ANALYSIS & PDE

Volume 9 No. 5 2016

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Analysis & PDE

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1019

MULTIDIMENSIONAL ENTIRE SOLUTIONS FOR AN ELLIPTIC SYSTEM MODELLING PHASE SEPARATION

NICOLA SOAVE AND ALESSANDRO ZILIO

For the system of semilinear elliptic equations

$$\Delta V_i = V_i \sum_{j\neq i} V_j^2, \quad V_i > 0 \text{ in } \mathbb{R}^N,$$

we devise a new method to construct entire solutions. The method extends the existence results already available in the literature, which are concerned with the 2-dimensional case, also to higher dimensions $N \geq 3$. In particular, we provide an explicit relation between orthogonal symmetry subgroups, optimal partition problems of the sphere, the existence of solutions and their asymptotic growth. This is achieved by means of new asymptotic estimates for competing systems and new sharp versions for monotonicity formulae of Alt–Caffarelli–Friedman type.

1. Introduction

The elliptic systems

$$\begin{cases} \Delta V_i = V_i \sum_{j\neq i} V_j^2, \\ V_i \geq 0, \end{cases} \quad \text{in } \mathbb{R}^N,\ i = 1, \dots, k, \tag{1-1}$$

which arise in the blow-up analysis of phase-separation phenomena in coupled Schrödinger equations, has attracted increasing attention in recent years, and by now many results concerning existence and qualitative properties of the solutions are available. For a detailed explanation about how (1-1) appears, we refer to [Berestycki et al. 2013a; 2013b; Soave and Zilio 2016]. We prove the existence of *N-dimensional solutions* to (1-1) in $\mathbb{R}^N$ for any $N \geq 2$. By this, we mean that we construct solutions in $\mathbb{R}^N$ which cannot be obtained from solutions in lower dimensions by adding a dependence on some "mute" variable. Our results extend the construction developed in [Berestycki et al. 2013b], which concerns the planar case $N = 2$. In this perspective, we mention that previous results contained in [Berestycki et al. 2013a; 2013b] only regard the existence of solutions in dimension $N = 1$ or 2, and the question of the existence in higher dimensions was up to now open.

In order to state our main results, we introduce some notation. We denote by $\mathcal{O}(N)$ the orthogonal group of $\mathbb{R}^N$ and by $\mathfrak{S}_k$ the symmetric group of permutations of $\{1, \dots, k\}$. Let us assume that there

The authors are partially supported through the project ERC Advanced Grant 2013 No. 339958 "Complex Patterns for Strongly Interacting Dynamical Systems — COMPAT". Zilio is also partially supported by the ERC Advanced Grant 2013 No. 321186 "ReaDi — Reaction–Diffusion Equations, Propagation and Modelling".
MSC2010: primary 35B06, 35B08, 35B53; secondary 35B40, 35J47.
Keywords: entire solutions of elliptic systems, Liouville theorem, nonlinear Schrödinger systems, Almgren monotonicity formula, optimal partition problems, equivariant solutions.

exists a homomorphism $h : \mathcal{G} \to \mathfrak{S}_k$, where $\mathcal{G} < \mathcal{O}(N)$ is a nontrivial subgroup. We define the *equivariant right action* of $\mathcal{G}$ on $H^1(\mathbb{R}^N, \mathbb{R}^k)$ in the following way:

$$\begin{aligned} \mathcal{G} \times H^1(\mathbb{R}^N, \mathbb{R}^k) &\to H^1(\mathbb{R}^N, \mathbb{R}^k), \\ (g, \boldsymbol{u}) &\mapsto g \cdot \boldsymbol{u} := (u_{(h(g))^{-1}(1)} \circ g, \dots, u_{(h(g))^{-1}(k)} \circ g), \end{aligned} \tag{1-2}$$

where $\circ$ denotes the usual composition of functions, and we used the vector notation $\boldsymbol{u} := (u_1, \dots, u_k)$. The set

$$H_{(\mathcal{G},h)} := \{\boldsymbol{u} \in H^1(\mathbb{R}^N, \mathbb{R}^k) : \boldsymbol{u} = g \cdot \boldsymbol{u} \ \text{ for all } \ g \in \mathcal{G}\}$$

is the subspace of the $(\mathcal{G}, h)$-equivariant functions.

Definition 1.1. For $k \in \mathbb{N}$, a nontrivial subgroup $\mathcal{G} < \mathcal{O}(N)$, and a homomorphism $h : \mathcal{G} \to \mathfrak{S}_k$, we write that the triplet $(k, \mathcal{G}, h)$ is admissible if there exists a $(\mathcal{G}, h)$-equivariant function $\boldsymbol{u}$ with the following properties:

(i) $u_i \geq 0$ and $u_i \not\equiv 0$ for every i;

(ii) $u_i u_j \equiv 0$ for every $i \neq j$;

(iii) there exist $g_2, \dots, g_k \in \mathcal{G}$ such that

$$u_i = u_1 \circ g_i \quad \text{for } \ i = 2, \dots, k.$$

Remark 1.2. Notice that, if $(k, \mathcal{G}, h)$ is admissible triplet, then all the $(\mathcal{G}, h)$-equivariant functions satisfy (iii) in the previous definition with the same symmetries g_i; indeed, by (iii) and equivariance we deduce that $(h(g_i))^{-1}(i) = 1$ for every i, so that any equivariant function satisfies

$$v_i = v_{(h(g_i))^{-1}(i)} \circ g_i = v_1 \circ g_i \quad \text{for all } \ i = 1, \dots, k. \tag{1-3}$$

This tells us that any equivariant function associated to an admissible triplet is completely determined by its first component: if we know that $\boldsymbol{v}$ is $(\mathcal{G}, h)$-equivariant and that $(k, \mathcal{G}, h)$ is an admissible triplet, then (1-3) holds true, and hence $v_2, \dots, v_k$ can be obtained by knowing v_1 and $g_2, \dots, g_k$.

We also underline the fact that there may exist symmetries in $\mathcal{G}$ whose corresponding permutation is the identity. In this case, these symmetries are imposed on the single components.

Finally, we observe that the definition of admissible triplet implicitly imposes several restrictions on $(k, \mathcal{G}, h)$. For instance, by (iii) we immediately deduce that h can never be the trivial homomorphism $\mathcal{G} \to \mathfrak{S}_k$, $g \mapsto \text{id}$ for all g. Moreover, we also deduce that $\mathcal{G}$ has at least k different elements.

Let $(k, \mathcal{G}, h)$ be an admissible triplet. We let $\Lambda_{(\mathcal{G},h)}$ be the set of those $\varphi \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$ such that

$$\varphi \text{ is the restriction on } \mathbb{S}^{N-1} \text{ of a } (\mathcal{G}, h)\text{-equivariant function fulfilling Definition 1.1(i)–(iii).} \tag{1-4}$$

We consider the minimization problem

$$\ell_{(k,\mathcal{G},h)} := \inf_{\varphi \in \Lambda_{(\mathcal{G},h)}} \frac{1}{k} \sum_{i=1}^{k} \left(\sqrt{\left(\frac{N-2}{2}\right)^2 + \frac{\int_{\mathbb{S}^{n-1}} |\nabla_\theta \varphi_i|^2}{\int_{\mathbb{S}^{n-1}} \varphi_i^2}} - \frac{N-2}{2} \right), \tag{1-5}$$

where ∇_θ denotes the tangential gradient on $\mathbb{S}^{N-1}$.

Theorem 1.3. *For any admissible pair* $(k, \mathcal{G}, h)$, *there exists a solution* $\boldsymbol{V}$ *of* (1-1) *with* k *components in* $\mathbb{R}^N$ *satisfying the following properties*:

- $\boldsymbol{V}$ *is* $(\mathcal{G}, h)$*-equivariant*;
- *we have*

$$\lim_{r\to+\infty} \frac{1}{r^{N-1+2\ell_{(k,\mathcal{G},h)}}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2 \in (0, +\infty). \tag{1-6}$$

Here and in the rest of the paper $B_r(x_0)$ denotes the ball of centre x_0 and radius r; when $x_0 = 0$, we simply write B_r for the sake of simplicity.

Since the theorem is quite general, we think that it is worthwhile to spend some time making some explicit examples. This will be done in Section 2.1. For the moment, we anticipate that with our result we can recover Theorems 1.3 and 1.6 in [Berestycki et al. 2013b], and moreover we can produce a wealth of new solutions existing only in dimensions $N \geq 3$.

We also observe that condition (1-6) establishes that the solution $\boldsymbol{V}$ grows at infinity, in quadratic mean, like the power $|x|^{\ell(k,\mathcal{G},h)}$. It is worth remarking that for any solution $\boldsymbol{V}$ to (1-1) it is possible to define the *growth rate* as the uniquely determined value $d \in (0, +\infty]$ such that

$$\lim_{r\to+\infty} \frac{1}{r^{N-1+2m}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2 = \begin{cases} +\infty & \text{if } m < d, \\ 0 & \text{if } m > d; \end{cases}$$

see Proposition 1.5 in [Soave and Terracini 2015] and its proof. Therein, it is also shown that $\boldsymbol{V}$ *has algebraic growth*, i.e., it satisfies the pointwise upper bound

$$V_1(x) + \cdots + V_k(x) \leq C(1 + |x|^\alpha) \quad \text{for all } x \in \mathbb{R}^N \tag{1-7}$$

for some $C, \alpha \geq 1$, if and only if its growth rate d is finite; we point out moreover that, as shown in [Soave and Zilio 2014], the system does indeed admit solutions with exponential (i.e., nonalgebraic) growth.

Theorem 1.3 not only specifies the growth rate of the function ($d = \ell(k, \mathcal{G}, h)$), but also states that, for this precise growth rate, the limit

$$\lim_{r\to+\infty} \frac{1}{r^{N-1+2d}} \int_{\partial B_r} \sum_{i=1}^{k} V_i^2$$

is positive and finite. In this perspective we can prove that the solutions of Theorem 1.3 have minimal growth rate among all the possible $(\mathcal{G}, h)$-equivariant solutions.

Theorem 1.4. *Let* $(k, \mathcal{G}, h)$ *be an admissible pair and let* $\boldsymbol{V}$ *be a* $(\mathcal{G}, h)$*-equivariant solution of* (1-1). *Then the growth rate of* $\boldsymbol{V}$ *is at least* $\ell(k, \mathcal{G}, h)$.

Both the proofs of Theorems 1.3 and 1.4 exploit the hidden relationship between the elliptic system (1-1) and optimal partition problems of type (1-5). This relationship arises for instance by means of the validity of the following modification of the celebrated Alt–Caffarelli–Friedman monotonicity formula, tailor-made for the study of $(\mathcal{G}, h)$-equivariant solutions.

For $\boldsymbol{V} \in H^1(\mathbb{R}^N, \mathbb{R}^k)$ and $i = 1, \dots, k$ we define

$$J_i(r) := \int_{B_r} \frac{|\nabla V_i|^2 + V_i^2 \sum_{j \neq i} V_j^2}{|x|^{N-2}}.$$

Proposition 1.5. *Let $(k, \mathcal{G}, h)$ be an admissible triplet. There exists a constant $C > 0$ depending only on N and $(k, \mathcal{G}, h)$ such that, for any $(\mathcal{G}, h)$-equivariant solution $\boldsymbol{V}$ of* (1-1), *the function*

$$r \mapsto \frac{1}{r^{2k\ell(k,\mathcal{G},h)}} e^{-Cr^{-1/2}} J_1(r) \cdots J_k(r)$$

is monotone nondecreasing for $r > 1$ (we recall that $\ell(k, \mathcal{G}, h)$ has been defined in (1-5)).

The expert reader will have already recognized the similarity with the original Alt–Caffarelli–Friedman monotonicity formula, proved in [Alt et al. 1984]; monotonicity formulae of Alt–Caffarelli–Friedman type for competing systems are key ingredients for the results in [Conti et al. 2005; Farina and Soave 2014; Noris et al. 2010; Soave and Terracini 2015; Soave and Zilio 2015; Wang 2014]. The previous result is, to our knowledge, the first example of a monotonicity formula under a symmetry constraint.

We review now the main known results regarding entire solutions of the system (1-1) which were already available, starting with the system with $k = 2$ components. The 1-dimensional problem was studied in [Berestycki et al. 2013a], where it is proved that there exists a solution satisfying the symmetry property $V_2(x) = V_1(-x)$, the monotonicity condition $V_1' > 0$ and $V_2' < 0$ in $\mathbb{R}$, and having at most linear growth, in the sense that there exists $C > 0$ such that

$$V_1(x) + V_2(x) \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}^N.$$

Up to translations, scaling and exchange of the components, this is the unique solution in dimension $N = 1$; see [Berestycki et al. 2013b, Theorem 1.1]. The linear growth is the minimal admissible growth for nonconstant positive solutions of (1-1). Indeed, in any dimension $N \geq 1$, if (V_1, V_2) is a *nonnegative* solution of (1-1) (which means that the condition $V_i > 0$ is replaced by $V_i \geq 0$) and satisfies the sublinear growth condition

$$V_1(x) + V_2(x) \leq C(1 + |x|^\alpha) \quad \text{in } \mathbb{R}^N$$

for some $\alpha \in (0, 1)$ and $C > 0$, then one of V_1 and V_2 is 0 and the other has to be constant. This *Liouville-type theorem* has been proved by B. Noris et al. [2010, Proposition 2.6].

Differently from the problem in $\mathbb{R}$, in dimension $N = 2$, and hence in any dimension $N \geq 2$, the system (1-1) with $k = 2$ has infinitely many "geometrically distinct" solutions, i.e., solutions which cannot be obtained from each other by means of rigid motions, scalings or exchange of the components; see [Berestycki et al. 2013b, Theorem 1.3; Soave and Zilio 2014, Theorems 1.1 and 1.5]. These solutions can be distinguished according to their growth rates and symmetry properties. In particular, Berestycki et al. [2013b] proved the existence of solutions having algebraic growth, while the results in [Soave and Zilio 2014] concern solutions having exponential growth in x that are periodic in y.

Regarding systems with several components, the aforementioned existence results admit analogous counterparts for any $k \geq 3$; see [Berestycki et al. 2013b, Theorem 1.6; Soave and Zilio 2014, Theorem 1.8].

It is important to stress that the proofs in [Berestycki et al. 2013b; Soave and Zilio 2014] use the fact that the problem is posed in dimension $N = 2$, and apparently cannot be extended to higher dimensions (see Remark 4.4 for a more detailed discussion).

In parallel to the existence results, great efforts have been devoted to the analysis of the 1-dimensional symmetry of solutions under suitable assumptions; this, as explained in [Berestycki et al. 2013a], is inspired by some analogy with the derivation of (1-1) and of the Allen–Chan equation, for which symmetry results in the spirit of the celebrated De Giorgi's conjecture have been widely studied. In this context, we recall that, assuming $k = 2$ and $N = 2$, A. Farina [2014] proved that, if (V_1, V_2) has algebraic growth and $\partial_2 V_1 > 0$ in $\mathbb{R}^2$, then (V_1, V_2) is 1-dimensional. In the higher-dimensional case $N \geq 2$ with $k = 2$, Farina and the first author proved a Gibbons-type conjecture for (1-1); see [Farina and Soave 2014]. Furthermore, K. Wang [2014; 2015], as a product of his main results, showed that any solution of (1-1) with $k = 2$ having linear growth is 1-dimensional. We mention also [Berestycki et al. 2013a, Theorem 1.8; 2013b, Theorem 1.12], which are now included in Wang's result.

As far as the 1-dimensional symmetry for systems with $k > 2$ is concerned, we refer to [Soave and Terracini 2015, Theorem 1.3], where the main results in [Farina and Soave 2014; Wang 2014; 2015] are extended to systems with many components by means of improved Liouville-type theorems for multicomponent systems, which relate the number of nontrivial components of a nonnegative solution of the first equation in (1-1) and its growth rate. In this perspective, Theorem 1.4 is the counterpart of [Soave and Terracini 2015, Theorem 1.7] in a $(\mathcal{G}, h)$-equivariant setting. As a product of these two results, we can also derive the following:

Corollary 1.6. *For $k, N \in \mathbb{N}$, let*

$$\mathcal{L}_k(\mathbb{S}^{N-1}) := \inf_{(\omega_1,\dots,\omega_k)\in\mathcal{P}_k} \sup_{i=1,\dots,k} \lambda_1(\omega_i),$$

where $\mathcal{P}_k$ is the set of partitions of $\mathbb{S}^{N-1}$ in k open disjoint and connected sets, and λ_1 denotes the first eigenvalue of the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$. Also, let $(k, \mathcal{G}, h)$ be any admissible triplet with $\mathcal{G} < \mathcal{O}(N)$. Then

$$\mathcal{L}_k(\mathbb{S}^{N-1}) \leq \ell(k, \mathcal{G}, h).$$

It is tempting to conjecture that equality holds for an appropriate choice of $(\mathcal{G}, h)$, at least for some values of k, N. Indeed, in light of the known results in the literature, this is the case for $k = 2$ and $k = 3$, for every N. For $k = 2$, the only (up to isometries) optimal partition for $\mathcal{L}_2(\mathbb{S}^{N-1}) = 1$ is the partition of the sphere into two equal spherical cups [Alt et al. 1984]. This is clearly also an optimal partition for $\ell(2, \mathcal{G}, h)$ if $\mathcal{G}$ is equal to the group generated by the reflection T with respect to a hyperplane through the origin and $h(T)$ is defined as the permutation exchanging the indices 1 and 2. In the case $k = 3$, an optimal partition for $\mathcal{L}_3(\mathbb{S}^{N-1}) = \frac{3}{2}\left(N - \frac{1}{2}\right)$ is the so-called $\boldsymbol{Y}$-partition (see [Helffer et al. 2010; Soave and Terracini 2015]) which is then optimal also for $\ell(3, \mathcal{G}, h)$ if $\mathcal{G}$ is equal to the group generated by the rotation R of angle $\frac{2}{3}\pi$ around the x_N axis and $h(R)$ is the permutation mapping 1 into 2, 2 into 3 and 3 into 1.

To conclude, we mention also the contribution of Wang and Wei [2014], who considered the fractional analogue of (1-1). Such problems exhibit new interesting phenomena with respect to the local case. Moreover, we observe that our results, as those in [Berestycki et al. 2013b], seem to be somehow connected with those in [Wei and Weth 2007], which concern finite energy decaying solutions of a different problem.

Structure of the paper. in Section 2 we recall some known results needed for the rest of work, and which permit us to show, in Section 2.1, several concrete applications of Theorem 1.3. Section 3 is devoted to the proof of the equivariant Alt–Caffarelli–Friedman monotonicity formula, Proposition 1.5; finally, in Section 4, we give the proofs of the other main results, Theorems 1.3 and 1.4.

2. Preliminaries and application of Theorem 1.3

We introduce some notation and review some known results. Let $\beta > 0$, and let $\boldsymbol{U}$ be a solution to

$$\begin{cases} \Delta U_i = \beta U_i \sum_{j\neq i} U_j^2 & \text{in } B_R, \\ U_i > 0 & \text{in } B_R. \end{cases} \tag{2-1}$$

For $0 < r < R$, we set

$$\begin{aligned} H(\boldsymbol{U}, r) &:= \frac{1}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^k U_i^2, \\ E(\boldsymbol{U}, r) &:= \frac{1}{r^{N-2}} \int_{B_r} \sum_{i=1}^k |\nabla U_i|^2 + \beta \sum_{1\le i<j\le k} U_i^2 U_j^2, \\ N(\boldsymbol{U}, r) &:= \frac{E(\boldsymbol{U}, r)}{H(\boldsymbol{U}, r)} \quad \text{(the Almgren frequency function).} \end{aligned}$$

Under the previous notation, by Proposition 5.2 in [Berestycki et al. 2013b] it is known that $N(\boldsymbol{U}, \cdot)$ is monotone nondecreasing for $0 < r < R$,

$$\frac{d}{dr} H(\boldsymbol{U}, r) = \frac{2}{r} E(\boldsymbol{U}, r) + \frac{2\beta}{r^{N-1}} \int_{B_r} \sum_{i<j} U_i^2 U_j^2 > 0,$$

and, for any such r,

$$\int_1^r 2\beta \frac{\int_{B_s} \sum_{i<j} U_i^2 U_j^2}{s^{N-1} H(\boldsymbol{U}, s)} \, ds \le N(\boldsymbol{U}, r). \tag{2-2}$$

The frequency function, also called Almgren's quotient, gives information about the behaviour of the solutions with respect to radial dilations. Indeed, the possibility of defining a growth rate for any solution to (1-1) is a direct consequence of the monotonicity of $N(\boldsymbol{V}, \cdot)$. We recall that, as proved in [Soave and Terracini 2015, Proposition 1.5], for any solution $\boldsymbol{V}$ to (1-1) there exists a value $d \in (0, +\infty]$ such that

$$\lim_{r\to+\infty} \frac{(1/r^{N-1}) \int_{\partial B_r} \sum_{i=1}^k V_i^2}{r^{2d'}} = \begin{cases} +\infty & \text{if } d' < d, \\ 0 & \text{if } d' > d, \end{cases} \tag{2-3}$$

and $d < +\infty$ if and only if $\boldsymbol{V}$ has algebraic growth. We write that d is the *growth rate of* $\boldsymbol{V}$, and it is remarkable that

$$d = \lim_{r \to +\infty} N(\boldsymbol{V}, r); \tag{2-4}$$

again see [Soave and Terracini 2015, Proposition 1.5] (the result is stated in [Soave and Terracini 2015] for solutions with algebraic growth, but its proof works also without this assumption). Notice that on the left-hand side of (2-3) we have the quadratic average of $\boldsymbol{V}$ on spheres of increasing radius divided by a power of r^2; thus the name *growth rate*.

In the previous discussion $\beta > 0$ was fixed. Let us now consider a sequence of parameters $\beta \to +\infty$ and a corresponding sequence $\{\boldsymbol{U}_\beta\}$ of solutions to (2-1). The asymptotic behaviour of the family $\{\boldsymbol{U}_\beta\}$ has been studied in [Berestycki et al. 2013a; Dancer et al. 2012; Noris et al. 2010; Soave and Zilio 2015; 2016; Tavares and Terracini 2012; Wei and Weth 2008] and many results are available. We only recall that, if the sequence is bounded in $L^\infty(B_R)$, then it is in turn uniformly bounded in $\operatorname{Lip}(B_R)$, and hence up to a subsequence it converges to a limit $\boldsymbol{U}$ in $\mathcal{C}^{0,\alpha}(B_R)$ and in $H^1_{\mathrm{loc}}(B_R)$ (see [Soave and Zilio 2015; Noris et al. 2010]). If $\boldsymbol{U} \not\equiv \boldsymbol{0}$, then $\boldsymbol{U}$ is Lipschitz continuous and $\{\boldsymbol{U} = \boldsymbol{0}\}$ has Hausdorff dimension $N - 1$. Moreover, $H(\boldsymbol{U}, r)$ is nondecreasing and is nonzero for every $r > 0$ (see [Tavares and Terracini 2012]).

An important application of this asymptotic theory lies in the possibility of defining blow-down limits of entire solutions to (1-1). We recall part of [Berestycki et al. 2013b, Theorem 1.4] ($k = 2$) and [Soave and Terracini 2015, Theorem 1.4] (k arbitrary). Let $\boldsymbol{V}$ be a solution to (1-1), and for any $R > 0$ let us define the *blow-down family*

$$\boldsymbol{V}_R(x) := \frac{1}{H(\boldsymbol{V}, R)^{1/2}} \boldsymbol{V}(Rx).$$

If $\boldsymbol{V}$ has algebraic growth, i.e., its growth rate $d = N(\boldsymbol{V}, +\infty)$ is finite, then $\{\boldsymbol{V}_R\}$ converges, in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$, as $R \to +\infty$ and up to a subsequence, to a homogeneous vector-valued function $\boldsymbol{V}_\infty$ with homogeneity degree d and such that

- the components $V_{i,\infty}$ are nonnegative and with disjoint support: $V_{i,\infty} V_{j,\infty} \equiv 0$ for every $i \neq j$;
- $V_{i,\infty} - V_{j,\infty}$ for any $i \neq j$ is harmonic in the interior of its support.

When $k = 2$, it then results that $(V_{1,\infty}, V_{2,\infty}) = (\Psi^+, \Psi^-)$, where Ψ is a homogenous harmonic polynomial in $\mathbb{R}^N$, and hence necessarily d is an integer.

2.1. ***A wealth of new solutions: applications of Theorem 1.3.*** We recall that, for any $k \geq 2$, problem (1-1) has several solutions in $\mathbb{R}^2$. Clearly, these are also solutions in higher dimensions, and up to now it was an open question whether or not there exist N-dimensional solutions of (1-1) in $\mathbb{R}^N$ with $N \geq 3$, i.e., solutions in $\mathbb{R}^N$ which cannot be obtained as solutions in $\mathbb{R}^{N-1}$ by adding a dependence on a variable. Theorem 1.3 gives a positive answer to these questions. In what follows we show how to use Theorem 1.3 as a recipe to construct entire solutions of (1-1).

A concrete example in $\mathbb{R}^3$ for $k = 2$. To start with a very concrete example, we focus on problem (1-1) in $\mathbb{R}^3$ with $k = 2$, and we examine the case where $\mathcal{G}$ is equal to the group of symmetries generated by the reflections T_1, T_2 and T_3 with respect to the planes $\{x = 0\}$, $\{y = 0\}$ and $\{z = 0\}$, respectively, and $h : \mathcal{G} \to \mathfrak{S}_k$ is defined on the generators of $\mathcal{G}$ by $h(T_i) = (1\ 2)$ for every i. We used here the standard notation $(1\ 2)$ to denote the cycle mapping 1 to 2, and 2 to 1. In order to check that this is an admissible triplet, we verify that

$$(u_1, u_2) = ((xyz)^+, (xyz)^-)$$

is a $(\mathcal{G}, h)$-equivariant function satisfying (i)–(iii) in Definition 1.1. For the equivariance, we explicitly observe that

$$\begin{aligned} T_i \cdot (u_1, u_2) &= (u_2 \circ T_i, u_1 \circ T_i) && \text{(see (1-2))} \\ &= (u_1, u_2) && \text{(by definition of } \boldsymbol{u}) \end{aligned}$$

for every i, and since $\mathcal{G}$ is generated by T_1, T_2, T_3, this is sufficient to conclude that $\boldsymbol{u}$ is $(\mathcal{G}, h)$-equivariant. Points (i) and (ii) in Definition 1.1 are straightforward, and (iii) is satisfied since $u_2 = u_1 \circ T_i$ for any i. As a consequence, by Theorem 1.3 there exists a $(\mathcal{G}, h)$-equivariant solution (V_1, V_2) of (1-1) in $\mathbb{R}^3$ with $k = 2$ having growth rate equal to $\ell(k, \mathcal{G}, h) = N(\boldsymbol{V}, +\infty)$ (we recall that the growth rate is always equal to the limit at infinity of the Almgren frequency function; see (2-4)). Since the symmetries of $\mathcal{G}$ involve the 3 variables, this solution cannot be obtained by a 2-dimensional solution adding the dependence of 1 variable: $V_1 - V_2$ is not constant since $\boldsymbol{V}$ has growth rate $\ell(2, \mathcal{G}, h) > 0$; moreover, thanks to the symmetries T_1, T_2, T_3, we have that the function $V_1 - V_2$ vanishes on the set $\{x = 0\} \cup \{y = 0\} \cup \{z = 0\}$. Since the projection of this set on any 2-dimensional subspace is equal to the entire subspace but $\boldsymbol{V}$ is nontrivial, we immediately deduce that the solution cannot be 2-dimensional.

In this particular case we can also explicitly compute $\ell(2, \mathcal{G}, h)$, in the following way: by minimality,

$$\ell(2, \mathcal{G}, h) \le \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta (xyz)^+|^2}{\int_{\mathbb{S}^2} |(xyz)^+|^2}} - \frac{1}{2}\right) + \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta (xyz)^-|^2}{\int_{\mathbb{S}^2} |(xyz)^-|^2}} - \frac{1}{2}\right),$$

and the right-hand side is equal to 3; indeed, since $\Phi := xyz$ is a homogeneous harmonic polynomial of degree 3, its angular part $\Phi|_{\mathbb{S}^2}$ solves

$$-\Delta_\theta \Phi|_{\mathbb{S}^2} = 12 \Phi|_{\mathbb{S}^2} \quad \text{in } \mathbb{S}^2,$$

and this permits us to carry out explicit computations. This means that Ψ (the blow-down limit) is a homogeneous harmonic polynomial of degree $\ell(2, \mathcal{G}, h) \le 3$. It is then necessary that $\Psi = \Phi = xyz$; to check this, we can simply consider all the homogeneous harmonic polynomials in $\mathbb{R}^3$ with degree at most 3, which have been classified, and observe that the only one being $(\mathcal{G}, h)$ equivariant is Φ. As a consequence, the degree of homogeneity of Ψ is $3 = \ell(2, \mathcal{G}, h)$.

General case in $\mathbb{R}^N$ with $k = 2$. The very same argument as before can be considered by taking any homogeneous harmonic polynomial Φ in $\mathbb{R}^N$ of degree $d \in \mathbb{N}$ with a nontrivial finite group of symmetries $\mathcal{G}$; by this we mean that there exists a group of symmetries with generators $T_1, \dots, T_m$ such that $\Phi^\pm \circ T_i = \Phi^\mp$. To any T_i we associate the cycle $(1\ 2)$. This induces a homomorphism $h : \mathcal{G} \to \mathfrak{S}_2$, and it is not difficult to check that $(2, \mathcal{G}, h)$ is an admissible triplet. Indeed, by assumption the pair $(u_1, u_2) = (\Phi^+, \Phi^-)$ fulfills (i)–(iii) in Definition 1.1, and is $(\mathcal{G}, h)$-equivariant: the equivariance follows by

$$\begin{aligned} T_i \cdot (u_1, u_2) &= (u_2 \circ T_i, u_1 \circ T_i) \quad \text{(see (1-2))} \\ &= (u_1, u_2) \end{aligned}$$

for any i. Points (i) and (ii) in Definition 1.1 are trivial, and (iii) is satisfied since $u_2 = u_1 \circ T_i$ for any i by assumption. If, as in the example above, the group $\mathcal{G}$ is chosen from the beginning so that the symmetries of $\mathcal{G}$ involve all the N variables, we obtain an N-dimensional solution to (1-1). Explicit cases where the previous argument is applicable are the following:

- At first, we show how we can recover Theorem 1.3 in [Berestycki et al. 2013b]. In dimension $N = 2$, we take $\Phi_d(x, y) := \mathfrak{Re}((x + iy)^d)$, with $d \in \mathbb{N}$. Then Φ_d is symmetric, in the previous sense, with respect to the group of symmetries generated by the reflections $T_1, \dots, T_d$ with respect to its nodal lines: $\Phi_d^{\pm} \circ T_i = \Phi_d^{\mp}$. By the previous argument, we find $(\mathcal{G}, h)$-equivariant solutions of the problem with growth rate $\ell(2, \mathcal{G}, h)$, which clearly are 2-dimensional. Reasoning as in our first example, it is not difficult in this case to check that $\ell(2, \mathcal{G}, h) = d$.

- Secondly, we construct infinitely many new solutions in $\mathbb{R}^3$. We take $\Phi_d(x, y) := \mathfrak{Re}((x + iy)^d)z$, with $d \in \mathbb{N}$. Let $T_1, \dots, T_d$ denote the reflections with respect to the nodal planes of $\mathfrak{Re}((x + iy)^d)$, and let T_z denote the reflection with respect to $\{z = 0\}$. Then $\Phi_d^{\pm} \circ T_i = \Phi_d^{\mp}$, so that the general argument above is applicable, and hence we find a $(\mathcal{G}, h)$-equivariant solution of (1-1) with growth rate $\ell(2, \mathcal{G}, h)$. As in the first example, since the nodal set of $V_1 - V_2$ has surjective projection on any 2-dimensional subspace, $\boldsymbol{V}$ is necessarily 3-dimensional. We can also check that $\ell(2, \mathcal{G}, h) = d + 1$. Since (Φ_d^+, Φ_d^-) is a $(\mathcal{G}, h)$-equivariant function, we have

$$\ell(2, \mathcal{G}, h) \le \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta \Phi_d^+|^2}{\int_{\mathbb{S}^2} |\Phi_d^+|^2}} - \frac{1}{2}\right) + \frac{1}{2}\left(\sqrt{\frac{1}{4} + \frac{\int_{\mathbb{S}^2} |\nabla_\theta \Phi_d^-|^2}{\int_{\mathbb{S}^2} |\Phi_d^-|^2}} - \frac{1}{2}\right).$$

As in the previous example, we can prove that the right-hand side is equal to $d+1$. On the other hand, using the blow-down theorem and explicitly observing that the only $(\mathcal{G}, h)$-equivariant homogeneous harmonic polynomial in $\mathbb{R}^3$ with degree less than or equal to $d + 1$ is Φ_d, we conclude that $\ell(2, \mathcal{G}, h) = d + 1$.

- We conclude with the observation that the previous constructions can be extended in any dimensions. For instance we can consider the harmonic polynomial $\Phi = x_1 \cdots x_N$, together with the symmetry group generated by the reflections $T_1, \dots, T_N$ with respect to the coordinate planes $\{x_i = 0\}$, $i = 1, \dots, N$; notice that $\Phi^{\pm} \circ T_i = \Phi^{\mp}$ for any i. In the same way we could consider the harmonic polynomial $\Psi = \mathfrak{Re}((x_1 + ix_2)^d)x_3 \cdots x_N$, together with symmetry group generated by the reflections $T_1, \dots, T_d$ with respect to the nodal hyperplanes of $\mathfrak{Re}((x_1 + ix_2)^d)$, and by $R_3, \dots, R_N$, reflections with respect to the coordinate planes $\{x_i = 0\}$, $i = 3, \dots, N$.

The case $k \ge 3$ in $\mathbb{R}^2$. For $k \ge 3$ components, we first show how to recover Theorem 1.6 in [Berestycki et al. 2013b]. We thus focus for the moment on the dimension $N = 2$. Let $k \ge 3$ and, for any $m \in \mathbb{N}$, let $d = \frac{1}{2}mk$. We denote by R_d the rotation of angle π/d, by T_y the reflection with respect to $\{y = 0\}$

(this corresponds to complex conjugation in $\mathbb{C}$), and we consider the group $\mathcal{G} < \mathcal{O}(N)$ generated by R_d and T_y. We define a homomorphism $h : \mathcal{G} \to \mathfrak{S}_k$ (the group of permutations of $\{1, \dots, k\}$) letting

$$h(R_d) := (1\ 2\ \cdots\ d) \quad \text{and} \quad h(T_y) : i \mapsto k + 2 - i,$$

where the indexes are counted modulus k. We can explicitly check that $(k, \mathcal{G}, h)$ is an admissible triplet. Let us consider the function

$$\begin{aligned} u_1 &:= \begin{cases} r^d \cos(d\theta) & \text{in } \bigcup_{i=0}^{m-1} R_d^{ik}(\{-\pi/2d < \theta < \pi/2d\}), \\ 0 & \text{otherwise}, \end{cases} \\ u_2 &:= u_1 \circ R_d, \\ &\ \vdots \\ u_k &:= u_{k-1} \circ R_d = u_1 \circ R_d^{k-1}. \end{aligned}$$

It is $(\mathcal{G}, h)$-equivariant, as

$$\begin{aligned} R_d \cdot \boldsymbol{u} &= (u_k \circ R_d, u_1 \circ R_d, \dots, u_{k-1} \circ R_d) = \boldsymbol{u}, \\ T_y \cdot \boldsymbol{u} &= (u_1 \circ T_y, u_k \circ T_y, u_{k-1} \circ T_y, \dots, u_3 \circ T_y, u_2 \circ T_y) = \boldsymbol{u}. \end{aligned}$$

It clearly satisfies (i) and (ii) in Definition 1.1, and for (iii) it is sufficient to note that $u_j = u_1 \circ R_d^{j-1}$ for every $j = 2, \dots, k$. By Theorem 1.3, we obtain a $(\mathcal{G}, h)$-equivariant solution $\boldsymbol{V}$ of (1-1); the fact that $\boldsymbol{V}$ is 2-dimensional follows again from the symmetries: if $\boldsymbol{V}$ were 1-dimensional, then we could say that $\bigcup_{i \neq j}\{V_i - V_j = 0\}$ is the union of straight parallel lines. But, on the other hand, $\{V_2 - V_3 = 0\} = R_d(\{V_1 - V_2 = 0\})$, which cannot be parallel whenever $d > 1$, i.e., whenever $k \geq 3$.

To complete the analogy with the results in [Berestycki et al. 2013b], we still would have to prove that $N(\boldsymbol{V}, +\infty) = \ell(k, \mathcal{G}, h)$ is equal to d. Since we are in dimension $N = 2$, this can be done by means of explicit computations, following the line of reasoning already adopted in the previous examples. We decided to not stress this point for the sake of brevity.

The general case $k \geq 3$ in $\mathbb{R}^3$. The case $k \geq 3$ and $N \geq 3$ is intrinsically more involved, and hence we focus on some particular examples given by the groups of symmetries of the Platonic polyhedra. Let us consider for instance the group $\mathcal{G}_4 < \mathcal{O}(N)$ associated to the tetrahedron $\mathcal{T}$. It is known that this group is isomorphic to $\mathfrak{S}_4$. The isomorphism h_4 is obtained labelling all the vertices of $\mathcal{T}$, and associating to any $g \in \mathcal{G}_4$ the permutation induced on the vertices themselves. In order to define the function φ satisfying (i)–(iii) of Definition 1.1, we first take a tetrahedron with barycentre 0, and define on a face A a positive function $\tilde{\varphi}_1$ that is 0 on ∂A and symmetric with respect to all the transformations in $\mathcal{G}_4$ leaving A invariant. By rotation, we can define $\tilde{\varphi}_2$, $\tilde{\varphi}_3$ and $\tilde{\varphi}_4$ on the remaining faces. Now, considering the radial projection of the tetrahedron into the unit sphere $\mathbb{S}^2$, we obtain a function $(\varphi_1, \dots, \varphi_4)$ whose 1-homogeneous extension is by construction $(\mathcal{G}_4, h_4)$-equivariant, and satisfies (i)–(iii) of Definition 1.1. Thus $(4, \mathcal{G}_4, h_4)$ is an admissible triplet, and Theorem 1.3 yields the existence of a $(\mathcal{G}_4, h_4)$-equivariant solution for the system with 4 components in $\mathbb{R}^3$. Since the symmetries of the tetrahedron involve the dependence on 3 variables, this solution is not 2-dimensional.

In a similar way, one can construct $(\mathcal{G}_6, h_6)$-equivariant solutions with respect to the group of symmetries of the cube $\mathcal{G}_6$ (isomorphic to a subgroup of $\mathfrak{S}_8$ through an isomorphism h_6) for systems with $k = 3$ or $k = 6$ components. To this purpose, we consider a cube with barycentre 0 in $\mathbb{R}^3$, and we define on a face A a positive function $\tilde{\varphi}_1$ that is 0 on ∂A and symmetric with respect to all the transformations in $\mathcal{G}_6$ leaving A invariant. By rotation, we can define $\tilde{\varphi}_2, \dots, \tilde{\varphi}_6$ on the remaining faces. Considering the radial projection of the cube onto the unit sphere $\mathbb{S}^2$, we obtain a function $(\varphi_1, \dots, \varphi_6)$ whose 1-homogeneous extension is $(\mathcal{G}_6, h_6)$-equivariant and satisfies (i)–(iii) of Definition 1.1. Theorem 1.3 then gives a 3-dimensional $(\mathcal{G}_6, h_6)$-equivariant solution to (1-1) with 6 components in $\mathbb{R}^3$. In order to obtain a 3-component $(\mathcal{G}_6, h_6)$-equivariant solution, we proceed as in the previous discussion replacing $\tilde{\varphi}_1$ with $\tilde{\psi}_1 = \tilde{\varphi}_1 + \tilde{\varphi}_4$, where φ_4 has support on the face opposite to A in the cube. By rotation, we determine $\tilde{\psi}_2$ and $\tilde{\psi}_3$, each of them supported on the union of two opposite faces. As before, we can then consider the radial projection onto $\mathbb{S}^2$, and afterwards its 1-homogeneous extension (ψ_1, ψ_2, ψ_3), which is $(\mathcal{G}_6, h_6)$-equivariant and satisfies (i)–(iii) of Definition 1.1. For the equivariance, we recall that any isometry of the cube is identified by the faces that three given adjacent faces are mapped to (this is why we could construct solutions with cubical symmetry for systems with 3 components). In conclusion, by Theorem 1.3 we obtain a $(\mathcal{G}_6, h_6)$-equivariant solution of (1-1) with $k = 3$ components.

Arguing in a similar way, we may also obtain equivariant solutions with respect to the symmetries of the octahedron for systems with $k = 4$ and $k = 8$ components, and so on.

3. An Alt–Caffarelli–Friedman monotonicity formula for equivariant solutions

In this section we aim at proving Proposition 1.5. We always suppose that $(k, \mathcal{G}, h)$ is an admissible triplet, according to Definition 1.1. Moreover, we often omit the phrase "up to a subsequence" for simplicity. The proof is divided into several steps, and, as usual when dealing with Alt–Caffarelli–Friedman monotonicity formulae for competing systems, is based upon a control on an "approximated" optimal partition problem on $\mathbb{S}^{N-1}$. For any $\boldsymbol{u} \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, we let

$$I_\beta(\boldsymbol{u}) := \frac{1}{k} \sum_{i=1}^{k} \gamma\left(\frac{\int_{\mathbb{S}^{n-1}} |\nabla_\theta u_i|^2 + \frac{1}{2}\beta u_i^2 \sum_{j \neq i} u_j^2}{\int_{\mathbb{S}^{n-1}} u_i^2} \right),$$

where

$$\gamma(t) := \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right).$$

We denote by $\hat{H}_{(\mathcal{G},h)}$ the subspace of $(\mathcal{G}, h)$-equivariant functions in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, and we introduce the optimal value

$$\ell_\beta(k, \mathcal{G}, h) := \inf_{\hat{H}_{(\mathcal{G},h)}} I_\beta.$$

In what follows, to keep the notation as simple as possible, we simply write ℓ and ℓ_β instead of $\ell(k, \mathcal{G}, h)$ and $\ell_\beta(k, \mathcal{G}, h)$, respectively.

Lemma 3.1. *Both ℓ and ℓ_β are positive and achieved (for all $\beta > 0$). It follows that $\ell_\beta \to \ell$ as $\beta \to +\infty$, and there exists a minimizer for ℓ_β, which solves*

$$\begin{cases} -\Delta_\theta u_{i,\beta} = \lambda_\beta u_{i,\beta} - \beta u_{i,\beta} \sum_{j \neq i} u_j^2 & \text{in } \mathbb{S}^{N-1}, \\ u_{i,\beta} > 0 & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} u_{i,\beta}^2 = 1 & \text{for all } i, \end{cases} \tag{3-1}$$

where $\lambda_\beta \geq 0$ and Δ_θ denotes the Laplace–Beltrami operator on $\mathbb{S}^{N-1}$. Moreover, $\boldsymbol{u}_\beta \rightharpoonup \varphi$ weakly in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$ and φ is a nonnegative minimizer for ℓ.

Proof. Restricting ourselves to the subset of functions in $\hat{H}_{(\mathcal{G},h)}$ whose components have prescribed $L^2(\mathbb{S}^{N-1})$-norm equal to 1, it is easy to check that the functional I_β is weakly lower semicontinuous and coercive. Since $\hat{H}_{(\mathcal{G},h)}$ is also weakly closed, the direct method of the calculus of variations ensures the existence of a minimizer $\boldsymbol{u}_\beta$ for ℓ_β, which can be assumed to be nonnegative. By the Palais principle of symmetric criticality (notice that I_β is invariant under the action of any symmetry in $\mathcal{O}(N)$), the Lagrange multipliers rule, and the strong maximum principle, it follows that $\boldsymbol{u}_\beta$ satisfies

$$\begin{cases} -\Delta_\theta u_{i,\beta} + \sum_{j \neq i} \frac{1}{2}(1 + \mu_{j,\beta}/\mu_{i,\beta}) \beta u_{i,\beta} u_{j,\beta}^2 = \lambda_{i,\beta} u_{i,\beta} & \text{in } \mathbb{S}^{N-1}, \\ u_{i,\beta} > 0 & \text{in } \mathbb{S}^{N-1}, \end{cases}$$

where

$$\mu_{i,\beta} := \gamma' \biggl(\int_{\mathbb{S}^{n-1}} |\nabla_\theta u_{i,\beta}|^2 + \tfrac{1}{2} \beta u_{i,\beta}^2 \sum_{j \neq i} u_{j,\beta}^2 \biggr).$$

The equation for $u_{i,\beta}$ is nothing but (3-1): indeed, thanks to the symmetries in $\hat{H}(\mathcal{G}, h)$ (see Remark 1.2), we have $\mu_{i,\beta} = \mu_{j,\beta}$ and $\lambda_{i,\beta} = \lambda_{j,\beta} \geq 0$ for every $i \neq j$. Finally, $\ell_\beta > 0$ since otherwise $\boldsymbol{u}_\beta \equiv \mathbf{0}$, in contradiction with the normalization condition.

As far as ℓ is concerned, we introduce an auxiliary functional $I_\infty : \hat{H}_{(\mathcal{G},h)} \to (0, +\infty]$, defined by

$$I_\infty(\boldsymbol{u}) := \begin{cases} (1/k) \sum_{i=1}^k \gamma \bigl(\int_{\mathbb{S}^{n-1}} |\nabla u_i|^2 / \int_{\mathbb{S}^{n-1}} u_i^2 \bigr) & \text{if } u_i u_j = 0 \text{ a.e. on } \mathbb{S}^{n-1} \text{ for any } i \neq j, \\ +\infty & \text{otherwise.} \end{cases}$$

It is easy to see that I_β is increasing in β and converges pointwise to I_∞, implying that I_∞ is a weakly lower semicontinuous functional in the weakly closed set $\hat{H}_{(\mathcal{G},h)}$, and that I_β Γ-converges to I_∞ in the weak H^1-topology. Moreover, since the family $\{I_\beta\}$ is equicoercive, any sequence $\{\boldsymbol{u}_\beta\}$ of minimizers for I_β converges to a minimizer $\boldsymbol{u}$ of I_∞. Finally, by definition, $\ell > \ell_\beta$ for every $\beta > 0$, whence $\ell > 0$ follows. □

Further properties of the sequence $\{\boldsymbol{u}_\beta\}$ are collected in the next two lemmas.

Lemma 3.2. *The sequence $\{\boldsymbol{u}_\beta\}$ is uniformly bounded in* $\mathrm{Lip}(\mathbb{S}^{N-1})$. *Moreover, the sequence (λ_β) is bounded.*

Proof. Let $\{\boldsymbol{u}_\beta\}$ be a sequence of minimizers for ℓ_β satisfying (3-1), weakly converging to a minimizer $\boldsymbol{u}$ for ℓ. As $I_\beta(\boldsymbol{u}_\beta) = \ell_\beta \leq \ell$, there exists $C > 0$ such that

$$\int_{\mathbb{S}^{N-1}} \beta u_{i,\beta}^2 \sum_{j \neq i} u_{j,\beta}^2 \leq C.$$

Moreover, by weak convergence, $\{\boldsymbol{u}_\beta\}$ is bounded in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. Therefore, testing the first equation in (3-1) against $u_{i,\beta}$, we deduce that $\{\lambda_\beta\}$ is a bounded sequence of positive numbers, and this implies, through a Brézis–Kato argument (see for instance [Tavares 2010, page 124] for a detailed proof and [Brézis and Kato 1979] for the original argument), that $\{\boldsymbol{u}_\beta\}$ is uniformly bounded in $L^\infty(\mathbb{S}^{N-1}, \mathbb{R}^k)$. By the main results in [Soave and Zilio 2015], we infer that $\{\boldsymbol{u}_\beta\}$ is uniformly bounded[1] in $\mathrm{Lip}(\mathbb{S}^{N-1})$. □

Lemma 3.3. *We have $\boldsymbol{u}_\beta \to \varphi$ strongly in the $H^1(\mathbb{S}^{N-1})$ topology, in $C^{0,\alpha}(\mathbb{S}^{N-1})$ for every $0 < \alpha < 1$, and*

$$\lim_{\beta\to+\infty} \beta \int_{\mathbb{S}^{N-1}} u_{i,\beta}^2 u_{j,\beta}^2 = 0.$$

Moreover, $\lambda_\beta \to \ell(\ell + N - 2)$ and

$$\begin{cases} -\Delta_\theta \varphi_i = \ell(\ell+N-2)\varphi_i & \text{in } \{\varphi_i > 0\}, \\ \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1. \end{cases}$$

Proof. Thanks to Lemma 3.2, we can simply apply Theorem 1.4 in [Noris et al. 2010]. To check that $\lambda_\beta \to \ell(\ell + N - 2)$, we observe that, by boundedness, $\lambda_\beta \to \lambda_\infty \geq 0$ as $\beta \to +\infty$. Therefore, recalling that $\boldsymbol{u}_\beta \rightharpoonup \varphi$ in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$, for $i = 1, \dots, k$ we have

$$\begin{cases} -\Delta_\theta \varphi_i = \lambda_\infty \varphi_i & \text{in } \{\varphi_i > 0\}, \\ \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1. \end{cases}$$

This implies that

$$\ell = \frac{1}{k} \sum_i \sqrt{\left(\frac{N-2}{2}\right)^2 + \int_{\mathbb{S}^{N-1}} |\nabla_\theta \varphi_i|^2} - \frac{N-2}{2} = \sqrt{\left(\frac{N-2}{2}\right)^2 + \lambda_\infty} - \frac{N-2}{2},$$

whence the thesis follows. □

The following result is the counterpart of Lemma 4.2 in [Wang 2014] in a $(\mathcal{G}, h)$-equivariant setting; see also Theorem 5.6 in [Berestycki et al. 2013b] for an analogous statement in dimension $N = 2$.

Lemma 3.4. *There exists a constant $C > 0$ such that*

$$\ell_\beta \geq \ell - C\beta^{-1/4}.$$

Before proving the lemma, we need a technical result. We recall that $\hat{H}_{(\mathcal{G},h)}$ denotes the set of $(\mathcal{G}, h)$-equivariant functions in $H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$.

[1] It is worth mentioning that the results in [Soave and Zilio 2015] are proved for the Laplace operator in the interior of subsets of $\mathbb{R}^N$, and their extension to a Riemannian setting presents some technical difficulties; the general extension of [Soave and Zilio 2015] to equations on manifolds will be the object a future contribution [Smit Vega Garcia et al. ≥ 2016]. We anticipate here the main argument: the key ingredients for the regularity results in [Soave and Zilio 2015] are elliptic estimates, an Almgren-type monotonicity formula and a sharp version of the Alt–Caffarelli–Friedman-type monotonicity formula. Thus, we need to extend these three tools for systems on $\mathbb{S}^{N-1}$. The elliptic theory is already available, as is the Almgren-type monotonicity formula (see for instance [Tavares and Terracini 2012, Section 7]). The Alt–Caffarelli–Friedman-type monotonicity formula represents the only obstruction, but it can be obtained by combining the results in [Teixeira and Zhang 2011] (an Alt–Caffarelli–Friedman-type monotonicity formula for scalar equations on Riemannian manifolds) and in [Soave and Zilio 2015] (the sharp version of the Alt–Caffarelli–Friedman-type monotonicity formula for systems in the euclidean space). Once these three tools are available, the proof proceeds as in [Soave and Zilio 2015].

Lemma 3.5. *Let $\boldsymbol{u} \in \hat{H}_{(\mathcal{G},h)}$. Then the function $\hat{\boldsymbol{u}}$, defined by*

$$\hat{u}_i = v_i^+ := \Big(u_i - \sum_{j \neq i} u_j\Big)^+,$$

also belongs to $\hat{H}_{(\mathcal{G},h)}$.

Proof. As $u_i \in H^1(\mathbb{S}^{N-1})$, it follows straightforwardly that $\hat{\boldsymbol{u}} \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. We have to show that it is also $(\mathcal{G}, h)$-equivariant, and to this end it is sufficient to show that $\boldsymbol{v}$ is $(\mathcal{G}, h)$-equivariant. This can be checked directly:

$$\begin{aligned} v_{(h(g))^{-1}(i)}(g(x)) &= u_{(h(g))^{-1}(i)}(g(x)) - \sum_{j \neq (h(g))^{-1}(i)} u_j(g(x)) = u_{(h(g))^{-1}(i)}(g(x)) - \sum_{j \neq i} u_{(h(g))^{-1}(j)}(g(x)) \\ &= v_i(x), \end{aligned}$$

where the last equality follows from the fact that $\boldsymbol{u}$ is $(\mathcal{G}, h)$-equivariant. □

Proof of Lemma 3.4. In order to simplify the notation, only in this proof we write ∇ and Δ instead of ∇_θ and Δ_θ, respectively. Let us consider the functions $\hat{\boldsymbol{u}}_\beta$, defined in Lemma 3.5. Since the components of $\hat{\boldsymbol{u}}_\beta$ have disjoint supports, we can use it as a competitor for ℓ. We aim at showing that $\hat{\boldsymbol{u}}_\beta$ is actually close enough to $\boldsymbol{u}_\beta$ in the energy sense, and in doing this we shall use many times the properties proved in Lemma 3.2. To be precise, we shall prove that there exists a constant $C > 0$ such that

$$1 - C\beta^{-1/2} \leq \int_{\mathbb{S}^{n-1}} \hat{u}_{i,\beta}^2 \leq 1 + C\beta^{-1/2}, \tag{3-2}$$

$$\int_{\mathbb{S}^{N-1}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}. \tag{3-3}$$

Before we continue, let us point out that second estimate can be derived from an analogous one: there exists $C > 0$ independent of β and $\bar{\delta} > 0$ such that, for almost any $\delta \in (0, \bar{\delta})$, we have

$$\int_{\{\hat{u}_{i,\beta} > \delta\}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4} + C\delta.$$

Indeed, if the previous estimate is satisfied,

$$\int_{\mathbb{S}^{N-1}} |\nabla \hat{u}_{i,\beta}|^2 = \int_{\{\hat{u}_{i,\beta} > 0\}} |\nabla \hat{u}_{i,\beta}|^2 = \lim_{\delta \to 0^+} \int_{\{\hat{u}_{i,\beta} > \delta\}} |\nabla \hat{u}_{i,\beta}|^2 \leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4}.$$

Notice that in principle the value $\bar{\delta}$ could depend on β, but this is not a problem since C is, on the contrary, a universal constant.

Pointwise bounds. The boundedness of $\{\boldsymbol{u}_\beta\}$ in $\mathrm{Lip}(\mathbb{S}^{N-1})$ (Lemma 3.2) implies that there exists a constant $C_1 > 0$ such that

$$\beta^{1/2} u_{i,\beta} u_{j,\beta} \leq C_1 \quad \text{for all } i \neq j. \tag{3-4}$$

The proof is a straightforward adaptation of the one in [Soave and Zilio 2016, Theorem 1.1], which regards the same estimate in subsets of $\mathbb{R}^N$.

As a consequence we have that, for each $\theta \in \mathbb{S}^{N-1}$ and each $\beta > 0$,

$$u_{i,\beta}(\theta) \geq 2kC_1^{1/2}\beta^{-1/4} \quad \text{for at most one index } i, \tag{3-5}$$

where C_1 is the same constant as appears in (3-4). Indeed, assuming the contrary, there would exist two distinct indices $i \neq j$ satisfying the previous inequality, and hence

$$4k^2C_1\beta^{-1/2} \leq u_{i,\beta}(\theta)u_{j,\beta}(\theta) \leq C_1\beta^{-1/2},$$

a contradiction.

Finally, we observe that

$$\hat{u}_{i,\beta}(\theta) = 0 \implies u_{i,\beta}(\theta) \leq 2k(k-1)C_1^{1/2}\beta^{-1/4}. \tag{3-6}$$

If not, we have that (3-5) holds for i, and moreover

$$2k(k-1)C_1^{1/2}\beta^{-1/4} \leq u_{i,\beta}(\theta) \leq \sum_{j\neq i} u_{j,\beta}(\theta) \leq (k-1)\max_{j\neq i} u_{j,\beta}(\theta);$$

hence there exist two indexes for which (3-5) is satisfied in θ, a contradiction.

Integral bounds for the Laplacian. We prove that there exists a constant $C > 0$ (independent of β) such that

$$\int_{\mathbb{S}^{N-1}} |\Delta u_{i,\beta}| \leq C. \tag{3-7}$$

Indeed, directly from the equation and the divergence theorem,

$$0 = \int_{\mathbb{S}^{N-1}} (-\Delta u_{i,\beta}) = \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta} - \beta u_{i,\beta} \sum_{j\neq i} u_j^2;$$

that is,

$$0 \leq \int_{\mathbb{S}^{N-1}} \beta u_{i,\beta} \sum_{j\neq i} u_{j,\beta}^2 = \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta} \leq C,$$

as the functions $u_{i,\beta}$ are bounded in $L^\infty(\mathbb{S}^{N-1})$ and $\{\lambda_\beta\}$ is bounded. Consequently,

$$\int_{\mathbb{S}^{N-1}} |\Delta u_{i,\beta}| \leq \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta} + \beta u_{i,\beta} \sum_{j\neq i} u_{j,\beta}^2 \leq C.$$

Integral bounds for the competition term. Using (3-5) and the computations in the previous point, we deduce that

$$\begin{aligned}\int_{\mathbb{S}^{N-1}} \beta \sum_{i\neq j} u_{i,\beta}^2 u_{j,\beta}^2 &\\ \leq \sum_{i\neq j} \biggl(\|u_{i,\beta}\|_{L^\infty(\{u_{i,\beta}\leq u_{j,\beta}\})} &\int_{\{u_{i,\beta}\leq u_{j,\beta}\}} \beta u_{i,\beta} u_{j,\beta}^2 + \|u_{j,\beta}\|_{L^\infty(\{u_{j,\beta}<u_{i,\beta}\})} \int_{\{u_{j,\beta}<u_{i,\beta}\}} \beta u_{j,\beta} u_{i,\beta}^2 \biggr) \\ \leq C\beta^{-1/4} \sum_{i=1}^{k} &\int_{\{u_{i,\beta}\leq u_{j,\beta}\}} \beta u_{i,\beta} \sum_{j\neq i} u_{j,\beta}^2 \leq C\beta^{-1/4}.\end{aligned}$$

Integral bounds for the normal derivatives. For analogous reasons, we can show that there exists a constant $C > 0$ and $\bar{\delta} > 0$ small enough such that, for almost every $\delta \in (0, \bar{\delta})$,

$$\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} |\partial_\nu \hat{u}_{i,\beta}| \le C.$$

Firstly, since the function $\hat{u}_{i,\beta}$ is regular for β fixed, the set $\partial\{\hat{u}_{i,\beta} > \delta\}$ is regular for almost every $\delta > 0$, by Sard's lemma. Moreover, since $\hat{u}_{i,\beta}$ is nonnegative and regular, if $\delta < \bar{\delta}$ is small enough then

$$\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} |\partial_\nu \hat{u}_{i,\beta}| = -\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu \hat{u}_{i,\beta}. \tag{3-8}$$

Hence, for almost every $\delta \in (0, \bar{\delta})$ the set $\partial\{\hat{u}_{i,\beta} > \delta\}$ is regular, and (3-8) holds. With this choice we are in position to apply the divergence theorem, and consequently

$$\left| \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu \hat{u}_{i,\beta} \right| = \left| \int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta \hat{u}_{i,\beta} \right| \le \int_{\{\hat{u}_{i,\beta}>\delta\}} \sum_{j=1}^{k} |\Delta u_{j,\beta}| \le C,$$

where C is independent of β by (3-7). With similar computations we also have the uniform estimate

$$\left| \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu u_{i,\beta} \right| \le C.$$

Estimates for the $L^2(\mathbb{S}^{N-1})$ norm. Thanks to (3-5) and (3-6), we have

$$\begin{aligned} \int_{\mathbb{S}^{n-1}} (\hat{u}_{i,\beta} - u_{i,\beta})^2 &= \int_{\{\hat{u}_{i,\beta}>0\}} (\hat{u}_{i,\beta} - u_{i,\beta})^2 + \int_{\{\hat{u}_{i,\beta}=0\}} (\hat{u}_{i,\beta} - u_{i,\beta})^2 \\ &= \int_{\{u_{i,\beta}>\sum_{j\neq i} u_{j,\beta}\}} \Bigl(\sum_{j\neq i} u_{j,\beta} \Bigr)^2 + \int_{\{\hat{u}_{i,\beta}=0\}} u_{i,\beta}^2 \le C\beta^{-1/2}, \end{aligned}$$

whence (3-2) follows.

Estimates for the $H^1(\mathbb{S}^{N-1})$ seminorm. As a last step, we wish to estimate the L^2 norm of $\nabla \hat{u}_{i,\beta}$. Since $\partial\{\hat{u}_{i,\beta} > \delta\}$ is regular, we can apply the divergence theorem, deducing that

$$\begin{aligned} \int_{\{\hat{u}_{i,\beta}>\delta\}} |\nabla \hat{u}_{i,\beta}|^2 &= \int_{\{\hat{u}_{i,\beta}>\delta\}} (-\Delta \hat{u}_{i,\beta}) \hat{u}_{i,\beta} + \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} (\partial_\nu \hat{u}_{i,\beta}) \hat{u}_{i,\beta} \\ &= \underbrace{\int_{\{\hat{u}_{i,\beta}>\delta\}} (-\Delta u_{i,\beta}) u_{i,\beta}}_{\text{(I)}} + \int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta u_{i,\beta} \sum_{j\neq i} u_{j,\beta} \\ &\qquad + \underbrace{\int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta \Bigl(\sum_{j\neq i} u_{j,\beta} \Bigr) \hat{u}_{i,\beta}}_{\text{(II)}} + \delta \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu \hat{u}_{i,\beta}. \end{aligned}$$

The first term (I) can be bounded, also recalling that $\lambda_\beta \geq 0$, using the equation

$$\begin{aligned}\int_{\{\hat{u}_{i,\beta}>\delta\}} (-\Delta u_{i,\beta})u_{i,\beta} &= \int_{\{\hat{u}_{i,\beta}>\delta\}} \lambda_\beta u_{i,\beta}^2 - \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \\ &\leq \int_{\mathbb{S}^{N-1}} \lambda_\beta u_{i,\beta}^2 - \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 + \int_{\mathbb{S}^{N-1}\setminus\{\hat{u}_{i,\beta}>\delta\}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 \\ &= \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \int_{\mathbb{S}^{N-1}\setminus\{\hat{u}_{i,\beta}>\delta\}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2.\end{aligned}$$

The term (II) can be expanded further as

$$\begin{aligned}&\int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta\Big(\sum_{j\neq i} u_{j,\beta}\Big)\hat{u}_{i,\beta} \\ &\qquad = -\int_{\{\hat{u}_{i,\beta}>\delta\}} \nabla\Big(\sum_{j\neq i} u_{j,\beta}\Big)\cdot\nabla\hat{u}_{i,\beta} + \delta\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu\Big(\sum_{j\neq i} u_{j,\beta}\Big) \\ &\qquad = \int_{\{\hat{u}_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\Delta\hat{u}_{i,\beta} - \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\partial_\nu\hat{u}_{i,\beta} + \delta\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu\Big(\sum_{j\neq i} u_{j,\beta}\Big).\end{aligned}$$

Recalling the previous computations, and using again (3-5), we have

$$\begin{aligned}&\int_{\{\hat{u}_{i,\beta}>\delta\}} |\nabla\hat{u}_{i,\beta}|^2 \\ &\quad\leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \int_{\mathbb{S}^{N-1}\setminus\{\hat{u}_{i,\beta}>\delta\}} \beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2 + \int_{\{\hat{u}_{i,\beta}>\delta\}} \Delta u_{i,\beta} \sum_{j\neq i} u_{j,\beta} + \int_{\{\hat{u}_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\Delta\hat{u}_{i,\beta} \\ &\qquad\qquad - \int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \Big(\sum_{j\neq i} u_{j,\beta}\Big)\partial_\nu\hat{u}_{i,\beta} + \delta\int_{\partial\{\hat{u}_{i,\beta}>\delta\}} \partial_\nu u_{i,\beta} \\ &\quad\leq \int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + C\beta^{-1/4} + C\delta,\end{aligned}$$

which, as already observed, implies (3-3).

With (3-2) and (3-3) we are in position to complete the proof. By minimality, $\ell \leq I_\infty(\hat{\boldsymbol{u}}_\beta)$ for every β, which gives

$$\begin{aligned}\ell &\leq \frac{1}{k}\sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{N-1}}|\nabla\hat{u}_{i,\beta}|^2}{\int_{\mathbb{S}^{N-1}}\hat{u}_{i,\beta}^2}\right) \leq \frac{1}{k}\sum_{i=1}^k \gamma\left(\frac{\int_{\mathbb{S}^{N-1}}|\nabla u_{i,\beta}|^2 + C\beta^{-1/4}}{1 - C\beta^{-1/2}}\right) \\ &\leq \frac{1}{k}\sum_{i=1}^k \gamma\left(\int_{\mathbb{S}^{N-1}} |\nabla u_{i,\beta}|^2 + \frac{1}{2}\beta u_{i,\beta}^2 \sum_{j\neq i} u_{j,\beta}^2\right) + C\beta^{-1/4} = \ell_\beta + C\beta^{-1/4}.\end{aligned}$$

□

The proof of Proposition 1.5 can be obtained in a somewhat usual way.

Sketch of the proof of Proposition 1.5. Arguing as in [Conti et al. 2005, Section 7], or [Noris et al. 2010, Lemma 2.5], or else [Soave and Zilio 2015, Theorem 3.14], it is possible to check that

$$\frac{d}{dr}\log\left(\frac{J_1(r)\cdots J_k(r)}{r^{2k\ell}}\right) = -\frac{2k\ell}{r} + \frac{2}{r}\sum_i \gamma\left(\frac{r^2\int_{\partial B_r}|\nabla u_i|^2 + \frac{1}{2}u_i^2\sum_{j\neq i}u_j^2}{\int_{\partial B_r}u_i^2}\right).$$

Changing variables in the integrals (see Theorem 3.14 in [Soave and Zilio 2015] for the details), we deduce that

$$\sum_i \gamma\left(\frac{r^2\int_{\partial B_r}|\nabla u_i|^2 + \frac{1}{2}u_i^2\sum_{j\neq i}u_j^2}{\int_{\partial B_r}u_i^2}\right) \geq k\ell_{r^2},$$

where ℓ_{r^2} denotes the optimal value ℓ_β for $\beta = r^2$. Coming back to the previous equation and using Lemma 3.4, we conclude that

$$\frac{d}{dr}\log\left(\frac{J_1(r)\cdots J_k(r)}{r^{2k\ell}}\right) \geq \frac{2k}{r}(\ell_{r^2} - \ell) \geq -2kCr^{-3/2}$$

and, integrating, the thesis follows. □

4. Construction of equivariant solutions

For an admissible triplet $(k, \mathcal{G}, h)$, we prove the existence of a $(\mathcal{G}, h)$-equivariant solution to (1-1) with k components. We partially follow the method introduced in [Berestycki et al. 2013b], which consists in two steps:

- firstly, we prove the existence of a sequence of $(\mathcal{G}, h)$-equivariant solutions $\boldsymbol{V}_R$, defined in balls of increasing radii $R \to +\infty$;
- secondly, we show that this sequence converges locally uniformly in $\mathbb{R}^N$ to a nontrivial solution.

With respect to [Berestycki et al. 2013b], we modify the construction conveniently choosing R from the beginning; this substantially simplifies the proof of the convergence of $\{\boldsymbol{V}_R\}$, and we refer to Remark 4.4 for more details. Finally, in the last part of the proof we characterize the growth of the solution using the Alt–Caffarelli–Friedman monotonicity formula for $(\mathcal{G}, h)$-equivariant solutions.

By Lemma 3.1, we know that the optimal value ℓ (see (1-5)) is achieved by a nonnegative $(\mathcal{G}, h)$-equivariant function $\varphi \in H^1(\mathbb{S}^{N-1}, \mathbb{R}^k)$. Differently from the previous section, we take

$$\int_{\mathbb{S}^{N-1}} \varphi_i^2 = \frac{1}{k} \iff \sum_{i=1}^k \int_{\mathbb{S}^{N-1}} \varphi_i^2 = 1. \tag{4-1}$$

This choice is possible, since the minimum problem for ℓ is invariant under scaling of type $t \mapsto t\varphi$ with $t \in \mathbb{R}$, and simplifies some computations.

Lemma 4.1. *For any $\beta > 0$ there exists a $(\mathcal{G}, h)$-equivariant solution $\{\boldsymbol{U}_\beta\}$ to the problem*

$$\begin{cases} \Delta U_{i,\beta} = \beta U_{i,\beta}\sum_{j\neq i} U_{j,\beta}^2 & \text{in } B_1, \\ U_{i,\beta} > 0 & \text{in } B_1, \\ U_{i,\beta} = \varphi_i & \text{on } \partial B_1 = \mathbb{S}^{N-1}. \end{cases}$$

Moreover,

(i) $U_{i,\beta}(0) = U_{j,\beta}(0)$ *for all* i, $j = 1, \dots, k$ *and* $\beta > 0$;

(ii) *letting*

$$\mathcal{E}_\beta(\boldsymbol{U}) = \int_{B_1} \sum_{i=1}^{k} |\nabla U_i|^2 + \beta \sum_{i<j} U_i^2 U_j^2,$$

the uniform estimate $\mathcal{E}_\beta(\boldsymbol{U}_\beta) \le \ell$ *holds*;

(iii) *there exists a Lipschitz continuous function* $\mathbf{0} \not\equiv \boldsymbol{U}_\infty$ *such that, up to a subsequence,* $\boldsymbol{U}_\beta \to \boldsymbol{U}_\infty$ *in* $\mathcal{C}^{0,\alpha}(B_1)$ *for every* $\alpha \in (0, 1)$ *and in* $H^1_{\mathrm{loc}}(B_1)$.

Proof. It is not difficult to check that the functional $\mathcal{E}_\beta$ admits a minimizer $\boldsymbol{U}_\beta$ in the H^1-weakly closed set of the $(\mathcal{G}, h)$-equivariant functions in $H^1(B_1, \mathbb{R}^k)$ with the prescribed boundary conditions. The fact that this minimizer solves the Euler–Lagrange equation is a consequence of Palais' principle of symmetric criticality. Property (i) follows straightforwardly by the equivariance (recall Remark 1.2). Concerning property (ii), we introduce the ℓ-homogeneous extension of φ, defined by

$$\phi(x) := |x|^\ell \varphi\Big(\frac{x}{|x|}\Big).$$

By minimality, $\mathcal{E}_\beta(\boldsymbol{U}_\beta) \le \mathcal{E}_\beta(\phi)$, so that it remains to check that $\mathcal{E}_\beta(\phi) \le \ell$. At first, since φ_i is an eigenfunction of $-\Delta_\theta$ on $\{\varphi_i > 0\}$ associated to the eigenvalue $\ell(\ell + N - 2)$, the function ϕ_i is harmonic in $\{\phi_i > 0\}$. Furthermore, by definition,

$$\sum_i \int_{\partial B_1} \phi_i^2 = 1$$

for every i. Therefore, using the Euler formula for homogeneous functions, we deduce that

$$\mathcal{E}_\beta(\phi) = \sum_i \int_{B_1} |\nabla \phi_i|^2 = \sum_i \int_{\{\phi_i > 0\} \cap B_1} |\nabla \phi_i|^2 = \sum_i \int_{\partial B_1 \cap \{\phi_i > 0\}} \phi_i \, \partial_\nu \phi_i = \ell \sum_i \int_{\partial B_1 \cap \{\phi_i > 0\}} \phi_i^2 = \ell.$$

It remains to prove (iii). By (ii) and the boundary conditions, the sequence $\{\boldsymbol{U}_\beta\}$ is bounded in $H^1(B_1)$, and hence it converges weakly to some limit $\boldsymbol{U}_\infty$. By compactness of the trace operator, $\boldsymbol{U}_\infty \not\equiv \mathbf{0}$. All the functions $\boldsymbol{U}_\beta$ are nonnegative, subharmonic and have the same boundary conditions, and hence by the maximum principle they are uniformly bounded in $L^\infty(B_1)$. This, as recalled in Section 2, implies the thesis. □

We plan to use the solutions of Lemma 4.1 in order to construct entire solutions to (1-1). Our method is based on a simple blow-up argument. For a positive radius r_β to be determined, we introduce

$$V_{i,\beta}(x) := \beta^{1/2} r_\beta U_{i,\beta}(r_\beta x).$$

By definition, $\boldsymbol{V}_\beta$ solves

$$\Delta V_{i,\beta} = V_{i,\beta} \sum_{j \ne i} V_{j,\beta}^2 \quad \text{in } B_{1/r_\beta}.$$

A convenient choice of r_β is suggested by the following statement.

Lemma 4.2. *For any fixed $\beta > 1$ there exists a unique $r_\beta > 0$ such that*

$$\int_{\partial B_1} \sum_{i=1}^k V_{i,\beta}^2 = 1.$$

Moreover $r_\beta \to 0$, and consequently $B_{1/r_\beta} \to \mathbb{R}^N$, in the sense that for any compact $K \subset \mathbb{R}^N$ we have $K \Subset B_{1/r_\beta}$ provided β is sufficiently large.

Proof. We have to find $r_\beta > 0$ such that $\beta r_\beta^2 H(\boldsymbol{U}_\beta, r_\beta) = 1$. The strict monotonicity of $H(\boldsymbol{U}_\beta, \cdot)$ (see Section 2) implies the strict monotonicity of the continuous function $r \mapsto \beta r^2 H(\boldsymbol{U}_\beta, r)$. By regularity, for any fixed β,

$$\lim_{r\to 0} \beta r^2 H(\boldsymbol{U}_\beta, r) = \lim_{r\to 0} \beta \frac{r^2}{r^{N-1}} \int_{\partial B_r} \sum_{i=1}^k U_{i,\beta}^2 = \beta \lim_{r\to 0} r^2 \cdot \sum_{i=1}^k U_{i,\beta}^2(0) = 0,$$

and, by the normalization (4-1), $\beta H(\boldsymbol{U}_\beta, 1) = \beta > 1$. This proves existence and uniqueness of r_β. If, by contradiction, $r_\beta \geq \bar r > 0$, then by Lemma 4.1(iii) and the monotonicity of $H(\boldsymbol{U}_\beta, \cdot)$ we would have

$$1 = \beta r_\beta^2 H(\boldsymbol{U}_\beta, r_\beta) \geq \beta \bar r^2 H(\boldsymbol{U}_\beta, \bar r) \geq \frac{\beta \bar r^2}{2} \frac{1}{\bar r^{N-1}} \int_{\partial B_{\bar r}} \sum_{i=1}^k U_{i,\infty} \geq \beta C,$$

which gives a contradiction for $\beta > 1/C$. In order to bound from below the second-to-last term, we recall that since $\mathbf{0} \not\equiv \boldsymbol{U}_\infty$ we have $H(\boldsymbol{U}_\infty, r) \neq 0$ for all $0 < r < 1$ (see Section 2). □

Lemma 4.3. *Up to a subsequence, $\boldsymbol{V}_\beta \to \boldsymbol{V}$ in $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^N)$, and $\boldsymbol{V}$ is an entire $(\mathcal{G}, h)$-equivariant solution of* (1-1) *with $N(\boldsymbol{V}, r) \leq \ell$ for every $r > 0$.*

Proof. Since $\mathcal{E}_\beta(\boldsymbol{U}_\beta) \leq \ell$ and $H(\boldsymbol{U}_\beta, 1) = 1$, by scaling and using the monotonicity of the Almgren quotient we have

$$N(\boldsymbol{V}_\beta, r) \leq N\Big(\boldsymbol{V}_\beta, \frac{1}{r_\beta}\Big) = N(\boldsymbol{U}_\beta, 1) \leq \frac{\mathcal{E}(\boldsymbol{U}_\beta)}{H(\boldsymbol{U}_\beta, 1)} \leq \ell \tag{4-2}$$

for every $0 < r < 1/r_\beta$, $\beta > 0$. Now let $r > 0$; then, for β sufficiently large,

$$\frac{d}{dr} \log H(\boldsymbol{V}_\beta, r) = \frac{2}{r} N_\beta(\boldsymbol{v}_\beta, r) + \frac{2}{r^{N-1} H(\boldsymbol{V}_\beta, r)} \int_{B_r} \sum_{i<j} V_i^2 V_j^2 \leq \frac{2\ell}{r} + \frac{2}{r^{N-1} H(\boldsymbol{V}_\beta, r)} \int_{B_r} \sum_{i<j} V_i^2 V_j^2.$$

Integrating the inequality for $r \in (1, R)$, and recalling (2-2), we infer that

$$\frac{H(\boldsymbol{V}_\beta, R)}{R^{2\ell}} \leq H(\boldsymbol{V}_\beta, 1) e^\ell = e^\ell \quad \text{for all } R \geq 1, \tag{4-3}$$

independently of β. By subharmonicity and standard elliptic estimates, we deduce that $\boldsymbol{V}_\beta$ converges in $\mathcal{C}^2(B_R)$ to some limit $\boldsymbol{V}^R$, and since R has been chosen arbitrarily, a diagonal selection gives convergence to an entire limit $\boldsymbol{V}$, which is clearly $(\mathcal{G}, h)$-equivariant. Since $\boldsymbol{V}$ solves (1-1) and

$$\int_{\partial B_1} \sum_{i=1}^k V_{i,\beta}^2 = 1 \qquad \text{and} \qquad V_{i,\beta}(0) = V_{j,\beta}(0) \quad \text{for all } i, j$$

(see Lemmas 4.1 and 4.2), all the components of $\boldsymbol{V}$ are nontrivial, and hence nonconstant. □

We now show that the growth rate of the solution is exactly equal to ℓ. In light of the upper bound on the Almgren quotient proved in the previous lemma, this is a consequence of Theorem 1.4.

Proof of Theorem 1.4. Let us assume by contradiction that there exists a $(\mathcal{G}, h)$-equivariant solution $\boldsymbol{V}$ with growth rate less than $\ell-\varepsilon$ for some $\varepsilon > 0$. By monotonicity it results $N(\boldsymbol{V}, r) \le N(\boldsymbol{V}, +\infty) \le \ell-\varepsilon$ for every $r > 0$. We consider the blow-down sequence

$$\boldsymbol{V}_R(x) = \frac{1}{\sqrt{H(\boldsymbol{V}, R)}}\boldsymbol{V}(Rx).$$

By Theorem 1.4 in [Soave and Terracini 2015], it converges in $\mathcal{C}^{0,\alpha}_{\text{loc}}(\mathbb{R}^N)$ to a limit $\boldsymbol{W}$, which is segregated, nonnegative, homogeneous with homogeneity degree $\delta := N(\boldsymbol{V}, +\infty) \le \ell-\varepsilon$, and such that $\Delta W_i = 0$ in $\{W_i > 0\}$. The uniform convergence entails the $(\mathcal{G}, h)$-equivariance, and hence the trace $\hat{\boldsymbol{w}}$ of $\boldsymbol{W}$ on the sphere $\mathbb{S}^{N-1}$ is an admissible competitor for ℓ, in the sense that $\ell \le I_\infty(\hat{\boldsymbol{w}})$ (I_∞ is defined in Lemma 3.1). The value $I_\infty(\hat{\boldsymbol{w}})$ can be computed explicitly; indeed, by harmonicity, homogeneity and symmetry, $\hat{w}_i$ is an eigenfunction of the Laplace–Beltrami operator $-\Delta_\theta$ on a subdomain of $\mathbb{S}^{N-1}$, associated to the eigenvalue $\delta(\delta+N-2)$. This, by definition, implies that $I_\infty(\hat{\boldsymbol{w}}) = \delta < \ell$, in contradiction with the minimality of ℓ. □

So far we proved the existence of a $(\mathcal{G}, h)$-equivariant solution having growth rate ℓ in the weak sense of (2-3). It remains to show that the stronger condition (1-6) holds. First we make the following remark.

Remark 4.4. Both Theorem 1.3 and [Berestycki et al. 2013b, Theorem 1.6] are based upon the same two-step procedure: construction of solutions in balls B_R of increasing radius, and passage to the limit as $R \to +\infty$. The main difference is in the fact that while in [Berestycki et al. 2013b] the authors prescribed the value of the functions on the boundary ∂B_R, we prescribed the value on ∂B_1, conveniently choosing r_β. This permits us to greatly simplify the proof of the convergence, since by the doubling property (4-3) the normalization on ∂B_1 is enough to have $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^N)$-convergence of our approximating sequence. In [Berestycki et al. 2013b, page 123], this compactness is proved in a different way, using fine tools such as Proposition 5.7 therein, which seems difficult to generalize to higher dimensions.

Lemma 4.5. *We have*

$$\lim_{r\to\infty} \frac{1}{r^{2\ell}} H(\boldsymbol{V}, r) \in (0, +\infty).$$

Proof. It is easy to prove that the limit exists and it is less than 1. Indeed

$$\frac{d}{dr}\log\frac{H(\boldsymbol{V}, r)}{r^{2\ell}} = \frac{H'(\boldsymbol{V}, r)}{H(\boldsymbol{V}, r)} - \frac{2\ell}{r} = \frac{2}{r}(N(\boldsymbol{V}, r)-\ell) \le 0,$$

and by construction $H(\boldsymbol{V}, 1) = 1$. Letting

$$L = \lim_{r\to\infty}\frac{H(\boldsymbol{V}, r)}{r^{2\ell}},$$

we are left to show that $L > 0$. Recalling that $N(\boldsymbol{V}, +\infty) = \ell$, we have

$$L = \lim_{r\to\infty}\left(\frac{E(\boldsymbol{V}, r)}{r^{2\ell}}\right)\cdot\lim_{r\to+\infty}\frac{H(\boldsymbol{V}, r)}{E(\boldsymbol{V}, r)} \ge \frac{1}{\ell}\liminf_{r\to\infty}\frac{E(\boldsymbol{V}, r)}{r^{2\ell}},$$

and the thesis follows if
$$\liminf_{r\to\infty} \frac{E(\boldsymbol{V},r)+H(\boldsymbol{V},r)}{r^{2\ell}} > 0.$$
To this aim, we note that with computations analogous to those in [Soave and Zilio 2016, conclusion of the proof of Theorem 1.5] we can prove that
$$\frac{E(\boldsymbol{V},r)+H(\boldsymbol{V},r)}{r^{2\ell}} \geq \frac{C}{r^{2\ell}}(J_1(r)\cdots J_k(r))^{1/k} = C\left(\frac{1}{r^{2\ell k}} J_1(r)\cdots J_k(r)\right)^{\frac{1}{k}},$$
where the integrals J_i are evaluated for the function $\boldsymbol{V}$. Since $\boldsymbol{V}$ is a $(\mathcal{G}, h)$-equivariant solution of (1-1), we are in position to apply the Alt–Caffarelli–Friedman monotonicity formula of Proposition 1.5, whence
$$\frac{E(\boldsymbol{V},r)+H(\boldsymbol{V},r)}{r^{2\ell}} \geq C(J_1(1)\cdots J_k(1))^{1/k} e^{Cr^{-1/2}} \geq C e^{Cr^{-1/2}}$$
for every $r > 1$. □

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Received 16 Jul 2015. Revised 5 Feb 2016. Accepted 29 Apr 2016.

NICOLA SOAVE: nicola.soave@gmail.com
Mathematisches Institut, Justus-Liebig-Universität Giessen, Arndtstrasse 2, 35392 Giessen, Germany

ALESSANDRO ZILIO: azilio@ehess.fr
Centre d'Analyse et de Mathématique Sociales, École des Hautes Études en Sciences Sociales, 190–198 Avenue de France, 75244 Paris CEDEX 13, France

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1043

TIME-PERIODIC APPROXIMATIONS OF THE EULER–POISSON SYSTEM NEAR LANE–EMDEN STARS

JUHI JANG

We show a long-time validity of the time-periodic linear approximations to the gravitational Euler–Poisson system near Lane–Emden equilibria for all relevant adiabatic exponents. To prove the result, we reformulate the problem in Lagrangian coordinates and use the weighted energy estimates together with Hardy inequalities.

