\} \cdot \text{sign } b^i(x). \tag{17}$$

In view of (17), it is evident that $\tilde{b}_\varepsilon^i D_i u$ converges as $\varepsilon \rightarrow 0$ to $b^i D_i u$ a.e. in Ω . We claim that it is possible to change \tilde{b}_ε^i such that the “corrected coefficients” b_ε^i satisfy

$$|b_\varepsilon^i D_i u| \leq |b^i D_i u| \quad \text{in } \Omega. \tag{18}$$

Indeed, if $|\tilde{b}_\varepsilon^i D_i u| \leq |b^i D_i u|$ in Ω then (18) holds with $b_\varepsilon^i \equiv \tilde{b}_\varepsilon^i$. Otherwise, consider a point $x^0 \in \Omega$, where $|\tilde{b}_\varepsilon^i(x^0)D_i u(x^0)| > |b^i(x^0)D_i u(x^0)|$.

- (a) Let $\tilde{b}_\varepsilon^i(x^0)D_i u(x^0) > b^i(x^0)D_i u(x^0) \geq 0$. In this case, we decrease all the coefficients $\tilde{b}_\varepsilon^i(x^0)$ corresponding to the positive summands such that the sums $b_\varepsilon^i D_i u$ and $b^i D_i u$ become equal.
- (b) Let $\tilde{b}_\varepsilon^i(x^0)D_i u(x^0) < b^i(x^0)D_i u(x^0) \leq 0$. In this case, we decrease all the coefficients $\tilde{b}_\varepsilon^i(x^0)$ corresponding to the negative summands such that the sums $b_\varepsilon^i D_i u$ and $b^i D_i u$ become equal.
- (c) Finally, let $\tilde{b}_\varepsilon^i(x^0)D_i u(x^0)$ and $b^i(x^0)D_i u(x^0)$ have different signs. In this case, we apply to $-b_\varepsilon^i(x^0)$ the arguments from case (a) or from case (b), respectively.

Due to construction, the “corrected sum” $b_\varepsilon^i D_i u$ also converges as $\varepsilon \rightarrow 0$ to $b^i D_i u$ a.e. in Ω , and the pointwise inequalities (15) hold true.

Finally, taking into account (18) and applying Lemma 2.5, we get (16). □

3. Gradient estimates near the boundary

Lemma 3.1. *Let $\mathcal{N} \subset \mathbb{R}_+^n$ be an open set, let $\gamma = v/\sqrt{n-1}$, let $\rho > 0$, and let*

$$\Pi_\rho = \{y \in \mathbb{R}^n : |y_i| < \rho \text{ for } i = 1, \dots, n-1; 0 < y_n < \gamma\rho\}.$$

We assume that $|\mathbf{b}| \in \mathcal{X}(\mathcal{N})$ and a function v satisfies the conditions

$$v \in \mathcal{W}_{\mathcal{X},\text{loc}}^2(\mathcal{N}), \quad v \geq 0 \quad \text{in } \Pi_\rho, \quad v \geq k = \text{constant} > 0 \quad \text{on } \partial\mathcal{N} \cap \bar{\Pi}_\rho.$$

Then

$$v \geq C_1 k - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} - C_3 \rho \|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \quad \text{in } \mathcal{N} \cap B_{\frac{\gamma\rho}{4}}(z),$$

where $z = (0, \dots, 0, \frac{1}{2}\gamma\rho)$, while $C_1 = \frac{1}{16}(1-\gamma^2)$, $C_2 = C_2(n, v, \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N}})$, and $C_3 = C_3(n, v, \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N}})$.

Proof. The proof is similar in spirit to [Apushkinskaya and Ural'tseva 1995, Lemma 1].

Consider the barrier function

$$\psi(y) = k \left(\left(1 - \frac{y_n}{\gamma\rho}\right)^2 - \frac{|y'|^2}{\rho^2} \right).$$

An elementary computation gives

$$\mathcal{L}\psi \leq k \left(\frac{2(n-1)}{\rho^2} v^{-1} - \frac{2}{\gamma^2 \rho^2} v \right) + |\mathbf{b}| |D\psi| \leq N_1(n, v) |\mathbf{b}| \frac{k}{\rho} \quad \text{in } \Pi_\rho.$$

Moreover, setting

$$S_1 = \{y \in \partial(\mathcal{N} \cap \Pi_\rho) : |y_i| = \rho \text{ for some } i = 1, \dots, n-1\},$$

$$S_2 = \{y \in \partial(\mathcal{N} \cap \Pi_\rho) : y_n = \gamma\rho\},$$

we have

$$\begin{aligned} \psi|_{S_1 \cup S_2} &\leq 0 \leq v, \\ \psi|_{\partial\mathcal{N} \cap \bar{\Pi}_\rho} &\leq k \leq v|_{\partial\mathcal{N} \cap \bar{\Pi}_\rho}. \end{aligned}$$

Applying inequality (11) in $\mathcal{N} \cap \Pi_\rho$ to the difference $\psi - v$, we obtain

$$\psi - v \leq N_0 \cdot \text{diam}(\Pi_\rho) \cdot \|(\mathcal{L}\psi - \mathcal{L}v)_+\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \quad \text{in } \mathcal{N} \cap \Pi_\rho,$$

and, consequently,

$$\begin{aligned} v &\geq k \left(\left(1 - \frac{\frac{3}{4}\gamma\rho}{\gamma\rho}\right)^2 - \frac{\gamma^2 \rho^2}{16\rho^2} \right) - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} - C_3 \rho \|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \\ &= \frac{1}{16}(1-\gamma^2)k - C_2 k \|\mathbf{b}\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} - C_3 \rho \|(\mathcal{L}v)_-\|_{\mathcal{X}, \mathcal{N} \cap \Pi_\rho} \quad \text{in } \mathcal{N} \cap B_{\frac{\gamma\rho}{4}}(z). \quad \square \end{aligned}$$

Our next statement is a version of [Nazarov 2012, Theorem 2.3].

Lemma 3.2. *Let $v \in \mathcal{W}_{\mathcal{X}, \text{loc}}^2(\Omega) \cap \mathcal{C}(\bar{\Omega})$, let $v|_{\partial\Omega} = 0$, and let $|\mathbf{b}| \in \mathcal{X}(\Omega)$. Suppose also that for all $\rho \leq \rho_* \leq 1$, the inequalities*

$$\|b^n\|_{\mathcal{X}, \mathcal{P}_\rho \cap \Omega} \leq \mathfrak{B}\sigma(\rho/\rho_*), \quad \|(\mathcal{L}v)_+\|_{\mathcal{X}, \mathcal{P}_\rho \cap \Omega} \leq \mathfrak{F}\sigma(\rho/\rho_*)$$

hold true. Here \mathfrak{B} and \mathfrak{F} are some positive constants, while the function σ belongs to \mathcal{D}_1 .

Then

$$\sup_{0 < x_n < \rho} \frac{v(0, x_n)}{x_n} \leq C_4 \left(\rho^{-1} \sup_{\mathcal{P}_\rho \cap \Omega} v + \mathfrak{F} \mathcal{J}_\sigma(\rho/\rho_*) \right) \quad \forall \rho \leq \rho_*. \tag{19}$$

Here the constant C_4 depends on $n, v, \mathfrak{B}, \sigma$, and on the moduli of continuity of $|\mathbf{b}'|$ in $\mathcal{X}(\mathcal{P}_{\rho_*} \cap \Omega)$, whereas \mathcal{J}_σ is a function defined by formula (2).

