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EMANUEL INDREI AND ANDREAS MINNE

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NONTRANSVERSAL INTERSECTION OF FREE AND FIXED BOUNDARIES FOR FULLY NONLINEAR ELLIPTIC OPERATORS IN TWO DIMENSIONS

EMANUEL INDREI AND ANDREAS MINNE

In the study of classical obstacle problems, it is well known that in many configurations, the free boundary intersects the fixed boundary tangentially. The arguments involved in producing results of this type rely on the linear structure of the operator. In this paper, we employ a different approach and prove tangential touch of free and fixed boundaries in two dimensions for fully nonlinear elliptic operators. Along the way, several n-dimensional results of independent interest are obtained, such as BMO-estimates, $C^{1,1}$-regularity up to the fixed boundary, and a description of the behavior of blow-up solutions.

1. Introduction

Optimal interior regularity results have recently been obtained for solutions to fully nonlinear free boundary problems [Figalli and Shahgholian 2014; Indrei and Minne 2015] via methods inspired by [Andersson et al. 2013]. Under further thickness assumptions, these results imply C^1-regularity of the free boundary. However, a description of the dynamics on how the free boundaries intersect the fixed boundary has remained an open problem for at least a decade in the fully nonlinear setting (although partial results have been obtained in [Matevosyan and Markowich 2004] under strong density and growth assumptions involving the solutions and a homogeneity assumption on the operator). On the other hand, extensive work has been carried out to investigate this question for the classical problem

$$\begin{cases} \Delta u = \chi_{u>0} & \text{in } B_1^+, \\ u \geq 0 & \text{in } B_1^+, \\ u = 0 & \text{on } \{x_n = 0\} \end{cases}$$

and its variations [Apushkinskaya and Uraltseva 1995; Shahgholian and Uraltseva 2003; Matevosyan 2005; Andersson et al. 2006; Andersson 2007]. The conclusions have varied as a function of the boundary data, but in the homogeneous case, it has been shown that the free boundary touches the fixed boundary tangentially. Dynamics of this type have also been the object of study in the classical dam problem [Caffarelli and Gilardi 1980; Alt and Gilardi 1982], which is a mathematical model describing the filtration of water through a porous medium split into wet and dry parts via a free boundary.

The methods utilized in establishing the above-mentioned results strongly rely on the linear structure of the operator, e.g., in arguments involving Green's functions and monotonicity formulas. In particular, the Alt–Caffarelli–Friedman and Weiss monotonicity formulas are frequently applied — tools only available

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in the setting of linear operators in divergence form; see [Petrosyan et al. 2012, Chapter 8]. Thus the tangential touch problem for fully nonlinear operators requires a different approach.

In this article, we prove nontransversal intersection of free and fixed boundaries in two dimensions for the problem

$$\begin{cases} F(D^2u) = \chi_\Omega & \text{a.e. in } B_1^+, \\ u = 0 & \text{on } B_1', \end{cases}$$

where $\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_2 > 0\} \subset \mathbb{R}^2_+$ and the free boundary is $\mathbb{R}^2_+ \cap \partial\Omega$. The starting point of our method is to first consider functions $u \in W^{2,n}(B_1^+)$ satisfying

$$\begin{cases} F(D^2u) = 1 & \text{a.e. in } B_1^+ \cap \Omega, \\ |D^2u| \le K & \text{a.e. in } B_1^+ \backslash \Omega, \\ u = 0 & \text{on } B_1', \end{cases} \tag{1}$$

where $\Omega \subset B_1^+$ is an (unknown) open set, $K > 0$, F is C^1 and satisfies standard structural assumptions (see Section 3).

Since by assumption D^2u is bounded in the complement of Ω, it follows that $F(D^2u)$ is bounded in B_1^+ and u is a strong L^n-solution to a fully nonlinear elliptic equation with bounded right-hand side [Caffarelli et al. 1996]. Under our structural assumptions on F, we have that L^n-solutions are also viscosity solutions, and it follows that $u \in W^{2,p}_{\text{loc}}(B_1^+)$ for all $p < \infty$ [Petrosyan et al. 2012]. If $u \ge 0$ and $\Omega = \{u \neq 0\}$, then since $D^2u = 0$ a.e. in the set $\{u = 0\}$, the Hessian condition in (1) is trivially satisfied; thus, (1) encodes the classical obstacle problem and likewise the equations $F(D^2u) = \chi_{u\neq 0}$, $F(D^2u) = \chi_{\nabla u \neq 0}$, and $F(D^2u) = \chi_{\{\nabla u\neq 0\}\cup\{u\neq 0\}}$ via the appropriate selection of Ω.

A heuristic description of our strategy is as follows: We consider

$$M := \limsup_{|x|\to 0} \frac{1}{x_n} \sup_{e\in \mathbb{S}^{n-2}\cap e_n^\perp} \partial_e u(x).$$

By extending interior $C^{1,1}$-results (see Section 3), it follows that M is finite, and we extract information on the nature of blow-up solutions by considering possible values for M. In particular, if $\{\nabla u \neq 0\} \cap \{x_n > 0\} \subset \Omega$ and the origin is a contact point, we show that either all blow-ups are of the form bx_n^2 if $M = 0$, or there is a sequence producing a blow-up having the form $ax_1x_n + bx_n^2$ if $M \neq 0$ (Theorem 2.1).

We then show that in $\mathbb{R}^2_+$, if $\Omega = (\{u \neq 0\} \cup \{\nabla u \neq 0\}) \cap \{x_2 > 0\}$ and $ax_1x_n + bx_n^2$ is a blow-up solution, then $\partial(\text{Int}\{u = 0\})$ stays away from the origin (Lemma 2.2) and this enables us to prove that blow-ups at the origin are unique (Theorem 2.4). Thereafter, a standard argument readily yields nontransversal intersection of the free and fixed boundaries at contact points (Theorem 2.5).

The rest of the paper is organized as follows: in the remainder of this section, we set up the problem and discuss relevant notation; Section 2 is the core of the paper where we rigorously develop the heuristics described above; Section 3 is devoted to the $C^{1,1}$-regularity up to the boundary of solutions, which follows as in [Indrei and Minne 2015] once a suitable BMO result is established. The results of Section 3 are used in Section 2. We have chosen to reverse the logical ordering of these sections in order to make the tangential touch section more accessible.