1. Introduction and formulation

One of the simplest fundamental hydrodynamical models for describing the motion of self-gravitating Newtonian inviscid gaseous stars is the compressible Euler–Poisson equations:

$$\begin{aligned} \partial_t \rho + \nabla\cdot(\rho \boldsymbol{u}) &= 0, \\ \rho(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla \boldsymbol{u}) + \nabla p &= -\rho\nabla\Phi, \\ \Delta\Phi &= 4\pi\rho, \end{aligned} \tag{1-1}$$

where $(t, x) \in \mathbb{R}_+ \times \mathbb{R}^3$ and ρ, $\boldsymbol{u}$ and p denote respectively the density, velocity and pressure of gas. Φ is the gravitational potential and it is related to the gas through the Poisson equation. We consider polytropic gases with equation of state given by

$$p = K\rho^{\gamma}, \tag{1-2}$$

where K is an entropy constant and $\gamma > 1$ is the adiabatic gas exponent. There are many interesting works available on the Euler–Poisson system (1-1); for instance, see [Luo et al. 2014; Makino and Ukai 1987; Nishida 1986] for the existence theory, [Makino 1992] for a nonexistence result and blowup, and [Deng et al. 2002; Jang 2008; 2014; Luo and Smoller 2008; 2009; Rein 2003] for the stability and instability theory. However, some important questions are still waiting to be answered. In this paper, we are interested in long-time radial solutions to (1-1) around the Lane–Emden equilibrium stars.

Supported in part by NSF grants DMS-1212142, DMS-1351898, DMS-1608494, and a von Neumann fellowship of the Institute for Advanced Study through the NSF grant DMS-1128155.
MSC2010: 35L80, 35Q05, 35R37.
Keywords: Euler–Poisson system, Lane–Emden solutions, physical vacuum boundary, time-periodic approximation.

The spherically symmetric solutions to the system (1-1) — $\rho(t,x)=\rho(t,r)$ and $\boldsymbol{u}(t,x)=u(t,r)x/r$, where $r=|x|$ — satisfy the equations

$$\begin{aligned} \rho_t+\frac{1}{r^2}(r^2\rho u)_r&=0,\\ \rho u_t+\rho u u_r+p_r+\frac{4\pi\rho}{r^2}\int_0^r \rho s^2\,ds&=0. \end{aligned} \tag{1-3}$$

Static solutions ($\rho_0(r)$, $u_0=0$) of (1-3) satisfy the ordinary differential equation

$$\frac{dp}{dr}+\frac{4\pi\rho}{r^2}\int_0^r \rho s^2\,ds=0, \tag{1-4}$$

which can be transformed into the famous Lane–Emden equation [Chandrasekhar 1938]. Nonnegative solutions of (1-4) can be characterized according to γ as follows [Chandrasekhar 1938; Lin 1997]: Letting $M(\rho)\equiv\int 4\pi s^2\rho(s)\,ds$ be the total mass of a star, if $\gamma>\frac{6}{5}$ and $M>0$ then there exists at least one compactly supported solution ρ such that $M(\rho)=M$. For $\gamma>\frac{4}{3}$, every solution is compactly supported and unique. If $\gamma=\frac{6}{5}$ and $M>0$, there is a unique solution ρ with infinite support. If $1<\gamma<\frac{6}{5}$, there are no stationary solutions with finite total mass. The compactly supported equilibria for $\frac{6}{5}<\gamma<2$ are called the Lane–Emden stars; see also Section 1B.

It is well known [Chandrasekhar 1938; Lin 1997] that the boundary behavior of compactly supported Lane–Emden solutions is characterized as follows:

$$\bar\rho(r)\sim(R-r)^{1/(\gamma-1)}\quad\text{for } r\sim R. \tag{1-5}$$

This boundary behavior is often referred to as physical vacuum [Liu and Yang 2000]. As far as the full dynamics of compressible flows involving physical vacuum is concerned, the degeneracy and the interaction with nonlinearity make the analysis nontrivial. Despite its physical importance, even local-in-time well-posedness of compressible Euler equations in the presence of physical vacuum was established only recently [Coutand and Shkoller 2012; Jang and Masmoudi 2009; 2015]. For more discussion on physical vacuum, we refer to [Jang and Masmoudi 2011] and for other problems involving vacuum see [Jang and Masmoudi 2012; Liu 1996; Liu and Yang 1997; Makino et al. 1986; Sideris 2014].

The goal of this article is to investigate a detailed structure of the solutions to (1-3) near the compactly supported Lane–Emden stars beyond the local existence. More specifically, we will construct the time-periodic linearized solutions and show the validity of such linear approximations in the fully nonlinear setting for large times for all $\frac{6}{5}<\gamma<2$. To this end, we will first introduce suitable Lagrangian coordinates in accordance with the recent advancement of physical vacuum, and formulate the problem in such Lagrangian coordinates.

1A. ***Lagrangian coordinates.*** Let $\eta(t,x)$ be the position of the gas particle x at time t, so that

$$\eta_t=\boldsymbol{u}(t,\eta(t,x))\ \text{ for } t>0 \quad\text{and}\quad \eta(0,x)=\eta_0 \quad\text{in } \Omega. \tag{1-6}$$

Here Ω is a compact smooth domain and $\eta_0:\Omega\to\Omega$ is a diffeomorphism with positive Jacobian determinant. For the purpose of this article, we take Ω as a ball, which corresponds to the support of a

Lane–Emden solution and the initial density. Our choice of η_0 will depend on the initial density profile and in fact, in our setup, the identity map will correspond to the equilibrium state. The following are the Lagrangian quantities:

$$\begin{aligned}
&\boldsymbol{v}(t,x) \equiv \boldsymbol{u}(t,\eta(t,x)), && \varrho(t,x) \equiv \rho(t,\eta(t,x)), && \Psi(t,x) \equiv \Phi(t,\eta(t,x)),\\
&A \equiv (D\eta)^{-1}, && J \equiv \det D\eta, && a \equiv JA.
\end{aligned}$$

We use Einstein's summation convention and the notation $F,_k$ to denote the k-th partial derivative of F. In this subsection, we use i, j, k, l, r, s to denote 1, 2, 3. The Euler–Poisson equations (1-1) read as

$$\begin{aligned}
&\varrho_t + \varrho A_i^j \boldsymbol{v}^i,_j = 0,\\
&\varrho \boldsymbol{v}_t^i + K A_i^k \varrho^\gamma,_k = -\varrho A_i^k \Psi,_k\,,\\
&A_i^k (A_i^l \Psi,_l),_k = 4\pi\varrho.
\end{aligned} \tag{1-7}$$

Since $J_t = JA_i^j v^i,_j$ we find that $\varrho J = \rho(0)J(0) = \rho_{\text{in}} \det D\eta_0$, where ρ_{in} is a given initial density function. For ρ_{in} exhibiting the same boundary behavior as $\bar\rho$ such that $\rho_{\text{in}}/\bar\rho$ is a smooth positive function, we choose η_0 so that

$$\varrho J = \rho_{\text{in}} \det D\eta_0 = \bar\rho, \tag{1-8}$$

where $\bar\rho$ is the equilibrium density profile of the Lane–Emden star given by (1-4). Existence of such an η_0 follows from the Dacorogna–Moser theorem [1990].

By using the relation $A_i^k = J^{-1} a_i^k$, we see that the system (1-7) is reduced to

$$\begin{aligned}
&\bar\rho \boldsymbol{v}_t^i + K a_i^k (\bar\rho^\gamma J^{-\gamma}),_k = -\bar\rho A_i^k \Psi,_k\,,\\
&A_i^k (A_i^l \Psi,_l),_k = 4\pi\bar\rho J^{-1},
\end{aligned} \tag{1-9}$$

along with

$$\eta_t^i = \boldsymbol{v}^i. \tag{1-10}$$

Now we introduce the equilibrium enthalpy

$$w \equiv K\bar\rho^{\gamma-1}. \tag{1-11}$$

We will work with the enthalpy w rather than the density $\bar\rho$, since w behaves like a distance function near the boundary regardless of the values of γ under the physical vacuum condition (1-5). This w will be treated as the weight function. By using the Piola identity $a_i^k,_k = 0$, we see that the system (1-9) takes the form

$$\begin{aligned}
&w^\alpha \boldsymbol{v}_t^i + (w^{1+\alpha} A_i^k J^{-1/\alpha}),_k = -w^\alpha A_i^k \Psi,_k\,,\\
&A_i^k (A_i^l \Psi,_l),_k = 4\pi K^{-\alpha} w^\alpha J^{-1},
\end{aligned} \tag{1-12}$$

where

$$\alpha \equiv \frac{1}{\gamma - 1}. \tag{1-13}$$

Here α has been introduced for notational convenience. We will use both α and γ, which are related through (1-13), in the equations and the estimates throughout the article. For instance, the range of the adiabatic exponents of our interest reads in terms of α as

$$\tfrac{6}{5} < \gamma < 2 \iff 1 < \alpha < 5.$$

For the spherically symmetric Euler–Poisson flows, it is convenient to introduce the expansion and contraction variable ξ as

$$\eta(t,x) \equiv \xi(t,r)x, \tag{1-14}$$

where $r = |x|$. Since $\eta_t = \xi_t\, x = \boldsymbol{v}$, we have $v(t,r) = r\,\xi_t$. Since $\partial_i = (x^i/r)\partial_r$, we can write

$$J = \xi^2(\xi + \xi_r r) \quad \text{and} \quad (D\eta)^{-1} = \frac{1}{\xi} I - \frac{\xi_r}{\xi(\xi+\xi_r r)r}\,(x^i x^j) \tag{1-15}$$

and hence A_i^k is given by

$$A_i^k = \frac{\delta_i^k}{\xi} - \frac{\xi_r\, x^k x^i}{\xi(\xi+\xi_r r)r}. \tag{1-16}$$

Now, for spherically symmetric functions, the gradient $A_i^k \partial_k$ is given by

$$A_i^k \partial_k = \frac{x^i}{r(\xi+\xi_r r)} \partial_r$$

and the Laplacian $A_i^k \partial_k (A_i^l \partial_l)$ is given by

$$A_i^k \partial_k (A_i^l \partial_l) = \frac{1}{(\xi+\xi_r r)(\xi r)^2} \partial_r \left(\frac{(\xi r)^2}{\xi + \xi_r r} \partial_r \right).$$

Thus the Poisson equation in (1-12) for spherically symmetric flows takes the form

$$\frac{1}{(\xi+\xi_r r)(\xi r)^2} \partial_r \left(\frac{(\xi r)^2}{\xi + \xi_r r} \Psi_r \right) = 4\pi K^{-\alpha} w^\alpha J^{-1}. \tag{1-17}$$

Based on (1-14), (1-23) and (1-16), we see that the momentum equation in (1-12) for spherically symmetric flows can be written as an equation for ξ:

$$w^\alpha \xi_{tt} + \frac{\xi^2}{r} \partial_r \big(w^{1+\alpha} (\xi^2(\xi+\xi_r r))^{-\gamma} \big) + \frac{w^\alpha}{r(\xi+\xi_r r)} \Psi_r = 0 \tag{1-18}$$

for $t \geq 0$ and $0 \leq r \leq R$, where R is the radius of the Lane–Emden star. We remark that no boundary conditions are necessary to construct smooth solutions for (1-18) due to the degenerate weights [Coutand and Shkoller 2012; Jang and Masmoudi 2015]. More detail on the Lagrangian formulation described in the above can be found in [Jang 2014].

Note that from (1-17) the potential term can be also written as

$$\frac{w^\alpha}{r(\xi+\xi_r r)} \Psi_r = \frac{w^\alpha}{\xi^2 r^3} \int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2\, ds = \frac{w^\alpha}{\xi^2 r^3} \int_{B(0,r)} \bar{\rho}\, dx.$$

This potential term has the right weight w^α and it is of lower order with respect to the differential structure. It looks harmless. However, the potential term plays an important role in the stability theory, as shown in [Jang 2014; Rein 2003]. Not surprisingly, we will show that it also has an impact on the validity time of the time-periodic linear approximations.

1B. ***Lane–Emden star configuration in the Lagrangian formulation.*** In this subsection, we will identify the Lane–Emden stars satisfying (1-4) in our Langrangian formulation. The static equilibria of the Euler–Poisson system under spherical symmetry governed by (1-18) can be found by setting $\xi \equiv 1$. It is clear that w satisfies the ordinary differential equation

$$\frac{1}{r}\partial_r(w^{1+\alpha}) + \frac{w^\alpha}{r^3}\int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2\,ds = 0, \tag{1-19}$$

or equivalently

$$w_{rr} + \frac{2}{r}w_r + \frac{4\pi}{(1+\alpha)K^\alpha} w^\alpha = 0. \tag{1-20}$$

This is the so-called Lane–Emden equation, which has been studied extensively. In particular, we recall the well-known existence result from [Chandrasekhar 1938; Lin 1997]: supplemented with the normalized boundary conditions

$$w(0) = 1 \quad \text{and} \quad w_r(0) = 0$$

for a given finite total mass M, there exist a ball-type solution w to the Lane–Emden equation (1-20) and a finite radius R when $1 < \alpha < 5$, or equivalently $\frac{6}{5} < \gamma < 2$, such that (i) $w > 0$ for $0 < r < R$ and $w(R) = 0$; (ii) $-\infty < w_r < 0$ for $0 < r < R$; (iii) w satisfies the physical vacuum condition (1-5). The Lane–Emden configuration w enjoys better regularity. The regularity results of w are summarized in Section 2A.

We next write (1-18) in a perturbation form around the equilibrium state given by $\xi = 1$ and $\xi_t = 0$. Letting $\xi \equiv 1 + \zeta$ with $|\zeta| \ll 1$, we obtain the equation for ζ as

$$w^\alpha \zeta_{tt} + \frac{(1+\zeta)^2}{r}\partial_r\big(w^{1+\alpha}((1+\zeta)^2(1+\zeta+\zeta_r r))^{-\gamma}\big) + \frac{w^\alpha}{(1+\zeta)^2 r^3}\int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2\,ds = 0. \tag{1-21}$$

1C. ***ψ formulation.*** We further introduce a variable ψ whose equation displays a better structure for the pressure gradient term in our coordinates. Let

$$\psi \equiv \zeta + \zeta^2 + \tfrac{1}{3}\zeta^3. \tag{1-22}$$

Then, since $d\psi/d\zeta = 1 + 2\zeta + \zeta^2 > 0$, by the inverse function theorem $\zeta = \zeta(\psi)$ can be regarded as a smooth function of ψ. Notice that

$$J = (1+\zeta)^2(1+\zeta+\zeta_r r) = 1 + \frac{1}{r^2}\big(r^3\big(\zeta+\zeta^2+\tfrac{1}{3}\zeta^3\big)\big)_r = 1 + \frac{1}{r^2}(r^3\psi)_r \tag{1-23}$$

and

$$\zeta_t = \frac{\psi_t}{(1+\zeta)^2} \quad \text{and} \quad \zeta_{tt} = \frac{\psi_{tt}}{(1+\zeta)^2} - \frac{2\psi_t^2}{(1+\zeta)^5}. \tag{1-24}$$

Thus (1-21) can be written in terms of ψ as

$$\frac{w^\alpha r^4\psi_{tt}}{(1+\zeta)^4}-\frac{2w^\alpha r^4\psi_t^2}{(1+\zeta)^7}+r^3\partial_r\left(w^{1+\alpha}\left(\left(1+\frac{1}{r^2}(r^3\psi)_r\right)^{-\gamma}-1\right)\right)\\ +\frac{1-(1+\zeta)^4}{(1+\zeta)^4}w^\alpha r\int_0^r\frac{4\pi}{K^\alpha}w^\alpha s^2\,ds=0. \tag{1-25}$$

Notice that (1-25) relies on the Lane–Emden equation (1-19).

Throughout the paper, we will use $\mathfrak{A}\lesssim\mathfrak{B}$ to denote that $\mathfrak{A}\le C\mathfrak{B}$ for a generic constant $C>0$. We will use big O notation to describe the leading order of small quantities.

2. Time-periodic linearized solutions and main result

In this section, we study the linearized Euler–Poisson system around compactly supported Lane–Emden stars for $\frac{6}{5}<\gamma<2$ (i.e., $1<\alpha<5$). We will first derive the linearized equation of (1-25). Notice that by Taylor's theorem, for sufficiently small ψ, the nonlinear pressure term in (1-25) can be written as

$$\left(1+\frac{1}{r^2}(r^3\psi)_r\right)^{-\gamma}=1-\frac{\gamma}{r^2}(r^3\psi)_r+h, \tag{2-1}$$

where h is a smooth function of $(1/r^2)(r^3\psi)_r$ and $h=O\big(|(1/r^2)(r^3\psi)_r|^2\big)$. Also, the ζ-related part of the last term in (1-25) can be written as

$$\frac{1-(1+\zeta)^4}{(1+\zeta)^4}=\frac{-4\zeta-6\zeta^2-4\zeta^3-\zeta^4}{(1+\zeta)^4}=\frac{-4\psi-2\zeta^2-\frac{8}{3}\zeta^3-\zeta^4}{(1+\zeta)^4}=-4\psi+f, \tag{2-2}$$

where f is a smooth function of ζ (and hence ψ) and $f=O(|\zeta|^2)=O(|\psi|^2)$ due to (1-22).

Then the linearized equation of (1-25) reads as

$$w^\alpha r^4\psi_{tt}-\gamma r^3\partial_r\left(w^{1+\alpha}\frac{1}{r^2}(r^3\psi)_r\right)+4r^3\partial_r(w^{1+\alpha})\psi=0, \tag{2-3}$$

where we have used (1-19). We will denote the last two terms by $L\psi$. A simple computation shows that

$$\begin{aligned}L\psi&=-\gamma r^3\partial_r\left(w^{1+\alpha}\frac{1}{r^2}(r^3\psi)_r\right)+4r^3\partial_r(w^{1+\alpha})\psi\\&=-\gamma(w^{1+\alpha}r^4\psi_r)_r+(4-3\gamma)r^3\partial_r(w^{1+\alpha})\psi.\end{aligned} \tag{2-4}$$

The associated eigenvalue problem is given by

$$L\psi=\lambda w^\alpha r^4\psi. \tag{2-5}$$

Then L is self-adjoint and hence λ is real. In fact, this eigenvalue problem was considered by Eddington [1918] to explain the luminosity variations of the Cepheid variables and Beyer [1995] studied the spectrum for L in $L^2((0,R),dr)$, which consists of simple eigenvalues $\lambda_1<\cdots<\lambda_n<\lambda_{n+1}<\cdots\to\infty$. See also Proposition 1 in [Makino 2015]. We recall that in [Lin 1997], the stability criterion was introduced based on the eigenvalues: w^α $(\sim\bar\rho)$ is called neutrally stable if $\lambda>0$ for all eigenvalues λ and unstable if

$\lambda < 0$ for some eigenvalue λ, and it was shown that w^α ($\sim\bar{\rho}$) is unstable for any $3 < \alpha < 5$ $\left(\frac{6}{5} < \gamma < \frac{4}{3}\right)$ and stable for $1 < \alpha < 3$ $\left(\frac{4}{3} < \gamma < 2\right)$ in the mass Lagrangian framework. In particular, for $1 < \alpha < 3$ $\left(\frac{4}{3} < \gamma < 2\right)$, the least eigenvalue λ_1 is positive.

Now fix a positive eigenvalue $\lambda = \lambda_n$ for some $\lambda_n > 0$ and an associated eigenfunction $\Psi = \Psi(r)$ of L:

$$L\Psi = \lambda w^\alpha r^4 \Psi. \tag{2-6}$$

We take Ψ that is bounded near both $r \sim 0$ and $r \sim R$, in particular $\Psi \in H$, where H is a Hilbert space with the norm

$$\|\Psi\|_H^2 \equiv \int_0^R w^{1+\alpha} r^4 (\Psi_r)^2 \, dr + \int_0^R w^\alpha r^4 \Psi^2 \, dr.$$

For more discussion on the existence of such Ψ, see [Makino 2015]. Then, for a given constant θ_0,

$$\psi_1(t, r) := \sin(\sqrt{\lambda} t + \theta_0)\Psi(r) \tag{2-7}$$

is a time-periodic solution to the linearized equation (2-3).

2A. ***The behavior of Ψ near the origin and near the boundary.*** Notice that Ψ satisfies

$$\lambda w^\alpha r^4 \Psi = -\gamma (w^{1+\alpha} r^4 \Psi')' + (4 - 3\gamma)(w^{1+\alpha})' r^3 \Psi. \tag{2-8}$$

We can deduce the regularity of Ψ from (2-8) based on the behavior of the Lane–Emden solution w. In what follows, we summarize the results from [Jang 2014] regarding w and Ψ.

Lemma 2.1 (regularity of w). *Let $1 < \alpha < 5$ be given and let w be a ball-type solution to the Lane–Emden equation* (1-20). *Then*:

(1) *w is analytic near the origin. Moreover*,

$$w(r) = 1 - br^2 + O(r^4), \quad r \sim 0,$$

for some positive constant $b > 0$. Also, $(\partial_r^{2k+1} w)(0) = 0$ for any nonnegative integer $k \geq 0$.

(2) *$\partial_r^i w$ is uniformly bounded on $(0, R)$ for each $0 \leq i \leq \alpha + 2$ and also $w^{(k-1)/2} \partial_r^{k+1} w$ is uniformly bounded on $(0, R)$ for each $1 \leq k \leq 2\alpha + 1$. In addition, w enjoys the integral regularity*

$$\int_0^R w^{\alpha+j} r^4 |\partial_r^{j+1} w|^2 \, dr < \infty$$

for each $0 \leq j < 3\alpha + 3$.

Lemma 2.2. *Let $\Psi \in H$ be the solution to* (2-8). *Ψ is analytic at $r = 0$ and, moreover, $\Psi = a + O(r)$ around the origin, where a is a constant.*

Lemma 2.3. *Let Ψ be the solution to* (2-8) *in H. Then*:

(1) *Ψ has the following integrability: for any $0 \leq \beta \leq \alpha$,*

$$\int_0^R w^{\alpha-\beta} r^4 \Psi^2 \, dr + \int_0^R w^{1+\alpha-\beta} r^4 (\Psi')^2 \, dr < \infty.$$

Moreover, for any $z > 1$,

$$\int_0^R w^{z-2} r^4 \Psi^2 \, dr < \infty.$$

(2) Ψ *has the following regularity*: *for* $1 \le k \le 2\alpha + 1$,

$$\int_0^R w^{1+\alpha+k} r^4 (\partial_r^{k+1} \Psi)^2 \, dr < \infty.$$

The proofs of Lemmas 2.1, 2.2 and 2.3 can be found in [Jang 2014]. Based on the above lemmas, we deduce that Ψ belongs to the function spaces of interest to us, namely it has a finite total initial energy for $1 < \alpha < 5$; see (2-12) and (2-14).

2B. ***Main result.*** We are interested in solutions (ψ, ψ_t) of (1-25) with the form

$$\psi(t, r; \epsilon) = \epsilon \psi_1(t, r) + \epsilon^2 \varphi(t, r; \epsilon), \tag{2-9}$$

where ψ_1 is a time-periodic linearized solution given in (2-7) and ϵ is a small positive parameter. For given initial data for $(\zeta, \zeta_t)|_{t=0}$ or $(\psi, \psi_t)|_{t=0}$ having a finite energy via (2-14), we can construct local-in-time solutions to (1-21) and hence to (1-25) for $0 < t < T$, where T is independent of ϵ, by the local existence theory [Coutand and Shkoller 2012; Jang and Masmoudi 2015; Luo et al. 2014]. We can set $\epsilon^2 \varphi(t, r; \epsilon) := \psi(t, r; \epsilon) - \epsilon \psi_1(t, r)$ to deduce that $\epsilon^2 \varphi$ is bounded in the corresponding energy norm. However, φ could be very large when ϵ is small. Our aim is to show that this does not happen, namely φ is bounded for all sufficiently small ϵ for all $0 < t < T$. In order to establish $\|\varphi\| = O(1)$, we will derive the uniform-in-ϵ estimates of φ. Let us first derive the equation for φ.

Plugging the ansatz (2-9) into (1-25), using the fact that ψ_1 solves (2-3), and also using (2-2), we obtain

$$\begin{aligned}
&\frac{w^\alpha r^4 \varphi_{tt}}{(1+\zeta)^4} + \frac{w^\alpha r^4 (\psi_1)_{tt}}{\epsilon} \left(\frac{1}{(1+\zeta)^4} - 1 \right) - \frac{2 w^\alpha r^4 |(\psi_1)_t + \epsilon \varphi_t|^2}{(1+\zeta)^7} \\
&\qquad + \frac{r^3}{\epsilon^2} \partial_r \left(w^{1+\alpha} \left(\left(1 + \frac{1}{r^2} (r^3 (\epsilon \psi_1 + \epsilon^2 \varphi))_r \right)^{-\gamma} - 1 + \gamma \frac{1}{r^2} (r^3 \epsilon \psi_1)_r \right) \right) \\
&\qquad\qquad - 4 w^\alpha r^4 \Phi(r) \varphi + w^\alpha r^4 \Phi(r) \frac{f}{\epsilon^2} = 0,
\end{aligned}$$

where $\Phi(r)$ is the prescribed function defined by

$$\Phi(r) \equiv \frac{1}{r^3} \int_0^r \frac{4\pi}{K^\alpha} w^\alpha s^2 \, ds = -\frac{(w^{1+\alpha})_r}{r w^\alpha} = -(1+\alpha) \frac{w_r}{r}. \tag{2-10}$$

Notice that $\Phi(r) > 0$ for each $0 < r < R$. By further using $(\psi_1)_{tt} = -\lambda \psi_1$ as well as (2-2), we arrive at

$$\begin{aligned}
&\frac{w^\alpha r^4 \varphi_{tt}}{(1+\zeta)^4} + 4\lambda w^\alpha r^4 \psi_1^2 + 4\lambda \epsilon w^\alpha r^4 \psi_1 \varphi - \lambda w^\alpha r^4 \psi_1 \frac{f}{\epsilon} - \frac{2 w^\alpha r^4 |(\psi_1)_t + \epsilon \varphi_t|^2}{(1+\zeta)^7} \\
&\qquad + \frac{r^3}{\epsilon^2} \partial_r \left(w^{1+\alpha} \left(\left(1 + \frac{1}{r^2} (r^3 (\epsilon \psi_1 + \epsilon^2 \varphi))_r \right)^{-\gamma} - 1 + \gamma \frac{1}{r^2} (r^3 \epsilon \psi_1)_r \right) \right) \\
&\qquad\qquad - 4 w^\alpha r^4 \Phi(r) \varphi + w^\alpha r^4 \Phi(r) \frac{f}{\epsilon^2} = 0.
\end{aligned} \tag{2-11}$$

We are concerned with the behavior of $(\varphi, \varphi_t)(t, r; \epsilon)$, the solutions of the initial value problem of (2-11) with given initial data $(\varphi, \varphi_t)(0, r; \epsilon) = (\varphi_0(r), \varphi_1(r))$. We remark that the appearance of ϵ in the denominator of the first and third lines in (2-11) is not harmful because $f = f(\psi) = O(|\psi|^2) = O(\epsilon^2)$. For the second line in (2-11) involving the second-order differential operator, at least formally, it is of order 1 with respect to ϵ. To make it rigorous, it needs to be treated very carefully. Notice that we have not decomposed it into the linear and nonlinear parts yet.

Motivated by the work on physical vacuum [Jang 2014; Jang and Masmoudi 2009; 2015], we consider the weighted energy norms: for $j \geq k \geq 0$,

$$\begin{aligned}\bar{\mathcal{E}}^{j,k} &\equiv \int_0^R w^{\alpha+k} r^4 |\partial_t^{j-k} \partial_r^k \varphi_t|^2\, dr + \int_0^R w^{1+\alpha+k} r^4 |\partial_t^{j-k} \partial_r^k \varphi_r|^2\, dr + \int_0^R w^{\alpha+k} r^4 |\partial_t^{j-k} \partial_r^k \varphi|^2\, dr \\ &\equiv \bar{\mathcal{E}}_t^{j,k} + \bar{\mathcal{E}}_r^{j,k} + \bar{\mathcal{E}}_0^{j,k}. \end{aligned} \tag{2-12}$$

Notice that the following relations hold:

$$\bar{\mathcal{E}}_t^{j,k} = \bar{\mathcal{E}}_r^{j,k-1} \quad \text{for } j \geq k \geq 1; \qquad \bar{\mathcal{E}}_0^{j,k} = \bar{\mathcal{E}}_t^{j-1,k} \quad \text{for } j \geq 1,\ j \geq k \geq 0. \tag{2-13}$$

We define the total energy $\bar{\mathcal{E}}$ by

$$\bar{\mathcal{E}}(t) \equiv \sum_{j=0}^{[\alpha]+4} \sum_{k=0}^{j} \bar{\mathcal{E}}^{j,k}(t), \tag{2-14}$$

where $[\alpha] = \max\{N \in \mathbb{Z} : N \leq \alpha\}$, so that $0 \leq \alpha - [\alpha] < 1$.

We also introduce the energy space

$$Z_\alpha = \left\{ (\varphi_0, \varphi_1) \;\Big|\; \sum_{k=0}^{[\alpha]+5} \int_0^R w^{\alpha+k} r^4 |\partial_r^k \varphi_0|^2\, dr + \sum_{k=0}^{[\alpha]+4} \int_0^R w^{\alpha+k} r^4 |\partial_r^k \varphi_1|^2\, dr < \infty \right\}.$$

We are now ready to state our main result.

Theorem 2.4. *For given initial data $(\varphi_0, \varphi_1) \in Z_\alpha$ independent of ϵ, let $(\varphi, \varphi_t) = (\varphi, \varphi_t)(t, r; \epsilon)$ be the solution of (2-11) with finite total energy for $0 < t \leq T$ satisfying $(\varphi, \varphi_t)(0, r; \epsilon) = (\varphi_0(r), \varphi_1(r))$. Then, if $1 < \alpha < 3$ $\left(\frac{4}{3} < \gamma < 2\right)$, there exists an $\epsilon_0 = O(1/T) > 0$ such that $\sup_{0<t\leq T} \bar{\mathcal{E}}(t) = O(1)$ for all $0 < \epsilon \leq \epsilon_0$, and, if $3 \leq \alpha < 5$ $\left(\frac{6}{5} < \gamma \leq \frac{4}{3}\right)$, there exists an $\epsilon_0 = O(1/e^{\kappa T}) > 0$ for some constant $\kappa > 0$ such that $\sup_{0<t\leq T} \bar{\mathcal{E}}(t) = O(1)$ for all $0 < \epsilon \leq \epsilon_0$.*

As a direct consequence of Theorem 2.4, we have $\|\psi - \epsilon\psi_1\|_{\bar{\mathcal{E}}} = O(\epsilon^2)$, which asserts the validity of the time-periodic linear approximations ψ_1 defined in (2-7) for the nonlinear solutions ψ to (1-25) having the form of (2-9). In fact, Theorem 2.4 recasts a recent work by Makino [2015], in which the time-periodic linear approximations were shown for γ for which $\gamma/(\gamma-1)$ is an integer and $\frac{6}{5} < \gamma < 2$ in a suitable weighted Sobolev space. More importantly, our theorem covers all the relevant exponents γ and it answers an open problem proposed in [Makino 2015]. We take a different approach: while in [Makino 2015], the Nash–Moser–Hamilton theory was used to prove the result, we use the weighted energy estimates that have been proven to be useful to study physical vacuum states of compressible flows [Coutand and Shkoller 2012; Jang and Masmoudi 2015].

The energy inequalities obtained in this article yield a rather concrete upper bound for the total energy involving ϵ, which gives an estimate for an upper bound for ϵ_0 as stated in the theorem. It is noteworthy to observe the qualitative difference on the upper bound ϵ_0 between $\frac{4}{3} < \gamma < 2$ and $\frac{6}{5} < \gamma \leq \frac{4}{3}$. We recall that $\frac{4}{3} < \gamma < 2$ corresponds to the stability regime of Lane–Emden stars and $\frac{6}{5} < \gamma \leq \frac{4}{3}$ to the instability regime [Deng et al. 2002; Jang 2008; 2014; Lin 1997; Rein 2003]. Our result indicates that for a given large time T, a small expansion (approximation) parameter ϵ in the instability regime needs to be taken much smaller than the ϵ in the stability regime in order to guarantee the validity of the expansion (approximation) ansatz (2-9). Even if the same $\lambda > 0$ is allowed to be chosen in (2-7), the set of small parameters ϵ to hold up the validity of such linear approximations could be very different depending on the value of the adiabatic exponent γ. Of course, this comparison and characterization deduced from the energy inequalities may not be optimal.

The estimates of φ obtained in the subsequent sections can be used to establish the existence of the solutions ψ to (1-25) of the form (2-9) with the corresponding initial data of the same expansion form having a finite total energy. We will not pursue this direction in detail in this article, but will make one comment. In this perspective, one can fix a small parameter ϵ first and then derive a lower bound on $T = T(\epsilon)$ that guarantees the existence of the solutions. Then Theorem 2.4 implies that $T = O(1/\epsilon)$ for $\gamma > \frac{4}{3}$ and $T = O(\ln(1/\epsilon))$ for $\frac{6}{5} < \gamma \leq \frac{4}{3}$. We observe that the lifespan of the solutions having finite total energy for a given small $\epsilon > 0$ may depend on whether γ falls into the stability regime or not. Again, this comparison may not be optimal; it would be an interesting problem to study the optimality of such lower bounds.

We can also consider the limit of $\epsilon \to 0$ and the convergence rate. Note that a maximal time T of the convergence of ψ to 0 (0 corresponds to the Lane–Emden stars) goes to infinity as $\epsilon \to 0$, namely the convergence to the equilibrium becomes global. And the rate of convergence may depend on whether the value of γ is in the stability regime or not. It is interesting to point out that a similar question was studied in a completely different context, Hilbert expansion from the Boltzmann theory [Guo et al. 2010; Guo and Jang 2010].

Finally, we remark that by no means does Theorem 2.4 imply a stability result in the usual sense, but it gives a set of initial data having the form (2-9) of which evolutions for later times stay in the same form. In particular, it was shown in [Jang 2014] that for $\frac{6}{5} < \gamma < \frac{4}{3}$ there exists a family of initial data for (1-21) leading to a nonlinear instability for the Lane–Emden equilibrium and thus there's no hope to show the stability result for general initial data. On the other hand, for $\gamma > \frac{4}{3}$, [Rein 2003] gives a nonlinear stability result based on a variational approach. However, the result of [Rein 2003] is conditional, in that the existence of the desired solutions was assumed without a proof. It still remains an interesting open problem to prove a complete stability result for the Euler–Poisson system for $\gamma > \frac{4}{3}$ and we hope that this work provides interesting evidence towards a satisfactory stability theory.

The rest of the paper will be devoted to the proof of Theorem 2.4. The proof consists of three parts. First we give the L^∞ bounds of functions in terms of our energy norms (2-12) by using Hardy inequalities. Then we derive the energy inequalities for nonlinear instant energies (4-1) by the weighed energy method. The estimates of the total energy involving spatial and mixed derivatives are obtained by elliptic estimates.

The embedding results will be used to close the weighted energy estimates as well as the elliptic estimates for the solutions of (2-11). The final step of the proof, solving differential inequalities, will be given in Section 7.

3. L^∞ bounds and embeddings

The goal of this section is to derive the L^∞ bounds of φ and its derivatives with suitable weights by using the energy norms introduced in (2-12) and (2-14). To this end, we will utilize the Hardy inequalities and embedding inequalities.

3A. *Hardy inequalities.* We recall the following version of the Hardy inequality:

Lemma 3.1 (Hardy inequality). *Let $k > 1$ be a given real number and let g be a function satisfying $\int_0^1 s^k(g^2 + g'^2)\,ds < \infty$. Then we have*

$$\int_0^1 s^{k-2} g^2\,ds \lesssim \int_0^1 s^k(g^2 + |g'|^2)\,ds.$$

For the proof of Lemma 3.1, we refer to [Kufner et al. 2007]. Since our energies involve different weights near the origin and near the boundary, we will utilize the localized version of the above Hardy inequalities as in [Jang 2014]. We begin by recalling the following results:

Lemma 3.2 [Jang 2014]. (1) *For any function u satisfying $\int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr < \infty$,*

$$\int_0^{R/2} r^2|u|^2\,dr \lesssim \int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr. \tag{3-1}$$

(2) *For any function u satisfying $\int_0^{3R/4} r^4|u_{rr}|^2\,dr + \int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr < \infty$,*

$$\int_0^{R/2} |u|^2\,dr \lesssim \int_0^{3R/4} r^4|u_{rr}|^2\,dr + \int_0^{3R/4} r^4|u_r|^2\,dr + \int_0^{3R/4} r^4|u|^2\,dr. \tag{3-2}$$

(3) *Let $a > 1$ be given. For any function v satisfying $\int_{R/4}^R w^a|v_r|^2\,dr + \int_{R/4}^R w^a|v|^2\,dr < \infty$,*

$$\int_{R/2}^R w^{a-2}|v|^2\,dr \lesssim \int_{R/4}^R w^a|v_r|^2\,dr + \int_{R/4}^R w^a|v|^2\,dr. \tag{3-3}$$

We can now derive Hardy embedding inequalities.

Lemma 3.3. *Let m be any nonnegative integer. Then*

$$\|u\|_{L^1}^2 \lesssim \sum_{k=0}^{2} \int_0^{3R/4} r^4|\partial_r^k u|^2\,dr + \sum_{k=0}^{m} \int_{R/4}^R w^{\alpha-[\alpha]+2m}|\partial_r^k u|^2\,dr. \tag{3-4}$$

Proof. Consider

$$\int_0^R |u|\,dr = \int_0^{R/2} |u|\,dr + \int_{R/2}^R |u_r|\,dr =: \text{(i)} + \text{(ii)}.$$

By Hölder's inequality and (3-2), we obtain

$$(\mathrm{i}) \lesssim \left(\int_0^{R/2} |u|^2\,dr\right)^{\frac12} \lesssim \left(\int_0^{3R/4} r^4|u|^2\,dr + \int_0^{3R/4} r^4|\partial_r u|^2\,dr + \int_0^{3R/4} r^4|\partial_r^2 u|^2\,dr\right)^{\frac12}.$$

For (ii), we first apply Hölder's inequality to get

$$(\mathrm{ii}) \le \left(\int_{R/2}^{R} w^{-\alpha+[\alpha]}\,dr\right)^{\frac12}\left(\int_{R/2}^{R} w^{\alpha-[\alpha]}|u|^2\,dr\right)^{\frac12}.$$

Notice that $\int_{R/2}^R w^{-\alpha+[\alpha]}\,dr < \infty$, since $0 \le \alpha-[\alpha] < 1$ and $w \sim R-r$ near $r = R$. We then apply the localized Hardy inequality (3-3) to the second term repeatedly to deduce the result. □

Lemma 3.4. *Let m be any nonnegative integer. Then*

$$\|u\|_\infty^2 \lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^k u|^2\,dr + \sum_{k=0}^{m+1}\int_{R/4}^{R} w^{\alpha-[\alpha]+2m}|\partial_r^k u|^2\,dr. \tag{3-5}$$

Proof. Notice that, since u is a function on the interval $(0, R)$, u is bounded by the $W^{1,1}$-norm:

$$\|u\|_\infty \lesssim \int_0^R |u|\,dr + \int_0^R |u_r|\,dr.$$

By applying (3-4) to each term, we obtain the desired result. □

3B. L^∞ *bounds.* A direct consequence of the above Hardy embedding inequalities is the validity of the boundedness assumption (4-9) within our energy space.

Lemma 3.5. (1)
$$|\varphi| + |\varphi_t| + |\varphi_{tt}| + \sum_{q=1}^{[\alpha]+2} |r^{\delta(q)} w^{(q-1)/2}\partial_t^{q+2}\varphi| \lesssim \bar{\mathcal{E}}^{1/2},$$

where $\delta(q) = 0$ for $q \le [\alpha]$, $\delta(q) = 1$ for $q = [\alpha]+1$, and $\delta(q) = 2$ for $q = [\alpha]+2$.

(2)
$$|\varphi_r| + |\varphi_{tr}| + \sum_{q=1}^{[\alpha]+2} |r^{\delta(q)} w^{q/2}\partial_t^{q+1}\partial_r\varphi| \lesssim \bar{\mathcal{E}}^{1/2},$$

where $\delta(q) = 0$ for $q \le [\alpha]$, $\delta(q) = 1$ for $q = [\alpha]+1$, and $\delta(q) = 2$ for $q = [\alpha]+2$.

Proof. We will present the details for the terms

$$\partial_t^3\varphi, \quad \partial_t\partial_r\varphi, \quad r^{\delta(2)}w^{1/2}\partial_t^4\varphi, \quad r^2 w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi.$$

Other terms can be treated in the same way. To see the boundedness of $\partial_t^3\varphi$, we apply (3-5) for $u = \partial_t^3\varphi$ with $m = [\alpha]+1$:

$$\|\partial_t^3\varphi\|_\infty^2 \lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^k\partial_t^3\varphi|^2\,dr + \sum_{k=0}^{[\alpha]+2}\int_{R/4}^{R} w^{\alpha-[\alpha]+2[\alpha]+2}|\partial_r^k\partial_t^3\varphi|^2\,dr.$$

Then, since w is bounded from below and above on $\left(0, \frac34 R\right)$ and r is bounded from below and above on $\left(\frac14 R, R\right)$, we deduce that the right-hand side is bounded by $\bar{\mathcal{E}}$.

To see the boundedness of $\partial_t\partial_r\varphi$, we apply (3-5) for $u=\partial_t\partial_r\varphi$ with $m=[\alpha]+2$:

$$\|\partial_t\partial_r\varphi\|_\infty^2 \lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^{k+1}\partial_t\varphi|^2\,dr+\sum_{k=0}^{[\alpha]+3}\int_{R/4}^{R} w^{\alpha-[\alpha]+2[\alpha]+4}|\partial_r^{k+1}\partial_t\varphi|^2\,dr.$$

It is easy to see that the right-hand side is bounded by $\bar{\mathcal{E}}$.

For the boundedness of $r^{\delta(2)}w^{1/2}\partial_t^3\varphi$, we divide into two cases: $2\le[\alpha]\le 4$ and $[\alpha]=1$. For the first case, $\delta(2)=0$. In this case, it suffices to show the boundedness of $w(\partial_t^4\varphi)^2$. By the Sobolev embedding,

$$\|w(\partial_t^4\varphi)^2\|_\infty \lesssim \int_0^R w(\partial_t^4\varphi)^2\,dr+\int_0^R |(w(\partial_t^4\varphi)^2)_r|\,dr.$$

Since w_r is bounded, by using the Cauchy–Schwarz inequality,

$$\|w(\partial_t^4\varphi)^2\|_\infty \lesssim \int_0^R |\partial_t^4\varphi|^2\,dr+\int_0^R w^2|\partial_r\partial_t^4\varphi|^2\,dr.$$

We now apply Hardy inequalities (3-2) and (3-3) to obtain

$$\begin{aligned}\|w(\partial_t^4\varphi)^2\|_\infty &\lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^k\partial_t^4\varphi|^2\,dr+\sum_{k=0}^{[\alpha]+1}\int_{R/4}^{R} w^{2+2[\alpha]}|\partial_r^k\partial_t^4\varphi|^2\,dr\\ &\lesssim \sum_{k=0}^{3}\int_0^{3R/4} r^4|\partial_r^k\partial_t^4\varphi|^2\,dr+\sum_{k=0}^{[\alpha]+1}\int_{R/4}^{R} w^{\alpha+k}|\partial_r^k\partial_t^4\varphi|^2\,dr,\end{aligned}$$

where we have used $w^{[\alpha]+1}\lesssim w^\alpha$. Notice that the right-hand side is bounded by $\bar{\mathcal{E}}$.

When $[\alpha]=1$, we have $\delta(2)=1$. In this case, it suffices to show that $r^2w(\partial_t^4\varphi)^2$ is bounded by $\bar{\mathcal{E}}$. Applying Sobolev embedding, the Cauchy–Schwarz inequality and Hardy inequalities, we obtain

$$\|r^2w(\partial_t^4\varphi)^2\|_\infty \lesssim \int_0^R |\partial_t^4\varphi|^2\,dr+\int_0^R r^4w^2|\partial_r\partial_t^4\varphi|^2\,dr \lesssim \sum_{k=0}^{2}\int_0^R r^4w^{\alpha+k}|\partial_r^k\partial_t^4\varphi|^2\,dr.$$

Since $[\alpha]=1$, the right-hand side is bounded by $\bar{\mathcal{E}}$.

To prove the boundedness of $r^2w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi$, we first apply Sobolev embedding and use the boundedness of w and w_r to obtain

$$\begin{aligned}\|r^2w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi\|_\infty &\lesssim \int_0^R r^2w^{[\alpha]/2}|\partial_t^{[\alpha]+3}\partial_r\varphi|\,dr\\ &\quad+\int_0^R rw^{([\alpha]+2)/2}|\partial_t^{[\alpha]+3}\partial_r\varphi|\,dr+\int_0^R r^2w^{([\alpha]+2)/2}|\partial_r^2\partial_t^{[\alpha]+3}\varphi|\,dr.\end{aligned}$$

By Hölder's inequality,

$$\|r^2w^{([\alpha]+2)/2}\partial_t^{[\alpha]+3}\partial_r\varphi\|_\infty^2 \lesssim \int_0^R r^2w^{\alpha-[\alpha]+[\alpha]}|\partial_t^{[\alpha]+3}\partial_r\varphi|^2\,dr+\int_0^R r^4w^{\alpha-[\alpha]+[\alpha]+2}|\partial_r^2\partial_t^{[\alpha]+3}\varphi|^2\,dr.$$

Notice that the second term in the right-hand side is $\bar{\mathcal{E}}_r^{[\alpha]+4,1}$. For the first term in the right-hand side we apply Hardy inequalities (3-1) and (3-3) to ensure that it is bounded by $\bar{\mathcal{E}}_r^{[\alpha]+3,0}$ and $\bar{\mathcal{E}}_r^{[\alpha]+4,1}$. $\square$

The results can be extended to other quantities involving more spatial derivatives. In the next lemma, we present the weighted L^∞ bounds of φ_{rr} and its time derivatives.

Lemma 3.6. *We have*

$$\sum_{q=0}^{[\alpha]+2} |w^{(q+1)/2} r^{\delta(q)} \partial_t^q \partial_r^2 \varphi| \lesssim \bar{\mathcal{E}}^{1/2}, \tag{3-6}$$

where $\delta(q)=0$ *for* $q\le[\alpha]$, $\delta(q)=1$ *for* $q=[\alpha]+1$, *and* $\delta(q)=2$ *for* $q=[\alpha]+2$.

Proof. The choice of $\delta(q)$ is clear because of (3-5). We will focus on the bound near the boundary. So we will assume that $\delta(q)=0$ and φ is supported in $\left(\frac{1}{4}R,\, R\right)$. We will use the $W^{1,1}$ bound for the squared quantity:

$$\begin{aligned}
\|w^{q+1}(\partial_t^q\partial_r^2\varphi)^2\|_\infty &\le \int_0^R w^{q+1}(\partial_t^q\partial_r^2\varphi)^2\,dr + \int_0^R |(w^{q+1}(\partial_t^q\partial_r^2\varphi)^2)_r|\,dr\\
&\lesssim \int_0^R w^{q+1}(\partial_t^q\partial_r^2\varphi)^2\,dr + \int_0^R w^{q}(\partial_t^q\partial_r^2\varphi)^2\,dr + \int_0^R w^{q+1}\partial_t^q\partial_r^2\varphi\partial_t^q\partial_r^3\varphi\,dr\\
&\lesssim \int_0^R w^{q}(\partial_t^q\partial_r^2\varphi)^2\,dr + \int_0^R w^{q+2}(\partial_t^q\partial_r^3\varphi)^2\,dr,
\end{aligned}$$

where we have used the Cauchy–Schwarz inequality and the boundedness of w. Applying the Hardy inequality (3-3), we obtain

$$\|w^{q+1}(\partial_t^q\partial_r^2\varphi)^2\|_\infty \lesssim \sum_{k=0}^{m+1}\int_0^R w^{q+2+2m}|\partial_t^q\partial_r^{k+2}\varphi|^2\,dr.$$

Choose $m=[\alpha]+2-q$. Then, since $w^{[\alpha]+2}\lesssim w^{\alpha+1}$ and $0\le k\le[\alpha]+3-q$,

$$\|w^{q+1}(\partial_t^q\partial_r^2\varphi)^2\|_\infty \lesssim \sum_{k=0}^{[\alpha]+3-q}\int_0^R w^{[\alpha]+2+[\alpha]+4-q}|\partial_t^q\partial_r^{k+2}\varphi|^2\,dr \lesssim \sum_{k=0}^{[\alpha]+3-q}\bar{\mathcal{E}}^{q+k+1,k+1}\lesssim\bar{\mathcal{E}}. \qquad \square$$

Remark 3.7. The strengths of the weights appearing for $\partial_t^{q+2}\varphi$, $\partial_t^{q+1}\partial_r\varphi$ and $\partial_t^q\partial_r^2\varphi$ in the previous lemmas depend on the number of spatial derivatives as well as the number of time derivatives. This is due to the energy structure of $\bar{\mathcal{E}}$.

4. The instant energy

In this section, we will introduce the various energies and establish the equivalence of the temporal instant energy and the total energy for (φ, φ_t).

Let $T>0$ be given such that the solutions to (1-21) or (2-11) satisfy the bound

$$\sup_{r\in(0,R)}|(\zeta\circ\psi)(t,r)| = \sup_{r\in(0,R)}\left|\zeta(\epsilon\psi_1(t,r)+\epsilon^2\varphi(t,r))\right| \le \tfrac{1}{4} \quad \text{for all } 0\le t\le T.$$

For each time $0 \le t \le T$, we introduce the following instant energies and the total energy for the solutions to the φ equation (2-11). The higher-order (temporal) instant energy is, for $j \ge 0$,

$$\mathcal{E}^j \equiv \int_0^R \frac{w^\alpha r^4 |\partial_t^j \varphi_t|^2}{(1+\zeta)^4}\,dr + \int_0^R \gamma \frac{w^{1+\alpha} J^{-\gamma-1} |(r^3 \partial_t^j \varphi)_r|^2}{r^2}\,dr - a(\gamma) \int_0^R 4 w^\alpha r^4 \Phi(r) |\partial_t^j \varphi|^2\,dr, \quad (4\text{-}1)$$

where J was defined in (1-23), and $a(\gamma) = 1$ for $\gamma > \frac{4}{3}$ and $a(\gamma) = 0$ otherwise. The total instant energy is

$$\mathcal{E}(t) \equiv \sum_{j=0}^{[\alpha]+4} \mathcal{E}^j(t). \quad (4\text{-}2)$$

A simple computation shows — see also the equivalent expressions for L in (2-4) —

$$-r^3 \partial_r \left(w^{1+\alpha} \frac{1}{r^2} (r^3 \psi)_r \right) = -(w^{1+\alpha} r^4 \psi_r)_r - 3r^3 \partial_r (w^{1+\alpha}) \psi. \quad (4\text{-}3)$$

Multiply this identity by ψ and integrate to obtain

$$\int_0^R \frac{w^{1+\alpha}}{r^2} |(r^3 \psi)_r|^2\,dr = \int_0^R w^{1+\alpha} r^4 |\psi_r|^2\,dr + \int_0^R 3 w^\alpha r^4 \Phi(r) \psi^2\,dr. \quad (4\text{-}4)$$

We observe that (4-4) gives another expression for the spatial part of the instant energy $\mathcal{E}^j$ if $J = 1$ throughout the domain for all time. However, it is not obvious we can guarantee the positiveness of $\mathcal{E}^j$ since J varies in time and radius. In the following lemma, we show the positivity of $\mathcal{E}^j$ and equivalence of the homogeneous energy $\bar{\mathcal{E}}^{j,0}$ for all sufficiently small $\epsilon > 0$.

Lemma 4.1. *Suppose that $\bar{\mathcal{E}}$ given in* (2-14) *is bounded for all* $0 \le t \le T$. *Then we have*

$$\mathcal{E}^j = \mathfrak{E}^j + \mathcal{R}^j, \quad (4\text{-}5)$$

where $\mathfrak{E}^j$ and $\mathcal{R}^j$ satisfy the estimates

(1) $(1 + \epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0} \lesssim \mathfrak{E}^j \lesssim (1 + \epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0}$,

(2) $|\mathcal{R}^j| \lesssim (\epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0}$,

(3) $|d\mathcal{R}^j/dt| \lesssim (\epsilon + \epsilon^2 \bar{\mathcal{E}}^{1/2}) \bar{\mathcal{E}}^{j,0}$,

for all sufficiently small $\epsilon > 0$.

Proof. To extract the positive part of $\mathcal{E}^j$, we will rewrite the spatial part similarly as in (4-4). To this end, from (4-3) we first obtain

$$\begin{aligned} -r^3 \partial_r \left(w^{1+\alpha} J^{-\gamma-1} \frac{1}{r^2} (r^3 \psi)_r \right) &= -(w^{1+\alpha} J^{-\gamma-1} r^4 \psi_r)_r \\ &\quad - 3r^3 J^{-\gamma-1} \partial_r (w^{1+\alpha}) \psi - 3r^3 \partial_r (J^{-\gamma-1}) w^{1+\alpha} \psi, \end{aligned} \quad (4\text{-}6)$$

which in turn yields the integral identity

$$\int_0^R \frac{w^{1+\alpha}J^{-\gamma-1}}{r^2}|(r^3\psi)_r|^2\,dr \\ = \int_0^R w^{1+\alpha}J^{-\gamma-1}r^4|\psi_r|^2\,dr + \int_0^R 3w^\alpha J^{-\gamma-1}r^4\Phi(r)\psi^2\,dr - \int_0^R 3r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}\psi^2\,dr. \tag{4-7}$$

By using (4-7), we write $\mathcal{E}^j$ as $\mathcal{E}^j \equiv \mathfrak{E}^j + \mathcal{R}^j$, where

$$\mathfrak{E}^j = \int_0^R \frac{w^\alpha r^4|\partial_t^j\varphi_t|^2}{(1+\zeta)^4}\,dr + \gamma\int_0^R w^{1+\alpha}J^{-\gamma-1}r^4|\partial_t^j\varphi_r|^2\,dr \\ + (3\gamma-4a(\gamma))\int_0^R w^\alpha J^{-\gamma-1}r^4\Phi(r)|\partial_t^j\varphi|^2\,dr,$$

$$\mathcal{R}^j = -3\gamma\int_0^R r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}|\partial_t^j\varphi|^2\,dr + 4a(\gamma)\int_0^R (J^{-\gamma-1}-1)w^\alpha r^4\Phi(r)|\partial_t^j\varphi|^2\,dr. \tag{4-8}$$

Since $3\gamma-4a(\gamma)>0$ for all γ, we now see that $\mathfrak{E}^j$ is positive for all γ. Moreover, by (1-23), (2-9), Taylor expansion and Lemma 3.5, we deduce the first result, which shows that $\mathfrak{E}^j$ is equivalent to $\bar{\mathcal{E}}^{j,0}$. The estimate of $\mathcal{R}^j$ follows similarly. Here we present the detail for the bound of $d\mathcal{R}^j/dt$. We start with the second term. The time derivative of the second term consists of the two terms

$$\int_0^R J^{-\gamma-2}J_t w^\alpha r^4\Phi(r)|\partial_t^j\varphi|^2\,dr, \qquad \int_0^R (J^{-\gamma-1}-1)w^\alpha r^4\Phi(r)\partial_t^j\varphi\partial_t^j\varphi_t\,dr.$$

Then, since $\Phi(r)<\infty$ and $|J^{-\gamma-2}J_t| \lesssim \epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$ and $|J^{-\gamma-1}-1| \lesssim \epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$ by Lemmas 3.6 and 3.5, we obtain the desired bounds in terms of $\bar{\mathcal{E}}^{j,0}$. On the other hand, the time derivative of the first integral of $\mathcal{R}^j$ consists of the two terms

$$\int_0^R r^3\partial_r\partial_t(J^{-\gamma-1})w^{1+\alpha}|\partial_t^j\varphi|^2\,dr, \qquad \int_0^R r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}\partial_t^j\varphi\partial_t^j\varphi_t\,dr.$$

By Lemmas 3.6 and 3.5, we see that $|w\partial_r\partial_t(J^{-\gamma-1})| \lesssim \epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$. Hence, by further using the localized Hardy inequality (3-1) near the origin, we have

$$\left|\int_0^R r^3\partial_r\partial_t(J^{-\gamma-1})w^{1+\alpha}|\partial_t^j\varphi|^2\,dr\right| \lesssim (\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2})\int_0^R r^2w^\alpha|\partial_t^j\varphi|^2\,dr \lesssim (\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2})\bar{\mathcal{E}}^{j,0}.$$

For the second term, we use $|w\partial_r(J^{-\gamma-1})| \lesssim \epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2}$ as well as the Cauchy–Schwarz inequality to get

$$\left|\int_0^R r^3\partial_r(J^{-\gamma-1})w^{1+\alpha}\partial_t^j\varphi\partial_t^j\varphi_t\,dr\right| \lesssim (\epsilon+\epsilon^2\bar{\mathcal{E}}^{1/2})\left(\int_0^R r^2w^\alpha|\partial_t^j\varphi|^2\,dr + \int_0^R r^4w^\alpha|\partial_t^j\varphi_t|^2\,dr\right).$$

We apply (3-1) to the first integral to obtain the desired bound. □

Lemma 4.1 implies that, if $\bar{\mathcal{E}}$ is bounded, a nonlinear instant energy $\mathcal{E}^j$ in (4-1) is equivalent to the homogenous energy $\bar{\mathcal{E}}^{j,0}$ given in (2-12) for all sufficiently small $\epsilon>0$.

The next goal is to derive the a priori estimates for $\mathcal{E}$ and $\bar{\mathcal{E}}$ under the assumption

$$|\varphi|+|\varphi_t|+|\varphi_{tt}|+\sum_{q=1}^{[\alpha]+2}|r^{\delta(q)}w^{(q-1)/2}\partial_t^{q+2}\varphi|+|\varphi_r|+|\varphi_{tr}| \\ +\sum_{q=1}^{[\alpha]+2}|r^{\delta(q)}w^{q/2}\partial_t^{q+1}\partial_r\varphi|+\sum_{q=0}^{[\alpha]+2}|w^{(q+1)/2}r^{\delta(q)}\partial_t^{q}\partial_r^{2}\varphi|\le M, \tag{4-9}$$

where M is a fixed constant. We recall that the validity of this assumption within the total energy $\bar{\mathcal{E}}$ was provided in Lemmas 3.5 and 3.6. The a priori estimates consist of two parts: the temporal energy estimates for $\mathcal{E}$, and the elliptic estimates to recover all other terms in $\bar{\mathcal{E}}$.

We start with the energy estimates of $\mathcal{E}$.

5. Weighted energy estimates

This section is devoted to the proof of this proposition:

Proposition 5.1. *Suppose that (φ, φ_t) satisfy* (2-11) *for $0\le t\le T$ and the corresponding total instant energy $\mathcal{E}$ is bounded. Moreover, we assume* (4-9). *Then $\mathcal{E}$ enjoys the energy inequality*

$$\frac{d}{dt}\mathcal{E}\lesssim\sqrt{\mathcal{E}}+(1-a(\gamma))\mathcal{E}+(\epsilon+\epsilon^2M)(\mathcal{E}+\sqrt{\bar{\mathcal{E}}}\sqrt{\mathcal{E}}), \tag{5-1}$$

where $a(\gamma)=1$ for $\gamma>\frac{4}{3}$ and $a(\gamma)=0$ otherwise, and $\epsilon>0$ is small enough.

Remark 5.2. $\mathcal{E}$ is positive for all sufficiently small ϵ due to Lemma 4.1, Hence $\sqrt{\mathcal{E}}$ is well defined in the right-hand side of (5-1).

Lemma 5.3 ($\mathcal{E}^0$). *Suppose that (φ, φ_t) satisfy* (2-11) *for $0\le t\le T$ and the corresponding total instant energy $\mathcal{E}$ is bounded. Moreover, we assume* (4-9). *Then*

$$\frac{d}{dt}\mathcal{E}^0\lesssim\sqrt{\mathcal{E}^0}+(1-a(\gamma))\mathcal{E}^0+(\epsilon+\epsilon^2M)(\mathcal{E}^0+\mathcal{E}^1), \tag{5-2}$$

where $a(\gamma)$ was introduced in the definition of $\mathcal{E}^0$.

Proof. We begin by multiplying (2-11) by φ and integrating over $(0, R)$:

$$\int_0^R\frac{w^\alpha r^4\varphi_{tt}}{(1+\zeta)^4}\varphi_t\,dr+\int_0^R\biggl(4\lambda w^\alpha r^4\psi_1^2+4\lambda\epsilon w^\alpha r^4\psi_1\varphi-\lambda w^\alpha r^4\psi_1\frac{f}{\epsilon}-\frac{2w^\alpha r^4|(\psi_1)_t+\epsilon\varphi_t|^2}{(1+\zeta)^7}\biggr)\varphi_t\,dr \\ +\int_0^R\frac{r^3}{\epsilon^2}\partial_r\biggl(w^{1+\alpha}\biggl(\Bigl(1+\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\Bigr)^{-\gamma}-1+\gamma\frac{1}{r^2}(r^3\epsilon\psi_1)_r\biggr)\biggr)\varphi_t\,dr \\ -\int_0^R 4w^\alpha r^4\Phi(r)\varphi\varphi_t\,dr+\int_0^R w^\alpha r^4\Phi(r)\frac{f}{\epsilon^2}\varphi_t\,dr=0.$$

We denote the left-hand side by $\sum_{k=1}^5 I_k$. The first term I_1 can be rewritten as

$$I_1=\frac{1}{2}\frac{d}{dt}\int_0^R\frac{w^\alpha r^4|\varphi_t|^2}{(1+\zeta)^4}\,dr+2\int_0^R\frac{w^\alpha r^4|\varphi_t|^2}{(1+\zeta)^5}\frac{(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^2}\,dr,$$

where we have used (1-24). For I_2, we use the boundedness of $(\psi_1)_t$ and $f = O(|\epsilon\psi_1 + \epsilon^2\varphi|^2)$ to deduce that

$$|I_2| \lesssim \sqrt{\mathcal{E}^0} + \epsilon\mathcal{E}^0 + \epsilon^2 \sup|\varphi_t|\mathcal{E}^0.$$

For I_3, we integrate by parts and use (2-1):

$$\begin{aligned}
I_3 &= -\int_0^R \frac{w^{1+\alpha}}{\epsilon^2}\left(\left(1+\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\right)^{-\gamma} - 1 + \gamma\frac{1}{r^2}(r^3\epsilon\psi_1)_r\right)(r^3\varphi_t)_r\,dr \\
&= -\int_0^R w^{1+\alpha}\left(-\frac{\gamma}{r^2}(r^3\varphi)_r + \frac{h}{\epsilon^2}\right)(r^3\varphi_t)_r\,dr \\
&= -\int_0^R w^{1+\alpha}\left(-\frac{\gamma}{r^2}J^{-\gamma-1}(r^3\varphi)_r + (J^{-\gamma-1}-1)\frac{\gamma}{r^2}(r^3\varphi)_r + \frac{h}{\epsilon^2}\right)(r^3\varphi_t)_r\,dr \\
&= \frac{\gamma}{2}\frac{d}{dt}\int_0^R w^{1+\alpha}J^{-\gamma-1}\frac{|(r^3\varphi)_r|^2}{r^2}\,dr + \frac{\gamma(\gamma+1)}{2}\int_0^R w^{1+\alpha}J^{-\gamma-2}J_t\frac{|(r^3\varphi)_r|^2}{r^2}\,dr \\
&\qquad \underbrace{-\int_0^R w^{1+\alpha}(J^{-\gamma-1}-1)\frac{\gamma}{r^2}(r^3\varphi)_r(r^3\varphi_t)_r\,dr}_{I_3^1} \underbrace{-\int_0^R w^{1+\alpha}\frac{h}{\epsilon^2}(r^3\varphi_t)_r\,dr}_{I_3^2}.
\end{aligned}$$

Since $J_t = 3\psi_t + r\psi_{tr} = 3(\epsilon(\psi_1)_t + \epsilon^2\varphi_t) + r(\epsilon(\psi_1)_{tr} + \epsilon^2\varphi_{tr})$, the commutator involving J_t is bounded by $(\epsilon+\epsilon^2 M)\mathcal{E}^0$. Notice that $|J^{-\gamma-1}-1| = O\big(\big|(1/r^2)(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\big|\big) \lesssim \epsilon + \epsilon^2 M$, so by the Cauchy–Schwarz inequality we see that

$$|I_3^1| \lesssim (\epsilon+\epsilon^2 M)(\mathcal{E}^0+\mathcal{E}^1).$$

Since $h = O\big(\big|(1/r^2)(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\big|^2\big)$, we have

$$|I_3^2| \lesssim \sqrt{\mathcal{E}^1} + \epsilon(\mathcal{E}^0+\mathcal{E}^1) + \epsilon^2 \sup\left|\frac{(r^3\varphi)_r}{r^2}\right|(\mathcal{E}^0+\mathcal{E}^1).$$

It is easy to see that

$$I_4 = -2\frac{d}{dt}\int_0^R w^\alpha r^4\Phi(r)\varphi^2\,dr \tag{5-3}$$

and also it satisfies

$$|I_4| \lesssim \mathcal{E}^0. \tag{5-4}$$

If $\gamma > \frac{4}{3}$, we will use (5-3) so that I_4 contributes to the energy. If $\gamma \le \frac{4}{3}$, then we will use the estimate (5-4), in which case the contribution of $\mathcal{E}^0$ in the right-hand side of the energy inequality will be of order 1.

For the last term, we obtain

$$|I_5| \lesssim \sqrt{\mathcal{E}^0} + \epsilon\mathcal{E}^0 + \epsilon^2 \sup|\varphi|\mathcal{E}^0.$$

This finishes the proof. □

As Lemma 5.3 indicates, the right-hand side of the energy inequality involves higher-order energy due to the nonlinearity and degeneracy, and thus the energy estimates cannot be closed at the physical energy level $\mathcal{E}^0$. This motivates us to go beyond $\mathcal{E}^0$.

The time differentiation of (2-11) yields

$$\begin{aligned}\frac{w^\alpha r^4\varphi_{ttt}}{(1+\zeta)^4}&-\frac{4w^\alpha r^4\varphi_{tt}(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^7}+8\lambda w^\alpha r^4\psi_1(\psi_1)_t+4\lambda\epsilon w^\alpha r^4(\psi_1)_t\varphi+4\lambda\epsilon w^\alpha r^4\psi_1\varphi_t\\&-\lambda w^\alpha r^4(\psi_1)_t\frac{f}{\epsilon}-\lambda w^\alpha r^4\psi_1\frac{f_t}{\epsilon}-\frac{4w^\alpha r^4((\psi_1)_t+\epsilon\varphi_t)((\psi_1)_{tt}+\epsilon\varphi_{tt})}{(1+\zeta)^7}\\&-\frac{14w^\alpha r^4((\psi_1)_t+\epsilon\varphi_t)^2(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^{10}}\\&-\gamma\frac{r^3}{\epsilon^2}\partial_r\left(w^{1+\alpha}\left(J^{-\gamma-1}\left(\frac{1}{r^2}\big(r^3(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)\big)_r\right)-\frac{1}{r^2}(r^3\epsilon(\psi_1)_t)_r\right)\right)\\&\qquad\qquad-4w^\alpha r^4\Phi(r)\varphi_t+w^\alpha r^4\Phi(r)\frac{f_t}{\epsilon^2}=0,\end{aligned}\tag{5-5}$$

where we have substituted J for its equivalent expression given in (1-23). We next present the estimates for $\mathcal{E}^1$.

Lemma 5.4 ($\mathcal{E}^1$). *Suppose that (φ,φ_t) satisfy* (2-11) *for $0\le t\le T$ and the corresponding total instant energy $\mathcal{E}$ is bounded. Moreover, we assume* (4-9). *Then*

$$\frac{d}{dt}\mathcal{E}^1\lesssim(1+\epsilon M)\sqrt{\mathcal{E}^1}+(1-a(\gamma))\mathcal{E}^1+(\epsilon+\epsilon^2M+\epsilon^4M^2)(\mathcal{E}^0+\mathcal{E}^1)+\epsilon\sqrt{\bar{\mathcal{E}}^{1,1}}\sqrt{\mathcal{E}^1}.\tag{5-6}$$

Proof. We multiply (5-5) by φ_{tt} and integrate it over $(0,R)$. We denote each integral by I_k for $1\le k\le 12$. We will estimate them term by term. I_1 forms an energy plus a commutator and thus I_1+I_2 can be written as

$$I_1+I_2=\frac{1}{2}\frac{d}{dt}\int_0^R\frac{w^\alpha r^4|\varphi_{tt}|^2}{(1+\zeta)^4}\,dr-\int_0^R\frac{2w^\alpha r^4\varphi_{tt}^2(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^7}\,dr,$$

where we have used (1-24). Note that the second term is bounded by $(\epsilon+\epsilon^2M)\mathcal{E}^1$ since $(\psi_1)_t$ is bounded and $|\varphi_t|\le M$ due to (4-9).

I_3 is a source term and it is easy to see that

$$|I_3|\lesssim\sqrt{\mathcal{E}^1}$$

due to the boundedness of ψ_1. For I_4 and I_5, we apply the Cauchy–Schwarz inequality to obtain

$$|I_4|+|I_5|\lesssim\epsilon(\mathcal{E}^0+\mathcal{E}^1).$$

In order to estimate I_6 and I_7, we recall (2-2) and that $f=O(|\epsilon\psi_1+\epsilon^2\varphi|^2)$. Then $f/\epsilon=O(\epsilon|\psi_1+\epsilon\varphi|^2)$ and $f_t/\epsilon=O\big(\epsilon(\psi_1+\epsilon\varphi)((\psi_1)_t+\epsilon\varphi_t)\big)$. Hence we deduce that

$$|I_6|+|I_7|\lesssim\epsilon\sqrt{\mathcal{E}^1}+(\epsilon^2+\epsilon^3M)(\mathcal{E}^0+\mathcal{E}^1).$$

I_8 and I_9 can be similarly estimated:

$$|I_8|\lesssim\sqrt{\mathcal{E}^1}+(\epsilon+\epsilon^2M)(\mathcal{E}^0+\mathcal{E}^1)\quad\text{and}\quad|I_9|\lesssim\epsilon\sqrt{\mathcal{E}^1}+(\epsilon^2+\epsilon^3M+\epsilon^4M^2)(\mathcal{E}^0+\mathcal{E}^1).$$

We next move onto I_{10}, which will give rise to another energy term. We first rewrite the fourth line in (5-5):

$$-\gamma\frac{r^3}{\epsilon^2}\partial_r\left(w^{1+\alpha}\left(J^{-\gamma-1}\left(\frac{1}{r^2}\big(r^3(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)\big)_r\right)-\frac{1}{r^2}(r^3\epsilon(\psi_1)_t)_r\right)\right)$$
$$=-\gamma r^3\partial_r\left(w^{1+\alpha}J^{-\gamma-1}\frac{1}{r^2}(r^3\varphi_t)_r\right)-\gamma r^3\partial_r\left(w^{1+\alpha}\frac{(J^{-\gamma-1}-1)}{\epsilon}\frac{1}{r^2}(r^3(\psi_1)_t)_r\right). \quad (5\text{-}7)$$

By replacing the fourth line using (5-7), we have two terms in I_{10}, denoted by I_{10}^1 and I_{10}^2. For I_{10}^1, we integrate by parts to obtain a perfect time derivative plus a commutator:

$$I_{10}^1=\frac{\gamma}{2}\frac{d}{dt}\int_0^R w^{1+\alpha}J^{-\gamma-1}\frac{1}{r^2}|(r^3\varphi_t)_r|^2\,dr+\frac{\gamma(\gamma+1)}{2}\int_0^R w^{1+\alpha}J^{-\gamma-2}J_t\frac{1}{r^2}|(r^3\varphi_t)_r|^2\,dr.$$

Note that $J_t=3\psi_t+r\psi_{tr}=3(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)+r(\epsilon(\psi_1)_{tr}+\epsilon^2\varphi_{tr})$. Thus the commutator is bounded by $(\epsilon+\epsilon^2M)\mathcal{E}^1$. For I_{10}^2, we first rewrite it as

$$\begin{aligned}I_{10}^2=&-\gamma\int_0^R r^3\varphi_{tt}\partial_r\left(w^{1+\alpha}\frac{J^{-\gamma-1}-1}{\epsilon}\frac{1}{r^2}(r^3(\psi_1)_t)_r\right)dr\\
=&-\gamma\int_0^R r^3\varphi_{tt}(w^{1+\alpha})_r\frac{J^{-\gamma-1}-1}{\epsilon}(3(\psi_1)_t+r(\psi_1)_{tr})\,dr\\
&+\gamma(\gamma+1)\int_0^R r^3\varphi_{tt}w^{1+\alpha}\frac{J^{-\gamma-2}J_r}{\epsilon}(3(\psi_1)_t+r(\psi_1)_{tr})\,dr\\
&-\gamma\int_0^R r^3\varphi_{tt}w^{1+\alpha}\frac{J^{-\gamma-1}-1}{\epsilon}(4(\psi_1)_{tr}+r(\psi_1)_{trr})\,dr\equiv I_{10}^{2,1}+I_{10}^{2,2}+I_{10}^{2,3}.\end{aligned}$$

For $I_{10}^{2,1}$ and $I_{10}^{2,3}$, we note that $(J^{-\gamma-1}-1)/\epsilon=-(\gamma+1)((r^3(\psi_1+\epsilon\varphi))_r)/r^2+\tilde{h}/\epsilon$, where $\tilde{h}=O\big(\big|(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r/r^2\big|^2\big)$, which yields $|(J^{-\gamma-1}-1)/\epsilon|\lesssim 1+\epsilon M$. Then, from Hölder's inequality and the regularity of ψ_1,

$$|I_{10}^{2,3}|\lesssim(1+\epsilon M)\left(\int_0^R w^\alpha r^4\varphi_{tt}^2\,dr\right)^{\frac12}\left(\int_0^R w^{\alpha+2}\big(r^2|(\psi_1)_{tr}|^2+r^4|(\psi_1)_{trr}|^2\big)\,dr\right)^{\frac12}\lesssim(1+\epsilon M)\sqrt{\mathcal{E}^1}.$$

For $I_{10}^{2,1}$, since $|(w^{1+\alpha})_r|\sim w^\alpha$, we apply the Hardy inequality near the boundary. Then, from the regularity of ψ_1, we obtain

$$|I_{10}^{2,1}|\lesssim(1+\epsilon M)\sqrt{\mathcal{E}^1}.$$

For $I_{10}^{2,2}$, we first note that $J_r/\epsilon=(4\psi_r+r\psi_{rr})/\epsilon=4((\psi_1)_r+\epsilon\varphi_r)+r((\psi_1)_{rr}+\epsilon\varphi_{rr})$. Then, from the regularity of ψ_1,

$$\begin{aligned}|I_{10}^{2,2}|\lesssim&\left(\int_0^R w^\alpha r^4\varphi_{tt}^2\,dr\right)^{\frac12}\left(\int_0^R w^{\alpha+2}(r^2|(\psi_1)_r|^2+r^4|(\psi_1)_{rr}|^2)\,dr\right)^{\frac12}\\
&+\epsilon\left(\int_0^R w^\alpha r^4\varphi_{tt}^2\,dr\right)^{\frac12}\left(\int_0^R w^{\alpha+2}(r^2|\varphi_r|^2+r^4|\varphi_{rr}|^2)\,dr\right)^{\frac12}\lesssim(1+\epsilon\sqrt{\bar{\mathcal{E}}^{1,1}})\sqrt{\mathcal{E}^1}.\end{aligned}$$

Next, it is easy to see that

$$I_{11} = -2\frac{d}{dt}\int_0^R w^\alpha r^4 \Phi(r)\varphi_t^2\,dr \quad \text{and} \quad |I_{11}| \lesssim \mathcal{E}^1. \tag{5-8}$$

If $\gamma > \frac{4}{3}$, then I_{11} will contribute to the energy via (5-8). If $\gamma \le \frac{4}{3}$, then we will use the estimate (5-8), in which case the contribution of $\mathcal{E}^1$ in the right-hand side of the energy inequality will be of order 1.

For the last term, since $f_t/\epsilon^2 = O\big((\psi_1+\epsilon\varphi)((\psi_1)_t+\epsilon\varphi_t)\big)$, we obtain

$$|I_{12}| \lesssim \sqrt{\mathcal{E}^1} + (\epsilon+\epsilon^2 M)(\mathcal{E}^0+\mathcal{E}^1).$$

This finishes the proof of the lemma. □

Lemmas 5.3 and 5.4 give rise to the energy inequality for $\mathcal{E}^0+\mathcal{E}^1$. However, the right-hand side involves M from the assumption (4-9) as well as $\bar{\mathcal{E}}^{1,1}$. In order to justify the assumption and to close the estimates, we will carry out the higher-order estimates.

The equations for $\partial_t^i\varphi_t,\ 1 \le i \le [\alpha]+4$, can be written in the form

$$\begin{aligned}
&\frac{w^\alpha r^4 \partial_t^i \varphi_{tt}}{(1+\zeta)^4} + \sum_{j=1}^{i} c_{1j} w^\alpha r^4 \partial_t^{i-j}\varphi_{tt}\partial_t^{j-1}\biggl(\frac{-4(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)}{(1+\zeta)^7}\biggr)\\
&\qquad + 8\lambda w^\alpha r^4 \partial_t^{i-1}(\psi_1(\psi_1)_t) + \sum_{j=0}^{i} c_{2j} 4\lambda\epsilon w^\alpha r^4 \partial_t^{i-j}\psi_1\partial_t^j\varphi - \sum_{j=0}^{i}\lambda w^\alpha r^4\partial_t^{i-j}\psi_1\frac{\partial_t^j f}{\epsilon}\\
&\qquad - \sum_{j=0}^{i} c_{2j} 2 w^\alpha r^4 \partial_t^{i-j}\bigl(((\psi_1)_t+\epsilon\varphi_t)^2\bigr)\partial_t^j\biggl(\frac{1}{(1+\zeta)^7}\biggr)\\
&\qquad - \gamma r^3\partial_r\biggl(w^{1+\alpha}J^{-\gamma-1}\frac{1}{r^2}(r^3\partial_t^i\varphi)_r\biggr) - 4w^\alpha r^4\Phi(r)\partial_t^i\varphi\\
&\qquad - \sum_{j=0}^{i-2} c_{3j}\gamma r^3\partial_r\biggl(w^{1+\alpha}\partial_t^{i-1-j}(J^{-\gamma-1})\frac{1}{r^2}(r^3\partial_t^j\varphi_t)_r\biggr)\\
&\qquad - \sum_{j=0}^{i-1} c_{3j}\gamma r^3\partial_r\biggl(w^{1+\alpha}\partial_t^{i-1-j}\biggl(\frac{J^{-\gamma-1}-1}{\epsilon}\biggr)\frac{1}{r^2}(r^3(\partial_t^j\psi_1)_t)_r\biggr) + w^\alpha r^4\Phi(r)\frac{\partial_t^i f}{\epsilon^2} = 0,
\end{aligned} \tag{5-9}$$

where c_{1j}, c_{2j} and c_{3j} are binomial coefficients. Notice that we have used (5-7) to write the elliptic, spatial part.