Remark 3.3. We recall that $0 \in \partial\Omega$.

Proof. First, we assume that $\rho \leq \bar{\rho}$, where $\bar{\rho} \leq \rho_*$ will be fixed later. Following [Nazarov 2012], we introduce the sequence of cylinders $\mathcal{P}_{\rho_k, h_k}$, with $k \geq 0$, where $\rho_k = 2^{-k}\rho$ and $h_k = \zeta_k \rho_k$, while the sequence $\zeta_k \downarrow 0$ will be chosen later.

We set $w_k = v - M_k x_n$, where the quantities M_k , with $k \geq 1$, are defined as

$$M_k = \sup_{\mathcal{P}_{\rho_k, h_{k-1}} \cap \Omega} \frac{v(x)}{\max\{x_n, h_k\}} \geq \sup_{\{\mathcal{P}_{\rho_k, h_{k-1}} \setminus \mathcal{P}_{\rho_k, h_k}\} \cap \Omega} \frac{v(x)}{x_n}.$$

It is easy to see that $w_k \leq 0$ on $\partial\Omega \cap \bar{\mathcal{P}}_{\rho_k, h_k}$, while the definition of M_k gives $w_k \leq 0$ on the top of the cylinder $\mathcal{P}_{\rho_k, h_k}$.

Let $x^0 \in \mathcal{P}_{\rho_k - h_k, h_k} \cap \Omega$. Taking into account Remark 2.4, we apply the so-called “boundary growth lemma” (see, for instance, [Ladyzhenskaya and Ural'tseva 1985, Lemma 2.5'], [Safonov 2010, Lemma 2.6] or [Nazarov 2012, Lemma 2.2]) to the (positive) function $M_k h_k - w_k$ in $\mathcal{P}_{h_k}(x^{0'}) \cap \Omega$. It gives for $x \in \mathcal{P}_{h_k/2, h_k}(x^{0'}) \cap \Omega$,

$$M_k h_k - w_k(x) \geq M_k h_k [\vartheta - N_2 \|\mathbf{b}\|_{\mathcal{X}, \mathcal{P}_{\rho_k} \cap \Omega}] - N_3 h_k \|(\mathcal{L}w_k)_+\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^{0'}) \cap \Omega}, \tag{20}$$

where $\vartheta = \vartheta(n, \nu, \sigma, \mathfrak{B}) \in (0, 1)$ and the positive constant N_2 depends on the same parameters as ϑ , whereas the positive constant N_3 is completely defined by the values of n, ν and \mathfrak{B} . We suppose that $\bar{\rho}$ is so small that the quantity in the square brackets is greater than $\vartheta/2$. Further, direct calculation shows that the assumptions of our lemma imply

$$\begin{aligned} \|(\mathcal{L}w_k)_+\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^{0'}) \cap \Omega} &\leq \|(\mathcal{L}v)_+\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^{0'}) \cap \Omega} + M_k \|b^n\|_{\mathcal{X}, \mathcal{P}_{h_k}(x^{0'}) \cap \Omega} \\ &\leq (\mathfrak{F} + M_k \mathfrak{B}) \sigma(\rho_k / \rho_*). \end{aligned}$$

Substituting the last inequality into (20) and taking the supremum with respect to x^0 , we obtain

$$\sup_{\mathcal{P}_{\rho_k - h_k, h_k} \cap \Omega} w_k \leq M_k h_k (1 - \vartheta/2 + N_2 \mathfrak{B} \sigma(\rho_k / \rho_*)) + N_3 h_k \mathfrak{F} \sigma(\rho_k / \rho_*).$$

Repeating previous arguments provides for all integers $m \leq \rho_k / h_k$ the inequalities

$$\sup_{\mathcal{P}_{\rho_k - mh_k, h_k} \cap \Omega} w_k \leq M_k h_k \left((1 - \vartheta/2)^m + N_2 \mathfrak{B} \frac{\sigma(\rho_k / \rho_*)}{\vartheta/2} \right) + N_3 h_k \mathfrak{F} \frac{\sigma(\rho_k / \rho_*)}{\vartheta/2}.$$

Setting $m = \lfloor \rho_{k+1} / h_k \rfloor$, we arrive at

$$\sup_{\mathcal{P}_{\rho_{k+1}, h_k} \cap \Omega} w_k \leq \frac{M_k h_k}{1 - \vartheta/2} \left(\exp\left(-\lambda \frac{\rho_{k+1}}{h_k}\right) + N_2 \mathfrak{B} \frac{\sigma(\rho_k / \rho_*)}{\vartheta/2} \right) + N_3 h_k \mathfrak{F} \frac{\sigma(\rho_k / \rho_*)}{(1 - \vartheta/2)\vartheta/2},$$

where $\lambda = -\ln(1 - \vartheta/2) > 0$.

Therefore, for $x \in \mathcal{P}_{\rho_{k+1}, h_k} \cap \Omega$,

$$\frac{w_k(x)}{\max\{x_n, h_{k+1}\}} \leq M_k \gamma_k + N_3 \mathfrak{F} \frac{\sigma(\rho_k / \rho_*)}{(1 - \vartheta/2)\vartheta/2} \cdot \frac{2\zeta_k}{\zeta_{k+1}}, \tag{21}$$

where

$$\gamma_k = \frac{1}{1 - \vartheta/2} \frac{2\zeta_k}{\zeta_{k+1}} \cdot \left(\exp\left(-\frac{\lambda}{2\zeta_k}\right) + N_2 \mathfrak{B} \frac{\sigma(\rho_k/\rho_*)}{\vartheta/2} \right).$$

Estimate (21) implies

$$\begin{aligned} M_{k+1} &\leq M_k(1 + \gamma_k) + N_3 \mathfrak{F} \frac{\sigma(\rho_k/\rho_*)}{(1 - \vartheta/2)\vartheta/2} \cdot \frac{2\zeta_k}{\zeta_{k+1}} \\ &\leq M_1 \cdot \prod_{j=1}^k (1 + \gamma_j) + 2N_3 \mathfrak{F} \cdot \sum_{j=1}^k \sigma(\rho_j/\rho_*) \frac{\zeta_j}{\zeta_{j+1}} \cdot \prod_{l=j}^{k-1} (1 + \gamma_l). \end{aligned}$$

We set $\zeta_k = 1/(k + k_0)$ and choose k_0 so large and $\bar{\rho}/\rho_*$ so small that $\gamma_1 \leq \frac{1}{2}$. Note that $k_0 = k_0(n, \nu, \sigma, \mathfrak{B})$, whereas $\bar{\rho}/\rho_*$ depends on the same parameters as k_0 and, in addition, on the moduli of continuity of $|\mathbf{b}'|$ in $\mathcal{X}(\mathcal{P}_{\rho_*} \cap \Omega)$.