Setup and notation. We study fully nonlinear elliptic partial differential equations of the form

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1^+ \cap \Omega, \\ |D^2u| \le K & \text{a.e. in } B_1^+ \backslash \Omega, \\ u = 0 & \text{on } B_1', \end{cases} \tag{2}$$

where $u : B_1^+ \to \mathbb{R}$ is assumed to be in $W^{2,n}(B_1^+)$, $\Omega \subseteq B_1^+$ is an open set, $B_1(x) = \{x \in \mathbb{R}^n : |x| < 1\}$, $B_r^+(x) = B_r(x) \cap \{x_n > 0\}$, $B_r'(x) = B_r(x) \cap \{x_n = 0\}$, and $B_r = B_r(0)$.

Furthermore, F is assumed to satisfy the following structural conditions:

(H1) $F(0, x) \equiv 0$.

(H2) F is uniformly elliptic with ellipticity constants λ_0, $\lambda_1 > 0$ such that

$$\mathcal{P}^-(M - N) \le F(M, x) - F(N, x) \le \mathcal{P}^+(M - N) \quad \forall x \in B_1^+,$$

where M and N are symmetric matrices and $\mathcal{P}^\pm$ are the Pucci operators

$$\mathcal{P}^-(M) := \inf_{\lambda_0 \operatorname{Id} \le N \le \lambda_1 \operatorname{Id}} \operatorname{Tr} NM \quad \text{and} \quad \mathcal{P}^+(M) := \sup_{\lambda_0 \operatorname{Id} \le N \le \lambda_1 \operatorname{Id}} \operatorname{Tr} NM.$$

(H3) $F(\cdot\,, x)$ is concave or convex for all $x \in B_1^+$.

(H4) $$|F(M, x) - F(M, y)| \le \bar{C}(|M| + 1)|x - y|^{\bar{\alpha}}$$

for some $\bar{\alpha} \in (0, 1]$ and x, $y \in B_1^+$.

Moreover, let

$$\beta(x, x^0) := \sup_{M \in \mathcal{S}} \frac{|F(M, x) - F(M, x^0)|}{|M| + 1},$$

where $\mathcal{S}$ is the space of $n \times n$ symmetric real valued matrices.

Points in $\mathbb{R}^n$ are generally denoted by x, x^0, y etc., while subscripts are used for components, i.e., $x = (x_1, \dots, x_n)$, scalar sequences, and functions. The notation x' is used for $(n-1)$-dimensional vectors. Similarly, ∇ and ∇' will be used, respectively, for the gradient and the gradient with respect to the first $n - 1$ variables. We will also use the following:

$\mathbb{R}^n_+$	is the upper half space $\{x \in \mathbb{R}^n : x_n > 0\}$;
Ω	is an open set in $\mathbb{R}^n_+$;
Γ	is the set $\mathbb{R}^n_+ \cap \partial\Omega$;
Γ_i	is the set $\mathbb{R}^n_+ \cap \partial \operatorname{Int}\{u = 0\}$;
$B_r(x^0)$	is the open ball $\{x \in \mathbb{R}^n : \lvert x - x^0 \rvert < r\}$;
$B_r^+(x^0)$	is the truncated open ball $\{x \in \mathbb{R}^n : \lvert x - x^0 \rvert < r,\ x_n > 0\}$;
$\partial B_r^+(x^0)$	is the topological boundary of $B_r^+(x^0)$ in $\mathbb{R}^n$;

B'_r	is the ball $\{x' \in \mathbb{R}^{n-1} : \lvert x'\rvert < r\}$;
$\mathbb{S}^{n-1}$	is the $(n-1)$-sphere $\{x \in \mathbb{R}^n : \lvert x\rvert = 1\}$;
$e^\perp$	is the vector space orthogonal to $e \in \mathbb{S}^{n-1}$;
$C^{k,\alpha}(\Omega)$	denotes the usual Hölder space;
$C^{k,\alpha}_{\mathrm{loc}}(\Omega)$	denotes the local Hölder space;
$W^{k,p}(\Omega)$	denotes the usual Sobolev space.

The term "blow-ups of u at x^0" will be used for limits of the form

$$\lim_{j\to\infty} \frac{u(x^0 + r_j x)}{r_j^2},$$

where r_j is a sequence such that $r_j \to 0^+$ as $j \to \infty$, and $\operatorname{Int}\{u=0\} = \{u=0\}^\circ$ means the interior of the set $\{u=0\} := \{x \in \mathbb{R}^n_+ : u(x) = 0\}$. Finally, $S(\psi)$ denotes the space of viscosity solutions corresponding to ψ and the ellipticity constants λ_0 and λ_1 in (H2); see [Caffarelli and Cabré 1995].

2. Main result

Our first result gives a natural dichotomy of blow-ups of solutions to (1) in any dimension.[1]

Theorem 2.1 (blow-up alternative). *Let u be a solution to (1) and suppose $\{\nabla u \neq 0\} \cap \{x_n > 0\} \subset \Omega$, $0 \in \overline{\{u \neq 0\}}$, and $\nabla u(0) = 0$. Then exactly one of the following holds*:

(i) *All blow-ups of u at the origin are of the form $u_0(x) = b x_n^2$ for some unique $b > 0$.*

(ii) *There exists a blow-up of u at the origin of the form*

$$u_0(x) = a x_1 x_n + b x_n^2$$

for $a \neq 0$, $b \in \mathbb{R}$.

Proof. Firstly, since $u(x',0) = 0$, it follows that $\partial_{x_i} u(x',0) = 0$ for all $i \in \{1,\dots,n-1\}$. By $C^{1,1}$-regularity (Theorem 3.1), there is a constant $C > 0$ such that

$$\left\lvert \frac{1}{x_n} \partial_{x_i} u(x',x_n) \right\rvert = \left\lvert \frac{1}{x_n} \big(\partial_{x_i} u(x',x_n) - \partial_{x_i} u(x',0)\big) \right\rvert \le C, \qquad x_n > 0.$$

Define

$$M := \limsup_{\substack{\lvert x\rvert \to 0 \\ x_n > 0}} \frac{1}{x_n} \sup_{e \in \mathbb{S}^{n-2} \cap e_n^\perp} \partial_e u(x).$$

In particular, $0 \le M \le C < \infty$ and there exists a sequence $x^j \to 0$ with $x_n^j > 0$ and directions $e_{x^j} \in \mathbb{S}^{n-2}$ such that

$$\lim_{j\to\infty} \frac{1}{x_n^j} \partial_{e_{x^j}} u(x^j) = M.$$

[1] Regularity results from Section 3 will be utilized in the proof of Theorem 2.1.