We record the high-order energy inequalities for the solutions to (5-9):

Lemma 5.5 ($\mathcal{E}^i$, $i \ge 2$). *Suppose that* (φ, φ_t) *satisfy* (2-11) *for* $0 \le t \le T$ *and the corresponding total instant energy* $\mathcal{E}$ *is bounded. Moreover, we assume* (4-9). *Then*

$$\begin{aligned}
\frac{d}{dt}\mathcal{E}^i &\lesssim (1+\epsilon M)\sqrt{\mathcal{E}^i} + (1-a(\gamma))\mathcal{E}^i\\
&\qquad + \sum_{k=1}^{i}(\epsilon+\epsilon^2 M)^k\sum_{j=0}^{i}\mathcal{E}^j + \sum_{k=1}^{i}(\epsilon+\epsilon^2M)^k\biggl(\sum_{j=0}^{i}\sum_{l=0}^{j}\sqrt{\bar{\mathcal{E}}^{j,l}}\biggr)\sqrt{\mathcal{E}^i}.
\end{aligned} \tag{5-10}$$

Proof. We multiply (5-9) by $\partial_t^i \varphi_t$ and integrate it over $(0, R)$. We denote each integral by J_k for $1 \le k \le 11$. As before, we will estimate them term by term. As in the case of I_1 in the previous lemma, the first term J_1 forms an energy plus a commutator:

$$J_1 = \frac{1}{2}\frac{d}{dt}\int_0^R \frac{w^\alpha r^4 |\partial_t^i \varphi_t|^2}{(1+\zeta)^4}\,dr + \int_0^R \frac{2w^\alpha r^4 |\partial_t^i \varphi_t|^2 (\epsilon(\psi_1)_t + \epsilon^2 \varphi_t)}{(1+\zeta)^7}\,dr,$$

where we have used (1-24). Note that the second term is bounded by $(\epsilon+\epsilon^2 M)\mathcal{E}^i$ since $(\psi_1)_t$ is bounded and $|\varphi_t| \le M$ due to (4-9). For J_2, we note that the second factor in the summation of the second term in the first line of (5-9) has the form

$$(\epsilon \partial_t^{j-k}(\psi_1)_t + \epsilon^2 \partial_t^{j-k}\varphi_t)(\epsilon(\psi_1)_t + \epsilon^2\varphi_t)^{k-1}, \quad 1 \le k \le j;$$

thus, since $|\zeta| \le \frac{1}{4}$, essentially J_2 consists of the following terms: for each $1 \le k \le j \le i$,

$$\begin{aligned}
&\int_0^R w^\alpha r^4 \partial_t^i \varphi_t \partial_t^{i-j+1}\varphi_t (\epsilon \partial_t^{j-k}(\psi_1)_t + \epsilon^2 \partial_t^{j-k}\varphi_t)(\epsilon(\psi_1)_t + \epsilon^2\varphi_t)^{k-1}\,dr \\
&\quad = \epsilon \int_0^R w^\alpha r^4 \partial_t^i \varphi_t \partial_t^{i-j+1}\varphi_t \partial_t^{j-k}(\psi_1)_t (\epsilon(\psi_1)_t + \epsilon^2\varphi_t)^{k-1}\,dr \\
&\qquad\qquad + \epsilon^2 \int_0^R w^\alpha r^4 \partial_t^i \varphi_t \partial_t^{i-j+1}\varphi_t \partial_t^{j-k}\varphi_t (\epsilon(\psi_1)_t + \epsilon^2\varphi_t)^{k-1}\,dr \\
&\quad = J_2^1 + J_2^2.
\end{aligned} \tag{5-11}$$

For J_2^1, we recall $(\psi_1)_{tt} = -\lambda\psi_1$ and hence $\partial_t^{j-k}(\psi_1)_t$ is a constant multiple of ψ_1 or $(\psi_1)_t$. By further recalling that ψ_1 and $(\psi_1)_t$ are bounded and $|\varphi_t| \lesssim M$, and by using the Cauchy–Schwarz inequality, we see that

$$|J_2^1| \lesssim \epsilon(\epsilon + \epsilon^2 M)^{k-1}(\mathcal{E}^i + \mathcal{E}^{i-j+1}).$$

For J_2^2, let $1 \le j \le \left[\frac{i}{2}\right] + 1$ first. Then

$$\begin{aligned}
|J_2^2| &= \epsilon^2 \left| \int_0^R w^{\alpha/2} r^2 \partial_t^i \varphi_t w^{(\alpha-j+k+1)/2} r^2 \partial_t^{i-j+1}\varphi_t w^{(j-k-1)/2}\partial_t^{j-k}\varphi_t (\epsilon(\psi_1)_t + \epsilon^2\varphi_t)^{k-1}\,dr \right| \\
&\le \epsilon^2 \sup\left| w^{(j-k-1)/2}\partial_t^{j-k}\varphi_t \right| (\epsilon + \epsilon^2 M)^{k-1}\sqrt{\mathcal{E}^i}\,\underbrace{\left(\int_0^R w^{\alpha-j+k+1} r^4 |\partial_t^{i-j+1}\varphi_t|^2\,dr \right)^{\frac{1}{2}}}_{J_2^{2,1}}.
\end{aligned}$$

By (4-9), $\sup\left|w^{(j-k-1)/2}\partial_t^{j-k}\varphi_t\right| \le M$. To estimate $J_2^{2,1}$, since $k \ge 1$ we first observe that $J_2^{2,1} \lesssim \mathcal{E}^i$ when $j = 1$, and $J_2^{2,1} \lesssim \mathcal{E}^{i-1}$ when $j = 2$. Now, when $2 \le j \le \left[\frac{i}{2}\right] + 1$ we apply the Hardy inequality (3-3) near the boundary $j-2$ times to obtain

$$\int_0^R w^{\alpha-j+k+1} r^4 |\partial_t^{i-j+1}\varphi_t|^2\,dr \lesssim \sum_{l=0}^{j-2}\int_0^R w^{\alpha-j+k+1+2(j-2)} r^4 |\partial_t^{i-j+1}\partial_r^l \varphi_t|^2\,dr \lesssim \sum_{l=0}^{j-2} \bar{\mathcal{E}}^{i-j+1+l,l}.$$

Now, for J_2^2, when there exist i and j such that $\left[\frac{i}{2}\right]+2 \le j \le i$ we write

$$\begin{aligned}|J_2^2| &= \epsilon^2\left|\int_0^R w^{\alpha/2}r^2\partial_t^i\varphi_t w^{(i-j)/2}\partial_t^{i-j+1}\varphi_t w^{(\alpha-i+j)/2}r^2\partial_t^{j-k}\varphi_t(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{k-1}\,dr\right|\\ &\le \epsilon^2 \sup\left|w^{(i-j)/2}\partial_t^{i-j+1}\varphi_t\right|(\epsilon+\epsilon^2M)^{k-1}\sqrt{\mathcal{E}^i}\left(\int_0^R w^{\alpha-i+j}r^4|\partial_t^{j-k}\varphi_t|^2\,dr\right)^{\frac{1}{2}}.\end{aligned}$$

Note that $\sup\left|w^{(i-j)/2}\partial_t^{i-j+1}\varphi_t\right| \le M$ due to (4-9). Let $J_2^{2,2}$ be the integral in the last term; we apply (3-3) $i-j$ times to get

$$J_2^{2,2} \lesssim \sum_{l=0}^{i-j}\int_0^R w^{\alpha-i+j+2(i-j)}r^4|\partial_t^{j-k}\partial_r^l\varphi_t|^2\,dr \lesssim \sum_{l=0}^{i-j}\bar{\mathcal{E}}^{j-k+l,l}.$$

We summarize the above estimates for J_2:

$$|J_2| \lesssim \sum_{1\le k,j\le i}(\epsilon+\epsilon^2M)^k\mathcal{E}^j + \sqrt{\mathcal{E}^i}\sum_{1\le k\le i}\epsilon^2M(\epsilon+\epsilon^2M)^{k-1}\left(\sum_{0\le l\le j\le i}\bar{\mathcal{E}}^{j,l}\right)^{\frac{1}{2}}.$$

Next, by using $(\psi_1)_{tt}=-\lambda\psi_1$ and the boundedness of ψ_1 and $(\psi_1)_t$, we easily deduce that

$$|J_3| \lesssim \sqrt{\mathcal{E}^i}.$$

Likewise, $\partial_t^{i-j}\psi_1$ in J_4 is a constant multiple of ψ_1 or $(\psi_1)_t$ and hence, by the Cauchy–Schwarz inequality, we obtain

$$|J_4| \lesssim \epsilon\sum_{j=0}^{i}\mathcal{E}^j.$$

To estimate J_5, we observe that $\partial_t^j f/\epsilon$ consists of terms like

$$\epsilon(\partial_t^{j-k}\psi_1+\epsilon\partial_t^{j-k}\varphi)(\partial_t^k\psi_1+\epsilon\partial_t^k\varphi)$$

for $0\le k\le j\le i$. The contribution coming from $\partial_t^{j-k}\psi_1\cdot\partial_t^k\psi_1$, $\partial_t^{j-k}\varphi\cdot\partial_t^k\psi_1$ or $\partial_t^{j-k}\psi_1\cdot\partial_t^k\varphi$ can be bounded by $\epsilon\sqrt{\mathcal{E}^i}+\epsilon^2\sum_{j=0}^i\mathcal{E}^j$. The remaining nonlinear part can be controlled similarly as done for J_2 by using L^∞ bounds and Hardy inequalities. By the boundedness of $\partial_t^{i-j}\psi_1$, it would suffice to estimate

$$\epsilon^3\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{j-k}\varphi\partial_t^k\varphi\,dr.$$

By symmetry of indices, we may assume $0\le k\le\left[\frac{j}{2}\right]$. If k is 0 or 1, then by (4-9) the integral is bounded by $\epsilon^3M(\mathcal{E}^i+\mathcal{E}^{j-k-1})$ with the understanding that $\mathcal{E}^{-1}=\mathcal{E}^0$. Suppose $2\le k\le\left[\frac{j}{2}\right]$. Then we get

$$\begin{aligned}\epsilon^3\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{j-k}\varphi\partial_t^k\varphi\,dr &= \epsilon^3\int_0^R w^{\alpha/2}r^2\partial_t^i\varphi_t w^{(\alpha-k+2)/2}r^2\partial_t^{j-k}\varphi w^{(k-2)/2}\partial_t^k\varphi\,dr\\ &\lesssim \epsilon^3\sup\left|w^{(k-2)/2}\partial_t^k\varphi\right|\sqrt{\mathcal{E}^i}\Bigg(\underbrace{\int_0^R w^{\alpha-k+2}r^4|\partial_t^{j-k}\varphi|^2\,dr}_{J_5^1}\Bigg)^{\frac{1}{2}}.\end{aligned}$$

Due to (4-9), $\sup\left|w^{(k-2)/2}\partial_t^k\varphi\right|\le M$. For J_5^1, we apply the Hardy inequality (3-3) $k-2$ times to obtain

$$J_5^1 \lesssim \sum_{l=0}^{k-2}\int_0^R w^{\alpha-k+2+2(k-2)}r^4|\partial_t^{j-k}\partial_r^l\varphi|^2\,dr \lesssim \sum_{l=0}^{k-2}\bar{\mathcal{E}}^{j-k-1+l,l}.$$

We have derived the estimate of J_5 as

$$|J_5|\lesssim \epsilon\sqrt{\mathcal{E}^i}+\epsilon^2\sum_{j=0}^{i}\mathcal{E}^j+\epsilon^3 M\biggl(\sum_{j=0}^{i}\mathcal{E}^j+\sqrt{\mathcal{E}^i}\biggl(\sum_{0\le l\le j\le i-3}\bar{\mathcal{E}}^{j,l}\biggr)^{\frac{1}{2}}\biggr).$$

We next estimate J_6. First let $j=0$. Then the third line of (5-9) essentially takes the following form

$$w^\alpha r^4(\partial_t^{i-k}(\psi_1)_t+\epsilon\partial_t^{i-k}\varphi_t)(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t),\quad 0\le k\le i.$$

We may assume $0\le k\le\left[\frac{i}{2}\right]$. As before, it is easy to see that the contribution coming from ψ_1 related terms is bounded by $\sqrt{\mathcal{E}^i}+\epsilon\sum_{j=0}^{i}\mathcal{E}^j$. The remaining nonlinear part can be controlled similarly as in the previous case by using (4-9) and Hardy inequality:

$$\epsilon^2\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-k}\varphi_t\partial_t^k\varphi_t\,dr\lesssim \epsilon^2 M\sqrt{\mathcal{E}^i}\biggl(\sum_{l=0}^{k-1}\bar{\mathcal{E}}^{i-k+l,l}\biggr)^{\frac{1}{2}}.$$

Now let $1\le j\le i$. Then the second time-differentiated term $\partial_t^j((1+\zeta)^{-7})$ consists of the terms

$$(\epsilon\partial_t^{j-m}(\psi_1)_t+\epsilon^2\partial_t^{j-m}\varphi_t)(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{m-1},\quad 1\le m\le j.$$

The term $\epsilon\partial_t^{j-m}(\psi_1)_t(\epsilon(\psi_1)_t+\epsilon^2\varphi_t)^{m-1}$ is bounded by $\epsilon(\epsilon+\epsilon^2 M)^{m-1}$ and thus, by the same argument as in the previous case, the corresponding integral in J_6 is bounded by

$$\begin{aligned}
&\epsilon(\epsilon+\epsilon^2M)^{m-1}\int_0^R w^\alpha r^4\partial_t^i\varphi_t(\partial_t^{i-j-k}(\psi_1)_t+\epsilon\partial_t^{i-j-k}\varphi_t)(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t)\,dr\\
&\quad\lesssim \epsilon(\epsilon+\epsilon^2M)^{m-1}\biggl(\sqrt{\mathcal{E}^i}+\epsilon(\mathcal{E}^k+\mathcal{E}^{i-j-k})+\epsilon^2\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-j-k}\varphi_t\partial_t^k\varphi_t\,dr\biggr)\\
&\quad\lesssim \epsilon(\epsilon+\epsilon^2M)^{m-1}\biggl(\sqrt{\mathcal{E}^i}+\epsilon(\mathcal{E}^k+\mathcal{E}^{i-j-k})+\epsilon^2M(\sqrt{\mathcal{E}^i}+\sqrt{\mathcal{E}^{i-j}})\biggl(\sum_{l=0}^{k-1}\bar{\mathcal{E}}^{i-j-k+l,l}\biggr)^{\frac{1}{2}}\biggr),
\end{aligned}$$

where we have expanded $\partial_t^{i-j}\bigl(((\psi_1)_t+\epsilon\varphi_t)^2\bigr)$ and assumed $k\le\left[\frac{1}{2}(i-j)\right]$. The last case is of the form, for $1\le m\le j$ and $k\le i-j$,

$$\epsilon^2\int_0^R w^\alpha r^4\partial_t^i\varphi_t\bigl(\partial_t^{i-j-k}(\psi_1)_t+\epsilon\partial_t^{i-j-k}\varphi_t\bigr)\bigl(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t\bigr)\partial_t^{j-m}\varphi_t\bigl(\epsilon(\psi_1)_t+\epsilon^2\varphi_t\bigr)^{m-1}\,dr,$$

which is bounded by

$$\begin{aligned}\epsilon^2(\epsilon+\epsilon^2 M)^{m-1}&\int_0^R w^\alpha r^4 \partial_t^i\varphi_t(\partial_t^{i-j-k}(\psi_1)_t+\epsilon\partial_t^{i-j-k}\varphi_t)(\partial_t^k(\psi_1)_t+\epsilon\partial_t^k\varphi_t)\partial_t^{j-m}\varphi_t\,dr\\ &\lesssim \epsilon^2(\epsilon+\epsilon^2 M)^{m-1}\bigg(\mathcal{E}^i+\mathcal{E}^{j-m}+\epsilon\underbrace{\int_0^R w^\alpha r^4\partial_t^i\varphi_t(\partial_t^{i-j-k}\varphi_t+\partial_t^k\varphi_t)\partial_t^{j-m}\varphi_t\,dr}_{J_6^1}\\ &\qquad\qquad+\epsilon^2\underbrace{\int_0^R w^\alpha r^4\partial_t^i\varphi_t\partial_t^{i-j-k}\varphi_t\partial_t^k\varphi_t\partial_t^{j-m}\varphi_t\,dr}_{J_6^2}\bigg),\end{aligned}$$

where we have used the boundedness of ψ_1 and $(\psi_1)_t$. The estimation of J_6^1 is similar to previous nonlinear terms. First, if $m=j$ then it is clear that $J_6^1\lesssim M(\mathcal{E}^i+\mathcal{E}^{i-j-k}+\mathcal{E}^k)$. So let $1\le m\le j-1$. If $1\le j\le\left[\frac{i}{2}\right]+1$, take the supremum of $w^{(j-m-1)/2}\partial_t^{j-m}\varphi_t$ and apply the Hardy inequality to deduce that

$$J_6^1\lesssim M\sqrt{\mathcal{E}^i}\bigg(\sum_{l=0}^{j-1}\bar{\mathcal{E}}^{i-j-k+l,l}+\bar{\mathcal{E}}^{k+l,l}\bigg)^{\frac{1}{2}}.$$

If $\left[\frac{i}{2}\right]+2\le j\le i$, then take the supremum of $w^{(i-j-1)/2}(\partial_t^{i-j-k}\varphi_t+\partial_t^k\varphi_t)$ when $j<i$, the supremum of φ_t when $j=i$, and apply the Hardy inequality to obtain $J_6^1\lesssim M\sqrt{\mathcal{E}^i}\big(\sum_{l=0}^{j-1}\bar{\mathcal{E}}^{i-j-k+l,l}+\bar{\mathcal{E}}^{k+l,l}\big)^{1/2}$. By the same argument as before, we deduce that

$$J_6^1\lesssim M\big(\mathcal{E}^i+\mathcal{E}^{j-m}+\sqrt{\mathcal{E}^i}\big(\sum_{l=0}^{i-j-1}\bar{\mathcal{E}}^{j-m+l,l}\big)^{1/2}\big).$$

It now remains to estimate J_6^2. Here, not only j but also k will matter. Let us start with $1\le j\le\left[\frac{i}{2}\right]+1$. Due to the symmetry of indices, we can assume that $k\le\left[\frac{1}{2}(i-j)\right]$. Notice that, if $m=j$ or $k=0$, then the last factor or the third factor is bounded by M and thus this reduces to the case that has been treated before. Let $1\le m\le j-1$ and $1\le k\le\left[\frac{1}{2}(i-j)\right]$. We write J_6^2 as

$$J_6^2=\int_0^R w^{\alpha/2}r^2\partial_t^i\varphi_t w^{(\alpha-k-j+m+2)/2}r^2\partial_t^{i-j-k}\varphi_t w^{(k-1)/2}\partial_t^k\varphi_t w^{(j-m-1)/2}\partial_t^{j-m}\varphi_t\,dr.$$

Hence by (4-9) we first see that

$$J_6^2\lesssim M^2\sqrt{\mathcal{E}^i}\bigg(\int_0^R w^{\alpha-k-j+m+2}r^4|\partial_t^{i-j-k}\varphi_t|^2\,dr\bigg)^{\frac{1}{2}}.$$

By applying the Hardy inequality (3-3) $j+k-2$ times to the last term we obtain

$$J_6^2\lesssim M^2\sqrt{\mathcal{E}^i}\bigg(\sum_{l=0}^{j+k-2}\bar{\mathcal{E}}^{i-j-k+l,l}\bigg)^{\frac{1}{2}}.$$

Now let $\left[\frac{i}{2}\right]+2\le j\le i$. If $j=i$ or $j=i-1$, then $k=0$ or $k=1$, and thus $J_6^2\lesssim M^2(\mathcal{E}^i+\mathcal{E}^{j-m})$. If $k=0$ or $k=i-j$, then this reduces to the previous case. So we assume $\left[\frac{i}{2}\right]+2\le j\le i-2$ and

$1 \le k \le i-j-1$. In this case, we have

$$J_6^2 = \int_0^R w^{\alpha/2} r^2 \partial_t^i \varphi_t w^{(i-j-k-1)/2} \partial_t^{i-j-k} \varphi_t w^{(k-1)/2} \partial_t^k \varphi_t w^{(\alpha-i+j+2)/2} r^2 \partial_t^{j-m} \varphi_t \, dr$$
$$\lesssim M^2 \sqrt{\mathcal{E}^i} \biggl(\sum_{l=0}^{i-j-2} \bar{\mathcal{E}}^{j-m+l,l} \biggr)^{\frac{1}{2}}.$$

We next move onto J_7, which will contribute to the energy. Integration by parts yields

$$J_7 = \gamma \int_0^R \partial_r (r^3 \partial_t^i \varphi_t) w^{1+\alpha} J^{-\gamma-1} \frac{1}{r^2} (r^3 \partial_t^i \varphi)_r \, dr$$
$$= \frac{\gamma}{2} \frac{d}{dt} \int_0^R w^{1+\alpha} J^{-\gamma-1} \frac{1}{r^2} |(r^3 \partial_t^i \varphi)_r|^2 \, dr + \frac{\gamma(\gamma+1)}{2} \int_0^R w^{1+\alpha} J^{-\gamma-2} J_t \frac{1}{r^2} |(r^3 \partial_t^i \varphi)_r|^2 \, dr,$$

where the commutator is bounded by $(\epsilon + \epsilon^2 M) \mathcal{E}^i$.

J_8 satisfies

$$J_8 = -2 \frac{d}{dt} \int_0^R w^\alpha r^4 \Phi(r) |\partial_t^i \varphi|^2 \, dr \quad \text{and} \quad |J_8| \lesssim \mathcal{E}^i.$$

If $\gamma > \frac{4}{3}$, the first expression will be used, so that J_8 can contribute to the energy. If $\gamma \le \frac{4}{3}$, then we will use the estimation, so the contribution of $\mathcal{E}^i$ in the right-hand side of the energy inequality will be of order 1.

Next, for J_9, by distributing the spatial derivative we write it as

$$-\frac{J_9}{\gamma} = \sum_{j=0}^{i-2} c_{3j} \int_0^R \partial_t^i \varphi_t r^3 (w^{1+\alpha})_r \partial_t^{i-1-j} (J^{-\gamma-1}) \frac{1}{r^2} (r^3 \partial_t^j \varphi_t)_r \, dr$$
$$+ \sum_{j=0}^{i-2} c_{3j} \int_0^R \partial_t^i \varphi_t r^3 w^{1+\alpha} \partial_t^{i-1-j} \partial_r (J^{-\gamma-1}) \frac{1}{r^2} (r^3 \partial_t^j \varphi_t)_r \, dr$$
$$+ \sum_{j=0}^{i-2} c_{3j} \int_0^R \partial_t^i \varphi_t r^3 w^{1+\alpha} \partial_t^{i-1-j} (J^{-\gamma-1}) (4 \partial_t^j \partial_r \varphi_t + r \partial_t^j \partial_r^2 \varphi_t) \, dr.$$

We denote the integrals in the above three summations by J_9^1, J_9^2, J_9^3. We start with J_9^1. Notice that $\partial_t^{i-1-j} (J^{-\gamma-1})$ consists of $(\partial_t^{i-j-1-k} J)(J_t)^k$ for $0 \le k \le i-j-2$, where

$$\partial_t^{i-j-1-k} J = 3 \bigl(\epsilon \partial_t^{i-j-1-k} (\psi_1) + \epsilon^2 \partial_t^{i-j-1-k} \varphi \bigr) + r \bigl(\epsilon \partial_t^{i-j-1-k} (\psi_1)_r + \epsilon^2 \partial_t^{i-j-1-k} \varphi_r \bigr). \tag{5-12}$$

Let $j+1 \le \left[\frac{i}{2}\right]$. Then $|w^{j/2} (1/r^2) (r^3 \partial_t^j \varphi_t)_r| \lesssim M$ by (4-9), and $|J_t|^k \lesssim (\epsilon + \epsilon^2 M)^k$. We also recall that $(w^{1+\alpha})_r = -r w^\alpha \Phi(r)$, where $\Phi(r)$ is bounded. Thus

$$|J_9^1| \lesssim (\epsilon + \epsilon^2 M)^k M \sqrt{\mathcal{E}^i} \biggl(\int_0^R w^{\alpha-j} r^4 |\partial_t^{i-j-1-k} J|^2 \, dr \biggr)^{\frac{1}{2}}.$$

From (5-12) we use the regularity of ψ_1 and apply the Hardy inequality to obtain

$$\begin{aligned}\int_0^R w^{\alpha-j} r^4 |\partial_t^{i-j-1-k} J|^2\,dr &\lesssim \epsilon^2+\epsilon^4 \int_0^R w^{\alpha-j} r^4 |\partial_t^{i-j-1-k}\varphi|^2\,dr + \epsilon^4\int_0^R w^{\alpha-j} r^6 |\partial_t^{i-j-1-k}\varphi_r|^2\,dr \\ &\lesssim \epsilon^2+\epsilon^4\sum_{l=0}^{j+1}\bar{\mathcal{E}}^{i-j-1-k+l,l}. \end{aligned}\tag{5-13}$$

Hence we have $|J_9^1|\lesssim \epsilon(\epsilon+\epsilon^2M)^k M\sqrt{\mathcal{E}^i}\big(1+\epsilon\big(\sum_{l=0}^{j+1}\bar{\mathcal{E}}^{i-j-1-k+l,l}\big)^{1/2}\big)$ for $j+1\le\big[\frac{i}{2}\big]$. Now suppose $\big[\frac{i}{2}\big]\le j\le i-2$. Then $|w^{(i-j-2-k)/2}\partial_t^{i-j-1-k}J|\lesssim \epsilon+\epsilon^2M$. Therefore, by further applying the Hardy inequality,

$$|J_9^1|\lesssim(\epsilon+\epsilon^2M)^{k+1}\sqrt{\mathcal{E}^i}\bigg(\int_0^R w^{\alpha-i+j+2+k}\frac{1}{r^2}|(r^3\partial_t^j\varphi_t)_r|^2\,dr\bigg)^{\frac12}\lesssim(\epsilon+\epsilon^2M)^{k+1}\sqrt{\mathcal{E}^i}\bigg(\sum_{l=0}^{i-j-2}\bar{\mathcal{E}}^{j+1+l,l}\bigg)^{\frac12}.$$

We next treat J_9^3. Let $j<\big[\frac12(i-3)\big]$. Then $|w^{(j+2)/2}(4\partial_t^j\partial_r\varphi_t+r\partial_t^j\partial_r^2\varphi_t)|\lesssim M$ by (4-9). Thus

$$|J_9^3|\lesssim(\epsilon+\epsilon^2M)^k M\sqrt{\mathcal{E}^i}\bigg(\int_0^R w^{\alpha-j}r^2|\partial_t^{i-j-1-k}J|^2\,dr\bigg)^{\frac12},$$

where we have used $|J_t|^k\lesssim(\epsilon+\epsilon^2M)^k$. Hence, this case is the same as in the previous case of J_9^1 (see (5-13)) except for the factor r^2 instead of r^4. The weight r^4 is recovered by applying the Hardy inequality (3-1) once. Notice that the Hardy inequality near the boundary is used multiple times in (5-13) and thus we obtain the same result as in J_9^1. Now suppose $\big[\frac12(i-3)\big]\le j\le i-2$. Then $|w^{(i-j-2-k)/2}\partial_t^{i-j-1-k}J|\lesssim\epsilon+\epsilon^2M$. Therefore, by further applying the Hardy inequality,

$$\begin{aligned}|J_9^3|&\lesssim(\epsilon+\epsilon^2M)^{k+1}M\sqrt{\mathcal{E}^i}\bigg(\int_0^R w^{\alpha-i+j+4+k}\big(r^2|\partial_t^j\partial_r\varphi_t|^2+r^4|\partial_t^j\partial_r^2\varphi_t|^2\big)\,dr\bigg)^{\frac12}\\ &\lesssim(\epsilon+\epsilon^2M)^{k+1}M\sqrt{\mathcal{E}^i}\bigg(\sum_{l=0}^{i-j-2}\bar{\mathcal{E}}^{j+2+l,l+1}\bigg)^{\frac12}.\end{aligned}$$

Now J_9^2 can be treated similarly to J_9^3 by considering $j\le\big[\frac12(i-3)\big]$ and $j>\big[\frac12(i-3)\big]$, since the nonlinear structure and number of spatial derivatives involved are essentially the same. We omit the details.

We next move onto J_{10}. As in J_9, we first distribute the spatial derivative to write

$$\begin{aligned}-\frac{J_{10}}{\gamma}=\sum_{j=0}^{i-1}c_{3j}&\int_0^R\partial_t^i\varphi_t r^3(w^{1+\alpha})_r\partial_t^{i-1-j}\bigg(\frac{J^{-\gamma-1}-1}{\epsilon}\bigg)\frac{1}{r^2}(r^3(\partial_t^j\psi_1)_t)_r\,dr\\ &+\sum_{j=0}^{i-1}c_{3j}\int_0^R\partial_t^i\varphi_t r^3 w^{1+\alpha}\partial_t^{i-1-j}\partial_r\bigg(\frac{J^{-\gamma-1}-1}{\epsilon}\bigg)\frac{1}{r^2}(r^3(\partial_t^j\psi_1)_t)_r\,dr\\ &+\sum_{j=0}^{i-1}c_{3j}\int_0^R\partial_t^i\varphi_t r^3 w^{1+\alpha}\partial_t^{i-1-j}\bigg(\frac{J^{-\gamma-1}-1}{\epsilon}\bigg)(4\partial_t^{j+1}(\psi_1)_r+\partial_t^{j+1}(\psi_1)_{rr})\,dr.\end{aligned}$$

We denote these summands by J_{10}^1, J_{10}^2 and J_{10}^3. Before we discuss further, we remark that, since $\partial_t^{j+1}\psi_1$ is a constant multiple of ψ_1 or $(\psi_1)_t$, the last factor in the integral doesn't lose derivatives at all and it is just a nice function with a desirable regularity in our weighted spaces. We will treat J_{10}^1 and J_{10}^3. Notice that $r^3(w^{1+\alpha})_r \lesssim r^4 w^\alpha$. We first consider $j=i-1$. Then, by recalling $|(J^{-\gamma-1}-1)/\epsilon| \lesssim 1+\epsilon M$ (see the estimation of I_{10}^2 in the previous lemma) and the regularity of ψ_1, we deduce that the integral is bounded by $(1+\epsilon M)\sqrt{\mathcal{E}^i}$. The same argument yields the same bound for the case $j=i-1$ of J_{10}^3. Now let $0\le j\le i-2$. Then $\partial_t^{i-1-j}((J^{-\gamma-1}-1)/\epsilon)$ consists of $(1/\epsilon)(\partial_t^{i-j-1-k}J)(J_t)^k$ for $0\le k\le i-j-2$, where $\partial_t^{i-j-1-k}J$ is given in (5-12). The estimates of J_{10}^1 and J_{10}^3 can be obtained in a similar way as in the previous case. The differences are the presence of $1/\epsilon$ and that the last factor in the integral is a given function in this case, which only makes the argument easier. As can be seen in (5-12) and (5-13), $\partial_t^{i-j-1-k}J/\epsilon$ is bounded by the total energy and the result will be $1/\epsilon$ times the corresponding estimates of J_9^1 and J_9^3. By the same argument, we can obtain the estimate of J_{10}^2 as $1/\epsilon$ times the corresponding estimates of J_9^2. In all cases, the leading order of the bounds is $\sqrt{\mathcal{E}^i}$, while the leading order for J_9 is $\epsilon(M\sqrt{\mathcal{E}^i}+\mathcal{E}^i)$.

Lastly, J_{11} can be estimated in the same way as in the case $j=i$ in J_5. The difference is the order of ϵ:

$$|J_{11}| \lesssim \sqrt{\mathcal{E}^i}+\epsilon\sum_{j=0}^{i}\mathcal{E}^j+\epsilon^2 M\biggl(\sum_{j=0}^{i}\mathcal{E}^j+\sqrt{\mathcal{E}^i}\biggl(\sum_{0\le l\le j\le i-3}\bar{\mathcal{E}}^{j,l}\biggr)^{\frac{1}{2}}\biggr).$$

This finishes the proof of the lemma. □

6. Elliptic estimates

Proposition 6.1. *Suppose that* (φ,φ_t) *satisfy* (2-11) *for* $0\le t\le T$ *and the corresponding total energy* $\bar{\mathcal{E}}$ *is bounded. Moreover, we assume* (4-9). *Then* $\bar{\mathcal{E}}$ *enjoys the estimates*

$$\bar{\mathcal{E}} \lesssim 1+(1+\epsilon^4M^2)\mathcal{E}+\epsilon^2(M^2+\bar{\mathcal{E}}+\epsilon^2M^2\bar{\mathcal{E}}) \tag{6-1}$$

for all sufficiently small $\epsilon>0$.

Notice that (6-1) is trivially obtained for $\bar{\mathcal{E}}^{j,0}$ for $0\le j\le[\alpha]+4$ because $\bar{\mathcal{E}}^{j,0}$ and $\mathcal{E}^j$ are equivalent. Moreover, due to (2-13), it suffices to estimate $\bar{\mathcal{E}}_r^{j,k}$ for $1\le k\le j\le[\alpha]+4$. We start with the simplest case $j=1$ and $k=1$ and then move onto the general case $j\ge 2$.

Lemma 6.2 ($\bar{\mathcal{E}}^{1,1}$). *Suppose that* (φ,φ_t) *satisfy* (2-11) *for* $0\le t\le T$ *and the corresponding total instant energy* $\mathcal{E}$ *is bounded. Moreover, we assume* (4-9). *Then there exists a constant* $C>0$ *such that*

$$\bar{\mathcal{E}}_r^{1,1} \lesssim 1+(1+\epsilon^4M^2)(\mathcal{E}^0+\mathcal{E}^1)+\epsilon^2(M^2+(1+\epsilon^2M^2)\bar{\mathcal{E}}_r^{1,1}).$$

Proof. In this case, because of (2-13), we only need to show that $\int_0^R w^{2+\alpha}r^4|\varphi_{rr}|^2\,dr$ is bounded by the temporal instant energy. By using (2-1) and (4-3), we rewrite (2-11) in the form

$$\begin{aligned}\gamma\bigl(w^{1+\alpha}r^4\varphi_r\bigr)_r = \frac{w^\alpha r^4\varphi_{tt}}{(1+\zeta)^4}+4\lambda w^\alpha r^4\psi_1^2+4\lambda\epsilon w^\alpha r^4\psi_1\varphi-\lambda w^\alpha r^4\psi_1\frac{f}{\epsilon}+w^\alpha r^4\Phi(r)\frac{f}{\epsilon^2}\\ -\frac{2w^\alpha r^4|(\psi_1)_t+\epsilon\varphi_t|^2}{(1+\zeta)^7}+(3\gamma-4)w^\alpha r^4\Phi(r)\varphi+r^3\biggl(w^{1+\alpha}\frac{h}{\epsilon^2}\biggr)_r.\end{aligned} \tag{6-2}$$

We will exploit the elliptic structure of the term in the left-hand side of (6-2). Square both sides of (6-2), divide them by $w^\alpha r^4$ and integrate the result over $(0, R)$ to get

$$\begin{aligned}\int_0^R \frac{\gamma^2}{w^\alpha r^4}|(w^{1+\alpha}r^4\varphi_r)_r|^2\,dr \\ \lesssim \int_0^R \frac{w^\alpha r^4|\varphi_{tt}|^2}{(1+\zeta)^8}\,dr + \int_0^R w^\alpha r^4\psi_1^4\,dr + \epsilon^2\int_0^R w^\alpha r^4\psi_1^2|\varphi|^2\,dr \\ + \int_0^R w^\alpha r^4\psi_1^2\left|\frac{f}{\epsilon}\right|^2 dr + \int_0^R w^\alpha r^4\left|\Phi(r)\frac{f}{\epsilon^2}\right|^2 dr + \int_0^R \frac{w^\alpha r^4|(\psi_1)_t+\epsilon\varphi_t|^4}{(1+\zeta)^{14}}\,dr \\ + \int_0^R w^\alpha r^4|\Phi(r)\varphi|^2\,dr + \int_0^R \frac{1}{w^\alpha r^4}\left|r^3\left(w^{1+\alpha}\frac{h}{\epsilon^2}\right)_r\right|^2 dr. \end{aligned} \tag{6-3}$$

We denote the integral in the left-hand side by I and each integral in the right-hand side by I_k for $1 \le k \le 8$. It is clear that

$$I_1 \lesssim \mathcal{E}^1, \quad I_2 \lesssim 1, \quad I_3 \lesssim \epsilon^2\mathcal{E}^0. \tag{6-4}$$

For I_4 and I_5, we recall that $f = O(|\epsilon\psi_1+\epsilon^2\varphi|^2)$. Then, by using the boundedness of ψ_1 and Φ as well as (4-9), we have

$$I_4 \lesssim \epsilon^2(1+\epsilon^4 M^2\mathcal{E}^0), \quad I_5 \lesssim 1+\epsilon^4 M^2\mathcal{E}^0.$$

Similarly, we obtain

$$I_6 \lesssim 1+\epsilon^4 M^2\mathcal{E}^1, \quad I_7 \lesssim \mathcal{E}^0.$$

The last term involves the full derivatives and it needs to be estimated carefully. Recall that

$$h = h\left(\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\right) = O\left(\left|\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r\right|^2\right),$$
$$\frac{1}{r^2}(r^3(\epsilon\psi_1+\epsilon^2\varphi))_r = 3(\epsilon\psi_1+\epsilon^2\varphi) + r(\epsilon(\psi_1)_r+\epsilon^2\varphi_r).$$

We then see that

$$r^3\left(w^{1+\alpha}\frac{h}{\epsilon^2}\right)_r = r^3 w^{1+\alpha}\frac{h^{(1)}}{\epsilon}\cdot\big(4((\psi_1)_r+\epsilon\varphi_r) + r((\psi_1)_{rr}+\epsilon\varphi_{rr})\big) + r^3(w^{1+\alpha})_r\frac{h}{\epsilon^2},$$

where $h^{(1)}$ means the first derivative of h with respect to the argument. By using the notation Φ given in (2-10), we write $(w^{1+\alpha})_r = -rw^\alpha\Phi(r)$ and, hence, we see that I_8 is bounded by

$$\begin{aligned} I_8 \lesssim \int_0^R w^{2+\alpha}r^2\left|\frac{h^{(1)}}{\epsilon}\right|^2|(\psi_1)_r+\epsilon\varphi_r|^2\,dr + \int_0^R w^{2+\alpha}r^4\left|\frac{h^{(1)}}{\epsilon}\right|^2|(\psi_1)_{rr}+\epsilon\varphi_{rr}|^2\,dr \\ + \int_0^R w^\alpha r^4|\Phi(r)|^2\left|\frac{h}{\epsilon^2}\right|^2 dr = I_8^1+I_8^2+I_8^3. \end{aligned}$$

Notice that $|h^{(1)}/\epsilon| \lesssim 1+\epsilon M$ and $|h/\epsilon^2| \lesssim 1+\epsilon^2 M^2$. It is easy to see that

$$I_8^2 \lesssim (1+\epsilon M)^2(1+\epsilon^2\bar{\mathcal{E}}_r^{1,1}).$$

For I_8^1 and I_8^3, we further employ the Hardy inequalities near the origin (3-1) and near the boundary (3-3) respectively to deduce that

$$I_8^1 + I_8^3 \lesssim (1+\epsilon^2 M^2)(1+\epsilon^2(\mathcal{E}^0 + \bar{\mathcal{E}}_r^{1,1})).$$

We now turn our attention to the integral I in the left-hand side of (6-3). First notice that

$$(w^{1+\alpha}r^4\varphi_r)_r = w^{1+\alpha}r^4\varphi_{rr} + 4w^{1+\alpha}r^3\varphi_r + (w^{1+\alpha})_r r^4\varphi_r = w^{1+\alpha}r^4\varphi_{rr} + 4w^{1+\alpha}r^3\varphi_r - w^\alpha r^5\Phi(r)\varphi_r.$$

Then I reads as

$$I = \gamma^2 \int_0^R w^\alpha r^4 \left| w\varphi_{rr} + \frac{4w\varphi_r}{r} - r\Phi(r)\varphi_r \right|^2 dr.$$

By expanding out terms, we see that

$$\begin{aligned}\frac{I}{\gamma^2} &= \int_0^R w^{2+\alpha}r^4|\varphi_{rr}|^2\,dr + 16\int_0^R w^{2+\alpha}r^2|\varphi_r|^2\,dr + \int_0^R w^\alpha r^6|\Phi(r)|^2|\varphi_r|^2\,dr \\ &\quad + \underbrace{8\int_0^R w^{2+\alpha}r^3\varphi_{rr}\varphi_r\,dr}_{I^1} \underbrace{-2\int_0^R w^{1+\alpha}r^5\Phi(r)\varphi_{rr}\varphi_r\,dr}_{I^2} - 8\int_0^R w^{1+\alpha}r^4\Phi(r)|\varphi_r|^2\,dr.\end{aligned}$$

For I^1 and I^2, we integrate by parts to get

$$\begin{aligned} I^1 &= -4\int_0^R (w^{2+\alpha})_r r^3|\varphi_r|^2\,dr - 12\int_0^R w^{2+\alpha}r^2|\varphi_r|^2\,dr \\ &= 4\frac{2+\alpha}{1+\alpha}\int_0^R w^{1+\alpha}r^4\Phi(r)|\varphi_r|^2\,dr - 12\int_0^R w^{2+\alpha}r^2|\varphi_r|^2\,dr, \\ I^2 &= -\int_0^R w^\alpha r^6|\Phi(r)|^2|\varphi_r|^2 + 5\int_0^R w^{1+\alpha}r^4\Phi(r)|\varphi_r|^2\,dr + \int_0^R w^{1+\alpha}r^5\Phi'(r)|\varphi_r|^2\,dr. \end{aligned}$$

Hence we obtain

$$\begin{aligned}&\int_0^R w^{2+\alpha}r^4|\varphi_{rr}|^2\,dr + 4\int_0^R w^{2+\alpha}r^2|\varphi_r|^2\,dr \\ &\qquad = \frac{I}{\gamma^2} + \underbrace{3\int_0^R w^{1+\alpha}r^4\Phi(r)|\varphi_r|^2\,dr - 4\frac{2+\alpha}{1+\alpha}\int_0^R w^{1+\alpha}r^4\Phi(r)|\varphi_r|^2\,dr - \int_0^R w^{1+\alpha}r^5\Phi'(r)|\varphi_r|^2\,dr}_{\lesssim \mathcal{E}^0}.\end{aligned}$$

It is clear that the last three terms in the right-hand side are bounded by the zeroth-order energy $\mathcal{E}^0$. Combining all the estimates, we deduce the result. This finishes the proof for the case of $j = 1$ and $k = 1$. □

We now turn into the cases $[\alpha]+4 \geq j \geq 2$. As in the case of $j = 1$, we will directly use the equation and take advantage of the elliptic estimates. What is subtle and interesting here is to capture the correct behavior of solutions in the normal direction ∂_r near the boundary.

Lemma 6.3 ($\bar{\mathcal{E}}_r^{j,k}$ for $1 \le k \le j,\ 2 \le j$). *Suppose that* (φ, φ_t) *satisfy* (2-11) *for* $0 \le t \le T$ *and the corresponding total instant energy* $\mathcal{E}$ *is bounded. Moreover, we assume* (4-9). *Then there exists a constant* $C > 0$ *such that*

$$\bar{\mathcal{E}}_r^{j,k} \lesssim (1+\epsilon^4 M^2)\sum_{l=0}^{j} \mathcal{E}^l + \bar{\mathcal{E}}_r^{j-1,k-1} + \epsilon^2\biggl(M^2 + (1+\epsilon^2 M^2)\sum_{m=1}^{j}\sum_{l=1}^{m} \bar{\mathcal{E}}_r^{m,l}\biggr).$$

Proof. Notice that because of (2-13), it suffices to show that each spatial energy term $\mathcal{E}_r^{j,k}$ for $1 \le k \le j$ satisfies the inequality. We will present the detail for $j = 2$; other cases follow by induction on j, k. When $k = 1$, the spatial energy term $\bar{\mathcal{E}}_r^{2,1}$ contains one temporal derivative and two spatial derivatives. The time differentiation of (6-2) is the place to start. Notice that the time derivative does not affect the weights at all since w and r do not change with time. Therefore, following the same procedure for $\bar{\mathcal{E}}_r^{1,1}$ in the previous lemma, we can deduce that

$$\bar{\mathcal{E}}_r^{2,1} \lesssim 1 + (1+\epsilon^4 M^2)(\mathcal{E}^0 + \mathcal{E}^1 + \mathcal{E}^2) + (1+\epsilon^2 M^2)(1 + \epsilon^2(\bar{\mathcal{E}}_r^{1,1} + \bar{\mathcal{E}}_r^{2,1})).$$

To deal with $\bar{\mathcal{E}}_r^{2,2}$, which contains three spatial derivatives, we will first derive the equation for φ_{rrr} from (6-2). By following the idea in [Jang 2014], first divide both sides of (6-2) by $r^3 w^\alpha$:

$$\begin{aligned}
&\gamma(wr\varphi_{rr} + (1+\alpha)w_r r\varphi_r + 4w\varphi_r)\\
&\quad= \frac{r\varphi_{tt}}{(1+\zeta)^4} + 4\lambda r\psi_1^2 + 4\lambda\epsilon r\psi_1\varphi - \lambda r\psi_1\frac{f}{\epsilon} + r\Phi(r)\frac{f}{\epsilon^2} - \frac{2r|(\psi_1)_t + \epsilon\varphi_t|^2}{(1+\zeta)^7} + (3\gamma-4)r\Phi(r)\varphi\\
&\qquad + w\frac{h^{(1)}}{\epsilon}\bigl(4((\psi_1)_r + \epsilon\varphi_r) + r((\psi_1)_{rr} + \epsilon\varphi_{rr})\bigr) + (1+\alpha)w_r\frac{h}{\epsilon^2}.
\end{aligned}$$

Then we take ∂_r of both sides of the above equation and move the terms involving φ_r into the right-hand side to get

$$\begin{aligned}
&\gamma(wr\varphi_{rrr} + (2+\alpha)w_r r\varphi_{rr} + 5w\varphi_{rr})\\
&\quad= -\gamma((5+\alpha)w_r\varphi_r + (1+\alpha)w_{rr}r\varphi_r)\\
&\qquad + \biggl(\frac{r\varphi_{tt}}{(1+\zeta)^4}\biggr)_r + 4\lambda(r\psi_1^2)_r + 4\lambda\epsilon(r\psi_1\varphi)_r - \lambda\biggl(r\psi_1\frac{f}{\epsilon}\biggr)_r + \biggl(r\Phi(r)\frac{f}{\epsilon^2}\biggr)_r\\
&\qquad - \biggl(\frac{2r|(\psi_1)_t + \epsilon\varphi_t|^2}{(1+\zeta)^7}\biggr)_r + (3\gamma-4)(r\Phi(r)\varphi)_r\\
&\qquad + \biggl(w\frac{h^{(1)}}{\epsilon}\bigl(4((\psi_1)_r + \epsilon\varphi_r) + r((\psi_1)_{rr} + \epsilon\varphi_{rr})\bigr) + (1+\alpha)w_r\frac{h}{\epsilon^2}\biggr)_r.
\end{aligned} \tag{6-5}$$

As in the previous lemma, we square both sides of (6-5), multiply by $w^{1+\alpha}r^2$ — here the choice of the multiplier $w^{1+\alpha}$ has been inspired by the analysis carried out in [Jang and Masmoudi 2015] — and integrate it over $(0, R)$ to obtain an integral inequality similar to (6-3). We denote the integral in the left-hand side by I and the integrals in the right-hand side by I_k, $1\le k\le 9$. For I_1 we apply the Hardy inequality near the origin (3-1) to overcome stronger weights near the origin:

$$I_1 \lesssim \int_0^R w^{1+\alpha} r^2(|w_r|^2+|rw_{rr}|^2)|\varphi_r|^2\,dr \lesssim \mathcal{E}^0+\bar{\mathcal{E}}_r^{1,1}.$$

For I_2, we obtain

$$\begin{aligned}I_2 &\lesssim \int_0^R \frac{w^{1+\alpha}r^4|\varphi_{ttr}|^2}{(1+\zeta)^8}\,dr+\int_0^R \frac{w^{1+\alpha}r^2|\varphi_{tt}|^2}{(1+\zeta)^8}\,dr+\int_0^R \frac{w^{1+\alpha}r^4|\varphi_{tt}(\epsilon(\psi_1)_r+\epsilon^2\varphi_r)|^2}{(1+\zeta)^{14}}\,dr\\ &\lesssim \bar{\mathcal{E}}_r^{2,0}+(\mathcal{E}^1+\bar{\mathcal{E}}_r^{2,0})+(\epsilon^2+\epsilon^4M^2)\mathcal{E}^1,\end{aligned}$$

where we have applied the Hardy inequality (3-1) to the second term. Next, by the regularity of ψ_1 and the Hardy inequality (3-1), we observe that

$$I_3+I_4\lesssim 1+\epsilon^2\mathcal{E}^0.$$

For I_5 and I_6, we note that $f=f(\epsilon\psi_1+\epsilon^2\varphi)=O(|\epsilon\psi_1+\epsilon^2\varphi|^2)$ and $f_r=f^{(1)}\cdot(\epsilon(\psi_1)_r+\epsilon^2\varphi_r)$. Hence

$$I_5+I_6\lesssim 1+\epsilon^4M^2\mathcal{E}^0.$$

Similarly, by using the Hardy inequality (3-1) and (4-9) we have

$$I_7\lesssim 1+\epsilon^4M^2(\mathcal{E}^0+\mathcal{E}^1).$$

Since $(r\Phi(r)\varphi)_r=\Phi(r)\varphi+r\Phi(r)'\varphi+r\Phi(r)\varphi_r$, by (3-1) for the first term again we see that

$$I_8\lesssim \mathcal{E}^0.$$

For I_9, we note that the last line of (6-5) can be written as follows:

$$\begin{aligned}&w\frac{h^{(1)}}{\epsilon}\big(5((\psi_1)_{rr}+\epsilon\varphi_{rr})+r((\psi_1)_{rrr}+\epsilon\varphi_{rrr})\big)+wh^{(2)}\big(4((\psi_1)_r+\epsilon\varphi_r)+r((\psi_1)_{rr}+\epsilon\varphi_{rr})\big)^2\\ &\qquad+(2+\alpha)w_r\frac{h^{(1)}}{\epsilon}\big(4((\psi_1)_r+\epsilon\varphi_r)+r((\psi_1)_{rr}+\epsilon\varphi_{rr})\big)+(1+\alpha)w_{rr}\frac{h}{\epsilon^2},\end{aligned}$$

where $h^{(2)}$ means the second derivative with respect to the argument. Thus I_9 includes φ_{rrr}, φ_{rr}, φ_r with different weights and it can be treated in a similar way as we did for I_8 of (6-3) in the previous lemma. We expand it out and apply the Hardy inequalities both near the origin (3-1) and near the boundary (3-3) to deduce that

$$I_9\lesssim (1+\epsilon M)^2(1+\epsilon^2(\mathcal{E}^0+\bar{\mathcal{E}}_r^{1,1}+\bar{\mathcal{E}}_r^{2,2})). \tag{6-6}$$

What follows now is the elliptic estimate for I coming from the first term in (6-5), which will give rise to the term $\bar{\mathcal{E}}_r^{2,2}$:

$$\begin{aligned} I &= \int_0^R w^{1+\alpha} r^2 |wr\varphi_{rrr} + (2+\alpha) w_r r \varphi_{rr} + 5w\varphi_{rr}|^2 \, dr \\ &= \int_0^R w^{3+\alpha} r^4 |\varphi_{rrr}|^2 \, dr + (2+\alpha)^2 \int_0^R w^{1+\alpha} r^4 |w_r|^2 |\varphi_{rr}|^2 \, dr + 25 \int_0^R w^{3+\alpha} r^2 |\varphi_{rr}|^2 \, dr \\ &\qquad + \underbrace{2(2+\alpha) \int_0^R w^{2+\alpha} r^4 w_r \varphi_{rrr} \varphi_{rr} \, dr}_{I^1} + \underbrace{10 \int_0^R w^{3+\alpha} r^3 \varphi_{rrr} \varphi_{rr} \, dr}_{I^2} \\ &\qquad + 10(2+\alpha) \int_0^R w^{2+\alpha} r^3 w_r |\varphi_{rr}|^2 \, dr. \end{aligned}$$

For I^1 and I^2, we integrate by parts to get

$$\begin{aligned} \frac{I^1}{2+\alpha} &= -\int_0^R (w^{2+\alpha})_r r^4 w_r |\varphi_{rr}|^2 \, dr - 4 \int_0^R w^{2+\alpha} r^3 w_r |\varphi_{rr}|^2 \, dr - \int_0^R w^{2+\alpha} r^4 w_{rr} |\varphi_{rr}|^2 \, dr \\ &= -(2+\alpha) \int_0^R w^{1+\alpha} r^4 |w_r|^2 |\varphi_{rr}|^2 \, dr - 4 \int_0^R w^{2+\alpha} r^3 w_r |\varphi_{rr}|^2 \, dr - \int_0^R w^{2+\alpha} r^4 w_{rr} |\varphi_{rr}|^2 \, dr, \\ I^2 &= -5(3+\alpha) \int_0^R w^{2+\alpha} w_r r^3 |\varphi_{rr}|^2 \, dr - 15 \int_0^R w^{3+\alpha} r^2 |\varphi_{rr}|^2 \, dr. \end{aligned}$$

Thus we obtain

$$\begin{aligned} \int_0^R w^{3+\alpha} r^4 |\varphi_{rrr}|^2 \, dr + 10 \int_0^R w^{3+\alpha} r^2 |\varphi_{rr}|^2 \, dr \\ = I - (\alpha - 3) \int_0^R w^{2+\alpha} w_r r^3 |\varphi_{rr}|^2 \, dr + (2+\alpha) \int_0^R w^{2+\alpha} r^4 w_{rr} |\varphi_{rr}|^2 \, dr. \end{aligned}$$

By noting $w_r = -r\Phi(r)/(1+\alpha)$, we see that the last two terms are bounded by $\bar{\mathcal{E}}_r^{1,1}$. By combining with all other estimates, we deduce that

$$\bar{\mathcal{E}}_r^{2,2} \lesssim 1 + (1+\epsilon^4 M^2)(\mathcal{E}^0 + \mathcal{E}^1) + \mathcal{E}^2 + \bar{\mathcal{E}}_r^{1,1} + (1+\epsilon^2 M^2)(1 + \epsilon^2(\mathcal{E}^0 + \bar{\mathcal{E}}_r^{1,1} + \bar{\mathcal{E}}_r^{2,2})).$$

By the previous lemma, the desired result follows and this finishes the proof of the case $j = 2$. Other cases can be done inductively: take ∂_r derivatives of (6-5), square it, multiply by appropriate weights depending on the number of spatial derivatives, and exploit the Hardy inequalities and the elliptic estimates. The procedure and the estimates are similar to the previous cases and we omit the details. □

The inequality (6-1) in Proposition 6.1 now follows from Lemmas 6.2 and 6.3 by considering a suitable linear combination of $\bar{\mathcal{E}}^{j,k}$ to absorb $\bar{\mathcal{E}}^{j-1,k-1}$ and $\epsilon^2 \sum_{m=1}^{j} \sum_{l=1}^{m} \bar{\mathcal{E}}_r^{m,l}$ into the left-hand side.

7. Proof of Theorem 2.4

Since $M \lesssim \bar{\mathcal{E}}^{1/2}$, (6-1) yields

$$\bar{\mathcal{E}} \lesssim 1 + \mathcal{E} + \epsilon^2 \bar{\mathcal{E}} + \epsilon^4 \bar{\mathcal{E}}^2.$$

Therefore, for all sufficiently small $\epsilon > 0$, we deduce that the total energy is bounded by the total temporal energy:

$$\bar{\mathcal{E}} \lesssim 1 + \mathcal{E}.$$

Now from the energy inequality (5-1) in Proposition 5.1, we obtain

$$\frac{d}{dt}\sqrt{\mathcal{E}} \lesssim 1 + (1-a(\gamma))\sqrt{\mathcal{E}} + (\epsilon + \epsilon^2 M)(\sqrt{\mathcal{E}} + \sqrt{\bar{\mathcal{E}}}) \lesssim 1 + (1-a(\gamma))\sqrt{\mathcal{E}} + \epsilon\sqrt{\mathcal{E}} + (\epsilon\sqrt{\mathcal{E}})^2. \tag{7-1}$$

First let $\gamma > \frac{4}{3}$, in which case $a(\gamma) = 1$. So the differential inequality becomes

$$\frac{d}{dt}\sqrt{\mathcal{E}} \lesssim 1 + \epsilon\sqrt{\mathcal{E}} + (\epsilon\sqrt{\mathcal{E}})^2,$$

which in turn gives rise to

$$\frac{d}{dt}(\epsilon\sqrt{\mathcal{E}} + 1) \lesssim \epsilon(\epsilon\sqrt{\mathcal{E}} + 1)^2.$$

Therefore, by solving this differential inequality, we deduce that

$$\sqrt{\mathcal{E}}(t) \lesssim \frac{\sqrt{\mathcal{E}(0)} + (\epsilon\sqrt{\mathcal{E}(0)} + 1)t}{1 - \epsilon(\epsilon\sqrt{\mathcal{E}(0)} + 1)t}.$$

Hence, in the case of $\gamma > \frac{4}{3}$, we conclude that $\sup_{0 \le t \le T} \sqrt{\mathcal{E}}(t)$ is bounded for all sufficiently small $\epsilon \le \epsilon_0$, where $\epsilon_0 = O(1/T)$.

Next let $\gamma \le \frac{4}{3}$. Then we need to solve

$$\frac{d}{dt}\sqrt{\mathcal{E}} \lesssim 1 + \sqrt{\mathcal{E}} + \epsilon^2(\sqrt{\mathcal{E}})^2.$$

Equivalently,

$$\frac{d}{dt}(\epsilon^2\sqrt{\mathcal{E}} + 1) \lesssim (\epsilon^2\sqrt{\mathcal{E}} + 1)^2 + \epsilon^2 - 1.$$

Let $k = \sqrt{1-\epsilon^2}$. Then

$$\left(\frac{1}{\epsilon^2\sqrt{\mathcal{E}} + 1 - k} - \frac{1}{\epsilon^2\sqrt{\mathcal{E}} + 1 + k}\right)\frac{d}{dt}(\epsilon^2\sqrt{\mathcal{E}} + 1) \lesssim 2k.$$

Thus

$$\sqrt{\mathcal{E}}(t) \lesssim \frac{\sqrt{\mathcal{E}}(0)((1+k)^2 e^{2kt} - \epsilon^2) + (1+k)(e^{2kt} - 1)}{(1+k)\big(1 + k - (1-k)e^{2kt} - \epsilon^2\sqrt{\mathcal{E}}(0)(e^{2kt} - 1)\big)}.$$

Notice that $1+k = 1+\sqrt{1-\epsilon^2} = O(1)$ and $1-k = \epsilon^2/(1+\sqrt{1-\epsilon^2}) = O(\epsilon^2)$. Therefore, we conclude, for $\gamma \le \frac{4}{3}$, that $\sup_{0 \le t \le T} \sqrt{\mathcal{E}}(t)$ is bounded for all sufficiently small $\epsilon \le \epsilon_1$, where $\epsilon_1 = O(1/e^{\kappa T})$ for some $\kappa > 0$.

Remark 7.1. If we fix a small $\epsilon > 0$ in the ansatz (2-9) instead of fixing a time T, then the above results would imply that (2-9) can be justified up to $t \le T = O(1/\epsilon)$ for $\gamma > \frac{4}{3}$ and $t \le T = O(|\ln \epsilon|)$ for $\gamma \le \frac{4}{3}$.

Acknowledgements

The author would like to thank Professor Tetu Makino for bringing her attention to this problem and stimulating discussion, and thank the referees for their helpful comments and suggestions, which have improved the presentation of the paper. She also acknowledges the support and hospitality of the Korea Institute for Advanced Study, where part of this work was done.

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Received 18 Sep 2015. Revised 17 Feb 2016. Accepted 30 Mar 2016.

JUHI JANG: juhijang@usc.edu
Department of Mathematics, University of Southern California, Los Angeles, CA 90089, United States

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1079

SHARP WEIGHTED NORM ESTIMATES BEYOND CALDERÓN–ZYGMUND THEORY

FRÉDÉRIC BERNICOT, DOROTHEE FREY AND STEFANIE PETERMICHL

We dominate nonintegral singular operators by adapted sparse operators and derive optimal norm estimates in weighted spaces. Our assumptions on the operators are minimal and our result applies to an array of situations, whose prototypes are Riesz transforms or multipliers, or paraproducts associated with a second-order elliptic operator. It also applies to such operators whose unweighted continuity is restricted to Lebesgue spaces with certain ranges of exponents (p_0, q_0) with $1 \leq p_0 < 2 < q_0 \leq \infty$. The norm estimates obtained are powers α of the characteristic used by Auscher and Martell. The critical exponent in this case is $\mathfrak{p} = 1 + p_0/q_0'$. We prove $\alpha = 1/(p - p_0)$ when $p_0 < p \leq \mathfrak{p}$ and $\alpha = (q_0 - 1)/(q_0 - p)$ when $\mathfrak{p} \leq p < q_0$. In particular, we are able to obtain the sharp A_2 estimates for nonintegral singular operators which do not fit into the class of Calderón–Zygmund operators. These results are new even in Euclidean space and are the first ones for operators whose kernel does not satisfy any regularity estimate.

1. Introduction

In the last ten years, it has been of great interest to obtain optimal operator norm estimates in Lebesgue spaces endowed with Muckenhoupt weights. One asks for the growth of the norm of certain operators, such as the Hilbert transform or the Hardy–Littlewood maximal function, with respect to a characteristic assigned to the weight. Originally, the main motivation for sharp estimates of this type came from certain important applications to partial differential equations. See for example Fefferman, Kenig and Pipher [Fefferman et al. 1991] and Astala, Iwaniec and Saksman [Astala et al. 2001]. Indeed, a long-standing regularity problem has been solved through the optimal weighted norm estimate of the Beurling–Ahlfors operator, a classical Calderón–Zygmund operator; see [Petermichl and Volberg 2002]. Since then, the area has been developing rapidly. Advances have greatly improved conceptual understanding of classical objects such as Calderón–Zygmund operators. The latter are now understood in several different ways, one of them being through pointwise control by so-called sparse operators; see, most recently, [Lacey 2015; Lerner and Nazarov 2015]. We bring this circle of ideas to the wide range of nonintegral singular operators, such as those considered in [Auscher and Martell 2007a]. Under minimal assumptions, we now demonstrate control by well-chosen sparse operators and derive optimal norm estimates in weighted spaces.

From a historic standpoint, Hunt, Muckenhoupt and Wheeden [Hunt et al. 1973] proved that the Hilbert transform is bounded on L^2_ω if and only if the weight ω satisfies the so-called A_2 condition. Then the

This project is partly supported by ANR projects AFoMEN no. 2011-JS01-001-01 and HAB no. ANR-12-BS01-0013, as well as the ERC project FAnFArE no. 637510. Petermichl is a member of IUF.
MSC2010: 42B20, 58J35.
Keywords: singular operators, weights.

extension for $p \in (1,\infty)$ of the class A_p for weights was made legitimate by the characterization of the Hardy–Littlewood maximal operator on L^p_ω. These classes, as well as the "dual classes" RH_q (describing a reverse Hölder property), originally in Euclidean space, are only defined in terms of volume of balls, so this entire theory has been extended to the doubling framework. Calderón–Zygmund operators have been proved to be bounded on L^p_ω if $\omega \in A_p$. More recently, the so-called A_p conjecture (which is now solved) was about the sharp dependence of this operator norm with respect to the A_p characteristic of the weight. This conjecture was solved by Petermichl and Volberg [2002] for the Beurling–Ahlfors transform, by Petermichl [2007; 2008] for the Hilbert and Riesz transforms (see also the alternative proof by Lacey, Petermichl and Reguera [Lacey et al. 2010]) and by Hytönen [2012] for arbitrary Calderón–Zygmund operators. The idea of dyadic shift [Petermichl 2000] and the seminal articles on two-weight questions of dyadic operators by Nazarov, Treil and Volberg [Nazarov et al. 1999; 2008] were very influential in this area at that point. While [Nazarov et al. 1999] influenced earlier proofs, [Nazarov et al. 2008] was important for later proofs. Recently Lerner [2010; 2013a; 2013b] has obtained an alternate proof of this result, by exploiting the notion of *local mean oscillation* rather than dyadic shift in order to control the norm of a Calderón–Zygmund operator by the norm of some specific operators, called *sparse operators*. Most recently, Lacey [2015] and Lerner and Nazarov [2015] gave another proof, which gets around the use of local oscillation through pointwise control.

Simultaneously, in recent years people were also interested in weighted estimates for nonintegral singular operators in a space of homogeneous type. Even on Euclidean space, Riesz transforms $\nabla L^{-1/2}$ may be considered in several situations where we do not have pointwise regularity estimates of an integral kernel, for example $L = -\operatorname{div}(A\nabla)$ with bounded coefficients A, or $L = -\Delta + V$ with some potential V. The situation is even more difficult if we are looking at Riesz transforms on bounded subsets (with Neumann–Dirichlet conditions), second-order elliptic operators on Lipschitz domains, and Riesz transforms on Riemannian manifolds, for example. For all such operators, there is only a range of exponents (p_0, q_0) where we have L^p estimates for the semigroup $(e^{-tL})_t$ and its gradient for $p \in (p_0, q_0)$. Weighted estimates for such operators are more delicate, naturally restricted to these same ranges of p. We refer the reader to [Auscher and Martell 2007a] for a recent survey about weighted estimates.

In this current work, we aim to combine these two fashionable problems and give a modern approach to singular nonkernel operators. This setting had been resistant to many of the ideas developed in recent years. Indeed, we are going to adapt the approach of Lacey [2015] in order to be able to deal with nonintegral singular operators. The main idea relies on defining a suitable maximal operator and then controlling the operator by *sparse operators* (whose definition is modified from the previous works). We describe our method in a very general setting given by a space of homogeneous type, equipped with a semigroup. However, we point out that even in the Euclidean case our results are new, since they do not rely on any pointwise regularity estimates of the kernel of the considered operators. Moreover, we modify the maximal operator that we are going to use: instead of the maximal truncated operator used by Lacey [2015], we use truncation in the "frequency" point of view (where the notion of "frequencies" has to be understood in terms of the semigroup). Simultaneously, we will use a slightly weaker notion of sparse operators; both of these facts will allow us to give a proof which is simpler than Lacey's proof. However, we are not able to recover the full A_2 result in its generality: indeed, the assumptions we need require that the considered operator satisfies a suitable decomposition in the frequency point of view (see Remark 1.5). As shown in Section 3, that covers the main prototypes of operators. It is interesting to observe that the proof of these sharp weighted estimates can be substantially simplified in our situation and extended to operators whose kernel does not satisfy any regularity estimate. Recently, Bui, Conde-Alonso, Duong and Hormozi [Bui et al. 2015] have extended Lerner's approach for operators with kernels having L^{p_0}-L^∞ regularity estimates (which corresponds to the case $q_0 = \infty$, as we will see in Section 3D). We emphasize that this work is the first one where we are able to consider the case $q_0 < \infty$ and where no regularity is required on the eventual "kernel", which (as shown in the examples in Section 3) will allow us to deal with various situations in terms of operators and ambient spaces.

1A. *The setting.* Let M be a locally compact, separable metric space equipped with a Borel measure μ that is finite on compact sets and strictly positive on any nonempty open set. For a measurable subset Ω of M, we shall often denote $\mu(\Omega)$ by $|\Omega|$.

For all $x \in M$ and $r > 0$, let $B(x,r)$ be the open ball for the metric d with centre x and radius r. For a ball B of radius r and $\lambda > 0$, denote by λB the ball concentric with B and with radius λr. We sometimes denote by $r(B)$ the radius of the ball B. Finally, we will use $u \lesssim v$ to say that there exists a constant C (independent of the important parameters) such that $u \le Cv$, and $u \simeq v$ to say that $u \lesssim v$ and $v \lesssim u$. Moreover, for a subset $\Omega \subset M$ of finite and nonvanishing measure and $f \in L^1_{\mathrm{loc}}(M,\mu)$, we denote by $⨍_\Omega f\,d\mu = (1/|\Omega|)\int_\Omega f\,d\mu$ the average of f on Ω. We let $\mathcal{M}$ be the uncentred Hardy–Littlewood maximal operator. For $p \in [1,\infty)$, we abbreviate by $\mathcal{M}_p$ the operator defined by $\mathcal{M}_p(f) := \big(\mathcal{M}(|f|^p)\big)^{1/p}$ for $f \in L^p_{\mathrm{loc}}(M,\mu)$.

We shall assume that (M,d,μ) satisfies the volume doubling property, that is,

$$|B(x,2r)| \lesssim |B(x,r)| \quad \text{for all } x \in M,\ r > 0. \tag{VD}$$

It follows that there exists $\nu > 0$ such that

$$|B(x,r)| \lesssim \left(\frac{r}{s}\right)^{\nu} |B(x,s)| \quad \text{for all } x \in M,\ r \ge s > 0, \tag{VD$_\nu$}$$

which implies

$$|B(x,r)| \lesssim \left(\frac{d(x,y)+r}{s}\right)^{\nu} |B(y,s)| \quad \text{for all } x,y \in M,\ r \ge s > 0.$$

An easy consequence of (VD) is that balls with a nonempty intersection and comparable radii have comparable measures.

Let us recall that, for $0 \le \theta < \frac{\pi}{2}$, a linear operator L with dense domain $\mathcal{D}_2(L)$ in $L^2(M,\mu)$ is called *θ-accretive* if the spectrum $\sigma(L)$ of L is contained in the closed sector $S_{\theta+} := \{\zeta \in \mathbb{C} : |\arg\zeta| \le \theta\} \cup \{0\}$, and $\langle Lg, g\rangle \in S_{\omega+}$ for all $g \in \mathcal{D}_2(L)$.

We suppose that there exists an unbounded operator L on $L^2(M,\mu)$ satisfying these assumptions:

Assumptions on L. *Let L be an injective, θ-accretive operator with dense domain $\mathfrak{D}_2(L)$ in $L^2(M,\mu)$, where $0 \le \theta < \frac{\pi}{2}$. We assume that there exist two exponents $1 \le p_0 < 2 < q_0 \le \infty$ such that, for all balls B_1, B_2 of radius $\sqrt{t}$,*

$$\|e^{-tL}\|_{L^{p_0}(B_1)\to L^{q_0}(B_2)} \lesssim |B_1|^{-1/p_0}|B_2|^{1/q_0}e^{-cd(B_1,B_2)^2/t}. \tag{1-1}$$

As a consequence, L is a maximal accretive operator on $L^2(M,\mu)$, and therefore has a bounded H^∞ functional calculus on $L^2(M,\mu)$. The assumption $\theta < \frac{\pi}{2}$ implies that $-L$ is the generator of an analytic semigroup $(e^{-tL})_{t>0}$ in $L^2(M,\mu)$ (see [Albrecht et al. 1996; Kato 1966] for definitions and further considerations). The last part in the assumption means that the considered semigroup satisfies L^{p_0}-L^{q_0} off-diagonal estimates, an extension of L^2-L^2 Davies–Gaffney estimates. In situations where pointwise heat kernel bounds fail (see below for examples), this has turned out to be an appropriate replacement.

In this work, we study weighted estimates for nonintegral singular operators satisfying some cancellation with respect to this operator. We consider a linear (or sublinear) operator T satisfying the following properties:

Assumptions. (a) *T is well-defined as a bounded operator in L^2.*

(b) (L^{p_0}-L^{q_0} off-diagonal estimates) *There exists $N_0 \in \mathbb{N}$ such that, for all integers $N \ge N_0$ and all balls B_1, B_2 of radius $\sqrt{t}$,*

$$\|T(tL)^N e^{-tL}\|_{L^{p_0}(B_1)\to L^{q_0}(B_2)} \lesssim |B_1|^{-1/p_0}|B_2|^{1/q_0}\left(1+\frac{d(B_1,B_2)^2}{t}\right)^{-\frac{\nu+1}{2}}.$$

(c) *There exists an exponent $p_1 \in [p_0, 2)$ such that, for all $x \in M$ and $r > 0$,*

$$\left(\fint_{B(x,r)} |Te^{-r^2L}f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \inf_{y\in B(x,r)} \mathcal{M}_{p_1}(Tf)(y) + \inf_{y\in B(x,r)} \mathcal{M}_{p_1}(f)(y).$$

Item (b) encodes the fact that the operator T has some cancellation property which interacts well with the cancellation of the considered semigroup. Item (c) is a property which allows us to get off-diagonal estimates for the low-frequency part of the operator T. We point out that (b) and (c) are the main assumptions and were already used in numerous works to replace the notion of Calderón–Zygmund operators (see, e.g., [Auscher 2007; Auscher et al. 2004] and references therein).

We will assume the above throughout the paper. We abbreviate the setting with (M,μ,L,T).

1B. *Results.* Consider the setting (M,μ,L,T) satisfying the above assumptions. Then we claim that such an operator satisfies weighted boundedness. Indeed, such an operator satisfies the three following properties:

- T is bounded on L^2.
- For every $r > 0$ and some integer N large enough, $T(I-e^{-r^2L})^N$ satisfies L^{p_0}-L^{q_0} off-diagonal estimates (outside the diagonal); see Corollary 4.2 for a precise statement.
- T satisfies the Cotlar-type inequality of Assumption (c) for some $p_1 < 2$.

We then know from [Auscher 2007, Theorems 1.1 and 1.2] — see also the earlier results in [Auscher and Martell 2007a; Blunck and Kunstmann 2003; Auscher et al. 2004] — that T is bounded in L^p for every $p \in (p_0, q_0)$. By [Auscher and Martell 2007a, Theorem 3.13], T also satisfies some weighted estimates: for every $p \in (p_0, q_0)$ and every weight $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$ (see Section 6 for a precise definition of this class of weights), the operator T is bounded in L^p_ω. However, it is not clear from these previous results how the quantity $\|T\|_{L^p_\omega \to L^p_\omega}$ depends on the weight ω. The methods used do not tend to give optimal estimates.

Our main result is the following:

Theorem 1.1. *Consider the setting (M, μ, L, T) as above. For $p \in (p_0, q_0)$, there exists a constant c_p such that, for every weight $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$,*

$$\|T\|_{L^p_\omega \to L^p_\omega} \leq c_p \big([\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}}\big)^\alpha,$$

with

$$\alpha := \max\left\{\frac{1}{p-p_0}, \frac{q_0-1}{q_0-p}\right\}. \tag{1-2}$$

In particular, by defining the specific exponent

$$\mathfrak{p} := 1 + \frac{p_0}{q_0'} \in (p_0, q_0),$$

we have $\alpha = 1/(p-p_0)$ if $p \in (p_0, \mathfrak{p}]$, and $\alpha = (q_0-1)/(q_0-p)$ if $p \in [\mathfrak{p}, q_0)$.

Remark 1.2. In the case $q_0 = p_0'$, we have $\mathfrak{p} = 2$ and obtain a sharp L^2_ω inequality with an exponent

$$\alpha = \frac{1}{2-p_0}.$$

Remark 1.3. If $p_0 = 1$ and $q_0 = \infty$, we obtain $\alpha = \max\{1, 1/(p-1)\}$ and so we reprove the A_2 conjecture for such operators. Note that we are then able to prove these sharp estimates in the case of the Riesz transform $T = \nabla L^{-1/2}$ in the situation where this operator does not fit the Calderón–Zygmund framework (there is no pointwise regularity estimate of the full kernel); see Section 3.

Remark 1.4. We also prove the optimality of such estimates (in terms of the growth with respect to the characteristic of the weight) for sparse operators, which are shown to control our operators. The optimality also still holds for the operator itself if we know some "lower off-diagonal" estimates.

Remark 1.5. On the Euclidean space $\mathbb{R}^\nu$, consider the canonical heat semigroup and an "arbitrary" Calderón–Zygmund operator T: a linear L^2-bounded operator with a kernel K satisfying the regularity estimates

$$|K(x,y) - K(z,y)| + |K(y,x) - K(z,x)| \lesssim \left(\frac{d(y,z)}{d(y,z)+d(x,y)}\right)^\varepsilon d(x,y)^{-\nu}$$

for some $\varepsilon > 0$ and all points x, y, z with $2d(y,z) \leq d(x,z)$. Then it is well known that T is L^p-bounded for every $p \in (1, \infty)$. Consequently, we can check that our Assumptions are satisfied for $p_0 = 1$ and any $q_0 < \infty$ as large as we want. Unfortunately, it is unclear to us if our approach could recover the optimal A_p estimates for arbitrary Calderón–Zygmund operators (which would correspond to $q_0 = \infty$). It appears that Assumption (b) describes an extra property on the operator T, a kind of suitable frequency decomposition or representation (as Fourier multipliers or paraproducts, for example). It is interesting to observe that, under this extra property, we are going to detail an "elementary" proof of the sharp weighted estimates (simpler than all the existing proofs, such as [Lerner 2013b; Lacey 2015]), which has also the very important property that is extends to nonintegral operators with no regularity property on the kernel.

We remark that this extra property already appeared in [Duong and McIntosoh 1999, Theorem 3], where boundedness of the maximal operator $T^{\#}$ (see Section 4 for the definition) in the case $q_0 = \infty$ was shown, and that this is also the only place where we are using it. See also [Duong and McIntosoh 1999, Remark, p. 251]. Moreover, as illustrated in Section 3, this extra property is satisfied for the main prototype of Calderón–Zygmund operators.

2. Notation and preliminaries on approximation operators

2A. ***Notation.*** For $p \in [1, \infty)$, a subset $E \subset M$ and a measure λ on E, we write $L^p(E, d\lambda)$ for the Lebesgue space, equipped with the norm

$$\|f\|_{L^p(E,d\lambda)} = \left(\int_E |f|^p \, d\lambda \right)^{\frac{1}{p}}.$$

For convenience, we forget E if $E = M$ is the whole space and λ if $\lambda = \mu$ is the underlying measure. Thus, L^p stands for $L^p(M, \mu)$. For a positive function ω, we write L^p_ω for the weighted Lebesgue space, equipped with the norm

$$\|f\|_{L^p_\omega} = \left(\int_M |f|^p \omega \, d\mu \right)^{\frac{1}{p}}.$$

For a positive function $\rho : M \to (0, \infty)$, we identify the function ρ with the measure $\rho \, d\mu$ in the sense that, for every measurable subset $E \subset M$, we use

$$\rho(E) = \int_E \rho \, d\mu.$$

For a ball B, we let $S_0(B) = 2B$ and $S_j(B) = 2^{j+1} B \setminus 2^j B$ for $j \geq 1$. By extending the average notion to coronas, we let

$$\fint_{S_j(B)} f \, d\mu = |2^j B|^{-1} \int_{S_j(B)} f \, d\mu.$$

2B. ***Operator estimates.*** The building blocks of our analysis will be the following operators derived from the semigroup $(e^{-tL})_{t>0}$. They serve as a replacement for Littlewood–Paley operators.

Two different classes of elementary operators will be needed: $(P_t)_{t>0}$, corresponding to an approximation of the identity at scale $\sqrt{t}$ commuting with the heat semigroup, and $(Q_t)_{t>0}$, which satisfies some extra cancellation with respect to L.

Definition 2.1. Let $N > 0$ and set $c_N = \int_0^{+\infty} s^N e^{-s}\, ds/s$. For $t > 0$, define

$$Q_t^{(N)} := c_N^{-1} (tL)^N e^{-tL} \tag{2-1}$$

and

$$P_t^{(N)} := \int_1^\infty Q_{st}^{(N)} \frac{ds}{s} = \phi_N(tL), \tag{2-2}$$

with $\phi_N(x) := c_N^{-1} \int_x^{+\infty} s^N e^{-s}\, ds/s$ for $x \geq 0$.

Remarks 2.2. Let $p \in [p_0, q_0]$ with $p < \infty$ and $N > 0$.

(i) Note that $P_t^{(1)} = e^{-tL}$ and $Q_t^{(1)} = tLe^{-tL}$. The two families of operators $(P_t^{(N)})_{t>0}$ and $(Q_t^{(N)})_{t>0}$ are related by

$$t\partial_t P_t^{(N)} = tL\phi_N'(tL) = -Q_t^{(N)}.$$

(ii) If N is an integer, then $Q_t^{(N)} = (-1)^N c_N^{-1} t^N \partial_t^N e^{-tL}$ and $P_t^{(N)} = p(tL)e^{-tL}$, where p is a polynomial of degree $N-1$ with $p(0) = 1$.

(iii) By L^p analyticity of the semigroup and (1-1), we know that, for every integer $N > 0$ and every $t > 0$, $P_t^{(N)}$ and $Q_t^{(N)}$ satisfy off-diagonal estimates at the scale $\sqrt{t}$. See the arguments in [Hofmann et al. 2011, Proposition 3.1], for example.

(iv) The operators $P_t^{(N)}$ and $Q_t^{(N)}$ are bounded in L^p, uniformly in $t > 0$. See [Auscher and Martell 2007b, Theorem 2.3], taking into account (iii).

Proposition 2.3 (Calderón reproducing formula). *Let $N > 0$ and $p \in (p_0, q_0)$. For every $f \in L^p$,*

$$\lim_{t \to 0^+} P_t^{(N)} f = f \quad \text{in } L^p, \tag{2-3}$$

$$\lim_{t \to +\infty} P_t^{(N)} f = 0 \quad \text{in } L^p, \tag{2-4}$$

and

$$f = \int_0^{+\infty} Q_t^{(N)} f \frac{dt}{t} \quad \text{in } L^p. \tag{2-5}$$

In particular, it follows that as L^p-bounded operators we have the decomposition

$$P_t^{(N)} = \mathrm{Id} - \int_0^t Q_s^{(N)} \frac{ds}{s}. \tag{2-6}$$

3. Examples and applications

Our assumptions on L hold for a large variety of second-order operators, for example uniformly elliptic operators in divergence form and Schrödinger operators with singular potentials on $\mathbb{R}^n$, or the Laplace–Beltrami operator on a Riemannian manifold. For more precise examples of L and references, see Section 3B, where we give some examples of singular integral operators T that fit into our setting. See also [Auscher and Martell 2006].

3A. ***Holomorphic functional calculus of L.*** Let $0 \le \theta < \sigma < \pi$, where θ denotes the angle of accretivity of L. Define the open sector in the complex plane of angle σ by

$$S_\sigma^o := \{z \in \mathbb{C} : z \neq 0,\ |\arg z| < \sigma\}.$$

Denote by $H(S_\sigma^o)$ the space of all holomorphic functions on S_σ^o, and let

$$H^\infty(S_\sigma^o) := \{\varphi \in H(S_\sigma^o) : \|\varphi\|_\infty < \infty\}.$$

By our assumptions, L has a bounded H^∞ functional calculus on L^2. Blunck and Kunstmann [2003] showed that, under the assumption (1-1), the functional calculus can be extended to L^p for $p \in (p_0, q_0)$.

We now obtain the following weighted version: Let $\sigma > \theta$ and let $\varphi \in H^\infty(S_\sigma^o)$. Set $T = \varphi(L)$. We check our Assumptions. Item (a) is a restatement of the fact that L has a bounded H^∞ functional calculus on L^2. Since T commutes with $e^{-r^2 L}$, we can obtain (c) as a consequence of (1-1) (we do not detail this here; similar estimates are done in the sequel). Finally, for large enough N, by adapting [Auscher et al. 2008, Lemma 3.6] one can show that $\varphi(L)(tL)^N e^{-tL}$ satisfies L^{q_0}-L^{q_0} off-diagonal estimates. Combining this with L^{p_0}-L^{q_0} off-diagonal estimates for e^{-tL} gives (b). We therefore have:

Theorem 3.1. *Let $p \in (p_0, q_0)$ and $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$. The operator L has a bounded holomorphic functional calculus in L^p_ω with, for every $\sigma > \theta$,*

$$\|\varphi(L)\|_{L^p_\omega \to L^p_\omega} \le c_{p,\sigma} \big([\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}}\big)^\alpha \|\varphi\|_\infty$$

for all $\varphi \in H^\infty(S_\sigma^o)$ and α as defined in (1-2).

3B. ***Riesz transforms.*** The L^p-boundedness of Riesz transforms on manifolds has been widely studied in recent years. We refer the reader to [Bernicot and Frey 2015] for recent work and references for more details about such operators.

Several situations fit into our setting; we can consider specific operators, or specific ambient spaces, or both. Let us give some examples; more can be studied, like Riesz transforms on bounded domains or those associated with Schrödinger operators.

Dirichlet forms. Let (M, d, μ) be a complete space of homogeneous type, as above. Consider a self-adjoint operator L on L^2 and the quadratic form $\mathscr{E}$ associated with L, that is,

$$\mathscr{E}(f, g) = \int_M f L g \, d\mu.$$

If $\mathscr{E}$ is a strongly local and regular Dirichlet form (see [Fukushima et al. 1994; Gyrya and Saloff-Coste 2011] for precise definitions) with a carré du champ structure, then, with Γ being equal to this carré du champ operator, assume that the Poincaré inequality (P_2) holds, that is,

$$\left(\fint_B \left| f - \fint_B f \, d\mu \right|^2 d\mu\right)^{\frac{1}{2}} \lesssim r \left(\fint_B d\Gamma(f, f)\right)^{\frac{1}{2}} \tag{P_2}$$

for every $f \in \mathscr{D}(\mathscr{E})$ and every ball $B \subset M$ with radius r.

If the heat semigroup $(e^{-tL})_{t>0}$ and its carré du champ $(\sqrt{t}\Gamma e^{-tL})_{t>0}$ satisfy L^{p_0}-L^{q_0} off-diagonal estimates, then it can be checked that our Assumptions are satisfied for the Riesz transform (see [Auscher et al. 2004])

$$\mathfrak{R} := \Gamma L^{-1/2} = c_k \Gamma\biggl(\int_0^\infty (tL)^k e^{-tL}\,\frac{dt}{\sqrt{t}}\biggr),$$

for some numerical constant c_k and every integer $k \geq 1$.[1]

In particular, for $p_0 = 1$ and $q_0 = \infty$, we get the following result:

Theorem 3.2. *Consider the Riesz transform $\mathfrak{R}$ in one of the following situations*:

- *Euclidean space or any doubling Riemannian manifold with bounded geometry and nonnegative Ricci curvature* (see [Li and Yau 1986]).
- *In a convex doubling subset of $\mathbb{R}^\nu$ with the Laplace operator associated with Neumann boundary conditions* (see [Wang and Yan 2013]).

Then, for every $p \in (1,\infty)$ and every weight $\omega \in A_p$, we have

$$\|\mathfrak{R}\|_{L^p_\omega \to L^p_\omega} \lesssim [\omega]^\alpha_{A_p} \quad \textit{with } \alpha = \max\Bigl\{1, \frac{1}{p-1}\Bigr\}.$$

Note that in these situations we only have Lipschitz regularity of the heat kernel; the full kernel of the Riesz transform does not satisfy any pointwise regularity estimate and so does not fit into the class of Calderón–Zygmund operators (as previously studied in [Lerner 2013b; Lacey 2015]).

Second-order divergence form operators. Consider a doubling Riemannian manifold (M, d, μ), equipped with the Riemannian gradient ∇ and its divergence operator $\operatorname{div} = \nabla^*$. To a complex, bounded, measurable, matrix-valued function $A = A(x)$, defined on M and satisfying the ellipticity (or accretivity) condition $\operatorname{Re}(A(x)) \geq \kappa I > 0$ a.e., we may define a second-order divergence form operator

$$L = L_A f := -\operatorname{div}(A\nabla f).$$

Then L is sectorial and satisfies the conservation property but may not be self-adjoint.

Assume that the Poincaré inequality (P_2) holds on (M, d, μ). If the semigroup $(e^{-tL})_{t>0}$ and its gradient $(\sqrt{t}\nabla e^{-tL})_{t>0}$ satisfy L^{p_0}-L^{q_0} off-diagonal estimates, then it can be checked that our Assumptions are satisfied for the Riesz transform

$$\mathfrak{R} := \nabla L^{-1/2} = c_k \int_0^\infty \nabla (tL)^k e^{-tL}\,\frac{dt}{\sqrt{t}}.$$

We refer the reader to [Auscher 2007] for a precise study in the Euclidean setting of the exponents p_0 and q_0 depending on the matrix-valued map A. For example, we have $p_0 = 1$ and $q_0 = \infty$ in dimension $\nu = 1$.

[1] It is known that the assumed Poincaré inequality (P_2) self-improves into a Poincaré inequality (P_{p_1}) for some $p_1 < 2$ (see [Keith and Zhong 2008]), which allows us to check Assumption (c).

3C. ***Paraproducts associated with L.*** Throughout this subsection we assume that the semigroup satisfies the conservation property, which means that $e^{-tL}1 = 1$ for every $t > 0$, as well as the fact that the semigroup is supposed to have a heat kernel with pointwise Gaussian bounds (which correspond to L^1-L^∞ estimates).

Paraproducts with a BMO function. In recent works [Bernicot 2012; Frey 2013], several paraproducts have been studied in the context of a semigroup. They allow us to have (as is well known in Euclidean space) a decomposition of the pointwise product with two paraproducts and a resonant term (we also refer the reader to [Bailleul et al. 2015] for some applications of such paraproducts in the context of paracontrolled calculus for solving singular PDEs). Moreover, BMO spaces adapted to such a framework have been the focus of numerous works, so it is natural (as in the Euclidean setting) to study the linear operator given by the paraproduct of a BMO function.