Now we observe that the first term in γ_k forms a convergent series. The same is true for the second term, since

$$\sum_{k=1}^{\infty} \sigma(2^{-k} \rho/\rho_*) \asymp \int_0^{\infty} \sigma(2^{-s} \rho/\rho_*) ds \asymp \mathcal{J}_{\sigma}(\rho/\rho_*).$$

Therefore, the infinite product $\Pi = \prod_k (1 + \gamma_k)$ also converges, and we obtain for $k > 1$, the inequality

$$\begin{aligned} M_k &\leq \Pi \cdot \left(M_1 + 2N_3 \mathfrak{F} \cdot \sum_{j=1}^k \sigma(\rho_j/\rho_*) \frac{\zeta_j}{\zeta_{j+1}} \right) \\ &\leq \Pi \cdot (M_1 + N_4(n, \nu, \sigma, \mathfrak{B}) \mathfrak{F} \mathcal{J}_{\sigma}(\rho/\rho_*)). \end{aligned} \tag{22}$$

Thus, all M_k are bounded. It remains only to note that

$$M_1 \leq \frac{1}{h_1} \sup_{\mathcal{P}_{\rho/2} \cap \Omega} v. \tag{23}$$

Combining (22) and (23), we arrive at

$$\sup_{0 < x_n < \rho/2} \frac{v(0, x_n)}{x_n} \leq N_5(n, \nu, \sigma, \mathfrak{B}) (\rho^{-1} \sup_{\mathcal{P}_{\rho/2} \cap \Omega} v + \mathfrak{F} \mathcal{J}_{\sigma}(\rho/\rho_*)). \tag{24}$$

Further, it is easy to find a majorant for $v(0, x_n)/x_n$ for any $x_n \in [\rho/2, \rho]$ since

$$\sup_{\rho/2 \leq x_n < \rho} \frac{v(0, x_n)}{x_n} \leq 2\rho^{-1} \sup_{\rho/2 \leq x_n < \rho} v(0, x_n) \leq 2\rho^{-1} \sup_{\mathcal{P}_{\rho} \cap \Omega} v. \tag{25}$$

Combining (24) and (25) implies (19) with $C_4 = \max\{N_5, 2\}$ for $\rho \leq \bar{\rho}$.

Now, we consider $\rho > \bar{\rho}$. If $x_n < \bar{\rho}$ then the estimate

$$\frac{v(0, x_n)}{x_n} \leq 2N_5 (\bar{\rho}^{-1} \sup_{\mathcal{P}_{\bar{\rho}} \cap \Omega} v + \mathfrak{F} \mathcal{J}_{\sigma}(\rho/\rho_*)) \tag{26}$$

follows from the above arguments. Otherwise, i.e., for $x_n \geq \bar{\rho}$, inequality (26) is especially true. Thus, for $\rho > \bar{\rho}$, we again arrive at (19) with $C_4 = \max\{N_5, 2\} \bar{\rho}^{-1}$. □

4. Main results

Recall that Ω satisfies the assumptions on page 442. Throughout this section, we shall suppose that \mathcal{L} is defined by (1), $|\mathbf{b}| \in \mathcal{X}(\Omega)$, and a function u satisfies the assumptions

$$u \in \mathcal{W}_{\mathcal{X}, \text{loc}}^2(\Omega) \cap \mathcal{C}(\bar{\Omega}), \quad \mathcal{L}u = 0 \quad \text{in } \Omega, \quad u|_{\partial\Omega \cap \bar{\mathcal{P}}_{\mathcal{R}_0}} = 0. \quad (27)$$

Theorem 4.1. *Let the inequality*

$$\sup_{x \in \mathcal{P}_{\mathcal{R}_0/2}} \|b^n\|_{\mathcal{X}, \mathcal{P}_\rho(x') \cap \Omega} \leq \mathfrak{B}\sigma(\rho/\mathcal{R}_0)$$

hold true for all $\rho \leq \frac{1}{2}\mathcal{R}_0$. Here \mathfrak{B} is a positive constant, and a function $\sigma \in \mathcal{D}_1$ satisfies

$$\mathcal{J}_\sigma(t) = o(\delta(t)) \quad \text{as } t \rightarrow 0. \quad (28)$$

Then, there exists a sufficiently small positive number R_0 completely defined by $n, \nu, \mathcal{R}_0, \mathfrak{B}$, by the functions σ, δ , and by the moduli of continuity of $|\mathbf{b}'|$ in $\mathcal{X}(\Omega)$ such that for any $r \in (0, \frac{1}{2}R_0)$, we have

$$\text{osc}_{\Omega \cap \mathcal{P}_{r/4}} \frac{u(x)}{x_n} \leq (1 - \varkappa\delta(r)) \text{osc}_{\Omega \cap \mathcal{P}_{2r}} \frac{u(x)}{x_n}. \quad (29)$$

Here the constant $\varkappa \in (0, 1)$ is completely determined by n, ν .

Proof. The proof will be divided into 3 steps.

Step 1: Our arguments are adapted from [Apushkinskaya and Ural'tseva 1995, Lemma 2; Ural'tseva 1996, Lemma 3]. Let us denote

$$m^\pm = \sup_{\Omega \cap \mathcal{P}_{2r}} \pm \frac{u(x)}{x_n}, \quad \omega = m^+ + m^- = \text{osc}_{\Omega \cap \mathcal{P}_{2r}} \frac{u(x)}{x_n}.$$

Since $u|_{\partial\Omega} = 0$, we have $m^\pm \geq 0$. Therefore, at least one of the numbers m^\pm is not less than $\frac{1}{2}\omega$, and both of the numbers m^\pm are less than ω .

Let $m^+ \geq \frac{1}{2}\omega$ for definiteness. Then we consider the nonnegative function $v(x) = m^+x_n - u(x)$ in $\Omega \cap \mathcal{P}_{2r}$; if $m^- > \frac{1}{2}\omega$ then we consider the function $v(x) = m^-x_n + u(x)$.

Due to the definition of δ , for any sufficiently small $r > 0$, we can find a point $x^* \in \partial\mathcal{P}_r \cap \partial\Omega$ such that $x_n^* = r\delta(r)$. Without loss of generality, we may assume that $x_1^* = r$ and $x_\tau^* = 0$ for $\tau = 2, \dots, n-1$.

Next we assign to x^* a local orthogonal coordinate system y_1, \dots, y_n such that

- (a) the y_1 -axis is directed along the projection of the vector $(x_1^*, \dots, x_{n-1}^*)$ onto a tangential hyperplane to $\partial\Omega$ at x^* ;
- (b) the y_2, \dots, y_{n-1} -axes are parallel to the x_2, \dots, x_{n-1} -axes, respectively;
- (c) the y_n -axis is directed inside Ω .

Due to the extremal property of x^* , the axes y_1, \dots, y_{n-1} lie in the supporting hyperplane to $\partial\Omega$ at x^* . Moreover, if x^* is a smooth point of $\partial\Omega$ then y_n is directed along the inward normal to $\partial\Omega$.

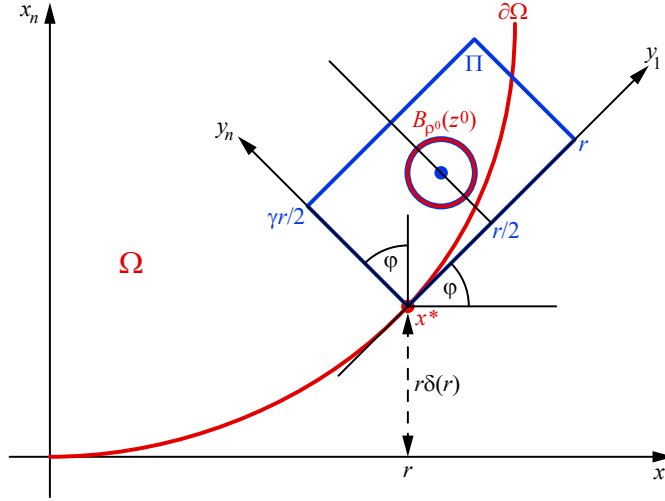


Figure 1. Schematic view of Π and $B_{\rho_0}(z^0)$.