Moreover, there exists $e \in \mathbb{S}^{n-2}$ such that (up to a subsequence) $e_{x^j} \to e$. Next note

$$\begin{aligned}\left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot e - M\right| &\le \left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot (e - e_{x^j})\right| + \left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot e_{x^j} - M\right| \\ &\le C|e - e_{x^j}| + \left|\frac{1}{x_n^j}\nabla' u(x^j)\cdot e_{x^j} - M\right| \to 0\end{aligned}$$

as $j \to \infty$. Thus, up to a rotation,

$$\lim_{j\to\infty} \frac{1}{x_n^j}\partial_{x_1} u(x^j) = M.$$

Thanks to uniform boundedness, consider a sequence $\{s_j\}$ such that $s_j \to 0^+$ and the corresponding blow-up procedure so that

$$u_j(x) := \frac{u(s_j x)}{s_j^2} \to u_0(x)$$

in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_+)$ for any $\alpha \in [0, 1)$, and u_0 satisfies

$$\begin{cases} F(D^2 u_0) = 1 & \text{a.e. in } \mathbb{R}^n_+ \cap \Omega_0, \\ |\nabla u_0| = 0 & \text{in } \mathbb{R}^n_+ \backslash \Omega_0, \\ u = 0 & \text{on } \mathbb{R}^{n-1}_+, \end{cases} \tag{3}$$

where $\Omega_0 = \{\nabla u_0 \neq 0\} \cap \{x_n > 0\}$ (via nondegeneracy). The definition of M implies

$$M \ge \lim_j \left|\frac{\partial_{x_i} u(s_j x)}{s_j x_n}\right| = \lim_j \left|\frac{\partial_{x_i} u_j(x)}{x_n}\right| = \left|\frac{\partial_{x_i} u_0(x)}{x_n}\right| \tag{4}$$

for all $i \in \{1, \dots, n-1\}$. In particular, let $v = \partial_{x_1} u_0$ so that in $\mathbb{R}^n_+$,

$$|v(x)| \le M x_n. \tag{5}$$

If $M = 0$, then (4) implies $\partial_{x_i} u_0 = 0$ for all $i \in \{1, \dots, n-1\}$ so that $u_0(x) = u_0(x_n)$. However, since $u_0(0) = |\nabla u_0(0)| = 0,\ 0 \in \overline{\{u_0 \neq 0\}}$ and u_0 satisfies (3), the uniform ellipticity of F readily implies

$$u_0(x) = b x_n^2$$

for some unique $b > 0$. This shows that if $M = 0$, then any blow-up at the origin is of the form in (i).

Now suppose $M > 0$. In order to prove (ii), we cook up a specific blow-up: let $r_j := |x^j|$ (recall that $\{x^j\}$ is the sequence achieving the lim sup in the definition of M) so that as before

$$u_j(x) := \frac{u(r_j x)}{r_j^2} \to u_0(x)$$

in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_+)$ for any $\alpha \in [0, 1)$, and u_0 satisfies (3), (4), and (5). Set $y^j = x^j / r_j \in \mathbb{S}^{n-1} \cap \{x_n > 0\}$, and note that along a subsequence, $y^j \to y \in \mathbb{S}^{n-1} \cap \{x_n \ge 0\}$. Moreover, by the choice of the sequence $\{x^j\}$

and the $C^{1,\alpha}$-convergence of u_j to u_0, if $y_n > 0$, then

$$\lim_j \frac{v(y^j)}{y_n^j} = \lim_j \frac{\partial_{x_1} u_j(y^j)}{y_n^j} = \lim_j \frac{\partial_{x_1} u(x^j)}{x_n^j} = M,$$

so that

$$v(y) = My_n, \tag{6}$$

and note that (6) also holds if $y_n = 0$. We consider several possibilities, keeping in mind that $M > 0$.

Case 1: If $y \in \Omega_0$, then by differentiating (3), we get the elliptic equation

$$a_{ij}\partial_{ij}(v(x) - Mx_n) = 0$$

for some measurable a_{ij}, and by (5), (6), and the maximum principle, it follows that $v(x) = Mx_n$ in the connected component of Ω_0 containing y, say Ω_0^y. If there exists $x \in \partial\Omega_0^y \cap \{x_n > 0\}$, then $Mx_n = v(x) = 0$, so we must have $M = 0$, a contradiction. Thus, $v(x) = Mx_n$ in $\mathbb{R}^n_+$ and by integrating,

$$u_0(x) = Mx_1x_n + h(x_2, \ldots, x_n).$$

Now, the $C^{2,\alpha}$-estimate up to the boundary given by the Krylov–Safonov theorem (see, e.g., Theorem 3.3) applied to $u_0(Rx)/R^2$ yields

$$\frac{|D^2u_0(x) - D^2u_0(y)|}{|x-y|^\alpha} \le \frac{C}{R^\alpha}, \quad x \ne y \in B_R^+,$$

and taking $R \to \infty$ implies that D^2u_0 is a constant matrix, and thus h is a second-order polynomial. Since u_0 vanishes on $\{x_n = 0\}$, it follows that

$$h(x_2, \ldots, x_n) = x_n \sum_{i \ne n} \alpha_i x_i + bx_n^2,$$

and so up to a rotation,

$$u_0(x) = ax_1x_n + bx_n^2,$$

with a or $b \ne 0$.

Case 2: If $y \in \partial\Omega_0 \cap \{x_n > 0\}$, then $My_n = v(y) = 0$, a contradiction.

Case 3: If $y \in \overline{\Omega}_0^c$, then for all but finitely many j, we have $y^j \in \Omega_0^c$ and since $\{\nabla u_0 \ne 0\} \subset \Omega_0$, it follows that $v(y^j) = 0$ if $j \ge N$ for some $N \in \mathbb{N}$. However, $y_n^j > 0$ and so

$$0 = \lim_j \frac{v(y^j)}{y_n^j} = M,$$

a contradiction.