Let us recall some definitions. A BMO_L function is a locally integrable function $f \in L^1_{\mathrm{loc}}$ such that

$$\|f\|_{\mathrm{BMO}_L} := \sup_B \left(\fint_B |f - e^{-r^2 L} f|^2 \, d\mu \right)^{\frac{1}{2}},$$

where we take the supremum over all balls B with radius $r > 0$. Such BMO spaces satisfy "standard" properties, such as the John–Nirenberg inequality and $T(1)$ theorem. In particular it is known (see [Bernicot and Zhao 2012; Bernicot and Martell 2015]) that, since the semigroup satisfies L^1-L^∞ off-diagonal estimates, the norm in BMO_L can be built through an L^p oscillation for any $p \in (1, \infty)$ and the corresponding norms are equivalent. For some integer k, the paraproduct under consideration is

$$\Pi_g(f) = \int_0^\infty Q_t^{(k)} (Q_t^{(k)} f \cdot P_t^{(k)} g) \, \frac{dt}{t}.$$

Using square function estimates, we then know that $Q_t^{(k)} g$ is uniformly bounded in L^∞ for $g \in \mathrm{BMO}_L$, so that Π_g is L^2-bounded. Assumptions (b) and (c) are also satisfied with $p_0 = 1$ and $q_0 = \infty$ (see details in the above references) and so we may apply Theorem 1.1 to the previous paraproduct for $g \in \mathrm{BMO}_L$.

Algebra property for fractional Sobolev spaces. Bernicot, Coulhon, and Frey [Bernicot et al. 2015] have used some paraproducts associated with such a framework involving a heat semigroup in order to study the algebra property for fractional Sobolev spaces. We refer to [Bernicot et al. 2015] for more details and references for other paraproducts associated with a semigroup. Then, up to some constant c_N, we have the product decomposition for two functions

$$fg = \Pi_g(f) + \Pi_f(g),$$

with the paraproduct defined by

$$\Pi_g(f) = \int_0^\infty Q_t^{(N)} f \cdot P_t^{(N)} g \, \frac{dt}{t}.$$

Fix a function $g \in L^\infty$; then, for $\alpha \in (0, 1)$, we are looking for the $\dot{L}^p_\alpha$-boundedness of Π_g, which corresponds the to L^p-boundedness of $T := L^{\alpha/2} \Pi_g L^{-\alpha/2}$. In [Bernicot et al. 2015], we gave different situations and criteria under which our Assumptions are satisfied. Mainly we considered the condition, introduced in [Auscher et al. 2004], that, for some $p \in (2,\infty]$,

$$\sup_{t>0} \big\| \sqrt{t}\, |\Gamma e^{-tL}| \big\|_{p\to p} < +\infty, \tag{G_p}$$

where Γ is the carré du champ associated with the operator L (and $|\Gamma\cdot|$ is its modulus). In this way, we may apply Theorem 1.1 to T and obtain a sharp algebra property for weighted fractional spaces, sharp with respect to the weight. We obtain the following estimates:

Theorem 3.3. *Let $(M,d,\mu,\mathscr{E})$ be a doubling metric measure Dirichlet space with a carré du champ* (see [Bernicot et al. 2015] for more details) *and assume that the heat semigroup e^{-tL} has a heat kernel with usual pointwise Gaussian estimates. For some $s \in (0,1)$ and $p \in (1,\infty)$, consider the following weighted Leibniz rule*: *for every weight ω and all functions $f, g \in \{h \in L^\infty : L^{s/2}(h) \in L^p_\omega\}$,*

$$\|L^{s/2}(fg)\|_{L^p_\omega} \lesssim c(\omega)\big(\|L^{s/2} f\|_{L^p_\omega} \|g\|_\infty + \|f\|_\infty \|L^{s/2} g\|_{L^p_\omega}\big). \tag{3-1}$$

(a) (3-1) *is valid for $p \in (1,2)$ and $s \in (0,1)$ with every weight $\omega \in A_p \cap \mathrm{RH}_{(2/p)'}$ and a constant*

$$c(\omega) = ([\omega]_{A_p}[\omega]_{\mathrm{RH}_{(2/p)'}})^\alpha \quad \textit{with} \quad \alpha := \max\Big\{\frac{1}{p-1}, \frac{1}{2-p}\Big\}.$$

(b) *Under (G_q) for some $q \in (2,\infty)$,* (3-1) *is valid for $p \in (1,q)$ and $s \in (0,1)$ with every weight $\omega \in A_p \cap \mathrm{RH}_{(q^-/p)'}$, where $q^- \in (p,q)$, and a constant*

$$c(\omega) = ([\omega]_{A_p}[\omega]_{\mathrm{RH}_{(q^-/p)'}})^\alpha \quad \textit{with} \quad \alpha := \max\Big\{\frac{1}{p-1}, \frac{q^- -1}{q^- - p}\Big\}.$$

(c) *Under (G_∞),* (3-1) *is valid for $p \in (1,\infty)$ and $s \in (0,1)$ with every weight $\omega \in A_p$ and a constant*

$$c(\omega) = [\omega]_{A_p}^\alpha \quad \textit{with} \quad \alpha := \max\Big\{\frac{1}{p-1}, 1\Big\}.$$

Other estimates can be obtained by combining the results of this paper with the other estimates of [Bernicot et al. 2015].

3D. ***Fourier multipliers.*** Let us also explain how we can recover the results of [Bui et al. 2015]. The main linear result [Bui et al. 2015, Theorem C] fits into our framework and corresponds to the particular case $q_0 = \infty$. Let us focus on the application to linear Fourier multipliers.

Consider a linear symbol m on $\mathbb{R}^\nu$ satisfying the Hörmander condition $M(s,l)$, which is

$$\sup_{R>0} \bigg(R^{s|\alpha|-\nu} \int_{R\le|\xi|\le 2R} |\partial_\xi^\alpha m(\xi)|^s \, d\xi \bigg)^{\frac{1}{s}} < \infty$$

for all $|\alpha| \le l$, some $s \in (1,2]$ and $l \in (\nu/s, \nu)$. To this symbol we associate the linear Fourier multiplier

$$T(f) = T_m(f) : x \mapsto \int e^{ix.\xi} m(\xi) \widehat{f}(\xi)\, d\xi.$$

For every $r \in (\nu/l, \infty)$, [Bui et al. 2015, Lemma 5.2] shows that the kernel of T satisfies some L^r-L^∞ regularity off-diagonal estimates. So consider a smooth function ψ such that $\widehat{\psi}$ is supported on

$B(0,4) \setminus B(0,1)$ and well-normalized with $\int_0^\infty \widehat{\psi}(t\xi)\,dt/t = 1$ for every ξ. Then, with the elementary operators

$$T_t(f) : x \mapsto \int e^{ix.\xi} m(\xi)\widehat{\psi}(t\xi)\widehat{f}(\xi)\,d\xi,$$

it can be proved that our Assumptions are satisfied for $p_0 = r$ and $q_0 = \infty$. Consequently, Theorem 1.1 allows us to regain [Bui et al. 2015, Theorem 5.3(a)]. Moreover, since T is self-adjoint, by duality we also deduce that the kernel of T satisfies some L^1-$L^{r'}$ off-diagonal estimates. Similarly, one can then show that our Assumptions are satisfied for $p_0 = 1$ and $q_0 = r'$. Consequently, Theorem 1.1 allows us to regain [Bui et al. 2015, Theorem 5.3(b)]. So we regain the same full result as [Bui et al. 2015, Theorem 5.3], with the exact same behaviour of the weighted estimates with respect to the weight.

The same comparison can be done for the linear part of their main result [Bui et al. 2015, Theorem C]. Under their assumptions (H1) and (H2), our Assumptions are satisfied with $q_0 = \infty$. We leave the details to the reader.

4. Unweighted boundedness of a certain maximal operator

Before introducing and studying a certain maximal operator related to T, we first explain some technical details of off-diagonal estimates.

4A. ***Off-diagonal estimates.*** We fix an integer $N > N_0$ (with N_0 as in our Assumptions) and write, for $t > 0$,

$$T_t := TQ_t^{(N)}.$$

Let $p \in (p_0, q_0)$. The Calderón reproducing formula (see Proposition 2.3) gives the identity

$$\mathrm{Id} = \int_0^\infty Q_t^{(N)}\,\frac{dt}{t}$$

in L^p. Since T is assumed to be sublinear, we can decompose the operator for $f \in L^p$ into

$$|T(f)| \le \int_0^\infty |T_t(f)|\,\frac{dt}{t}. \tag{4-1}$$

Fix $t > 0$ and the elementary operator T_t. From Assumption (b) we know that T_t satisfies L^{p_0}-L^{q_0} off-diagonal estimates at the scale $\sqrt{t}$. Then consider a ball B of radius $r > 0$ with $r \le \sqrt{t}$ and its dilated ball $\widetilde{B} := (\sqrt{t}/r)B$. We have $B \subset \widetilde{B}$ and $|\widetilde{B}| \lesssim (\sqrt{t}/r)^\nu |B|$, so

$$\left(\fint_B |T_t f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \left(\frac{\sqrt{t}}{r}\right)^{\frac{\nu}{q_0}} \sum_{j\ge 0} 2^{-j(\nu+1)}\left(\fint_{S_j(\widetilde{B})} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}. \tag{4-2}$$

Lemma 4.1. *Consider three parameters $r, \varepsilon, t > 0$. Let $N \in \mathbb{N}$ with $N > \max\{\frac{3}{2}\nu + 1,\ N_0\}$.*

(1) *If $r^2 < \varepsilon < t$, we have, for every ball B_r of radius r and the dilated ball $B_{\sqrt{\varepsilon}} = (\sqrt{\varepsilon}/r)B_r$,*

$$\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t(I - e^{-r^2L})^N f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}} \sum_{l\ge 0} 2^{-l(\nu+1)}\left(\fint_{2^l B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

(2) *If $t < \varepsilon < r^2$, we have, for every ball B_r of radius r, every $j \geq 3$, every ball $B_{\sqrt{\varepsilon}}$ of radius $\sqrt{\varepsilon}$ included in $S_j(B_r)$, and every function f supported on B_r,*

$$\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t(I-e^{-r^2L})^N f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim 2^{-j(\nu+1)}\left(\frac{t}{r^2}\right)^{\frac{1}{2}}\left(\fint_{B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

The same estimates are true for $T_t(I-P^{(N)}_{r^2})$ in place of $T_t(I-e^{-r^2L})^N$.

Proof. Consider the first case, $r^2 < \varepsilon < t$. We show the result for $T_t(I-P^{(N)}_{r^2})$, and then explain how to modify the proof in the case of $T_t(I-e^{-r^2L})^N$. By the definition of $P^{(N)}_{r^2}$, we have

$$I - P^{(N)}_{r^2} = \int_0^{r^2} Q_s^{(N)}\,\frac{ds}{s}.$$

Hence,

$$\left(\fint_{B_{\sqrt{\varepsilon}}} |T_t(I-P^{(N)}_{r^2})f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \int_0^{r^2}\left(\fint_{B_{\sqrt{\varepsilon}}} |T_tQ_s^{(N)}f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\frac{ds}{s}.$$

Note that $T_t = TQ_t^{(N)}$ and $Q_t^{(N)}Q_s^{(N)} = (s/(s+t))^N Q^{(2N)}_{s+t}$ (up to a numerical constant) as well as $s+t \simeq t$. Using Assumption (b) and (4-2) for $s < r^2$ with $r^2 < \varepsilon < t$, we obtain that

$$\begin{aligned}\left(\fint_{B_{\sqrt{\varepsilon}}} |T_tQ_s^{(N)}f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim \left(\frac{s}{t}\right)^N\left(\fint_{B_{\sqrt{\varepsilon}}} |T_{s+t}f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\\ &\lesssim \left(\frac{s}{t}\right)^N\left(\frac{t}{\varepsilon}\right)^{\frac{\nu}{2q_0}}\sum_{l\geq 0}2^{-l(\nu+1)}\left(\fint_{2^lB_{\sqrt{t}}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}},\end{aligned}$$

where we used that $s+t \lesssim t$. Since $s < r^2 < \varepsilon$, we can estimate $(s/t)^N(t/\varepsilon)^{\nu/2q_0}$ by $(s/t)^{N-\nu/2q_0}$, and then deduce that, for $k \geq 0$ such that $2^kr \simeq \sqrt{t}$,

$$\begin{aligned}\left(\frac{s}{t}\right)^{N-\frac{\nu}{2q_0}}2^{-l(\nu+1)}\left(\fint_{2^lB_{\sqrt{t}}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}} &\lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}}\left(\frac{s}{t}\right)^{\frac{N}{2}-\frac{\nu}{2q_0}}2^{-l(\nu+1)}\left(\fint_{2^lB_{\sqrt{t}}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \left(\frac{r^2}{t}\right)^{\frac{N}{2}}\left(\frac{s}{r^2}\right)^{\frac{\nu+1}{2}}2^{-(l+k)(\nu+1)}\left(\fint_{2^{l+k}B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}},\end{aligned}$$

where we used that N is sufficiently large that $\frac{1}{2}N - \nu/(2q_0) > \frac{1}{2}(\nu+1)$. We then conclude by summing over l and integrating over $s \in (0, r^2)$.

In the second case, when $t < \varepsilon < r^2$, we follow the same reasoning: with $\tau = \max\{s,t\}$,

$$\begin{aligned}\left(\fint_{B_{\sqrt{\varepsilon}}} |T_tQ_s^{(N)}f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim \left(\frac{\min\{s,t\}}{\tau}\right)^{N-\frac{\nu}{2q_0}}\left(\frac{\tau}{2^{2j}r^2}\right)^{\frac{\nu+1}{2}}\left(\frac{r^2}{\tau}\right)^{\frac{\nu}{2p_0}}\left(\fint_{B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \left(\frac{\min\{s,t\}}{\max\{s,t\}}\right)^{\frac{N}{2}-\frac{\nu}{2q_0}}\left(\frac{t}{r^2}\right)^{\frac{1}{2}}2^{-j(\nu+1)}\left(\fint_{B_r} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}},\end{aligned}$$

where we used that $N > \nu + 1$. We may now integrate over s and obtain the desired result.

The modifications required for the case $T_t(I - e^{-r^2L})^N$ are straightforward. We first observe that

$$(I - e^{-r^2L})^N = \left(\int_0^{r^2} Le^{-sL}\,ds\right)^N = \int_0^{Nr^2} \alpha(s)(sL)^N e^{-sL}\,\frac{ds}{s}$$

with

$$\alpha(s) := s^{1-N}\big|\{(s_1, \dots, s_N) \in (0, r^2)^N : s_1 + \cdots + s_N = s\}\big| \lesssim 1.$$

Define $\psi_s^{(N)}(L) := \alpha(s)(sL)^N e^{-sL}$. Then

$$(I - e^{-r^2L})^N = \int_0^{Nr^2} \psi_s^{(N)}(L)\,\frac{ds}{s}.$$

Now we can also write $Q_t^{(N)}\psi_s^{(N)}(L) = (\min\{s,t\}/\max\{s,t\})^N\Theta_{s,t}$ with some operator $\Theta_{s,t}$ satisfying L^{p_0}-L^{q_0} off-diagonal estimates, and conclude as above. □

Considering the particular case $\varepsilon = r^2$, we may integrate over t the two inequalities of Lemma 4.1 and, from (4-1), deduce the following result:

Corollary 4.2. *For an integer $N > \max\{\frac{3}{2}\nu + 1, N_0\}$ and $r > 0$, $T(I - e^{-r^2L})^N$ satisfies L^{p_0}-L^{q_0} (strictly) off-diagonal estimates at the scale $r > 0$: if B_1 and B_2 are two balls of radius $r > 0$ with $d(B_1, B_2) > 4r$, then for every function f supported on B_1 we have*

$$\left(\fint_{B_2} |T(I - e^{-r^2L})^N f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} \lesssim \left(1 + \frac{d(B_1, B_2)}{r}\right)^{-(\nu+1)}\left(\fint_{B_1} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.$$

4B. *Maximal operator.* We now fix an integer $N > \max\{\frac{3}{2}\nu + 1,\ N_0\}$ (and all the implicit constants may depend on it).

Definition 4.3. Define the maximal operator $T^\#$ of T by

$$T^\# f(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \left(\fint_B \left| T\int_{r(B)^2}^{\infty} Q_t^{(N)} f\,\frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}}, \quad x \in M,$$

for $f \in L^{q_0}_{\mathrm{loc}}$.

By definition of $P_t^{(N)} := \int_1^\infty Q_{st}^{(N)}\,ds/s$, we then have

$$T^\# f(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \left(\fint_B |TP^{(N)}_{r(B)^2} f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}, \quad x \in M,$$

for $f \in L^{q_0}_{\mathrm{loc}}$.

Lemma 4.4. *Consider a sequence $(u_\varepsilon)_\varepsilon$ of L^2 functions which converges (in L^2) to some function $u \in L^2$ when ε tends to 0. Then, for almost every $x \in M$, we have*

$$|u(x)| \le \liminf_{\varepsilon \to 0} \fint_{B(x,\varepsilon)} |u_\varepsilon|\,d\mu.$$

Proof. Due to the Lebesgue differentiation lemma, we know that

$$|u(x)| \leq \liminf_{\varepsilon \to 0} \fint_{B(x,\varepsilon)} |u| \, d\mu.$$

Then we split, as follows:

$$\fint_{B(x,\varepsilon)} |u| \, d\mu \leq \fint_{B(x,\varepsilon)} |u_\varepsilon| \, d\mu + \fint_{B(x,\varepsilon)} |u - u_\varepsilon| \, d\mu.$$

The second part is pointwise bounded by $\mathcal{M}[u_\varepsilon - u](x)$, which converges in L^2 to 0 (due to the L^2-boundedness of the maximal function), which allows us to conclude the proof. □

As a consequence of the previous lemma with the L^2-boundedness of T and Proposition 2.3, we deduce the following result:

Corollary 4.5. *For every function $f \in L^2$ we have, almost everywhere,*

$$|T(f)| \leq T^{\#}(f).$$

Proposition 4.6. *The sublinear operator $T^{\#}$ is of weak type (p_0, p_0) and is bounded in L^p for every $p \in (p_0, 2]$.*

Remark 4.7. In the definition of the maximal operator, the previous boundedness still holds if we replace the average on the ball B by any average on λB for some constant $\lambda > 1$. In this case, the implicit constants will depend on λ.

Proof. We proceed in two steps:

Step 1 (L^2-boundedness of $T^{\#}$). We first claim that $T^{\#}$ satisfies the following Cotlar-type inequality ($p_1 \in [p_0, 2)$ is introduced in our Assumptions):

$$T^{\#} f(x) \lesssim \mathcal{M}_{p_1}(Tf)(x) + \mathcal{M}_{p_1} f(x), \quad x \in M. \tag{4-3}$$

Indeed,

$$T^{\#} f(x) = \sup_{\substack{B \text{ ball} \\ B \ni x}} \left(\fint_B |T P^{(N)}_{r(B)^2} f|^{q_0} \, d\mu \right)^{\frac{1}{q_0}}$$

and, since N is an integer, we have by Remark 2.2(ii), with $r = r(B)$, that

$$P^{(N)}_{r^2} = p(r^2 L) e^{-r^2 L}$$

with p a polynomial function. We then factor as

$$T P^{(N)}_{r^2} = (T e^{-r^2 L/2})(p(r^2 L) e^{-r^2 L/2}).$$

By Assumption (c), $T e^{-r^2 L/2}$ satisfies some L^{p_1}-L^{q_0} estimates and, by Assumption (b) and Lemma 4.1, both $T(I - P^{(N)}_{r^2})$ and $p(r^2 L) e^{-r^2 L/2}$ satisfy L^{p_1}-L^{p_1} off-diagonal estimates at the scale r. We may compose these two estimates in order to obtain similar estimates as Assumption (c) for $T P^{(N)}_{r^2}$ and then directly obtain (4-3).

This in particular implies that $T^{\#}$ is bounded on L^2, since T is bounded on L^2 by assumption, and the Hardy–Littlewood maximal operator $\mathcal{M}_{p_1}$ is bounded on L^2 as $p_1 < 2$.

In the second step, we now use the extrapolation method of [Auscher 2007; Blunck and Kunstmann 2003] to show that $T^{\#}$ is of weak type (p_0, p_0), which by interpolation with the L^2-boundedness will conclude the proof of the proposition.

Step 2 (weak type (p_0, p_0) of $T^{\#}$). We apply [Auscher 2007, Theorem 1.1] (see also [Blunck and Kunstmann 2003]). As shown in Step 1, $T^{\#}$ is bounded on L^2. By assumption, we know that $(e^{-tL})_{t>0}$ satisfies L^{p_0}-L^2 off-diagonal estimates. It remains to show that $T^{\#}(I-e^{-tL})^N$ satisfies L^{p_0}-L^2 off-diagonal estimates (not including the diagonal), where we will use (for convenience, but it could be chosen differently) the same integer N as the one defining the maximal operator, which is chosen sufficiently large. More precisely, for a ball $B \subseteq M$ of radius r and a function $b \in L^{p_0}$ with $\operatorname{supp} b \subseteq B$, we will show that

$$|2^{j+1}B|^{-1/2} \|T^{\#}(I-e^{-r^2L})^N b\|_{L^2(S_j(B))} \lesssim c(j)|B|^{-1/p_0}\|b\|_{L^{p_0}(B)}, \quad j \geq 3, \tag{4-4}$$

with coefficients $c(j)$ satisfying $\sum_{j\geq 2} c(j)2^{\nu j} < \infty$.

For $x \in M$ and $\varepsilon > 0$, denote by $B_{x,\varepsilon}$ a ball of radius $\sqrt{\varepsilon}$ containing x. Then recall that $T_t = TQ_t^{(N)}$ and

$$T^{\#}(I-e^{-r^2L})^N b(x) \leq \sup_{\varepsilon>0} \left(\fint_{B_{x,\varepsilon}} \left| T \int_{\varepsilon}^{\infty} Q_t^{(N)}(I-e^{-r^2L})^N b \, \frac{dt}{t} \right|^{q_0} d\mu \right)^{\frac{1}{q_0}}.$$

Let $x \in S_j(B)$ for $j \geq 3$ and consider first the case $r^2 < \varepsilon$. Applying Lemma 4.1(1), we then deduce that

$$\left(\fint_{B_{x,\varepsilon}} |T_t(I-e^{-r^2L})^N b|^{q_0} \, d\mu \right)^{\frac{1}{q_0}} \lesssim \left(\frac{r^2}{t} \right)^{\frac{N}{2}} \left(1 + \frac{d(B, B_{x,\varepsilon})^2}{t} \right)^{-\frac{\nu+1}{2}} \left(\fint_B |b|^{p_0} \, d\mu \right)^{\frac{1}{p_0}}.$$

Since $\varepsilon < t$, it follows that either $2^j r \geq \sqrt{t}$, in which case $d(B, B_{x,\varepsilon}) \simeq 2^j r$, or $2^j r \leq \sqrt{t}$, in which case $d(B, B_{x,\varepsilon}) \leq 2\sqrt{t}$. So, in both situations, we have

$$1 + \frac{d(B, B_{x,\varepsilon})^2}{t} \simeq 1 + \frac{4^j r^2}{t}.$$

Consequently, we get

$$\left(\fint_{B_{x,\varepsilon}} |T_t(I-e^{-r^2L})^N b|^{q_0} \, d\mu \right)^{\frac{1}{q_0}} \lesssim \left(\frac{r^2}{t} \right)^{\frac{N}{2}} \left(1 + \frac{4^j r^2}{t} \right)^{-\frac{\nu+1}{2}} \left(\fint_B |b|^{p_0} \, d\mu \right)^{\frac{1}{p_0}}.$$

We then have to integrate along $t \in (\varepsilon, \infty)$ and we split the integral into two parts, depending on whether $t < 4^j r^2$ or $t > 4^j r^2$. We then obtain that

$$
\begin{aligned}
\biggl(\fint_{B_{x,\varepsilon}}\biggl|\int_\varepsilon^\infty T_t(I-e^{-r^2L})^N b\,\frac{dt}{t}\biggr|^{q_0}\,d\mu\biggr)^{\frac{1}{q_0}}&\\
&\lesssim \biggl(\int_\varepsilon^{4^jr^2} 2^{-j(\nu+1)}\Bigl(\frac{t}{r^2}\Bigr)^{\frac14}\,\frac{dt}{t}+\int_{4^jr^2}^\infty\Bigl(\frac{r^2}{t}\Bigr)^{\frac N2}\,\frac{dt}{t}\biggr)\biggl(\fint_B|b|^{p_0}\,d\mu\biggr)^{\frac{1}{p_0}}\\
&\lesssim 2^{-j(\nu+1/2)}\biggl(\fint_B|b|^{p_0}\,d\mu\biggr)^{\frac{1}{p_0}},
\end{aligned}
$$

which corresponds to the desired estimate (4-4) with $c(j)=2^{-j(\nu+1/2)}$.

Consider now the case $\varepsilon\le r^2$. Again, let $x\in S_j(B)$ for $j\ge 3$. We split the corresponding part of $T^\#(I-e^{-r^2L})^N b(x)$ into

$$
\sup_{\varepsilon<r^2}\biggl(\fint_{B_{x,\varepsilon}}|T(I-e^{-r^2L})^N b|^{q_0}\,d\mu\biggr)^{\frac{1}{q_0}}+\sup_{\varepsilon<r^2}\biggl(\fint_{B_{x,\varepsilon}}\biggl|\int_0^\varepsilon T_t(I-e^{-r^2L})^N b\,\frac{dt}{\sqrt{t}}\biggr|^{q_0}\,d\mu\biggr)^{\frac{1}{q_0}}
=: I_1(x)+I_2(x). \tag{4-5}
$$

Let $\widetilde{S}_j(B)$ be a slightly enlarged annulus such that $B_{x,\varepsilon}\subseteq\widetilde{S}_j(B)$ for $x\in S_j(B)$. We estimate the first term $I_1(x)$ in (4-5) against the maximal function, localized in $\widetilde{S}_j(B)$ due to the restriction of the supremum to small ε and the assumption $j\ge 3$. This gives, for $x\in S_j(B)$,

$$
I_1(x)\lesssim \mathcal{M}_{q_0}\bigl(\mathbb{1}_{\widetilde{S}_j(B)}T(I-e^{-r^2L})^N b\bigr)(x).
$$

By Hölder's inequality and Kolmogorov's lemma (see, e.g., [Duoandikoetxea 2001, Lemma 5.16]) for $\mathcal{M}_{q_0}$, we have

$$
\begin{aligned}
|2^{j+1}B|^{-1/2}\|I_1\|_{L^2(S_j(B))}&\lesssim |2^{j+1}B|^{-1/2}\bigl\|\mathcal{M}_{q_0}\bigl(\mathbb{1}_{\widetilde{S}_j(B)}T(I-e^{-r^2L})^N b\bigr)\bigr\|_{L^2(2^jB)}\\
&\lesssim |2^{j+1}B|^{-1/q_0}\|T(I-e^{-r^2L})^N b\|_{L^{q_0}(\widetilde{S}_j(B))}.
\end{aligned}
$$

By Corollary 4.2, we know that $T(I-e^{-r^2L})^N$ satisfies L^{p_0}-L^{q_0} (strictly) off-diagonal estimates at the scale r, thus giving (4-4) for this part with coefficients $c(j)=2^{-j(\nu+1)}$.

For I_2, on the other hand, we can directly estimate, using Lemma 4.1(2),

$$
|B_{x,\varepsilon}|^{-1/q_0}\|T_t(I-e^{-r^2L})^N b\|_{L^{q_0}(B_{x,\varepsilon})}\lesssim 2^{-j(\nu+1)}\Bigl(\frac{t}{r^2}\Bigr)^{\frac12}\biggl(\fint_B|b|^{p_0}\,d\mu\biggr)^{\frac{1}{p_0}}.
$$

Therefore, we may then integrate over $t\in(0,\varepsilon)$. By taking the supremum over $\varepsilon\in(0,r^2)$ and over x, and using Minkowski's inequality, we obtain (4-4) also for I_2 with coefficients $c(j)=2^{-j(\nu+1)}$. □

5. Boundedness of the maximal operator by sparse operators

As done in previous works (see for example [Petermichl 2007; Hytönen 2012; Lerner 2010; 2013a; 2013b; Lacey 2015]), the analysis will involve a discrete stopping-time argument that relies on nice properties associated with a dyadic structure, which is by now well-known in the context of doubling space. We first recall the main results and then, by using this structure, we detail the stopping-time argument to bound the maximal operator $T^{\#}$ by some specific operators, called *sparse operators*.

5A. ***Preliminaries and reminder on dyadic analysis.*** We first recall several results about the construction of adjacent dyadic systems (see [Christ 1990; Sawyer and Wheeden 1992; Hytönen and Kairema 2012] for more details).

Definition 5.1. Let us fix some constants $0 < c_0 \le C_0 < \infty$ and $\delta \in (0,1)$. A *dyadic system* (with parameters c_0, C_0, δ) is a family of open subsets $(Q^l_\alpha)_{\alpha \in \mathscr{A}_l, l \in \mathbb{Z}}$ satisfying the following properties:

- For every $l \in \mathbb{Z}$, the ambient space M is covered (up to a set with vanishing measure) by the disjoint union of the subsets at scale l, that is, there exists Z_l with $\mu(Z_l) = 0$ such that
$$M = \bigsqcup_{\alpha \in \mathscr{A}_l} Q^l_\alpha \sqcup Z_l.$$
- If $l \ge k$, $\alpha \in \mathscr{A}_k$ and $\beta \in \mathscr{A}_l$ then either $Q^l_\beta \subseteq Q^k_\alpha$ or $Q^k_\alpha \cap Q^l_\beta = \varnothing$.
- For every $l \in \mathbb{Z}$ and $\alpha \in \mathscr{A}_l$, there exists a point z^l_α with
$$B(z^l_\alpha, c_0\delta^l) \subseteq Q^l_\alpha \subseteq B(z^l_\alpha, C_0\delta^l) =: B(Q^l_\alpha). \tag{5-1}$$

For a cube Q^k_α, $k \in \mathbb{Z}$, $\alpha \in \mathscr{A}_k$, we call the unique cube Q^{k-1}_β, $\beta \in \mathscr{A}_{k-1}$, for which $Q^k_\alpha \subseteq Q^{k-1}_\beta$ the *parent* of Q^k_α. We denote the parent of $Q \in \mathscr{D}$ by Q^a and call Q a *child* of Q^a.

We refer the reader to [Hytönen and Kairema 2012] for a variant where the negligible Z_l does not appear if the subsets are not necessarily assumed to be open. We also refer to a very recent survey by Lerner and Nazarov [2015] about dyadic structures and how they are used for proving weighted estimates of singular operators.

Then we have the following result (see [Hytönen and Kairema 2012] and references therein):

Theorem 5.2. *There exist constants c_0, C_0, δ, finite constants $K = K(c_0, C_0, \delta)$ and $\rho = \rho(c_0, C_0, \delta)$, as well as a finite collection of families $\mathscr{D}^b$, $b = 1, 2, \dots, K$, where each $\mathscr{D}^b$ is a dyadic system* (*with parameters c_0, C_0, δ*) *with the following extra property*: *for every ball $B = B(x, r) \subseteq M$, there exists $b \in \{1, \dots, K\}$ and $Q \in \mathscr{D}^b$ with*
$$B \subseteq Q \quad \textit{and} \quad \operatorname{diam}(Q) \le \rho r. \tag{5-2}$$

We define
$$\mathscr{D} := \bigcup_{b=1}^{K} \mathscr{D}^b,$$
and call a cube Q a *dyadic cube* whenever $Q \in \mathscr{D}$.

For every dyadic set $Q \in \mathscr{D}$, we let $\ell(Q) := \delta^k$, where the integer k is determined by
$$\delta^{k+1} \le \operatorname{diam}(Q) < \delta^k.$$

This result means that in typical situations it is sufficient to consider a dyadic system instead of the whole collection of balls.

Definition 5.3. Given one of the previous dyadic systems $\mathscr{D}^k$ and a nonnegative weight $h \in L^1_{\mathrm{loc}}$, we define its corresponding maximal operator, weighted by h, by

$$\mathcal{M}_h^{\mathscr{D}^k}[f](x) := \sup_{x \in Q \in \mathscr{D}^k} \left(\frac{1}{h(Q)} \int_Q |f|\, h\, d\mu \right), \quad x \in M,$$

for every $f \in L^1_{\mathrm{loc}}(h\, d\mu)$.

Lemma 5.4. *Uniformly in $k \in \{1, \dots, K\}$ and in the weight h, the maximal operator $\mathcal{M}_h^{\mathscr{D}^k}$ is of weak type $(1, 1)$ and strong type (p, p) for the measure $h\, d\mu$ for every $p \in (1, \infty]$.*

We refer the reader to [Lerner and Nazarov 2015, Theorem 15.1] for a detailed proof of this result and more details. For completeness, we give a short proof here.

Proof. Since $\mathcal{M}_h^{\mathscr{D}_k}$ is L^∞-bounded (and so $L^\infty(h\, d\mu)$-bounded), it suffices by interpolation to check its weak $L^1(h\, d\mu)$-boundedness.

Fix a function $f \in L^1(h\, d\mu)$. For every $\lambda > 0$, we consider the set

$$\Omega_\lambda := \{x \in M : \mathcal{M}_h^{\mathscr{D}^k}[f](x) > \lambda\}.$$

Due to the properties of the dyadic system, there exists a collection $\mathscr{Q} := (P)_{P \in \mathscr{Q}} \subset \mathscr{D}^k$ of dyadic sets such that $\Omega_\lambda = \bigcup_{P \in \mathscr{Q}} P$ (up to a subset of measure zero) and such that each $P \in \mathscr{Q}$ is maximal in Ω_λ and, for every $P \in \mathscr{Q}$,

$$\frac{1}{h(P)} \int_P |f| h\, d\mu > \lambda.$$

Due to the maximality, the dyadic sets $P \in \mathscr{Q}$ are pairwise disjoint and so we conclude that

$$h(\Omega_\lambda) = \sum_{P \in \mathscr{Q}} h(P) \le \lambda^{-1} \sum_{P \in \mathscr{Q}} \int_P |f| h\, d\mu \le \lambda^{-1} \|f\|_{L^1(h\, d\mu)},$$

which leads to weak $L^1(h\, d\mu)$-boundedness, uniformly with respect to h. □

We will also need the weak type of a slight modification of the previous maximal function.

Lemma 5.5. *Fix $k \in \{1, \dots, K\}$ and consider the maximal function*

$$\mathcal{M}^*[f](x) := \sup_{x \in Q \in \mathscr{D}^k} \inf_{y \in Q} \mathcal{M}[f](y), \quad x \in M,$$

for every $f \in L^1_{\mathrm{loc}}(h\, d\mu)$. It follows that $\mathcal{M}^*[f] = \mathcal{M}[f]$ almost everywhere. Consequently, the maximal operator $\mathcal{M}^*$ is of weak type $(1, 1)$ and strong type (p, p) for every $p \in (1, \infty]$.

Proof. Indeed, since the quantity $\inf_{y \in Q} \mathcal{M} f(y)$ is decreasing with respect to Q, it follows that

$$\mathcal{M}^*[f](x) = \lim_{\substack{x \in Q \\ \mathrm{diam}(Q) \to 0}} \inf_{y \in Q} \mathcal{M}[f](y) = \mathcal{M}[f](x),$$

where we have used the Lebesgue differentiation lemma, which implies the last equality for almost every $x \in M$. □

5B. ***Upper estimates of the maximal operator with sparse operators.*** From the previous subsection we know that we have several dyadic grids $\mathcal{D}^b$ for $b \in \{1, \dots, K\}$. In the sequel, we define $\mathcal{D} := \bigcup_{b=1}^K \mathcal{D}^b$ and call any element of $\mathcal{D}$ a *dyadic set*.

Definition 5.6 (sparse collection). A collection of dyadic sets $\mathcal{S} := (P)_{P\in\mathcal{S}} \subset \mathcal{D}$ is said to be *sparse* if for each $P \in \mathcal{S}$ one has

$$\sum_{Q\in \mathrm{ch}_{\mathcal{S}}(P)} \mu(Q) \le \tfrac{1}{2}\mu(P), \tag{5-3}$$

where $\mathrm{ch}_{\mathcal{S}}(P)$ is the collection of $\mathcal{S}$-children of P, namely the maximal elements of $\mathcal{S}$ that are strictly contained in P.

For a dyadic cube $Q \in \mathcal{D}$, we denote by $5Q$ its neighbourhood

$$5Q := \{x \in M : d(x, Q) \le 4\ell(Q)\}.$$

Theorem 5.7. *Consider an exponent $p \in (p_0, q_0)$. There exists a constant $C > 0$ such that, for all $f \in L^p$ and $g \in L^{p'}$, both supported in $5Q_0$ for some $Q_0 \in \mathcal{D}$, there exists a sparse collection $\mathcal{S} \subset \mathcal{D}$ (depending on f and g) with*

$$\left|\int_{Q_0} Tf\cdot g\, d\mu\right| \le C\sum_{P\in\mathcal{S}} \mu(P)\left(\fint_{5P} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \left(\fint_{5P} |g|^{q_0'}\, d\mu\right)^{\frac{1}{q_0'}}.$$

A careful examination of the proof shows that, indeed, if the initial ball Q_0 belongs to the dyadic grid $\mathcal{D}^b$ for some $b \in \{1, \dots, K\}$, then the whole sparse collection $\mathcal{S}$ belongs to the same dyadic grid $\mathcal{D}^b$. However, it will be important in Proposition 6.4 (to prove sharp weighted estimates for sparse operators) to play with the different dyadic grids.

Proof. Let $p \in (p_0, q_0)$. Suppose $f \in L^p$ and $g \in L^{p'}$, supported in $5Q_0$ for a dyadic set $Q_0 \in \mathcal{D}$. Fix the parameter $b \in \{1, \dots, K\}$ such that $Q_0 \in \mathcal{D}^b$. For some large enough constant η (which will be fixed later), define the subset

$$E = \left\{x \in Q_0 \;\middle|\; \max\{\mathcal{M}^*_{Q_0,p_0} f(x), T^\#_{Q_0} f(x)\} > \eta\left(\fint_{5Q_0} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}}\right\},$$

where both $\mathcal{M}^*_{Q_0,p_0}$ and $T^\#_{Q_0}$ are defined relative to the initial subset $Q_0 \in \mathcal{D}^b$ as follows: for every $x \in Q_0$,

$$\mathcal{M}^*_{Q_0,p_0}[f](x) := \sup_{\substack{x\in Q\subset Q_0\\ Q\in\mathcal{D}^b}} \inf_{y\in Q} \mathcal{M}_{p_0}[f](y)$$

and

$$T^\#_{Q_0} f(x) = \sup_{\substack{x\in Q\subset Q_0\\ Q\in\mathcal{D}^b}} \left(\fint_Q \left| T\int_{\ell(Q)^2}^\infty Q_t^{(N)}(f)\, \frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}}.$$

We extend both $\mathcal{M}^*_{Q_0,p_0}$ and $T^{\#}_{q_0}$ by 0 outside Q_0.

Due to the properties of dyadic subsets, we know that every $Q \in \mathscr{D}^b$ is contained in a ball with radius equivalent to $\ell(Q)$. Thus, up to some implicit constants, $\mathcal{M}^*_{Q_0,p_0}$ is bounded by the Hardy–Littlewood maximal function $\mathcal{M}_{p_0}$ (see Lemma 5.5) and $T^{\#}_{Q_0}$ is controlled by the maximal operator $T^{\#}$. So Proposition 4.6 yields that both $\mathcal{M}^*_{Q_0,p_0}$ and $T^{\#}_{Q_0}$ are of weak type (p_0, p_0).

Then it follows that $\mu(E) \lesssim (1/\eta)\mu(Q_0)$. So, if η is chosen large enough, then we know that E is an open proper subset of Q_0. In the sequel, all the implicit constants will only depend on the ambient space. For convenience, we only emphasize the dependence relative to η, which will be useful later to show how η can be fixed.

Consider a maximal dyadic covering of E, which is a collection of dyadic subsets $(B_j)_j \subset \mathscr{D}^b$ such that

- the collection covers E: $E = \bigsqcup_j B_j$, up to a set of null measure, with disjointness of the dyadic cubes;
- the dyadic cubes are maximal, in the sense that $B_j^a \cap E^c \neq \varnothing$ for every j, where we recall that B_j^a is the parent of B_j.

Since $\mu(B_j) \le \mu(E) \lesssim \eta^{-1}\mu(Q_0)$, if η is chosen large enough then, using the doubling property of the measure μ, we deduce that we also have

$$\mu(B_j^a) \le \mu(Q_0).$$

Due to the properties of the dyadic system, we then deduce that B_j^a is included in Q_0, and so the maximality of B_j yields

$$\max\left\{\inf_{y\in B_j^a} \mathcal{M}_{p_0}[f](y),\ \left(\fint_{B_j^a}\left|T\int_{\ell(B_j^a)^2}^{\infty} Q_t^{(N)}(f)\,\frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}}\right\} \le \eta\left(\fint_{5Q_0} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}. \tag{5-4}$$

We first initialize the collection $\mathcal{S} := \{Q_0\}$, which we are going to build in a recursive way. For $B \in \mathscr{D}$, define the operator T_B by

$$T_B f := T\int_0^{\ell(B)^2} Q_t^{(N)}(f\mathbb{1}_{5B})\,\frac{dt}{t}.$$

Step 1. In this step, we aim to show that, for some numerical constant C_0,

$$\left|\int_{Q_0} Tf\cdot g\,d\mu\right| \le C_0\eta|Q_0|\left(\fint_{Q_0}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\left(\fint_{Q_0}|g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}} + \sum_j\left|\int_{B_j} T_{B_j} f\cdot g\,d\mu\right|. \tag{5-5}$$

Seeking that, write

$$\left|\int_{Q_0} Tf\cdot g\,d\mu\right| \le \left|\int_{Q_0\setminus E} Tf\cdot g\,d\mu\right| + \left|\int_E Tf\cdot g\,d\mu\right|.$$

For the first part, notice that $|Tf(x)| \le T^{\#}_{Q_0} f(x) \le \eta\big(\fint_{5Q_0} |f|^{p_0}\, d\mu\big)^{1/p_0}$ for a.e. $x \in Q_0 \setminus E$ by definition of E. Hence

$$\left|\int_{Q_0\setminus E} Tf \cdot g\, d\mu\right| \le \eta \mu(Q_0) \left(\fint_{5Q_0} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}} \left(\fint_{Q_0} |g|^{q_0'}\, d\mu\right)^{\frac{1}{q_0'}}.$$

For the part on E, we use the covering to obtain

$$\begin{aligned}\left|\int_E Tf \cdot g\, d\mu\right| &\le \sum_j \left|\int_{B_j} T_{B_j} f \cdot g\, d\mu\right| + \left|\sum_j \int_{B_j} (T - T_{B_j}) f \cdot g\, d\mu\right| \\ &\le \sum_j \left|\int_{B_j} T_{B_j} f \cdot g\, d\mu\right| + \sum_j \mu(B_j) \left(\fint_{B_j} |(T - T_{B_j}) f|^{q_0}\, d\mu\right)^{\frac{1}{q_0}} \left(\fint_{B_j} |g|^{q_0'}\, d\mu\right)^{\frac{1}{q_0'}}.\end{aligned}$$

The first sum enters into the recursion and is acceptable in view of (5-5). For the second sum, we have

$$\left|(T - T_{B_j}) f\right| \le \left|T \int_{\ell(B_j)^2}^{\infty} Q_t^{(N)}(f)\, \frac{dt}{t}\right| + \left|T \int_0^{\ell(B_j)^2} Q_t^{(N)}(\mathbb{1}_{(5B_j)^c} f)\, \frac{dt}{t}\right|. \tag{5-6}$$

Using the doubling property, we can estimate the first term against the maximal operator and get

$$\begin{aligned}\left(\fint_{B_j} \left|T \int_{\ell(B_j)^2}^{\infty} Q_t^{(N)} f\, \frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}} &\lesssim \left(\fint_{B_j^a} \left|T \int_{\ell(B_j)^2}^{\infty} Q_t^{(N)} f\, \frac{dt}{t}\right|^{q_0}\right)^{\frac{1}{q_0}} \\ &\lesssim \inf_{z \in B_j^a} T^{\#} f(z) + \left(\fint_{B_j^a} \left|T \int_{\ell(B_j)^2}^{\ell(B_j^a)^2} Q_t^{(N)} f\, \frac{dt}{t}\right|^{q_0}\right)^{\frac{1}{q_0}}.\end{aligned}$$

By the maximality of the dyadic cubes B_j, we know that B_j^a intersects E^c; hence, from (5-4), we have

$$\inf_{z \in B_j^a} T^{\#} f(z) \le \eta \left(\fint_{5Q_0} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}}.$$

Moreover, we also know that for every dyadic set B_j we have

$$\inf_{y \in B_j^a} \mathcal{M}_{p_0}[f](y) \le \eta \left(\fint_{5Q_0} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}},$$

which yields in particular that

$$\begin{aligned}\left(\fint_{B_j^a} \left|T \int_{\ell(B_j)^2}^{\ell(B_j^a)^2} Q_t^{(N)} f\, \frac{dt}{t}\right|^{q_0}\right)^{\frac{1}{q_0}} &\lesssim \int_{\ell(B_j)^2}^{\ell(B_j^a)^2} \left(\fint_{B_j^a} |T Q_t^{(N)} f|^{q_0}\, d\mu\right)^{\frac{1}{q_0}} \frac{dt}{t} \\ &\lesssim \eta \left(\fint_{5Q_0} |f|^{p_0}\, d\mu\right)^{\frac{1}{p_0}}.\end{aligned}$$

We do not detail this last inequality, since it is a simpler particular case of the next one.

For the second term in (5-6), we use the L^{p_0}-L^{q_0} off-diagonal estimates for $T_t = TQ_t^{(N)}$ from Assumption (b). We have that

$$(5B_j)^c \subset \bigcup_{k=2}^{\infty} S_k(B_j),$$

and can therefore decompose

$$\begin{aligned}\left(\fint_{B_j}\left|T\int_0^{\ell(B_j)^2} Q_t^{(N)}(f\mathbb{1}_{(5B_j)^c})\,\frac{dt}{t}\right|^{q_0} d\mu\right)^{\frac{1}{q_0}} &\leq \int_0^{\ell(B_j)^2}\left(\fint_{B_j}|T_t(f\mathbb{1}_{(5B_j)^c})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\frac{dt}{t}\\ &\leq \sum_{k\geq 2}\int_0^{\ell(B_j)^2}\left(\fint_{B_j}|T_t(f\mathbb{1}_{S_k(B_j)})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\frac{dt}{t}.\end{aligned}$$

For fixed $t \in (0, \ell(B_j)^2)$ we know that T_t satisfies L^{p_0}-L^{q_0} off-diagonal estimates at the scale $\sqrt{t}$. We then cover $S_k(B_j)$ by balls of radius $\sqrt{t}$, with a finite overlap property (by the doubling property of the measure). We then deduce that these balls R satisfy

$$d(R, B_j) \geq \ell(B_j) \quad \text{and} \quad d(R, B_j) \simeq d(S_k(B_j), B_j) \simeq 2^k \ell(B_j).$$

Moreover, the number of these balls needed to cover $S_k(B_j)$ is controlled by

$$\#\{R\} \lesssim \left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\nu}. \tag{5-7}$$

By summing over such a covering, we get

$$\begin{aligned}\left(\fint_{B_j}|T_t(f\mathbb{1}_{S_k(B_j)})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim \sum_R\left(1+\frac{d(R,B_j)^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\fint_R |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \left(1+\frac{4^k\ell(B_j)^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\frac{\nu}{p_0}}|2^kB_j|^{-\frac{1}{p_0}}\sum_R\left(\int_R |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.\end{aligned}$$

By Hölder's inequality with the bounded overlap property of the collection $\{R\}$ with (5-7), we then have

$$\sum_R\left(\int_R |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}} \lesssim \left(\int_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\frac{\nu}{p_0'}},$$

hence

$$\begin{aligned}\left(\fint_{B_j}|T_t(f\mathbb{1}_{S_k(B_j)})|^{q_0}\,d\mu\right)^{\frac{1}{q_0}} &\lesssim \left(1+\frac{4^k\ell(B_j)^2}{t}\right)^{-\frac{\nu+1}{2}}\left(\frac{2^k\ell(B_j)}{\sqrt{t}}\right)^{\nu}\left(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \left(\frac{\sqrt{t}}{2^k\ell(B_j)}\right)\left(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}.\end{aligned}$$

We therefore get

$$\begin{aligned}\left(\fint_{B_j}\left|T\int_0^{\ell(B_j)^2} Q_t^{(N)}(f\mathbb{1}_{(5B_j)^c})\,\frac{dt}{t}\right|^{q_0}d\mu\right)^{\frac{1}{q_0}} &\lesssim \sum_{k=2}^{\infty}\left(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\int_0^{\ell(B_j)^2}\left(\frac{\sqrt{t}}{2^k\ell(B_j)}\right)\frac{dt}{t}\\ &\lesssim \sum_{k=2}^{\infty}2^{-k}\left(\fint_{S_k(B_j)}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \sup_{k\geq 2}\left(\fint_{2^kB_j}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\\ &\lesssim \inf_{z\in B_j^a}\mathcal{M}_{p_0}f(z)\lesssim \eta\left(\fint_{5Q_0}|f|^{p_0}\right)^{\frac{1}{p_0}}, \qquad (5\text{-}8)\end{aligned}$$

where we used (5-4).

On the other hand,

$$\left(\fint_{B_j}|g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}}\leq \inf_{z\in B_j}\mathcal{M}_{q_0'}g(z),$$

and, using $\bigcup_j B_j = E$, Kolmogorov's inequality, the fact that $\mu(E)\lesssim\mu(Q_0)$ (since η will be chosen larger than 1) and supp $g\subseteq 5Q_0$,

$$\sum_j \mu(B_j)\inf_{z\in B_j}\mathcal{M}_{q_0'}[g](z)\leq\int_E \mathcal{M}_{q_0'}[g](z)\,d\mu(z)\lesssim\mu(E)^{1-1/q_0'}\big\||g|^{q_0'}\big\|_1^{\frac{1}{q_0'}}\lesssim\mu(Q_0)\left(\fint_{5Q_0}|g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}}.$$

Therefore, putting all the estimates together, we have shown that

$$\sum_j\mu(B_j)\left(\fint_{B_j}|(T-T_{B_j})f|^{q_0}\,d\mu\right)^{\frac{1}{q_0}}\left(\fint_{B_j}|g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}}\lesssim\eta\mu(Q_0)\left(\fint_{5Q_0}|f|^{p_0}\right)^{\frac{1}{p_0}}\left(\fint_{5Q_0}|g|^{q_0'}\right)^{\frac{1}{q_0'}},$$

where the implicit constant only depends on the ambient space through previous numerical constants. This concludes the proof of (5-5).

Step 2 (recursion and conclusion). Starting from the initial dyadic cube Q_0, we have built a collection of dyadic cubes $(Q_1^j)_j$ such that

$$\left|\int_{Q_0}Tf\cdot g\,d\mu\right|\leq C_0\eta\mu(Q_0)\left(\fint_{5Q_0}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\left(\fint_{5Q_0}|g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}}+\sum_j\left|\int_{Q_1^j}T_{Q_1^j}f^j\cdot g^j\,d\mu\right|,$$

where f^j and g^j are both supported in $5Q_1^j$ and are pointwise bounded by f and g, respectively. Moreover, the following properties hold:

(a) *Small measure*: for some numerical constant $\widetilde{K}$,

$$\sum_j\mu(Q_1^j)\leq\frac{\widetilde{K}}{\eta}\mu(Q_0).$$

(b) *Disjointness and covering*: $(Q_1^j)_j$ are pairwise disjoint and included in Q_0.

We then add all these cubes to the collection $\mathcal{S}$, and rename $\mathcal{S} = \mathcal{S} \cup \bigcup_j \{Q_1^j\}$. And we iterate the procedure. For every cube Q_1^j, there exists a collection of dyadic cubes $(Q_2^{j,k})_k$ such that

$$\left|\int_{Q_1^j} Tf^j \cdot g^j \, d\mu\right| \\ \leq C_0 \eta \mu(Q_1^j) \left(\fint_{5Q_1^j} |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \left(\fint_{5Q_1^j} |g|^{q_0'} \, d\mu \right)^{\frac{1}{q_0'}} + \sum_k \left| \int_{Q_2^{j,k}} T_{Q_2^{j,k}} f^{j,k} \cdot g^{j,k} \, d\mu \right|,$$

with the properties that $f^{j,k}$ and $g^{j,k}$ are pointwise bounded by f and g, and also:

(a) *Small measure*: $$\sum_k \mu(Q_2^{j,k}) \leq \frac{\widetilde{K}}{\eta} \mu(Q_1^j).$$

(b) *Disjointness and covering*: $(Q_2^{j,k})_k$ are pairwise disjoint and included in Q_1^j.

We then add all these cubes to the collection $\mathcal{S}$, to obtain $\mathcal{S} = \mathcal{S} \cup \bigcup_{j,k} \{Q_2^{j,k}\}$. We iterate this reasoning, which allows us to build the collection $\mathcal{S}$ with the property that

$$\left|\int_{Q_0} Tf \cdot g \, d\mu\right| \leq C_0 \eta \sum_{Q \in \mathcal{S}} \mu(Q) \left(\fint_{5Q} |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \left(\fint_{5Q} |g|^{q_0'} \, d\mu \right)^{\frac{1}{q_0'}}.$$

Indeed, it is easy to check that the remainder term at the i-th step is an integral over a subset of measure which tends to 0 as i goes to ∞. So for fixed $f \in L^p$ and $g \in L^{p'}$ with $p' < \infty$, the remainder term also tends to 0.

It remains for us to check that this collection $\mathcal{S}$ is sparse.

So consider $Q \in \mathcal{S}$. By the disjointness property of the selected dyadic cubes, it is clear that any child $\overline{Q} \in \mathrm{ch}_{\mathcal{S}}(Q)$ has been selected (strictly) after Q and in the collection $\mathcal{S}_Q$ *generated* by Q. Using the smallness property of the measure in the algorithm, we know that summing over all the cubes R selected strictly after Q in the collection generated by Q gives us

$$\sum_{R \in \mathcal{S}_Q} \mu(R) = \sum_{l \geq 1} \left(\frac{\widetilde{K}}{\eta} \right)^l \mu(Q) \leq \frac{\widetilde{K}}{\eta - \widetilde{K}} \mu(Q).$$

We then deduce that, by choosing η large enough, the selected collection is sparse. □

6. Boundedness of a sparse operator

Definition 6.1 (A_p weight). A measurable function $\omega : M \to (0, \infty)$ is an A_p weight for some $p \in (1, \infty)$ if

$$[\omega]_{A_p} := \sup_{\text{ball } B} \left(\fint_B \omega \, d\mu \right) \left(\fint_B \omega^{1-p'} \, d\mu \right)^{p-1} < \infty,$$

with p' the conjugate exponent $p' = p/(p-1)$. For $p = 1$, we extend this notion with the characteristic constant

$$[\omega]_{A_1} := \sup_{\text{ball } B} \left(\fint_B \omega \, d\mu \right) \left(\operatorname*{ess\,inf}_{x \in B} \omega(x) \right)^{-1}.$$

Definition 6.2 (RH_q weight). A measurable function $\omega : M \to (0, \infty)$ is an RH_q weight for some $q \in (1, \infty)$ if

$$[\omega]_{\mathrm{RH}_q} := \sup_{\text{ball } B} \left(\fint_B \omega^q \, d\mu \right)^{\frac{1}{q}} \left(\fint_B \omega \, d\mu \right)^{-1} < \infty.$$

For $q = \infty$, we extend this notion with the characteristic constant

$$[\omega]_{\mathrm{RH}_\infty} := \sup_{\text{ball } B} \left(\operatorname*{ess\,sup}_{x \in B} \omega(x) \right) \left(\fint_B \omega \, d\mu \right)^{-1}.$$

We recall some well-known properties of the weight.

Lemma 6.3. (a) *If $p \in (1, \infty)$ and ω is a weight, then $\omega \in A_p$ if and only if $\omega^{1-p'} \in A_{p'}$ with*

$$[\omega^{1-p'}]_{A_{p'}} = [\omega]_{A_p}^{p'-1}.$$

(b) (see [Johnson and Neugebauer 1991]) *If $q \in [1, \infty]$, $s \in [1, \infty)$ and ω is a weight, then $\omega \in A_q \cap \mathrm{RH}_s$ if and only if $\omega^s \in A_{s(q-1)+1}$ with*

$$[\omega^s]_{A_{s(q-1)+1}} \leq [\omega]_{A_q}^s [\omega]_{\mathrm{RH}_s}^s.$$

We prove the following sharp weighted estimates for the "sparse" operators:

Proposition 6.4. *Let $p_0, q_0 \in [1, \infty]$ be two exponents with $p_0 < q_0$, and let $p \in (p_0, q_0)$. Suppose that S is a bounded operator on L^p and that there exists a constant $c > 0$ such that for all $f \in L^p$ and $g \in L^{p'}$ there exists a sparse collection $\mathcal{S}$ with*

$$|\langle S(f), g \rangle| \leq c \sum_{P \in \mathcal{S}} \left(\fint_{5P} |f|^{p_0} \, d\mu \right)^{\frac{1}{p_0}} \left(\fint_{5P} |g|^{q_0'} \, d\mu \right)^{\frac{1}{q_0'}} \mu(P).$$

Denote

$$r := \left(\frac{q_0}{p} \right)' \left(\frac{p}{p_0} - 1 \right) + 1 \quad \textit{and} \quad \delta := \min\{q_0', \; p_0(r-1)\}.$$

Then there exists a constant $C = C(S, p, p_0, q_0)$ such that, for every weight $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$, the operator S is bounded on L^p_ω with

$$\|S\|_{L^p_\omega \to L^p_\omega} \leq C \left([\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}} \right)^{\alpha},$$

with

$$\alpha := \frac{1}{\delta} \left(\frac{q_0}{p} \right)' = \max\left\{ \frac{1}{p - p_0}, \; \frac{q_0 - 1}{q_0 - p} \right\}.$$

In particular, by defining the specific exponent

$$\mathfrak{p} := 1 + \frac{p_0}{q_0'} \in (p_0, q_0),$$

we have $\alpha = 1/(p - p_0)$ *if* $p \in (p_0, \mathfrak{p}]$, *and* $\alpha = (q_0 - 1)/(q_0 - p)$ *if* $p \in [\mathfrak{p}, q_0)$.

Remark 6.5. The property $p_0 < \mathfrak{p}$ is equivalent to the condition $p_0 < q_0$, and the property $\mathfrak{p} < q_0$ is also equivalent to the condition $p_0 < q_0$. So the assumption guarantees us that

$$p_0 < \mathfrak{p} < q_0.$$

We note that, using extrapolation theory (as developed in [Auscher and Martell 2007a, Theorem 4.9]) and by tracking the behaviour of implicit constants with respect to the weights, a sharp weighted estimate for one particular exponent $p \in (p_0, q_0)$ allows us to get the sharp weighted estimates for all the exponents in the range $p \in (p_0, q_0)$. Here we are going to detail a proof which directly gives the weighted estimates for all such exponents.

Remarks 6.6. (1) In the case where $q_0 = p_0'$, it is $\mathfrak{p} = 2$ and we obtain sharp weighted estimates with the power

$$\alpha = \max\left\{\frac{1}{p - p_0}, \frac{1}{p + p_0 - pp_0}\right\}.$$

(2) In particular, in the situation where $p_0 = 1$ and $q_0 = \infty$, we recover the "usual" sharp behaviour, dictated by the A_2 conjecture, with the power

$$\alpha = \max\left\{1, \frac{1}{p - 1}\right\}.$$

(3) In the case $q_0 = \infty$, we obtain

$$\alpha = \max\{1, (p - p_0)^{-1}\},$$

which is the same exponent as in [Bui et al. 2015] and allows us to regain their result (the linear part) as explained in Section 3D.

Remark 6.7. For a weight ω, we know (see Lemma 6.3 and [Auscher and Martell 2007a, Lemma 4.4]) that

$$\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'} \iff \sigma := \omega^{1-p'} \in A_{p'/q_0'} \cap \mathrm{RH}_{(p_0'/p')'}.$$

These are also equivalent to

$$\omega^{(q_0/p)'} \in A_r$$

with $r := (q_0/p)'(p/p_0 - 1) + 1$. We have the estimates on the characteristic constants

$$[\omega^{(q_0/p)'}]_{A_r} \lesssim \left([\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}}\right)^{(q_0/p)'} \quad \text{and} \quad [\sigma]_{A_{p'/q_0'}} [\sigma]_{\mathrm{RH}_{(p_0'/p')'}} \lesssim \left([\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}}\right)^{p'-1}.$$

Proof of Proposition 6.4. Let us define three weights,

$$\sigma := \omega^{1-p'}, \quad u := \sigma^{(p_0'/p')'} \quad \text{and} \quad v := \omega^{(q_0/p)'}.$$

Then $u = v^{1-r'}$ with

$$r := \left(\frac{q_0}{p}\right)'\left(\frac{p}{p_0} - 1\right) + 1.$$

Combining the previous remark with Lemma 6.3, the fact that $\omega \in A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$ yields that $v \in A_r$ and so

$$\sup_{\text{ball } B}\left(\fint_B v\,d\mu\right)\left(\fint_B u\,d\mu\right)^{r-1} \le [v]_{A_r} \lesssim [\omega]^{(q_0/p)'},$$

where we set

$$[\omega] := [\omega]_{A_{p/p_0}}[\omega]_{\mathrm{RH}_{(q_0/p)'}},$$

the characteristic constant of the weight ω in the class $A_{p/p_0} \cap \mathrm{RH}_{(q_0/p)'}$. Using the comparison between dyadic subsets with balls and the doubling property of the measure μ, we then deduce that

$$\sup_{Q\in\mathscr{D}}\left(\fint_Q v\,d\mu\right)\left(\fint_Q u\,d\mu\right)^{r-1} \lesssim [v]_{A_r} \lesssim [\omega]^{(q_0/p)'}. \tag{6-1}$$

We know that the dual space (with respect to the measure $d\mu$) of L^p_ω is $L^{p'}_\sigma$. So the desired L^p_ω-boundedness of S is equivalent to the inequality

$$|\langle S(f), g\rangle| \lesssim [\omega]^\alpha \|f\|_{L^p_\omega}\|g\|_{L^{p'}_\sigma}. \tag{6-2}$$

Let us fix two functions $f \in L^p_\omega$ and $g \in L^{p'}_\sigma$. Then, by assumption, there exists a sparse collection $\mathscr{S}$ such that

$$|\langle S(f), g\rangle| \le c\sum_{P\in\mathscr{S}}\left(\fint_{5P}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\left(\fint_{5P}|g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}}\mu(P).$$

For every $P \in \mathscr{S}$, we know that there exists a dyadic cube $\overline{P}$ such that $5P \subset \overline{P}$ and $\mu(\overline{P}) \lesssim \mu(5P)$. We split $\mathscr{S}$ into K collections $(\mathscr{S}_k)_{k=1,\dots,K}$ for which $\overline{P} \in \mathscr{D}^k$. Each collection $\mathscr{S}_k$ is still sparse, since it is a subcollection of $\mathscr{S}$.

We now fix $k \in \{1, \dots, K\}$. For every $P \in \mathscr{S}_k$, we set $E_P \subset P$ to be the set of all $x \in P$ which are not contained in any $\mathscr{S}_k$-child of P. By the sparseness property of $\mathscr{S}_k$, we then have

$$\mu(P) \le 2\mu(E_P)$$

and the sets $(E_P)_{P\in\mathscr{S}_k}$ are pairwise disjoint.

So we have

$$|\langle S(f), g\rangle| \lesssim \sum_{k=1}^{K}\sum_{P\in\mathscr{S}_k}\left(\fint_{\overline{P}}|f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}}\left(\fint_{\overline{P}}|g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}}\mu(E_P). \tag{6-3}$$

We then change the measure with the weight u as follows:

$$\left(\fint_{\overline{P}} |f|^{p_0}\,d\mu\right)^{\frac{1}{p_0}} = \left(\fint_{\overline{P}} |u^{-1/p_0} f|^{p_0} u\,d\mu\right)^{\frac{1}{p_0}}$$
$$= \left(\frac{1}{u(\overline{P})}\int_{\overline{P}} |u^{-1/p_0} f|^{p_0} u\,d\mu\right)^{\frac{1}{p_0}} \left(\fint_{\overline{P}} u\,d\mu\right)^{\frac{1}{p_0}}. \tag{6-4}$$

Similarly, we have

$$\left(\fint_{\overline{P}} |g|^{q_0'}\,d\mu\right)^{\frac{1}{q_0'}} = \left(\fint_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v\,d\mu\right)^{\frac{1}{q_0'}}$$
$$= \left(\frac{1}{v(\overline{P})}\int_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v\,d\mu\right)^{\frac{1}{q_0'}} \left(\fint_{\overline{P}} v\,d\mu\right)^{\frac{1}{q_0'}}. \tag{6-5}$$

Set $\alpha := \delta^{-1}(q_0/p)'$, with $\delta := \min\{q_0', p_0(r-1)\}$ and $\beta := 1/p_0 - (r-1)/q_0'$. We note that $\beta \le 0$ is equivalent to $p \ge \mathfrak{p}$ and is also equivalent to $\delta = q_0'$; whereas $\beta \ge 0$ is equivalent to $p \le \mathfrak{p}$ and to $\delta = p_0(r-1)$. We are first going to detail the end of the proof in the case $\beta \le 0$ and then explain that the situation $\beta \ge 0$ is very similar.

Step 1 (the case $p \ge \mathfrak{p}$, i.e., $\beta \le 0$). Putting the two last estimates, (6-4) and (6-5), into (6-3) yields

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \sum_{P\in\mathscr{S}_k} \left(\frac{1}{u(\overline{P})}\int_{\overline{P}} |u^{-1/p_0} f|^{p_0} u\,d\mu\right)^{\frac{1}{p_0}}$$
$$\times \left(\frac{1}{v(\overline{P})}\int_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v\,d\mu\right)^{\frac{1}{q_0'}} \left(\fint_{\overline{P}} u\,d\mu\right)^{\beta} \mu(E_P), \tag{6-6}$$

where we used that

$$\left(\fint_{\overline{P}} u\,d\mu\right)^{\frac{r-1}{\delta}} \left(\fint_{\overline{P}} v\,d\mu\right)^{\frac{1}{\delta}} \lesssim [\omega]^{\delta^{-1}(q_0/p)'}, \tag{6-7}$$

which comes from (6-1).

Since $\beta \le 0$ and $E_P \subset P \subset \overline{P}$ with $\mu(E_p) \ge \frac{1}{2}\mu(P) \ge c_\nu \mu(\overline{P})$, where c_ν is a constant only dependent on the doubling property of μ and constants of the dyadic system, we deduce that

$$\left(\fint_{\overline{P}} u\,d\mu\right)^{\beta} \le c_\nu^{-\beta} \left(\fint_{E_P} u\,d\mu\right)^{\beta}.$$

Then let us define two other weights,

$$\varpi := \sigma v^{p'/q_0'} \quad \text{and} \quad \rho := \omega u^{p/p_0}. \tag{6-8}$$

Since $u = v^{1-r'}$, an easy computation yields

$$u^{-\beta} \varpi^{1/p'} \rho^{1/p} = \sigma^{1/p'} \omega^{1/p} = 1. \tag{6-9}$$

By Hölder's inequality with $\gamma := 1/(1-\beta) \in [0,1]$ and the relation

$$1 = \frac{\gamma}{p} + \frac{\gamma}{p'} + (1-\gamma),$$

we have

$$\mu(E_P) = \int_{E_P} (u^{-\beta} \varpi^{1/p'} \rho^{1/p})^\gamma \, d\mu \le u(E_P)^{-\beta\gamma} \varpi(E_P)^{\gamma/p'} \rho(E_P)^{\gamma/p}. \tag{6-10}$$

Hence,

$$\left(\fint_{E_P} u \, d\mu \right)^\beta \mu(E_P) = u(E_P)^\beta \mu(E_P)^{1-\beta} \le \varpi(E_P)^{1/p'} \rho(E_P)^{1/p}.$$

So, coming back to (6-6) we then deduce that

$$\begin{aligned} |\langle S(f), g\rangle| \lesssim [\omega]^\alpha \sum_{k=1}^K \sum_{P \in \mathcal{S}_k} \left(\frac{1}{u(\overline{P})} \int_{\overline{P}} |u^{-1/p_0} f|^{p_0} u \, d\mu \right)^{\frac{1}{p_0}} \\ \times \left(\frac{1}{v(\overline{P})} \int_{\overline{P}} |v^{-1/q_0'} g|^{q_0'} v \, d\mu \right)^{\frac{1}{q_0'}} \varpi(E_P)^{1/p'} \rho(E_P)^{1/p}. \end{aligned}$$

With the dyadic weighted maximal function (see Lemma 5.4 for its definition) and since $E_P \subset P \subset \overline{P}$, we deduce that

$$\begin{aligned} &|\langle S(f), g\rangle| \\ &\lesssim [\omega]^\alpha \sum_{k=1}^K \sum_{P \in \mathcal{S}_k} \inf_{E_P} \mathcal{M}_u^{\mathcal{D}^k} \left(|u^{-1/p_0} f|^{p_0} \right)^{1/p_0} \inf_{E_P} \mathcal{M}_v^{\mathcal{D}^k} \left(|v^{-1/q_0'} g|^{q_0'} \right)^{1/q_0'} \varpi(E_P)^{1/p'} \rho(E_P)^{1/p} \\ &\lesssim [\omega]^\alpha \sum_{k=1}^K \sum_{P \in \mathcal{S}_k} \left(\int_{E_P} \mathcal{M}_u^{\mathcal{D}^k} \left(|u^{-1/p_0} f|^{p_0} \right)^{p/p_0} \rho \, d\mu \right)^{\frac{1}{p}} \left(\int_{E_P} \mathcal{M}_v^{\mathcal{D}^k} \left(|v^{-1/q_0'} g|^{q_0'} \right)^{p'/q_0'} \varpi \, d\mu \right)^{\frac{1}{p'}}. \end{aligned}$$

By Hölder's inequality and using the disjointness of the collection $(E_P)_{P \in \mathcal{S}_k}$, one gets

$$|\langle S(f), g\rangle| \lesssim [\omega]^\alpha \sum_{k=1}^K \left(\int \mathcal{M}_u^{\mathcal{D}^k} \left(|u^{-1/p_0} f|^{p_0} \right)^{p/p_0} \rho \, d\mu \right)^{\frac{1}{p}} \left(\int \mathcal{M}_v^{\mathcal{D}^k} \left(|v^{-1/q_0'} g|^{q_0'} \right)^{p'/q_0'} \varpi \, d\mu \right)^{\frac{1}{p'}}.$$

Since $p \in (p_0, q_0)$, the dyadic maximal function $\mathcal{M}_u^{\mathcal{D}^k}$ is $L^{p/p_0}(u\, d\mu)$-bounded (uniformly in the weight u; see Lemma 5.4) and similarly for the weight v, hence

$$|\langle S(f), g\rangle| \lesssim [\omega]^\alpha \left(\int |u^{-1/p_0} f|^p \rho \, d\mu \right)^{\frac{1}{p}} \left(\int |v^{-1/q_0'} g|^{p'} \varpi \, d\mu \right)^{\frac{1}{p'}}.$$

Due to the definition (6-8) of ρ and ϖ, we conclude

$$|\langle S(f), g\rangle| \lesssim [\omega]^\alpha \left(\int |f|^p \omega \, d\mu \right)^{\frac{1}{p}} \left(\int |g|^{p'} \sigma \, d\mu \right)^{\frac{1}{p'}},$$

which corresponds to (6-2).

Step 2 (the case $p \leq \mathfrak{p}$, i.e., $\beta \geq 0$). In this situation, (6-7) still holds and, due to the choice of δ, it yields (instead of (6-6))

$$|\langle S(f), g\rangle| \lesssim [\omega]^{\alpha} \sum_{k=1}^{K} \sum_{P \in \mathcal{S}_k} \left(\frac{1}{u(\bar{P})} \int_{\bar{P}} |u^{-1/p_0} f|^{p_0} u \, d\mu\right)^{\frac{1}{p_0}} \times \left(\frac{1}{v(\bar{P})} \int_{\bar{P}} |v^{-1/q_0'} g|^{q_0'} v \, d\mu\right)^{\frac{1}{q_0'}} \left(\fint_{\bar{P}} v \, d\mu\right)^{\bar{\beta}} \mu(E_P), \quad (6\text{-}11)$$

with

$$\bar{\beta} := \frac{1}{q_0'} - \frac{1}{\delta} = \frac{1}{q_0'} - \frac{1}{p_0(r-1)} = -(r-1)\beta.$$

In particular, since we are in the situation $\beta \geq 0$, we know that $\bar{\beta} \leq 0$. We can then reproduce a similar reasoning as in the first step, using the inequality

$$\left(\fint_{\bar{P}} v \, d\mu\right)^{\bar{\beta}} \lesssim \left(\fint_{E_P} v \, d\mu\right)^{\bar{\beta}}.$$

We use the same weights ϖ and ρ as defined in (6-8), and the exact same computations allow us to conclude since, by definition, $u = v^{1-r'}$, which implies

$$u^{-\beta} = v^{-\beta(1-r')} = v^{-\bar{\beta}}. \qquad \square$$

7. Sharpness of the weighted estimates for the "sparse operators"

We are going to show that the exponents we obtained previously are sharp for sparse operators. We do so only for dimension $n = 1$, since higher-dimensional cases follow through minor modifications.

So let us consider the Euclidean space $\mathbb{R}$, equipped with its natural metric and measure. We first state some easy estimates on specific weights. For $p > 1$, the weight $w_\alpha : x \mapsto |x|^\alpha$ belongs to A_p if and only if $-1 < \alpha < p - 1$. One has

$$[w_{-1+\varepsilon}]_{A_p} \sim \varepsilon^{-1} \quad \text{and} \quad [w_{p-1-\varepsilon}]_{A_p} \sim \varepsilon^{-(p-1)}$$

as $\varepsilon \to 0$.

On the other hand, if $s > 1$ then $w_{-1/s+\varepsilon}$ is critical for RH_s. When $\varepsilon \to 0$,

$$[w_{-1/s+\varepsilon}]_{\mathrm{RH}_s} \sim \varepsilon^{-1/s}.$$

Having these sharp estimates, we are now going to prove the optimality of Proposition 6.4. Consider the particular sparse collection $\mathcal{S}$ of those dyadic intervals contained in $[0, 1]$ that contain 0, namely $\mathcal{S} = \{I_n := [0, 2^{-n}] : n \in \mathbb{N}\}$. Then $\mathcal{S}$ is a sparse collection. We consider sharpness in the inequality

$$\sum_{I \in \mathcal{S}} |I| \langle |f|^{p_0}\rangle_I^{1/p_0} \langle |g|^{q_0'}\rangle_I^{1/q_0'} \lesssim \Phi([\omega]_{p_0,q_0,p}) \|f\|_{L^p_\omega} \|g\|_{L^{p'}_\sigma}, \tag{7-1}$$

where $1 \leq p_0 < 2 < q_0 \leq \infty$ are fixed and, to simplify the notation, we denote by $\langle \cdot \rangle_I$ the average on the interval I.

Proposition 7.1. *For $p \in (p_0, q_0)$, there exist functions f and g such that, asymptotically as $r \to \infty$, the power function $\Phi(r) = r^\alpha$ is the best possible choice, where $\alpha = 1/(p - p_0)$ if $p \in (p_0, \mathfrak{p}]$ and $\alpha = (q_0 - 1)/(q_0 - p)$ if $p \in [\mathfrak{p}, q_0)$.*

Notice that for $q_0 = \infty$ the above sum corresponds to the pointwise-defined operator

$$Sf = \sum_{I \subseteq \mathscr{D}[0,1], 0 \in I} \langle |f^{p_0}| \rangle_I^{1/p_0} \chi_I$$

tested against g.

For convenience, we also will use the following notation (introduced in [Auscher and Martell 2007a]): for a weight ω,

$$[\omega]_{p_0,q_0,p} := [\omega]_{A_{p/p_0}} [\omega]_{\mathrm{RH}_{(q_0/p)'}}.$$

Proof. Let $p \in (p_0, \mathfrak{p}]$. Consider functions $f_\varepsilon := x \mapsto x^{-1/p_0+\varepsilon} \chi_{[0,1]}$ and $g_\varepsilon := x \mapsto x^{-1/p_0'+\varepsilon} \chi_{[0,1]}$. One calculates, for $I_n = [0, 2^{-n}]$ with $n \geqslant 0$, that

$$\langle |f_\varepsilon|^{p_0} \rangle_{I_n}^{1/p_0} = \frac{2^{n/p_0 - n\varepsilon}}{(p_0 \varepsilon)^{1/p_0}} \sim \varepsilon^{-1/p_0} 2^{-n\varepsilon} 2^{n/p_0}$$

and

$$\langle |g_\varepsilon|^{q_0'} \rangle_{I_n}^{1/q_0'} = \frac{2^{n/p_0' - n\varepsilon}}{(1 - q_0'/p_0' + q_0' \varepsilon)^{1/q_0'}} \sim 2^{-n\varepsilon} 2^{n/p_0'}$$

by noticing that $q_0'/p_0' < 1$.

Hence we obtain, for the left-hand side of (7-1),

$$\varepsilon^{-1/p_0} \sum_{n=0}^{\infty} 2^{-2n\varepsilon} = \varepsilon^{-1/p_0} \frac{1}{1 - \left(\frac{1}{4}\right)^\varepsilon} \sim \varepsilon^{-1/p_0} \varepsilon^{-1}.$$

Choose the weight $\omega_\varepsilon = w_{p/p_0 - 1 - \varepsilon} := x \mapsto x^{p/p_0 - 1 - \varepsilon}$, which is critical for A_{p/p_0}, with

$$[\omega_\varepsilon]_{A_{p/p_0}} \sim \varepsilon^{-(p/p_0 - 1)} \quad \text{as } \varepsilon \to 0.$$

We also notice that ω_ε is a power weight of positive exponent and therefore $[\omega_\varepsilon]_{\mathrm{RH}_{(q_0/p)'}} \sim 1$ as $\varepsilon \to 0$. Thus, $[\omega_\varepsilon]_{p_0,q_0,p} \sim \varepsilon^{-(p/p_0 - 1)}$ and $[\omega_\varepsilon]_{p_0,q_0,p}^{1/(p-p_0)} \sim \varepsilon^{-1/p_0}$. We calculate

$$\|f_\varepsilon\|_{L^p_{\omega_\varepsilon}} = \left(\int_0^1 x^{-1+(p-1)\varepsilon} \, dx \right)^{\frac{1}{p}} \sim \varepsilon^{-1/p}.$$

With $\sigma_\varepsilon = \omega_\varepsilon^{1-p'}$ we calculate

$$\|g_\varepsilon\|_{L^{p'}_{\sigma_\varepsilon}} = \left(\int_0^1 x^{(2p'-1)\varepsilon - 1} \, dx \right)^{\frac{1}{p'}} \sim \varepsilon^{-1/p'}.$$

Gathering the information gives $\varepsilon^{-1/p} \varepsilon^{-1/p'} \varepsilon^{-1/p_0}$ on the right-hand side and $\varepsilon^{-1} \varepsilon^{-1/p_0}$ on the left, showing that the choice of Φ cannot be improved for this range of p.

Now let $p \in [\mathfrak{p}, q_0)$. To treat this range, we apply what we have found before to the modified exponents $1 \le q_0' < 2 < p_0' \le \infty$. We have seen examples of sharpness for the sum

$$\sum_{I \in \mathcal{S}} |I| \langle |f|^{q_0'} \rangle_I^{1/q_0'} \langle |g|^{p_0} \rangle_I^{1/p_0} \sim [\omega]_{q_0', p_0', s}^{1/(s-q_0')} \|f\|_{L^s_\omega} \|g\|_{L^{s'}_\sigma}$$

when $q_0' \le s \le \mathfrak{p}(q_0', p_0')$. Indeed, with $f_\varepsilon := x \mapsto x^{-1/q_0'+\varepsilon} \chi_{[0,1]}$, $g_\varepsilon := x \mapsto x^{-1/q_0+\varepsilon} \chi_{[0,1]}$ and $\omega_\varepsilon := x \mapsto |x|^{s/q_0'-1-\varepsilon}$ we obtain that the left-hand side is of order $\varepsilon^{-1}\varepsilon^{-1/q_0'}$, and $\|f_\varepsilon\|_{L^s_{\omega_\varepsilon}} \sim \varepsilon^{-1/s}$ and $\|g_\varepsilon\|_{L^{s'}_\sigma} \sim \varepsilon^{-1/s'}$. Now observe that $[\mathfrak{p}(q_0', p_0')]' = \mathfrak{p}(p_0, q_0)$. Note also that, therefore, $\mathfrak{p}(p_0, q_0) \le s' \le q_0$. Using this for $s' = p$, it remains to calculate $[\sigma_\varepsilon]_{p_0,q_0,p}^{(q_0-1)/(q_0-p)}$, where $\sigma_\varepsilon = \omega_\varepsilon^{1-p}$:

$$\sigma_\varepsilon(x) = |x|^{(p'/q_0'-1-\varepsilon)(1-p)} = |x|^{-1/(q_0/p)'+(p-1)\varepsilon}.$$

This weight is of negative exponent and critical for $\mathrm{RH}_{(q_0/p)'}$ with $[\sigma_\varepsilon]_{p_0,q_0,p} \sim \varepsilon^{-1/(q_0/p)'}$. Therefore, $[\sigma_\varepsilon]_{p_0,q_0,p}^{(q_0-1)/(q_0-p)} \sim \varepsilon^{-1/q_0'}$. Gathering the information, we obtain that the left-hand side is of order $\varepsilon^{-1}\varepsilon^{-1/q_0'}$ and of order $\varepsilon^{-1/q_0'}\varepsilon^{-1/p}\varepsilon^{-1/p'}$ when using $\Phi(r) = r^\alpha$, showing that the estimate cannot be improved. □

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Received 5 Oct 2015. Revised 19 Feb 2016. Accepted 30 Mar 2016.

FRÉDÉRIC BERNICOT: frederic.bernicot@univ-nantes.fr
Laboratoire Jean Leray, CNRS - Universite de Nantes, 2, rue de la Houssinière, 44322 Nantes Cedex 3, France

DOROTHEE FREY: d.frey@tudelft.nl
Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, Netherlands

STEFANIE PETERMICHL: stefanie.petermichl@math.univ-toulouse.fr
Institut de Mathématiques de Toulouse, Université Paul Sabatier, 118 Route de Narbonne, 31062 Toulouse Cedex 9, France

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1115

EXISTENCE, UNIQUENESS AND OPTIMAL REGULARITY RESULTS FOR VERY WEAK SOLUTIONS TO NONLINEAR ELLIPTIC SYSTEMS

MIROSLAV BULÍČEK, LARS DIENING AND SEBASTIAN SCHWARZACHER

We establish existence, uniqueness and optimal regularity results for very weak solutions to certain nonlinear elliptic boundary value problems. We introduce structural asymptotic assumptions of Uhlenbeck type on the nonlinearity, which are sufficient and in many cases also necessary for building such a theory. We provide a unified approach that leads qualitatively to the same theory as the one available for linear elliptic problems with continuous coefficients, e.g., the Poisson equation.

The result is based on several novel tools that are of independent interest: local and global estimates for (non)linear elliptic systems in weighted Lebesgue spaces with Muckenhoupt weights, a generalization of the celebrated div-curl lemma for identification of a weak limit in border line spaces and the introduction of a Lipschitz approximation that is stable in weighted Sobolev spaces.