Setting $\gamma = \nu/\sqrt{n-1}$, we consider in y -coordinates the cylinder

$$\Pi := \{y \in \mathbb{R}^n : |y_1 - \frac{1}{2}r| < \frac{1}{2}r, |y_\tau| < \frac{1}{2}r, 0 < y_n < \frac{1}{2}\gamma r\},$$

and the ball $B_{\rho_0}(z^0)$ with $\rho_0 = \frac{1}{8}\gamma r$ and $z^0 = (\frac{1}{2}r, 0, \dots, 0, \frac{1}{4}\gamma r)$.

It should be emphasized that from now on, all considerations will be carried out in x -coordinates.

We claim that

$$B_{\rho_0}(z^0) \subset \Omega. \tag{30}$$

Indeed, assume that (30) fails. Then there is a point $\hat{x} \in B_{\rho_0}(z^0)$ satisfying (in x -coordinates) the inequalities

$$F(\hat{x}') \geq \hat{x}_n \geq z_n^0 - \rho_0. \tag{31}$$

Since $\hat{x} \in B_{\rho_0}(z^0)$, it is clear that $|\hat{x}'| \leq 2r$ and

$$F(\hat{x}') \leq 2r\delta(2r).$$

On the other hand, denoting by φ the angle between the x_n - and y_n -axes (see Figure 1), we conclude that

$$z_n^0 - \rho_0 = r\delta(r) + \frac{1}{2}r \sin \varphi + \frac{1}{4}\gamma r \cos \varphi - \frac{1}{8}\gamma r \geq \frac{1}{8}\gamma r (2 \cos \varphi - 1).$$

Thus (31) is transformed into

$$\gamma(2 \cos \varphi - 1) \leq 16\delta(2r). \tag{32}$$

In view of (10) and Lemma 2.1, one can choose R_0 so small that $\delta_1(R_0) \leq \frac{3}{4}$. It guarantees for all $r \leq \frac{1}{2}R_0$, the inequalities

$$\cos \varphi = \frac{1}{\sqrt{1 + \tan^2 \varphi}} \geq \frac{1}{\sqrt{1 + \delta_1^2(r)}} \geq \frac{1}{\sqrt{1 + \delta_1^2(R_0)}} \geq \frac{4}{5}. \tag{33}$$

Now, combining (33) and (32), we get a contradiction with relation (10) provided $\delta(R_0)$ is small enough. The proof of (30) is complete.

Step 2: With (30) at hand, we observe that

$$\inf\{x_n : x \in \Omega \cap \Pi\} \geq r\delta(r).$$

On the other hand, the condition $u = 0$ for $x \in \partial\Omega \cap \Pi$ gives the estimate

$$v = m^+ x_n \geq \frac{1}{2}\omega x_n \quad \text{on } \partial\Omega \cap \Pi.$$

Hence,

$$v \geq \frac{1}{2}\omega r\delta(r) =: k_0 \quad \text{on } \partial\Omega \cap \Pi. \tag{34}$$

So, we can apply Lemma 3.1 to the function v in cylinder Π . This gives the estimate

$$\inf_{B_{\rho_0}(z^0)} v \geq (k_0(C_1 - C_2\|\mathbf{b}\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}}) - C_3\omega r\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+,$$

where C_1, C_2 and C_3 are the constants from Lemma 3.1. Decreasing R_0 , if necessary, we may assume that $\|\mathbf{b}\|_{\mathcal{X},\Omega \cap \mathcal{P}_{R_0}} \leq C_1/(2C_2)$. Thus, we arrive at

$$\inf_{B_{\rho_0}(z^0)} v \geq (k_0\frac{1}{2}C_1 - C_3\omega r\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+ =: k_1. \tag{35}$$

Consider now an arbitrary point $\tilde{z} = (\tilde{z}', \frac{1}{4}r + \frac{1}{8}\rho_0)$ such that $|\tilde{z}'| \leq \frac{1}{4}r$. Observe also that $B_{\rho_0}(\tilde{z}) \subset \Omega$, otherwise we get a contradiction with the definition of $\delta(r)$.

We claim that

$$\inf_{B_{\rho_0/8}(\tilde{z})} v \geq (k_0\tilde{C}_1 - \tilde{C}_2\omega r\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+, \tag{36}$$

where $\tilde{C}_1 = \tilde{C}_1(n, \nu)$, whereas \tilde{C}_2 is determined completely by n, ν , and $\|\mathbf{b}\|_{\mathcal{X},\Omega}$. Indeed, due to the convexity of Ω , for l running from 1 to a finite number $\mathfrak{N} = \mathfrak{N}(n, \nu)$ chosen so that

$$\frac{4}{3\rho_0}|z^0 - \tilde{z}| \leq \mathfrak{N} \leq \frac{2}{\rho_0}|z^0 - \tilde{z}|, \tag{37}$$

and for points $z^{[l]} := z^0 - (l/\mathfrak{N})(z^0 - \tilde{z})$, we have $B_{\rho_0}(z^{[l]}) \subset \Omega$. It should be emphasized that the lower and the upper bounds in (37) do not depend on r .

In view of (35), we can compare in $\mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \rho_0)$ the function v with the standard barrier function

$$w(x) = k_1 \frac{|x - z^{[1]}|^{-s} - \rho_0^{-s}}{(\frac{1}{8}\rho_0)^{-s} - \rho_0^{-s}}.$$

If $s = n\nu^{-2}$ then elementary calculation guarantees the estimates

$$\begin{aligned} \mathcal{L}w \leq |\mathbf{b}||Dw| &\leq c(n, \nu)k_1|\mathbf{b}|\rho_0^{-1} && \text{in } \mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \rho_0), \\ w(x) &= k_1 \leq v(x) && \text{on the sphere } |x - z^{[1]}| = \frac{1}{8}\rho_0, \\ w(x) &= 0 \leq v(x) && \text{on the sphere } |x - z^{[1]}| = \rho_0. \end{aligned}$$

Applying the maximum principle (11) in $\mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \rho_0)$ to the difference $w - v$ gives us the inequality

$$v(x) \geq (k_1(w(x) - 2cN_0\|\mathbf{b}\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}}) - N_0\frac{1}{4}\gamma r\omega\|b^n\|_{\mathcal{X},\Omega \cap \mathcal{P}_{2r}})_+.$$

Since $B_{\rho_0/8}(z^{[2]}) \subset \mathcal{B}(z^{[1]}, \frac{1}{8}\rho_0, \frac{7}{8}\rho_0)$, the evident bound $w \geq \theta(n, \nu)$ holds true in $B_{\rho_0/8}(z^{[2]})$.