Case 4: If $y \in \partial\Omega_0 \cap \{x_n = 0\}$, by differentiating (3) in Ω_0, it can be seen that for $r > 0$ (to be picked later), v satisfies

$$Lv = 0 \ \text{ in } B_r(y)^+ \cap \Omega_0,$$

where $L = F_{ij}(D^2u_0)\partial_{ij}$ is elliptic. Since $u_0 \in C^{1,1}(B_r^+(y))$, it follows that the $F_{ij}(D^2u_0)$ are bounded and measurable on $B_r^+(y)$.

We know that $Mx_n - v(x) \geq 0$ in $\mathbb{R}^n_+$, and if equality holds everywhere, $u_0(x) = ax_1x_n + bx_n^2$ just as in Case 1. If there is a point z where strict inequality holds, that is, $Mz_n - v(z) > 0$, then we can choose a ball $B_r^+(y)$ so that, by continuity of v, we have $v(x) < Mx_n$ in a neighborhood $B_s(z)$, where z is a boundary point of $B_r^+(y)$. Note that this strict inequality necessarily occurs on $\partial B_r^+(y) \cap \{x_n > 0\}$ since both v and Mx_n are zero on the hyperplane $\{x_n = 0\}$. Now choose a smooth nonnegative (but not identically zero) function ϕ supported on $B_s(z)$ small enough such that $Mx_n - \phi(x) \geq v(x)$ and $Mx_n - \phi(x) > 0$ in $\mathbb{R}^n_+$ (this can be done since $B_s(z)$ is some distance away from the hyperplane $\{x_n = 0\}$). Then if

$$\begin{cases} Lw = 0 & \text{in } B_r^+(y), \\ w = Mx_n - \phi & \text{on } \partial B_r^+(y), \end{cases}$$

we have that $w > 0$ in $B_r^+(y)$ by the strong maximum principle since $Mx_n - \phi(x) > 0$. In particular, $w > v = 0$ on $\partial\Omega$, and since $v \leq w$ on $\partial B_r^+(y)$, the strong maximum principle again gives $w > v$ in $B_r^+(y) \cap \Omega$. Note also by linearity that $w = Mx_n - h$, where h solves

$$\begin{cases} Lh = 0 & \text{in } B_r^+(y), \\ h = \phi & \text{on } \partial B_r^+(y). \end{cases}$$

Once more, the strong maximum principle shows that $h > 0$ in $B_r^+(y)$, so the boundary Harnack comparison principle implies that $cx_n \leq h(x)$ in $B_{r/2}^+(y)$, where $c > 0$ depends on ellipticity constants and ϕ. With this,

$$M = \lim_{j\to\infty} \frac{v(y^j)}{y_n^j} \leq \limsup_{\substack{x_n \to 0^+ \\ x \in B_{r/4}^+(y)}} \frac{w(x)}{x_n} \leq \lim_{\substack{x_n \to 0^+ \\ x \in B_{r/4}^+(y)}} \frac{Mx_n - cx_n}{x_n} = M - c,$$

a contradiction. □

The next lemma shows that in two dimensions, if (ii) in Theorem 2.1 occurs, then $\Gamma_i = \mathbb{R}^n_+ \cap \partial\,\mathrm{Int}\{u = 0\}$ stays away from the origin.

Lemma 2.2. *Let u be a solution to (1) with $\Omega = \big(\{u \neq 0\} \cup \{\nabla u \neq 0\}\big) \cap \{x_2 > 0\} \subset \mathbb{R}^2_+$. If there exists $\{r_j\} \subset \mathbb{R}^+$ such that $r_j \to 0$ as $j \to \infty$ and*

$$u_j(x) := \frac{u(r_jx)}{r_j^2} \to u_0(x) = ax_1x_2 + bx_2^2$$

in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_+)$ as $j \to \infty$ for $a \neq 0$, $b \in \mathbb{R}$, then there exists $\delta \in (0,1)$ such that $B_\delta^+ \cap \Gamma_i = \varnothing$.

Proof. We may assume $a > 0$. Set $v_j := \partial_1 u_j$ and let $R > 2$, $\mu \in \big(0, \frac{1}{4}\big)$, and $\delta \in \big(0, \frac{1}{4}\big)$. Then select $j_0 = j_0(R, \mu, \delta) > 0$ such that for all $j \geq j_0$,

$$|\nabla u_j(x)| > 0, \quad x \in B_R^+ \setminus B_\delta^+, \tag{7}$$

$$v_j(x) > 0, \quad x \in B_R^+ \cap \{x_2 \geq \mu\} \tag{8}$$

(the two-dimensional setting is crucial for (7)). Consider $z \in \partial B_1 \cap \{x_2 = 0\}$ and note that

$$B^+_{3/4}(z) \subset B^+_R \setminus B^+_\delta.$$

Thanks to (7), u_j satisfies $F(D^2u_j) = 1$ in $B^+_{3/4}(z)$ for all $j \geq j_0$. $C^{2,\alpha}$-estimates up to the boundary (see Theorem 3.3) imply

$$\sup_j \|u_j\|_{C^{2,\alpha}\left(B^+_{3/4}(z)\right)} < \infty.$$

Thus, along a subsequence, $v_j \to ax_2 =: v$ in $C^{0,1}$ ($C^{2,\alpha}$ is compactly contained in $C^{1,1}$) and so

$$c_j := \sup_{\substack{x,y \in B^+_{3/4}(z) \\ x \neq y}} \frac{|(v_j(x) - v_j(y)) - (v(x) - v(y))|}{|x-y|} \to 0.$$

In particular, since $v_j(x_1, 0) = v(x_1, 0) = 0$, it follows that

$$\frac{|v_j(x) - ax_2|}{x_2} \leq c_j,$$

and so

$$v_j(x) \geq (a - c_j)x_2.$$

Now we select j large such that $v_j(x) \geq 0$ on ∂B_1. Note that $Lv_j = 0$ in $B^+_1 \cap \Omega(u_j)$, where L is an elliptic second-order operator obtained by differentiating (1). Indeed, u_j satisfies

$$\begin{cases} F(D^2u_j) = 1 & \text{a.e. in } B^+_{1/r_j} \cap \Omega(u_j), \\ u_j = 0 & \text{on } B'_{1/r_j}, \end{cases}$$

where $\Omega(u_j)$ is the dilated set Ω/r_j, and without loss we may assume $r_j < \frac{1}{2}$.