1. Introduction

We study the following nonlinear problem: for a given n-dimensional domain $\Omega \subset \mathbb{R}^n$ with $n \geq 2$, a given $f : \Omega \to \mathbb{R}^{n\times N}$ with $N \in \mathbb{N}$ arbitrary and a given mapping $A : \Omega \times \mathbb{R}^{n\times N} \to \mathbb{R}^{n\times N}$, find $u : \Omega \to \mathbb{R}^N$ satisfying

$$\begin{aligned} -\operatorname{div}(A(x, \nabla u)) &= -\operatorname{div} f \quad && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{1-1}$$

Owing to a significant number of problems originating in various applications, it is natural to require that A is a Carathéodory mapping, satisfying the natural coercivity, growth and (strict) monotonicity conditions. It means that

$$A(\,\cdot\,, \eta) \text{ is measurable for any fixed } \eta \in \mathbb{R}^{n\times N}, \tag{1-2}$$

$$A(x, \cdot\,) \text{ is continuous for almost all } x \in \Omega, \tag{1-3}$$

Bulíček's work is supported by the project LL1202 financed by the Ministry of Education, Youth and Sports of the Czech Republic and by the University Center for Mathematical Modeling, Applied Analysis and Computational Mathematics (MathMAC). Schwarzacher thanks the program PRVOUK P47, financed by the Charles University in Prague. Bulíček is a member of the Nečas Center for Mathematical Modeling.
MSC2010: 35D99, 35J57, 35J60, 35A01.
Keywords: nonlinear elliptic systems, weighted estimates, existence, uniqueness, very weak solution, monotone operator, div-curl-biting lemma, weighted space, Muckenhoupt weights.

and there exist positive constants c_1 and c_2 such that for almost all $x \in \Omega$ and all $\eta_1, \eta_2 \in \mathbb{R}^{n\times N}$

$$
\begin{aligned}
c_1|\eta_1|^2 - c_2 &\le A(x,\eta_1)\cdot\eta_1 && \text{(coercivity)}, && \text{(1-4)}\\
|A(x,\eta_1)| &\le c_2(1+|\eta_1|) && \text{(growth)}, && \text{(1-5)}\\
0 &\le (A(x,\eta_1)-A(x,\eta_2))\cdot(\eta_1-\eta_2) && \text{(monotonicity)}. && \text{(1-6)}
\end{aligned}
$$

If for all $\eta_1 \neq \eta_2$ the inequality (1-6) is strict, then A is said to be strictly monotone.

Under the assumptions (1-2)–(1-6), it is standard to show (with the help of the Minty method [1963]) that, for any $f \in L^2(\Omega;\mathbb{R}^{n\times N})$, there exists $u \in W_0^{1,2}(\Omega;\mathbb{R}^N)$ that solves (1-1) in the sense of distribution. In addition if A is strictly monotone, then this solution is unique in the class of $W_0^{1,2}(\Omega;\mathbb{R}^N)$-weak solutions.

An important question that immediately arises is whether such a result can be extended to a more general setting. Namely,

$$
\begin{aligned}
&\textit{whether for any } f \in L^q(\Omega;\mathbb{R}^{n\times N}) \textit{ with } q\in(1,\infty)\\
&\textit{there exists a (unique) } u \in W_0^{1,q}(\Omega;\mathbb{R}^N) \textit{ solving (1-1) in the weak sense.}
\end{aligned}
\qquad (\mathcal{Q})
$$

If $q \neq 2$, then we call the problem of existence and uniqueness to (1-1) beyond the natural pairing. If $q > 2$ and $f \in L^q(\Omega;\mathbb{R}^{n\times N})$, then $f \in L^2(\Omega;\mathbb{R}^{n\times N})$ as well, and the standard monotone operator theory in the duality pairing provides a $W_0^{1,2}(\Omega;\mathbb{R}^N)$ solution to (1-1). Thus, in this case, $(\mathcal{Q})$ calls only for improvement of the integrability of ∇u. If $q < 2$, then the considered question is more challenging as the existence of an object with which to start any kind of analysis is unclear. This is the reason why, for $1 < q < 2$, $W_0^{1,q}(\Omega;\mathbb{R}^N)$-solutions are called *very weak solutions*.

Our general aim is to establish, for a given $f \in L^q(\Omega;\mathbb{R}^{n\times N})$ with $q \in (1,\infty)\setminus 2$, the existence of a (unique) $W_0^{1,q}(\Omega;\mathbb{R}^N)$ solution to (1-1)–(1-6), i.e., to give the affirmative answer to $(\mathcal{Q})$. However, for general operators, this is not possible due to the following two reasons:

(i) the way how the nonlinearity $A(x,\eta)$ depends on η,

(ii) the way how the nonlinearity $A(x,\eta)$ depends on x.

We shall discuss each of these points from two perspectives: the available counterexamples and so far established affirmative results (that were rather sporadic and had several limitations).

First, we consider (1-1) with A depending only on η. If $q \ge 2$, then there always exists a (unique) weak solution and the only difficult part is to obtain appropriate a priori estimates in the space $W_0^{1,q}(\Omega;\mathbb{R}^N)$. On the one hand, for general operators, such a priori estimates are not true for large $q \gg 2$. This follows from the counterexamples due to Nečas [1977] and Sverák and Yan [2002], where they found a mapping A that does not depend on x and satisfies[1] (1-2)–(1-6) and showed that the corresponding unique weak solution is not in $\mathscr{C}^1$ or is even unbounded for smooth f. This directly contradicts the general theory for $q \gg 2$. The singular behavior of solutions in the above-mentioned counterexamples is due to the fact that the mapping A depends highly nonlinearly on the *vectorial* variable η. On the other hand,

[1]Not only does the mapping A satisfy (1-2)–(1-6), it has even more structure. It is given as a derivative of a uniformly convex smooth potential F, which makes the counterexamples even stronger.

if $q \in [2, 2+\varepsilon)$, then the $W_0^{1,q}(\Omega; \mathbb{R}^N)$ theory can be built for general mappings fulfilling only (1-2)–(1-6), where $\varepsilon > 0$ depends on c_1 and c_2. For such q, it is known that, if $f \in L^q(\Omega; \mathbb{R}^{n\times N})$, then there exists a solution $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ to (1-1). Such a result can be obtained by using the reverse Hölder inequality (see, e.g., [Giaquinta 1983]) and holds also for more general growth conditions, including operators of p-Laplacian type. For the p-Laplacian itself, $A(x, \eta) := |\eta|^{p-2}\eta$ with $p \in (1, \infty)$, various positive results are known for large exponents (in this case $q \in (p, \infty)$ or even BMO estimates) [Iwaniec 1983; Caffarelli and Peral 1998; Diening et al. 2012]. The theory is built on the seminal works of Uraltseva [1968] (the scalar case) and Uhlenbeck [1977] (the vectorial case).

For $q < 2$, the situation is even more delicate. In this case, the existence of any solution is not straightforward at all. Indeed, a general existence theory for operators satisfying (1-2)–(1-6) alone might be impossible to get. Up to now, the only general result holds for $q \in (2-\varepsilon, 2+\varepsilon)$ with ε depending only on c_1 and c_2 and A being uniformly monotone and also uniformly Lipschitz continuous, i.e., for all $\eta_1, \eta_2 \in \mathbb{R}^{n\times N}$ and almost all $x \in \Omega$,

$$|A(x, \eta_1) - A(x, \eta_2)| \le c_2|\eta_1 - \eta_2|. \tag{1-7}$$

In this case, we know that for all $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ there exists a unique solution $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ to (1-1) whenever $q \in (2-\varepsilon, 2+\varepsilon)$ [Bulíček 2012], and we also recall [Greco et al. 1997] for the result in the so-called grand Lebesgue spaces $L^{(2)}(\Omega)$. Moreover, for a general operator satisfying only (1-4)–(1-5), it may be shown with the help of the technique developed in [Bulíček 2012] that any very weak solution to (1-1) satisfies the uniform estimate

$$\int_\Omega |\nabla u|^q \,\mathrm{d}x \le C(c_1, c_2, q, \Omega) \int_\Omega |f|^q \,\mathrm{d}x \quad \text{for all } q \in (2-\varepsilon, 2+\varepsilon). \tag{1-8}$$

However, any existence theory for q “away” from 2 is either missing or impossible.

More positive results are available in the scalar case $N = 1$ (and even for a more general class of operators including the p-Laplacian) but for the *smoother* right-hand side, i.e., the case when $f \in W^{1,1}(\Omega; \mathbb{R}^n)$ or at least $f \in \mathrm{BV}(\Omega; \mathbb{R}^n)$. Then the existence of a very weak solution is known; see the pioneering works [Boccardo and Gallouët 1992; Stampacchia 1965]. Furthermore, one can study further qualitative properties of such a solution [Mingione 2013]. Moreover, in case $f \in W^{1,1}(\Omega; \mathbb{R}^n)$, the uniqueness of a solution can be shown in the class of *entropy* solutions [Bénilan et al. 1995; Boccardo et al. 1996; Dal Maso et al. 1997; 1999]. On the other hand, in case $f \in \mathrm{BV}(\Omega; \mathbb{R}^n)$, or more precisely if $\operatorname{div} f$ is only a Radon measure, the uniqueness is not known. An exception is the case when $\operatorname{div} f$ is a finite sum of Dirac measures. In that case, the study on isolated singularities by Serrin implies the uniqueness for very general nonlinear operators including the p-Laplace equation; see [Serrin 1965; Friedman and Véron 1986] and references therein. To conclude this part, we would like to emphasize that all results for smoother right-hand side surely do not cover the full generality of the result we would like to have, which may be easily seen in the framework of the Sobolev embedding. Indeed, if $f \in W^{1,1}(\Omega; \mathbb{R}^n)$, then $f \in L^{n/(n-1)}(\Omega; \mathbb{R}^n)$ and we see that the case $q \in (2, n'(p-1))$ remains untouched even in the scalar case.[2]

[2]Throughout the paper, we use the notation of dual exponents $q' := q/(q-1)$.

The second obstacle, related to (ii), is the possible discontinuity of the operator with respect to the spatial variable. To demonstrate this in more detail, we consider the *linear* problem

$$\begin{aligned} -\operatorname{div}(a(x)\nabla u) &= -\operatorname{div} f && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{1-9}$$

with a uniformly elliptic matrix a. Note here that (1-9) is a particular case of (1-1) with $A(x,\eta) := a(x)\eta$ and A fulfilling (1-2)–(1-6) with $N = 1$. In case a is continuous and Ω is a $\mathscr{C}^1$-domain, one can use the singular operator theory and show that for any $f \in L^q(\Omega;\mathbb{R}^n)$ there exists a unique weak solution $u \in W_0^{1,q}(\Omega)$ to (1-9) [Dolzmann and Müller 1995, Lemma 2]. This can be weakened to the case when a has coefficients with vanishing mean oscillations; see [Iwaniec and Sbordone 1998] or [Di Fazio 1996]. However, the same is not true in the case that a is uniformly elliptic with general measurable coefficients. Even worse, it was shown by Serrin [1964] that for any $q \in (1,2)$ and $f \equiv 0$ there exists an elliptic matrix a with measurable coefficients such that one can find a distributive solution (called a *pathological solution*) $v \in W_0^{1,q}(\Omega) \setminus W_0^{1,2}(\Omega)$ that satisfies (2-5). These pathological solutions should be excluded as only the zero function itself is the natural solution, which of course is the unique weak solution $u \in W_0^{1,2}(\Omega)$ in case $f \equiv 0$. This indicates that any reasonable theory for $q \in (1,2)$ must be able to avoid the existence of such pathological solutions.

Thus, to get a theory for all $q \in (1,\infty)$, the counterexamples mentioned above indicate that we need to assume more structural assumptions on A, which we shall describe in detail in the next section, where we recall our problem, introduce the structural assumptions on A and formulate the main results of this paper.

2. Results

As discussed above, we study the problem (1-1) with a mapping A fulfilling (1-2)–(1-6). Further, inspired by the counterexamples recalled in the previous section and also by the available positive results, we shall assume in what follows that the mapping A is *asymptotically Uhlenbeck*; i.e., we will assume that there exists a continuous mapping $\tilde{A} : \overline{\Omega} \to \mathbb{R}^{n\times N} \times \mathbb{R}^{n\times N}$ fulfilling the following:

$$\begin{array}{l} \text{for all } \varepsilon > 0, \text{ there exists } k > 0 \text{ such that,} \\ \text{for almost all } x \in \Omega \text{ and all } \eta \in \mathbb{R}^{n\times N} \text{ satisfying } |\eta| \ge k, \end{array} \quad |A(x,\eta) - \tilde{A}(x)\eta| \le \varepsilon|\eta|. \tag{2-1}$$

This assumption combined with (1-4)–(1-6) implies that $\tilde{A}$ necessarily satisfies

$$c_1|\eta|^2 \le \tilde{A}(x)\eta\cdot\eta \le c_2|\eta|^2 \quad \text{for all } \eta \in \mathbb{R}^{n\times N}. \tag{2-2}$$

Although the above assumption might seem to be restrictive, it enables us to cover many cases used in applications. The prototypical example is of the form

$$A(x,\eta) = a(x,|\eta|)\eta \quad \text{with } \lim_{\lambda\to\infty} a(x,\lambda) = \tilde{a}(x), \text{ where } \tilde{a} \in \mathscr{C}(\overline{\Omega}). \tag{2-3}$$

Note that a may be measurable with respect to x and the required continuity must hold only for $\tilde{a}$. The assumptions (1-4)–(1-6) are met if a is strictly positive and bounded and if the function $a(x,\lambda)\lambda$ is nondecreasing with respect to λ for almost all $x \in \Omega$. The fact that, besides (1-2)–(1-6), we will not assume anything more than (2-1) makes our approach general.

Moreover, to obtain the uniqueness of the solution, we will consider a stronger version of (2-1). Namely, we shall assume that A is *strongly asymptotically Uhlenbeck*; i.e., we will assume that there exists a continuous mapping $\tilde{A} : \overline{\Omega} \to \mathbb{R}^{n\times N} \times \mathbb{R}^{n\times N}$ fulfilling the following:

$$\begin{array}{l}\text{for all } \varepsilon > 0, \text{ there exists } k > 0 \text{ such that,}\\ \text{for almost all } x \in \Omega \text{ and all } \eta \in \mathbb{R}^{n\times N} \text{ satisfying } |\eta| \ge k,\end{array} \quad \left| \frac{\partial A(x,\eta)}{\partial \eta} - \tilde{A}(x) \right| \le \varepsilon. \tag{2-4}$$

Concerning the example (2-3), the condition (2-4) follows if $a(x,\lambda)$ is differentiable with respect to λ for $\lambda \gg 1$ and $\lim_{\lambda\to\infty}|a'(x,\lambda)\lambda| = 0$. This includes the approximations for the p-Laplace operator

$$\begin{aligned} a(x,|\eta|) &= \max\{\mu, |\eta|^{p-2}\} \qquad \text{for } p \in (1,2),\\ a(x,|\eta|) &= \min\{\mu^{-1}, |\eta|^{p-2}\} \quad \text{for } p \in (2,\infty), \end{aligned}$$

which are (for small μ) arbitrary close to the original setting.

The first main result of the paper giving the answer to ($\mathcal{Q}$) is the following:

Theorem 2.1. *Let* Ω *be a bounded* $\mathscr{C}^1$*-domain and* A *satisfy* (1-2)–(1-6) *and* (2-1). *Then for any* $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ *with* $q \in (1,\infty)$, *there exists* $u \in W_0^{1,q}(\Omega; \mathbb{R}^{n\times N})$ *such that*

$$\int_\Omega A(x,\nabla u)\cdot\nabla\varphi \,\mathrm{d}x = \int_\Omega f\cdot\nabla\varphi\,\mathrm{d}x \quad \textit{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega;\mathbb{R}^N). \tag{2-5}$$

Moreover, every very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ *to* (2-5) *with some* $\tilde{q} > 1$ *satisfies*

$$\int_\Omega |\nabla\tilde{u}|^q \,\mathrm{d}x \le C(A,q,\Omega)\left(1 + \int_\Omega |f|^q\,\mathrm{d}x\right). \tag{2-6}$$

In addition, if A *is strictly monotone and strongly asymptotically Uhlenbeck, i.e.,* (2-4) *holds, then the solution is unique in any class* $W_0^{1,\tilde{q}}(\Omega;\mathbb{R}^N)$ *with* $\tilde{q} > 1$.

Notice here that (2-5) is nothing else than the weak formulation of (1-1). Next, we would like to emphasize the novelty of the above result. First, to derive the estimate (2-6), one can use the comparison of (2-5) with the system with $A(x,\eta)$ replaced by $\tilde{A}(x)\eta$ to end up with (2-6) provided that the left-hand side of (2-6) is finite a priori. From this point of view, the a priori estimate (2-6) is indeed clear. On the other hand, and what is not obvious, is that (2-6) holds for *all very weak solutions* to (2-5) that belong to some $W_0^{1,\tilde{q}}(\Omega;\mathbb{R}^N)$ for some $\tilde{q} > 1$.

Second, Theorem 2.1 implies that *we can construct solutions for the whole range* $q \in (1,\infty)$, which makes the existence theory identical to the theory for linear operators with continuous coefficients since we know that the linear theory is not true for $q = 1$ or $q = \infty$.

Third, Theorem 2.1 provides *the uniqueness of the very weak solution* for vector-valued nonlinear elliptic systems without any additional qualitative properties of a solution, e.g., the entropy inequality. In particular, the result of Theorem 2.1 directly leads to the uniqueness of a solution when div f is a general vector-valued Radon measure. As this is of independent interest, we formulate this result in the following corollary, where we shall denote by the symbol $\mathcal{M}(\Omega;\mathbb{R}^N)$ the space of $\mathbb{R}^N$-valued Radon measures.

Corollary 2.2. *Let Ω be a bounded $\mathscr{C}^1$-domain and A satisfy* (1-2)–(1-6) *and* (2-1). *Then for any $f \in \mathcal{M}(\Omega; \mathbb{R}^N)$, there exists $u \in W_0^{1,n'-\varepsilon}(\Omega; \mathbb{R}^{n\times N})$ with arbitrary $\varepsilon > 0$ such that*

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = \langle f, \varphi \rangle \quad \text{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^N). \tag{2-7}$$

Moreover, every very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ to (2-7) *with some $\tilde{q} > 1$ satisfies for all $q \in (1, n')$*

$$\int_\Omega |\nabla \tilde{u}|^q \, \mathrm{d}x \le C(A, q, \Omega)(1 + \|f\|_{\mathcal{M}}^q). \tag{2-8}$$

In addition, if A is strictly monotone and strongly asymptotically Uhlenbeck, i.e., (2-4) *holds, then the solution is unique in any class $W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with $\tilde{q} > 1$.*

Although Theorem 2.1 gives the final answer to ($\mathscr{Q}$), it is actually a consequence of the following stronger result. It shows the existence of a solution that is optimally smooth with respect to the right-hand side in weighted spaces. For $p \in [1, \infty)$, we denote by $\mathscr{A}_p$ the Muckenhoupt class of nonnegative weights on $\mathbb{R}^n$ (see Section 3 for the precise definition) and define the weighted Lebesgue space $L^p_\omega(\Omega) := \{f \in L^1(\Omega); \int_\Omega |f|^p \omega \, \mathrm{d}x < \infty\}$. Then we have the following result.

Theorem 2.3. *Let Ω be a bounded $\mathscr{C}^1$-domain, A satisfy* (1-2)–(1-6) *and* (2-1) *and $f \in L^{p_0}_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ for some $p_0 \in (1, \infty)$ and $\omega_0 \in \mathscr{A}_{p_0}$. Then there exists a $u \in W_0^{1,1}(\Omega; \mathbb{R}^N)$ solving* (2-5) *such that for all $p \in (1, \infty)$ and all weights $\omega \in \mathscr{A}_p$ the estimate*

$$\int_\Omega |\nabla u|^p \omega \, \mathrm{d}x \le C(A_p(\omega), \Omega, A, p)\left(1 + \int_\Omega |f|^p \omega \, \mathrm{d}x\right) \tag{2-9}$$

holds whenever the right-hand side is finite. Moreover, every very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ to (2-5) *with some $\tilde{q} > 1$ satisfies* (2-9). *In addition, if A is strictly monotone and strongly asymptotically Uhlenbeck, i.e.,* (2-4) *holds, then the solution is unique in any class $W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with $\tilde{q} > 1$.*

Clearly, Theorem 2.1 is an immediate consequence of Theorem 2.3. Observe that (2-9) is an optimal existence result with respect to the weighted spaces. It cannot be generalized to more general weights, which is demonstrated by the theory for the Laplace equation in the whole $\mathbb{R}^n$, where one can prove that (2-9) holds in general if and only if $\omega \in \mathscr{A}_p$. This follows from the singular integral representation of the solution and the fundamental result of Muckenhoupt [1972] on the continuity of the maximal function in weighted spaces.

At this point, we wish to present the following corollary of Theorem 2.3. It shows that if $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ the solution constructed by Theorem 2.3 implies an estimate in terms of a Hilbert space that therefore inherits the spirit of duality. Denoting by Mf the Hardy–Littlewood maximal function (see the Section 3 for the precise definition), we have the following corollary.

Corollary 2.4. *Let Ω be a bounded $\mathscr{C}^1$-domain and A satisfy* (1-2)–(1-6) *and* (2-1). *Then for any $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ with $q \in (1, 2]$, there exists $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ satisfying* (2-5). *Moreover, any very weak solution $\tilde{u} \in W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with some $\tilde{q} > 1$ fulfilling* (2-5) *satisfies the estimate*

$$\int_\Omega \frac{|\nabla \tilde{u}|^2}{(1 + Mf)^{2-q}} \, \mathrm{d}x \le C(A, q, \Omega, f)\left(1 + \int_\Omega |f|^q \, \mathrm{d}x\right). \tag{2-10}$$

As mentioned above, the estimate (2-10) preserves the natural duality pairing in terms of weighted L^2 spaces, and as will be seen in the proof, the estimate (2-10) plays the key role in the convergence analysis of approximate solutions to the desired one. Indeed, the weighted L^2 integrability is the key property of the system, and we wish to emphasize that the only L^q-a priori information (with $q < 2$) does not seem to be sufficient to pass to the limit with the nonlinearity of approximating sequences. The reason for such a speculation is that all known methods for identification of the weak limit in the nonlinearity $A(\nabla u)$ are based on the identification of the "weak" limit of $A(\nabla u) \cdot \nabla u$ on "large" sets. However, having only L^q-estimates with $q < 2$, any identification of this type is impossible. On the other hand, we believe (based on the result of the paper) that the key estimate should reflect the duality pairing with possibly Muckenhoupt weight exactly as in (2-10). Having such an estimate, the new technique developed in the paper allows us to reconstruct the nonlinearity, although it is governed by a weakly converging subsequence only. It highly relies on the weighted theory that allows us to use the *weighted biting div-curl lemma*; see Theorem 2.6. To support the conjecture about the only possible choice of estimates in the weighted spaces preserving the duality pairing and reflecting the right-hand side, we quote the recent result [Bulíček and Schwarzacher 2016]. Here the theory for general operators with measurable coefficients and having a p-Laplacian-like structure is developed for all $q \in (p-\varepsilon, p]$ with $\varepsilon > 0$ depending only on the nonlinearity. Observe that the L^q-estimates for these p-Laplacian-like operators and $q \in (p-\varepsilon, p]$ have been known for some time [Lewis 1993; Greco et al. 1997] but the existence even in that case was not possible. Moreover, we wish to mention that the proof for the a priori estimates by Lewis [1993] already relied on the characterization of Muckenhoupt weights via the maximal operator. Therefore, we strongly believe that the effort to establish the very weak solution for the p-Laplace problem should not be blindly focused on obtaining L^q-estimates for $q < p$ but we should rather focus on the weighted L^p-estimates.

Next, we formulate new results that are on the one hand essential for the proof of Theorems 2.3 and 2.1 but on the other hand of independent interest in the fields of harmonic analysis and the compensated compactness theory. These results are mainly related to two critical problems: first to the a priori estimate (2-9) and second to the stability of the nonlinearity $A(x, \nabla u)$ under the weak convergence of ∇u. To solve the first problem, we use the linear system as a comparison to provide (2-9). The weighted theory for linear problems is known for $\Omega = \mathbb{R}^n$ in the case of constant coefficients (see, e.g., [Coifman and Fefferman 1974, p. 244]) but seems to be missing for bounded domains and linear operators continuously depending on x. Therefore, another essential contribution of this paper is the following theorem.

Theorem 2.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded $\mathscr{C}^1$-domain, $\omega \in \mathcal{A}_p$ for some $p \in (1, \infty)$ be arbitrary and $\tilde{A} \in \mathscr{C}(\overline{\Omega}; \mathbb{R}^{n\times N\times n\times N})$ satisfy for all $z \in \mathbb{R}^{n\times N}$ and all $x \in \overline{\Omega}$*

$$c_1|\eta|^2 \le \tilde{A}(x)\eta\cdot\eta \le c_2|\eta|^2 \tag{2-11}$$

with some positive constants c_1 and c_2. Then for any $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$, there exists unique $v \in W^{1,1}_0(\Omega; \mathbb{R}^N)$ solving

$$\int_\Omega \tilde{A}(x)\nabla v(x)\cdot\nabla\varphi(x)\,\mathrm{d}x = \int_\Omega f(x)\cdot\nabla\varphi(x)\,\mathrm{d}x \quad \textit{for all } \varphi \in \mathscr{C}^{0,1}_0(\Omega; \mathbb{R}^N) \tag{2-12}$$

and fulfilling

$$\int_\Omega |\nabla v|^p \omega \,\mathrm{d}x \le C(\Omega, \mathcal{A}_p(\omega), p, c_1, c_2) \int_\Omega |f|^p \omega \,\mathrm{d}x. \tag{2-13}$$

In addition, if $\bar{v} \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ *for some* $q > 1$ *fulfills* (2-12), *then* $\bar{v} = v$.

We wish to point out that we include natural local weighted estimates in the interior as well as on the boundary that are certainly of independent interest (see Lemmas 5.1 and 5.2).

The second obstacle we have to deal with is an identification of the weak limit, and for this purpose, we invent a generalization of the celebrated div-curl lemma.

Theorem 2.6 (weighted, biting div-curl lemma). *Let* $\Omega \subset \mathbb{R}^n$ *be an open bounded set. Assume that for some* $p \in (1, \infty)$ *and given* $\omega \in \mathcal{A}_p$ *we have a sequence of vector-valued measurable functions* $(a^k, b^k)_{k=1}^\infty : \Omega \to \mathbb{R}^n \times \mathbb{R}^n$ *such that*

$$\sup_{k \in \mathbb{N}} \int_\Omega |a^k|^p \omega + |b^k|^{p'} \omega \,\mathrm{d}x < \infty. \tag{2-14}$$

Furthermore, assume that, for every bounded sequence $\{c^k\}_{k=1}^\infty$ *from* $W_0^{1,\infty}(\Omega)$ *that fulfills*

$$\nabla c^k \rightharpoonup^* 0 \quad \text{weakly}^* \text{ in } L^\infty(\Omega),$$

there holds

$$\lim_{k \to \infty} \int_\Omega b^k \cdot \nabla c^k \,\mathrm{d}x = 0, \tag{2-15}$$

$$\lim_{k \to \infty} \int_\Omega a_i^k \partial_{x_j} c^k - a_j^k \partial_{x_i} c^k \,\mathrm{d}x = 0 \quad \text{for all } i, j = 1, \dots, n. \tag{2-16}$$

Then there exists a subsequence (a^k, b^k) *that we do not relabel, and there exists a nondecreasing sequence of measurable subsets* $E_j \subset \Omega$ *with* $|\Omega \setminus E_j| \to 0$ *as* $j \to \infty$ *such that*

$$a^k \rightharpoonup a \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^n), \tag{2-17}$$

$$b^k \rightharpoonup b \qquad \text{weakly in } L^1(\Omega; \mathbb{R}^n), \tag{2-18}$$

$$a^k \cdot b^k \omega \rightharpoonup a \cdot b \omega \quad \text{weakly in } L^1(E_j) \text{ for all } j \in \mathbb{N}. \tag{2-19}$$

The original version of this lemma, first invented by Murat [1978; 1981] and Tartar [1978; 1979], was designed to identify many types of nonlinearities appearing in many types of partial differential equations. However, they assumed stronger assumptions on a^k and b^k than (2-15)–(2-16), which lead to (2-19) for $E_j \equiv \Omega$. To be more specific, they did not assume weighted spaces and considered $\omega \equiv 1$ and they required that (2-15) hold for any c^k converging weakly in $W^{1,p}$ and (2-16) for any c^k converging weakly in $W^{1,p'}$. The first result more in the spirit of Theorem 2.6 is due to Conti et al. [2011], who worked with $\omega \equiv 1$ and kept (2-15)–(2-16) but assumed the equi-integrability of the sequence $a^k \cdot b^k$. Such a result is then based on the proper use of the Lipschitz approximation of Sobolev functions introduced in [Acerbi and Fusco 1984], which we shall use here as well. The first use of the biting version of this result is in [Bulíček 2015], where the very similar technique for identification of the nonlinearity as in this paper is used but yet without the presence of Muckenhoupt weights. In this paper, we finally use the full strength of the *weighted biting div-curl lemma*, which is able to cover a borderline case in two ways: the integrability assumptions on a^k and b^k are minimal with respect to Lebesgue spaces (2-14) and the convergence assumptions (2-15)–(2-16) on $\operatorname{div}(b^k)$ and $\operatorname{curl}(a^k)$ are minimal. In addition, exactly this version of the div-curl lemma was one of the key results of this manuscript used in the recent paper [Bulíček and Schwarzacher 2016] to treat the p-Laplacian problem.

The proof of Theorem 2.6 relies on the original proof but is completed by using the Chacon biting lemma [Brooks and Chacon 1980; Ball and Murat 1989] and also a very improved Lipschitz approximation method in the framework of weighted spaces, which is yet another essential result of the paper.

Theorem 2.7 (Lipschitz approximation). *Let $\Omega \subset \mathbb{R}^n$ be an open set with Lipschitz boundary. Let $g \in W_0^{1,1}(\Omega; \mathbb{R}^N)$. Then for all $\lambda > 0$, there exists a* Lipschitz truncation $g^\lambda \in W_0^{1,\infty}(\Omega; \mathbb{R}^N)$ *such that*

$$g^\lambda = g \quad \text{and} \quad \nabla g^\lambda = \nabla g \qquad \text{in } \{M(\nabla g) \le \lambda\}, \tag{2-20}$$

$$|\nabla g^\lambda| \le |\nabla g| \chi_{\{M(\nabla g) \le \lambda\}} + C\lambda \chi_{\{M(\nabla g) > \lambda\}} \quad \text{almost everywhere}. \tag{2-21}$$

Further, if $\nabla g \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ for some $1 \le p < \infty$ and $\omega \in \mathcal{A}_p$, then

$$\begin{aligned} \int_\Omega |\nabla g^\lambda|^p \omega \,\mathrm{d}x &\le C(\mathcal{A}_p(\Omega), \Omega, N, p) \int_\Omega |\nabla g|^p \omega \,\mathrm{d}x, \\ \int_\Omega |\nabla (g - g^\lambda)|^p \omega \,\mathrm{d}x &\le C(\mathcal{A}_p(\Omega), \Omega, N, p) \int_{\Omega \cap \{M(\nabla g) > \lambda\}} |\nabla g|^p \omega \,\mathrm{d}x. \end{aligned} \tag{2-22}$$

This result has its origin in the paper [Acerbi and Fusco 1988]. The approach was considerably improved and successfully used for the existence theory in the context of fluid mechanics; see, e.g., [Frehse et al. 2000; Diening et al. 2008; 2013; Diening 2013] or [Breit et al. 2012; 2013] for divergence-free Lipschitz approximation. However, these results do not contain the weighted estimates (2-22) and for this reason we also provide its proof in this paper.

Finally, for the sake of completeness, we present straightforward generalizations of the above results. First, we establish the theory for the nonhomogeneous Dirichlet problem.

Theorem 2.8. *Let Ω be a bounded $\mathscr{C}^1$-domain, A satisfy* (1-2)–(1-6) *and* (2-1), $f \in L^{p_0}_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ *and $u_0 \in W^{1,1}(\Omega; \mathbb{R}^N)$ be such that $\nabla u_0 \in L^{p_0}_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ for some $p_0 \in (1, \infty)$ and $\omega_0 \in \mathcal{A}_{p_0}$. Then there exists a solution u of* (2-5) *such that $u - u_0 \in W_0^{1,1}(\Omega; \mathbb{R}^{n\times N})$, and for all $p \in (1, \infty)$ and all weights $\omega \in \mathcal{A}_p$, the estimate*

$$\int_\Omega |\nabla u|^p \omega \,\mathrm{d}x \le C(A_p(\omega), \Omega, A, p)\left(1 + \int_\Omega (|f|^p + |\nabla u_0|^p)\omega \,\mathrm{d}x\right) \tag{2-23}$$

holds whenever the right hand side is finite. Moreover, every very weak solution u of (2-5) *fulfilling $\tilde{u} - u_0 \in W_0^{1,\tilde{q}}(\Omega, \mathbb{R}^N)$ with some $\tilde{q} > 1$ satisfies* (2-23). *In addition, if A is strictly monotone and strongly asymptotically Uhlenbeck, i.e.,* (2-4) *holds, then the solution is unique in any class $W^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ with $\tilde{q} > 1$.*

Second, we remark that, for the theory for (1-1), the assumptions (2-1)–(2-4) are not necessary and can be weakened.

Remark 2.9. At this point, we wish to discuss possible relaxations of the conditions (2-1) and (2-4) that might be useful for further application of the theory developed here. The proofs of existence or uniqueness do not require that the matrix $A(x, \eta)$ converge uniformly to a continuous target matrix $\tilde{A}(x)$ but rather that the two matrices are "close" for values $|\eta| > k$ for some k. Indeed, it is possible to quantify the necessary closeness in accordance with the ellipticity and continuity parameters of $\tilde{A}(x)$ and $\partial\Omega$. A different relaxation of (2-1) and (2-4) could be done in a nonpointwise manner by replacing the pointwise asymptotic conditions by asymptotic conditions in terms of vanishing mean oscillations (VMO).

We conclude this section by highlighting the essential novelties of this paper:

(1) A complete unified $W_0^{1,q}(\Omega; \mathbb{R}^N)$-theory for nonlinear elliptic systems with the asymptotic Uhlenbeck structure satisfying (1-2)–(1-6), (2-1) and (2-4) has been developed in such a way that the theory is identical with that for linear operators with continuous coefficients: Theorems 2.1 and 2.8. Moreover, the new estimate suitable for numerical purposes is established in Corollary 2.4.

(2) A maximal regularity in weighted spaces of any very weak solution is established as well as its uniqueness, which in particular leads to the uniqueness of very weak solutions to the problems with measure right-hand side: Theorem 2.3 for the nonlinear case and Theorem 2.5 for the linear setting.

(3) A new tool in harmonic analysis, the Lipschitz approximation method in weighted spaces, is developed: Theorem 2.7.

(4) A new tool for identification of a weak limit of the nonlinear operator, the biting weighted div-curl lemma, is invented: Theorem 2.6. Such a tool has a potential to improve the known methods in compensated compactness theory in significant manner.

To summarize, this paper proposes a new way to attack more general elliptic problems than those discussed in Section 2. Indeed, it seems that the only missing point in the analysis of more general problems, e.g., the p-Laplace equation, is the *formal* a priori estimates beyond the duality pairing. Once such a priori estimates are available, one can follow the method introduced in this paper and gain an existence and uniqueness theory for general problems beyond the natural duality. Indeed, the first step in this direction was already done in [Bulíček and Schwarzacher 2016], where more general operators having the p structure are treated.

The structure of the paper is somewhat in reversed order. After introducing some auxiliary tools and some necessary notation in Section 3, we first prove the main Theorems 2.1 and 2.3 in Section 4. For that result, we use the (technical) theorems, which are each independently proved in Sections 5–8. Finally Section 9 is dedicated to the proofs of the corollaries.

3. Auxiliary tools

3A. ***Muckenhoupt weights and the maximal function.*** We start this part by recalling the definition of the Hardy–Littlewood maximal function. For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define

$$Mf(x) := \sup_{R>0} \fint_{B_R(x)} |f(y)| \,\mathrm{d}y \quad \text{with} \fint_{B_R(x)} |f(y)| \,\mathrm{d}y := \frac{1}{|B_R(x)|} \int_{B_R(x)} |f(y)| \,\mathrm{d}y,$$

where $B_R(x)$ denotes a ball with radius R centered at $x \in \mathbb{R}^n$. We shall use similar notation for vector- or tensor-valued functions as well. Note here that we could replace balls in the definition of the maximal function by cubes with sides parallel to the axes without any change. We will also use in what follows the standard notion for Lebesgue and Sobolev spaces. Further, we say that $\omega : \mathbb{R}^n \to \mathbb{R}$ is a weight if it is a measurable function that is almost everywhere finite and positive. For such a weight and arbitrary measurable $\Omega \subset \mathbb{R}^n$, we denote the space $L^p_\omega(\Omega)$ with $p \in [1, \infty)$ as

$$L^p_\omega(\Omega) := \left\{ u : \Omega \to \mathbb{R}^n;\ \|f\|_{L^p_\omega} := \left(\int_\Omega |u(x)|^p \omega(x)\, \mathrm{d}x \right)^{1/p} < \infty \right\}.$$

Note that our weights are defined on the whole space $\mathbb{R}^n$. Next, for $p \in [1, \infty)$, we say that a weight ω belongs to the Muckenhoupt class $\mathcal{A}_p$ if and only if there exists a positive constant A such that for every ball $B \subset \mathbb{R}^n$

$$\left(\fint_B \omega \, \mathrm{d}x \right) \left(\fint_B \omega^{-(p'-1)} \, \mathrm{d}x \right)^{1/(p'-1)} \le A \qquad \text{if } p \in (1, \infty), \tag{3-1}$$

$$M\omega(x) \le A\omega(x) \quad \text{if } p = 1. \tag{3-2}$$

In what follows, we denote by $A_p(\omega)$ the smallest constant A for which the inequality (3-1) or (3-2) holds. Due to the celebrated result of Muckenhoupt [1972], we know that $\omega \in \mathcal{A}_p$ is for $1 < p < \infty$ equivalent to the existence of a constant A' such that for all $f \in L^p(\mathbb{R}^n)$

$$\int |Mf|^p \omega \, \mathrm{d}x \le A' \int |f|^p \omega \, \mathrm{d}x. \tag{3-3}$$

Further, if $p \in [1, \infty)$ and $\omega \in \mathcal{A}_p$, then we have an embedding $L^p_\omega(\Omega) \hookrightarrow L^1_{\mathrm{loc}}(\Omega)$ since for all balls $B \subset \mathbb{R}^n$

$$\fint_B |f| \, \mathrm{d}x \le \left(\fint_B |f|^p \omega \, \mathrm{d}x \right)^{1/p} \left(\fint_B \omega^{-(p'-1)} \, \mathrm{d}x \right)^{1/p'} \le (\mathcal{A}_p(\omega))^{1/p} \left(\frac{1}{\omega(B)} \int_B |f|^p \omega \, \mathrm{d}x \right)^{1/p}.$$

In particular, the distributional derivatives of all $f \in L^p_\omega$ are well defined. Next, we summarize some properties of Muckenhoupt weights in the following lemma.

Lemma 3.1 [Turesson 2000, Lemma 1.2.12]. *Let $\omega \in \mathcal{A}_p$ for some $p \in [1, \infty)$. Then $\omega \in \mathcal{A}_q$ for all $q \ge p$. Moreover, there exists $s = s(p, A_p(\omega)) > 1$ such that $\omega \in L^s_{\mathrm{loc}}(\mathbb{R}^n)$ and we have the reverse Hölder inequality, i.e.,*

$$\left(\fint_B \omega^s \, \mathrm{d}x \right)^{1/s} \le C(n, A_p(\omega)) \fint_B \omega \, \mathrm{d}x. \tag{3-4}$$

Further, if $p \in (1, \infty)$, then there exists $\sigma = \sigma(p, A_p(\omega)) \in [1, p)$ such that $\omega \in \mathcal{A}_\sigma$. In addition, $\omega \in \mathcal{A}_p$ is equivalent to $\omega^{-(p'-1)} \in \mathcal{A}_{p'}$.

In the paper, we also use the following improved embedding $L^p_\omega(\Omega) \hookrightarrow L^q_{\mathrm{loc}}(\Omega)$ valid for all $\omega \in \mathcal{A}_p$ with $p \in (1, \infty)$ and some $q \in [1, p)$ depending only on $A_p(\omega)$. Such an embedding can be deduced by a direct application of Lemma 3.1. Indeed, since $\omega \in \mathcal{A}_p$, we have $\omega^{-(p'-1)} \in \mathcal{A}_{p'}$. Thus, using Lemma 3.1, there exists $s = s(A_p(\omega)) > 1$ such that

$$\left(\fint_B \omega^{-s(p'-1)}\,\mathrm{d}x\right)^{1/s} \le C(A_p(\omega)) \fint_B \omega^{-(p'-1)}\,\mathrm{d}x.$$

Consequently, for $q := sp/(p+s-1) \in (1, p)$, we can use the Hölder inequality to deduce that

$$\begin{aligned}\left(\fint_B |f|^q\,\mathrm{d}x\right)^{1/q} &\le \left(\fint_B |f|^p \omega\,\mathrm{d}x\right)^{1/p} \left(\fint_B \omega^{-s(p'-1)}\,\mathrm{d}x\right)^{1/(sp')} \\ &\le C(A_p(\omega)) \left(\frac{1}{\omega(B)} \int_B |f|^p \omega\,\mathrm{d}x\right)^{1/p}, \end{aligned} \tag{3-5}$$

which implies the desired embedding.

The next result makes another link between the maximal function and $\mathscr{A}_p$-weight.

Lemma 3.2 [Torchinsky 1986, p. 229–230; Turesson 2000, p. 5]. *Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be such that $Mf < \infty$ almost everywhere in $\mathbb{R}^n$. Then for all $\alpha \in (0, 1)$, we have $(Mf)^\alpha \in \mathscr{A}_1$. Furthermore, for all $p \in (1, \infty)$ and all $\alpha \in (0, 1)$, there holds $(Mf)^{-\alpha(p-1)} \in \mathscr{A}_p$.*

We would also like to point out that the maximum $\omega_1 \vee \omega_2$ and minimum $\omega_1 \wedge \omega_2$ of two $\mathscr{A}_p$-weights are again $\mathscr{A}_p$-weights. For $p = 2$, we even have $A_2(\omega_1 \wedge \omega_2) \le A(\omega_1) + A_2(\omega_2)$, which follows from the simple computation

$$\begin{aligned}\fint_B \omega_1 \wedge \omega_2\,\mathrm{d}x \fint_B \frac{1}{\omega_1 \wedge \omega_2}\,\mathrm{d}x &\le \left[\left(\fint_B \omega_1\,\mathrm{d}x\right) \wedge \left(\fint_B \omega_2\,\mathrm{d}x\right)\right] \fint_B \frac{1}{\omega_1} + \frac{1}{\omega_2}\,\mathrm{d}x \\ &\le A_2(\omega_1) + A_2(\omega_2). \end{aligned} \tag{3-6}$$

3B. ***Convergence tools.*** The results recalled in the previous sections shall give us a direct method for a priori estimates for an approximative problem (1-1). However, to identify the limit correctly, we use Theorem 2.6, which is based on the following biting lemma.

Lemma 3.3 (Chacon's biting lemma [Ball and Murat 1989]). *Let Ω be a bounded domain in $\mathbb{R}^n$, and let $\{v^n\}_{n=1}^\infty$ be a bounded sequence in $L^1(\Omega)$. Then there exists a nondecreasing sequence of measurable subsets $E_j \subset \Omega$ with $|\Omega \setminus E_j| \to 0$ as $j \to \infty$ such that $\{v^n\}_{n\in\mathbb{N}}$ is precompact in the weak topology of $L^1(E_j)$, for each $j \in \mathbb{N}$.*

Note here that precompactness of v^n is equivalent to the following: for every $j \in \mathbb{N}$ and every $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $A \subset E_j$ with $|A| \le \delta$ and all $n \in \mathbb{N}$

$$\int_A |v^n|\,\mathrm{d}x \le \varepsilon. \tag{3-7}$$

3C. ***L^q-theory for linear systems with continuous coefficients.*** The starting point for getting all a priori estimates in the paper is the following:

Lemma 3.4 [Dolzmann and Müller 1995, Lemma 2]. *Let Ω be a $\mathscr{C}^1$-domain and $\boldsymbol{B} \in \mathscr{C}(\overline{\Omega}, \mathbb{R}^{n\times N\times n\times N})$ be a continuous, elliptic tensor that satisfies for all $\eta \in \mathbb{R}^{n\times N}$ and all $x \in \overline{\Omega}$*

$$c_1|\eta|^2 \le \boldsymbol{B}(x)\eta \cdot \eta \le c_2|\eta|^2 \tag{3-8}$$

for some $c_1, c_2 > 0$. Then for any $f \in L^q(\Omega; \mathbb{R}^{n\times N})$ with $q \in (1,\infty)$, there exists unique $w \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ solving

$$-\operatorname{div}(\boldsymbol{B}\nabla w) = -\operatorname{div} f \quad \text{in } \Omega$$

in the sense of distribution. Moreover, there exists a constant C depending only on $\boldsymbol{B}$, q and the shape of Ω such that

$$\|\nabla w\|_{L^q(\Omega)} \le C(\boldsymbol{B}, q, \Omega)\|f\|_{L^q(\Omega)}. \tag{3-9}$$

4. Proof of Theorems 2.1 and 2.3

First, it is evident that Theorem 2.1 directly follows from Theorem 2.3 by setting $\omega \equiv 1$, which is surely an $\mathcal{A}_p$-weight. Therefore, we focus on the proof of Theorem 2.3. We split the proof into several steps. We start with the uniform estimates, which heavily rely on Theorem 2.5, then provide the existence proof, for which we use the result of Theorem 2.6, and finally show the uniqueness of the solution, again based on Theorem 2.5.

4A. ***Uniform estimates.*** We start the proof by showing the uniform estimate (2-9) for arbitrary $u \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ with $q > 1$ solving (2-5). Without loss of generality, we can restrict ourselves to the case $q \in (1,2)$. First, we consider the case when $f \in L^2_\omega(\Omega; \mathbb{R}^{n\times N})$ with some weight $\omega \in \mathcal{A}_2$. For $j \in \mathbb{N}$, we define the auxiliary weight $\omega_j := \omega \wedge j(1 + M|\nabla u|)^{q-2}$. Then it follows from Lemma 3.2 and the fact that $q \in (1,2)$ that $w_j \in \mathcal{A}_2$. Moreover, we have

$$A_2(\omega_j) \le A_2(\omega) + A_2(j(1+M|\nabla u|)^{q-2}) = A_2(\omega) + A_2((1+M|\nabla u|)^{q-2}) \le C(u,\omega)$$

and also that $\nabla u, f \in L^2_{\omega_j}(\Omega; \mathbb{R}^{n\times N})$. Next, using (2-5), we see that for all $\varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^N)$

$$\int_\Omega \tilde{A}(x)\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega (f - A(x,\nabla u) + \tilde{A}(x)\nabla u)\cdot\nabla\varphi\,\mathrm{d}x. \tag{4-1}$$

Since the right-hand side belongs to $L^2_{\omega_j}(\Omega; \mathbb{R}^{n\times N})$, we can use Theorem 2.5 and the assumptions (1-5) and (2-2) to get the estimate

$$\begin{aligned}
\int_\Omega |\nabla u|^2\omega_j\,\mathrm{d}x &\le C(\tilde{A}, A_2(\omega_j), \Omega, c_1, c_2)\int_\Omega |f - A(x,\nabla u) + \tilde{A}(x)\nabla u|^2\omega_j\,\mathrm{d}x \\
&\le C(\tilde{A}, u, \omega, \Omega, c_1, c_2)\left(\int_\Omega |f|^2\omega_j\,\mathrm{d}x + \int_\Omega |A(x,\nabla u) - \tilde{A}(x)\nabla u|^2\omega_j\,\mathrm{d}x\right) \\
&\le C(\tilde{A}, u, \omega, \Omega, c_1, c_2)\int_\Omega (|f|^2 + k^2)\omega_j\,\mathrm{d}x \\
&\qquad + C(\tilde{A}, u, \omega, \Omega, c_1, c_2)\int_{\{|\nabla u|\ge k\}} \frac{|A(x,\nabla u) - \tilde{A}(x)\nabla u|^2}{|\nabla u|^2}|\nabla u|^2\omega_j\,\mathrm{d}x.
\end{aligned}$$

Finally, we set

$$\varepsilon^2 := \frac{1}{2C(\tilde{A}, u, \omega, \Omega, c_1, c_2)}$$

and according to (2-1) we can find k such that

$$\frac{|A(x,\nabla u)-\tilde{A}(x)\nabla u|^2}{|\nabla u|^2}\le \frac{1}{2C(\tilde{A},u,\omega,\Omega,c_1,c_2)},$$

provided that $|\nabla u|\ge k$. Inserting this inequality above, we deduce that

$$\int_\Omega |\nabla u|^2\omega_j\,\mathrm{d}x \le C(\tilde{A},u,\omega,\Omega,c_1,c_2)\int_\Omega (|f|^2+k^2)\omega_j\,\mathrm{d}x + \frac{1}{2}\int_\Omega |\nabla u|^2\omega_j\,\mathrm{d}x.$$

Since we already know that $\nabla u\in L^2_{\omega_j}(\Omega;\mathbb{R}^{n\times N})$ and k is fixed independently of j, we can absorb the last term into the left-hand side to get

$$\int_\Omega |\nabla u|^2\omega_j\,\mathrm{d}x \le C(\tilde{A},u,\omega,\Omega,c_1,c_2)\int_\Omega (|f|^2+1)\omega_j\,\mathrm{d}x.$$

Next, we let $j\to\infty$ in the above inequality. For the right-hand side, we use the fact that $\omega_j\le\omega$, and for the left-hand side, we use the monotone convergence theorem (notice here that $\omega_j\nearrow\omega$ since $M|\nabla u|<\infty$ almost everywhere) to obtain

$$\int_\Omega |\nabla u|^2\omega\,\mathrm{d}x \le C(\tilde{A},u,\omega,\Omega,c_1,c_2)\biggl(1+\int_\Omega |f|^2\omega\,\mathrm{d}x\biggr).$$

Although this estimate is not uniform yet, since the right-hand side still depends on the A_2 constant of $(1+M|\nabla u|)^{q-2}$, it implies that $\nabla u\in L^2_\omega(\Omega;\mathbb{R}^{n\times N})$ for the original weight ω. Therefore, we can reiterate this procedure; i.e., going back to (4-1) and applying Theorem 2.5, we find that

$$\begin{aligned}\int_\Omega |\nabla u|^2\omega\,\mathrm{d}x &\le C(\tilde{A},A_2(\omega),\Omega,c_1,c_2)\int_\Omega |f-A(x,\nabla u)+\tilde{A}(x)\nabla u|^2\omega\,\mathrm{d}x\\ &\le C(\tilde{A},A_2(\omega),\Omega,c_1,c_2)\int_\Omega (|f|^2+k)\omega\,\mathrm{d}x\\ &\qquad + C(\tilde{A},A_2(\omega),\Omega,c_1,c_2)\int_{\{|\nabla u|\ge k\}}\frac{|A(x,\nabla u)-\tilde{A}(x)\nabla u|^2}{|\nabla u|^2}|\nabla u|^2\omega\,\mathrm{d}x.\end{aligned}$$

Since we already know that $\nabla u\in L^2_\omega(\Omega;\mathbb{R}^{n\times N})$, we can use the same procedure as above and absorb the last term into the left-hand side to get

$$\int_\Omega |\nabla u|^2\omega\,\mathrm{d}x \le C(c_1,c_2,A_2(\omega),\Omega,\tilde{A})\biggl(1+\int_\Omega |f|^2\omega\,\mathrm{d}x\biggr). \tag{4-2}$$

We would like to emphasize that the constant C in (4-2) depends on ω only through its A_2-constant. Therefore, by the *miracle of extrapolation* [Cruz-Uribe et al. 2006, Theorem 3.1] (see also [Rubio de Francia 1984]) applied to the couples $(\nabla u, f)$, we can extend this estimate valid for all $\mathcal{A}_2$-weights to all $\mathcal{A}_p$-weights. In particular, we find that

$$\int_\Omega |\nabla u|^p\omega\,\mathrm{d}x \le C(c_1,c_2,A_p(\omega),\Omega,\tilde{A})\biggl(1+\int_\Omega |f|^p\omega\,\mathrm{d}x\biggr)\quad\text{for all } 1<p<\infty \text{ and } \omega\in\mathcal{A}_p,$$

which is just (2-9) from our claim.

4B. ***Existence of a solution.*** Let $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ with some $p \in (1,\infty)$ and $\omega \in \mathcal{A}_p$ be arbitrary. Then according to (3-5), there exists some $q_0 \in (1,2)$ such that $L^p_\omega(\Omega) \hookrightarrow L^{q_0}(\Omega)$. Therefore, defining $\omega_0 := (1+Mf)^{q_0-2}$, we can use Lemma 3.2 to obtain that $\omega_0 \in \mathcal{A}_2$ and it is evident that $f \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$.

The construction of the solution is based on a proper approximation of the right-hand-side f and a limiting procedure. We first extend f outside of Ω by zero and define $f^k := f\chi_{\{|f|<k\}}$. Then f^k are bounded functions, $|f^k| \nearrow |f|$ and

$$f^k \to f \quad \text{strongly in } L^2_{\omega_0} \cap L^{q_0}(\mathbb{R}^n; \mathbb{R}^{n\times N}). \tag{4-3}$$

For such an approximative f^k, we can use the standard monotone operator theory to find a solution $u^k \in W^{1,2}_0(\Omega; \mathbb{R}^N)$ fulfilling

$$\int_\Omega A(x, \nabla u^k)\cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f^k \cdot \nabla\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in W^{1,2}_0(\Omega; \mathbb{R}^N). \tag{4-4}$$

Hence, we can use the already proven estimate (2-9) to deduce that

$$\begin{aligned}\int_\Omega |\nabla u^k|^2 \omega_0 \,\mathrm{d}x &\le C(c_1, c_2, A_2(\omega_0), \Omega, \tilde{A})\Big(1 + \int_\Omega |f^k|^2\omega_0 \,\mathrm{d}x\Big)\\ &\le C(c_1, c_2, q_0, f, A_2(\omega_0), \tilde{A})\Big(1 + \int_\Omega |f|^2\omega_0 \,\mathrm{d}x\Big)\\ &\le C(c_1, c_2, \Omega, \tilde{A}, f, \omega).\end{aligned} \tag{4-5}$$

Using the estimate (4-5), the reflexivity of the corresponding spaces, the embedding $L^2_{\omega_0}(\Omega) \hookrightarrow L^{q_0}(\Omega)$ and the growth assumption (1-5), we can pass to a subsequence (still denoted by u^k) such that

$$u^k \rightharpoonup u \qquad \text{weakly in } W^{1,q_0}_0(\Omega; \mathbb{R}^N), \tag{4-6}$$

$$\nabla u^k \rightharpoonup \nabla u \quad \text{weakly in } L^2_{\omega_0} \cap L^{q_0}(\Omega; \mathbb{R}^{n\times N}), \tag{4-7}$$

$$A(x, \nabla u^k) \rightharpoonup \bar{A} \qquad \text{weakly in } L^2_{\omega_0} \cap L^{q_0}(\Omega; \mathbb{R}^{n\times N}). \tag{4-8}$$

Next, using (4-5)–(4-7), the weak lower semicontinuity and the unique identification of the limit u in $W^{1,1}(\Omega)$, we obtain

$$\int_\Omega |\nabla u|^2\omega_0 \,\mathrm{d}x \le C(c_1, c_2, A_2(\omega_0), \Omega, \tilde{A})\Big(1 + \int_\Omega |f|^2\omega_0 \,\mathrm{d}x\Big). \tag{4-9}$$

The last step is to show that u is a solution to our problem, i.e., that it satisfies (2-5). Using (4-4), (4-3) and (4-8), it follows that

$$\int_\Omega \bar{A}\cdot\nabla\varphi \,\mathrm{d}x = \int_\Omega f\cdot\nabla\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in \mathscr{C}^{0,1}_0(\Omega; \mathbb{R}^N). \tag{4-10}$$

Hence, to complete the existence part of the proof of Theorem 2.3, it remains to show that

$$\bar{A}(x) = A(x, \nabla u(x)) \quad \text{in } \Omega. \tag{4-11}$$

To do so, we use[3] Theorem 2.6. We denote $a^k := \nabla u^k$ and $b^k := A(x, \nabla u^k)$. By using (4-5) and (1-5), we find that (2-14) is satisfied with the weight ω_0. Also the assumption (2-15) holds, which follows from (4-3), (4-4) and (4-10). Finally, (2-16) is valid trivially since a^k is a gradient. Therefore, Theorem 2.6 can be applied, which implies the existence of a nondecreasing sequence of measurable sets E_j such that $|\Omega \setminus E_j| \to 0$ and

$$A(x, \nabla u^k) \cdot \nabla u^k \omega_0 \rightharpoonup \bar{A} \cdot \nabla u \omega_0 \qquad \text{weakly in } L^1(E_j). \tag{4-12}$$

For any $B \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$, we have that $B\omega_0$ and also $A(\,\cdot\,, B)\omega_0$ belong to $L^2_{1/\omega_0}(\Omega; \mathbb{R}^{n\times N})$, and therefore using (4-7) and (4-8), we can observe that

$$(A(x, \nabla u^k) - A(x, B)) \cdot (\nabla u^k - B)\omega_0 \rightharpoonup (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \quad \text{weakly in } L^1(E_j). \tag{4-13}$$

Due to the monotonicity of A, we see that the term on the left-hand side is nonnegative and consequently its weak limit is nonnegative as well and we have that

$$\int_{E_j} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,\mathrm{d}x \geq 0 \quad \text{for all } B \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N}) \text{ and all } j \in \mathbb{N}. \tag{4-14}$$

Therefore, it follows that

$$\int_{\Omega} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,\mathrm{d}x \geq \int_{\Omega\setminus E_j} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,\mathrm{d}x,$$

and letting $j \to \infty$ (note that the integral is well defined due to (4-7) and (4-8)) and using the fact that $|\Omega \setminus E_j| \to 0$ as $j \to \infty$ and the Lebesgue dominated convergence theorem, we obtain

$$\int_{\Omega} (\bar{A} - A(x, B)) \cdot (\nabla u - B)\omega_0 \,\mathrm{d}x \geq 0 \quad \text{for all } B \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N}).$$

Hence, setting $B := \nabla u - \varepsilon G$ where $G \in L^\infty(\Omega; \mathbb{R}^{n\times N})$ is arbitrary and dividing by ε, we get

$$\int_{\Omega} (\bar{A} - A(x, \nabla u - \varepsilon G)) \cdot G\omega_0 \,\mathrm{d}x \geq 0 \quad \text{for all } G \in L^\infty(\Omega; \mathbb{R}^{n\times N}).$$

Finally, using the Lebesgue dominated convergence theorem, the assumption (1-5) and the continuity of A with respect to the second variable, we can let $\varepsilon \to 0_+$ to deduce

$$\int_{\Omega} (\bar{A} - A(x, \nabla u)) \cdot G\omega_0 \,\mathrm{d}x \geq 0 \quad \text{for all } G \in L^\infty(\Omega; \mathbb{R}^{n\times N}).$$

Since ω_0 is strictly positive almost everywhere in Ω, the relation (4-11) easily follows by setting, e.g.,

$$G := -\frac{\bar{A} - A(x, \nabla u)}{1 + |\bar{A} - A(x, \nabla u)|}.$$

Thus, (4-10) follows and u is a very weak solution.

[3] Although Theorem 2.6 is formulated for vector-valued functions, it is an easy extension to use it also for matrix-valued functions, which is the case here.

4C. ***Uniqueness.*** Let $u_1, u_2 \in W_0^{1,q}(\Omega; \mathbb{R}^N)$ with $q > 1$ be two very weak solutions to (2-5) for some given $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$, where $p \in (1, \infty)$ and $\omega \in \mathcal{A}_p$. Then it directly follows that

$$\int_\Omega (A(x, \nabla u_1) - A(x, \nabla u_2)) \cdot \nabla\varphi \,\mathrm{d}x = 0 \quad \text{for all } \varphi \in \mathcal{C}_0^{0,1}(\Omega; \mathbb{R}^{n\times N}). \tag{4-15}$$

First, consider the case that $f \in L^2(\Omega; \mathbb{R}^{n\times N})$. Then using the result of the previous part, we see that $u_1, u_2 \in W_0^{1,2}(\Omega; \mathbb{R}^N)$, and therefore due to the growth assumption (1-5), we see that (4-15) is valid for all $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^N)$. Consequently, the choice $\varphi := u_1 - u_2$ is admissible, and due to the strict monotonicity of A, we conclude that $\nabla u_1 = \nabla u_2$ almost everywhere in Ω and due to the zero trace also that $u_1 = u_2$.

Thus, it remains to discuss the case $f \notin L^2(\Omega; \mathbb{R}^{n\times N})$. But since $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ with $p > 1$ and ω being the $\mathcal{A}_p$-weight, we can deduce that $f \in L^{p_0}(\Omega; \mathbb{R}^{n\times N})$ for some $p_0 > 1$; see (3-5). Consequently, following Lemma 3.2, we can define the $\mathcal{A}_2$-weight $\omega_0 := (1+Mf)^{p_0-2}$ and we get that $f \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$. Therefore, the weighted a priori estimates imply that $\nabla u_i \in L^2_{\omega_0}(\Omega; \mathbb{R}^{n\times N})$ for $i = 1, 2$. Hence, defining a new weight $w^n := 1 \wedge (n\omega_0)$, which is bounded, we also get that for each n the solutions satisfy $\nabla u_i \in L^2_{\omega^n}(\Omega; \mathbb{R}^{n\times N})$. Moreover, we have the estimate $A_2(\omega^n) \le A_2(1) + A_2(n\omega_0) = 1 + A_2(\omega_0) \le C(f)$. Hence, rewriting the identity (4-15) into the form

$$\int_\Omega \tilde{A}(x)(\nabla u_1 - \nabla u_2) \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega \big(\tilde{A}(x)\nabla u_1 - A(x, \nabla u_1) - (\tilde{A}(x)\nabla u_2 - A(x, \nabla u_2))\big) \cdot \nabla\varphi \,\mathrm{d}x, \tag{4-16}$$

which is valid for all $\varphi \in \mathcal{C}_0^{0,1}(\Omega; \mathbb{R}^{n\times N})$, we can use Theorem 2.5 to obtain

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \omega^n \,\mathrm{d}x \le C \int_\Omega |\tilde{A}(x)\nabla u_1 - A(x, \nabla u_1) - (\tilde{A}(x)\nabla u_2 - A(x, \nabla u_2))|^2 \omega^n \,\mathrm{d}x \tag{4-17}$$

with some constant C independent of n. Moreover, due to the properties of the solution and ω^n, we can deduce that the integral appearing on the right-hand side is finite. In order to continue, we first recall the following algebraic result, whose proof can be found at the end of this subsection.

Lemma 4.1. *Let A fulfill* (1-4), (1-5), (2-1) *and* (2-4). *Then for every $\delta > 0$, there exists C such that for all $x \in \Omega$ and all $\eta_1, \eta_2 \in \mathbb{R}^{n\times N}$*

$$|A(x, \eta_1) - A(x, \eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| \le \delta|\eta_1 - \eta_2| + C(\delta). \tag{4-18}$$

Next, using the estimate (4-18) in (4-17), we find that for all $\delta > 0$

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \omega^n \,\mathrm{d}x \le C \int_\Omega \delta|\nabla u_1 - \nabla u_2|^2 \omega^n + C(\delta)\omega^n \,\mathrm{d}x. \tag{4-19}$$

Thus, setting $\delta := 1/(2C)$, we can deduce that

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \omega^n \,\mathrm{d}x \le C(\delta) \int_\Omega \omega^n \,\mathrm{d}x \le C, \tag{4-20}$$

where the last inequality follows from the fact that Ω is bounded and $\omega^n \le 1$. Hence, letting $n \to \infty$ in (4-20), using that $\omega^n \nearrow 1$ (which follows from the fact that $\omega_0 > 0$ almost everywhere) and using the monotone convergence theorem, we find that

$$\int_\Omega |\nabla u_1 - \nabla u_2|^2 \,\mathrm{d}x \le C.$$

Hence, we see that $u_1 - u_2 \in W_0^{1,2}(\Omega; \mathbb{R}^N)$. In addition, using (4-18) again,

$$\begin{aligned}\int_\Omega &|A(x,\nabla u_1) - A(x,\nabla u_2)|^2 \,\mathrm{d}x \\ &\le 2\int_\Omega |A(x,\nabla u_1) - \tilde{A}(x)\nabla u_1 - A(x,\nabla u_2) + \tilde{A}(x)\nabla u_2|^2 \,\mathrm{d}x + 2\int_\Omega |\tilde{A}(x)\nabla u_1 - \tilde{A}(x)\nabla u_2|^2 \,\mathrm{d}x \\ &\le C\left(1 + \int_\Omega |\nabla u_1 - \nabla u_2|^2 \,\mathrm{d}x\right) \le C.\end{aligned}$$

Therefore, (4-15) holds for all $\varphi \in W_0^{1,2}(\Omega; \mathbb{R}^{n\times N})$ and consequently also for $\varphi := u_1 - u_2$ and the strict monotonicity finishes the proof of the uniqueness. It remains to prove Lemma 4.1.

Proof of Lemma 4.1. Let δ be given and fixed. According to (2-1) and (2-4), we can find $k > 0$ (depending on δ) such that for all $x \in \Omega$ and all $|\eta| \ge k$

$$\frac{|A(x,\eta) - \tilde{A}(x)\eta|}{|\eta|} + \left|\frac{\partial A(x,\eta)}{\partial \eta} - \tilde{A}(x)\right| \le \frac{\delta}{4}. \tag{4-21}$$

To prove (4-18), we shall discus all possible cases of values η_1 and η_2. Recall here that δ and k are already fixed.

The case $|\eta_1| \le 2k$ and $|\eta_2| \le 2k$. In this case, we can simply use (1-5) to show that

$$|A(x,\eta_1) - A(x,\eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| \le C(1 + |\eta_1| + |\eta_2|) \le C(1+4k)$$

and (4-18) follows.

The case $|\eta_1| \le 2k$ and $|\eta_2| > 2k$. In this case, we again use (1-5), which combined with (4-21) leads to

$$\begin{aligned}|A(x,\eta_1) - A(x,\eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| &\le C(1+|\eta_1|) + \left|\frac{\tilde{A}(x)\eta_2 - A(x,\eta_2)}{|\eta_2|}\right| |\eta_2| \le C(1+2k) + \frac{\delta|\eta_2|}{2} \\ &\le C(1+2k+|\eta_1|) + \frac{\delta|\eta_2 - \eta_1|}{2} \le C(1+4k) + \delta|\eta_2 - \eta_1|.\end{aligned}$$

Therefore, (4-18) holds. Moreover, the case $|\eta_1| \ge 2k$ and $|\eta_2| \le 2k$ is treated similarly.

The case $|\eta_1| > 2k$ and $|\eta_2| > 2k$. First, let us also assume that

$$|\eta_2| \le 2|\eta_1 - \eta_2| \quad \text{and} \quad |\eta_1| \le 2|\eta_1 - \eta_2|. \tag{4-22}$$

In this setting, we use (4-21) to conclude

$$\begin{aligned}|A(x,\eta_1) - A(x,\eta_2) - \tilde{A}(x)(\eta_1 - \eta_2)| &\le \left|\frac{\tilde{A}(x)\eta_1 - A(x,\eta_1)}{|\eta_1|}\right| |\eta_1| + \left|\frac{\tilde{A}(x)\eta_2 - A(x,\eta_2)}{|\eta_2|}\right| |\eta_2| \\ &\le \frac{\delta}{4}(|\eta_1| + |\eta_2|) \le \delta|\eta_1 - \eta_2|,\end{aligned}$$

which again directly implies (4-18). Finally, it remains to discuss the case when at least one of the inequalities in (4-22) does not hold. For simplicity, we consider only the case when $|\eta_1| > 2|\eta_1 - \eta_2|$ since the second case can be treated similarly. First of all, using the assumption on η_1 and η_2, we deduce that for all $t \in [0, 1]$

$$|t\eta_2 + (1-t)\eta_1| = |\eta_1 - t(\eta_1 - \eta_2)| \geq |\eta_1| - t|\eta_1 - \eta_2| \geq |\eta_1| - |\eta_1 - \eta_2| \geq \frac{|\eta_1|}{2} \geq k.$$

Hence, since any convex combination of η_1 and η_2 is outside of the ball or radius k, we can use the assumption (4-21) to conclude

$$\begin{aligned}
&|A(x,\eta_2) - A(x,\eta_1) - \tilde{A}(x)(\eta_2 - \eta_1)| \\
&\qquad = \left| \int_0^1 \frac{\mathrm{d}}{\mathrm{d}t}\big(A(x, t\eta_2 + (1-t)\eta_1) - \tilde{A}(x)(t\eta_2 + (1-t)\eta_1)\big)\,\mathrm{d}t \right| \\
&\qquad = \left| \int_0^1 \left(\frac{\partial A(x, t\eta_2 + (1-t)\eta_1)}{\partial(t\eta_2 + (1-t)\eta_1)} - \tilde{A}(x) \right)(\eta_2 - \eta_1)\,\mathrm{d}t \right| \leq \int_0^1 \frac{\delta}{4}|\eta_2 - \eta_1|\,\mathrm{d}t \leq \delta|\eta_2 - \eta_1|
\end{aligned}$$

and (4-18) follows. □

5. Proof of Theorem 2.5

We start the proof by getting the a priori estimate in the standard nonweighted Lebesgue spaces, which is available due to Lemma 3.4. Let us fix a ball Q_0 such that $\Omega \subset Q_0$. Since $\omega \in \mathcal{A}_p$, we can use (3-5) to show that for some $\tilde{q} > 1$ we have $L^p_\omega(Q_0) \hookrightarrow L^{\tilde{q}}(Q_0)$. Thus, $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ implies that $f \in L^{\tilde{q}}(\Omega; \mathbb{R}^{n\times N})$. The starting point of further analysis is the use of Lemma 3.4, which leads to the existence of a unique solution $u \in W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$ to (2-12) with the a priori bound

$$\left(\int_\Omega |\nabla u|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}} \leq C(A, \tilde{q}, \Omega)\left(\int_\Omega |f|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}}.$$

Consequently, using (3-5), we deduce

$$\left(\frac{1}{|Q_0|}\int_\Omega |\nabla u|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}} \leq C(A, p, \Omega, \mathcal{A}_p(\omega))\left(\frac{1}{\omega(Q_0)}\int_\Omega |f|^p \omega\,\mathrm{d}x\right)^{1/p}. \tag{5-1}$$

It remains to prove the a priori estimate (2-13). We divide the proof into several steps. In the first one, we shall prove the local (in Ω) estimates. Then we extend such a result up to the boundary, and finally we combine them to get Theorem 2.5.

5A. *Interior estimates.* This part is devoted to the estimates that are local in Ω; i.e., we shall prove the following:

Lemma 5.1. *Let $B \subset \mathbb{R}^n$ be a ball, $\omega \in \mathcal{A}_p$ arbitrary with some $p \in (1, \infty)$ and $A \in L^\infty(2B; \mathbb{R}^{n\times N\times n\times N})$ arbitrary satisfying*

$$c_1|\eta|^2 \leq A(x)\eta\cdot\eta \leq c_2|\eta|^2 \quad \textit{for all } x \in 2B \textit{ and all } \eta \in \mathbb{R}^{n\times N}.$$

Then there exists $\delta > 0$ depending only on p, c_1, c_2 and $A_p(\omega)$ such that, if

$$|A(x) - A(y)| \le \delta \quad \text{for all } x, y \in 2B,$$

then for arbitrary $f \in L^p_\omega(2B; \mathbb{R}^{n\times N})$ and $u \in W^{1,\tilde{q}}(2B; \mathbb{R}^N)$ with some $\tilde{q} > 1$ satisfying

$$\int_{2B} A(x)\nabla u(x) \cdot \nabla\varphi(x)\,\mathrm{d}x = \int_{2B} f(x) \cdot \nabla\varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in \mathscr{C}^{0,1}_0(2B; \mathbb{R}^N),$$

the following holds:

$$\left(\fint_B |\nabla u|^p \omega\,\mathrm{d}x\right)^{1/p} \le C\left(\fint_{2B} |f|^p \omega\,\mathrm{d}x\right)^{1/p} + C\left(\fint_{2B} \omega\,\mathrm{d}x\right)^{1/p}\left(\fint_{2B} |\nabla u|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}}, \tag{5-2}$$

where the constant C depends only on p, c_1, c_2 and $A_p(\omega)$.

Proof. First, we introduce some more notation. For ω, we denote $\omega(S) := \int_S \omega\,\mathrm{d}x$. Next, using Lemma 3.1, we can find $q \in (1, \tilde{q})$ such that $\omega \in \mathscr{A}_{p/q}$. Note here that $u \in W^{1,q}(2B; \mathbb{R}^N)$, which follows from the fact that $2B$ is bounded. In what follows, we fix such q and introduce the centered maximal operator with power q

$$(M_q(g))(x) := \sup_{r>0}\left(\fint_{B_r(x)} |g|^q\,\mathrm{d}y\right)^{1/q}.$$

Since $M_q(g) = (M(|g|^q))^{1/q}$, we see from the definition and the choice of q (which leads to $\omega \in \mathscr{A}_{p/q}(\mathbb{R}^n)$) that the operator M_q is bounded in $L^p_\omega(\mathbb{R}^n)$. We shall also use the restricted maximal operator

$$(M_q^{<\rho}(g))(x) = \sup_{\rho\ge r>0}\left(\fint_{B_r(x)} |g|^q\,\mathrm{d}y\right)^{1/q},$$

and it directly follows that for every Lebesgue point x of g

$$|g(x)| \le (M_q^{<\rho}(g))(x) \le (M_q(g))(x).$$

The inequality (5-2) will be proven using the proper estimates on the level sets for $|\nabla u|$ defined through

$$O_\lambda := \{x \in \mathbb{R}^n;\ M_q(\chi_{2B}\nabla u)(x) > \lambda\}.$$

Please observe that O_λ are open. Next, we use the Calderón–Zygmund decomposition. Thus, for fixed $\lambda > 0$ and $x \in B \cap Q_\lambda$, using the continuity of the integral with respect to the integration domain, we can find a ball $Q_{r_x}(x)$ such that

$$\lambda^q < \fint_{Q_{r_x}(x)} |\chi_{2B}\nabla u|^q\,\mathrm{d}x \le 2\lambda^q \quad \text{and} \quad \fint_{Q_r(x)} |\chi_{2B}\nabla u|^q\,\mathrm{d}x \le 2\lambda^q \quad \text{for all } r \ge r_x. \tag{5-3}$$

Next, using the Besicovich covering theorem, we can extract a countable subset $Q_i := Q_{r_i}(x_i)$ such that the Q_i have finite intersection, i.e., there exists a constant C depending only on n such that for all $i \in \mathbb{N}$

$$\#\{j \in \mathbb{N};\ Q_i \cap Q_j \ne \varnothing\} \le C.$$

In addition, it follows from the construction that

$$O_\lambda \cap B = \bigcup_{i\in\mathbb{N}} (Q_i \cap B). \tag{5-4}$$

Then we set

$$\Lambda := \left(\fint_{2B} |\nabla u|^q \, \mathrm{d}x \right)^{1/q},$$

and it directly follows that for any $Q \subset \mathbb{R}^n$

$$\left(\fint_{Q} |\chi_{2B} \nabla u|^q \, \mathrm{d}x \right)^{1/q} \le \left(\frac{|2B|}{|Q|} \right)^{1/q} \Lambda.$$

Consequently, assuming that $\lambda \ge 2^{2n}\Lambda$, we can deduce for every Q_i that

$$2^{2n}\Lambda \le \lambda < \left(\fint_{Q_i} |\chi_{2B} \nabla u|^q dx \right)^{1/q} \le \left(\frac{|2B|}{|Q_i|} \right)^{1/q} \Lambda = 2^{2n/q} \left(\frac{|B|}{|2Q_i|} \right)^{1/q} \Lambda.$$

Since $q \ge 1$, this inequality directly leads to $|2Q_i| \le |B|$. Therefore, using the fact that $Q_i = Q_{r_i}(x_i)$ with some $x_i \in B$, we observe that $2Q_i \subset 2B$. Moreover, it is evident that for some constant C depending only on the dimension n

$$|Q_i| \le C(n)|Q_i \cap B|. \tag{5-5}$$

Since $\omega \in \mathcal{A}_p$, the above relation implies (see, e.g., [Stein 1993, §V.1.7])

$$\omega(Q_i) \le C(n, A_p(\omega))\omega(Q_i \cap B). \tag{5-6}$$

Next, for arbitrary $\varepsilon > 0$ and $k \ge 1$, we introduce the redistributional set

$$U^\lambda_{\varepsilon,k} := O_{k\lambda} \cap \{x \in \mathbb{R}^n;\, M_q(f\chi_{2B})(x) \le \varepsilon\lambda\}.$$

Finally, we shall assume the following (recall that δ comes from the assumption of Lemma 5.1):

there exists $k \ge 1$ depending only on c_1, c_2, n, p, and $A_p(\omega)$ such that for all $\varepsilon \in (0, 1)$ and all $\lambda \ge 2^{2^n}\Lambda$

$$|Q_i \cap U^\lambda_{\varepsilon,k} \cap B| \le C(c_1, c_2, n)(\varepsilon+\delta)|Q_i|. \tag{5-7}$$

We postpone the proof of (5-7) and continue assuming that it holds true with fixed k such that (5-7) is valid. Hence, using (5-7), the Hölder inequality and the reverse Hölder inequality (which follows for $\mathcal{A}_p$-weights from (3-4)) and (5-6), we obtain for some $r > 1$ depending only on n, p and $A_p(\omega)$

$$\begin{aligned}\omega(Q_i \cap U^\lambda_{\varepsilon,k} \cap B) &\le C(n)|Q_i| \left(\fint_{Q_i} \omega^r \, \mathrm{d}x \right)^{1/r} \left(\frac{|Q_i \cap U^\lambda_{\varepsilon,k} \cap B|}{|Q_i|} \right)^{1/r'} \\ &\le C(n, p, A_p(\omega), c_1, c_2)(\varepsilon+\delta)^{1/r'} \omega(Q_i) \le C(n, p, A_p(\omega), c_1, c_2)(\varepsilon+\delta)^{1/r'} \omega(Q_i \cap B).\end{aligned}$$

By using the finite intersection property of the Q_i, we find

$$\omega(U^\lambda_{\varepsilon,k} \cap B) \le C(n, A_p(\omega), c_1, c_2)(\varepsilon+\delta)^{1/r'} \omega(O_\lambda \cap B). \tag{5-8}$$

Finally, using the Fubini theorem, we obtain

$$\int_B |\nabla u|^p \omega \,\mathrm{d}x = p\int_0^\infty \omega(\{(\nabla u)\chi_B > \lambda\})\lambda^{p-1}\,\mathrm{d}\lambda \le \Lambda^p\omega(B) + p\int_\Lambda^\infty \lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda. \tag{5-9}$$

Therefore, to get the estimate (5-2), we need to estimate the last term on the right-hand side. To do so, we use the definition of $U^\lambda_{\varepsilon,k}$ and the substitution theorem, which leads for all $m > k\Lambda$ to

$$\begin{aligned}\int_{k\Lambda}^m \lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda &\le \int_{k\Lambda}^m \lambda^{p-1}\omega(U^{\lambda/k}_{\varepsilon,k}\cap B)\,\mathrm{d}\lambda + \int_{k\Lambda}^m \lambda^{p-1}\omega\Big(\Big\{M_q(f\chi_{2B}) > \varepsilon\frac{\lambda}{k}\Big\}\Big)\,\mathrm{d}\lambda\\
&\overset{(5\text{-}8)}{\le} C(\varepsilon+\delta)^{1/r'}\int_{k\Lambda}^m \lambda^{p-1}\omega(O_{\lambda/k}\cap B)\,\mathrm{d}\lambda + \frac{k^p}{p\varepsilon^p}\int_{\mathbb{R}^n}|M_q(f\chi_{2B})|^p\omega\,\mathrm{d}x\\
&\le C(p,q,\varepsilon,A_p(\omega))\int_{2B}|f|^p\omega\,\mathrm{d}x + Ck^p(\varepsilon+\delta)^{1/r'}\int_\Lambda^{m/k}\lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda\\
&\le C(p,q,\varepsilon,A_p(\omega))\int_{2B}|f|^p\omega\,\mathrm{d}x + Ck^p(\varepsilon+\delta)^{1/r'}\int_\Lambda^{k\Lambda}\lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda\\
&\qquad + Ck^p(\varepsilon+\delta)^{1/r'}\int_{k\Lambda}^m\lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda,\end{aligned}$$

where we used the fact that $\omega\in\mathcal{A}_{p/q}$. Finally, assuming (note that k is already fixed by (5-7), and at this point, we fix the maximal value of δ arising in the assumption of Lemma 5.1) that δ is so small that $Ck^p\delta^{1/r'}\le\frac{1}{8}$, we can find $\varepsilon\in(0,1)$ such that $Ck^p(\varepsilon+\delta)^{1/r'}\le\frac{1}{2}$. Consequently, we absorb the last term into the left-hand side, and letting $m\to\infty$, we find that

$$\int_{k\Lambda}^\infty \lambda^{p-1}\omega(O_\lambda\cap B)\,\mathrm{d}\lambda \le C(k,p,q,A_p(\omega))\Big(\int_{2B}|f|^p\omega\,\mathrm{d}x + \Lambda^p\omega(B)\Big).$$

Substituting this into (5-9), we find (5-2). To finish the proof, it remains to find $k\ge 1$ such that (5-7) holds.

Hence, assume that $Q_i\cap B\cap U^\lambda_{\varepsilon,k}\ne\varnothing$. Then it follows from the definition of $U^\lambda_{\varepsilon,k}$ that

$$\Big(\fint_{2Q_i}|f|^q\,\mathrm{d}x\Big)^{1/q}\le 2^n\varepsilon\lambda. \tag{5-10}$$

For $\lambda\ge 2^{2n}\Lambda$ (which implies $2Q_i\subset 2B$), we compare the original problem with

$$\begin{aligned}-\operatorname{div}(A_i\nabla h) &= 0 \quad \text{in } 2Q_i,\\ h &= u \quad \text{on } \partial(2Q_i),\end{aligned} \tag{5-11}$$

where the matrix A_i is defined as $A_i := A(x_i)$. Lemma 3.4 ensures the existence of such a solution (just consider $u-h$ with zero boundary data). Moreover, the matrix A_i is constant and elliptic and therefore we have the local $L^\infty - L^1$ estimate for h, i.e.,

$$\sup_{(3/2)Q_i}|\nabla h|\le C\fint_{2Q_i}|\nabla h|\,\mathrm{d}x, \tag{5-12}$$

where the constant C depends only on n, c_1 and c_2. Further, since u solves our original problem, we find

$$\begin{aligned} -\operatorname{div}(A_i\nabla(u-h)) &= -\operatorname{div}((A-A_i)\nabla u - f) && \text{in } 2Q_i,\\ u-h &= 0 && \text{on } \partial 2Q_i. \end{aligned}$$

Therefore, we can use Lemma 3.4 to observe

$$⨍_{2Q_i} |\nabla(u-h)|^q \,\mathrm{d}x \le C ⨍_{2Q_i} |A-A_i|^q |\nabla u|^q \,\mathrm{d}x + C ⨍_{2Q_i} |f|^q \,\mathrm{d}x \le C(\varepsilon^q+\delta^q)\lambda^q, \tag{5-13}$$

where for the second inequality we used (5-3), (5-10) and the assumption that $|A(x)-A(y)|\le\delta$ for all $x, y\in B$. Then using the definition of Q_i, we see that, for all $y\in Q_i$ and all $r>r_i/2$, we have that $B_r(y)\subset B_{3r}(x_i)$ and $Q_i\subset B_{3r}(x_i)$. Consequently,

$$⨍_{B_r(y)} |\chi_{2B}\nabla u|^q \,\mathrm{d}x \le 3^n ⨍_{B_{3r}(x_i)} |\chi_{2B}\nabla u|^q \,\mathrm{d}x \le 6^n\lambda^q,$$

where we used (5-3). Choosing $k\ge 6^n$ and assuming that $\varepsilon,\delta\le 1$, we get by the previous estimate, the sublinearity of the maximal operator and the weak Harnack inequality (5-12) that for all $x\in Q_i\cap\{M_q(\nabla u)>k\lambda\}$

$$\begin{aligned} M_q(\nabla u)(x) &= M_q^{<r_i/2}(\nabla u)(x) \le M_q^{<r_i/2}(\nabla h)(x) + M_q^{<r_i/2}(\nabla u-\nabla h)(x)\\ &\le C\biggl(⨍_{2Q_i}|\nabla h|^q\,\mathrm{d}x\biggr)^{1/q} + M_q^{<r_i/2}(\nabla u-\nabla h)(x) \le C\lambda + M_q^{<r_i/2}(\nabla u-\nabla h)(x). \end{aligned}$$

Hence, setting $k:=\max\{C+1, 6^n\}$, we can use the weak L^q-estimate for the maximal functions and the estimate (5-13) to conclude

$$\begin{aligned} |\{M_q(\nabla u)>k\lambda\}\cap Q_i| \le |\{M_q^{<r_i/2}(\nabla u-\nabla h)\ge\lambda\}\cap Q_i| &\le \frac{C}{\lambda^q}\int_{2Q_i}|\nabla(u-h)|^q\,\mathrm{d}x\\ &\le C(\varepsilon+\delta)|Q_i|, \end{aligned}$$

which finishes the proof of (5-7) and Lemma 5.1. □

5B. ***Estimates near the boundary.*** In this part, we generalize the result from the previous paragraph and extend its validity also to the neighborhood of the boundary.