Decreasing R_0 , if necessary, we ensure that $\|b\|_{\mathcal{X}, \Omega \cap \mathcal{P}_{R_0}} \leq (4cN_0)^{-1}\theta$. This implies

$$\inf_{B_{\rho_0/8}(z^{[2]})} v(x) \geq \left(\frac{1}{2}k_1\theta - N_0\frac{1}{4}\gamma r\omega\|b^n\|_{\mathcal{X}, \Omega \cap \mathcal{P}_{2r}}\right)_+ =: k_2.$$

Repeating this procedure for $\mathcal{B}(z^{[l]}, \frac{1}{8}\rho_0, \rho_0)$ and $l = 2, \dots, \mathfrak{N}$, we arrive at (36) with $\tilde{C}_1 = (\frac{1}{2}\theta)^{\mathfrak{N}}$ and

$$\tilde{C}_2 = N_0\frac{1}{4}\gamma \cdot \frac{1 - (\frac{1}{2}\theta)^{\mathfrak{N}}}{1 - \frac{1}{2}\theta}.$$

Furthermore, it is clear that

$$(k_0\tilde{C}_1 - \tilde{C}_2r\omega\|b^n\|_{\mathcal{X}, \Omega \cap \mathcal{P}_{2r}})_+ \geq \omega r \left(\frac{1}{2}\tilde{C}_1\delta(r) - \tilde{C}_2\mathfrak{B}\sigma(r/R_0)\right)_+,$$

while inequalities (3) and (4) guarantee that

$$\sigma(r/R_0) \leq \frac{\mathcal{J}_\sigma(r)}{R_0}.$$

Decreasing R_0 again and taking into account the assumption (28) and the above inequalities, we can transform (36) into the form

$$\inf_{B_{\rho_0/8}(\tilde{z})} v \geq \frac{1}{4}\tilde{C}_1\omega r\delta(r) =: \tilde{k}. \tag{38}$$

Step 3: Now, we take a small $\eta > 0$, define the set

$$\mathcal{A}_\eta := \mathcal{B}(\tilde{z}, \frac{1}{8}\rho_0, \tilde{z}_n) \cap \Omega \cap \{x \in \mathcal{P}_{R_0} : F(x') + \eta < x_n < R_0\}$$

and introduce in \mathcal{A}_η the barrier function

$$W(x) = \mu\tilde{k} \frac{|x - \tilde{z}|^{-s} - (\tilde{z}_n)^{-s}}{(\frac{1}{8}\rho_0)^{-s} - (\tilde{z}_n)^{-s}},$$

where $s = nv^{-2}$ and $0 < \mu \leq 1$.

Notice that $D(Du) \in \mathcal{X}(\mathcal{A}_\eta)$. Using Lemma 2.8, we construct the family of operators \mathcal{L}_ε satisfying $\|\mathcal{L}_\varepsilon u\|_{\mathcal{X}, \mathcal{A}_\eta} \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Arguing in the spirit of the proof of Lemma 4.2 [Ladyzhenskaya and Ural'tseva 1988], we define $v_1(x)$ and $v_2(x)$ as solutions of the problems

$$\begin{cases} \mathcal{L}_\varepsilon v_1 = b_\varepsilon^i D_i W & \text{in } \mathcal{A}_\eta, \\ v_1 = v & \text{on } \partial\mathcal{A}_\eta, \end{cases} \quad \begin{cases} \mathcal{L}_\varepsilon v_2 = b_\varepsilon^i D_i W - b_\varepsilon^n m^+ & \text{in } \mathcal{A}_\eta, \\ v_2 = 0 & \text{on } \partial\mathcal{A}_\eta. \end{cases}$$

It is well known (see, for instance, [Krylov 2008, Chapter 6]) that $D(Dv_1)$ and $D(Dv_2)$ belong to the space $\text{BMO}_{\text{loc}}(\mathcal{A}_\eta)$. Moreover, the John–Nirenberg theorem [1961] (see also [Duoandikoetxea 2001, §4, Chapter 6]) implies that $D(Dv_i)$, with $i = 1, 2$, belong to the Orlicz space $L_{\Phi, \text{loc}}(\mathcal{A}_\eta)$ with $\Phi(\xi) = e^\xi - \xi - 1$. So, taking into account the property (iii), we may conclude that $v_i \in \mathcal{W}_{\mathcal{X}, \text{loc}}^2(\mathcal{A}_\eta)$, with $i = 1, 2$.

Furthermore, in view of (38) and by direct calculation, we have the inequalities

$$\begin{aligned} \mathcal{L}_\varepsilon W &\leq b_\varepsilon^i D_i W && \text{in } \mathcal{A}_\eta, \\ W(x) = \mu \tilde{k} &\leq v(x) = v_1(x) && \text{on the sphere } |x - \tilde{z}| = \frac{1}{8} \rho_0, \\ W(x) = 0 &\leq v(x) = v_1(x) && \text{on } \partial \mathcal{A}_\eta \cap \{x \in \mathbb{R}^n : |x - \tilde{z}| = \tilde{z}_n\}. \end{aligned}$$

On the rest of $\partial \mathcal{A}_\eta$, we have $x_n = F(x') + \eta$ and, consequently, $\text{dist}\{x, \partial \Omega\} \leq \eta$. Since $u \in C(\bar{\Omega})$, the latter inequality implies the estimate $u \leq H(\eta)$ there, and therefore,

$$v_1(x) = v(x) = m^+ x_n - u \geq \frac{1}{2} \omega x_n - H(\eta),$$

where H is a nonnegative function tending to zero as $\eta \rightarrow 0$.

In addition, it is easy to verify that

$$W(x) \leq \mu N_6(n, \nu) \tilde{C}_1 \omega \delta(r) x_n \quad \text{in } \bar{\mathcal{B}}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n).$$

Choosing $\mu = \min\{1, (2N_6 \tilde{C}_1)^{-1}\}$, we get

$$v_1(x) \geq W(x) - H(\eta) \quad \text{on } \partial \mathcal{A}_\eta.$$

The maximum principle (11) applied to the difference $W - H(\eta) - v_1$ in \mathcal{A}_η provides the inequality

$$v_1(x) \geq W(x) - H(\eta) \geq \mu N_7(n, \nu) \tilde{C}_1 \omega \delta(r) (\tilde{z}_n - |x - \tilde{z}|) - H(\eta).$$

It follows from the last inequality with $x = (\tilde{z}', x_n) \in \Omega$ and $0 < x_n \leq \tilde{z}_n - \frac{1}{8} \rho_0 = \frac{1}{4} r$ that

$$v_1(\tilde{z}', x_n) \geq N_8(n, \nu) \omega \delta(r) x_n - H(\eta). \tag{39}$$

Next, we look for a majorant for v_2 . With this aim in view, we extend the coefficients a_ε^{ij} continuously and the coefficients b_ε^i by zero to the whole annulus $\mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n)$, and denote by $\tilde{v}_2(x)$ the solution of the problem

$$\begin{aligned} \mathcal{L}_\varepsilon \tilde{v}_2 &= \begin{cases} (\mathcal{L}_\varepsilon v_2)_+ & \text{in } \mathcal{A}_\eta, \\ 0 & \text{in } \mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n) \setminus \mathcal{A}_\eta, \end{cases} \\ \tilde{v}_2 &= 0 \quad \text{on } \partial \mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n). \end{aligned}$$

The maximum principle guarantees

$$v_2 \leq \tilde{v}_2 \quad \text{in } \mathcal{A}_\eta. \tag{40}$$

Direct computations show that for $\rho \leq \frac{1}{4} r$ the barrier function W satisfies in the set

$$\mathcal{E}_\rho := \mathcal{P}_\rho(\tilde{z}', 0) \cap \mathcal{B}(\tilde{z}, \frac{1}{8} \rho_0, \tilde{z}_n)$$

the following inequalities

$$\begin{aligned} |D_n W| &\leq |DW| \leq N_9(n, \nu) \mu \frac{\tilde{k}}{r} \leq N_9 \omega \delta(r), \\ |D' W| &\leq N_9 \mu \frac{\tilde{k} \rho}{r^2} \leq N_9 \omega \frac{\delta(r) \rho}{r}. \end{aligned}$$