Since v_j vanishes on $\partial\Omega(u_j)$ and is nonnegative on ∂B^+_1, the maximum principle implies $v_j > 0$ in $B^+_1 \cap \Omega(u_j)$ (note that v_j is not identically zero by (8)). If $\Gamma_i(u_j) \cap B^+_\delta \neq \varnothing$, consider a ball N in the interior of $\{u_j = 0\} \cap B^+_\delta$. For $t \in \mathbb{R}$, let $N_t = N + te_1$. Note that by taking t negative, we can move N_t to the left so that eventually $N_t \subset B^+_1 \setminus B^+_\delta$. Consider the strip $S = \bigcup_{t\in\mathbb{R}} N_t$. The next claim is that there exists a ball in the set $(S \cap B^+_1) \setminus B^+_\delta$ such that $u_j \neq 0$ in this ball: if not, then for each point $z \in (S \cap B^+_1) \setminus B^+_\delta$, there exists a sequence $\{z_k\} \subset \{u_j = 0\}$ such that $z_k \to z$; by continuity, $u_j(z) = 0$, so $u_j = 0$ in $(S \cap B^+_1) \setminus B^+_\delta$, and therefore the gradient also vanishes there, a contradiction to (7). Denote the ball by $E \subset \Omega(u_j)$ and note that $u_j < 0$ on E since for each $z \in E$, there exists $t_z > 0$ such that $z + e_1 t_z \in \{u_j = 0\}$ and $v_j > 0$ in $B^+_1 \cap \Omega(u_j)$. Thus, $E \subset \Omega(u_j) \cap \{u_j < 0\}$. Now move E to the right until the first time it touches $\{u_j = 0\}$, and let y be the contact point.

If $\nabla u_j(y) = 0$, we immediately obtain a contradiction via Hopf's lemma. Thus we may assume $\nabla u_j(y) \neq 0$, which implies $y \in \Omega(u_j)$, whence $v_j(y) > 0$ (recall that $v_j > 0$ in $\Omega(u_j)$). By continuity, $v_j > 0$ in $B_r(y)$ for some $r > 0$, so in particular $v_j(y + te_1) > 0$ for all $t > 0$ small. Since $\{y + te_1 : t \in (0, r)\} \subset \Omega(u_j)$, we know $t_* := \sup\{t > 0 : y + te_1 \in \Omega(u_j)\}$ is positive. Note that $y + te_1$

will eventually enter N as t gets larger. However,

$$u_j(y+t_*e_1)-u_j(y)=\int_0^{t_*} v_j(y+se_1)\,ds>0,$$

and this implies $0=u_j(y+t_*e_1)>u_j(y)=0$, a contradiction. Thus $\Gamma_i(u_j)\cap B_\delta^+=\varnothing$ and the result follows. □

Before proving uniqueness of blow-ups and tangential touch, we require one more lemma.

Lemma 2.3. *Let u be a solution to (1) with $\Omega=\big(\{u\neq 0\}\cup\{\nabla u\neq 0\}\big)\cap\{x_n>0\}$. If $s\in(0,1]$ and $(B_s^+\setminus\Omega)^\circ=\varnothing$, then $|B_s^+\setminus\Omega|=0$.*

Proof. Since $u\in W^{2,n}(B_1^+)$, it follows that $D^2u=0$ a.e. on $B_s^+\setminus\Omega$. Let $Z:=\{D^2u=0\}\cap(B_s^+\setminus\Omega)$ and note that $|Z|=|B_s^+\setminus\Omega|$. Thus if $Z\subset(B_s^+\setminus\Omega)^\circ$, then the result follows. Let $x^0\in Z$ and suppose $x^0\notin(B_s^+\setminus\Omega)^\circ$. Then consider a sequence of points $x^j\to x^0$ such that $u(x^j)\neq 0$ and let $r_j:=|x^0-x^j|$. Nondegeneracy (see, e.g., [Indrei and Minne 2015, Lemma 3.1]) implies that for j large,

$$\sup_{\partial B_{r_j}(x^0)}\frac{u}{r_j^2}\geq c>0,$$

or in other words,

$$\sup_{\partial B_1(0)}\frac{u(x^0+r_jx)}{r_j^2}\geq c>0.$$

Now for each j large enough, let $y^j\in\partial B_1(0)$ be the element achieving the supremum in the previous expression; note that since

$$u(x^0)=|\nabla u(x^0)|=|D^2u(x^0)|=0,$$

we have

$$u(x^0+r_jy^j)=o(r_j^2),$$

a contradiction. □

Theorem 2.1, Lemma 2.2, and Lemma 2.3 imply uniqueness of blow-ups in two dimensions.

Theorem 2.4 (uniqueness of blow-ups). *Let u be a solution to (1) with $\Omega=\big(\{u\neq 0\}\cup\{\nabla u\neq 0\}\big)\cap\{x_2>0\}\subset\mathbb{R}^2_+$. If $0\in\overline{\{u\neq 0\}}$ and $\nabla u(0)=0$, then all blow-up limits u_0 of u at the origin are of the form*

$$u_0(x)=ax_1x_2+bx_2^2,$$

where $a,b\in\mathbb{R}$ with at least one of them nonzero.

Proof. We divide the proof into two cases.

Case 1: $0\in\overline{\Gamma}_i$. Lemma 2.2 implies the nonexistence of a blow-up u_0 of u of the form

$$ax_1x_2+bx_2^2,$$

$a\neq 0$, $b\in\mathbb{R}$, from which it follows that (i) holds in Theorem 2.1. Note that b is uniquely determined by the equation.

Case 2: $0 \notin \overline{\Gamma}_i$. In this case, there exists $\delta > 0$ such that $\Gamma_i \cap B_\delta^+ = \varnothing$. Since $0 \in \overline{\{u \neq 0\}}$ (by assumption), it follows that $B_\delta^+ \not\subset \{u = 0\}^\circ$ and as $\Gamma_i \cap B_\delta^+ = \varnothing$, we may conclude that $\{u = 0\}^\circ \cap B_\delta^+ = \varnothing$. Thus the hypotheses of Lemma 2.3 are satisfied, and by applying the lemma, we obtain that $F(D^2 u) = 1$ a.e. in B_δ^+. Therefore $u \in C^{2,\alpha}(B_{\delta/2}^+)$ and the blow-up limit u_0 is uniquely given by

$$\begin{aligned}\lim_{r\to 0} \frac{u(rx)}{r^2} &= \lim_{r\to 0} \frac{u(0) + \nabla u(0)\cdot rx + \langle rx, D^2 u(0) rx\rangle + o(r^2)}{r^2} \\ &= \langle x, D^2 u(0) x\rangle = a x_1 x_2 + b x_2^2.\end{aligned}$$

The last equality follows from the boundary condition. Furthermore, u_0 solves the same equation as u, so

$$F(D^2 u_0) = F(D^2 u(0)) = 1$$

and thus a and b cannot both be zero due to (H1). $\square$

If blow-ups are unique and of the form given above, it is rather standard to show that the free boundary touches the fixed boundary tangentially (see, e.g., [Petrosyan et al. 2012, Chapter 8]). The proof is included for completeness.