Lemma 5.2. *Let $\Omega\subset\mathbb{R}^n$ be a domain with $\mathscr{C}^1$ boundary, $\omega\in\mathscr{A}_p$ be arbitrary with some $p\in(1,\infty)$ and $A\in L^\infty(\Omega;\mathbb{R}^{n\times N\times n\times N})$ be arbitrary satisfying*

$$c_1|\eta|^2 \le A(x)\eta\cdot\eta \le c_2|\eta|^2 \quad \textit{for all } x\in 2B \textit{ and all } \eta\in\mathbb{R}^{n\times N}.$$

Then there exists $r^*>0$ and $\delta>0$ depending only on Ω, p, c_1, c_2 and $A_p(\omega)$ such that, if

$$\sup_{x,y\in\Omega;\,|x-y|\le r^*} |A(x)-A(y)| \le \delta,$$

then for arbitrary $f \in L^p_\omega(\Omega; \mathbb{R}^{n\times N})$ *and* $u \in W^{1,\tilde{q}}_0(\Omega; \mathbb{R}^N)$ *with some* $\tilde{q} > 1$ *satisfying*

$$\int_\Omega A\nabla u \cdot \nabla\varphi \,\mathrm{d}x = \int_\Omega f \cdot \nabla\varphi \,\mathrm{d}x \quad \text{for all } \varphi \in \mathscr{C}^{0,1}_0(\Omega; \mathbb{R}^N), \tag{5-14}$$

we have for all $x_0 \in \overline{\Omega}$ *and all* $r \le r^*$ *the estimate*

$$\fint_{B_r(x_0)\cap\Omega} |\nabla u|^p \omega \,\mathrm{d}x \le \fint_{B_{2r}(x_0)\cap\Omega} C|f|^p \omega \,\mathrm{d}x + \fint_{B_{2r}(x_0)\cap\Omega} \omega \,\mathrm{d}x \left(\fint_{B_{2r}(x_0)\cap\Omega} C|\nabla u|^{\tilde{q}} \,\mathrm{d}x \right)^{p/\tilde{q}}. \tag{5-15}$$

First notice that in case $B_{2r}(x_0) \subset \Omega$ the inequality (5-15) follows from Lemma 5.1. Therefore, we focus only on the behavior near the boundary. Hence, let $x_0 \in \partial\Omega$ be arbitrary. Since $\Omega \in \mathscr{C}^1$, we know that there exist $\alpha, \beta > 0$ and $r_0 > 0$ such that (after a possible change of coordinates)

$$B^+_{r_0} := \{(x', x_n);\, |x'| < \alpha,\ a(x') - \beta < x_n < a(x')\} \subset \Omega,$$
$$B^-_{r_0} := \{(x', x_n);\, |x'| < \alpha,\ a(x') < x_n < a(x') + \beta\} \subset \Omega^c.$$

Here, we abbreviated $(x_1, \dots, x_n) := (x', x_n)$. Moreover, we know that for all $r \le r_0/2$ it holds that $B_{2r}(x_0) \cap \Omega \subset B^+_{r_0}$ and $B_{2r}(x_0) \cap \Omega^c \subset B^-_{r_0}$. In addition, we have $a \in \mathscr{C}^1([-\alpha, \alpha]^{n-1})$ and $\nabla a(0) \equiv 0$. For later purposes, we also denote

$$B_{r_0} := B^+_{r_0} \cup B^-_{r_0} \cup \{(x, x_n);\, |x'| < \alpha,\ a(x') = x_n\}$$

and define a mapping $T : B^+_{r_0} \to B^-_{r_0}$ as

$$T(x', x_n) := (x', 2a(x') - x_n) \quad \text{with } J(x) := \nabla T(x), \text{ i.e., } (J(x))_{ij} := \partial_{x_j}(T(x))_i.$$

It directly follows from the definition that $|\det J(x)| \equiv 1$ and also that T and T^{-1} are $\mathscr{C}^1$ mappings. Finally, we extend all quantities into $B^-_{r_0}$ as follows:

$$\tilde{u}(x) := \begin{cases} u(x) & \text{for } x \in B^+_{r_0}, \\ -u(T^{-1}(x)) & \text{for } x \in B^-_{r_0}, \end{cases}$$
$$\tilde{A}(x) := \begin{cases} A(x) & \text{for } x \in B^+_{r_0}, \\ J(T^{-1}x)A(T^{-1}x)J^T(T^{-1}x) & \text{for } x \in B^-_{r_0}, \end{cases}$$
$$\tilde{f}(x) := \begin{cases} f(x) & \text{for } x \in B^+_{r_0}, \\ -J(T^{-1}x)f(T^{-1}(x)) & \text{for } x \in B^-_{r_0}, \end{cases}$$
$$\widetilde{\omega}(x) := \begin{cases} \omega(x) & \text{for } x \in B^+_{r_0}, \\ \omega(T^{-1}(x)) & \text{for } x \in B^-_{r_0}. \end{cases}$$

It also directly follows from the definition and the fact that u has zero trace on $\partial\Omega$ that $\tilde{u} \in W^{1,q}(B_{r_0}; \mathbb{R}^N)$. Finally, we show that for all $\varphi \in \mathscr{C}^{0,1}_0(B_{r_0}; \mathbb{R}^N)$ the following identity holds:

$$\int_{B_{r_0}} \tilde{A}\nabla\tilde{u} \cdot \nabla\varphi \,\mathrm{d}x = \int_{B_{r_0}} \tilde{f} \cdot \nabla\varphi \,\mathrm{d}x. \tag{5-16}$$

For this, we observe that for any $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0}^-; \mathbb{R}^N)$ and $\hat{\varphi} := \varphi \circ T \in \mathscr{C}_0^{0,1}(B_{r_0}^+; \mathbb{R}^N)$

$$\begin{aligned}\int_{B_{r_0}^-} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi \,\mathrm{d}x &= \int_{B_{r_0}^-} \left(\tilde{A}_{ij}^{\mu\nu}(x)\frac{\partial\tilde{u}^\nu(x)}{\partial x_j} - \tilde{f}_i^\mu(x)\right)\frac{\partial\varphi^\mu(x)}{\partial x_i}\,\mathrm{d}x \\ &= \int_{B_{r_0}^-} \left(-\tilde{A}_{ij}^{\mu\nu}(x)\frac{\partial(u^\nu(T^{-1}x))}{\partial x_j} - \tilde{f}_i^\mu(x)\right)\frac{\partial(\hat{\varphi}^\mu(T^{-1}(x)))}{\partial x_i}\,\mathrm{d}x \\ &= \int_{B_{r_0}^-} \left(-\tilde{A}_{ij}^{\mu\nu}(x)\frac{\partial u^\nu(T^{-1}x)}{\partial(T^{-1}(x))_k}J_{kj}^{-1}(T^{-1}(x)) - \tilde{f}_i^\mu(x)\right)\frac{\partial\hat{\varphi}^\mu(T^{-1}(x))}{\partial(T^{-1}(x))_m}J_{mi}^{-1}(T^{-1}(x))\,\mathrm{d}x \\ &= \int_{B_{r_0}^+} \left(-\tilde{A}_{ij}^{\mu\nu}(Tx)\frac{\partial u^\nu(x)}{\partial x_k}J_{kj}^{-1}(x)J_{mi}^{-1}(x) - \tilde{f}_i^\mu(Tx)J_{mi}^{-1}(x)\right)\frac{\partial\hat{\varphi}^\mu(x)}{\partial x_m}\,\mathrm{d}x \\ &= -\int_{B_{r_0}^+} (A(x)\nabla u(x) - f(x)) \cdot \nabla\hat{\varphi}(x)\,\mathrm{d}x.\end{aligned}$$

In particular, for all $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0}^+; \mathbb{R}^N)$

$$\int_{B_{r_0}^-} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla(\varphi \circ T^{-1})\,\mathrm{d}x = -\int_{B_{r_0}^+} (A\nabla u - f) \cdot \nabla\varphi\,\mathrm{d}x. \tag{5-17}$$

Thus, if we define for $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0}; \mathbb{R}^N)$ the function

$$\bar{\varphi} := \begin{cases} \varphi \circ T^{-1} & \text{on } B_{r_0}^-, \\ \varphi & \text{on } B_{r_0}^+, \end{cases}$$

then $\bar{\varphi} \in \mathscr{C}_0^{0,1}(B_{r_0}; \mathbb{R}^N)$ and (5-17) implies

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\bar{\varphi}\,\mathrm{d}x = 0.$$

Therefore,

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi\,\mathrm{d}x = \int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla(\varphi - \bar{\varphi})\,\mathrm{d}x = \int_{B_{r_0}^-} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla(\varphi - \bar{\varphi})\,\mathrm{d}x.$$

Using (5-17) again, we get

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi\,\mathrm{d}x = -\int_{B_{r_0}^+} (A\nabla u - f) \cdot \nabla((\varphi - \bar{\varphi}) \circ T^{-1})\,\mathrm{d}x.$$

Since $(\varphi - \bar{\varphi}) \circ T^{-1} = 0$ on $\partial\Omega$, we finally deduce with the help of (5-14) that

$$\int_{B_{r_0}} (\tilde{A}\nabla\tilde{u} - \tilde{f}) \cdot \nabla\varphi\,\mathrm{d}x = 0$$

for all $\varphi \in \mathscr{C}_0^{0,1}(B_{r_0}; \mathbb{R}^N)$, which proves (5-16).

Consequently, we see that (5-16) holds, and therefore, we shall apply the local result stated in Lemma 5.1. To do so, we need to check the assumptions. First, the ellipticity of $\tilde{A}$ can be shown directly from the definition and the fact that J is a regular matrix. Moreover, the constant of ellipticity of $\tilde{A}$ depends only on the same constant for A and on the shape of Ω. Further, to be able to use (5-2), we need to show small oscillations of $\tilde{A}$. Since T is $\mathscr{C}^1$,

$$\begin{aligned}\sup_{x,y\in B^-_{r_0}} |\tilde{A}(x)-\tilde{A}(y)| &\le \sup_{x,y\in B^+_{r_0}} |J(x)A(x)J^T(x)-J(y)A(y)J^T(y)| \\ &\le C \sup_{x,y\in B^+_{r_0}} |A(x)-A(y)| + C \sup_{x,y\in B^+_{r_0}} |J(x)-J(y)|.\end{aligned}$$

Similarly, we can also deduce that

$$\begin{aligned}\sup_{x\in B^-_{r_0},\, y\in B^+_{r_0}} |\tilde{A}(x)-\tilde{A}(y)| &\le \sup_{x,y\in B^+_{r_0}} |J(x)A(x)J^T(x)-A(y)| \\ &\le C \sup_{x,y\in B^+_{r_0}} |A(x)-A(y)| + C \sup_{x\in B^+_{r_0}} |J(x)A(x)J^T(x)-A(x)| \\ &\le C \sup_{x,y\in B^+_{r_0}} |A(x)-A(y)| + C \sup_{x\in B^+_{r_0}} |\nabla a(x')|.\end{aligned}$$

Therefore, due to the continuity of J and the fact that $\nabla a(0)=0$, we see that for any $\delta>0$ we can find $r^*>0$ such that

$$C \sup_{x,y\in B^+_{r^*}} |J(x)-J(y)| + C \sup_{x\in B^+_{r^*}} |\nabla a(x')| < \frac{\delta}{2}.$$

Thus, assuming that

$$\sup_{x,y\in\Omega;\, C|x-y|\le r^*} |A(x)-A(y)| \le \frac{\delta}{2},$$

we can conclude that

$$\sup_{x,y\in B_{r^*}} |\tilde{A}(x)-\tilde{A}(y)| \le \delta.$$

We find $\delta>0$ and fix r^* such that all assumptions of Lemma 5.1 are satisfied and we consequently have

$$\left(\fint_{B_{r^*}(x_0)} |\nabla\tilde{u}|^p \widetilde{\omega}\,\mathrm{d}x\right)^{1/p} \le C\left(\fint_{B_{2r^*}(x_0)} |\tilde{f}|^p \widetilde{\omega}\,\mathrm{d}x\right)^{1/p} + C\left(\fint_{B_{2r^*}(x_0)} \widetilde{\omega}\,\mathrm{d}x\right)^{1/p}\left(\fint_{B_{2r^*}(x_0)} |\nabla\tilde{u}|^{\tilde{q}}\,\mathrm{d}x\right)^{1/\tilde{q}}$$

and (5-15) follows directly.

5C. ***Global estimates.*** Finally, we focus on the proof of Theorem 2.5. Recall that the ball Q_0 is a superset of Ω. Since A is continuous, we can find for any $\delta>0$ some r^* such that

$$\sup_{x,y\in\Omega;\, |x-y|\le r^*} |A(x)-A(y)| \le \delta.$$

Therefore on any sufficiently small ball, we can use the estimate (5-15). Since Ω has $\mathscr{C}^1$ boundary, we can find a finite covering of Ω by balls B_i of radius at most equal to r^* such that $|B_i\cap\Omega|\ge c|B_i|$. Then it follows from (5-15) and (5-1) that

$$\int_\Omega |\nabla u|^p \omega \,\mathrm{d}x \le C \int_\Omega |f|^p \omega \,\mathrm{d}x + C \sum_i \frac{\omega(2B_i)}{|2B_i|^{p/\tilde q}} \left(\int_\Omega |\nabla u|^{\tilde q} \,\mathrm{d}x \right)^{p/\tilde q}$$
$$\le C \int_\Omega |f|^p \omega \,\mathrm{d}x + C(p, \tilde q, A, \Omega)\omega(Q_0) \left(\int_\Omega |\nabla u|^{\tilde q} \,\mathrm{d}x \right)^{p/\tilde q} \le C(A, \Omega, A_p(\omega)) \int_\Omega |f|^p \omega \,\mathrm{d}x,$$

which finishes the proof of Theorem 2.5.

6. Proof of Theorem 2.6

We start the proof by observing that (2-14) leads to the estimate

$$\int_\Omega |a^k \cdot b^k| \omega \,\mathrm{d}x \le \int_\Omega |a^k|^p \omega + |b^k|^{p'} \omega \,\mathrm{d}x \le C.$$

Consequently, we can use Lemma 3.3 to conclude that there is a nondecreasing sequence of measurable sets $E_j \subset \Omega$ fulfilling $|\Omega \setminus E_j| \to 0$ as $j \to \infty$ such that for any $j \in \mathbb{N}$ and any $\varepsilon > 0$ there exists a $\delta > 0$ such that for each $U \subset E_j$ fulfilling $|U| \le \delta$

$$\sup_{k \in \mathbb{N}} \int_U |a^k \cdot b^k| \omega \,\mathrm{d}x \le \sup_{k \in \mathbb{N}} \int_U |a^k|^p \omega + |b^k|^{p'} \omega \,\mathrm{d}x \le \varepsilon. \tag{6-1}$$

Consequently, for any E_j, we can extract a subsequence that we do not relabel such that

$$a^k \cdot b^k \omega \rightharpoonup \overline{a \cdot b\omega} \quad \text{weakly in } L^1(E_j), \tag{6-2}$$

where $\overline{a \cdot b\omega}$ denotes in our notation the weak limit. Further, since $L^p_\omega(\Omega)$ and $L^{p'}_\omega(\Omega)$ are reflexive, we can pass to a (nonrelabeled) subsequence with

$$\begin{aligned} a_k &\rightharpoonup a \quad \text{weakly in } L^p_\omega(\Omega; \mathbb{R}^n), \\ b_k &\rightharpoonup b \quad \text{weakly in } L^{p'}_\omega(\Omega; \mathbb{R}^n). \end{aligned} \tag{6-3}$$

Our goal is to show that

$$\overline{a \cdot b\omega} = a \cdot b\omega \quad \text{almost everywhere in } \Omega. \tag{6-4}$$

Indeed, if this is the case, then it follows that not only a subsequence but the whole sequence fulfills (6-2).

Since $\omega \in \mathcal{A}_p$, we can find by (3-5) some $q > 1$ such that $L^p_\omega(\Omega) \hookrightarrow L^q(\Omega)$. This implies

$$a^k \rightharpoonup a \quad \text{weakly in } L^q(\Omega; \mathbb{R}^n). \tag{6-5}$$

Moreover, since the mapping $g \mapsto g\omega^{1/s}$ is an isometry from $L^s_\omega(\Omega)$ to $L^s(\Omega)$, we also have

$$a^k \omega^{1/p} \rightharpoonup a\omega^{1/p} \quad \text{weakly in } L^p(\Omega; \mathbb{R}^n), \tag{6-6}$$
$$b^k \omega^{1/p'} \rightharpoonup b\omega^{1/p'} \quad \text{weakly in } L^{p'}(\Omega; \mathbb{R}^n). \tag{6-7}$$

Then, extending a^k by zero outside Ω, we can introduce d^k such that

$$\Delta d^k = a^k \quad \text{in } \mathbb{R}^n;$$

i.e., we set $d^k := a^k * G$, where G denotes the Green function of the Laplace operator on the whole $\mathbb{R}^n$. Then, using (6-5), we see that

$$d^k \rightharpoonup d \quad \text{weakly in } W^{2,q}_{\mathrm{loc}}(\mathbb{R}^n; \mathbb{R}^n), \tag{6-8}$$

where

$$\Delta d = a \quad \text{in } \mathbb{R}^n.$$

In addition, using (2-14) and the weighted theory for Laplace equation on $\mathbb{R}^n$ [Coifman and Fefferman 1974, p. 244], we can deduce

$$\nabla^2 d^k \rightharpoonup \nabla^2 d \quad \text{weakly in } L^p_\omega(\mathbb{R}^n; \mathbb{R}^{n\times n\times n}). \tag{6-9}$$

Hence, to show (6-4), it is enough to check whether

$$b^k \cdot (a^k - \nabla \operatorname{div} d^k)\omega \rightharpoonup b \cdot (a - \nabla \operatorname{div} d)\omega \quad \text{weakly in } L^1(E_j), \tag{6-10}$$

$$b^k \cdot \nabla(\operatorname{div} d^k)\omega \rightharpoonup b \cdot \nabla(\operatorname{div} d)\omega \qquad \text{weakly in } L^1(E_j), \tag{6-11}$$

for all $j \in \mathbb{N}$.

First, we focus on (6-10). Assume for a moment that we know

$$\lim_{k\to\infty} \int_\Omega |a^k - a + \nabla(\operatorname{div}(d - d^k))|\tau \,\mathrm{d}x = 0 \tag{6-12}$$

for all nonnegative $\tau \in \mathscr{D}(\Omega)$. Then for any $\varphi \in L^\infty(E_j)$,

$$\begin{aligned}\lim_{k\to\infty} \int_{E_j} & b^k \cdot (a^k - \nabla \operatorname{div} d^k)\omega\varphi \,\mathrm{d}x \\ &= \lim_{k\to\infty} \int_{E_j} b^k \cdot (a - \nabla \operatorname{div} d)\omega\varphi \,\mathrm{d}x + \lim_{k\to\infty} \int_{E_j} b^k \cdot (a^k - a + \nabla \operatorname{div}(d - d^k))\omega\varphi \,\mathrm{d}x \\ &\overset{(6\text{-}7)}{=} \int_{E_j} b \cdot (a - \nabla \operatorname{div} d)\omega\varphi \,\mathrm{d}x + \lim_{k\to\infty} \int_{E_j} b^k \cdot (a^k - a + \nabla \operatorname{div}(d - d^k))\omega\varphi \,\mathrm{d}x\end{aligned}$$

and (6-10) follows provided that the second limit in the above formula vanishes. However, we first notice that (for a subsequence) (6-12) implies that

$$b^k \cdot (a^k - a + \nabla \operatorname{div}(d - d^k))\omega\varphi \to 0 \quad \text{almost everywhere in } \Omega. \tag{6-13}$$

Second, using (6-8) and (6-6), we see that for any $U \subset E_j$

$$\int_U |b^k \cdot (a^k - a + \nabla \operatorname{div}(d - d^k))\omega\varphi| \,\mathrm{d}x \le C\|\varphi\|_\infty \|a^k - a\|_{L^p_\omega(\Omega)} \left(\int_U |b^k|^{p'} \omega \,\mathrm{d}x \right)^{1/p'}.$$

Then the equi-integrability (6-1) also guarantees the equi-integrability of the sequence (6-13), and consequently, the Vitali theorem leads to

$$\lim_{k\to\infty} \int_{E_j} b^k \cdot (a^k - a + \nabla \operatorname{div}(d - d^k))\omega\varphi \,\mathrm{d}x = 0,$$

which finishes the proof of (6-10) provided we show (6-12). First, it follows from (2-16) and (6-5) that for a subsequence that we do not relabel $\partial_{x_i} a_j^k - \partial_{x_j} a_i^k \to \partial_{x_i} a_j - \partial_{x_j} a_i$ strongly in $(W_0^{1,r}(\Omega))^*$ for all $i, j = 1, \dots, n$. Therefore, by the regularity theory for Poisson's equation, we find that

$$\partial_{x_i} d_j^k - \partial_{x_j} d_i^k \to \partial_{x_i} d_j - \partial_{x_j} d_i \quad \text{strongly in } W^{1,r}_{\mathrm{loc}}(\Omega) \tag{6-14}$$

for all $i, j = 1, \dots, n$ and all $r \in [1, q)$, where $q > 1$ comes from (6-5). Moreover, using the definition of d^k,

$$a_j^k - \partial_{x_j} \operatorname{div} d^k = \sum_{m=1}^{n} \partial^2_{x_m^2} d_j^k - \partial_{x_j} \partial_{x_m} d_m^k = \sum_{m=1}^{n} \partial_{x_m}(\partial_{x_m} d_j^k - \partial_{x_j} d_m^k),$$

and with the help of (6-14), we see that (6-12) directly follows and the proof of (6-10) is complete.

The rest of this section is devoted to the most difficult part of the proof, which is the validity of (6-11). For simplicity, we denote $e^k := \operatorname{div} d^k$, and due to (6-8) and (6-9),

$$e^k \rightharpoonup e \qquad \text{weakly in } W^{1,q}_{\mathrm{loc}}(\mathbb{R}^n), \tag{6-15}$$

$$\nabla e^k \rightharpoonup \nabla e \quad \text{weakly in } L^p_\omega(\mathbb{R}^n; \mathbb{R}^n), \tag{6-16}$$

where $e = \operatorname{div} d$. Since we are interested only in the convergence result in Ω, we localize e^k by a proper cutting outside Ω. To be more precise on the ball B (recall that it is a ball such that $\Omega \subsetneq B$), we set

$$e_B^k := e^k \tau$$

with $\tau \in \mathcal{D}(B)$ being identically one in Ω. In addition, we can observe that

$$e_B^k \rightharpoonup e_B \qquad \text{weakly in } W_0^{1,q}(B), \tag{6-17}$$

$$\nabla e_B^k \rightharpoonup \nabla e_B \quad \text{weakly in } L^p_\omega(B; \mathbb{R}^n). \tag{6-18}$$

Indeed, the relation (6-17) is a trivial consequence of (6-15), and for the validity of (6-18), it is enough to show that

$$\int_B |\nabla e_B^k|^p \omega \, \mathrm{d}x \le C.$$

Since $|\nabla e_B^k| \le C|\nabla e^k| + C|e^k - (e^k)_B| + C|(e^k)_B|$, where e_B^k denotes the mean value of e^k over B, it follows from (6-15) and (6-16) that we just need to estimate the term involving $|e^k - (e^k)_B|$. But using the pointwise estimate $|e^k - (e^k)_B| \le C(B) M(\nabla e^k)$,

$$\int_B |e_B^k - (e^k)_B|^p \omega \, \mathrm{d}x \le C \int_{\mathbb{R}^n} |M(\nabla e^k)|^p \omega \, \mathrm{d}x \le C A_p(\omega) \int_{\mathbb{R}^n} |\nabla e^k|^p \omega \, \mathrm{d}x \le C,$$

where we used the properties of $\mathcal{A}_p$-weights. Finally, since $e_B^k \in W_0^{1,1}(B)$, we can apply the Lipschitz approximation (Theorem 2.7), which implies that for arbitrary fixed $\lambda > \lambda_0$ and for any k we find the Lipschitz approximation of e_B^k on the set B and denote it by $e_B^{k,\lambda}$. Then thanks to Theorem 2.7, for any λ, we can find a subsequence (that is not relabeled) such that

$$\nabla e_B^{k,\lambda} \rightharpoonup^* \nabla e_B^\lambda \quad \text{weakly}^* \text{ in } L^\infty(B; \mathbb{R}^n), \tag{6-19}$$

$$\nabla e_B^{k,\lambda} \rightharpoonup \nabla e_B^\lambda \quad \text{weakly in } L^p_\omega(B; \mathbb{R}^n), \tag{6-20}$$

$$e_B^{k,\lambda} \to e_B^\lambda \quad \text{strongly in } \mathscr{C}(B). \tag{6-21}$$

Please notice that we do not have any a priori knowledge of how the limit e_B^λ can be found; we just know that it exists.

In the next step, we identify the weak limit of $b^k \cdot \nabla e_B^{k,\lambda}$. Due to (6-3) and (6-19), we see that this sequence is equi-integrable and consequently poses a weakly converging (in the topology of L^1) subsequence. Therefore, to identify it, it is enough to show that for all $\eta \in \mathscr{D}(\Omega)$

$$\lim_{k\to\infty} \int_\Omega b^k \cdot \nabla e_B^{k,\lambda} \eta \,\mathrm{d}x = \int_\Omega b \cdot \nabla e_B^\lambda \eta \,\mathrm{d}x.$$

However, using (2-15), (6-19) and (6-21), we can deduce that

$$\lim_{k\to\infty} \int_\Omega b^k \cdot \nabla e_B^{k,\lambda} \eta \,\mathrm{d}x = \lim_{k\to\infty} \int_\Omega b^k \cdot (\nabla e_B^{k,\lambda} - \nabla e_B^\lambda) \eta \,\mathrm{d}x + \int_\Omega b \cdot \nabla e_B^\lambda \eta \,\mathrm{d}x = \int_\Omega b \cdot \nabla e_B^\lambda \eta \,\mathrm{d}x$$

and therefore

$$b^k \cdot \nabla e_B^{k,\lambda} \rightharpoonup b \cdot \nabla e_B^\lambda \quad \text{weakly in } L^1(\Omega). \tag{6-22}$$

Finally, let $\varphi \in L^\infty(E_j)$ be arbitrary and $C := C(\|\varphi\|_\infty)$. Then we check the validity of (6-11) as follows:

$$\begin{aligned}
\lim_{k\to\infty} \left| \int_{E_j} (b^k \cdot \nabla(\operatorname{div} d^k) - b \cdot \nabla(\operatorname{div} d)) \omega\varphi \,\mathrm{d}x \right| &= \lim_{k\to\infty} \left| \int_{E_j} (b^k \cdot \nabla e_B^k - b \cdot \nabla e_B) \omega\varphi \,\mathrm{d}x \right| \\
&\le \lim_{k\to\infty} \left| \int_{E_j} (b^k \cdot \nabla e_B^{k,\lambda} - b \cdot \nabla e_B^\lambda) \omega\varphi \,\mathrm{d}x \right| + C \limsup_{k\to\infty} \int_{E_j} |b^k| |\nabla(e_B^k - e_B^{k,\lambda})| \omega \,\mathrm{d}x \\
&\qquad + \left| \int_{E_j} b \cdot \nabla(e_B - e_B^\lambda) \omega\varphi \,\mathrm{d}x \right| \\
&\le \lim_{k\to\infty} \left| \int_{E_j} \frac{(b^k \cdot \nabla e_B^{k,\lambda} - b \cdot \nabla e_B^\lambda)\varphi\omega}{1+\varepsilon\omega} \,\mathrm{d}x \right| + C \limsup_{k\to\infty} \left| \int_{E_j} \frac{\varepsilon\omega^2 (|b^k||\nabla e_B^{k,\lambda}| + |b||\nabla e_B^\lambda|)}{1+\varepsilon\omega} \,\mathrm{d}x \right| \\
&\qquad + C \limsup_{k\to\infty} \int_{E_j} |b^k| |\nabla(e_B^k - e_B^{k,\lambda})| \omega \,\mathrm{d}x + \left| \int_{E_j} b \cdot \nabla(e_B - e_B^\lambda) \omega\varphi \,\mathrm{d}x \right| \\
&\le C \limsup_{k\to\infty} \int_{E_j} \frac{\varepsilon\omega^2 |b^k||\nabla e_B^{k,\lambda}|}{1+\varepsilon\omega} + C \limsup_{k\to\infty} \int_{E_j} |b^k| |\nabla(e_B^k - e_B^{k,\lambda})| \omega \,\mathrm{d}x \\
&\qquad + \left| \int_{E_j} b \cdot \nabla(e_B - e_B^\lambda) \omega\varphi \,\mathrm{d}x \right| + C \int_{E_j} \frac{\varepsilon\omega^2 |b||\nabla e_B^\lambda|}{1+\varepsilon\omega} \,\mathrm{d}x =: (\mathrm{I}) + (\mathrm{II}) + (\mathrm{III}) + (\mathrm{IV}),
\end{aligned} \tag{6-23}$$

where the last identity follows from (6-22) since $\varphi\omega/(1+\varepsilon\omega)$ is a bounded function whenever $\varepsilon > 0$. In the next step, we show that all terms on the right-hand side vanish when we let $\varepsilon \to 0_+$ and $\lambda \to \infty$. To do so, we first observe that thanks to Theorem 2.7 and the weak lower semicontinuity

$$\nabla e_B^{k,\lambda} \rightharpoonup \nabla e_B^{\lambda} \quad \text{weakly in } L^p_\omega(\Omega; \mathbb{R}^n), \tag{6-24}$$

$$e_B^{k,\lambda} \rightharpoonup e_B^{\lambda} \quad \text{weakly in } W^{1,q}(\Omega), \tag{6-25}$$

$$\int_\Omega |\nabla e_B^\lambda|^q + |\nabla e_B^\lambda|^p \omega \, \mathrm{d}x \le C \liminf_{k\to\infty} \int_B |\nabla e_B^k|^q + |\nabla e_B^k|^p \omega \, \mathrm{d}x \le C. \tag{6-26}$$

Therefore, applying the Hölder inequality, we have the estimate

$$\int_{E_j} |b||\nabla e_B^\lambda| \omega \, \mathrm{d}x \le C.$$

Consequently, using the Lebesgue dominated convergence theorem (and also the fact that ω is finite almost everywhere), we deduce

$$\lim_{\varepsilon\to 0_+} (\mathrm{IV}) = C \lim_{\varepsilon\to 0_+} \int_{E_j} |b||\nabla e_B^\lambda| \frac{\varepsilon\omega^2}{1+\varepsilon\omega} \, \mathrm{d}x = 0. \tag{6-27}$$

For the second term involving ε the key property is the uniform equi-integrability of b^k stated in (6-1). Indeed, applying the Hölder inequality and (6-26) we have

$$\begin{aligned}
\lim_{\varepsilon\to 0_+} (\mathrm{I}) &= C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \int_{E_j} |b^k||\nabla e_B^{k,\lambda}| \frac{\varepsilon\omega^2|\varphi|}{1+\varepsilon\omega} \, \mathrm{d}x \\
&\le C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \left(\int_{E_j} |b^k|^{p'} \omega \frac{\varepsilon\omega}{1+\varepsilon\omega} \, \mathrm{d}x \right)^{1/p'} \left(\int_{E_j} |\nabla e_B^{k,\lambda}|^p \omega \, \mathrm{d}x \right)^{1/p} \\
&\le C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \left(\int_{E_j \cap \{\omega > \lambda\}} |b^k|^{p'} \omega \frac{\varepsilon\omega}{1+\varepsilon\omega} \, \mathrm{d}x \right)^{1/p'} \\
&\qquad + C \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \left(\int_{E_j \cap \{\omega \le \lambda\}} |b^k|^{p'} \omega \frac{\varepsilon\omega}{1+\varepsilon\omega} \, \mathrm{d}x \right)^{1/p'} \\
&\le C \limsup_{k\to\infty} \left(\int_{E_j \cap \{\omega > \lambda\}} |b^k|^{p'} \omega \, \mathrm{d}x \right)^{1/p'}.
\end{aligned}$$

Since $|\{\omega > \lambda\}| \le C/\lambda$, we can use (6-1) and let $\lambda \to \infty$ in the last inequality to deduce

$$\limsup_{\lambda\to\infty} \limsup_{\varepsilon\to 0_+} \limsup_{k\to\infty} \int_{E_j} |b^k||\nabla e_B^{k,\lambda}| \frac{\varepsilon\omega^2|\varphi|}{1+\varepsilon\omega} \, \mathrm{d}x = 0. \tag{6-28}$$

Next, we let $\lambda \to \infty$ in all remaining terms on the right-hand side of (6-23). Using (2-22) and the Hölder inequality,

$$\begin{aligned}\limsup_{\lambda\to\infty}(\mathrm{II}) &= C \limsup_{\lambda\to\infty}\limsup_{k\to\infty}\int_{E_j}|b^k||\nabla(e_B^k - e_B^{k,\lambda})|\omega\,\mathrm{d}x\\ &= C \limsup_{\lambda\to\infty}\limsup_{k\to\infty}\int_{E_j\cap\{M(\nabla e_B^k)>\lambda\}}|b^k||\nabla(e_B^k - e_B^{k,\lambda})|\omega\,\mathrm{d}x\\ &\le C \limsup_{\lambda\to\infty}\limsup_{k\to\infty}\biggl(\int_{E_j\cap\{M(\nabla e_B^k)>\lambda\}}|b^k|^{p'}\omega\,\mathrm{d}x\biggr)^{1/p'} = 0,\end{aligned}\tag{6-29}$$

where the last inequality follows from the fact that $|\{M(\nabla e_B^k) > \lambda\}| \le C/\lambda$ and (6-1). Finally, we are left to show

$$\lim_{\lambda\to\infty}(\mathrm{III}) = \lim_{\lambda\to\infty}\left|\int_{E_j} b\cdot\nabla(e_B - e_B^\lambda)\omega\varphi\,\mathrm{d}x\right| = 0.\tag{6-30}$$

However, to get (6-30), it is enough to show that

$$\nabla e_B^\lambda \rightharpoonup \nabla e_B \quad \text{weakly in } L^p_\omega(\Omega;\mathbb{R}^n).$$

Due to (6-26), we however have that there is some $\overline{e_B} \in W^{1,q}(\Omega)$ such that

$$\begin{aligned} e_B^\lambda &\rightharpoonup \overline{e_B} && \text{weakly in } W^{1,q}(\Omega),\\ \nabla e_B^\lambda &\rightharpoonup \nabla\overline{e_B} && \text{weakly in } L^p_\omega(\Omega;\mathbb{R}^n).\end{aligned}$$

Hence, due to the uniqueness of the weak limit, it is enough to check that $\bar e_B = e_B$. To do so, we use the compact embedding $W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^1(\Omega)$ to get

$$\begin{aligned}\|\bar e_B - e_B\|_1 &= \lim_{\lambda\to\infty}\int_\Omega |e_B^\lambda - e_B|\,\mathrm{d}x = \lim_{\lambda\to\infty}\lim_{k\to\infty}\int_\Omega |e_B^{k,\lambda} - e_B^k|\,\mathrm{d}x\\ &= \lim_{\lambda\to\infty}\lim_{k\to\infty}\int_{\Omega\cap\{M(\nabla e_B^k)>\lambda\}}|e_B^{k,\lambda} - e_B^k|\,\mathrm{d}x\\ &\le \lim_{\lambda\to\infty}\lim_{k\to\infty}\|e_B^{k,\lambda} - e_B^k\|_q|\Omega\cap\{M(\nabla e_B^k)>\lambda\}|^{1/q'} \le C\lim_{\lambda\to\infty}\lambda^{-1/q'} = 0,\end{aligned}$$

and consequently (6-30) holds. Hence, using (6-27)–(6-30) in (6-23), we deduce (6-11) and the proof is complete.

7. Proof of Theorem 2.7

This part of the paper is devoted to the proof of Theorem 2.7. All statements except (2-22) are already contained in [Diening et al. 2013, Theorem 13] (see also [Diening 2013] for a survey on the Lipschitz truncation). The first inequality of (2-22) follows directly from the second one, so it is enough to prove the second estimate.

It follows from (2-20) and (2-21) that

$$\begin{aligned}\|\nabla(g - g^\lambda)\|_{L^p_\omega} &\le \|\nabla(g - g^\lambda)\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega}\\ &\le \|\nabla g\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega} + c\|\lambda\chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega}.\end{aligned}$$

We need to control the second term in the last estimate. Let us consider the open set $\{M(\nabla g) > \lambda\}$. For every $x \in \{M(\nabla g) > \lambda\}$, there exists a ball $B_{r(x)}(x)$ with

$$\lambda < \fint_{B_r(x)} |\nabla g| \, dx \le 2\lambda. \tag{7-1}$$

These balls cover $\{M(\nabla g) > \lambda\}$. Next, using the Besicovich covering theorem, we can extract from this cover a countable subset B_i that is locally finite, i.e.,

$$\#\{j \in \mathbb{N};\, B_i \cap B_j \neq \varnothing\} \le C(n). \tag{7-2}$$

Using (7-1) and (7-2), we have the estimate

$$\begin{aligned}
\|\lambda \chi_{\{M(\nabla g)>\lambda\}}\|^p_{L^p_\omega} &= \lambda^p \omega(\{M(\nabla g) > \lambda\}) \le \sum_i \lambda^p \omega(B_i) \\
&\le \sum_i \biggl(\fint_{B_i} |\nabla g| \, dx\biggr)^p \omega(B_i) \le \sum_i \fint_{B_i} |\nabla g|^p \omega \, dx \biggl(\fint_{B_i} \omega^{-(p'-1)} \, dx\biggr)^{1/(p'-1)} \omega(B_i) \\
&\le \mathcal{A}_p(\omega) \sum_i \int_{B_i} |\nabla g|^p \omega \, dx \le C(n) \mathcal{A}_p(\omega) \int_{\{M(\nabla g)>\lambda\}} |\nabla g|^p \omega \, dx.
\end{aligned}$$

This directly leads to the inequality

$$\|\lambda \chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega} \le C(n) \mathcal{A}_p(\omega)^{1/p} \|\nabla g \chi_{\{M(\nabla g)>\lambda\}}\|_{L^p_\omega},$$

which proves the desired estimate (2-22).

8. Proof of Theorem 2.8

We present only a sketch of the proof here since all steps were already justified in the proof of Theorem 2.3. Hence, to obtain the a priori estimate (2-23), we observe that

$$\int_\Omega \tilde{A}(x)(\nabla u - \nabla u_0) \cdot \nabla \varphi \, dx = \int_\Omega \bigl(f - \tilde{A}(x)\nabla u_0 + \tilde{A}(x)\nabla u - A(x, \nabla u)\bigr) \cdot \nabla \varphi \, dx,$$

which by the use of Theorem 2.5 (note here that $u - u_0$ has zero trace) and (2-1) leads to

$$\begin{aligned}
\int_\Omega |\nabla u - \nabla u_0|^p \omega \, dx &\le C \int_\Omega \bigl(|f|^p + |\nabla u_0|^p + |\tilde{A}(x)\nabla u - A(x, \nabla u)|^p\bigr) \omega \, dx \\
&\le C(\varepsilon) \int_\Omega (|f|^p + |\nabla u_0|^p + 1) \omega \, dx + \varepsilon \int_\Omega |\nabla u|^p \omega \, dx.
\end{aligned}$$

Consequently, choosing ε small enough and using the triangle inequality, we find (2-9). The existence is then identically the same; we just also need to approximate u_0 by a sequence of smooth functions such that

$$u_0^k \to u_0 \quad \text{strongly in } W^{1,\tilde{q}}(\Omega; \mathbb{R}^N).$$

Finally, for the uniqueness proof, we use a similar procedure and see that if u_1 and u_2 are two solutions then

$$\int_\Omega \tilde{A}(x)(\nabla u_1 - \nabla u_2) \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega \big(\tilde{A}(x)(\nabla u_1 - \nabla u_2) + A(x, \nabla u_1) - A(x, \nabla u_2)\big) \cdot \nabla \varphi \, \mathrm{d}x,$$

and since $u_1 - u_2 \in W_0^{1,\tilde{q}}(\Omega; \mathbb{R}^N)$, we may now follow step by step the proof of Theorem 2.3.

9. Proofs of corollaries

Proof of Corollary 2.2. The proof of Corollary 2.2 is rather straightforward. Indeed, for a given measure $f \in \mathcal{M}(\Omega; \mathbb{R}^N)$, we can use the classical theory and find $v \in W_0^{1,n'-\varepsilon}(\Omega; \mathbb{R}^N)$ for all $\varepsilon > 0$ solving

$$\int_\Omega \nabla v \cdot \nabla \varphi \, \mathrm{d}x = \langle f, \varphi \rangle \quad \text{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^N).$$

Then it follows that u is a solution to (2-7) if and only if it solves

$$\int_\Omega A(x, \nabla u) \cdot \nabla \varphi \, \mathrm{d}x = \int_\Omega \nabla v \cdot \nabla \varphi \, \mathrm{d}x \quad \text{for all } \varphi \in \mathscr{C}_0^{0,1}(\Omega; \mathbb{R}^N). \tag{9-1}$$

Thus, we can now apply Theorem 2.1 with $f := \nabla v$ and all statements in Corollary 2.2 directly follow. □

Proof of Corollary 2.4. We show that Corollary 2.4 can be directly proved by using Theorem 2.3. Indeed, by setting

$$\omega := (1 + Mf)^{q-2} = (M(1 + |f|))^{q-2},$$

where we extended f by zero outside Ω, we can use Lemma 3.2 to deduce that $\omega \in \mathcal{A}_2$ provided that $|q - 2| < 1$. Since $q \in (1, 2)$, we always have $|q - 2| < 1$ and therefore $\omega \in \mathcal{A}_2$. Consequently, we can construct a solution u according to Theorem 2.3. Next, using (2-9) and the continuity of the maximal function, we can deduce

$$\begin{aligned}
\int_\Omega \frac{|\nabla u|^2}{(1 + Mf)^{2-q}} \, \mathrm{d}x &= \int_\Omega |\nabla u|^2 \omega \, \mathrm{d}x \le C(A_2(\omega), \Omega)\left(1 + \int_\Omega |f|^2 \omega \, \mathrm{d}x\right) \\
&= C(A_2(\omega), \Omega)\left(1 + \int_\Omega \frac{|f|^2}{(1 + Mf)^{2-q}} \, \mathrm{d}x\right) \\
&\le C(A_2(\omega), \Omega)\left(1 + \int_\Omega (Mf)^q \, \mathrm{d}x\right) \le C(f, \Omega, q)\left(1 + \int_\Omega |f|^q \, \mathrm{d}x\right),
\end{aligned}$$

which is nothing else than (2-10). □

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Received 21 Oct 2015. Revised 11 Feb 2016. Accepted 30 Mar 2016.

MIROSLAV BULÍČEK: mbul8060@karlin.mff.cuni.cz
Mathematical Institute, Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, 186 75 Prague, Czech Republic

LARS DIENING: ldiening@uni-osnabrueck.de
Institut für Mathematik, Universität Osnabrück, Albrechtstraße 28a, D-49076 Osnabrück, Germany

SEBASTIAN SCHWARZACHER: schwarz@karlin.mff.cuni.cz
Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University in Prague, Sokolovská 83, 186 75 Prague, Czech Republic

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1153

ON POLYNOMIAL CONFIGURATIONS IN FRACTAL SETS

KEVIN HENRIOT, IZABELLA ŁABA AND MALABIKA PRAMANIK

We show that subsets of $\mathbb{R}^n$ of large enough Hausdorff and Fourier dimension contain polynomial patterns of the form

$$(x,\ x+A_1 y, \dots,\ x+A_{k-1}y,\ x+A_k y+Q(y)e_n), \quad x \in \mathbb{R}^n,\ y \in \mathbb{R}^m,$$

where A_i are real $n \times m$ matrices, Q is a real polynomial in m variables and $e_n = (0, \dots, 0, 1)$.

1. Introduction

We investigate the presence of point configurations in subsets of $\mathbb{R}^n$ which are large in a certain sense. When E is a subset of $\mathbb{R}^n$ of positive Lebesgue measure, a consequence of the Lebesgue density theorem is that E contains a similar copy of any finite set. A more difficult result of Bourgain [1986] states that sets of positive upper density in $\mathbb{R}^n$ contain, up to isometry, all large enough dilates of the set of vertices of any fixed nondegenerate $(n-1)$-dimensional simplex. In a different setting, Roth's theorem [1953] in additive combinatorics states that subsets of $\mathbb{Z}$ of positive upper density contain nontrivial 3-term arithmetic progressions.

When a subset $E \subset \mathbb{R}$ is only assumed to have a positive Hausdorff dimension, a direct analogue of Roth's theorem is impossible. Indeed Keleti [1999] has constructed a set of full dimension in [0, 1] not containing the vertices of any nondegenerate parallelogram, and in particular not containing any nontrivial 3-term arithmetic progression. Maga [2011] has since extended this construction to dimensions $n \geqslant 2$. The work of Łaba and Pramanik [2009] and its multidimensional extension by Chan et al. [2016] circumvent these obstructions under additional assumptions on the set E, which we now describe.

When E is a compact subset of $\mathbb{R}^n$, Frostman's lemma [Wolff 2003, Chapter 8] essentially states that its Hausdorff dimension is equal to

$$\dim_{\mathscr{H}} E = \sup\Big\{\alpha \in [0,n) : \sup_{x\in\mathbb{R}^n,\, r>0} \mu(B(x,r))r^{-\alpha} < \infty \ \text{ for some } \ \mu \in \mathcal{M}(E)\Big\},$$

where $\mathcal{M}(E)$ is the space of probability measures supported on E. On the other hand, the Fourier dimension of E is

$$\dim_{\mathscr{F}} E = \sup\Big\{\beta \in [0,n) : \sup_{\xi\in\mathbb{R}^n} |\hat{\mu}(\xi)|(1+|\xi|)^{-\beta/2} < \infty \ \text{ for some } \ \mu \in \mathcal{M}(E)\Big\}.$$

This work was supported by NSERC Discovery grants 22R80520 and 22R82900.
MSC2010: primary 11B30; secondary 28A80.
Keywords: configurations in fractals, additive combinatorics.

It is well-known that we have $\dim_{\mathcal{F}}(E) \leqslant \dim_{\mathcal{H}}(E)$ for every compact set E, with strict inequality in many instances, and we call E a Salem set when equality holds. There are various known constructions of Salem sets [Salem 1951; Kaufman 1981; Bluhm 1996; 1998; Kahane 1985; Łaba and Pramanik 2009; Hambrooke 2016], several of which [Körner 2011; Chen 2016] also produce sets with prescribed Hausdorff and Fourier dimensions $0 < \beta \leqslant \alpha < n$.

In a very abstract setting, one may ask whether it is possible to find translation-invariant patterns of the form

$$\Phi(x, y) = (x, x + \varphi_1(y), \dots, x + \varphi_k(y)) \tag{1-1}$$

in the product set $E \times \cdots \times E$, where the $\varphi_j : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$ are certain shift functions. When $n + m \geqslant (k + 1)n$, the map Φ is often a submersion of an open subset of $\mathbb{R}^{n+m}$ onto $\mathbb{R}^{(k+1)n}$, and then one can find a pattern of the desired kind in E via the implicit function theorem. A natural restriction is therefore to assume that $m < kn$ in this multidimensional setting. Chan et al. [2016] studied the case where the maps $\varphi_j(y) = A_j y$ are linear for matrices $A_j \in \mathbb{R}^{n\times m}$, generalizing the study of Łaba and Pramanik for 3-term arithmetic progressions, under the following technical assumption:

Definition 1.1. Let $n, k, m \geqslant 1$ and suppose that $m = (k - r)n + n'$ with $1 \leqslant r < k$ and $0 \leqslant n' < n$. We say that the system of matrices $A_1, \dots, A_k \in \mathbb{R}^{n\times m}$ is nondegenerate when

$$\operatorname{rk} \begin{bmatrix} A_{j_1}^{\mathsf{T}} & \cdots & A_{j_{k-r+1}}^{\mathsf{T}} \\ I_{n\times n} & \cdots & I_{n\times n} \end{bmatrix} = (k - r + 1)n$$

for every set of indices $\{j_1, \dots, j_{k-r+1}\} \subset \{0, \dots, k\}$, with the convention that $A_0 = 0_{n\times n}$.

Requirements similar to the above arise when analyzing linear patterns by ordinary Fourier analysis in additive combinatorics [Roth 1954], and there is a close link with the modern definition of linear systems of complexity one [Gowers and Wolf 2010]. The main result of [Chan et al. 2016] gives a fractal analogue of the multidimensional Szemerédi theorem [Furstenberg and Katznelson 1978] for nondegenerate linear systems when the Frostman measure has both dimensional and Fourier decay. We only state it in the case where n divides m for simplicity.

Theorem 1.2 (Chan, Łaba and Pramanik). *Let $n, k, m \geqslant 1$, $D \geqslant 1$ and $\alpha, \beta \in (0, n)$. Suppose that E is a compact subset of $\mathbb{R}^n$ and μ is a probability measure supported on E such that*[1]

$$\mu(B(x, r)) \leqslant D r^{\alpha} \quad \textit{and} \quad |\hat{\mu}(\xi)| \leqslant D(n - \alpha)^{-D}(1 + |\xi|)^{-\beta/2}$$

for all $x \in \mathbb{R}^n$, $r > 0$ and $\xi \in \mathbb{R}^n$. Suppose that $(A_1, \dots, A_k)$ is a nondegenerate system of $n \times m$ matrices in the sense of Definition 1.1. Assume finally that $m = (k - r)n$ with $1 \leqslant r < k$ and, for some $\varepsilon \in (0, 1)$,

$$\left\lceil \frac{k}{2} \right\rceil n \leqslant m < kn, \quad \frac{2(kn-m)}{k+1} + \varepsilon \leqslant \beta < n, \quad n - c_{n,k,m,\varepsilon,D,(A_i)} \leqslant \alpha < n$$

[1] In fact, this theorem was proved in [Chan et al. 2016] under the more restrictive condition $|\hat{\mu}(\xi)| \leqslant D(1 + |\xi|)^{-\beta/2}$ for a fixed constant D. However, by examining the proof there, one can see that the constant $D = D_\alpha$ may be allowed to grow polynomially in $n - \alpha$, as was the case in the original argument of [Łaba and Pramanik 2009].

for a sufficiently small constant $c_{n,k,m,\varepsilon,D,(A_i)} > 0$. *Then, for every collection of strict subspaces* $V_1, \dots, V_q$ *of* $\mathbb{R}^{n+m}$, *there exists* $(x, y) \in \mathbb{R}^{n+m} \setminus V_1 \cup \dots \cup V_q$ *such that*

$$(x, x + A_1 y, \dots, x + A_k y) \in E^{k+1}.$$

Note that the Hausdorff dimension α is required to be large enough with respect to the constants involved in the dimensional and Fourier decay bounds for the Frostman measure. A construction due to Shmerkin [2015] shows that the dependence of α on the constants cannot be removed.

In practice, Salem set constructions provide a family of fractal sets indexed by α, and it is often possible to verify the conditions of Theorem 1.2 for α close to n; this was done in a number of cases in [Łaba and Pramanik 2009]. The requirement of Fourier decay of the measure μ serves as an analogue of the notion of pseudorandomness in additive combinatorics [Tao and Vu 2006], under which we expect a set to contain the same density of patterns as a random set of the same size.

In this work we consider a class of polynomial patterns, which generalizes that of Theorem 1.2. We aim to obtain results similar in spirit to the Furstenberg–Sárközy theorem [Sárközy 1978; Furstenberg 1977] in additive combinatorics, which finds patterns of the form $(x, x + y^2)$ in dense subsets of $\mathbb{Z}$. A deep generalization of this result is the multidimensional polynomial Szemerédi theorem in ergodic theory of [Bergelson and Leibman 1996; Bergelson and McCutcheon 2000] (see also [Bergelson et al. 2008, Section 6.3]), which handles patterns of the form (1-1) where each shift function φ_j is an integer polynomial vector with zero constant term. By contrast, the class of patterns we study includes only one polynomial term, which should satisfy certain nondegeneracy conditions. We are also forced to work with a dimension $n \geqslant 2$, and all these limitations are due to the inherent difficulty in analyzing polynomial patterns through Fourier analysis. On the other hand, we are able to relax the Fourier decay condition on the fractal measure needed in Theorem 1.2.

Theorem 1.3. *Let* $n, m, k \geqslant 2$, $D \geqslant 1$ *and* $\alpha, \beta \in (0, n)$. *Suppose that* E *is a compact subset of* $\mathbb{R}^n$ *and* μ *is a probability measure supported on* E *such that*

$$\mu(B(x, r)) \leqslant D r^{\alpha} \quad \text{and} \quad |\hat{\mu}(\xi)| \leqslant D(n - \alpha)^{-D}(1 + |\xi|)^{-\beta/2}$$

for all $x \in \mathbb{R}^n$, $r > 0$ *and* $\xi \in \mathbb{R}^n$. *Suppose that* $(A_1, \dots, A_k)$ *is a nondegenerate system of real* $n \times m$ *matrices in the sense of Definition 1.1. Let* Q *be a real polynomial in* m *variables such that* $Q(0) = 0$ *and the Hessian of* Q *does not vanish at zero. Assume furthermore that, for a constant* $\beta_0 \in (0, n)$,

$$(k - 1)n < m < kn, \quad \beta_0 \leqslant \beta < n, \quad n - c_{\beta_0,n,k,m,D,(A_i),Q} < \alpha < n$$

for a sufficiently small constant $c_{\beta_0,n,k,m,D,(A_i),Q} > 0$. *Then, for every collection* $V_1, \dots, V_q$ *of strict subspaces of* $\mathbb{R}^{m+n}$, *there exists* $(x, y) \in \mathbb{R}^{n+m} \setminus (V_1 \cup \dots \cup V_q)$ *such that*

$$(x,\ x + A_1 y, \dots,\ x + A_{k-1} y,\ x + A_k y + Q(y) e_n) \in E^{k+1}, \tag{1-2}$$

where $e_n = (0, \dots, 0, 1)$.

Our argument broadly follows the transference strategy devised by Łaba and Pramanik [2009] and its extension by Chan and these two authors [Chan et al. 2016]. However, the case of polynomial configurations requires a more delicate treatment of the singular integrals arising in the analysis. The weaker condition on β is obtained by exploiting restriction estimates for fractal measures due to Mitsis [2002] and Mockenhaupt [2000]. A more detailed outline of our strategy can be found in Section 3. By the method of this paper, one can also obtain an analogue of Theorem 1.2 with the same relaxed condition on the exponent β, and we state this version precisely in Section 9.

For concreteness's sake, we highlight the lowest-dimensional situation handled by Theorem 1.3. When $k = n = 2$ and $m = 3$, this theorem allows us to detect patterns of the form

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + A_1 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}, \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + A_2 \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} + \begin{bmatrix} 0 \\ Q(y_1, y_2, y_3) \end{bmatrix}$$

for matrices $A_1, A_2 \in \mathbb{R}^{2\times 3}$ of full rank such that $A_1 - A_2$ has full rank and for a nondegenerate quadratic form Q in three variables. We may additionally impose that $y_1, y_2, y_3 \in \mathbb{R} \setminus \{0\}$ by setting $V_i = \{(x, y) \in \mathbb{R}^5 : y_i = 0\}$ in Theorem 1.3. For example, when $A_1 = \left[\begin{smallmatrix} 1&0&0 \\ 0&1&0 \end{smallmatrix}\right]$, $A_2 = \left[\begin{smallmatrix} 0&0&1 \\ 1&0&0 \end{smallmatrix}\right]$ and $Q(y) = |y|^2$, we can detect the configuration

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 + y_3 \\ x_2 + y_1 + y_1^2 + y_2^2 + y_3^2 \end{bmatrix}$$

with $y_1, y_2, y_3 \in \mathbb{R} \setminus \{0\}$. However, we cannot detect the configuration

$$(x, x + y, x + y^2), \quad x \in \mathbb{R},\ y \in \mathbb{R} \setminus \{0\},$$

for then we have $n = m = 1$ and $k = 2$, and the condition $m > (k-1)n$ is not satisfied.

Note also that, in the statement of Theorem 1.3, one may add a linear term in variables $y_1, \ldots, y_m$ to the polynomial Q without affecting the assumptions on it. This allows for some flexibility in satisfying the matrix nondegeneracy conditions of Definition 1.1, since one may alter the last line of A_k at will. For example, the degenerate system of matrices $\left[\begin{smallmatrix} 1&0&0 \\ 0&1&0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0&0&1 \\ 0&0&0 \end{smallmatrix}\right]$ and the polynomial $Q(y) = |y|^2$ give rise to the configuration

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \end{bmatrix}, \quad \begin{bmatrix} x_1 + y_3 \\ x_2 + |y|^2 \end{bmatrix}.$$

Rewriting $|y|^2 = y_1 + y_2 + y_3 + Q_1(y)$, we see that Q_1 still has nondegenerate Hessian at zero and the configuration is now associated to the system of matrices $\left[\begin{smallmatrix} 1&0&0 \\ 0&1&0 \end{smallmatrix}\right]$, $\left[\begin{smallmatrix} 0&0&1 \\ 1&1&1 \end{smallmatrix}\right]$, which is easily seen to be nondegenerate. One possible explanation for this curious phenomenon is that, by comparison with the setting of Theorem 1.2, we have an extra variable at our disposition, since $m > (k-1)n \geqslant \left\lceil \frac{1}{2}k \right\rceil n$.

Finally, we note that there is a large body of literature on configurations in fractal sets where Fourier decay assumptions are not required. Here, the focus is often on finding a large variety (in a specified quantitative sense) of certain types of configurations. A well-known conjecture of Falconer [Wolff 2003, Chapter 9] states that when a compact subset E of $\mathbb{R}^n$ has Hausdorff dimension at least $\frac{1}{2}n$, its set of distances $\Delta(E) = \{|x - y| : x, y \in E\}$ must have positive Lebesgue measure. This can be phrased in terms of E containing configurations $\{x, y\}$ with $|x - y| = d$ for all $d \in \Delta(E)$, where $\Delta(E)$ is "large". Wolff [1999] and Erdoğan [2005; 2006] proved that the distance set $\Delta(E)$ has positive Lebesgue measure for $\dim_{\mathscr{H}} E > \frac{1}{2}n + \frac{1}{3}$, and Mattila and Sjölin [1999] showed that it contains an open interval for $\dim_{\mathscr{H}} E > \frac{1}{2}(n+1)$. More recently, Orponen [2015] proved using very different methods that $\Delta(E)$ has upper box dimension 1 if E is s-Ahlfors–David regular with $s \geqslant 1$. There is a rich literature generalizing these results to other classes of configurations, such as triangles [Greenleaf and Iosevich 2012], simplices [Grafakos et al. 2015; Greenleaf et al. 2014a], and sequences of vectors with prescribed consecutive lengths [Bennett et al. 2015; Greenleaf et al. 2014b].

In a sense, the configurations studied in these references enjoy a greater degree of directional freedom, which ensures that they are not avoided by sets of full Hausdorff dimension. By contrast, a Fourier decay assumption is necessary to locate 3-term progressions in a fractal set of full Hausdorff dimension (as mentioned earlier) and, in light of recent work of Máthé [2012], it is likely that a similar assumption is needed to find polynomial patterns of the form (1-2). It is, however, possible that our nondegeneracy assumptions are not optimal, or that special cases of our results could be proved without Fourier decay assumptions.[2] Loosely speaking, we would expect that configurations with more degrees of freedom are less likely to require Fourier conditions, but the specifics are far from understood and we do not feel that we have sufficient data to attempt to make a conjecture in this direction.

2. Notation

We define the following standard spaces of complex-valued functions and measures:

$$\begin{aligned}
\mathscr{C}(\mathbb{R}^d) &= \{\text{continuous functions on } \mathbb{R}^d\},\\
\mathscr{S}(\mathbb{R}^d) &= \{\text{Schwartz functions on } \mathbb{R}^d\},\\
\mathscr{C}_c^\infty(\mathbb{R}^d) &= \{\text{smooth compactly supported functions on } \mathbb{R}^d\},\\
\mathscr{C}_{c,+}^\infty(\mathbb{R}^d) &= \{\text{nonnegative smooth compactly supported functions on } \mathbb{R}^d\},\\
\mathscr{M}^+(\mathbb{R}^d) &= \{\text{finite nonnegative Borelian measures on } \mathbb{R}^d\}.
\end{aligned}$$

Similar notation is employed for functions on $\mathbb{T}^d$. We write $e(x) = e^{2i\pi x}$ for $x \in \mathbb{R}$. We let $\mathscr{L}$ denote either the Lebesgue measure on $\mathbb{R}^d$ or the normalized Haar measure on $\mathbb{T}^d$. We let $\mathrm{d}\sigma$ denote generically the Euclidean surface measure on a submanifold of $\mathbb{R}^d$. When f is a function on an abelian group G and t is an element of G, we denote the t-shift of f by $T^t f(x) = f(x+t)$. When A is a matrix we denote its transpose by $A^\intercal$. We also write $[n] = \{1, \dots, n\}$ for an integer n and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

3. Broad scheme

In this section we introduce the basic objects that we will work with in this paper. We also state the intermediate propositions corresponding to the main steps of our argument, and we derive Theorem 1.3 from them at the outset.

[2] After this article was first submitted for publication, a result of this type was indeed proved by Iosevich and Liu [2016].

We fix a compact set $E \subset \mathbb{R}^n$ and a probability measure μ supported on E. For technical reasons, we suppose that $E \subset \left[-\frac{1}{16}, \frac{1}{16}\right]^n$. We fix two exponents $0 < \beta \leqslant \alpha < n$, as well as two constants D, $D_\alpha \geqslant 1$, where the subscript in the second constant indicates that it is allowed to vary with α. We assume that the measure μ verifies the following dimensional and Fourier decay conditions:

$$\mu(B(x,r)) \leqslant Dr^\alpha \qquad (x \in \mathbb{R}^n,\ r > 0), \tag{3-1}$$

$$|\hat{\mu}(\xi)| \leqslant D_\alpha (1 + |\xi|)^{-\beta/2} \quad (\xi \in \mathbb{R}^n). \tag{3-2}$$

We suppose that the second constant involved blows up (if at all) at most polynomially as α tends to n:

$$D_\alpha \lesssim (n-\alpha)^{-O(1)}. \tag{3-3}$$

We also let $k \geqslant 3$ and we consider smooth functions $\varphi_1, \dots, \varphi_k : \Omega \subset \mathbb{R}^m \to \mathbb{R}^n$, where Ω is an open neighborhood of zero. We are interested in locating the pattern

$$\Phi(x,y) = (x, x+\varphi_1(y), \dots, x+\varphi_k(y)) \tag{3-4}$$

in E^{k+1}. While this abstract notation is sometimes useful, in practice we work with the maps

$$(\varphi_1(y), \dots, \varphi_k(y)) = (A_1 y, \dots, A_{k-1} y, A_k y + Q(y) e_n), \tag{3-5}$$

where $(A_1, \dots, A_k)$ is a nondegenerate system of $n \times m$ matrices in the sense of Definition 1.1 and $Q \in \mathbb{R}[y_1, \dots, y_m]$ is such that $Q(0) = 0$ and the Hessian of Q does not vanish at zero. We also fix a smooth cutoff $\psi \in \mathscr{C}^\infty_{c,+}(\mathbb{R}^m)$ supported on Ω such that $\psi \geqslant 1$ on a small box $[-c, c]^m$ and the Hessian of Q is bounded away from zero on the support of ψ. This cutoff is used in Definition 3.2 below. We take the opportunity here to state an equivalent form of Definition 1.1 when $m \geqslant (k-1)n$.

Definition 3.1. If $m \geqslant (k-1)n$, we say that the system of matrices $(A_i)_{1 \leqslant i \leqslant k}$ with $A_i \in \mathbb{R}^{n \times m}$ is nondegenerate when, for every $1 \leqslant j \leqslant k$, and writing $[k] = \{i_1, \dots, i_{k-1}, j\}$, the matrices

$$[A_1^\mathsf{T} \ \dots \ \hat{A}_j^\mathsf{T} \ \dots \ A_k^\mathsf{T}], \quad [(A_{i_1}^\mathsf{T} - A_j^\mathsf{T}) \ \dots \ (A_{i_{k-1}}^\mathsf{T} - A_j^\mathsf{T})]$$

(where the hat indicates omission) have rank $(k-1)n$.

We also state a few notational conventions applied throughout the article. When $(A_1, \dots, A_k)$ is a system of $n \times m$ matrices, we define the $kn \times m$ matrix $\boldsymbol{A}$ by $\boldsymbol{A}^\mathsf{T} = [A_1^\mathsf{T} \cdots A_k^\mathsf{T}]$. Unless mentioned otherwise, we allow every implicit or explicit constant in the article to depend on the integers n, k, m, the constant D, the matrices A_i and the polynomial Q, and the cutoff function ψ. This convention is already in effect in the propositions stated later in this section.

We start by defining a multilinear form which plays a central role in our argument.

Definition 3.2 (configuration form). For functions $f_0, \dots, f_k \in \mathscr{S}(\mathbb{R}^n)$, we let

$$\Lambda(f_0, \dots, f_k) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f_0(x) f_1(x+\varphi_1(y)) \cdots f_k(x+\varphi_k(y)) \, \mathrm{d}x \, \psi(y) \, \mathrm{d}y.$$

In Section 4, we show that the multilinear form has the following convenient Fourier expression:

Proposition 3.3. *For measurable functions $F_0, \dots, F_k$ on $\mathbb{R}^n$ and K on $\mathbb{R}^{nk}$, we let*

$$\Lambda^*(F_0, \dots, F_k; K) = \int_{(\mathbb{R}^n)^k} F_0(-\xi_1 - \dots - \xi_k) F_1(\xi_1) \cdots F_k(\xi_k) K(\boldsymbol{\xi}) \, \mathrm{d}\boldsymbol{\xi} \tag{3-6}$$

whenever the integral is absolutely convergent or the integrand is nonnegative. For all $f_0, \dots, f_k \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\Lambda(f_0, \dots, f_k) = \Lambda^*(\hat{f}_0, \dots, \hat{f}_k; J),$$

where J is the oscillatory integral of Definition 4.1.

We may extend the configuration operator to measures whenever we have absolute convergence of the dual form:

Definition 3.4. When $\lambda_0, \dots, \lambda_k \in \mathcal{M}^+(\mathbb{R}^n)$ are such that $\Lambda^*(|\hat{\lambda}_0|, \dots, |\hat{\lambda}_k|; |J|) < \infty$, we define

$$\Lambda(\lambda_0, \dots, \lambda_k) = \Lambda^*(\hat{\lambda}_0, \dots, \hat{\lambda}_k; J).$$

When $\lambda_j \in \mathcal{S}(\mathbb{R}^n)$, this is compatible with Definition 3.2 by Proposition 3.3.

The next step, carried out in Section 5, is to obtain bounds for the dual multilinear form evaluated at the Fourier–Stieltjes transform of the fractal measure μ. Such bounds hold only in certain ranges of α, β and under certain restrictions on n, k, m.

Proposition 3.5. *Let $\beta_0 \in (0, n)$ and suppose that, for a constant $c > 0$ small enough with respect to n, k and m,*

$$(k-1)n < m < kn, \quad \beta_0 \leqslant \beta < n, \quad n - c\beta_0 \leqslant \alpha < n. \tag{3-7}$$

Then

$$\Lambda^*(|\hat{\mu}|, \dots, |\hat{\mu}|; |J|) \lesssim_{\beta_0} (n-\alpha)^{-O(1)}. \tag{3-8}$$

Recalling Definition 3.4, we see that $\Lambda(\mu, \dots, \mu)$ is well-defined under the conditions (3-7). In practice, we will need slight variants of Proposition 3.5, which are discussed in Section 5. In the same section, we obtain singular integral bounds for bounded functions of compact support.

Proposition 3.6. *Suppose that $m > (k-1)n$. Then there exists $\varepsilon \in (0, 1)$ depending at most on n, k and m such that the following holds: for functions $f_0, \dots, f_k \in \mathcal{C}_c^\infty(\mathbb{R}^n)$ with support in $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$,*

$$|\Lambda(f_0, \dots, f_k)| \lesssim \prod_{0 \leqslant j \leqslant k} \|\hat{f}_j\|_\infty^\varepsilon \cdot \|f_j\|_\infty^{1-\varepsilon}.$$

In Section 6, we construct a measure detecting polynomial configurations, by exploiting the finiteness of the singular integral in (3-8) and the uniform decay of the fractal measure.

Proposition 3.7. *Let $\beta_0 \in (0, n)$ and suppose that (3-7) holds. Then there exists a measure $\nu \in \mathcal{M}^+(\mathbb{R}^{n+m})$ such that*

- $\|\nu\| = \Lambda(\mu, \dots, \mu)$,
- *ν is supported on the set of $(x, y) \in \mathbb{R}^n \times \Omega$ such that $(x, x + \varphi_1(y), \dots, x + \varphi_k(y)) \in E^{k+1}$,*
- *$\nu(H) = 0$ for every hyperplane $H < \mathbb{R}^{n+m}$.*

In Section 7, we show how to obtain a positive mass of polynomial configurations in sets of positive density, through the singular integral bound of Proposition 3.6 and the arithmetic regularity lemma from additive combinatorics.

Proposition 3.8. *Suppose that $m > (k-1)n$. Then, uniformly for every function $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that* $\operatorname{Supp} f \subset \left[-\frac{1}{8}, \frac{1}{8}\right]^n$, $0 \leqslant f \leqslant 1$ *and* $\int f = \tau \in (0, 1]$, *we have*

$$\Lambda(f, \dots, f) \gtrsim_\tau 1.$$

In Section 8, we show how to obtain a positive mass of configurations by a transference argument, by which the fractal measure μ is replaced by a mollified version of itself which is absolutely continuous with bounded density, allowing us to invoke Proposition 3.8.

Proposition 3.9. *Let $\beta_0 \in (0, n)$ and suppose that*

$$(k-1)n < m < kn, \quad \beta_0 \leqslant \beta < n, \quad n - c(\beta_0) \leqslant \alpha < n$$

for a sufficiently small constant $c(\beta_0) > 0$. Then

$$\Lambda(\mu, \dots, \mu) > 0.$$

At this stage we have stated all the necessary ingredients to prove the main theorem.

Proof of Theorem 1.3. We may assume that $E \subset \left[-\frac{1}{16}, \frac{1}{16}\right]^n$ after a translation and dilation, which does not affect the assumptions on μ, (A_i) and Q except for the introduction of constant factors in bounds. By Proposition 3.7, there exists a measure $\nu \in \mathcal{M}^+(\mathbb{R}^{n+m})$ with mass $\Lambda(\mu, \dots, \mu)$ supported on

$$X = \left\{(x, y) \in \mathbb{R}^n \times \Omega : (x, x + A_1 y, \dots, x + A_{k-1} y, x + A_k y + Q(y) e_n) \in E^{k+1}\right\}$$

and such that $\nu(V_i) = 0$ for every collection of hyperplanes $V_1, \dots, V_q$ of $\mathbb{R}^{n+m}$. We have therefore proven the result if we can show that $\|\nu\| = \Lambda(\mu, \dots, \mu) > 0$, for then $\nu(X \smallsetminus (V_1 \cup \dots \cup V_q)) > 0$ and the set $X \smallsetminus (V_1 \cup \dots \cup V_q)$ cannot be empty. We may apply Proposition 3.9 to obtain precisely this conclusion when α is close enough to n with respect to β_0 (and the other implicit parameters n, k, m, D, $\boldsymbol{A}$, Q). □

To conclude this outline, we comment briefly on the role that the Fourier decay hypothesis plays in our argument. Using the restriction theory of fractals, the assumption (3-2) is used together with the ball condition (3-1) in Appendix B to deduce that $\|\hat{\mu}\|_{2+\varepsilon} < \infty$ for an arbitrary $\varepsilon > 0$, provided that α is close enough to n (depending on ε). The Hausdorff dimension condition (3-1) alone does yield information on the average Fourier decay of μ, via the energy formula [Wolff 2003, Chapter 8], but this type of estimate seems to be insufficient to establish the boundedness of the singular integrals we encounter. Section 5 on singular integral bounds and Section 7 on absolutely continuous estimates only use the Fourier moment bound above. On the other hand, the estimation of degenerate configurations in Section 6 and the transference argument of Section 8 exploit in an essential way the assumption of uniform Fourier decay.

4. Counting operators and Fourier expressions

In this section we describe the various types of pattern-counting operators and singular integrals that arise in trying to detect translation-invariant patterns in the fractal set of the introduction. First, we define an oscillatory integral which arises naturally in the Fourier expression of the configuration form in Definition 3.2.

Definition 4.1 (oscillatory integral). For $\boldsymbol{\xi} \in (\mathbb{R}^n)^k$ and $\theta \in \mathbb{R}^m$ we define

$$J_\theta(\boldsymbol{\xi}) = \int_{\mathbb{R}^m} e[(\theta + A^{\mathsf{T}}\boldsymbol{\xi}) \cdot y + \xi_{kn} Q(y)]\psi(y)\,\mathrm{d}y, \qquad J = J_0.$$

We now derive the dual expression of the configuration form announced in Section 3.

Proof of Proposition 3.3. By inserting the Fourier expansions of $f_1, \dots, f_k$ and using Fubini, we have

$$\begin{aligned}\Lambda(f_0, \dots, f_k) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} f_0(x) f_1(x + \varphi_1(y)) \cdots f_k(x + \varphi_k(y))\,\mathrm{d}x\,\psi(y)\,\mathrm{d}y \\ &= \int_{(\mathbb{R}^n)^k} \hat{f}_1(\xi_1) \cdots \hat{f}(\xi_k) \int_{\mathbb{R}^n} f_0(x) e[(\xi_1 + \cdots + \xi_k) \cdot x]\,\mathrm{d}x \\ &\qquad\qquad \times \int_{\mathbb{R}^m} e[\xi_1 \cdot \varphi_1(y) + \cdots + \xi_k \cdot \varphi_k(y)]\psi(y)\,\mathrm{d}y\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k.\end{aligned}$$

Recalling Definition 4.1 and the choice (3-5), we deduce that

$$\Lambda(f_0, \dots, f_k) = \int_{(\mathbb{R}^n)^k} \hat{f}_0(-\xi_1 - \cdots - \xi_k) \hat{f}_1(\xi_1) \cdots \hat{f}_k(\xi_k) J(\boldsymbol{\xi})\,\mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k. \qquad \square$$

We single out a useful bound for the configuration operator, typically used when the λ_i are either the measure μ or a mollified version of it.

Proposition 4.2. *For measures $\lambda_0, \dots, \lambda_k \in \mathcal{M}^+(\mathbb{R}^n)$, we have*

$$|\Lambda(\lambda_0, \dots, \lambda_k)| \leqslant \prod_{j=0}^{k} \|\hat{\lambda}_j\|_\infty^\varepsilon \cdot \Lambda^*(|\hat{\lambda}_0|^{1-\varepsilon}, \dots, |\hat{\lambda}_k|^{1-\varepsilon}; |J|),$$

where the left-hand side is absolutely convergent if the right-hand side is finite.

Proof. This follows from Definition 3.4 and the successive bounds

$$\begin{aligned}|\Lambda^*(\hat{\lambda}_0, \dots, \hat{\lambda}_k; J)| &\leqslant \int_{(\mathbb{R}^n)^k} |\hat{\lambda}_0(\xi_1 + \cdots + \xi_k)|\,|\hat{\lambda}_1(\xi_1)| \cdots |\hat{\lambda}_k(\xi_k)|\,|J(\boldsymbol{\xi})|\,\mathrm{d}\boldsymbol{\xi} \\ &\leqslant \prod_{j=0}^{k} \|\hat{\lambda}_j\|_\infty^\varepsilon \int_{(\mathbb{R}^n)^k} |\hat{\lambda}_0(\xi_1 + \cdots + \xi_k)|^{1-\varepsilon} |\hat{\lambda}_1(\xi_1)|^{1-\varepsilon} \cdots |\hat{\lambda}_k(\xi_k)|^{1-\varepsilon} |J(\boldsymbol{\xi})|\,\mathrm{d}\boldsymbol{\xi}. \quad \square\end{aligned}$$

In some instances we will need a slightly more general multilinear form, as follows.

Definition 4.3 (smoothed configuration form). For functions $f_0, \dots, f_k \in \mathcal{S}(\mathbb{R}^n)$ and $F \in \mathcal{S}(\mathbb{R}^{n+m})$, let

$$\Lambda(f_0, \dots, f_k; F) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} F(x, y) f_0(x) f_1(x + \varphi_1(y)) \cdots f_k(x + \varphi_k(y))\, \mathrm{d}x\, \psi(y)\, \mathrm{d}y. \tag{4-1}$$

Proposition 4.4. *For functions $f_0, \dots, f_k \in \mathcal{S}(\mathbb{R}^n)$ and $F \in \mathcal{S}(\mathbb{R}^{n+m})$, we have*

$$\Lambda(f_0, \dots, f_k; F) = \int_{\mathbb{R}^n \times \mathbb{R}^m} \hat{F}(\kappa, \theta) \int_{(\mathbb{R}^n)^k} \hat{f}_0(-\kappa - \xi_1 - \cdots - \xi_k) \prod_{j=1}^{k} \hat{f}_j(\xi_j) J_\theta(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\kappa\, \mathrm{d}\theta.$$

Proof. By inserting the Fourier expansions of $F, f_1, \dots, f_k$ and using Fubini, we obtain

$$\begin{aligned}
\Lambda(f_0, \dots, f_k; F) &= \int_{\mathbb{R}^n} \int_{\mathbb{R}^m} F(x, y) f_0(x) f_1(x + \varphi_1(y)) \cdots f_k(x + \varphi_k(y))\, \mathrm{d}x\, \psi(y)\, \mathrm{d}y \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^m} \hat{F}(\kappa, \theta) \int_{(\mathbb{R}^n)^k} \hat{f}_1(\xi_1) \cdots \hat{f}(\xi_k) \int_{\mathbb{R}^n} f_0(x) e[(\kappa + \xi_1 + \cdots + \xi_k) \cdot x]\, \mathrm{d}x \\
&\qquad \times \int_{\mathbb{R}^m} e[\theta \cdot y + \xi_1 \cdot \varphi_1(y) + \cdots + \xi_k \cdot \varphi_k(y)] \psi(y)\, \mathrm{d}y\, \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_k\, \mathrm{d}\kappa\, \mathrm{d}\theta \\
&= \int_{\mathbb{R}^n \times \mathbb{R}^m} \hat{F}(\kappa, \theta) \int_{(\mathbb{R}^n)^k} \hat{f}_0(-\kappa - \xi_1 - \cdots - \xi_k) \hat{f}_1(\xi_1) \cdots \hat{f}_k(\xi_k) J_\theta(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\kappa\, \mathrm{d}\theta. \qquad \square
\end{aligned}$$

5. Bounding the singular integral

This section is devoted to the central task of bounding the singular integral (3-6) when the kernel K involved is the oscillatory integral J_θ from Definition 4.1. We will rely crucially on the following decay estimate:

Proposition 5.1. *Assuming that the neighborhood Ω of zero has been chosen small enough, we have*

$$|J_\theta(\boldsymbol{\xi})| \lesssim (1 + |\boldsymbol{A}^{\mathsf{T}} \boldsymbol{\xi} + \theta|)^{-m/2} \quad (\boldsymbol{\xi} \in (\mathbb{R}^n)^k,\ \theta \in \mathbb{R}^m). \tag{5-1}$$

Proof. By Definition 4.1, we have $J_\theta(\boldsymbol{\xi}) = I(\boldsymbol{A}^{\mathsf{T}} \boldsymbol{\xi} + \theta, \xi_{kn})$, where

$$I(\gamma, \gamma_{m+1}) = \int_{\mathbb{R}^m} e(\gamma \cdot y + \gamma_{m+1} Q(y)) \psi(y)\, \mathrm{d}x.$$

Consider the hypersurface $S = \{(y, Q(y)) : y \in \mathrm{Supp}(\psi)\}$ of $\mathbb{R}^{m+1}$; then our assumptions on Q mean that S has nonzero Gaussian curvature. Observe that I is the Fourier transform of $\tilde{\psi}\, \mathrm{d}\sigma_S$, where σ_S is the surface measure on S and $\tilde{\psi}$ is a smooth function with the same support as ψ. Therefore it satisfies the decay estimate [Stein 1993, Chapter VIII]

$$|I(\gamma, \gamma_{m+1})| \lesssim (1 + |\gamma| + |\gamma_{m+1}|)^{-m/2}$$

uniformly in $(\gamma, \gamma_{m+1}) \in \mathbb{R}^{m+1}$, which concludes the proof. $\square$

The main result of this section is a bound on the singular integral for functions in L^s for a range of s depending on n, m, k. In practice we will apply the proposition below when s is close to 2, which requires the parameter m' to be larger than $(k-1)n$, and when the functions F_i are powers of $|\hat{\mu}|$ or bounded functions supported on $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$.

Proposition 5.2. *Let* $1+\frac{1}{k} < s < k+1$ *and* $m' > 0$, *and write*

$$K_{\theta,m'}(\boldsymbol{\xi}) = (1+|\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\xi}+\theta|)^{-m'/2} \quad (\boldsymbol{\xi} \in (\mathbb{R}^n)^k, \theta \in \mathbb{R}^m).$$

Let $F_0, \dots, F_k$ *be nonnegative measurable functions on* $\mathbb{R}^n$. *Provided that*

$$m' > 2kn - \frac{2(k+1)}{s}n, \tag{5-2}$$

we have, *uniformly in* $\theta \in \mathbb{R}^m$,

$$\Lambda^*(F_0, \dots, F_k; K_{\theta,m'}) \lesssim_{s,m'} \|F_0\|_s \cdots \|F_k\|_s.$$

The first step towards the proof of this proposition is to bound moments of the kernels $K_{\theta,m'}$ on certain subspaces. Consider the $k+1$ linear maps $(\mathbb{R}^n)^k \to \mathbb{R}^n$ given by

$$\boldsymbol{\xi} \mapsto -(\xi_1+\dots+\xi_k) =: \xi_0, \quad \boldsymbol{\xi} \mapsto \xi_j \quad (1 \leqslant j \leqslant k).$$

For every $0 \leqslant j \leqslant k$ and $\eta \in \mathbb{R}^n$, the set $\{\boldsymbol{\xi} \in (\mathbb{R}^n)^k : \xi_j = \eta\}$ is an affine subspace of $(\mathbb{R}^n)^k$ of dimension $(k-1)n$. Recall that $\boldsymbol{A}^{\mathsf{T}} : \mathbb{R}^{nk} \to \mathbb{R}^m$, so that in the regime $m \geqslant (k-1)n$ we expect $(1+|\boldsymbol{A}^{\mathsf{T}}\cdot|)^{-1}$ to have bounded moments of order $q > (k-1)n$ on each of the subspaces $\{\xi_j = \eta\}$, under reasonable nondegeneracy conditions on the matrix $\boldsymbol{A}$. As the next lemma shows, what is needed is precisely the content of Definition 3.1.