So, in view of (15) and (10), we have for all $\rho \leq \frac{1}{4}r$, the bounds

$$\begin{aligned} \|(\mathcal{L}_\varepsilon \tilde{v}_2)_+\|_{\mathcal{X}, \varepsilon_\rho} &\leq \|b^n\|_{\mathcal{X}, \varepsilon_\rho} (m^+ + \|D_n W\|_{\infty, \varepsilon_\rho}) + \|b'\|_{\mathcal{X}, \varepsilon_\rho} \|D'W\|_{\infty, \varepsilon_\rho} \\ &\leq N_{10}(n, \nu) \omega \left(\mathfrak{B}\sigma\left(\frac{\rho}{\mathcal{R}_0}\right) + \frac{\delta(r)}{r} \rho \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \right). \end{aligned}$$

Since the function $\rho \mapsto (\mathfrak{B}\sigma(\rho/\mathcal{R}_0) + (\delta(r)/r)\rho\|b'\|_{\mathcal{X}, \mathcal{A}_\eta})$ satisfies the Dini condition at zero, there exist the uniquely defined function $\sigma_1 \in \mathcal{D}_1$ and a constant \mathfrak{B}_1 such that

$$\mathfrak{B}\sigma\left(\frac{\rho}{\mathcal{R}_0}\right) + \frac{\delta(r)}{r} \rho \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} = \mathfrak{B}_1 \sigma_1\left(\frac{4\rho}{r}\right).$$

Thus, we may apply Lemma 3.2 to the function \tilde{v}_2 . It gives for $\rho = \frac{1}{4}r$ the estimate

$$\sup_{0 < x_n < r/4} \frac{\tilde{v}_2(\tilde{z}', x_n)}{x_n} \leq C_4 \left(\left(\frac{1}{4}r\right)^{-1} \sup_{\varepsilon_{r/4}} \tilde{v}_2 + N_{10} \omega \mathfrak{B}_1 \mathcal{J}_{\sigma_1}(1) \right). \tag{41}$$

It is easy to see that

$$\mathfrak{B}_1 \mathcal{J}_{\sigma_1}(1) = \mathfrak{B} \mathcal{J}_\sigma\left(\frac{r}{4\mathcal{R}_0}\right) + \frac{1}{4} \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta}.$$

Furthermore, applying (11) to \tilde{v}_2 and to the operator \mathcal{L}_ε in $\mathcal{B}(\tilde{z}, \frac{1}{8}\rho_0, \tilde{z}_n)$, we obtain

$$\sup_{\varepsilon_{r/4}} \tilde{v}_2 \leq \sup_{\mathcal{B}(\tilde{z}, \rho_0/8, \tilde{z}_n)} \tilde{v}_2 \leq N_{11}(n, \nu, \|b\|_{\mathcal{X}, \Omega}) \omega r \left(\mathfrak{B}\sigma\left(\frac{r}{\mathcal{R}_0}\right) + \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \right).$$

Substitution of the above estimates in (41) and consideration of (3) provide

$$\sup_{0 < x_n < r/4} \frac{\tilde{v}_2(\tilde{z}', x_n)}{x_n} \leq N_{12} \omega \left(\mathfrak{B} \mathcal{J}_\sigma\left(\frac{r}{\mathcal{R}_0}\right) + \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \right), \tag{42}$$

where the constant N_{12} depends only on n, ν and $\|b\|_{\mathcal{X}, \Omega}$.

Taking into account the inequality (5), the assumption (28), and the evident relation $\|b'\|_{\mathcal{X}, \mathcal{A}} = o(1)$ as $r \rightarrow 0$, we decrease R_0 such that the property

$$\mathfrak{B} \mathcal{J}_\sigma\left(\frac{r}{\mathcal{R}_0}\right) + \delta(r) \|b'\|_{\mathcal{X}, \mathcal{A}_\eta} \leq \frac{N_8}{2N_{12}} \delta(r) \tag{43}$$

holds true for all $r \leq R_0$.

Finally, combining (39)–(40) with (42)–(43), we arrive at the estimate

$$v_1(\tilde{z}', x_n) - v_2(\tilde{z}', x_n) \geq \frac{1}{2} N_8 \omega \delta(r) x_n - H(\eta) \tag{44}$$

for $r \leq R_0$ and $x = (\tilde{z}', x_n) \in \Omega$ with $x_n \in [F(\tilde{z}') + \eta, \frac{1}{4}r]$.

Considering in \mathcal{A}_η the function $v_3(x) = v(x) - v_1(x) + v_2(x)$, one can easily see that

$$\mathcal{L}_\varepsilon v_3 = -\mathcal{L}_\varepsilon u \rightarrow 0 \quad \text{in } \mathcal{X}(\mathcal{A}_\eta) \text{ as } \varepsilon \rightarrow 0.$$

In addition, $v_3 = 0$ on $\partial\mathcal{A}_\eta$. Applying the maximum principle (11) to $\pm v_3$ and to the operator \mathcal{L}_ε , we obtain that the difference $v_1(x) - v_2(x)$ converges to $v(x)$ uniformly in \mathcal{A}_η . Therefore, passing in (44) first to the limit as $\varepsilon \rightarrow 0$ and then as $\eta \rightarrow 0$, we get

$$\frac{v(x)}{x_n} \geq \frac{1}{2} N_8 \omega \delta(r) \tag{45}$$

for $r \leq R_0$ and $x = (\tilde{z}', x_n) \in \Omega$ with $x_n \in [F(\tilde{z}'), \frac{1}{4}r]$.

Since \tilde{z}' can be chosen arbitrarily with only $|\tilde{z}'| \leq \frac{1}{4}r$, the estimate (45) gives (29) with $\varkappa = \frac{1}{2}N_8$. \square

Theorem 4.2 (main theorem). *Let the assumptions of Theorem 4.1 hold, and suppose*

$$\delta(r) = \max_{|x'| \leq r} \frac{F(x')}{|x'|}$$

does not satisfy the Dini condition at zero.

Then for any function u satisfying (27), the equality

$$\frac{\partial u}{\partial \mathbf{n}}(0) = 0$$

holds true.

Proof. Consider the sequence $r_k = 8^{-k} R_0$, with $k \geq 0$, where R_0 is the constant from Theorem 4.1.

Applying Theorem 4.1 to u guarantees for $k \geq 0$ the inequalities

$$\operatorname{osc}_{\Omega \cap \mathcal{P}_{r_{k+1}}} \frac{u(x)}{x_n} \leq (1 - \varkappa \delta(\frac{1}{2}r_k)) \operatorname{osc}_{\Omega \cap \mathcal{P}_{r_k}} \frac{u(x)}{x_n} \leq \operatorname{osc}_{\Omega \cap \mathcal{P}_{R_0}} \frac{u(x)}{x_n} \cdot \prod_{j=0}^k (1 - \varkappa \delta(\frac{1}{2}r_j)).$$

Since

$$\sum_{j=0}^{\infty} \ln(1 - \varkappa \delta(\frac{1}{2}r_j)) \asymp - \sum_{j=0}^{\infty} \delta(\frac{1}{2}r_j) \asymp - \int_0^{r_0} \frac{\delta(r)}{r} dr = -\infty,$$

we have

$$\prod_{j=0}^k (1 - \varkappa \delta(\frac{1}{2}r_j)) \rightarrow 0 \quad \text{as } k \rightarrow \infty.$$

We recall also that Lemma 3.2 implies the finiteness of the quantity $\operatorname{osc}_{\Omega \cap \mathcal{P}_{R_0}} (u(x)/x_n)$.