Theorem 2.5 (tangential touch). *Let u be a solution to (1) with $\Omega = \big(\{u \neq 0\} \cup \{\nabla u \neq 0\}\big) \cap \{x_2 > 0\} \subset \mathbb{R}^2_+$. Then there exists a constant $r_0 > 0$ and a modulus of continuity $\omega_u(r)$ such that*

$$\Gamma(u) \cap B_{r_0}^+ \subset \big\{x : x_2 \leq \omega_u(|x|)|x|\big\}$$

if $0 \in \overline{\Gamma(u)}$, where $\Gamma(u) := \partial\Omega \cap \mathbb{R}^2_+$.

Proof. By Theorem 2.4, the blow-up of u at the origin is not identically zero and is given by $u_0(x) = a x_1 x_2 + b x_2^2$. In particular, $\Gamma(u_0) = \varnothing$. It suffices to show that for any $\epsilon > 0$, there exists $\rho_\epsilon = \rho_\epsilon(u) > 0$ such that

$$\Gamma(u) \cap B_{\rho_\epsilon}^+ \subset B_{\rho_\epsilon}^+ \setminus \mathcal{C}_\epsilon,$$

where $\mathcal{C}_\epsilon := \{x_2 > \epsilon |x_1|\}$. Suppose not. Then there exists a solution u to (1) satisfying the hypotheses of the theorem and $\epsilon > 0$ such that for all $k \in \mathbb{N}$, there exists

$$x^k \in \Gamma(u) \cap B_{1/k}^+ \cap \mathcal{C}_\epsilon.$$

Let $r_k := |x^k|$ and $y^k := x^k / r_k \in \partial B_1 \cap \mathcal{C}_\epsilon$. Note that along a subsequence

$$y^k \to y \in \partial B_1 \cap \mathcal{C}_\epsilon.$$

Define

$$u_k(x) := \frac{u(r_k x)}{r_k^2}$$

so that $u_k \to u_0$ in $C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^n_+)$ (along a subsequence). In particular, $y \in \Gamma(u_0)$ by nondegeneracy, which contradicts that $\Gamma(u_0) = \varnothing$. $\square$

3. $C^{1,1}$-regularity up to the boundary

We now show BMO-estimates as well as $C^{1,1}$-regularity up to the fixed boundary of solutions to (2).

Theorem 3.1 ($C^{1,1}$-regularity). *Let $f \in C^{\alpha}(B_1^+)$ be a given function and $\Omega \subseteq B_1^+$ a domain such that $u : B_1^+ \to \mathbb{R}$ is a $W^{2,n}$-solution of (2). Assume F satisfies (H1)–(H4). Then there exists a constant C depending on $\|u\|_{W^{2,n}(B_1^+)}$, $\|f\|_{C^{\alpha}(B_1^+)}$, and universal constants such that*

$$|D^2u| \le C \quad \text{a.e. in } B_{1/2}^+.$$

There are three key tools needed to prove this theorem. The first two are $C^{2,\alpha}$- and $W^{2,p}$-estimates up to the boundary for the following classical fully nonlinear problem

$$\begin{cases} F(D^2u, x) = f(x) & \text{a.e. in } B_1^+, \\ u = 0 & \text{on } B_1', \end{cases} \tag{9}$$

and the last involves BMO-estimates. The $C^{2,\alpha}$- and $W^{2,p}$-estimates are well-known [Wang 1992; Safonov 1994; Winter 2009; Krylov 1982]. We have been unable to find a reference for the BMO-estimates and thus provide a proof, which is an adaptation of the interior case. For convenience, we record the following estimates; see, e.g., [Winter 2009, Theorem 4.3; Safonov 1994, Theorem 7.1]. Recall the definition of β,

$$\beta(x, x^0) := \sup_{M \in \mathcal{S}} \frac{|F(M, x) - F(M, x^0)|}{|M| + 1}.$$

Theorem 3.2 ($W^{2,p}$-regularity). *Let u be a $W^{2,p}$-viscosity solution to (9) and $f \in L^p(B_1^+)$ for $n \le p \le \infty$. If $\beta(x^0, y) \le \beta_0$ in $B_r^+(x^0) \cap B_1^+$ for all $x^0 \in B_1^+$ and $0 < r \le r_0$, where β_0 and r_0 are universal constants, then $u \in W^{2,p}(B_{1/2}^+)$ and*

$$\|u\|_{W^{2,p}(B_{1/2}^+)} \le C\big(\|u\|_{L^\infty(B_1^+)} + \|f\|_{L^p(B_1^+)}\big),$$

where $C = C(n, \lambda_0, \lambda_1, \bar{\alpha}, \bar{C}, p) > 0$.

Theorem 3.3 ($C^{2,\alpha}$-regularity). *Let u be a $W^{2,n}$-viscosity solution to (9) and $f \in C^{\bar{\alpha}}(B_1^+)$. Then if $\beta(x^0, y) \le \beta_0$ in $B_r^+(x^0) \cap B_1^+$ for all $x^0 \in B_1^+$ and $0 < r \le r_0$, where β_0 and r_0 are universal constants, then $u \in C^{2,\alpha}(B_{1/2}^+)$ and*

$$\|u\|_{C^{2,\alpha}(B_{1/2}^+)} \le C\big(\|u\|_{L^\infty(B_1^+)} + \|f\|_{C^{\bar{\alpha}}(B_1^+)}\big),$$

where $C = C(n, \lambda_0, \lambda_1, \bar{\alpha}, \bar{C}) > 0$.

The next results are technical tools utilized in the proof of the BMO-estimate (i.e., Proposition 3.6). The first is an approximation lemma; see, e.g., [Wang 1992, Lemma 1.4].