Proposition 5.3. *Let* $0 \leqslant j \leqslant k$ *and suppose that* $m \geqslant (k-1)n$. *Then for* $q > (k-1)n$ *we have*, *uniformly in* $\eta \in \mathbb{R}^n$ *and* $\theta \in \mathbb{R}^m$,

$$\int_{\xi_j=\eta} (1+|\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\xi}+\theta|)^{-q}\, \mathrm{d}\sigma(\boldsymbol{\xi}) \lesssim_q 1.$$

Proof. First note that the assumptions of Definition 3.1 mean that $\boldsymbol{A}^{\mathsf{T}}$ is injective on $\{\boldsymbol{\xi} : \xi_j = 0\}$ for $0 \leqslant j \leqslant k$. To see that, observe that the conditions

$$\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\xi} = 0,\ \xi_j = 0 \implies \boldsymbol{\xi} = 0 \quad (0 \leqslant j \leqslant k)$$

can be put in matrix form

$$\begin{bmatrix} A_1^{\mathsf{T}} & \cdots & A_j^{\mathsf{T}} & \cdots & A_k^{\mathsf{T}} \\ 0 & \cdots & I_{n\times n} & \cdots & 0 \end{bmatrix} \boldsymbol{\xi} = 0 \implies \boldsymbol{\xi} = 0 \quad (1 \leqslant j \leqslant k),$$

$$\begin{bmatrix} A_1^{\mathsf{T}} & \cdots & A_k^{\mathsf{T}} \\ I_{n\times n} & \cdots & I_{n\times n} \end{bmatrix} \boldsymbol{\xi} = 0 \implies \boldsymbol{\xi} = 0.$$

Since $m+n \geqslant kn$, the $(m+n) \times kn$ matrices above have empty kernel if and only if they have rank kn, a set of conditions which is easily seen to be equivalent to that of Definition 3.1.

Now let

$$I = \int_{\xi_j=\eta} (1+|\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\xi}+\theta|)^{-q}\, \mathrm{d}\sigma(\xi).$$

We parametrize the affine subspace $\{\xi_j = \eta\}$ by $\boldsymbol{\xi} = \boldsymbol{R}\boldsymbol{\xi}' + \boldsymbol{\xi}_\eta$, where $\boldsymbol{\xi}'$ runs over $(\mathbb{R}^n)^k$, $\boldsymbol{\xi}_\eta \in (\mathbb{R}^n)^k$ is picked so that $(\boldsymbol{\xi}_\eta)_j = \eta$, and $\boldsymbol{R} \in O(\mathbb{R}^{kn})$ is a rotation mapping the subspace $\mathbb{R}^{(k-1)n}$ to $\{\xi_j = 0\}$. We obtain

$$I = \int_{\mathbb{R}^{(k-1)n}} (1 + |\boldsymbol{A}^\mathsf{T}\boldsymbol{R}\boldsymbol{\xi}' + \boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}_\eta + \theta|)^{-q}\, \mathrm{d}\boldsymbol{\xi}',$$

and we write $\boldsymbol{B} = \boldsymbol{A}^\mathsf{T}\boldsymbol{R} \in \mathbb{R}^{m\times kn}$, which is injective on $\mathbb{R}^{(k-1)n}$. Consider the orthogonal decomposition $\boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}_\eta + \theta = \boldsymbol{B}\boldsymbol{\xi}_{\eta,\theta} + \gamma$ with $\boldsymbol{\xi}_{\eta,\theta} \in \mathbb{R}^{(k-1)n}$ and $\gamma \in (\boldsymbol{B}(\mathbb{R}^{(k-1)n}))^\perp$, and observe that, by Pythagoras and injectivity,

$$|\boldsymbol{B}\boldsymbol{\xi}' + \boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}_\eta + \theta| = |\boldsymbol{B}(\boldsymbol{\xi}' + \boldsymbol{\xi}_{\eta,\theta}) + \gamma| \geqslant |\boldsymbol{B}(\boldsymbol{\xi}' + \boldsymbol{\xi}_{\eta,\theta})| \gtrsim |\boldsymbol{\xi}' + \boldsymbol{\xi}_{\eta,\theta}|.$$

Via the change of variables $\boldsymbol{\xi}' \leftarrow \boldsymbol{\xi}' + \boldsymbol{\xi}_{\eta,\theta}$,

$$I \lesssim \int_{\mathbb{R}^{(k-1)n}} (1 + |\boldsymbol{\xi}'|)^{-q}\, \mathrm{d}\boldsymbol{\xi}',$$

which is bounded for $q > (k-1)n$, uniformly in $\eta \in \mathbb{R}^n$. □

Proposition 5.4. *Let $F_0, \dots, F_k$ be nonnegative measurable functions on $\mathbb{R}^n$. Let $\tau \in (0,1)$ and let $p, p' \in (1, +\infty)$ be parameters with $\frac{1}{p} + \frac{1}{p'} = 1$. Let $H \geqslant 0$ be a parameter and suppose that K is a nonnegative measurable function on $\mathbb{R}^{nk}$ such that*

$$\int_{\xi_j = \eta} K(\boldsymbol{\xi})^{p'}\, \mathrm{d}\sigma(\boldsymbol{\xi}) \leqslant H \quad (\eta \in \mathbb{R}^n,\ 0 \leqslant j \leqslant k).$$

Then

$$\Lambda^*(F_0, \dots, F_k; K) \leqslant H^{1/p'} \prod_{j=0}^{k} \left(\int_{\mathbb{R}^n} F_j(\eta)^{\tau p(k+1)/k}\, \mathrm{d}\eta \right)^{\frac{k}{k+1}\frac{1}{p}} \left(\int_{\mathbb{R}^n} F_j(\eta)^{(1-\tau)p'(k+1)}\, \mathrm{d}\eta \right)^{\frac{1}{k+1}\frac{1}{p'}}.$$

Proof. We write $I = \Lambda^*(F_0, \dots, F_k; K)$. By a first application of Hölder,

$$\begin{aligned} I &= \int_{(\mathbb{R}^n)^k} \prod_{j=0}^{k} F_j(\xi_j)^{\tau + (1-\tau)} K(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi} \\ &\leqslant \left(\int_{(\mathbb{R}^n)^k} \left(\prod_{j=0}^{k} F_j(\xi_j) \right)^{\tau p} \mathrm{d}\boldsymbol{\xi} \right)^{\frac{1}{p}} \times \left(\int_{(\mathbb{R}^n)^k} \left(\prod_{j=0}^{k} F_j(\xi_j) \right)^{(1-\tau)p'} K(\boldsymbol{\xi})^{p'}\, \mathrm{d}\boldsymbol{\xi} \right)^{\frac{1}{p'}} \\ &=: (I_1)^{1/p} \times (I_2)^{1/p'}. \end{aligned} \tag{5-3}$$

We can rewrite I_1 as follows:

$$I_1 = \int_{(\mathbb{R}^n)^k} \prod_{j=0}^{k} F_j(\xi_j)^{\tau p}\, \mathrm{d}\boldsymbol{\xi} = \int_{(\mathbb{R}^n)^k} \prod_{i=0}^{k} \Bigg(\prod_{\substack{0 \leqslant j \leqslant k \\ j \neq i}} F_j(\xi_j)^{\tau p} \Bigg)^{\frac{1}{k}} \mathrm{d}\boldsymbol{\xi}.$$

By Hölder, we can then reduce to integrals each involving only k of the ξ_j:

$$I_1 \leqslant \prod_{i=0}^{k}\Bigl(\int_{(\mathbb{R}^n)^k}\prod_{\substack{0\leqslant j\leqslant k\\ j\neq i}} F_j(\xi_j)^{\tau p(k+1)/k}\,\mathrm{d}\boldsymbol{\xi}\Bigr)^{\frac{1}{k+1}}.$$

Recall that $\xi_0=\xi_1+\cdots+\xi_k$, so that after appropriate changes of variables each inner integral splits and we have

$$I_1 \leqslant \prod_{i=0}^{k}\Bigl(\prod_{\substack{0\leqslant j\leqslant k\\ j\neq i}}\int_{\mathbb{R}^n} F_j(\eta)^{\tau p(k+1)/k}\,\mathrm{d}\eta\Bigr)^{\frac{1}{k+1}} = \prod_{j=0}^{k}\Bigl(\int_{\mathbb{R}^n} F_j(\eta)^{\tau p(k+1)/k}\,\mathrm{d}\eta\Bigr)^{\frac{k}{k+1}}. \tag{5-4}$$

To treat the integral I_2, we separate variables by Hölder, and then integrate along slices [Nicolaescu 2011]:

$$\begin{aligned}
I_2 &= \int_{(\mathbb{R}^n)^k}\prod_{j=0}^{k} F_j(\xi_j)^{(1-\tau)p'}K(\xi)^{p'}\,\mathrm{d}\boldsymbol{\xi}\\
&\leqslant \prod_{j=0}^{k}\Bigl(\int_{(\mathbb{R}^n)^k} F_j(\xi_j)^{(1-\tau)p'(k+1)}K(\boldsymbol{\xi})^{p'}\,\mathrm{d}\boldsymbol{\xi}\Bigr)^{\frac{1}{k+1}}\\
&= \prod_{j=0}^{k}\Bigl(\int_{\eta\in\mathbb{R}^n} F_j(\eta)^{(1-\tau)p'(k+1)}\Bigl(\int_{\xi_j=\eta}K(\boldsymbol{\xi})^{p'}\,\mathrm{d}\sigma(\boldsymbol{\xi})\Bigr)\,\mathrm{d}\eta\Bigr)^{\frac{1}{k+1}}.
\end{aligned}$$

Inside each inner integral we use the fiber moment condition, so that eventually

$$I_2 \leqslant H\prod_{j=0}^{k}\Bigl(\int_{\mathbb{R}^n} F_j(\eta)^{(1-\tau)p'(k+1)}\,\mathrm{d}\eta\Bigr)^{\frac{1}{k+1}}. \tag{5-5}$$

The proof is finished upon inserting (5-4) and (5-5) into (5-3). □

It remains to determine the parameters (τ, p) in Proposition 5.4 that lead to a bound involving a single L^s norm.

Corollary 5.5. *Suppose that $1+\frac{1}{k} < s < k+1$. Then there exist unique parameters $\tau\in(0,1)$ and $p\in(1,\infty)$ depending on k and s such that*

$$s=\frac{k+1}{k}p\tau=(k+1)p'(1-\tau), \tag{5-6}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, and for such (τ, p) we have

$$\frac{k+1}{s}=\frac{k}{p}+\frac{1}{p'}, \tag{5-7}$$

$$\frac{1}{p'}=\frac{1}{k-1}\Bigl(k-\frac{k+1}{s}\Bigr). \tag{5-8}$$

Proof. Starting from (5-6), and dividing by $\frac{k+1}{k}p$ in the first identity and by $\frac{k+1}{k}pp'$ in the second, we obtain the equivalent identities

$$\tau = \frac{k}{k+1}\frac{s}{p} \quad \text{and} \quad \Big(\frac{k}{p}+\frac{1}{p'}\Big)\tau = \frac{k}{p}. \tag{5-9}$$

Inserting the left-hand expression of τ in the right-hand identity, we deduce the relation (5-7). This is easily solved in p and p', and one finds that

$$\frac{1}{p} = \frac{1}{k-1}\Big(\frac{k+1}{s}-1\Big) \quad \text{and} \quad \frac{1}{p'} = \frac{1}{k-1}\Big(k-\frac{k+1}{s}\Big),$$

which in particular recovers (5-8). It can be checked that $\frac{1}{p} \in (0, 1)$ under the given conditions on s. Inserting this value of $\frac{1}{p}$ in the first identity of (5-9), we find that

$$\tau = \frac{k}{k-1}\Big(1-\frac{s}{k+1}\Big),$$

which again lies in $(0, 1)$ for the given range of s. □

Proof of Proposition 5.2. Apply Proposition 5.4 with $K(\boldsymbol{\xi}) = (1+|\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\xi}+\theta|)^{-m'/2}$ and the choice of parameters (τ, p) from Corollary 5.5. By (5-7), this gives

$$|\lambda^*(F_0, \dots, F_k; K)| \leqslant H^{1/p'} \prod_{j=0}^{k} (\|F_j\|_s^s)^{(k/p+1/p')/(k+1)} = H^{1/p'} \prod_{j=0}^{k} \|F_j\|_s,$$

where $H = \max_j \sup_{\eta,\theta} \int_{\xi_j=\eta} (1+|\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\xi}+\theta|)^{-p'm'/2}\, \mathrm{d}\sigma(\boldsymbol{\xi})$. Via Proposition 5.3 and (5-8), we have $H \lesssim_{s,m'} 1$ provided that

$$m' > \frac{2(k-1)n}{p'} = 2\Big(k-\frac{k+1}{s}\Big)n. \qquad \square$$

From Proposition 5.2, we now derive useful bounds on the dual form Λ^*, which are needed to develop the results of Sections 6–8. In the course of the proof, we refer to a restriction estimate from Appendix B, which states essentially that $\hat{\mu}$ is in $L^{2+\varepsilon}$ when β remains bounded away from zero and α is close enough to n. Recall the notation $T^{\kappa} f = f(\kappa + \cdot\,)$ from Section 2.

Proposition 5.6. *Let $\beta_0 \in (0, n)$ and suppose that, for a constant $c > 0$ small enough with respect to n, k and m,*

$$(k-1)n < m < kn, \quad \beta_0 \leqslant \beta < n, \quad n - c\beta_0 \leqslant \alpha < n.$$

Then there exists $\varepsilon \in (0, 1)$ depending at most on n, k and m such that

$$\sup_{(\kappa,\theta)\in\mathbb{R}^n\times\mathbb{R}^m} \Lambda^*(T^{\kappa}|\hat{\mu}|^{1-\varepsilon}, |\hat{\mu}|^{1-\varepsilon}, \dots, |\hat{\mu}|^{1-\varepsilon}; |J_\theta|^{1-\varepsilon}) < \infty,$$

$$\Lambda^*(|\hat{\mu}|^{1-\varepsilon}, \dots, |\hat{\mu}|^{1-\varepsilon}; |J|) \lesssim_{\beta_0} (n-\alpha)^{-O(1)}.$$

Proof. Let ε, $\delta \in (0, 1)$ be parameters. Recalling the majoration (5-1), we apply Proposition 5.2 to $F_0 = T^{\kappa}|\hat{\mu}|^{1-\varepsilon}$ and $F_i = |\hat{\mu}|^{1-\varepsilon}$ for $i \geqslant 1$, with parameters $m' = (1-\varepsilon)m$ and $s = (2+\delta)/(1-\varepsilon)$. The condition (5-2) is fulfilled when $m > (k-1)n$ and ε, δ are small enough with respect to n, k, m. We obtain, uniformly in $\kappa \in \mathbb{R}^n$ and $\theta \in \mathbb{R}^m$,

$$\Lambda^*(T^\kappa |\hat{\mu}|^{1-\varepsilon}, |\hat{\mu}|^{1-\varepsilon}, \dots, |\hat{\mu}|^{1-\varepsilon}; |J_\theta|^{1-\varepsilon}) \lesssim_{\varepsilon,s} \| |\hat{\mu}|^{1-\varepsilon} \|_s^{k+1} = \|\hat{\mu}\|_{2+\delta}^{(1-\varepsilon)(k+1)}.$$

By Proposition B.3 and (3-3), we conclude that

$$\Lambda^*(T^\kappa |\hat{\mu}|^{1-\varepsilon}, |\hat{\mu}|^{1-\varepsilon}, \dots, |\hat{\mu}|^{1-\varepsilon}; |J_\theta|^{1-\varepsilon}) \lesssim_{\varepsilon,\delta,\beta_0} (n-\alpha)^{-O(1)},$$

and the second bound follows since $|J| \lesssim |J|^{1-\varepsilon}$. □

Proof of Proposition 3.6. Let $\varepsilon \in (0,1)$ be a parameter. By Proposition 4.2 and (5-1), we have

$$|\Lambda(f_0, \dots, f_k)| \leqslant \prod_{0 \leqslant j \leqslant k} \|\hat{f}_j\|_\infty^\varepsilon \cdot \Lambda^*\big(|\hat{f}_0|^{1-\varepsilon}, \dots, |\hat{f}_k|^{1-\varepsilon}; (1+|\boldsymbol{A}^\mathsf{T} \cdot |)^{-m/2}\big). \tag{5-10}$$

For ε small enough with respect to n, k, m, we may apply Proposition 5.2 with $s = 2/(1-\varepsilon)$ and $m' = m$, together with Plancherel:

$$\begin{aligned}\Lambda^*(|\hat{f}_0|^{1-\varepsilon}, \dots, |\hat{f}_k|^{1-\varepsilon}; (1+|\boldsymbol{A}^\mathsf{T} \cdot |)^{-m/2}) &\lesssim \textstyle\prod_{j=0}^k \| |\hat{f}_j|^{1-\varepsilon} \|_{2/(1-\varepsilon)} \\ &= \textstyle\prod_{j=0}^k \|\hat{f}_j\|_2^{1-\varepsilon} \\ &= \textstyle\prod_{j=0}^k \|f_j\|_2^{1-\varepsilon} \\ &\leqslant \textstyle\prod_{j=0}^k \|f_j\|_\infty^{1-\varepsilon},\end{aligned}$$

where we used the assumption $\mathrm{Supp}(f_j) \subset \big[-\frac{1}{2}, \frac{1}{2}\big]^n$ in the last line. Inserting this bound in (5-10) finishes the proof. □

6. The configuration measure

In this section, we aim to construct the measure $\nu \in \mathcal{M}^+(\mathbb{R}^{n+m})$ specified in Proposition 3.7. We make extensive use of the singular integral bounds derived in the previous section. Our treatment is similar to that of [Chan et al. 2016], but we work in a more abstract setting. We assume throughout this section that the dimensionality conditions (3-7) are met, so that singular integral bounds are available.

We start with the proper definition of ν, which is the content of the next proposition (recall Definition 3.2 and Proposition 3.3). We define an extra shift function $\varphi_0 = 0$ for notational convenience.

Proposition 6.1. *Define the functional ν at $F \in \mathcal{S}(\mathbb{R}^{n+m})$ by*

$$\langle \nu, F \rangle = \lim_{\varepsilon \to 0} \Lambda(\mu_\varepsilon, \dots, \mu_\varepsilon; F),$$

where $\mu_\varepsilon = \mu * \phi_\varepsilon$ for an approximate identity ϕ_ε with $\phi \in \mathcal{C}_{c,+}^\infty(\mathbb{R}^n)$. Then ν is well-defined and we have, for every $F \in \mathcal{S}(\mathbb{R}^{n+m})$,

$$\langle \nu, F \rangle = \Lambda^*(\hat{\mu}, \dots, \hat{\mu}; \hat{F}) \quad \text{and} \quad |\langle \nu, F \rangle| \leqslant \|F\|_\infty \Lambda(\mu, \dots, \mu),$$

where the integrals defined by the right-hand sides converge absolutely. Therefore ν extends by density to a positive bounded linear operator on $\mathcal{C}_c(\mathbb{R}^{n+m})$.

Proof. By Proposition 4.4, we have

$$\Lambda(\mu_\varepsilon, \dots, \mu_\varepsilon; F) = \int_{\mathbb{R}^{n+m}} \hat{F}(\kappa, \theta) \int_{(\mathbb{R}^n)^k} \hat{\mu}(-\kappa - \xi_1 - \cdots - \xi_k) \prod_{j=1}^{k} \hat{\mu}(\xi_j) J_\theta(\boldsymbol{\xi}) h_\varepsilon(\boldsymbol{\xi}, \kappa)\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\kappa\, \mathrm{d}\theta,$$

where $h_\varepsilon(\boldsymbol{\xi}, \kappa) = \hat{\phi}(-\varepsilon(\kappa + \xi_1 + \cdots + \xi_k)) \prod_{j=1}^{k} \hat{\phi}(\varepsilon \xi_j)$. Since h_ε is bounded by 1 in absolute value and tends to 1 pointwise as $\varepsilon \to 0$, the limit of $\Lambda(\mu_\varepsilon, \dots, \mu_\varepsilon; F)$ as $\varepsilon \to 0$ exists and equals $\Lambda^*(\hat{\mu}, \dots, \hat{\mu}; \hat{F})$ by dominated convergence, since we have uniform boundedness of

$$\begin{aligned}\int_{\mathbb{R}^{n+m}} |\hat{F}(\kappa, \theta)| \int_{(\mathbb{R}^n)^k} |\hat{\mu}(\kappa + \xi_1 + \cdots + \xi_k)| \prod_{j=1}^{k} |\hat{\mu}(\xi_j)|\, |J_\theta(\boldsymbol{\xi})|\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\kappa\, \mathrm{d}\theta \\ \leqslant \sup_{(\kappa,\theta) \in \mathbb{R}^n \times \mathbb{R}^m} \Lambda^*(|T^\kappa \hat{\mu}|, |\hat{\mu}|, \dots, |\hat{\mu}|; |J_\theta|) \times \int_{\mathbb{R}^{n+m}} |\hat{F}(\kappa, \theta)|\, \mathrm{d}\kappa\, \mathrm{d}\theta < \infty,\end{aligned}$$

via Proposition 5.6 and the majorations $|J_\theta| \lesssim |J_\theta|^{1-\varepsilon}$ and $|\hat{\mu}| \leqslant |\hat{\mu}|^{1-\varepsilon}$. Recalling Definitions 3.2 and 4.3 and using the positivity of μ_ε, we also have

$$|\langle \nu, F \rangle| = \lim_{\varepsilon \to 0} |\Lambda(\mu_\varepsilon, \dots, \mu_\varepsilon; F)| \leqslant \|F\|_\infty \varlimsup_{\varepsilon \to 0} \Lambda(\mu_\varepsilon, \dots, \mu_\varepsilon).$$

By Fourier inversion (Proposition 3.3) and another instance of the dominated convergence theorem, exploiting the finiteness of $\Lambda^*(|\hat{\mu}|, \dots, |\hat{\mu}|; |J|)$ provided by Proposition 5.6, we obtain

$$|\langle \nu, F \rangle| \leqslant \|F\|_\infty \varlimsup_{\varepsilon \to 0} \Lambda^*(\hat{\mu}_\varepsilon, \dots, \hat{\mu}_\varepsilon; J) = \|F\|_\infty \Lambda^*(\hat{\mu}, \dots, \hat{\mu}; J).$$

This last quantity equals $\|F\|_\infty \Lambda(\mu, \dots, \mu)$ by Definition 3.4. □

Proposition 6.2. *When defined, the measure ν of Proposition 6.1 is supported on the compact set*

$$X = \{(x, y) \in E \times \operatorname{Supp} \psi : (x, x + \varphi_1(y), \dots, x + \varphi_k(y)) \in E^{k+1}\}.$$

Proof. We can rewrite $X = (E \times \operatorname{Supp} \psi) \cap \Phi^{-1}(E^{k+1})$, where Φ is the smooth map defined by (3-4), so that X is closed and bounded, and therefore compact. Since its complement X^c is open, it is enough to show that $\langle \nu, F \rangle = 0$ for every $F \in \mathscr{C}^\infty_{c,+}(\mathbb{R}^{n+m})$ such that $\operatorname{Supp} F \subset X^c$. By compactness we know that there exists $c > 0$ such that

$$\max_{0 \leqslant j \leqslant k} d(x + \varphi_j(y), E) \geqslant c > 0 \quad \text{for all } (x, y) \in \operatorname{Supp} F \cap (\mathbb{R}^n \times \operatorname{Supp} \psi).$$

On the other hand,

$$\langle \nu, F \rangle = \lim_{\varepsilon \to 0} \int_{\operatorname{Supp} F \cap (\mathbb{R}^n \times \operatorname{Supp} \psi)} F(x, y) \prod_{j=0}^{k} \mu_\varepsilon(x + \varphi_j(y))\, \mathrm{d}x\, \psi(y)\, \mathrm{d}y.$$

For ε small enough, since μ_ε is supported on $E + B(0, C\varepsilon)$ for a certain $C > 0$, the integrand above is always zero. □

Proposition 6.3. *We have $\|\nu\| = \Lambda(\mu, \dots, \mu)$.*

Proof. Consider the compact set X from Proposition 6.2, and the larger compact set

$$Y = \{(x, y) \in \mathbb{R}^n \times \operatorname{Supp}\psi : d(x + \varphi_j(y), E) \leqslant 1 \text{ for } 0 \leqslant j \leqslant k\}.$$

Pick a smoothed ball indicator $F \in \mathscr{C}^\infty_{c,+}(\mathbb{R}^{n+m})$ such that $F = 1$ on Y. Since ν is supported on $X \subset Y$, we have

$$\nu(\mathbb{R}^{n+m}) = \langle \nu, F\rangle = \lim_{\varepsilon\to 0} \int_{\mathbb{R}^n\times\operatorname{Supp}\psi} F(x, y) \prod_{j=0}^{k} \mu_\varepsilon(x + \varphi_j(y))\, \mathrm{d}x\, \psi(y)\, \mathrm{d}y.$$

Since $(x, y) \mapsto \prod_{j=0}^{k} \mu_\varepsilon(x + \varphi_j(y))$ is supported on Y for ε small enough, we therefore have

$$\nu(\mathbb{R}^{n+m}) = \lim_{\varepsilon\to 0} \Lambda(\mu_\varepsilon, \dots, \mu_\varepsilon).$$

By the same reasoning as in the end of the proof of Proposition 6.1, again using $\Lambda^*(|\hat\mu|, \dots, |\hat\mu|; |J|) < \infty$ provided by Proposition 5.6, we find eventually that $\|\nu\| = \Lambda(\mu, \dots, \mu)$. □

We now turn to the last expected feature of the configuration measure ν, which is that it has zero mass on any hyperplane.

Proposition 6.4. *We have $\nu(H) = 0$ for every hyperplane H of $\mathbb{R}^{n+m}$.*

Proof. Consider a hyperplane $H < \mathbb{R}^{n+m}$ and a rotation $R \in O_{n+m}(\mathbb{R})$ such that $H = R(\mathbb{R}^{n+m-1} \times \{0\})$. Consider parameters $L \geqslant 1$ and $\delta \in (0, 1]$. We consider a Schwartz function F_δ of the form

$$F_\delta \circ R = \chi\Big(\frac{\cdot}{L}\Big)\,\Xi\Big(\frac{\cdot}{\delta}\Big), \tag{6-1}$$

where $\chi \in \mathscr{S}(\mathbb{R}^{n+m-1})$ and $\Xi \in \mathscr{S}(\mathbb{R})$ are nonnegative with $\chi \geqslant 1$ on $[-1, 1]^{n+m-1}$ and $\Xi(0) \geqslant 1$. Writing $H_L = R([-L, L]^{n+m-1} \times \{0\})$, we therefore have $\nu(H_L) \leqslant \langle \nu, F_\delta\rangle$, and it is enough to show that $\langle \nu, F_\delta\rangle$ tends to 0 as $\delta \to 0$ for every fixed $L \geqslant 1$. By Proposition 6.1, writing $\gamma = (\kappa, \theta) \in \mathbb{R}^n \times \mathbb{R}^m$, we have

$$\langle \nu, F_\delta\rangle = \int_{\mathbb{R}^{n+m}} \int_{(\mathbb{R}^n)^k} \hat F_\delta(\gamma) \hat\mu(-\kappa - \xi_1 - \cdots - \xi_k) \prod_{j=1}^{k} \hat\mu(\xi_j) J_\theta(\boldsymbol{\xi})\, \mathrm{d}\boldsymbol{\xi}\, \mathrm{d}\gamma. \tag{6-2}$$

We assume that χ and Ξ have been chosen so that their Fourier transforms are supported on centered balls of radius 1, which is certainly possible. Recalling (6-1), we therefore have, for every $(u, v) \in \mathbb{R}^{n+m-1} \times \mathbb{R}$,

$$|\hat F_\delta \circ R(u, v)| = |\widehat{F_\delta \circ R}(u, v)| \lesssim L^{n+m-1} \cdot 1_{|u|\leqslant L^{-1}} \cdot \delta \cdot 1_{|v|\leqslant \delta^{-1}}. \tag{6-3}$$

We next show how to obtain some uniform γ-decay from the other factor in the integrand of (6-2). By (3-2) and (5-1), since $\beta \leqslant n \leqslant m$, we have

$$\begin{aligned}
|\hat\mu(\kappa+\xi_1+\cdots+\xi_k)| \prod_{j=1}^{k} |\hat\mu(\xi_j)|\,|J_\theta(\boldsymbol{\xi})| &\lesssim_\alpha (1+|\kappa+\xi_1+\cdots+\xi_k|)^{-\beta/2} \prod_{j=1}^{k} (1+|\xi_j|)^{-\beta/2} (1+|\boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}+\theta|)^{-m/2} \\
&\lesssim_\alpha (1+|\kappa+\xi_1+\cdots+\xi_k|+|\xi_1|+\cdots+|\xi_k|+|\boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}+\theta|)^{-\beta/2}.
\end{aligned}$$

Using this in conjunction with the triangle inequality and the decompositions $\theta = (\boldsymbol{A}^{\mathsf{T}}\boldsymbol{\xi} + \theta) - \sum_{j=1}^{k} A_j^{\mathsf{T}} \xi_j$ and $\kappa = (\kappa + \xi_1 + \cdots + \xi_k) - \sum_{j=1}^{k} \xi_j$, we deduce that

$$|\hat{\mu}(\kappa + \xi_1 + \cdots + \xi_k)| \prod_{j=1}^{k} |\hat{\mu}(\xi_j)|\,|J_\theta(\boldsymbol{\xi})| \lesssim_\alpha (1 + |\kappa| + |\theta|)^{-\beta/2} \asymp (1 + |\gamma|)^{-\beta/2}. \tag{6-4}$$

Let $\varepsilon \in (0, 1)$ be the small parameter in the statement of the proposition. At this point we have two parametrizations $\gamma = (\kappa, \theta) = R(u, v)$ with $(\kappa, \theta) \in \mathbb{R}^n \times \mathbb{R}^m$ and $(u, v) \in \mathbb{R}^{n+m-1} \times \mathbb{R}$. By integrating in (u, v) coordinates in (6-2) and bounding $\hat{F}_\delta(\gamma)$ via (6-3), we obtain

$$|\langle \nu, F_\delta \rangle| \lesssim \int_{\mathbb{R}^{n+m-1} \times \mathbb{R}} 1_{|u| \leqslant L^{-1}} \cdot L^{n+m-1} \cdot \delta \cdot 1_{|v| \leqslant \delta^{-1}} \left(\int_{(\mathbb{R}^n)^k} |\hat{\mu}(\kappa + \xi_1 + \cdots + \xi_k)| \prod_{j=1}^{k} |\hat{\mu}(\xi_j)|\,|J_\theta(\boldsymbol{\xi})| \,\mathrm{d}\boldsymbol{\xi} \right) \mathrm{d}u \,\mathrm{d}v.$$

By pulling out an ε-th power of the inner integrand and using (6-4), we infer that

$$\begin{aligned} |\langle \nu, F_\delta \rangle| &\lesssim_\alpha \int_{\mathbb{R}^{n+m-1}} L^{n+m-1} \cdot 1_{|u| \leqslant L^{-1}} \int_{\mathbb{R}} \delta \cdot 1_{|v| \leqslant \delta^{-1}} \cdot (1 + |(u, v)|)^{-\varepsilon\beta/2} \\ &\qquad \left(\int_{(\mathbb{R}^n)^k} |\hat{\mu}(\kappa + \xi_1 + \cdots + \xi_k)|^{1-\varepsilon} \prod_{j=1}^{k} |\hat{\mu}(\xi_j)|^{1-\varepsilon} |J_\theta(\boldsymbol{\xi})|^{1-\varepsilon} \,\mathrm{d}\boldsymbol{\xi} \right) \mathrm{d}u \,\mathrm{d}v \\ &\lesssim \sup_{(\kappa,\theta) \in \mathbb{R}^n \times \mathbb{R}^m} \Lambda^*(|T^\kappa \hat{\mu}|^{1-\varepsilon}, |\hat{\mu}|^{1-\varepsilon}, \ldots, |\hat{\mu}|^{1-\varepsilon}; |J_\theta|^{1-\varepsilon}) \times \delta \int_{|v| \leqslant \delta^{-1}} (1 + |v|)^{-\varepsilon\beta/2} \,\mathrm{d}v. \end{aligned}$$

The supremum above is finite by Proposition 5.6 and for ε small enough the last factor is bounded by $\delta^{\varepsilon\beta/2}$. Therefore $\langle \nu, F_\delta \rangle \to 0$ as $\delta \to 0$, as was to be shown. ☐

Proof of Proposition 3.7. It suffices to combine Propositions 6.1–6.4, recalling that we assumed (3-7) in this section. ☐

7. Absolutely continuous estimates

In this section we verify that absolutely continuous estimates are available when the shifts in (3-4) are given by polynomial vectors and the singular integral converges. We work with the notation of abstract shift functions.

The strategy, as in the regularity proof of Roth's theorem [Tao 2014], is to use the U^2 arithmetic regularity lemma to decompose a nonnegative bounded function into an almost-periodic component, an L^2 error, and a part which is Fourier-small. The precise version of the regularity lemma that we need is found in Appendix A. To neglect the contribution of Fourier-small functions, we use the fact that the counting operator is controlled by the Fourier L^∞ norm for bounded functions, in the sense of Proposition 3.6. To show that the pattern count for almost-periodic functions is high, we need uniform lower bounds for certain Bohr sets of almost-periods, the proof of which will occupy subsequent parts of this section. We define a Bohr set of $\mathbb{T}^n$ of a frequency set $\Gamma \subset \mathbb{Z}^n$, radius $\delta \in \left(0, \frac{1}{2}\right]$ and dimension $d = |\Gamma| < \infty$ by

$$B = B(\Gamma, \delta) = \{x \in \mathbb{T}^n : \|\xi \cdot x\| \leqslant \delta \ \text{ for all } \ \xi \in \Gamma\}. \tag{7-1}$$

We first prove the following conditional version of Proposition 3.8:

Proposition 7.1. *Suppose that $m > (k-1)n$ and, uniformly for every Bohr set B of $\mathbb{T}^n$ of dimension d and radius $\delta > 0$,*

$$\mathscr{L}\big\{y \in [-c, c]^m : \varphi_1(y), \dots, \varphi_k(y) \in B\big\} \gtrsim_{d,\delta} 1.$$

Then, for every function $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ supported on $\left[-\frac{1}{8}, \frac{1}{8}\right]^n$ such that $0 \leqslant f \leqslant 1$ and $\int f = \tau \in (0, 1]$, we have

$$\Lambda(f, \dots, f) \gtrsim_\tau 1.$$

Proof. We let $\kappa : (0, 1]^3 \to (0, 1]$ be a decay function and $\varepsilon \in (0, 1]$ be a parameter, both to be determined later. Write the decomposition of Proposition A.2 with respect to ε and κ as $f = f_1 + f_2 + f_3 = g + f_3$. Note that $f_1, g \geqslant 0$ and f_1, f_2, f_3, g are supported in $\left[-\frac{1}{4}, \frac{1}{4}\right]^n$ and uniformly bounded by 2 in absolute value. Expanding $f = g + f_3$ by multilinearity and using Proposition 3.6 together with the Fourier bound on f_3 in (A-5), we obtain

$$\Lambda(f, \dots, f) = \Lambda(g, \dots, g) + O\Big(\sum \Lambda(*, \dots, f_3, \dots, *)\Big) = \Lambda(g, \dots, g) + O(\kappa(\varepsilon, d^{-1}, \delta)^{\varepsilon'}) \tag{7-2}$$

for an $\varepsilon' \in (0, 1)$ depending at most on n, k, m. Recall that we assumed that ψ is at least 1 on a box $[-c, c]^m$ in Section 3, and let

$$E = \{y \in [-c, c]^m : \varphi_1(y), \dots, \varphi_k(y) \in B\}, \tag{7-3}$$

where B is the Bohr set of Proposition A.2. For reasons that shall be clear later, we first restrict integration to the set E, using the nonnegativity of g:

$$\Lambda(g, \dots, g) \geqslant \int_E \left(\int_{\mathbb{R}^n} g \cdot T^{\varphi_1(y)} g \cdots T^{\varphi_k(y)} g \, \mathrm{d}\mathscr{L}\right) \mathrm{d}y.$$

Next, we focus on the decomposition $g = f_1 + f_2$ and exploit the L^2 bound on f_2 in (A-5) by Cauchy–Schwarz in the inner integral:

$$\begin{aligned}
\Lambda(g, \dots, g) &\geqslant \int_E \left(\int_{\mathbb{R}^n} g \cdot T^{\varphi_1(y)} g \cdots T^{\varphi_k(y)} g \, \mathrm{d}\mathscr{L}\right) \mathrm{d}y \\
&\geqslant \int_E \left(\int_{\mathbb{R}^n} f_1 \cdot T^{\varphi_1(y)} f_1 \cdots T^{\varphi_k(y)} f_1 \, \mathrm{d}\mathscr{L} - \sum \int_{\mathbb{R}^n} * \cdots T^{\varphi_j(y)} f_2 \cdots * \, \mathrm{d}\mathscr{L}\right) \mathrm{d}y \\
&\geqslant \int_E \left(\int_{\mathbb{R}^n} f_1 \cdot T^{\varphi_1(y)} f_1 \cdots T^{\varphi_k(y)} f_1 \, \mathrm{d}\mathscr{L} - O(\varepsilon)\right) \mathrm{d}y.
\end{aligned}$$

Finally, we use the almost-periodicity estimate for f_1 in (A-5) and the definition (7-3) of E to replace the shifts of f_1 by itself:

$$\Lambda(g, \dots, g) \geqslant \int_E \left(\int_{[-\frac{1}{2}, \frac{1}{2}]^n} f_1^{k+1} \, \mathrm{d}\mathscr{L} - O(\varepsilon)\right) \mathrm{d}y.$$

By nesting of $L^p\big([-\frac{1}{2},\frac{1}{2}]^n\big)$ norms and the nonnegativity of f_1, we infer that

$$\Lambda(g,\dots,g) \geqslant \int_E \bigg(\bigg(\int_{[-\frac{1}{2},\frac{1}{2}]^n} f_1 \,\mathrm{d}\mathscr{L}\bigg)^{k+1} - O(\varepsilon)\bigg)\,\mathrm{d}y = \mathscr{L}(E)\cdot(\tau^{k+1} - O(\varepsilon)).$$

Choosing $\varepsilon = c\tau^{k+1}$ with a small $c>0$, and recalling (7-2) and the assumption on E, we obtain

$$\Lambda(f,\dots,f) \geqslant c(\delta,d^{-1})\tau^{k+1} - O(\kappa(c\tau^{k+1},d^{-1},\delta)^{\varepsilon'}).$$

Choosing $\kappa(\varepsilon,d^{-1},\delta) = c'\cdot(c(\delta,d^{-1})\varepsilon)^{1/\varepsilon'}$, recalling that $d,\delta^{-1} \lesssim_{\varepsilon,\kappa} 1 \lesssim_\tau 1$, we obtain

$$\Lambda(f,\dots,f) \geqslant \tfrac{1}{2}c(\delta,d^{-1})\tau^{k+1} \gtrsim_\tau 1. \qquad \square$$

It remains to determine a lower bound on the measure of the intersection of preimages of a Bohr set by the shift functions. This can be done when the shift functions are polynomial vectors, by reduction to a known diophantine approximation problem, and in fact there will be a series of intermediate reductions. We let d denote the L^∞ metric on $\mathbb{R}^n$ or $\mathbb{R}$ and we define

$$\|x\|_{\mathbb{T}^n} = d(x,\mathbb{Z}^n) = \max_{1\leqslant i\leqslant n} d(x_i,\mathbb{Z})$$

for $x\in\mathbb{R}^n$. In all subsequent propositions in this section we also liberate the letters n, k, m from their usual meaning, and we indicate the dependencies of implicit constants on all parameters. Our objective is to prove the following statement:

Proposition 7.2. *Let $t,m,n,l,d \geqslant 1$. Let $Q_1,\dots,Q_t:\mathbb{R}^m\to\mathbb{R}^n$ be polynomial vectors with components of degree at most l and such that $Q_i(0)=0$ for all $i\in[t]$. For $\xi_1,\dots,\xi_d\in\mathbb{R}^n$, we have*

$$\mathscr{L}\big\{y\in[-c,c]^m : \|Q_i(y)\cdot\xi_j\|_{\mathbb{T}} < \varepsilon \text{ for all } (i,j)\in[t]\times[d]\big\} \gtrsim_{\varepsilon,l,m,t,d,n} 1.$$

Our first reduction is to a finite system of conditions on monomials modulo 1.

Proposition 7.3. *Let $l,m\geqslant 1$ and $X=\{0,\dots,l\}^m\smallsetminus\{0\}$. For every $I\in X$, let $d_I\in\mathbb{N}_0$ and $\xi_I\in\mathbb{R}^{d_I}$. Then*[3]

$$\mathscr{L}\big\{y\in[-c,c]^m : \|y^I\xi_I\|_{\mathbb{T}^{d_I}} \leqslant \varepsilon \text{ for all } I\in X\big\} \gtrsim_{\varepsilon,l,m,(d_I)} 1.$$

Proof that Proposition 7.3 implies Proposition 7.2. Let $X=\{0,\dots,l\}^m\smallsetminus\{0\}$ and write $Q_i=\sum_{k\in[n]}Q_{ik}e_k$ with $Q_{ik}=\sum_{I\in X}a_I^{(ik)}y^I$. For every $I\in X$ we define $d_I=t+d+n$ and $\xi_I=(a_I^{(ik)}\xi_{jk})_{(i,j,k)}\in\mathbb{T}^{t+d+n}$, to make the following observation:

$$\begin{aligned}
&\|Q_i(y)\cdot\xi_j\|_{\mathbb{T}} \leqslant \varepsilon \quad \text{for all } (i,j)\in[t]\times[d]\\
&\qquad\iff \bigg\|\sum_{k\in[n]}\sum_{I\in X} a_I^{(ik)}y^I\xi_{jk}\bigg\|_{\mathbb{T}} \leqslant \varepsilon \qquad \text{for all } (i,j)\in[t]\times[d]\\
&\qquad\Longleftarrow \quad \|y^I a_I^{(ik)}\xi_{jk}\|_{\mathbb{T}} \leqslant \frac{\varepsilon}{nl^m} \quad \text{for all } (i,j,k)\in[t]\times[d]\times[n],\ I\in X\\
&\qquad\iff \quad \|y^I\xi_I\|_{\mathbb{T}^{d_I}} \leqslant \frac{\varepsilon}{nl^m} \quad \text{for all } I\in X.
\end{aligned}$$

[3]Here and in the sequel we set $\mathbb{R}^0=\{0\}$ and $\|0\|_{\mathbb{T}^0}=0$, so that the conditions involving a space $\mathbb{R}^0$ are void.

Applying Proposition 7.3 with $\varepsilon \leftarrow \varepsilon/nl^m$ and (d_I, ξ_I) as above, we find a lower bound on the quantity under study which depends only on ε, l, m, t, d, n. □

Our second reduction consists in a straightforward induction, which reduces the dimension of the problem to 1.

Proposition 7.4. *Let $l \geqslant 1$, $d_1, \dots, d_l \in \mathbb{N}_0$ and $\xi_1 \in \mathbb{R}^{d_1}, \dots, \xi_l \in \mathbb{R}^{d_l}$. We have*

$$\mathscr{L}\big\{y \in [-c, c] : \|y^j \xi_j\|_{\mathbb{T}^{d_j}} \leqslant \varepsilon \textit{ for all } j \in [l]\big\} \gtrsim_{\varepsilon,l,(d_i)} 1.$$

Proof that Proposition 7.4 implies Proposition 7.3. We induct on $m \geqslant 1$, the case $m = 1$ being precisely Proposition 7.4. Assume that we have proven the estimate for dimensions less than or equal to m, and write a tuple $I \in \{0, \dots, l\}^{m+1} \smallsetminus \{0\}$ as $I = (J, i_{m+1})$ with $J \in \{0, \dots, l\}^m$ and $i_{m+1} \geqslant 0$. We distinguish the conditions involving y_{m+1} or not by Fubini:

$$\begin{aligned}
&\mathscr{L}\big\{y \in [-c, c]^{m+1} : \|y^I \xi_I\|_{\mathbb{T}^{d_I}} \leqslant \varepsilon \text{ for all } I \in X\big\} \\
&\quad = \int_{[-c,c]^{m+1}} 1\big[\|y^J y_{m+1}^{i_{m+1}} \xi_I\|_{\mathbb{T}^{d_I}} \leqslant \varepsilon \text{ for all } (J, i_{m+1}) = I \in X\big]\, \mathrm{d}y_1 \cdots \mathrm{d}y_m\, \mathrm{d}y_{m+1} \\
&\quad = \int_{[-c,c]^m} 1\big[\|y^J \xi_I\|_{\mathbb{T}^{d_I}} \leqslant \varepsilon \text{ for all } (J, 0) = I \in X\big] \\
&\qquad\qquad \int_{[-c,c]} 1\big[\|y_{m+1}^{i_{m+1}} \cdot y^J \xi_I\|_{\mathbb{T}^{d_I}} \leqslant \varepsilon \text{ for all } (J, i_{m+1}) = I \in X : i_{m+1} \geqslant 1\big]\, \mathrm{d}y_{m+1}\, \mathrm{d}y_1 \cdots \mathrm{d}y_m.
\end{aligned}$$

By first applying the induction hypothesis with $m = 1$ at fixed $y_1, \dots, y_m$, and then by applying another instance of the induction hypothesis, we find that this quantity is indeed bounded from below by a positive constant depending only on ε, l, m and (d_I). □

Our final reduction is a simple discretization argument, which reduces the problem to the following known diophantine approximation estimate (see also [Green and Tao 2009, Proposition A.2; Baker 1986, Chapter 7]).

Proposition 7.5 [Lyall and Magyar 2011, Proposition B.2]. *Let $l \geqslant 1$ and $d_1, \dots, d_l \in \mathbb{N}_0$. Let $\alpha_i \in \mathbb{R}^{d_i}$ for $i = 1, \dots, l$ and $N \geqslant 1$. We have*

$$N^{-1}\#\big\{|n| \leqslant N : \|n^j \alpha_j\|_{\mathbb{T}^{d_j}} \leqslant \varepsilon \textit{ for all } j \in [l]\big\} \gtrsim_{\varepsilon,l,(d_j)} 1.$$

Proof that Proposition 7.5 implies Proposition 7.4. Consider a scale $N \geqslant 1$ going to infinity. Write each $|y| \leqslant c$ as $y = (n + u)/N$ with $n \in \mathbb{Z}$ and $u \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, so that $y^j = n^j/N^j + O_l(1/N)$ for every $j \in [l]$. For N large enough with respect to (ξ_j), ε and l, we therefore have

$$\|y^j \xi_j\|_{\mathbb{T}} \leqslant \varepsilon \Longleftarrow \left\|n^j \frac{\xi_j}{N^j}\right\|_{\mathbb{T}} \leqslant \frac{\varepsilon}{2}.$$

This yields

$$\begin{aligned}\mathscr{L}\big\{y \in [-c, c] : \|y^j \xi_j\|_{\mathbb{T}^{d_j}} \leqslant \varepsilon \text{ for all } j \in [l]\big\} \\ \geqslant \sum_{|n| \leqslant cN/2} \mathscr{L}\left\{ y = \frac{n+u}{N} : |u| \leqslant \frac{1}{2},\ \left\| n^j \frac{\xi_j}{N^j} \right\|_{\mathbb{T}^{d_j}} \leqslant \frac{\varepsilon}{2} \text{ for all } j \in [l] \right\} \\ \geqslant N^{-1} \# \left\{ |n| \leqslant \frac{cN}{2} : \left\| n^j \frac{\xi_j}{N^j} \right\|_{\mathbb{T}^{d_j}} \leqslant \varepsilon \text{ for all } j \in [l] \right\}.\end{aligned}$$

Applying Proposition 7.5 concludes the proof. □

To conclude this section we may now derive the absolutely continuous estimates stated in Section 3.

Proof of Proposition 3.8. It suffices to combine Propositions 7.1 and 7.2, recalling the shape (3-5) of our shift functions. □

8. The transference argument

This section is concerned with proving that $\Lambda(\mu, \dots, \mu) > 0$, by the transference argument of [Łaba and Pramanik 2009] exploiting the pseudorandomness of the fractal measure μ as $\alpha \to n$. We start by recalling the decomposition of [Chan et al. 2016, Section 6] of the fractal measure μ into a bounded smooth part (a mollified version of μ) and a Fourier-small part (the difference with the first part). This is the part of the argument where one lets α tend to n in a certain sense, and then the Fourier tail exhibits very strong, exponential-type decay in $n - \alpha$.

Proposition 8.1. *There exists a constant $C_1 > 0$ depending at most on n and D, and a decomposition $\mu = \mu_1 + \mu_2$, where $\mu_1 = f \,\mathrm{d}\mathscr{L}$, $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, $0 \leqslant f \leqslant C_1$, $\int f = 1$, $\operatorname{Supp} f \subset \left[-\frac{1}{8}, \frac{1}{8}\right]^n$, $|\hat{\mu}_i| \leqslant 2|\hat{\mu}|$ for $i \in \{1, 2\}$ and*

$$\|\hat{\mu}_2\|_\infty \lesssim (n-\alpha)^{-O(1)} e^{-\beta/(2+\beta)(n-\alpha)}.$$

Proof. Let $L \geqslant 1$ be a parameter. Consider a cutoff $\phi \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ such that $\int \phi = 1$, $\operatorname{Supp} \phi \subset B\big(0, \frac{1}{16}\big)$ and $0 \leqslant \phi \leqslant C_0$ for a certain $C_0 = C_0(n) > 0$, and define $\phi_L = L^n \phi(L\cdot)$. Let $f = \mu * \phi_L$ and consider the decomposition $\mu = \mu_1 + \mu_2$ with $\mu_1 = f \,\mathrm{d}\mathscr{L}$ and $\mu_2 = \mu - \mu_1$. We can already infer that $f \geqslant 0$, $\int f = 1$, $|\hat{\mu}_i| \leqslant 2|\hat{\mu}|$ for $i = 1, 2$ and $\operatorname{Supp} \mu_1 \subset \left[-\frac{1}{8}, \frac{1}{8}\right]^n$, since we assumed that $E \subset \left[-\frac{1}{16}, \frac{1}{16}\right]^n$ in Section 3.

Next, we show that f is bounded. Since ϕ_L has support in $B(0, 1/(16L))$, by (3-1) we have

$$f(x) = \int_{B(x, 1/(16L))} \phi_L(x-y) \,\mathrm{d}\mu(y) \leqslant \|\phi_L\|_\infty \cdot \mu\Big[B\Big(x, \frac{1}{16L}\Big)\Big] \leqslant C_0 D L^{n-\alpha}.$$

Choosing $L = e^{1/(n-\alpha)}$, we deduce that

$$\|f\|_\infty \leqslant C_0 D e =: C_1.$$

Finally, we bound the Fourier transform of $\hat{\mu}_2$. Observe that, for every $\xi \in \mathbb{R}^n$,

$$\hat{\mu}_2(\xi) = \hat{\mu}(\xi)\bigg(1 - \hat{\phi}\Big(\frac{\xi}{L}\Big)\bigg). \tag{8-1}$$

Since $\int \phi = 1$, we always have $|1-\hat{\phi}(\xi/L)| \leqslant 2$. On the other hand, since ϕ has support in $B\big(0, \frac{1}{16}\big)$, we have

$$\left|1-\hat{\phi}\left(\frac{\xi}{L}\right)\right| = \left|\int_{B(0,1/16)} \phi(x)\left(1-e\left(\frac{\xi \cdot x}{L}\right)\right) \mathrm{d}x\right| \lesssim \frac{|\xi|}{L}.$$

By inserting these two last bounds in (8-1), we obtain

$$|\hat{\mu}_2(\xi)| \lesssim \min\left(1, \frac{|\xi|}{L}\right)|\hat{\mu}(\xi)|.$$

Consequently, by (3-2) and (3-3) we have

$$|\hat{\mu}_2(\xi)| \lesssim (n-\alpha)^{-O(1)} \min\left(1, \frac{|\xi|}{L}\right) \min(1, |\xi|^{-\beta/2}).$$

By considering separately the ranges $|\xi| \geqslant L^{2/(2+\beta)}$ and $|\xi| \leqslant L^{2/(2+\beta)}$, we find that

$$|\hat{\mu}_2(\xi)| \lesssim (n-\alpha)^{-O(1)} L^{-\beta/(2+\beta)}.$$

Recalling our choice of L, we have

$$|\hat{\mu}_2(\xi)| \lesssim (n-\alpha)^{-O(1)} e^{-\beta/(2+\beta)(n-\alpha)}. \qquad \square$$

We now establish the positivity of $\Lambda(\mu, \dots, \mu)$, using the previous decomposition, with the main contribution from the absolutely continuous part estimated by Proposition 3.8, and the other contributions bounded away by Proposition 5.6.

Proof of Proposition 3.9. We consider the decomposition $\mu = \mu_1 + \mu_2$ from Proposition 8.1, and expand by multilinearity in

$$\Lambda(\mu, \dots, \mu) = C_1^{-(k+1)} \Lambda(\mu_1/C_1, \dots, \mu_1/C_1) + O\Big(\sum \Lambda(*, \dots, \mu_2, \dots, *)\Big),$$

where the sum is over $2^{k+1}-1$ terms and the stars denote measures equal to either μ_1 or μ_2. By Proposition 3.8, we deduce that, for a certain constant $c > 0$, we have

$$\Lambda(\mu, \dots, \mu) \geqslant c - O\Big(\sum \Lambda(*, \dots, \mu_2, \dots, *)\Big).$$

By Proposition 4.2, we have, furthermore, for any $\varepsilon \in (0, 1)$,

$$\Lambda(\mu, \dots, \mu) \geqslant c - O\big(\|\hat{\mu}_2\|_\infty^\varepsilon \Lambda^*(|\hat{\mu}|^{1-\varepsilon}, \dots, |\hat{\mu}|^{1-\varepsilon}; |J|)\big).$$

By taking ε to be that appearing in Proposition 5.6, and inserting the Fourier bound on μ_2 from Proposition 8.1, we find that

$$\Lambda(\mu, \dots, \mu) \geqslant c - O_{\beta_0}\big((n-\alpha)^{-O(1)} e^{-\varepsilon \cdot \beta_0/(2+\beta_0)(n-\alpha)}\big),$$

where we used the monotonicity of $x/(2+x)$. This can be made positive for $\alpha \geqslant n - c(\beta_0, \varepsilon)$ with $c(\beta_0, \varepsilon) > 0$ small enough. $\square$

9. Revisiting the linear case

In this section we indicate how the method of this article may be modified to obtain the following extension of Theorem 1.2, which allows for any positive exponent of Fourier decay for the fractal measure. For simplicity we only treat the case where n divides m, which already covers all the geometric applications discussed in [Chan et al. 2016].

Theorem 9.1. *Let $n, k, m \geqslant 1$, $D \geqslant 1$ and $\alpha, \beta \in (0, n)$. Suppose that E is a compact subset of $\mathbb{R}^n$ and μ is a probability measure supported on E such that*

$$\mu(B(x,r)) \leqslant Dr^{\alpha} \quad \text{and} \quad |\hat{\mu}(\xi)| \leqslant D(n-\alpha)^{-D}(1+|\xi|)^{-\beta/2}$$

for all $x \in \mathbb{R}^n$, $r > 0$ and $\xi \in \mathbb{R}^n$. Suppose that $(A_1, \dots, A_k)$ is a nondegenerate system of $n \times m$ matrices in the sense of Definition 1.1. Assume finally that $m = (k-r)n$ with $1 \leqslant r < k$ and, for some $\beta_0 \in (0, n)$,

$$\tfrac{1}{2}(k-1)n < m < kn, \quad \beta_0 \leqslant \beta < n, \quad n - c_{n,k,m,\beta_0,D,(A_i)} \leqslant \alpha < n$$

for a sufficiently small constant $c_{n,k,m,\beta_0,D,(A_i)} > 0$. Then, for every collection of strict subspaces $V_1, \dots, V_q$ of $\mathbb{R}^{n+m}$, there exists $(x, y) \in \mathbb{R}^{n+m} \smallsetminus V_1 \cup \dots \cup V_q$ such that

$$(x,\ x + A_1 y, \dots, x + A_k y) \in E^{k+1}.$$

Note that the condition on m is equivalent to that of Theorem 1.2. We only sketch the proof of Theorem 9.1, since it follows by a straightforward adaption of the methods of this paper, with the only difference lying in the treatment of the singular integral.

We start by stating a slight generalization of Hölder's inequality that was already used (for $l = k+1$, $r = k$) in the proof of Proposition 5.4. We write $\binom{[l]}{r}$ for the set of subsets of $[l]$ of size r.

Proposition 9.2. *Let $(X, \mathfrak{M}, \lambda)$ be a measure space and let $1 \leqslant r \leqslant l$. For measurable functions $F_1, \dots, F_l : X \to \mathbb{C}$, we have*

$$\int_X \prod_{j \in [l]} |F_j| \, \mathrm{d}\lambda \leqslant \prod_{S \in \binom{[l]}{r}} \left[\int_X \prod_{j \in S} |F_j|^{l/r} \, \mathrm{d}\lambda \right]^{1/\binom{l}{r}}.$$

Proof. First observe that, for arbitrary real numbers $a_1, \dots, a_l \geqslant 0$, we have

$$\prod_{j \in [l]} a_j = \prod_{S \in \binom{[l]}{r}} \left(\prod_{j \in S} a_j \right)^{1/\binom{l-1}{r-1}}.$$

Next, let $I = \int_X \prod_{j \in [l]} |F_j| \, \mathrm{d}\lambda$ and apply Hölder's inequality in

$$I = \int_X \prod_{S \in \binom{[l]}{r}} \left(\prod_{j \in S} |F_j| \right)^{1/\binom{l-1}{r-1}} \mathrm{d}\lambda \leqslant \prod_{S \in \binom{[l]}{r}} \left[\int_X \left(\prod_{j \in S} |F_j| \right)^{\binom{l}{r}/\binom{l-1}{r-1}} \mathrm{d}\lambda \right]^{1/\binom{l}{r}}.$$

A quick computation shows that $\binom{l}{r} / \binom{l-1}{r-1} = l/r$, which concludes the proof. ☐

We now place ourselves under the assumptions of Theorem 9.1, and in particular we assume that the matrices $A_1, \dots, A_k$ are nondegenerate in the sense of Definition 1.1. We also write $A_0 = 0_{n\times n}$ throughout. This matches the framework of this paper except that now $Q = 0$.

We fix a smooth cutoff $\psi \in \mathscr{C}^\infty_{c,+}(\mathbb{R}^n)$ which is at least 1 on a box $[-c, c]^n$. We define the oscillatory integral

$$J(\boldsymbol{\xi}) = \int_{\mathbb{R}^n} e(\boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}\cdot y)\psi(y)\,\mathrm{d}y = \hat{\psi}(-\boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}). \tag{9-1}$$

The counting operators are now defined by[4]

$$\begin{aligned}\Lambda(f_0, \dots, f_k) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^m} f_0(x) f_1(x + A_1 y)\cdots f_k(x + A_k y)\,\mathrm{d}x\ \ \psi(y)\,\mathrm{d}y,\\ \Lambda^*(F_0, \dots, F_k; J) &= \int_{(\mathbb{R}^n)^k} F_0(-\xi_1 - \cdots - \xi_k)F_1(\xi_1)\cdots F_k(\xi_k)J(\boldsymbol{\xi})\,\mathrm{d}\boldsymbol{\xi},\end{aligned}$$

for functions $f_i, F_i \in \mathscr{S}(\mathbb{R}^n)$, and we have $\Lambda(f_0, \dots, f_k) = \Lambda^*(\hat{f}_0, \dots, \hat{f}_k; J)$ as before.

Since we assumed that $\psi \in \mathscr{C}^\infty_c(\mathbb{R}^m)$, it follows from (9-1) that

$$|J(\boldsymbol{\xi})| \lesssim_N (1 + |\boldsymbol{A}^\mathsf{T}\boldsymbol{\xi}|)^{-N} \tag{9-2}$$

for every $N > 0$. Via some matricial considerations (as in [Chan et al. 2016, Lemma 3.2]), it can be checked that Definition 1.1 is equivalent to the requirement that $\boldsymbol{A}^\mathsf{T} : \mathbb{R}^{kn} \to \mathbb{R}^{kn-rn}$ is injective on each subspace of the form

$$\{\boldsymbol{\xi} \in (\mathbb{R}^n)^k : (\xi_j)_{j\in S} = \boldsymbol{\eta}\},$$

where S is a subset of $\{0, \dots, k\}$ of size r and $\boldsymbol{\eta} \in \mathbb{R}^{rn}$, and we write $\xi_0 = -(\xi_1 + \cdots + \xi_k)$ as before. Now consider an arbitrary subset S of $\{0, \dots, k\}$ of size r. By (9-2) one quickly deduces that

$$\int_{(\xi_j)_{j\in S}=\boldsymbol{\eta}} |J(\boldsymbol{\xi})|^q\,\mathrm{d}\sigma(\boldsymbol{\xi}) \lesssim_q 1 \quad (q > 0,\ \boldsymbol{\eta} \in (\mathbb{R}^n)^r), \tag{9-3}$$

in the same manner as in the proof of Proposition 5.3.

In our linear setting one may naturally obtain a better range of m for which the multilinear form Λ^* is controlled by L^s norms. The next proposition demonstrates this, and it is applicable to our problem only when $(k+1)/r > 2$, or equivalently $m = (k-r)n > \frac{1}{2}(k-1)n$.

Proposition 9.3. *We have*

$$|\Lambda^*(F_0, \dots, F_k; J)| \lesssim \|F_0\|_{(k+1)/r}\cdots\|F_k\|_{(k+1)/r}.$$

[4] In fact, one could work without cutoff functions in the y variable, as was done in [Chan et al. 2016], which simplifies the estimates somewhat. Here we keep smooth cutoffs to stay closer to the framework of the article.

Proof. Write $I=\Lambda^*(F_0,\dots,F_k;J)$ and $[0,k]=\{0,\dots,k\}$ for the purpose of this proof. By Proposition 9.2, we have

$$I \leqslant \int_{(\mathbb{R}^n)^k} \prod_{j=0}^{k} (F_j(\xi_j)\cdot|J(\boldsymbol{\xi})|^{1/(k+1)})\,\mathrm{d}\boldsymbol{\xi} \leqslant \prod_{S\in\binom{[0,k]}{r}} \left(\int_{(\mathbb{R}^n)^k} \prod_{j\in S} F_j(\xi_j)^{(k+1)/r}|J(\boldsymbol{\xi})|^{1/r}\,\mathrm{d}\boldsymbol{\xi}\right)^{1/\binom{k+1}{r}}.$$

Integrating along slices, and invoking (9-3), we obtain

$$\begin{aligned} I &\leqslant \prod_{S\in\binom{[0,k]}{r}} \left(\int_{(\mathbb{R}^n)^r} \prod_{j\in S} F_j(\eta_j)^{(k+1)/r}\left(\int_{(\xi_j)_{j\in S}=\boldsymbol{\eta}} |J(\boldsymbol{\xi})|^{1/r}\,\mathrm{d}\sigma(\boldsymbol{\xi})\right)\mathrm{d}\boldsymbol{\eta}\right)^{1/\binom{k+1}{r}} \\ &\lesssim \prod_{S\in\binom{[0,k]}{r}} \left(\int_{(\mathbb{R}^n)^r} \prod_{j\in S} F_j(\eta_j)^{(k+1)/r}\,\mathrm{d}\boldsymbol{\eta}\right)^{1/\binom{k+1}{r}}. \end{aligned}$$

Therefore each inner integral splits and we have

$$I \lesssim \prod_{S\in\binom{[0,k]}{r}} \left(\prod_{j\in S}\int_{\mathbb{R}^n} F_j(\eta)^{(k+1)/r}\,\mathrm{d}\boldsymbol{\eta}\right)^{1/\binom{k+1}{r}} = \prod_{j\in[0,k]} \left(\int_{\mathbb{R}^n} F_j(\eta)^{(k+1)/r}\,\mathrm{d}\boldsymbol{\eta}\right)^{\binom{k}{r-1}/\binom{k+1}{r}}.$$

Since $\binom{k+1}{r}/\binom{k}{r-1}=(k+1)/r$, it follows that $I \leqslant \prod_{j\in[0,k]}\|F_j\|_{(k+1)/r}$, as was to be shown. □

With Proposition 9.3 in hand, it is a simple matter to adapt the rest of the argument in this paper. In fact, one would need a slight variant of that proposition involving a shift θ, as in the case of Proposition 5.2. From such a proposition one may deduce the natural analogues of Propositions 5.6 and 3.6, which will impose the same conditions on α and β, and a distinct condition $m>\frac{1}{2}(k-1)n$ on m. With these singular integral bounds in hand, the arguments of Sections 6–8 go through essentially unchanged, and one obtains Theorem 9.1 by the process described at the end of Section 3.

Appendix A: The arithmetic regularity lemma

In this section, we derive a version of the U^2 arithmetic regularity lemma, following Tao's argument [2014], with minor twists to accommodate functions defined over $\mathbb{R}^n$ instead of $\mathbb{T}^n$. This set of ideas itself originates in [Bourgain 1986], albeit in a rather different language. We include the complete proof since the exact result we need is not stated in a convenient form in the literature.

We defined a Bohr set of $\mathbb{T}^n$ of a frequency set $\Gamma\subset\mathbb{Z}^n$, radius $\delta\in\left(0,\frac{1}{2}\right]$ and dimension $d=|\Gamma|<\infty$ in (7-1). We define the dilate of a Bohr set B of a frequency set Γ and radius δ by a factor $\rho\in(0,1]$ as $B(\Gamma,\delta)_\rho=B(\Gamma,\rho\delta)$. Note that $B(\Gamma,\delta)=\phi^{-1}(2\delta\cdot Q)$ for the cube $Q=\left[-\frac{1}{2},\frac{1}{2}\right]^n$ and the morphism $\phi:\mathbb{R}^n\to\mathbb{T}^d$, $x\mapsto(\xi\cdot x)_{\xi\in\Gamma}$. We can find a cube covering of the form $Q\subset\bigcup_{t\in T}(t+\delta\cdot Q)$ with $|T|=\lceil 1/\delta\rceil^d\leqslant(2/\delta)^d$, and therefore

$$1=|\phi^{-1}(Q)|\leqslant\sum_{t\in T}|\phi^{-1}(t+\delta\cdot Q)|.$$

By the pigeonhole principle, there exists $t \in T$ such that $|\phi^{-1}(t+\delta\cdot Q)| \geqslant \left(\tfrac{1}{2}\delta\right)^d$ and, since $\phi^{-1}(t+\delta\cdot Q) - \phi^{-1}(t+\delta\cdot Q) \subset B$, we deduce that

$$|B| = |B(\Gamma,\delta)| \geqslant \left(\tfrac{1}{2}\delta\right)^d \quad \text{for all } \delta \in \left(0, \tfrac{1}{2}\right]. \tag{A-1}$$

Now consider the tent function $\Delta(x) = (1-|x|)^+$ on $\mathbb{R}$, which is 1-Lipschitz, bounded by 1 everywhere, and bounded from below by $\frac{1}{2}$ on $\left[-\frac{1}{2}, \frac{1}{2}\right]$. For any Bohr set B, we define functions $\phi_B, \nu_B : \mathbb{T}^n \to \mathbb{C}$ by

$$\phi_B(x) = \Delta\Big(\frac{1}{\delta}\sup_{\xi\in\Gamma}\|\xi\cdot x\|\Big), \qquad \nu_B = \frac{\phi_B}{\int \phi_B},$$

so that $\int \nu_B = 1$ and $\frac{1}{2}1_{B_{1/2}} \leqslant \nu_B \leqslant 1_B$. The function ν_B is essentially a smoothed normalized indicator function of the Bohr set B, and its most important properties are summarized in the following proposition:

Proposition A.1. *For any Bohr set B of frequency set $\Gamma \subset \mathbb{Z}^n$ and radius $\delta \in \left(0, \frac{1}{2}\right]$, we have*

$$\|\nu_B\|_\infty \lesssim \left(\tfrac{1}{4}\delta\right)^{-d}, \tag{A-2}$$

$$\|T^t\nu_B - \nu_B\|_\infty \lesssim \left(\tfrac{1}{4}\delta\right)^{-d}\rho \quad \textit{for } t \in B_\rho,\ \rho \in (0,1], \tag{A-3}$$

$$\hat{\nu}_B(\xi) = 1 + O(\delta) \quad \textit{for } \xi \in \Gamma. \tag{A-4}$$

Proof. Note that $\int \phi_B \geqslant \frac{1}{2}|B_{1/2}| \geqslant \frac{1}{2}\left(\frac{1}{4}\delta\right)^d$ by (A-1), which implies the first estimate. For every $x, t \in \mathbb{T}^n$, we also have

$$|\nu_B(x+t) - \nu_B(x)| \leqslant 2\left(\tfrac{1}{4}\delta\right)^{-d}\Big|\Delta\Big(\frac{1}{\delta}\sup_{\xi\in\Gamma}\|\xi\cdot(x+t)\|\Big) - \Delta\Big(\frac{1}{\delta}\sup_{\xi\in\Gamma}\|\xi\cdot x\|\Big)\Big|.$$

When $t \in B_\rho$, we have $\|\xi\cdot t\| \leqslant \rho\delta$ for every $\xi \in \Gamma$, and therefore $|\nu_B(x+t) - \nu_B(x)| \lesssim \left(\frac{1}{4}\delta\right)^{-d}\rho$ since Δ is 1-Lipschitz, and we have established the second estimate. To obtain the third, consider $\xi \in \Gamma$ and observe that, since ν_B is supported on B and $\|\xi\cdot x\| \leqslant \delta$ for $x \in B$, we have

$$\hat{\nu}_B(\xi) = \int_B \nu_B(x) e(-\xi\cdot x)\,\mathrm{d}x = (1+O(\delta))\int \nu_B = 1 + O(\delta). \qquad \square$$

Proposition A.2. *Let $\varepsilon \in (0,1]$ be a parameter and let $\kappa : (0,1]^3 \to (0,1]$ be a decay function. Suppose that $f \in \mathscr{C}_c^\infty(\mathbb{R}^n)$ is such that $0 \leqslant f \leqslant 1$ and $\operatorname{Supp} f \subset \left[-\frac{1}{8}, \frac{1}{8}\right]^n$. Then there exists a decomposition $f = f_1 + f_2 + f_3$ with $f_i \in \mathscr{C}_c^\infty(\mathbb{R}^n)$, $\operatorname{Supp} f_i \subset \left[-\frac{1}{4}, \frac{1}{4}\right]^n$, $\|f_i\|_\infty \leqslant 1$, $f_1 \geqslant 0$, $f_1 + f_2 \geqslant 0$, $\int f_1 = \int f$ as well as a Bohr set B of dimension $d \lesssim_{\varepsilon,\kappa} 1$ and radius $\delta \gtrsim_{\varepsilon,\kappa} 1$ such that*

$$\|T^t f_1 - f_1\|_\infty \leqslant \varepsilon \quad \textit{for all } t \in B, \qquad \|f_2\|_2 \leqslant \varepsilon, \qquad \|\hat{f}_3\|_{L^\infty(\mathbb{R}^n)} \leqslant \kappa(\varepsilon, d^{-1}, \delta). \tag{A-5}$$

Proof. We initially consider f as defined on the torus $\mathbb{T}^n$, by identification with its 1-periodization from the cube $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Consider sequences of positive real numbers

$$\tfrac{1}{2} \geqslant \delta_0 \geqslant \delta_1 \geqslant \cdots \geqslant \delta_i \geqslant \cdots \quad \text{and} \quad 1 \geqslant \eta_1 \geqslant \cdots \geqslant \eta_i \geqslant \cdots$$

to be determined later. We define sequences of frequency sets Γ_i and Bohr sets B_i of dimension d_i, and measures ν_i, inductively for $i \geqslant 0$ by

$$\Gamma_{i+1} = \Gamma_i \cup \{|\hat{f}| \geqslant \eta_{i+1}\} \cup \bigcup_{j=0}^{i} \{|\hat{\nu}_j| \geqslant \eta_{i+1}\}, \quad B_{i+1} = B(\Gamma_{i+1}, \delta_{i+1}), \quad \nu_{i+1} = \nu_{B_{i+1}}. \tag{A-6}$$

We initialize with $\Gamma_0 = \{e_1, \dots, e_n\}$, $\delta_0 \leqslant \frac{1}{8}$, $B_0 = B(\Gamma_0, \delta_0)$ and $\nu_0 = \nu_{B_0}$, so that $d_0 = n$ and, by the definition (7-1) of Bohr sets, we have $B_i \subset \left[-\frac{1}{8}, \frac{1}{8}\right]^n$ for all i. Note that, by Chebyshev, we also have a dimension bound

$$d_{i+1} \leqslant d_i + \frac{\|\hat{f}\|_2^2}{\eta_{i+1}^2} + \sum_{j=0}^{i} \frac{\|\hat{\nu}_j\|_2^2}{\eta_{i+1}^2}.$$

By Plancherel and the bound (A-2), it follows that

$$d_i \lesssim_{\delta_0, \dots, \delta_{i-1}, d_{i-1}, \eta_i} 1 \quad (i \geqslant 1). \tag{A-7}$$

We start by finding a piece of the Fourier expansion of f which is small in L^2. To this end observe that

$$\sum_{i=0}^{k} \sum_{\Gamma_{i+2} \smallsetminus \Gamma_i} |\hat{f}|^2 \leqslant 2\|\hat{f}\|_2^2 = 2\|f\|_2^2 \leqslant 2.$$

By Chebyshev's bound, it follows that

$$\#\left\{0 \leqslant i \leqslant k : \sum_{\Gamma_{i+2} \smallsetminus \Gamma_i} |\hat{f}|^2 \geqslant \frac{\varepsilon^2}{2}\right\} \leqslant \frac{4}{\varepsilon^2}.$$

Choosing $k = \lceil 4/\varepsilon^2 \rceil$, we obtain the existence of an index $0 \leqslant i \leqslant k$ such that

$$\sum_{\Gamma_{i+2} \smallsetminus \Gamma_i} |\hat{f}|^2 \leqslant \tfrac{1}{2}\varepsilon^2. \tag{A-8}$$

We now decompose f into three pieces $f_1, f_2, f_3 : \mathbb{T}^n \to \mathbb{C}$ defined by

$$f = f * \nu_i + (f * \nu_{i+1} - f * \nu_i) + (f - f * \nu_{i+1}) = f_1 + f_2 + f_3.$$

Since f takes values in $[0, 1]$ and $\int \nu_i = 1$, the functions f_1, f_2, f_3 take values in $[-1, 1]$ by simple convolution bounds. It is also clear that f_1 and $f_1 + f_2$ are nonnegative and $\int f_1 = \int f$.

Let us first analyze the L^2-small piece. By Plancherel and (A-8), we have

$$\begin{aligned} \|f * \nu_{i+1} - f * \nu_i\|_2^2 &= \sum_{m \in \mathbb{Z}^n} |\hat{f}(m)|^2 |\hat{\nu}_{i+1}(m) - \hat{\nu}_i(m)|^2 \\ &\leqslant \frac{\varepsilon^2}{2} + \sum_{m \in \Gamma_i \cup (\mathbb{Z}^n \smallsetminus \Gamma_{i+2})} |\hat{f}(m)|^2 |\hat{\nu}_{i+1}(m) - \hat{\nu}_i(m)|^2. \end{aligned} \tag{A-9}$$

For $m \in \Gamma_i \subset \Gamma_{i+1}$, by (A-4) we have $|\hat{\nu}_{i+1}(m) - \hat{\nu}_i(m)| \lesssim \delta_{i+1} + \delta_i$. For $m \notin \Gamma_{i+2}$, the definition (A-6) of Γ_{i+2} implies that $|\hat{\nu}_i(m)| \leqslant \eta_{i+2}$ and $|\hat{\nu}_{i+1}(m)| \leqslant \eta_{i+2}$. Inserting these bounds into (A-9), we obtain

$$\|f * \nu_{i+1} - f * \nu_i\|_2^2 \leqslant \tfrac{1}{2}\varepsilon^2 + O(\delta_i + \delta_{i+1} + \eta_{i+2}) \leqslant \varepsilon^2, \tag{A-10}$$

provided that $\delta_j, \eta_j \leqslant c\varepsilon^2$ for all j.

Next, let us focus on the almost-periodic piece. Introducing a parameter $\rho_i \in (0, 1]$, we deduce from (A-3) that, for $t \in B_{\rho_i}$, we have

$$\|T^t f * \nu_i - f * \nu_i\|_\infty \leqslant \|f\|_1 \|T^t \nu_i - \nu_i\|_\infty \lesssim_n \delta_i^{-d_i} \rho_i \leqslant \varepsilon, \tag{A-11}$$

choosing $\rho_i = c_n \varepsilon \delta_i^{d_i}$. We write $\tilde{\delta}_i = \rho_i \delta_i$, and from (A-7) we see that $\tilde{\delta}_i$ depends at most on $n, \varepsilon, \delta_0, \dots, \delta_i$ and $\eta_1, \dots, \eta_i$.

Finally, we consider the Fourier-small piece. By Fourier inversion,

$$\|(f - f * \nu_{i+1})^\wedge\|_{l^\infty(\mathbb{Z}^n)} = \sup_{m \in \mathbb{Z}^n} |\hat{f}(m)||1 - \hat{\nu}_{i+1}(m)|.$$

For $m \in \Gamma_{i+1}$, we have $|1 - \hat{\nu}_{i+1}(m)| \lesssim \delta_{i+1}$ by (A-4), while for $m \notin \Gamma_{i+1}$, the definition (A-6) of Γ_{i+1} shows that $|\hat{f}(m)| \leqslant \eta_{i+1}$. Therefore

$$\|(f - f * \nu_{i+1})^\wedge\|_{l^\infty(\mathbb{Z}^n)} \lesssim \delta_{i+1} + \eta_{i+1} \leqslant c\kappa(\varepsilon, d_i^{-1}, \tilde{\delta}_i) \tag{A-12}$$

for a small constant $c > 0$ provided that we choose the δ_j and η_j recursively satisfying

$$\max(\delta_{i+1}, \eta_{i+1}) = c \min(\kappa(\varepsilon, d_i^{-1}, \tilde{\delta}_i), \varepsilon^2).$$

At this stage we have obtained the desired bounds (A-5) over $\mathbb{T}^n$ and for a Bohr set $\widetilde{B}_i = B_i(\Gamma_i, \tilde{\delta}_i)$, and from (A-7) and the construction of the δ_i it follows that $d_i \lesssim_{\varepsilon,\kappa} 1$ and $\delta_i \gtrsim_{\varepsilon,\kappa} 1$.

To finish the proof we now consider the functions f_1, f_2, f_3 as functions on $\mathbb{R}^n$ supported on $\left[-\frac{1}{2}, \frac{1}{2}\right]^n$. Since f and the Bohr sets measures ν_i are supported on $\left[-\frac{1}{8}, \frac{1}{8}\right]^n$, the convolutions $f * \nu_i$ over $\mathbb{T}^n$ may be readily interpreted as convolutions over $\mathbb{R}^n$, and the functions f_i are supported on $\left[-\frac{1}{4}, \frac{1}{4}\right]^n$. The properties (A-10) and (A-11) are readily viewed as holding over $\mathbb{R}^n$, thus we only need to verify that f_3 has the appropriate Fourier decay at real frequencies. We claim that, since f_3 has support in $\left[-\frac{1}{4}, \frac{1}{4}\right]^n$, we have $\|\hat{f}_3\|_{L^\infty(\mathbb{R}^n)} \lesssim \|\hat{f}_3\|_{l^\infty(\mathbb{Z}^n)}$ and, by taking the constant c in (A-12) small enough, we obtain the desired Fourier decay estimate. To prove this claim, consider a smooth bump function χ equal to 1 on $\left[-\frac{1}{4}, \frac{1}{4}\right]^n$. For $\xi \in \mathbb{R}^n$, expanding f as a Fourier series yields

$$\hat{f}_3(\xi) = \int_{[-1/4,1/4]^n} f_3(x)\chi(x)e(-\xi \cdot x)\,\mathrm{d}x = \sum_{k \in \mathbb{Z}^n} \hat{f}_3(k) \int_{\mathbb{R}^n} \chi(x)e((k - \xi) \cdot x)\,\mathrm{d}x = \sum_{k \in \mathbb{Z}^n} \hat{f}_3(k)\hat{\chi}(\xi - k).$$

Using the smoothness of χ, it follows that, uniformly in $\xi \in \mathbb{R}^n$,

$$|\hat{f}_3(\xi)| \lesssim \|\hat{f}_3\|_{l^\infty(\mathbb{Z}^n)} \sum_{k \in \mathbb{Z}^n} (1 + |\xi - k|)^{-(n+1)} \lesssim \|\hat{f}_3\|_{l^\infty(\mathbb{Z}^n)}. \qquad \square$$

Appendix B: Uniform restriction estimates for fractal measures

In this section we obtain restriction estimates for fractal measures satisfying dimensionality and Fourier decay conditions, with uniformity in all the parameters involved. We liberate μ, α and β from their usual meaning and track dependencies on all parameters, such as the dimension n. To facilitate our quoting of the literature, we first recall the functional equivalences in Tomas's T^*T argument [Wolff 2003, Chapter 7].

Fact B.1. *Suppose that $\mu \in \mathcal{M}^+(\mathbb{R}^n)$ and $p \in (1, +\infty]$, and that p' is given by $\frac{1}{p} + \frac{1}{p'} = 1$. Let $R > 0$. The following statements are equivalent*:

$$\|\hat{f}\|_{L^2(\mathrm{d}\mu)} \leqslant R\|f\|_{L^{p'}(\mathbb{R}^n)} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^n), \tag{B-1}$$

$$\|\widehat{g\,\mathrm{d}\mu}\|_{L^p(\mathbb{R}^n)} \leqslant R\|g\|_{L^2(\mathrm{d}\mu)} \quad \text{for all } g \in L^2(\mathrm{d}\mu). \tag{B-2}$$

We now fix two exponents $0 < \beta \leqslant \alpha \leqslant n$ and two constants $A, B \geqslant 1$, and we restrict our attention to probability measures μ on $\mathbb{R}^n$ satisfying

$$\mu(B(x, r)) \leqslant Ar^\alpha \qquad (x \in \mathbb{R}^n, r > 0), \tag{B-3}$$

$$|\hat{\mu}(\xi)| \leqslant B(1 + |\xi|)^{-\beta/2} \quad (\xi \in \mathbb{R}^n). \tag{B-4}$$

We define the critical exponent

$$p_0 = 2 + \frac{4(n-\alpha)}{\beta}, \tag{B-5}$$

so that the Mitsis–Mockenhaupt restriction theorem [Mitsis 2002; Mockenhaupt 2000] states that each of the inequalities in Fact B.1 holds for $p > p_0$ for a certain constant $R = R(A, B, \alpha, \beta, p, n)$. We wish to use (B-2) with $g \equiv 1$ and $p = 2 + \delta$ with a fixed small $\delta > 0$, which is possible when α is close enough to n by (B-5), but to be useful this requires some uniformity in α. The constants in [Mitsis 2002; Mockenhaupt 2000] can be given explicit expressions in terms of the parameters involved, and in fact one could likely adapt the version of Mockenhaupt's argument in [Łaba and Pramanik 2009, Proposition 4.1], to relax the condition $\beta > \frac{2}{3}$ there to $\beta > 0$. We provide instead a direct derivation from the estimate of [Bak and Seeger 2011], which includes explicit constants.

Proposition B.2. *Let $\beta_0 \in (0, n)$. There exists $C_{n,\beta_0} > 0$ such that, when $\beta \geqslant \beta_0$, the estimate* (B-1) *holds for $p \geqslant p_0$ with $R = C_{n,\beta_0} \max(A, B)^{p_0/2p}$.*

Proof. Apply [Bak and Seeger 2011, Equation (1.5)], replacing $a \leftarrow \alpha,\ b \leftarrow \frac{1}{2}\beta,\ d \leftarrow n,\ p \leftarrow p'$, so that $q = 2p/p_0$; and note that α and β belong to the compact interval $[\beta_0, n]$. Since $q \geqslant 2$ for $p \geqslant p_0$, by nesting of $L^s(\mathrm{d}\mu)$ norms this yields

$$\begin{aligned}\|\hat{f}\|_{L^2(\mathrm{d}\mu)} \leqslant \|\hat{f}\|_{L^q(\mathrm{d}\mu)} &\leqslant (C_{n,\beta_0})^{2/q} A^{1/q\cdot 2/p_0} B^{1/q\cdot(1-2/p_0)}\|f\|_{L^{p'}(\mathbb{R}^n)} \\ &\leqslant C_{n,\beta_0}\max(A, B)^{p_0/(2p)}\|f\|_{L^{p'}(\mathbb{R}^n)}.\end{aligned}$$

□

Alternatively, one may choose to track down the dependencies on constants in Mitsis's simpler argument [2002], which would lead to a similar estimate for the constant R in (B-1), up to a harmless (for our argument) factor $(p - p_0)^{-1}$. Via Proposition B.2, it is now possible to bound the moments of $\hat{\mu}$ of order slightly larger than 2 when α is close enough to n, with only a moderate dependency of constants on α.