Thus, taking into account that $u|_{\partial\Omega \cap \mathcal{P}_{R_0}} = 0$, we get

$$\left| \frac{\partial u}{\partial \mathbf{n}}(0) \right| = \left| \lim_{x_n \rightarrow 0} \frac{u(0, x_n)}{x_n} \right| \leq \lim_{k \rightarrow \infty} \left| \operatorname{osc}_{\Omega \cap \mathcal{P}_{r_k}} \frac{u(x)}{x_n} \right| = 0. \quad \square$$

Acknowledgements

The authors would like to thank the anonymous referee for essential comments.

This work was supported by the Russian Foundation of Basic Research (RFBR) through the grant 15-01-07650 and by the St. Petersburg State University grant 6.38.670.2013.

References

- [Aleksandrov 1960] A. D. Aleksandrov, “Certain estimates for the Dirichlet problem”, *Dokl. Akad. Nauk SSSR* **134** (1960), 1001–1004. In Russian; translated in *Soviet Math. Dokl.* **1** (1961), 1151–1154. [MR 26 #5290](#) [Zbl 0100.09303](#)
- [Aleksandrov 1963] A. D. Aleksandrov, “Uniqueness conditions and bounds for the solution of the Dirichlet problem”, *Vestnik Leningrad. Univ. Ser. Mat. Meh. Astronom.* **18**:3 (1963), 5–29. In Russian. [MR 29 #1434](#)
- [Alvarado 2011] R. Alvarado, *Topics in harmonic analysis and partial differential equations: extension theorems and geometric maximum principles*, master’s thesis, University of Missouri, 2011, available at <http://tinyurl.com/hzgw8t>.
- [Alvarado et al. 2011] R. Alvarado, D. Brigham, V. Maz’ya, M. Mitrea, and E. Ziadé, “On the regularity of domains satisfying a uniform hour-glass condition and a sharp version of the Hopf–Oleinik boundary point principle”, *Problem. Mat. Anal.* **57** (2011), 3–68. In Russian; translated in *J. Math. Sci. New York* **176**:3 (2011), 281–360. [MR 2012j:35060](#) [Zbl 1290.35046](#)
- [Apushkinskaya and Nazarov 2000] D. E. Apushkinskaya and A. I. Nazarov, “The Dirichlet problem for quasilinear elliptic equations in domains with smooth closed edges”, *Problem. Mat. Anal.* **21** (2000), 3–29. In Russian; translated in *J. Math. Sci. New York* **105**:5 (2001), 2299–2318. [MR 2002m:35061](#) [Zbl 0987.35043](#)
- [Apushkinskaya and Ural’tseva 1995] D. E. Apushkinskaya and N. N. Ural’tseva, “On the behavior of the free boundary near the boundary of the domain”, *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)* **221** (1995), 5–19, 253. In Russian; translated in *J. Math. Sci. (New York)* **87**:2 (1997), 3267–3276. [MR 96m:35340](#) [Zbl 0888.35141](#)
- [Bakel’man 1961] I. J. Bakel’man, “On the theory of quasilinear elliptic equations”, *Sibirsk. Mat. Ž.* **2** (1961), 179–186. In Russian. [MR 23 #A3900](#) [Zbl 0100.30503](#)
- [Duoandikoetxea 2001] J. Duoandikoetxea, *Fourier analysis*, Graduate Studies in Mathematics **29**, American Mathematical Society, Providence, RI, 2001. [MR 2001k:42001](#) [Zbl 0969.42001](#)
- [Giraud 1932] G. Giraud, “Generalisation des problèmes sur les opérations du type elliptique”, *Bull. Sci. Math. (2)* **56** (1932), 316–352. [JFM 58.0494.02](#)
- [Giraud 1933] G. Giraud, “Problèmes de valeurs à la frontière relatifs à certaines données discontinues”, *Bull. Soc. Math. France* **61** (1933), 1–54. [MR 1504997](#) [Zbl 0007.11502](#)
- [Himčenko 1970] B. N. Himčenko, “On the behavior of solutions of elliptic equations near the boundary of a domain of type $A^{(1)}$ ”, *Dokl. Akad. Nauk SSSR* **193** (1970), 304–305. In Russian; translated in *Soviet Math. Dokl.* **11** (1970), 943–944. [MR 42 #8071](#) [Zbl 0206.40604](#)
- [Hopf 1952] E. Hopf, “A remark on linear elliptic differential equations of second order”, *Proc. Amer. Math. Soc.* **3** (1952), 791–793. [MR 14,280b](#) [Zbl 0048.07802](#)
- [John and Nirenberg 1961] F. John and L. Nirenberg, “On functions of bounded mean oscillation”, *Comm. Pure Appl. Math.* **14** (1961), 415–426. [MR 24 #A1348](#) [Zbl 0102.04302](#)
- [Kamynin and Himčenko 1975] L. I. Kamynin and B. N. Himčenko, “Theorems of Giraud type for a second order elliptic operator that is weakly degenerate near the boundary”, *Dokl. Akad. Nauk SSSR* **224**:4 (1975), 752–755. In Russian; translated in *Soviet Math. Dokl.* **16**:5 (1975), 1287–1291. [MR 53 #1019](#) [Zbl 0345.35046](#)
- [Kamynin and Himčenko 1977] L. I. Kamynin and B. N. Himčenko, “Theorems of Giraud type for second order equations with a weakly degenerate non-negative characteristic part”, *Sibirsk. Mat. Ž.* **18**:1 (1977), 103–121, 238. In Russian; translated in *Sib. Math. J.* **18** (1977), 76–91. [MR 55 #10831](#) [Zbl 374.35013](#)
- [Kantorovich and Akilov 1982] L. V. Kantorovich and G. P. Akilov, *Functional analysis*, 2nd ed., Pergamon Press, Oxford-Elmsford, N.Y., 1982. [MR 83h:46002](#) [Zbl 0484.46003](#)
- [Keldysh and Lavrent’ev 1937] M. V. Keldysh and M. A. Lavrent’ev, “On the uniqueness of the Neumann problem”, *Dokl. Akad. Nauk SSSR* **16**:3 (1937), 151–152. In Russian.
- [Korn 1901] A. Korn, *Lehrbuch der Potentialtheorie, II: Allgemeine Theorie des Logarithmischen Potentials und der Potentialfunctionen in der Ebene*, F. Dümmler, Berlin, 1901. [JFM 31.0728.04](#)
- [Krylov 2008] N. V. Krylov, *Lectures on elliptic and parabolic equations in Sobolev spaces*, Graduate Studies in Mathematics **96**, American Mathematical Society, Providence, RI, 2008. [MR 2009k:35001](#) [Zbl 1147.35001](#)