Lemma 3.4 (approximation). *Let $\epsilon > 0$, $u \in W^{2,p}(B_1^+(x^0))$, and let v solve*

$$\begin{cases} F(D^2v, x^0) = a & \text{in } B_{1/2}^+(x^0), \\ v = u & \text{on } \partial B_{1/2}^+(x^0). \end{cases}$$

Then there exists $\delta > 0$ *and* $\eta > 0$ *such that if*

$$\beta(x,x^0) := \sup_{M\in\mathcal{S}} \frac{|F(M,x)-F(M,x^0)|}{|M|+1} \le \eta$$

and $|f(x)-a| \le \delta$ *a.e. for* $f(x) := F(D^2u(x),x)$ *in* $B_1^+(x^0)$, *then*

$$|u-v| \le \epsilon \quad \text{in } B_{1/2}^+.$$

Lemma 3.5. *Let* $u \in W^{2,n}(B_1^+)$ *satisfy* $|F(D^2u(x),x)| \le \delta$ *a.e. in* B_1^+ *for* δ *as in Lemma 3.4. Moreover, assume* $|u| \le 1$ *and that* $\beta(x,y)$ *satisfies (H4). Then there exists a universal constant* $\rho > 0$ *and second-order polynomials* P_{k,x^0} *for any* $k \in \mathbb{N}_0$ *and* $x^0 \in B_{1/2}^+$ *such that*

$$\begin{aligned}
|D^2P_{k,x^0} - D^2P_{k-1,x^0}| &\le C_0(n,\lambda_0,\lambda_1),\\
F(D^2P_{k,x^0},x^0) &= 0,\\
|u(x) - P_{k,x^0}(x)| &\le \rho^{2k} \quad \text{inside } B^+_{\min(\rho^k,1)}(x^0).
\end{aligned}$$

Proof. For $k=0$ and $k=-1$, the statement is true for $P_{k,x^0}(x) \equiv 0$ by assumption (recall (H1)). If we assume it is true up to some k, define

$$\begin{aligned}
u_k &:= \frac{u(\rho^k x + x^0) - P_{k,x^0}(\rho^k x + x^0)}{\rho^{2k}},\\
F_k(M,x) &:= F\big(M + D^2P_{k,x^0}, \rho^k x + x^0\big), \quad x \in B_1 \cap \{x_n > -x_n^0/\rho^k\}.
\end{aligned}$$

Then $|F_k(D^2u_k,x)| = \big|F\big((D^2u)(\rho^k x + x^0), \rho^k x + x^0\big)\big| \le \delta$ a.e. Also,

$$\begin{aligned}
\beta_k(x,0) &= \sup_{M\in\mathcal{S}} \frac{|F_k(M,x)-F_k(M,0)|}{|M|+1}\\
&= \sup_{M\in\mathcal{S}} \frac{\big|F\big(M+D^2P_{k,x^0},\rho^k x + x^0\big) - F\big(M + D^2P_{k,x^0},x^0\big)\big|}{|M|+1}\\
&= \sup_{M\in\mathcal{S}} \frac{|F(M,\rho^k x + x^0) - F(M,x^0)|}{|M-D^2P_{k,x^0}|+1}\\
&= \sup_{M\in\mathcal{S}} \frac{|F(M,\rho^k x + x^0) - F(M,x^0)|}{|M|+1}\,\frac{|M|+1}{|M-D^2P_{k,x^0}|+1}\\
&\le \beta(\rho^k x + x^0, x^0) \sup_{M\in\mathcal{S}} \frac{|M|+1}{|M-D^2P_{k,x^0}|+1}\\
&\le \bar{C}\rho^{\bar{\alpha}k} \sup_{M\in\mathcal{S}} \frac{|M|+1}{\big||M|-|D^2P_{k,x^0}|\big|+1}\\
&\le \bar{C}\rho^{\bar{\alpha}k}\big(|D^2P_{k,x^0}|+1\big),
\end{aligned}$$

where the last inequality follows from a calculation of the maximum of the function

$$\frac{x+1}{|x-a|+1}, \quad x, a \geq 0.$$

However, from the induction hypothesis,

$$|D^2 P_{k,x^0}| \leq \sum_{j=1}^{k} \left|D^2 P_{j-1,x^0} - D^2 P_{j,x^0}\right| \leq C_0 k,$$

so

$$\bar{C} \rho^{\bar{\alpha} k} \big(|D^2 P_{k,x^0}| + 1\big) \leq \bar{C} \rho^{\bar{\alpha} k} C_0 k \leq \eta$$

if ρ is chosen small enough (depending only on universal constants) and η as in Lemma 3.4. Thus $|v_k - u_k| \leq \epsilon$ in $B_{1/2} \cap \{x : x_n > -x_n^0/\rho^k\}$ by Lemma 3.4, where v_k solves

$$\begin{cases} F_k(D^2 v_k, 0) = 0 & \text{in } B_{1/2} \cap \{x : x_n > -x_n^0/\rho^k\}, \\ v_k = u_k & \text{on } \partial\big(B_{1/2} \cap \{x : x_n > -x_n^0/\rho^k\}\big). \end{cases}$$

Since

$$\|v_k\|_{L^\infty(B_{1/2} \cap \{x : x_n > -x_n^0/\rho^k\})} \leq \|u_k\|_{L^\infty(B_{1/2} \cap \{x : x_n > -x_n^0/\rho^k\})} \leq 1$$

by the maximum principle, Theorem 3.3 gives

$$\|v_k\|_{C^{2,\alpha}(B_{1/4} \cap \{x : x_n > -x_n^0/\rho^k\})} \leq C_0. \tag{10}$$

Now define $\hat{P}_{k,x^0}$ as the second-order Taylor expansion of v_k at the origin, and note that $F_k(D^2 \hat{P}_{k,x^0}, 0) = F_k(D^2 v_k(0), 0) = 0$. Then

$$|v_k - \hat{P}_{k,x^0}| \leq C_0 \rho^{2+\alpha} \quad \text{in } B_\rho \cap \{x : x_n > -x_n^0/\rho^k\}$$

for $\rho < \frac{1}{4}$, which gives

$$|u_k - \hat{P}_{k,x^0}| \leq |u_k - v_k| + |v_k - \hat{P}_{k,x^0}| \leq \epsilon + C_0 \rho^{2+\alpha} \quad \text{in } B_\rho \cap \{x : x_n > -x_n^0/\rho^k\}.$$