- [Ladyzhenskaya and Ural'tseva 1985] O. A. Ladyzhenskaya and N. N. Ural'tseva, “Estimates of the Hölder constant for functions satisfying a uniformly elliptic or uniformly parabolic quasilinear inequality with unbounded coefficients”, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **147** (1985), 72–94, 204. In Russian; translated in *J. Soviet Math.* **37:1** (1987), 837–851. MR 87e:35090 Zbl 0596.35128
- [Ladyzhenskaya and Ural'tseva 1988] O. A. Ladyzhenskaya and N. N. Ural'tseva, “Estimates on the boundary of the domain of first derivatives of functions satisfying an elliptic or a parabolic inequality”, *Trudy Mat. Inst. Steklov.* **179** (1988), 102–125, 243. In Russian; translated in *Proc. Steklov Inst. Math.* **179**, (1989), 109–135. MR 89k:35083 Zbl 0707.35023
- [Lichtenstein 1924] L. Lichtenstein, “Neue Beiträge zur Theorie der linearen partiellen Differentialgleichungen zweiter Ordnung vom elliptischen Typus”, *Math. Z.* **20:1** (1924), 194–212. MR 1544673
- [Lieberman 1985] G. M. Lieberman, “Regularized distance and its applications”, *Pacific J. Math.* **117:2** (1985), 329–352. MR 87j:35101 Zbl 0535.35028
- [Mikayelyan and Shahgholian 2015] H. Mikayelyan and H. Shahgholian, “Hopf’s lemma for a class of singular/degenerate PDEs”, *Ann. Acad. Sci. Fenn. Math.* **40:1** (2015), 475–484. MR 3329156 Zbl 1326.35156
- [Nadirashvili 1983] N. S. Nadirashvili, “On the question of the uniqueness of the solution of the second boundary value problem for second-order elliptic equations”, *Mat. Sb. (N.S.)* **122(164):3** (1983), 341–359. In Russian; translated in *Math. USSR-Sb.* **50:2** (1985), 325–341. MR 85f:35070 Zbl 0563.35026
- [Nazarov 2001] A. I. Nazarov, “Estimates for the maximum of solutions of elliptic and parabolic equations in terms of weighted norms of the right-hand side”, *Algebra i Analiz* **13:2** (2001), 151–164. In Russian; translated in *St. Petersburg Math. J.* **13:2** (2002), 269–279. MR 2002j:35049 Zbl 0994.35032
- [Nazarov 2005] A. I. Nazarov, “The maximum principle of A. D. Aleksandrov”, *Sovrem. Mat. Prilozh.* **29** (2005), 129–145. In Russian; translated in *J. Math. Sci. New York* **142:3** (2007), 2154–2171. MR 2011b:35051 Zbl 1157.35019
- [Nazarov 2012] A. I. Nazarov, “A centennial of the Zaremba–Hopf–Oleinik lemma”, *SIAM J. Math. Anal.* **44:1** (2012), 437–453. MR 2888295 Zbl 1243.35031
- [Nazarov and Uraltseva 2009] A. I. Nazarov and N. N. Uraltseva, “Qualitative properties of solutions to elliptic and parabolic equations with unbounded lower-order coefficients”, preprint, 2009, available at <http://www.mathsoc.spb.ru/preprint/2009/09-05.ps.gz>.
- [Neumann 1888] C. Neumann, “Ueber die Methode des arithmetischen Mittels, zweite Abhandlung”, *Leipz. Abh.* **14** (1888), 565–726. JFM 20.1015.01
- [Oleinik 1952] O. A. Oleinik, “On properties of solutions of certain boundary problems for equations of elliptic type”, *Mat. Sbornik N.S.* **30(72):3** (1952), 695–702. In Russian. MR 14,280a Zbl 0046.10401
- [Safonov 2008] M. V. Safonov, “Boundary estimates for positive solutions to second order elliptic equations”, preprint, 2008. arXiv 0810.0522
- [Safonov 2010] M. V. Safonov, “Non-divergence elliptic equations of second order with unbounded drift”, pp. 211–232 in *Nonlinear partial differential equations and related topics*, edited by A. A. Arkhipova and A. I. Nazarov, Amer. Math. Soc. Transl. Ser. 2 **229**, Amer. Math. Soc., Providence, RI, 2010. MR 2011i:35040 Zbl 1208.35028
- [Sweers 1997] G. Sweers, “Hopf’s lemma and two-dimensional domains with corners”, *Rend. Istit. Mat. Univ. Trieste* **28:suppl.** (1997), 383–419. MR 99e:35042 Zbl 0892.35032
- [Tolksdorf 1983] P. Tolksdorf, “On the Dirichlet problem for quasilinear equations in domains with conical boundary points”, *Comm. Partial Differential Equations* **8:7** (1983), 773–817. MR 85g:35053 Zbl 0515.35024
- [Ural'tseva 1996] N. N. Ural'tseva, “ C^1 regularity of the boundary of a noncoincident set in a problem with an obstacle”, *Algebra i Analiz* **8:2** (1996), 205–221. In Russian; translated in *St. Petersburg Math. J.* **8:2** (1997), 241–253. MR 97m:35105 Zbl 0869.35116
- [Widman 1967] K.-O. Widman, “Inequalities for the Green function and boundary continuity of the gradient of solutions of elliptic differential equations”, *Math. Scand.* **21** (1967), 17–37. MR 39 #621 Zbl 0164.13101
- [Zaremba 1910] S. Zaremba, “Sur un problème mixte relatif à l’ équation de Laplace”, *Krakau Anz. (A)* (1910), 313–344. JFM 41.0854.12

Received 2 Apr 2015. Revised 29 Sep 2015. Accepted 16 Dec 2015.

DARYA E. APUSHKINSKAYA: darya@math.uni-sb.de

Department of Mathematics, Saarland University, P.O. Box 151150, Saarbrücken 66041, Germany

and

St. Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg 199034, Russia

ALEXANDER I. NAZAROV: al.il.nazarov@gmail.com

St. Petersburg Department, Steklov Institute, Fontanka 27, St. Petersburg 191023, Russia

and

St. Petersburg State University, 7/9 Universitetskaya nab., St. Petersburg 199034, Russia

Analysis & PDE

msp.org/apde

EDITORS

EDITOR-IN-CHIEF

Patrick Gérard

patrick.gerard@math.u-psud.fr

Université Paris Sud XI

Orsay, France

BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France nicolas.burq@math.u-psud.fr	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Yuval Peres	University of California, Berkeley, USA peres@stat.berkeley.edu
Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Wilhelm Schlag	University of Chicago, USA schlag@math.uchicago.edu
Vaughan Jones	U.C. Berkeley & Vanderbilt University vaughan.f.jones@vanderbilt.edu	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
László Lempert	Purdue University, USA lempert@math.purdue.edu	András Vasy	Stanford University, USA andras@math.stanford.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbb@math.mit.edu	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpms.cam.ac.uk		

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor


See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US \$235/year for the electronic version, and \$430/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow[®] from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 9 No. 2 2016

Resonances for large one-dimensional “ergodic” systems FRÉDÉRIC KLOPP	259
On characterization of the sharp Strichartz inequality for the Schrödinger equation JIN-CHENG JIANG and SHUANGLIN SHAO	353
Future asymptotics and geodesic completeness of polarized T^2 -symmetric spacetimes PHILIPPE G. LEFLOCH and JACQUES SMULEVICI	363
Obstacle problem with a degenerate force term KAREN YERESSIAN	397
A counterexample to the Hopf–Oleinik lemma (elliptic case) DARYA E. APUSHKINSKAYA and ALEXANDER I. NAZAROV	439
Ground states of large bosonic systems: the Gross–Pitaevskii limit revisited PHAN THÀNH NAM, NICOLAS ROUGERIE and ROBERT SEIRINGER	459
Nontransversal intersection of free and fixed boundaries for fully nonlinear elliptic operators in two dimensions EMANUEL ÎNDREI and ANDREAS MINNE	487
Correction to the article Scattering threshold for the focusing nonlinear Klein–Gordon equation SLIM IBRAHIM, NADER MASMOUDI and KENJI NAKANISHI	503