For $\rho^\alpha \leq \frac{1}{2C_0}$ and $\epsilon \leq \frac{1}{2}\rho^2$, we get

$$|u_k - \hat{P}_{k,x^0}| \leq \rho^2 \quad \text{in } B_\rho \cap \{x : x_n > -x_n^0/\rho^k\},$$

or, in other words,

$$|u - P_{k+1,x^0}| \leq \rho^{2(k+1)} \quad \text{in } B^+_{\rho^{k+1}}(x^0)$$

for

$$P_{k+1,x^0}(x) := P_{k,x^0}(x) + \rho^{2k} \hat{P}_{k,x^0}\left(\frac{x - x^0}{\rho^k}\right).$$

Also, since $F_k(D^2 \hat{P}_{k,x^0}, 0) = 0$, by (10) we have

$$\begin{gathered} F(D^2 P_{k+1,x^0}, x^0) = F\big(D^2 P_{k,x^0} + D^2 \hat{P}_{k,x^0}, x^0\big) = F_k(D^2 \hat{P}_{k,x^0}, 0) = 0, \\ |D^2 P_{k+1,x^0} - D^2 P_{k,x^0}| = |D^2 \hat{P}_{k,x^0}| = |D^2 v_k(0)| \leq C_0. \end{gathered}$$

□

Proposition 3.6 (BMO-estimate). *Let u be a solution to (9), f bounded, and P_{k,x^0} and ρ be as in Lemma 3.5. Then*

$$\fint_{B^+_{\rho^k/2}(x^0)} |D^2u(y) - D^2P_{k,x^0}|^2 \le C, \quad x^0 \in \overline{B}^+_{1/2},$$

if ρ is smaller than a constant that depends only on $\|u\|_{W^{2,p}(B_1)}$, f, $\overline{C}$ in (H4), and universal constants.

Proof. Let $x^0 \in \overline{B}^+_{1/2}$ and define

$$v(x) := u\Big(\frac{x}{R}\Big) \quad \text{and} \quad G(M,x) := \frac{1}{R^2}F\Big(R^2M, \frac{x}{R}\Big)$$

for $R = R(\overline{C}, f, K, \delta)$ ($\overline{C}$ as in (H4)) chosen so that $|G(D^2v, x)| \le \delta$ in B^+_R for δ as in Lemma 3.4. Note also that

$$\beta_G(x,y) := \sup_{M \in \mathcal{S}} \frac{|G(M,x) - G(M,y)|}{|M|+1}$$

satisfies (H4). Then v solves

$$\begin{cases} G(D^2v, x) = f(x/R)/R^2 & \text{a.e. in } B^+_R \cap (R\Omega), \\ |D^2v| \le K/R^2 & \text{a.e. in } B^+_R \backslash (R\Omega), \\ v = 0 & \text{on } B'_R, \end{cases}$$

and there is a polynomial $\tilde{P}_{k,x^0}$ for which $G(D^2\tilde{P}_{k,x^0}, Rx^0) = 0$, and a constant $\tilde{\rho}$ such that

$$|v(x) - \tilde{P}_{k,x^0}(x)| \le \tilde{\rho}^{2k}, \quad x \in B^+_{\tilde{\rho}^k}(Rx^0),$$

that is,

$$|u(x) - P_{k,x^0}(x)| \le R^2\rho^{2k}, \quad x \in B^+_{\rho^k}(x^0),$$

for $P_{k,x^0}(x) := \tilde{P}_{k,x^0}(Rx)$ and $\rho^k := \tilde{\rho}^k/R$. Note also that

$$F(D^2P_{k,x^0}, x^0) = F\Big(R^2D^2\tilde{P}_{k,x^0}, \frac{Rx^0}{R}\Big) = R^2G(D^2\tilde{P}_{k,x^0}, Rx^0) = 0.$$

In particular, for

$$u_k(x) := \frac{u(\rho^kx + x^0) - P_{k,x^0}(\rho^kx + x^0)}{\rho^{2k}},$$
$$F_k(M,x) := F(M + D^2P_{k,x^0}, \rho^kx + x^0),$$

and β_k as in the proof of Lemma 3.5, we have $|u_k| \le R^2$, $\beta_k(x,y) \le \eta$ and $|F_k(u_k, x)| \le C$. Therefore we can apply Theorem 3.2 to deduce

$$\|u_k\|_{W^{2,p}(B_{1/2} \cap \{x_n \ge -x^0/\rho^k\})} \le C,$$

or

$$\fint_{B^+_{\rho^k/2}(x^0)} |D^2u(x) - D^2P_{k,x^0}|^p \, dx \le C. \qquad \square$$

From this it is straightforward to show that if u is a function satisfying (2), there exists a second-order polynomial $P_{r,x^0}(x)$ with $F(D^2 P_{r,x^0}, x^0) = f(x^0)$ such that

$$\sup_{r\in(0,1/4)} \fint_{B_r^+(x^0)} |D^2u(y) - D^2 P_{r,x^0}|^2\, dy \le C,$$

where $x^0 \in \overline{B}^+_{1/2}(0)$. The proof of $C^{1,1}$-regularity now follows as in [Indrei and Minne 2015] up to minor modifications (see also [Figalli and Shahgholian 2014]). The idea is that $D^2 P_{r,x^0}(x)$ provides a suitable approximation to $D^2u(x^0)$ and one may consider two cases: first, if $D^2 P_{r,x^0}(x)$ stays bounded in r, then one can show that $D^2u(x^0)$ is also bounded by a constant depending only on the initial ingredients; next, if $D^2 P_{r,x^0}(x)$ blows up in r, one can show that the set

$$A_r(x^0) := \frac{(B_r^+(x^0)\backslash\Omega) - x^0}{r} = B_1\backslash\big((\Omega - x^0)/r\big) \cap \{y : y_n > -x_n^0/r\}$$

decays fast enough to ensure yet again a bound on $D^2u(x^0)$.

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EMANUEL INDREI: egi@cmu.edu
Center for Nonlinear Analysis, Carnegie Mellon University, Pittsburgh, PA 15213, United States

ANDREAS MINNE: minne@kth.se
Department of Mathematics, KTH Royal Institute of Technology, 100 44 Stockholm, Sweden

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