Proposition B.3. *Let $\delta \in (0, 1)$ and $\beta_0 \in (0, n)$. Suppose that μ is a probability measure satisfying* (3-1) *and* (3-2). *Then, uniformly for $n - \frac{1}{4}\delta\beta_0 \leqslant \alpha < n$ and $\beta_0 \leqslant \beta < n$, we have*

$$\|\hat{\mu}\|_{2+\delta} \lesssim_{\beta_0,n} D_\alpha^{1/2}.$$

Proof. We consider the exponent $p = 2 + \delta$. Recalling (B-5), we have $p \geqslant p_0$ in the stated range of α. We can therefore invoke Proposition B.2 with $A = D \asymp 1$ and $B = D_\alpha$, so that the extension inequality (B-2) holds for $g \equiv 1$ with $R \lesssim_{\beta_0,n} D_\alpha^{1/2}$. □

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Received 25 Nov 2015. Revised 29 Mar 2016. Accepted 29 Apr 2016.

KEVIN HENRIOT: khenriot@math.ubc.ca
Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver BC V6T 1Z2, Canada

IZABELLA ŁABA: ilaba@math.ubc.ca
Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver BC V6T 1Z2, Canada

MALABIKA PRAMANIK: malabika@math.ubc.ca
Department of Mathematics, University of British Columbia, 1984 Mathematics Road, Vancouver BC V6T 1Z2, Canada

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1185

FREE PLURIHARMONIC FUNCTIONS ON NONCOMMUTATIVE POLYBALLS

GELU POPESCU

We study free k-pluriharmonic functions on the noncommutative regular polyball $\boldsymbol{B}_{\boldsymbol{n}}$, $\boldsymbol{n}=(n_1,\dots,n_k)\in\mathbb{N}^k$, which is an analogue of the scalar polyball $(\mathbb{C}^{n_1})_1\times\cdots\times(\mathbb{C}^{n_k})_1$. The regular polyball has a universal model $\boldsymbol{S}:=\{\boldsymbol{S}_{i,j}\}$ consisting of left creation operators acting on the tensor product $F^2(H_{n_1})\otimes\cdots\otimes F^2(H_{n_k})$ of full Fock spaces. We introduce the class $\boldsymbol{\mathcal{T}}_{\boldsymbol{n}}$ of k-multi-Toeplitz operators on this tensor product and prove that $\boldsymbol{\mathcal{T}}_{\boldsymbol{n}}=\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{\text{-SOT}}$, where $\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}$ is the noncommutative polyball algebra generated by $\boldsymbol{S}$ and the identity. We show that the bounded free k-pluriharmonic functions on $\boldsymbol{B}_{\boldsymbol{n}}$ are precisely the noncommutative Berezin transforms of k-multi-Toeplitz operators. The Dirichlet extension problem on regular polyballs is also solved. It is proved that a free k-pluriharmonic function has continuous extension to the closed polyball $\boldsymbol{B}_{\boldsymbol{n}}^-$ if and only if it is the noncommutative Berezin transform of a k-multi-Toeplitz operator in $\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{-\|\cdot\|}$.

We provide a Naimark-type dilation theorem for direct products $\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$ of unital free semigroups, and use it to obtain a structure theorem which characterizes the positive free k-pluriharmonic functions on the regular polyball with operator-valued coefficients. We define the noncommutative Berezin (resp. Poisson) transform of a completely bounded linear map on $C^*(\boldsymbol{S})$, the C^*-algebra generated by $\boldsymbol{S}_{i,j}$, and give necessary and sufficient conditions for a function to be the Poisson transform of a completely bounded (resp. completely positive) map. In the last section of the paper, we obtain Herglotz–Riesz representation theorems for free holomorphic functions on regular polyballs with positive real parts, extending the classical result as well as the Korányi–Pukánszky version in scalar polydisks.

Introduction

A multivariable operator model theory and a theory of free holomorphic functions on polydomains which admit universal operator models have been recently developed in [Popescu 2013; 2016]. An important feature of these theories is that they are related, via noncommutative Berezin transforms, to the study of

Research supported in part by NSF grant DMS 1500922.
MSC2010: primary 47A13, 47A56; secondary 47B35, 46L52.
Keywords: noncommutative polyball, Berezin transform, Poisson transform, Fock space, multi-Toeplitz operator, Naimark dilation, completely bounded map, pluriharmonic function, free holomorphic function, Herglotz–Riesz representation.

the operator algebras generated by the universal models as well as to the theory of functions in several complex variables. These results played a crucial role in our work on the curvature invariant [Popescu 2015a], the Euler characteristic [Popescu 2014], and the group of free holomorphic automorphisms on noncommutative regular polyballs [Popescu 2015b].

The main goal of the present paper is to continue our investigation along these lines and to study the class of free k-pluriharmonic functions of the form

$$F(\boldsymbol{X}) = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} a_{(\boldsymbol{\alpha};\boldsymbol{\beta})} \boldsymbol{X}_{1,\alpha_1}\cdots \boldsymbol{X}_{k,\alpha_k} \boldsymbol{X}_{1,\beta_1}^* \cdots \boldsymbol{X}_{k,\beta_k}^*, \quad a_{(\boldsymbol{\alpha};\boldsymbol{\beta})}\in\mathbb{C},$$

where the series converge in the operator norm topology for any $\boldsymbol{X} = \{X_{i,j}\}$ in the regular polyball $\boldsymbol{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. The results of this paper will play an important role in the hyperbolic geometry of noncommutative polyballs [Popescu ≥ 2016]. To present our results we need some notation and preliminaries on regular polyballs and their universal models.

Throughout this paper, $B(\mathcal{H})$ stands for the algebra of all bounded linear operators on a Hilbert space $\mathcal{H}$. We let $B(\mathcal{H})^{n_1} \times_c \cdots \times_c B(\mathcal{H})^{n_k}$, where $n_i \in \mathbb{N} := \{1, 2, \dots\}$, be the set of all tuples $\boldsymbol{X} := (X_1, \dots, X_k)$ in $B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ with the property that the entries of $X_s := (X_{s,1}, \dots, X_{s,n_s})$ commute with the entries of $X_t := (X_{t,1}, \dots, X_{t,n_t})$ for any $s, t \in \{1, \dots, k\}$, $s \neq t$. Note that the operators $X_{s,1}, \dots, X_{s,n_s}$ do not necessarily commute. Let $\boldsymbol{n} := (n_1, \dots, n_k)$ and define the polyball

$$\boldsymbol{P_n}(\mathcal{H}) := [B(\mathcal{H})^{n_1}]_1 \times_c \cdots \times_c [B(\mathcal{H})^{n_k}]_1,$$

where

$$[B(\mathcal{H})^n]_1 := \{(X_1, \dots, X_n) \in B(\mathcal{H})^n : \|X_1X_1^* + \cdots + X_nX_n^*\| < 1\}, \quad n \in \mathbb{N}.$$

If A is a positive invertible operator, we write $A > 0$. The *regular polyball* on the Hilbert space $\mathcal{H}$ is defined by

$$\boldsymbol{B_n}(\mathcal{H}) := \{\boldsymbol{X} \in \boldsymbol{P_n}(\mathcal{H}) : \boldsymbol{\Delta_X}(I) > 0\},$$

where the *defect mapping* $\boldsymbol{\Delta_X} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\boldsymbol{\Delta_X} := (\mathrm{id} - \Phi_{X_1}) \circ \cdots \circ (\mathrm{id} - \Phi_{X_k})$$

and $\Phi_{X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by

$$\Phi_{X_i}(Y) := \sum_{j=1}^{n_i} X_{i,j} Y X_{i,j}^*, \quad Y \in B(\mathcal{H}).$$

Note that if $k = 1$ then $\boldsymbol{B_n}(\mathcal{H})$ coincides with the noncommutative unit ball $[B(\mathcal{H})^{n_1}]_1$. We remark that the scalar representation of the (*abstract*) *regular polyball* $\boldsymbol{B_n} := \{\boldsymbol{B_n}(\mathcal{H}) : \mathcal{H} \text{ is a Hilbert space}\}$ is $\boldsymbol{B_n}(\mathbb{C}) = \boldsymbol{P_n}(\mathbb{C}) = (\mathbb{C}^{n_1})_1 \times \cdots \times (\mathbb{C}^{n_k})_1$.

Let H_{n_i} be an n_i-dimensional complex Hilbert space with orthonormal basis $e_1^i, \dots, e_{n_i}^i$. We consider the *full Fock space* of H_{n_i}, defined by $F^2(H_{n_i}) := \mathbb{C}1 \oplus \bigoplus_{p\geq 1} H_{n_i}^{\otimes p}$, where $H_{n_i}^{\otimes p}$ is the (Hilbert) tensor product of p copies of H_{n_i}. Let $\mathbb{F}_{n_i}^+$ be the unital free semigroup on n_i generators $g_1^i, \dots, g_{n_i}^i$ and the identity g_0^i. Set $e_\alpha^i := e_{j_1}^i \otimes \cdots \otimes e_{j_p}^i$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $e_{g_0^i}^i := 1 \in \mathbb{C}$. The length of $\alpha \in \mathbb{F}_{n_i}^+$ is defined by $|\alpha| := 0$ if $\alpha = g_0^i$ and $|\alpha| := p$ if $\alpha = g_{j_1}^i \cdots g_{j_p}^i$ with $j_1, \ldots, j_p \in \{1, \ldots, n_i\}$. We define the *left creation operator* $S_{i,j}$ acting on the Fock space $F^2(H_{n_i})$ by setting $S_{i,j} e_\alpha^i := e_{g_j^i \alpha}^i$, $\alpha \in \mathbb{F}_{n_i}^+$, and the operator $\boldsymbol{S}_{i,j}$ acting on the Hilbert tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ by setting

$$\boldsymbol{S}_{i,j} := \underbrace{I \otimes \cdots \otimes I}_{i-1 \text{ times}} \otimes S_{i,j} \otimes \underbrace{I \otimes \cdots \otimes I}_{k-i \text{ times}},$$

where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$. We define $\boldsymbol{S} := (\boldsymbol{S}_1, \ldots, \boldsymbol{S}_k)$, where $\boldsymbol{S}_i := (\boldsymbol{S}_{i,1}, \ldots, \boldsymbol{S}_{i,n_i})$, or write $\boldsymbol{S} := \{\boldsymbol{S}_{i,j}\}$. The noncommutative Hardy algebra $\boldsymbol{F}_{\boldsymbol{n}}^\infty$ (resp. the polyball algebra $\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}$) is the weakly closed (resp. norm closed) nonselfadjoint algebra generated by $\{\boldsymbol{S}_{i,j}\}$ and the identity. Similarly, we define the *right creation operator* $R_{i,j} : F^2(H_{n_i}) \to F^2(H_{n_i})$ by setting $R_{i,j} e_\alpha^i := e_{\alpha g_j^i}^i$ for $\alpha \in \mathbb{F}_{n_i}^+$, and the corresponding operator $\boldsymbol{R}_{i,j}$ acting on $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$. The polyball algebra $\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}$ is the norm closed nonselfadjoint algebra generated by $\{\boldsymbol{R}_{i,j}\}$ and the identity.

We proved in [Popescu 2016] (in a more general setting) that $\boldsymbol{X} \in B(\mathcal{H})^{n_1} \times \cdots \times B(\mathcal{H})^{n_k}$ is a *pure* element in the regular polyball $\boldsymbol{B}_{\boldsymbol{n}}(\mathcal{H})^-$, i.e., $\lim_{q_i \to \infty} \Phi_{X_i}^{q_i}(I) = 0$ in the weak operator topology, if and only if there is a Hilbert space $\mathcal{D}$ and a subspace $\mathcal{M} \subset F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \mathcal{D}$ invariant under each operator $\boldsymbol{S}_{i,j} \otimes I$ such that $X_{i,j}^* = (\boldsymbol{S}_{i,j}^* \otimes I)|_{\mathcal{M}^\perp}$, under an appropriate identification of $\mathcal{H}$ with $\mathcal{M}^\perp$. The k-tuple $\boldsymbol{S} := (\boldsymbol{S}_1, \ldots, \boldsymbol{S}_k)$, where $\boldsymbol{S}_i := (\boldsymbol{S}_{i,1}, \ldots, \boldsymbol{S}_{i,n_i})$, is an element in the regular polyball $\boldsymbol{B}_{\boldsymbol{n}}\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)^-$ and plays the role of *left universal model* for the abstract polyball $\boldsymbol{B}_{\boldsymbol{n}}^- := \{\boldsymbol{B}_{\boldsymbol{n}}(\mathcal{H})^- : \mathcal{H}$ is a Hilbert space$\}$. The existence of the universal model will play an important role in this paper, since it will make the connection between noncommutative function theory, operator algebras, and complex function theory in several variables.

Brown and Halmos [1963] showed that a bounded linear operator T on the Hardy space $H^2(\mathbb{D})$ is a Toeplitz operator if and only if $S^* T S = T$, where S is the unilateral shift. Expanding on this idea, a study of noncommutative multi-Toeplitz operators on the full Fock space with n generators $F^2(H_n)$ was initiated in [Popescu 1989; 1995] and has had an important impact in multivariable operator theory and the structure of free semigroup algebras (see [Davidson and Pitts 1998; Davidson et al. 2001; 2005; Popescu 2006; 2009; Kennedy 2011; 2013]).

In Section 1, we introduce and study the class $\boldsymbol{\mathcal{T}}_{\boldsymbol{n}}$, $\boldsymbol{n} := (n_1, \ldots, n_k) \in \mathbb{N}^k$, of k-multi-Toeplitz operators. A bounded linear operator T on the tensor product $F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k})$ of full Fock spaces is called a k-multi-Toeplitz operator with respect to the *right universal model* $\boldsymbol{R} = \{\boldsymbol{R}_{i,j}\}$ if

$$\boldsymbol{R}_{i,s}^* T \boldsymbol{R}_{i,t} = \delta_{st} T, \quad s, t \in \{1, \ldots, n_i\},$$

for every $i \in \{1, \ldots, k\}$. We associate with each k-multi-Toeplitz operator T a formal power series in several variables and show that we can recapture T from its noncommutative "Fourier series". Moreover, we characterize the noncommutative formal power series which are Fourier series of k-multi-Toeplitz operators (see Theorems 1.5 and 1.6). Using these results, we prove that the set of all k-multi-Toeplitz operators on $\bigotimes_{i=1}^k F^2(H_{n_i})$ coincides with

$$\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{\text{-SOT}} = \operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{\text{-WOT}},$$

where $\mathcal{A}_n$ is the noncommutative polyball algebra.

In Section 2, we characterize the bounded free k-pluriharmonic functions on regular polyballs. We prove that a function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a bounded free k-pluriharmonic function if and only if there is a k-multi-Toeplitz operator $A \in \mathcal{T}_n$ such that $F(\boldsymbol{X}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}[A]$ for $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, where $\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}$ is the noncommutative Berezin transform at $\boldsymbol{X}$ (see Section 1 for the definition). In this case, $A = \text{SOT-}\lim_{r\to 1} F(r\boldsymbol{S})$ and there is a completely isometric isomorphism of operator spaces

$$\Phi : \mathbf{PH}^\infty(\boldsymbol{B_n}) \to \mathcal{T}_n, \quad \Phi(F) := A,$$

where $\mathbf{PH}^\infty(\boldsymbol{B_n})$ is the operator space of all bounded free k-pluriharmonic functions on the polyball.

The Dirichlet extension problem [Hoffman 1962] on noncommutative regular polyballs is solved. We show that a mapping $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a free k-pluriharmonic function which has continuous extension (in the operator norm topology) to the closed polyball $\boldsymbol{B_n}(\mathcal{H})^-$, and write $F \in \mathbf{PH}^c(\boldsymbol{B_n})$, if and only if there exists a k-multi-Toeplitz operator $A \in \overline{\operatorname{span}}\{\mathcal{A}_n^* \mathcal{A}_n\}^{\|\cdot\|}$ such that $F(\boldsymbol{X}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}[A]$ for $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. In this case, $A = \lim_{r\to 1} F(r\boldsymbol{S})$, where the convergence is in the operator norm, and the map

$$\Phi : \mathbf{PH}^c(\boldsymbol{B_n}) \to \overline{\operatorname{span}}\{\mathcal{A}_n^* \mathcal{A}_n\}^{\|\cdot\|}, \quad \Phi(F) := A,$$

is a completely isometric isomorphism of operator spaces.

In Section 3, we provide a Naimark-type dilation theorem [1943] for direct products $\boldsymbol{F_n^+} := \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ of free semigroups. We show that a map $K : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{E})$ is a positive semidefinite left k-multi-Toeplitz kernel on $\boldsymbol{F_n^+}$ if and only if there exists a k-tuple of *commuting row isometries* $\boldsymbol{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ — i.e., the nonselfadjoint algebra $\operatorname{Alg}(V_i)$ commutes with $\operatorname{Alg}(V_s)$ for any $i, s \in \{1, \ldots, k\}$ with $i \neq s$ — such that

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = P_{\mathcal{E}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}\big|_{\mathcal{E}}, \quad \boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F_n^+},$$

and $\mathcal{K} = \bigvee_{\boldsymbol{\omega} \in \boldsymbol{F_n^+}} \boldsymbol{V}_{\boldsymbol{\omega}} \mathcal{E}$. In this case, the minimal dilation is unique up to isomorphism. Here, we use the notation $\boldsymbol{V}_{\boldsymbol{\sigma}} := V_{1,\sigma_1} \cdots V_{k,\sigma_k}$ if $\boldsymbol{\sigma} = (\sigma_1, \ldots, \sigma_k) \in \boldsymbol{F_n^+}$, and $V_{i,\sigma_i} := V_{i,j_1} \cdots V_{i,j_p}$ if $\sigma_i = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $V_{i,g_0^i} := I$. For more information on kernels in various noncommutative settings we refer the reader to the work of Ball and Vinnikov [2003] (see also [Ball et al. 2016] and the references therein).

We prove a Schur-type result [1918], which states that a free k-pluriharmonic function F on the polyball $\boldsymbol{B_n}$ is positive if and only if a certain right k-multi-Toeplitz kernel Γ_{F_r} associated with the mapping $\boldsymbol{S} \mapsto F(r\boldsymbol{S})$ is positive semidefinite for any $r \in [0, 1)$. Our Naimark-type result for positive semidefinite right k-multi-Toeplitz kernels on $\boldsymbol{F_n^+}$ is used to provide a structure theorem for positive free k-pluriharmonic functions. We show that a free k-pluriharmonic function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ with $F(0) = I$ is positive if and only if it has the form

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*,$$

where $\boldsymbol{V} = (V_1, \ldots, V_k)$ is a k-tuple of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ and $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$ is the reverse of $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k)$, i.e., $\tilde{\alpha}_i = g_{i_k}^i \cdots g_{i_1}^i$ if $\alpha_i = g_{i_1}^i \cdots g_{i_k}^i \in \mathbb{F}_{n_i}^+$. The general case, when $F(0) \geq 0$, is also considered. As a consequence of these results, we obtain a structure theorem for positive k-harmonic functions on the regular polydisk included in $[B(\mathcal{H})]_1 \times_c \cdots \times_c [B(\mathcal{H})]_1$, which extends the corresponding classical result in scalar polydisks [Rudin 1969].

In Section 4, we define the *free pluriharmonic Poisson kernel* on the regular polyball $\boldsymbol{B_n}$ by setting

$$\mathcal{P}(\boldsymbol{R}, X) := \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} \boldsymbol{R}^*_{1,\tilde{\alpha}_1}\cdots \boldsymbol{R}^*_{k,\tilde{\alpha}_k}\boldsymbol{R}_{1,\tilde{\beta}_1}\cdots \boldsymbol{R}_{k,\tilde{\beta}_k}\otimes X_{1,\alpha_1}\cdots X_{k,\alpha_k}X^*_{1,\beta_1}\cdots X^*_{k,\beta_k}$$

for any $\boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology. Given a completely bounded linear map $\mu : \operatorname{span}\{\boldsymbol{\mathcal{R}}^*_{\boldsymbol{n}}\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\} \to B(\mathcal{E})$, we introduce the *noncommutative Poisson transform* of μ to be the map $\boldsymbol{\mathcal{P}}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X}) := \hat{\mu}[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})], \quad \boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H}),$$

where the completely bounded linear map

$$\hat{\mu} := \mu\otimes \mathrm{id} : \operatorname{span}\{\boldsymbol{\mathcal{R}}^*_{\boldsymbol{n}}\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\}^{-\|\cdot\|} \otimes_{\min} B(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$$

is uniquely defined by $\hat{\mu}(A\otimes Y) := \mu(A)\otimes Y$ for any $A\in \operatorname{span}\{\boldsymbol{\mathcal{R}}^*_{\boldsymbol{n}}\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\}$ and $Y\in B(\mathcal{H})$. We remark that, in the particular case when $n_1 = \cdots = n_k = 1$, $\mathcal{H} = \mathcal{K} = \mathbb{C}$, $\boldsymbol{X} = (X_1, \dots, X_k) \in \mathbb{D}^k$, and μ is a complex Borel measure on $\mathbb{T}^k$ (which can be seen as a bounded linear functional on $C(\mathbb{T}^k)$), we have that the noncommutative Poisson transform of μ coincides with the classical Poisson transform of μ [Rudin 1969].

In Section 4, we give necessary and sufficient conditions for a function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ to be the noncommutative Poisson transform of a completely bounded linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$, where $C^*(\boldsymbol{R})$ is the C^*-algebra generated by the operators $\boldsymbol{R}_{i,j}$. In this case, we show that there exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of *doubly commuting row isometries* acting on a Hilbert space $\mathcal{K}$, i.e., $C^*(V_i)$ commutes with $C^*(V_j)$ if $i \neq j$, and bounded linear operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ such that

$$F(\boldsymbol{X}) = (W_1^*\otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2\otimes I), \quad \boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H}),$$

where

$$C_{\boldsymbol{X}}(\boldsymbol{V}) := (I\otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^k (I - V_{i,1}\otimes X^*_{i,1} - \cdots - V_{i,n_i}\otimes X^*_{i,n_i})^{-1}.$$

In particular, we obtain necessary and sufficient conditions for a function $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ to be the noncommutative Poisson transform of a completely positive linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$. In this case, we have the representation

$$F(\boldsymbol{X}) = (W^*\otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W\otimes I), \quad \boldsymbol{X}\in \boldsymbol{B_n}(\mathcal{H}).$$

In Section 5, we introduce the *noncommutative Herglotz–Riesz transform* of a completely positive linear map $\mu : \operatorname{span}\{\boldsymbol{\mathcal{R}}^*_{\boldsymbol{n}}\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}\} \to B(\mathcal{E})$ as the map $\boldsymbol{H}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{H}\mu)(\boldsymbol{X}) := \hat{\mu}\left(2\prod_{i=1}^k (I - \boldsymbol{R}^*_{i,1}\otimes X_{i,1} - \cdots - \boldsymbol{R}^*_{i,n_i}\otimes X_{i,n_i})^{-1} - I\right)$$

for $\boldsymbol{X} := (X_1, \dots, X_n) \in \boldsymbol{B_n}(\mathcal{H})$. The main result of this section provides necessary and sufficient conditions for a function f from the polyball $\boldsymbol{B_n}(\mathcal{H})$ to $B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ to admit a Herglotz–Riesz-type representation [Herglotz 1911; Riesz 1911], i.e.,

$$f(\boldsymbol{X}) = (\boldsymbol{H}\mu)(\boldsymbol{X}) + i\Im f(0), \quad \boldsymbol{X} \in \boldsymbol{B}_n(\mathcal{H}),$$

where $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ is a completely positive linear map with the property that $\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F_n^+}$. In this case, we show that there exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of doubly commuting row isometries on a Hilbert space $\mathcal{K}$ and a bounded linear operator $W : \mathcal{E} \to \mathcal{K}$ such that

$$f(\boldsymbol{X}) = (W^* \otimes I)\left(2\prod_{i=1}^{k}(I - V_{i,1}^* \otimes X_{i,1} - \cdots - V_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)(W \otimes I) + i\Im f(0)$$

and $W^* \boldsymbol{V}_{\boldsymbol{\alpha}}^* \boldsymbol{V}_{\boldsymbol{\beta}} W = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F_n^+}$.

We remark that, in the particular case when $n_1 = \cdots = n_k = 1$, we obtain an operator-valued extension of the integral representation for holomorphic functions with positive real parts in polydisks [Korányi and Pukánszky 1963].

1. k-multi-Toeplitz operators on tensor products of full Fock spaces

In this section, we introduce the class $\boldsymbol{\mathcal{T}_n}$ of k-multi-Toeplitz operators on tensor products of full Fock spaces. We associate with each k-multi-Toeplitz operator T a formal power series in several variables and show that we can recapture T from its noncommutative Fourier series. Moreover, we characterize the noncommutative formal power series which are Fourier series of k-multi-Toeplitz operators and prove that $\boldsymbol{\mathcal{T}_n} = \operatorname{span}\{\boldsymbol{\mathcal{A}_n^*}\boldsymbol{\mathcal{A}_n}\}^{\text{-SOT}}$, where $\boldsymbol{\mathcal{A}_n}$ is the noncommutative polyball algebra.

First, we recall (see [Popescu 1999; 2016]) some basic properties for a class of noncommutative Berezin-type transforms [1972] associated with regular polyballs. Let $\boldsymbol{X} = (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})^-$ with $X_i := (X_{i,1}, \dots, X_{i,n_i})$. We use the notation $X_{i,\alpha_i} := X_{i,j_1} \cdots X_{i,j_p}$ if $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$ and $X_{i,g_0^i} := I$. The *noncommutative Berezin kernel* associated with any element $\boldsymbol{X}$ in the noncommutative polyball $\boldsymbol{B_n}(\mathcal{H})^-$ is the operator

$$\boldsymbol{K_X} : \mathcal{H} \to F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_k}) \otimes \overline{\boldsymbol{\Delta_X}(I)(\mathcal{H})}$$

defined by

$$\boldsymbol{K_X} h := \sum_{\beta_i \in \mathbb{F}_{n_i}^+, i=1,\dots,k} e_{\beta_1}^1 \otimes \cdots \otimes e_{\beta_k}^k \otimes \boldsymbol{\Delta}_X(I)^{1/2} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^* h, \quad h \in \mathcal{H},$$

where the defect operator $\boldsymbol{\Delta_X}(I)$ was defined in the introduction. A very important property of the Berezin kernel is that $\boldsymbol{K_X} X_{i,j}^* = (\boldsymbol{S}_{i,j}^* \otimes I)\boldsymbol{K_X}$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. The *Berezin transform at* $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ is the map $\boldsymbol{\mathcal{B}_X} : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{H})$ defined by

$$\boldsymbol{\mathcal{B}_X}[g] := \boldsymbol{K_X^*}(g \otimes I_{\mathcal{H}}) f \boldsymbol{K_X}, \quad g \in B\left(\bigotimes_{i=1}^{k} F^2(H_{n_i})\right).$$

If g is in the C^*-algebra $C^*(\boldsymbol{S})$ generated by $\boldsymbol{S}_{i,1}, \dots, \boldsymbol{S}_{i,n_i}$, where $i \in \{1, \dots, k\}$, we define the Berezin transform at $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})^-$ by

$$\boldsymbol{\mathcal{B}}_X[g] := \lim_{r\to 1} \boldsymbol{K}^*_{rX}(g \otimes I_{\mathcal{H}})\boldsymbol{K}_{rX}, \quad g \in C^*(\boldsymbol{S}),$$

where the limit is in the operator norm topology. In this case, the Berezin transform at $\boldsymbol{X}$ is a unital completely positive linear map such that

$$\boldsymbol{\mathcal{B}}_X(\boldsymbol{S}_{\boldsymbol{\alpha}}\boldsymbol{S}^*_{\boldsymbol{\beta}}) = \boldsymbol{X}_{\boldsymbol{\alpha}}\boldsymbol{X}^*_{\boldsymbol{\beta}}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k},$$

where $\boldsymbol{S}_{\boldsymbol{\alpha}} := \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k}$ if $\boldsymbol{\alpha} := (\alpha_1, \dots, \alpha_k) \in \mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$.

The Berezin transform will play an important role in this paper. More properties concerning noncommutative Berezin transforms and multivariable operator theory on noncommutative balls and polydomains can be found in [Popescu 1999; 2013; 2016]. For basic results on completely positive and completely bounded maps we refer the reader to [Paulsen 1986; Pisier 2001; Effros and Ruan 2000].

Definition 1.1. Let $\mathcal{E}$ be a Hilbert space. A bounded linear operator $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is called k-multi-Toeplitz with respect to the universal model $\boldsymbol{R} := (\boldsymbol{R}_1, \dots, \boldsymbol{R}_k)$, where $\boldsymbol{R}_i := (\boldsymbol{R}_{i,1}, \dots, \boldsymbol{R}_{i,n_i})$, if

$$(I_{\mathcal{E}} \otimes \boldsymbol{R}^*_{i,s})A(I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t}) = \delta_{st}A, \quad s, t \in \{1, \dots, n_i\},$$

for every $i \in \{1, \dots, k\}$.

A few more notations are necessary. If $\omega, \gamma \in \mathbb{F}^+_n$, we say that $\gamma <_r \omega$ if there is $\sigma \in \mathbb{F}^+_n \setminus \{g_0\}$ such that $\omega = \sigma\gamma$. In this case, we set $\omega \setminus_r \gamma := \sigma$. Similarly, we say that $\gamma <_l \omega$ if there is $\sigma \in \mathbb{F}^+_n \setminus \{g_0\}$ such that $\omega = \gamma\sigma$ and set $\omega \setminus_l \gamma := \sigma$. We denote by $\tilde{\alpha}$ the reverse of $\alpha \in \mathbb{F}^+_n$, i.e., $\tilde{\alpha} = g_{i_k} \cdots g_{i_1}$ if $\alpha = g_{i_1} \cdots g_{i_k} \in \mathbb{F}^+_n$. Notice that $\gamma <_r \omega$ if and only if $\tilde{\gamma} <_l \tilde{\omega}$. In this case we have $(\omega \setminus_r \gamma)^\sim = \tilde{\omega} \setminus_l \tilde{\gamma}$. We say that ω is *right comparable* with γ, and write $\omega \sim_{\mathrm{rc}} \gamma$, if any one of the conditions $\omega <_r \gamma$, $\gamma <_r \omega$ or $\omega = \gamma$ holds. In this case, we define

$$c^+_r(\omega, \gamma) := \begin{cases} \omega \setminus_r \gamma & \text{if } \gamma <_r \omega, \\ g_0 & \text{if } \omega <_r \gamma \text{ or } \omega = \gamma, \end{cases} \qquad \text{and} \qquad c^-_r(\omega, \gamma) := \begin{cases} \gamma \setminus_r \omega & \text{if } \omega <_r \gamma, \\ g_0 & \text{if } \gamma <_r \omega \text{ or } \omega = \gamma. \end{cases}$$

Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ be in $\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$. We say that $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are right comparable, and write $\boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}$, if for each $i \in \{1, \dots, k\}$, any one of the conditions $\omega_i <_r \gamma_i$, $\gamma_i <_r \omega_i$ or $\omega_i = \gamma_i$ holds. In this case, we define

$$c^+_r(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c^+_r(\omega_1, \gamma_1), \dots, c^+_r(\omega_k, \gamma_k)) \quad \text{and} \quad c^-_r(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c^-_r(\omega_1, \gamma_1), \dots, c^-_r(\omega_k, \gamma_k)). \tag{1-1}$$

Similarly, we say that $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are *left comparable*, and write $\boldsymbol{\omega} \sim_{\mathrm{lc}} \boldsymbol{\gamma}$, if $\tilde{\boldsymbol{\omega}} \sim_{\mathrm{rc}} \tilde{\boldsymbol{\gamma}}$. The definitions of $c^+_l(\boldsymbol{\omega}, \boldsymbol{\gamma})$ and $c^-_l(\boldsymbol{\omega}, \boldsymbol{\gamma})$ are now clear. Note that

$$c^+_r(\boldsymbol{\omega}, \boldsymbol{\gamma})^\sim = c^+_l(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\gamma}}) \quad \text{and} \quad c^-_r(\boldsymbol{\omega}, \boldsymbol{\gamma})^\sim = c^-_l(\tilde{\boldsymbol{\omega}}, \tilde{\boldsymbol{\gamma}}).$$

For each $m \in \mathbb{Z}$, we set $m^+ := \max\{m, 0\}$ and $m^- := \max\{-m, 0\}$.

Lemma 1.2. *Let $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)$ and $\boldsymbol{\beta}=(\beta_1,\dots,\beta_k)$ be k-tuples in $\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$ such that $\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+$ for $i\in\{1,\dots,k\}$ with $|\alpha_i|=m_i^-$, $|\beta_i|=m_i^+$ and $m_i\in\mathbb{Z}$. If $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_k)$ and $\boldsymbol{\omega}=(\omega_1,\dots,\omega_k)$ are k-tuples in $\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, then the inner product*

$$\left\langle \boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*(e_{\gamma_1}^1\otimes\cdots\otimes e_{\gamma_k}^k),\, e_{\omega_1}^1\otimes\cdots\otimes e_{\omega_k}^k\right\rangle$$

is different from zero if and only if $\boldsymbol{\omega}\sim_{\mathrm{rc}}\boldsymbol{\gamma}$ and $(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)=(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))$.

Proof. Under the conditions of the lemma, $\boldsymbol{S}_{i,\alpha_i}\boldsymbol{S}_{j,\beta_j}^*=\boldsymbol{S}_{j,\beta_j}^*\boldsymbol{S}_{i,\alpha_i}$ for any $i,\ j\in\{1,\dots,k\}$, $\alpha_i\in\mathbb{F}_{n_i}^+$ and $\beta_j\in\mathbb{F}_{n_j}^+$. Note that the inner product is different from zero if and only if $\beta_i\omega_i=\alpha_i\gamma_i$ for any $i\in\{1,\dots,k\}$. Let $m_i\in\mathbb{Z}$ and assume that $|\alpha_i|=m_i^->0$. Then $\beta_i=g_0^i$ and, consequently, $\omega_i=\alpha_i\gamma_i$. This shows that $\gamma_i<_r\omega_i$, $c_r^+(\omega_i,\gamma_i)=\alpha_i$ and $c_r^-(\omega_i,\gamma_i)=g_0^i$. In the case when $|\beta_i|=m_i^+>0$, we have $\alpha_i=g_0^i$ and $\beta_i\omega_i=\gamma_i$. Consequently, $\omega_i<_r\gamma_i$, $c_r^+(\omega_i,\gamma_i)=g_0^i$ and $c_r^-(\omega_i,\gamma_i)=\beta_i$. When $\alpha_i=\beta_i=g_0^i$, we have $\omega_i=\gamma_i$. Therefore, the scalar product above is different from zero if and only if $\boldsymbol{\omega}\sim_{\mathrm{rc}}\boldsymbol{\gamma}$ and $(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)=(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))$. □

If $\beta_i,\gamma_i\in\mathbb{F}_{n_i}^+$ and, for each $i\in\{1,\dots,k\}$, $\beta_i<_\ell\gamma_i$ or $\beta_i=\gamma_i$, then we write $\boldsymbol{\beta}\le_\ell\boldsymbol{\gamma}$.

Lemma 1.3. *Given a k-tuple $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, the sequence*

$$\{\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*(e_{\gamma_1}^1\otimes\cdots\otimes e_{\gamma_k}^k)\}$$

consists of orthonormal vectors if $\alpha_i,\ \beta_i\in\mathbb{F}_{n_i}^+$, $i\in\{1,\dots,k\}$ with $m_i\in\mathbb{Z}$, $|\alpha_i|=m_i^-$, $|\beta_i|=m_i^+$ and $\boldsymbol{\beta}\le_\ell\boldsymbol{\gamma}$.

Proof. First, note that $\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*(e_{\gamma_1}^1\otimes\cdots\otimes e_{\gamma_k}^k)\neq 0$ if and only if $S_{i,\beta_i}^*(e_{\gamma_i}^i)\neq 0$ for each $i\in\{1,\dots,k\}$, which is equivalent to $\beta_i<_\ell\gamma_i$ or $\beta_i=\gamma_i$. Therefore, $\boldsymbol{\beta}\le_\ell\boldsymbol{\gamma}$.

Fix $i\in\{1,\dots,k\}$ and $\gamma_i\in\mathbb{F}_{n_i}^+$. We prove that the sequence $\{S_{i,\alpha_i}S_{i,\beta_i}^*e_{\gamma_i}^i\}$ consists of orthonormal vectors if $\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+$ have the following properties:

(i) If $|\alpha_i|>0$ then $\beta_i=g_0^i$, and if $|\beta_i|>0$ then $\alpha_i=g_0^i$.

(ii) $\beta_i\le_\ell\gamma_i$.

Indeed, let (α_i,β_i) and (α_i',β_i') be two distinct pairs with the above-mentioned properties. First, we consider the case when $g_0^i\neq\beta_i<_\ell\gamma_i$. Then $\alpha_i=g_0^i$ and, consequently, $S_{i,\alpha_i}S_{i,\beta_i}^*e_{\gamma_i}^i=e_{\gamma_i\backslash_\ell\beta_i}^i$. Similarly, if $g_0^i\neq\beta_i'<_\ell\gamma_i$ then $\alpha_i'=g_0^i$ and, consequently, $S_{i,\alpha_i'}S_{i,\beta_i'}^*e_{\gamma_i}^i=e_{\gamma_i\backslash_\ell\beta_i'}^i$. Since $(\alpha_i,\beta_i)\neq(\alpha_i',\beta_i')$, we must have $\beta_i\neq\beta_i'$, which implies $e_{\gamma_i\backslash_\ell\beta_i}^i\perp e_{\gamma_i\backslash_\ell\beta_i'}^i$. On the other hand, if $\beta_i'=g_0^i$ then $\alpha_i'\in\mathbb{F}_{n_i}^+$ and $S_{i,\alpha_i'}S_{i,\beta_i'}^*e_{\gamma_i}^i=e_{\alpha_i'}e_{\gamma_i}\perp e_{\gamma_i\backslash_\ell\beta_i'}^i$. It follows that $S_{i,\alpha_i'}S_{i,\beta_i'}^*e_{\gamma_i}^i\perp S_{i,\alpha_i'}S_{i,\beta_i'}^*e_{\gamma_i}^i$.

The second case is when $\beta_i=g_0^i$. Then $\alpha_i\in\mathbb{F}_{n_i}^+$ and $S_{i,\alpha_i}S_{i,\beta_i}^*e_{\gamma_i}^i=e_{\alpha_i}e_{\gamma_i}$. As we saw above, $S_{i,\alpha_i'}S_{i,\beta_i'}^*e_{\gamma_i}^i$ is equal to either $e_{\alpha_i'}e_{\gamma_i}$ (when $\beta_i'=g_0^i$) or $e_{\gamma_i\backslash_\ell\beta_i'}^i$ (when $g_0^i\neq\beta_i'<_\ell\gamma_i$). In each case, we have $S_{i,\alpha_i'}S_{i,\beta_i'}^*e_{\gamma_i}^i\perp S_{i,\alpha_i'}S_{i,\beta_i'}^*e_{\gamma_i}^i$, which completes the proof of our assertion. Using this result one can easily complete the proof of the lemma. □

We associate with each k-multi-Toeplitz operator $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ a formal power series

$$\varphi_A(\boldsymbol{S}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*,$$

where the coefficients are given by

$$\langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, \ell\rangle := \langle A(h \otimes x), \ell \otimes y\rangle, \quad h, \ell \in \mathcal{E}, \tag{1-2}$$

and $x := x_1 \otimes \cdots \otimes x_k$, $y = y_1 \otimes \cdots \otimes y_k$ with

$$\begin{cases} x_i = e_{\beta_i}^i \text{ and } y_i = 1 & \text{if } m_i \geq 0, \\ x_i = 1 \text{ and } y_i = e_{\alpha_i}^i & \text{if } m_i < 0, \end{cases} \tag{1-3}$$

for every $i \in \{1, \dots, k\}$.

The next result shows that a k-multi-Toeplitz operator is uniquely determined by is Fourier series.

Theorem 1.4. *If A, B are k-multi-Toeplitz operators on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, then $A = B$ if and only if the corresponding formal Fourier series $\varphi_A(\boldsymbol{S})$ and $\varphi_B(\boldsymbol{S})$ are equal. Moreover, $Aq = \varphi_A(\boldsymbol{S})q$ for any vector-valued polynomial*

$$q = \sum_{\substack{\omega_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\omega_i| \leq p_i}} h_{(\omega_1,\dots,\omega_k)} \otimes e_{\omega_1}^1 \otimes \cdots \otimes e_{\omega_k}^k,$$

where $h_{(\omega_1,\dots,\omega_k)} \in \mathcal{E}$ and $(p_1, \dots, p_k) \in \mathbb{N}^k$.

Proof. Let $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ be k-tuples in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, and let $h, h' \in \mathcal{E}$. Since A is a k-multi-Toeplitz operator on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, we have

$$\begin{aligned}
&\langle A(h \otimes e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k), h' \otimes e_{\omega_1}^1 \otimes \cdots \otimes e_{\omega_k}^k\rangle \\
&\qquad = \big\langle A(I_{\mathcal{E}} \otimes \boldsymbol{R}_{1,\tilde{\gamma}_1} \cdots \boldsymbol{R}_{k,\tilde{\gamma}_k})(h \otimes 1), (I_{\mathcal{E}} \otimes \boldsymbol{R}_{1,\tilde{\omega}_1} \cdots \boldsymbol{R}_{k,\tilde{\omega}_k})(h' \otimes 1)\big\rangle \\
&\qquad = \begin{cases} \langle A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h, h'\rangle & \text{if } \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

where $c_r^+(\boldsymbol{\omega}, \boldsymbol{\gamma})$ and $c_r^-(\boldsymbol{\omega}, \boldsymbol{\gamma})$ are defined by (1-1). Consequently,

$$A(h \otimes e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k) = \sum_{\substack{\boldsymbol{\omega} = (\omega_1,\dots,\omega_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \\ \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}}} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h \otimes e_{\omega_1}^1 \otimes \cdots \otimes e_{\omega_k}^k$$

is a vector in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Hence, we deduce that, for each $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, the series

$$\sum_{\substack{\boldsymbol{\omega} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+ \\ \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}}} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))}^* A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} \tag{1-4}$$

is WOT-convergent. Due to Lemma 1.3, given $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, the sequence $\{\boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^* (e_{\gamma_1}^1 \otimes \cdots \otimes e_{\gamma_k}^k)\}$, where $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $i \in \{1, \dots, k\}$ with $m_i \in \mathbb{Z}$, $|\alpha_i| = m_i^-$,

$|\beta_i| = m_i^+$ and $\boldsymbol{\beta} \leq_\ell \boldsymbol{\gamma}$, consists of orthonormal vectors. Note that, in this case, we also have $\boldsymbol{\alpha} \sim_{\rm rc} \boldsymbol{\beta}$ and, consequently, $A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} = A_{(c_r^+(\boldsymbol{\alpha},\boldsymbol{\beta});c_r^-(\boldsymbol{\alpha},\boldsymbol{\beta}))}$. Hence, and taking into account that the series (1-4) is WOT-convergent, we deduce that

$$\begin{aligned}
&\varphi_A(\boldsymbol{S})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k})\\
&\qquad := \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h \otimes \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1}\cdots \boldsymbol{S}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k})
\end{aligned}$$

is a convergent series in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ be k-tuples in $\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$. According to Lemma 1.2, the inner product

$$\left\langle \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1}\cdots \boldsymbol{S}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}),\, e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \right\rangle$$

is different from zero if and only if $\boldsymbol{\omega} \sim_{\rm rc} \boldsymbol{\gamma}$ and $(\alpha_1, \dots, \alpha_k, \beta_1, \dots, \beta_k) = (c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma}); c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))$. Now, using (1-4), one can see that

$$\begin{aligned}
&\left\langle \varphi_A(\boldsymbol{S})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}),\, h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \right\rangle\\
&\quad = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} \begin{aligned}[t] &\langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, h'\rangle\\ &\times \left\langle \boldsymbol{S}_{1,\alpha_1}\cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1}\cdots \boldsymbol{S}^*_{k,\beta_k}(e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}),\, e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \right\rangle \end{aligned}\\
&\quad = \begin{cases} \langle A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h, h'\rangle & \text{if } \boldsymbol{\omega} \sim_{\rm rc} \boldsymbol{\gamma},\\ 0 & \text{otherwise,} \end{cases}\\
&\quad = \langle A(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}),\, h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \rangle
\end{aligned}$$

for any $h, h' \in \mathcal{E}$, and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ in $\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$, which shows that $Aq = \varphi_A(\boldsymbol{S})q$ for any vector-valued polynomial in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Therefore, if the formal Fourier series $\varphi_A(\boldsymbol{S})$ and $\varphi_B(\boldsymbol{S})$ are equal, then $A = B$. $\square$

When $\mathcal{G}$ is a Hilbert space, $C_{(\alpha;\beta)} \in B(\mathcal{G})$, and the series

$$\Sigma_1 := \sum_{m\in\mathbb{Z}, m<0} \sum_{\substack{\alpha,\beta\in\mathbb{F}^+_n\\ |\alpha|=m^-,\, |\beta|=m^+}} C_{(\alpha;\beta)} \quad \text{and} \quad \Sigma_2 := \sum_{m\in\mathbb{Z}, m\geq 0} \sum_{\substack{\alpha,\beta\in\mathbb{F}^+_n\\ |\alpha|=m^-,\, |\beta|=m^+}} C_{(\alpha;\beta)}$$

are convergent in the operator topology, we say that the series

$$\sum_{m\in\mathbb{Z}} \sum_{\substack{\alpha,\beta\in\mathbb{F}^+_n\\ |\alpha|=m^-,\, |\beta|=m^+}} C_{(\alpha;\beta)} := \Sigma_1 + \Sigma_2$$

is convergent in the operator topology. In what follows, we show how we can recapture the k-multi-Toeplitz operators from their Fourier series. Moreover, we characterize the formal series which are Fourier series of k-multi-Toeplitz operators. Let $\mathcal{P}$ denote the set of all vector-valued polynomials in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, i.e., each $p \in \mathcal{P}$ has the form

$$q = \sum_{\substack{\omega_i \in \mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\} \\ |\omega_i| \le p_i}} h_{(\omega_1,\dots,\omega_k)} \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k},$$

where $h_{(\omega_1,\dots,\omega_k)} \in \mathcal{E}$ and $(p_1, \dots, p_k) \in \mathbb{N}^k$.

Theorem 1.5. *Let $\{A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\}$ be a family of operators in $B(\mathcal{E})$, where $\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+$, $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$, $m_i \in \mathbb{Z}$ and $i \in \{1, \dots, k\}$. Then*

$$\varphi(\boldsymbol{S}) := \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i \in \mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\} \\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1} \cdots \boldsymbol{S}^*_{k,\beta_k}$$

is the formal Fourier series of a k-multi-Toeplitz operator on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$ if and only if the following conditions are satisfied:

(i) *For each $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, the series*

$$\sum_{\substack{\boldsymbol{\omega} \in \mathbb{F}_{n_1}^+ \times\cdots\times \mathbb{F}_{n_k}^+ \\ \boldsymbol{\omega} \sim_{\mathrm{rc}} \boldsymbol{\gamma}}} A^*_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))}$$

is WOT-convergent.

(ii) *If $\mathcal{P}$ is the set of all vector-valued polynomials in $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$, then*

$$\sup_{r\in[0,1)} \sup_{p\in\mathcal{P}, \|p\|\le 1} \|\varphi(r\boldsymbol{S})p\| < \infty.$$

Moreover, if there is a k-multi-Toeplitz operator $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ such that $\varphi(\boldsymbol{S}) = \varphi_A(\boldsymbol{S})$, then the following statements hold:

(a) $$\varphi(r\boldsymbol{S}) := \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i \in \mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\} \\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes r^{\sum_{i=1}^k(|\alpha_i|+|\beta_i|)} \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1} \cdots \boldsymbol{S}^*_{k,\beta_k}$$

is convergent in the operator norm topology, and its sum, which does not depend on the order of the series, is an operator in

$$\operatorname{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|},$$

where $\boldsymbol{\mathcal{A}_n}$ is the polyball algebra.

(b) $A = \text{SOT-}\lim_{r\to 1} \varphi(r\boldsymbol{S})$ *and*

$$\|A\| = \sup_{r\in[0,1)} \|\varphi(r\boldsymbol{S})\| = \lim_{r\to 1} \|\varphi(r\boldsymbol{S})\| = \sup_{q\in\mathcal{P}, \|q\|\le 1} \|\varphi(\boldsymbol{S})q\|.$$

Proof. First, we assume that $A \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is a k-multi-Toeplitz operator and $\varphi(\boldsymbol{S}) = \varphi_A(\boldsymbol{S})$, where the coefficients $A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}$ are given by (1-2) and (1-3). Note that (i) follows from the proof of Theorem 1.4. Moreover, from the same proof and Lemma 1.3 we have $\varphi_A(\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})$ and, consequently, $\varphi_A(r\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})$, $r\in[0,1)$, are vectors in $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ and

$$\lim_{r\to 1}\varphi_A(r\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})=\varphi_A(\boldsymbol{S})(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})=A(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k}) \tag{1-5}$$

for any $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_k)$ in $\mathbb{F}^+_{n_1}\times\cdots\times\mathbb{F}^+_{n_k}$ and $h\in\mathcal{E}$. Note also that, due to (i) and Lemma 1.3, we have $\varphi_A(r_1\boldsymbol{S}_1,\dots,r_k\boldsymbol{S}_k)(h\otimes e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k})\in\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ for any $r_i\in[0,1)$, $i\in\{1,\dots,k\}$. Now, we show that the series

$$\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes\left(\prod_{i=1}^k r_i^{|\alpha_i|+|\beta_i|}\right)\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}^*_{1,\beta_1}\cdots\boldsymbol{S}^*_{k,\beta_k}$$

is convergent in the operator norm topology and its sum is in $\operatorname{span}\{f^*g : f,g\in B(\mathcal{E})\otimes_{\min}\boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$, where $\boldsymbol{\mathcal{A}_n}$ is the polyball algebra. We denote the series above by $\varphi_A(r_1\boldsymbol{S}_1,\dots,r_k\boldsymbol{S}_k)$. Since A is a k-multi-Toeplitz operator, it is also a 1-multi-Toeplitz operator with respect to $R_k:=(R_{k,1},\dots,R_{k,n_k})$, the right creation operators on the Fock space $F^2(H_{n_k})$. Applying Theorem 1.4 to 1-multi-Toeplitz operators, we deduce that A has a unique Fourier representation

$$\psi_A(S_k):=\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k}\\ |\alpha_k|=m_k^-,\,|\beta_k|=m_k^+}} C_{(\alpha_k;\beta_k)}\otimes S_{k,\alpha_k}S^*_{k,\beta_k},$$

where $C_{(\alpha_k;\beta_k)}\in B\big(\mathcal{E}\otimes\bigotimes_{i=1}^{k-1}F^2(H_{n_i})\big)$. Moreover, we can prove that, for any $r_k\in[0,1)$,

$$\psi_A(r_kS_k):=\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k}\\ |\alpha_k|=m_k^-,\,|\beta_k|=m_k^+}} r_k^{|\alpha_k|+|\beta_k|}C_{(\alpha_k;\beta_k)}\otimes S_{k,\alpha_k}S^*_{k,\beta_k} \tag{1-6}$$

is convergent in the operator norm topology. Indeed, since $\psi_A(S_k)$ is the Fourier representation of the 1-multi-Toeplitz operator A with respect to $R_k:=(R_{k,1},\dots,R_{k,n_k})$, item (i) implies, in the particular case when $\gamma_k=g_0^k$, that $\sum_{\alpha_k\in\mathbb{F}^+_{n_k}}C^*_{(\alpha_k;g_0^k)}C_{(\alpha_k;g_0^k)}$ is WOT-convergent. Since A^* is also a 1-multi-Toeplitz operator, we can similarly deduce that the series $\sum_{\beta_k\in\mathbb{F}^+_{n_k}}C_{(g_0^k;\beta_k)}C^*_{(g_0^k;\beta_k)}$ is WOT-convergent. Since $S_{k,1},\dots,S_{k,n_k}$ are isometries with orthogonal ranges, we have

$$\Bigg\|\sum_{\alpha_k\in\mathbb{F}^+_{n_k},|\alpha_k|=m}C_{(\alpha_k;g_0^k)}\otimes r_k^{|\alpha_k|}S_{k,\alpha_k}\Bigg\|=r_k^m\Bigg\|\sum_{\alpha_k\in\mathbb{F}^+_{n_k}}C^*_{(\alpha_k;g_0^k)}C_{(\alpha_k;g_0^k)}\Bigg\|^{1/2},$$

$$\Bigg\|\sum_{\beta_k\in\mathbb{F}^+_{n_k},|\beta_k|=m}C_{(g_0^k;\beta_k)}\otimes r_k^{|\alpha_k|}S^*_{k,\beta_k}\Bigg\|=r_k^m\Bigg\|\sum_{\alpha_k\in\mathbb{F}^+_{n_k}}C_{(g_0^k;\beta_k)}C^*_{(g_0^k;\beta_k)}\Bigg\|^{1/2},$$

for any $m\in\mathbb{N}$. Now, it is clear that the series defining $\psi_A(r_kS_k)$ is convergent in the operator norm topology and, consequently, $\psi_A(r_kS_k)$ belongs to

$$\operatorname{span}\left\{f^*g : f,g\in B\left(\mathcal{E}\otimes\bigotimes_{i=1}^{k-1}F^2(H_{n_i})\right)\otimes_{\min}\mathcal{A}_{n_k}\right\}^{-\|\cdot\|},$$

where $\mathcal{A}_{n_k}$ is the noncommutative disc algebra generated by $S_{k,1}, \dots, S_{k,n_k}$ and the identity. For each $i \in \{1, \dots, k\}$, we set $\mathcal{E}_i := \mathcal{E} \otimes F^2(H_{n_1}) \otimes \cdots \otimes F^2(H_{n_i})$.

The next step in our proof is to show that

$$\psi_A(r_k S_k) = \mathcal{B}^{\text{ext}}_{r_k S_k}[A] := (I_{\mathcal{E}_{k-1}} \otimes K^*_{r_k S_k})(A \otimes I_{F^2(H_{n_k})})(I_{\mathcal{E}_{k-1}} \otimes K_{r_k S_k}), \tag{1-7}$$

where $K_{r_k S_k} : F^2(H_{n_k}) \to F^2(H_{n_k}) \otimes \mathcal{D}_{r_k S_k} \subset F^2(H_{n_k}) \otimes F^2(H_{n_k})$ is the noncommutative Berezin kernel defined by

$$K_{r_k S_k}\xi := \sum_{\beta_k \in \mathbb{F}^+_{n_k}} e^k_{\beta_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}\xi, \quad \xi \in F^2(H_{n_k}),$$

and $\mathcal{D}_{r_k S_k} := \overline{\Delta_{r_k S_k}(I)(F^2(H_{n_k}))}$. Let $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_k)$ and $\boldsymbol{\omega} = (\omega_1, \dots, \omega_k)$ be k-tuples in $\mathbb{F}^+_{n_1} \times \cdots \times \mathbb{F}^+_{n_k}$, set $q := \max\{|\gamma_k|, |\omega_k|\}$, and define the operator

$$Q_q := \sum_{m_k \in \mathbb{Z}, |m_k| \le q} \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-, \, |\beta_k| = m_k^+}} C_{(\alpha_k; \beta_k)} \otimes S_{k,\alpha_k} S^*_{k,\beta_k}.$$

Since $\psi_A(S_k)p = Ap$ for any polynomial $p \in \mathcal{P}$, a careful computation reveals that

$$\begin{aligned}
&\left\langle \mathcal{B}^{\text{ext}}_{r_k S_k}[A](h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \right\rangle \\
&\quad = \left\langle (A \otimes I_{F^2(H_{n_k})})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes K_{r_k S_k}(e^k_{\gamma_k})), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes K_{r_k S_k}(e^k_{\omega_k}) \right\rangle \\
&\quad = \Bigg\langle (A \otimes I_{F^2(H_{n_k})}) \Bigg(\sum_{\alpha_k \in \mathbb{F}^+_{n_k}} h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}) \Bigg), \\
&\qquad\qquad \sum_{\beta_k \in \mathbb{F}^+_{n_k}} h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \Bigg\rangle \\
&\quad = \Bigg\langle \sum_{\alpha_k \in \mathbb{F}^+_{n_k}} A(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}) \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \\
&\qquad\qquad \sum_{\beta_k \in \mathbb{F}^+_{n_k}} h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \otimes \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \Bigg\rangle \\
&\quad = \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\substack{\alpha_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m}} \sum_{\substack{\beta_k \in \mathbb{F}^+_{n_k} \\ |\beta_k| = p}} \left\langle A(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \right\rangle \\
&\qquad\qquad \times \left\langle \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \right\rangle \\
&\quad = \sum_{m=0}^{q} \sum_{p=0}^{q} \sum_{\substack{\alpha_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m}} \sum_{\substack{\beta_k \in \mathbb{F}^+_{n_k} \\ |\beta_k| = p}} \left\langle Q_q(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \right\rangle \\
&\qquad\qquad \times \langle \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \rangle \\
&\quad = \sum_{m=0}^{\infty} \sum_{p=0}^{\infty} \sum_{\substack{\alpha_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m}} \sum_{\substack{\beta_k \in \mathbb{F}^+_{n_k} \\ |\beta_k| = p}} \left\langle Q_q(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes e^k_{\alpha_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes e^k_{\beta_k} \right\rangle \\
&\qquad\qquad \times \left\langle \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\alpha_k}(e^k_{\gamma_k}), \Delta_{r_k S_k}(I)^{1/2} S^*_{k,\beta_k}(e^k_{\omega_k}) \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \big\langle (Q_q \otimes I_{F^2(H_{n_k})})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^{k-1}_{\gamma_{k-1}} \otimes K_{r_k S_k}(e^k_{\gamma_k})), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^{k-1}_{\omega_{k-1}} \otimes K_{r_k S_k}(e^k_{\omega_k}) \big\rangle \\
&= \big\langle \mathcal{B}^{\mathrm{ext}}_{r_k S_k}[Q_q](h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \big\rangle \\
&= \sum_{m_k \in \mathbb{Z}, |m_k| \le q} \; \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-, \, |\beta_k| = m_k^+}} \big\langle (C_{(\alpha_k;\beta_k)} \otimes r_k^{|\alpha_k|+|\beta_k|} S_{k,\alpha_k} S^*_{k,\beta_k})(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \big\rangle \\
&= \big\langle \psi_A(r_k S_k)(h \otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}), h' \otimes e^1_{\omega_1} \otimes \cdots \otimes e^k_{\omega_k} \big\rangle
\end{aligned}
$$

for any $h, h' \in \mathcal{E}$. Consequently, (1-7) holds for any $r_k \in [0, 1)$. Hence, and using the fact the noncommutative Berezin kernel $K_{r S_k}$ is an isometry, we deduce that

$$
\|\psi_A(r_k S_k)\| \le \|A\|, \quad r_k \in [0, 1).
$$

Moreover, one can show that

$$
A = \text{SOT-}\lim_{r_k \to 1} \psi_A(r_k S_k).
$$

Indeed, due to (i) (for 1-multi-Toeplitz operators), we have $\|\psi_A(r_k S_k) p - \psi_A(S_k) p\| \to 0$ as $r_k \to 1$ for any polynomial $p \in \mathcal{E}_{k-1} \otimes F^2(H_{n_k})$ with coefficients in $\mathcal{E}_{k-1}$. Since $\psi_A(S_k) p = Ap$ and $\|\psi_A(r_k S_k)\| \le \|A\|$ for any $r_k \in [0, 1)$, an approximation argument proves our assertion.

Now, we prove that the coefficients $C_{(\alpha_k;\beta_k)} \in B\big(\mathcal{E} \otimes \bigotimes_{i=1}^{k-1} F^2(H_{n_i})\big)$ of the Fourier series $\psi_A(S_k)$ are 1-multi-Toeplitz operators with respect to $R_{k-1} := (R_{k-1,1}, \ldots, R_{k-1,n_{k-1}})$. For each $i \in \{1, \ldots, k-1\}$, $s, t \in \{1, \ldots, n_i\}$, and any vector-valued polynomial $p \in \mathcal{E} \otimes \bigotimes_{i=1}^{k} F^2(H_{n_i})$ with coefficients in $\mathcal{E}$, Theorem 1.4 implies

$$
\begin{aligned}
\sum_{m_k \in \mathbb{Z}} \; \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-, \, |\beta_k| = m_k^+}} & \big[(I_{\mathcal{E}_{k-2}} \otimes R^*_{i,s}) C_{(\alpha_k;\beta_k)} (I_{\mathcal{E}_{k-2}} \otimes R_{i,t}) \otimes S_{k,\alpha_k} S^*_{k,\beta_k} \big](p) \\
&= (I_{\mathcal{E}} \otimes \boldsymbol{R}^*_{i,s}) \psi_A(S_k) (I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t})(p) \\
&= (I_{\mathcal{E}} \otimes \boldsymbol{R}^*_{i,s}) A (I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t})(p) \\
&= \delta_{st} A(p) = \delta_{st} \psi_A(S_k)(p) \\
&= \delta_{st} \sum_{m_k \in \mathbb{Z}} \; \sum_{\substack{\alpha_k, \beta_k \in \mathbb{F}^+_{n_k} \\ |\alpha_k| = m_k^-, \, |\beta_k| = m_k^+}} (C_{(\alpha_k;\beta_k)} \otimes S_{k,\alpha_k} S^*_{k,\beta_k})(p).
\end{aligned}
$$

Hence, we deduce that

$$
(I_{\mathcal{E}_{k-2}} \otimes R^*_{i,s}) C_{(\alpha_k;\beta_k)} (I_{\mathcal{E}_{k-2}} \otimes R_{i,t}) = \delta_{st} C_{(\alpha_k;\beta_k)}
$$

for any $i \in \{1, \ldots, k-1\}$ and $s, t \in \{1, \ldots, n_i\}$, which proves that $C_{(\alpha_k;\beta_k)}$ is a 1-multi-Toeplitz operator with respect to $R_{k-1} := (R_{k-1,1}, \ldots, R_{k-1,n_{k-1}})$. Consequently, similarly to the first part of the proof, $C_{(\alpha_k;\beta_k)}$ has a Fourier representation

$$
\psi_{(\alpha_k;\beta_k)}(S_{k-1}) := \sum_{m_{k-1} \in \mathbb{Z}} \; \sum_{\substack{\alpha_{k-1}, \beta_{k-1} \in \mathbb{F}^+_{n_{k-1}} \\ |\alpha_{k-1}| = m^-_{k-1}, \, |\beta_{k-1}| = m^+_{k-1}}} C_{(\alpha_{k-1},\alpha_k;\beta_{k-1},\beta_k)} \otimes S_{k-1,\alpha_{k-1}} S^*_{k-1,\beta_{k-1}}, \tag{1-8}
$$

where $C_{(\alpha_{k-1},\alpha_k;\beta_{k-1},\beta_k)} \in B(\mathcal{E}_{k-2})$. Moreover, as above, one can prove that, for any $r_{k-1} \in [0, 1)$, the series $\psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1})$ is convergent in the operator norm topology, and its limit is an element in

$$\overline{\operatorname{span}}\{f^*g : f, g \in B(\mathcal{E}_{k-2}) \otimes_{\min} \mathcal{A}_{n_{k-1}}\}^{\|\cdot\|},$$

where $\mathcal{A}_{n_{k-1}}$ is the noncommutative disc algebra generated by $S_{k-1,1}, \dots, S_{k-1,n_{k-1}}$ and the identity. We also have

$$\lim_{r_{k-1}\to 1} \psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1})p = C_{(\alpha_k;\beta_k)}p$$

for any vector-valued polynomial $p \in \mathcal{E}_{k-2} \otimes F^2(H_{n_{k-1}})$. As in the first part of the proof, setting

$$\mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}}[u] := (I_{\mathcal{E}_{k-2}} \otimes K^*_{r_{k-1}S_{k-1}})(u \otimes I_{F^2(H_{n_{k-1}})})(I_{\mathcal{E}_{k-2}} \otimes K_{r_{k-1}S_{k-1}}), \quad u \in B(\mathcal{E}_{n-1}),$$

one can prove that

$$\psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1}) = \mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}}[C_{(\alpha_k;\beta_k)}] \quad \text{and} \quad \|\psi_{(\alpha_k,\beta_k)}(r_{k-1}S_{k-1})\| \le \|C_{(\alpha_k;\beta_k)}\| \tag{1-9}$$

for any $r_{k-1} \in [0, 1)$. Moreover, we can also show that

$$C_{(\alpha_k;\beta_k)} = \text{SOT-}\lim_{r_{k-1}\to 1} \psi_{(\alpha_k;\beta_k)}(r_{k-1}S_{k-1}).$$

Now, due to (1-6), (1-7), (1-8) and (1-9), we obtain

$$\begin{aligned}
&([\mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}} \otimes \mathrm{id}_{B(F^2(H_{n_k}))}] \circ \mathcal{B}^{\text{ext}}_{r_kS_k})[A] \\
&\quad = \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k} \\ |\alpha_k|=m_k^-,\ |\beta_k|=m_k^+}} \mathcal{B}^{\text{ext}}_{r_{k-1}S_{k-1}}[C_{(\alpha_k;\beta_k)}] \otimes r_k^{|\alpha_k|+|\beta_k|} S_{k,\alpha_k}S^*_{k,\beta_k} \\
&\quad = \sum_{m_k\in\mathbb{Z}} \sum_{m_{k-1}\in\mathbb{Z}} \sum_{\substack{\alpha_k,\beta_k\in\mathbb{F}^+_{n_k} \\ |\alpha_k|=m_k^-,\ |\beta_k|=m_k^+}} \sum_{\substack{\alpha_{k-1},\beta_{k-1}\in\mathbb{F}^+_{n_{k-1}} \\ |\alpha_{k-1}|=m_{k-1}^-,\ |\beta_{k-1}|=m_{k-1}^+}} r_k^{|m_k|} r_{k-1}^{|m_{k-1}|} C_{(\alpha_{k-1},\alpha_k;\beta_{k-1},\beta_k)} \\
&\qquad\qquad\qquad\qquad \otimes S_{k-1,\alpha_{k-1}}S^*_{k-1,\beta_{k-1}} \otimes S_{k,\alpha_k}S^*_{k,\beta_k},
\end{aligned}$$

where the series are convergent in the operator norm topology. Continuing this process, one can prove that there are some operators $C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \in B(\mathcal{E})$ such that the series $\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)$ given by

$$\sum_{m_k\in\mathbb{Z}} \cdots \sum_{m_1\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\ i\in\{1,\dots,k\} \\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}} r_k^{|m_k|} \cdots r_1^{|m_1|} C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1} \cdots \boldsymbol{S}^*_{k,\beta_k}$$

is convergent in the operator norm topology and

$$\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k) = [\mathcal{B}^{\text{ext}}_{r_1S_1} \otimes \mathrm{id}_{B(\bigotimes_{i=2}^k F^2(H_{n_i}))}] \circ [\mathcal{B}^{\text{ext}}_{r_2S_2} \otimes \mathrm{id}_{B(\bigotimes_{i=3}^k F^2(H_{n_i}))}] \circ \cdots \circ \mathcal{B}^{\text{ext}}_{r_kS_k}[A]. \tag{1-10}$$

Since the noncommutative Berezin kernels $K_{r_iS_i}$, $i \in \{1, \dots, k\}$, are isometries, we deduce that

$$\|\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)\| \le \|A\|, \quad r_i \in [0, 1).$$

Note that the coefficients of the k-multi-Toeplitz operator $\varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)$ satisfy the relation

$$\langle r_k^{|m_k|} \cdots r_1^{|m_1|} C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, \ell\rangle = \langle \varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)(h\otimes x), (\ell\otimes y)\rangle, \tag{1-11}$$

where x, y are defined as in (1-3). Since A is a k-multi-Toeplitz operator, so is $Y_{r_k} := \mathcal{B}^{\text{ext}}_{r_k S_k}[A] = \psi_A(r_k S_k)$ and, iterating the argument, we deduce that

$$Y_{r_2,\dots,r_k} := [\mathcal{B}^{\text{ext}}_{r_2 S_2} \otimes \mathrm{id}_{B(\bigotimes_{i=3}^k F^2(H_{n_i}))}] \circ \cdots \circ \mathcal{B}^{\text{ext}}_{r_k S_k}[A]$$

is a k-multi-Toeplitz operator. In particular, $Y_{r_2,\dots,r_k}$ is a 1-multi-Toeplitz operator with respect to $R_1 := (R_{1,1}, \dots, R_{1,n_1})$. Applying the first part of the proof to $Y_{r_2,\dots,r_k}$, we deduce that

$$\text{SOT-}\lim_{r_1\to 1} [\mathcal{B}^{\text{ext}}_{r_1 S_1} \otimes \mathrm{id}_{B(\bigotimes_{i=2}^k F^2(H_{n_i}))}][Y_{r_2,\dots,r_k}] = Y_{r_2,\dots,r_k}.$$

Continuing this process, we obtain

$$\text{SOT-}\lim_{r_k\to 1} \cdots \text{SOT-}\lim_{r_1\to 1} [\mathcal{B}^{\text{ext}}_{r_1 S_1} \otimes \mathrm{id}_{B(\bigotimes_{i=2}^k F^2(H_{n_i}))}][Y_{r_2,\dots,r_k}] = A.$$

Consequently, using (1-10), (1-11) and (1-2), we deduce that

$$\langle C_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} h, \ell\rangle = \langle A(h\otimes x), \ell\otimes y\rangle = \langle A_{(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)} h, \ell\rangle,$$

which shows that $\varphi_A(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k) = \varphi(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)$ for any $r_i \in [0, 1)$. Hence, we obtain

$$\varphi_A(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k) = \sum_{m_k\in\mathbb{Z}} \cdots \sum_{m_1\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}^+_{n_i},\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} r_k^{|m_k|} \cdots r_1^{|m_1|} \times A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}^*_{1,\beta_1} \cdots \boldsymbol{S}^*_{k,\beta_k},$$

where the series are convergent in the operator norm topology. Moreover, due to (1-10), we have

$$\|\varphi_A(r_1\boldsymbol{S}_1, \dots, r_k\boldsymbol{S}_k)\| \le \|A\|, \quad r_i \in [0, 1).$$

Due to (1-5), we have

$$\lim_{r\to 1} \varphi_A(r\boldsymbol{S})(h\otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}) = A(h\otimes e^1_{\gamma_1} \otimes \cdots \otimes e^k_{\gamma_k}).$$

Since $\|\varphi_A(r\boldsymbol{S}_1, \dots, r\boldsymbol{S}_k)\| \le \|A\|$, an approximation argument shows that

$$\text{SOT-}\lim_{r\to 1} \varphi_A(r\boldsymbol{S}_1, \dots, r\boldsymbol{S}_k) = A. \tag{1-12}$$

Let $\epsilon > 0$ and choose a vector-valued polynomial $q \in \mathcal{P}$ with $\|q\| = 1$ and $\|Aq\| > \|A\| - \epsilon$. Due to (1-12), there is $r_0 \in (0, 1)$ such that $\|\varphi_A(r_0\boldsymbol{S}_1, \dots, r_0\boldsymbol{S}_k)q\| > \|A\| - \epsilon$. Hence, we deduce that $\sup_{r\in[0,1)} \|\varphi_A(r\boldsymbol{S}_1, \dots, r\boldsymbol{S}_k)\| = \|A\|$.

Now, let $r_1, r_2 \in [0, 1)$ with $r_1 < r_2$. We already proved that $g(\boldsymbol{S}) := \varphi_A(r_2\boldsymbol{S}_1, \dots, r_2\boldsymbol{S}_k)$ is in $\operatorname{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$. Due to the von Neumann-type inequality [1951] from [Popescu 2016], we have $\|g(r\boldsymbol{S})\| \le \|g(\boldsymbol{S})\|$ for any $r \in [0, 1)$. In particular, setting $r = r_1/r_2$, we deduce that

$$\|\varphi_A(r_1\boldsymbol{S}_1, \dots, r_1\boldsymbol{S}_k)\| \le \|\varphi_A(r_2\boldsymbol{S}_1, \dots, r_2\boldsymbol{S}_k)\|.$$

It is clear that $\lim_{r\to 1}\|\varphi_A(r\boldsymbol{S}_1,\dots,r\boldsymbol{S}_k)\| = \|A\|$. On the other hand, since $Aq=\varphi_A(\boldsymbol{S})q$ for any vector-valued polynomial $q\in\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$, we deduce that $\|A\|=\sup_{q\in\mathcal{P},\|q\|\le 1}\|\varphi(\boldsymbol{S})q\|$.

Now we prove the converse of the theorem. Let $\{A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\}$ be a family of operators in $B(\mathcal{E})$, where $\alpha_i,\ \beta_i\in\mathbb{F}_{n_i}^+,\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+,\ m_i\in\mathbb{Z}$ and $i\in\{1,\dots,k\}$, and assume that conditions (i) and (ii) hold. Note that, due to (i), $\varphi(\boldsymbol{S})p$ and $\varphi(r\boldsymbol{S})p,\ r\in[0,1)$, are vectors in $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ and

$$\lim_{r\to 1}\varphi(r\boldsymbol{S})p=\varphi(\boldsymbol{S})p$$

for any $p\in\mathcal{P}$. Since $\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(r\boldsymbol{S})p\|<\infty$, there exists a unique bounded linear operator $A_r\in B\big(\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ such that $A_rp=\varphi(r\boldsymbol{S})p$ for any $p\in\mathcal{P}$. If $f\in\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ and $\{p_m\}$ is a sequence of polynomials $p_m\in\mathcal{P}$ such that $p_m\to f$ as $m\to\infty$, we set $A_r(f):=\lim_{m\to\infty}\varphi(r\boldsymbol{S})p_m$. Note that the definition is valid. On the other hand, note that

$$\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(\boldsymbol{S})p\|<\infty.$$

Indeed, this follows from the facts that $\lim_{r\to 1}\varphi(r\boldsymbol{S})p=\varphi(\boldsymbol{S})p$ and $\sup_{p\in\mathcal{P},\|p\|\le 1}\|\varphi(r\boldsymbol{S})p\|<\infty$. Consequently, there is a unique operator $A\in B\big(\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ such that $Ap=\varphi(\boldsymbol{S})p$ for any $p\in\mathcal{P}$. Since $\lim_{r\to 1}A_rp=\lim_{r\to 1}\varphi(r\boldsymbol{S})p=\varphi(\boldsymbol{S})p=Ap$ and $\sup_{r\in[0,1)}\|A_r\|<\infty$, we deduce that $A=$ SOT-$\lim_{r\to 1}A_r$.

Now we show that A is a k-multi-Toeplitz operator. First, note that $\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*$ is a k-multi-Toeplitz operator for any $\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\}$ with $m_i\in\mathbb{Z},\ |\alpha_i|=m_i^-$ and $|\beta_i|=m_i^+$. It is enough to check this on monomials of the form $h\otimes e_{\gamma_1}^1\otimes\cdots\otimes e_{\gamma_k}^k$. Consequently,

$$(I_{\mathcal{E}}\otimes\boldsymbol{R}_{i,s}^*)\varphi(r\boldsymbol{S})(I_{\mathcal{E}}\otimes\boldsymbol{R}_{i,t})p=\delta_{st}\varphi(r\boldsymbol{S})p,\quad s,t\in\{1,\dots,n_i\},$$

for any $p\in\mathcal{P}$ and every $i\in\{1,\dots,k\}$. Hence, A_r has the same property. Taking $r\to 1$, we conclude that A is a k-multi-Toeplitz operator. On the other hand, if $x:=x_1\otimes\cdots\otimes x_k,\ y=y_1\otimes\cdots\otimes y_k$ satisfy (1-3) and $h,\ \ell\in\mathcal{E}$, we have

$$\begin{aligned}
\langle A(h\otimes x),\ell\otimes y\rangle&=\lim_{r\to 1}\langle A_r(h\otimes x),\ell\otimes y\rangle\\
&=\lim_{r\to 1}\langle\varphi(r\boldsymbol{S})(h\otimes x),\ell\otimes y\rangle\\
&=\lim_{r\to 1}\big\langle r^{\sum_{i=1}^k|\alpha_i|+|\beta_i|}A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}h,\ell\big\rangle\\
&=\langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}h,\ell\rangle.
\end{aligned}$$

Therefore,

$$\varphi(\boldsymbol{S}):=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\}\\|\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}}A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*$$

is the formal Fourier series of the k-multi-Toeplitz operator A on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. □

Theorem 1.6. *Let $\{A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\}$ be a family of operators in $B(\mathcal{E})$, where $\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+$, $|\alpha_i|=m_i^-$, $|\beta_i|=m_i^+$, $m_i\in\mathbb{Z}$ and $i\in\{1,\dots,k\}$, and let*

$$\varphi(\boldsymbol{S}):=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes \boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}\boldsymbol{S}_{1,\beta_1}^*\cdots\boldsymbol{S}_{k,\beta_k}^*$$

be the associated formal Fourier series. Then $\varphi(\boldsymbol{S})$ is the formal Fourier series of a k-multi-Toeplitz operator A on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$ if and only if the series defining $\varphi(r\boldsymbol{S})$ is convergent in the operator norm topology for any $r\in[0,1)$ and

$$\sup_{r\in[0,1)}\|\varphi(r\boldsymbol{S})\|<\infty.$$

Moreover, if A is a k-multi-Toeplitz operator on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$, then $\varphi(r\boldsymbol{S})=\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A]$ and

$$\text{SOT-}\lim_{r\to 1}\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A]=A,\quad \textit{where } \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[u]:=(I_{\mathcal{E}}\otimes\boldsymbol{K}_{r\boldsymbol{S}}^*)(u\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_{\mathcal{E}}\otimes\boldsymbol{K}_{r\boldsymbol{S}}),\ u\in B(\mathcal{E}_k),$$

and $\boldsymbol{K}_{r\boldsymbol{S}}$ is the noncommutative Berezin kernel associated with $r\boldsymbol{S}\in\boldsymbol{B_n}\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$.

Proof. Assume that $\varphi(\boldsymbol{S})$ is the formal Fourier series of a k-multi-Toeplitz operator A on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. Then Theorem 1.5 implies that $\varphi(r\boldsymbol{S})$ is convergent in the operator norm topology and

$$\|A\|=\sup_{r\in[0,1)}\|\varphi(r\boldsymbol{S})\|.$$

We recall that the noncommutative Berezin kernel associated with $r\boldsymbol{S}\in\boldsymbol{B_n}\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is defined on $\bigotimes_{i=1}^k F^2(H_{n_i})$ with values in $\bigotimes_{i=1}^k F^2(H_{n_i})\otimes\mathcal{D}_{r\boldsymbol{S}}\subset\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)\otimes\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$, where

$$\mathcal{D}_{r\boldsymbol{S}}:=\overline{\boldsymbol{\Delta}_{r\boldsymbol{S}}(I)\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg)}.$$

Let $\boldsymbol{\gamma}=(\gamma_1,\dots,\gamma_k)$, $\boldsymbol{\omega}=(\omega_1,\dots,\omega_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, set $q:=\max\{|\gamma_1|,\dots|\gamma_k|,|\omega_1|,\dots,|\omega_k|\}$, and define the operator

$$\Gamma_q:=\sum_{m_1\in\mathbb{Z},|m_1|\le q}\cdots\sum_{m_k\in\mathbb{Z},|m_k|\le q}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_k}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k,\beta_1,\dots,\beta_k)}\otimes\boldsymbol{S_\alpha}\boldsymbol{S_\beta^*},$$

where we use the notation $\boldsymbol{S_\alpha}:=\boldsymbol{S}_{1,\alpha_1}\cdots\boldsymbol{S}_{k,\alpha_k}$ if $\boldsymbol{\alpha}:=(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$. We also set $e_{\boldsymbol{\alpha}}:=e_{\alpha_1}^1\otimes\cdots\otimes e_{\alpha_k}^k$. Note that

$$\begin{aligned}
&\langle\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A](h\otimes e_{\boldsymbol{\gamma}}),h'\otimes e_{\boldsymbol{\omega}}\rangle\\
&\ =\big\langle(I_{\mathcal{E}}\otimes\boldsymbol{K}_{r\boldsymbol{S}}^*)(A\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_{\mathcal{E}}\otimes\boldsymbol{K}_{r\boldsymbol{S}})(h\otimes e_{\boldsymbol{\gamma}}),h'\otimes e_{\boldsymbol{\omega}}\big\rangle\\
&\ =\bigg\langle(A\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})\sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} h\otimes e_{\boldsymbol{\alpha}}\otimes\boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} h'\otimes e_{\boldsymbol{\beta}}\otimes\boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\bigg\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \left\langle A(h\otimes e_{\boldsymbol{\alpha}})\otimes \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, h'\otimes e_{\boldsymbol{\beta}}\otimes \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \langle A(h\otimes e_{\boldsymbol{\alpha}}), h'\otimes e_{\boldsymbol{\beta}}\rangle \left\langle \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle \\
&= \sum_{m_1\in\mathbb{Z},|m_1|\le q}\cdots \sum_{m_k\in\mathbb{Z},|m_k|\le q} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_k}^+,\, i\in\{1,\ldots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} \langle \Gamma_q(h\otimes e_{\boldsymbol{\alpha}}), h'\otimes e_{\boldsymbol{\beta}}\rangle \left\langle \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle \\
&= \sum_{\boldsymbol{\alpha}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \sum_{\boldsymbol{\beta}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+} \langle \Gamma_q(h\otimes e_{\boldsymbol{\alpha}}), h'\otimes e_{\boldsymbol{\beta}}\rangle \left\langle \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\alpha}}^*(e_{\boldsymbol{\gamma}}),\, \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)^{1/2}\boldsymbol{S}_{\boldsymbol{\beta}}^*(e_{\boldsymbol{\omega}})\right\rangle \\
&= \left\langle (I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(\Gamma_q\otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_{\mathcal{E}}\otimes \boldsymbol{K}_{r\boldsymbol{S}})(h\otimes e_{\boldsymbol{\gamma}}), h'\otimes e_{\boldsymbol{\omega}}\right\rangle \\
&= \langle \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\mathrm{ext}}[\Gamma_q](h\otimes e_{\boldsymbol{\gamma}}), h'\otimes e_{\boldsymbol{\omega}}\rangle \\
&= \sum_{m_1\in\mathbb{Z},|m_1|\le q}\cdots \sum_{m_k\in\mathbb{Z},|m_k|\le q} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_k}^+,\, i\in\{1,\ldots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} \left\langle \left(A_{(\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_k)}\otimes r^{\sum_{i=1}^k(|\alpha_i|+|\beta_i|)}\boldsymbol{S}_{\boldsymbol{\alpha}}\boldsymbol{S}_{\boldsymbol{\beta}}^*\right)(h\otimes e_\gamma), h'\otimes e_{\boldsymbol{\omega}}\right\rangle \\
&= \langle \varphi_A(rS_1,\ldots,rS_k)(h\otimes e_{\boldsymbol{\gamma}}), h'\otimes e_{\boldsymbol{\omega}}\rangle.
\end{aligned}
$$

Consequently, we obtain

$$
\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\mathrm{ext}}[A] = \varphi_A(r\boldsymbol{S}_1,\ldots,r\boldsymbol{S}_k), \quad r\in[0,1),
$$

which proves the second part of the theorem.

To prove the converse, assume that $\{A_{(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_k)}\}$ is a family of operators in $B(\mathcal{E})$, where $\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+$, $|\alpha_i|=m_i^-$, $|\beta_i|=m_i^+$, $m_i\in\mathbb{Z}$ and $i\in\{1,\ldots,k\}$, and let $\varphi(\boldsymbol{S})$ be the associated formal Fourier series. We also assume that $\varphi(r\boldsymbol{S})$ is convergent in the operator norm topology for each $r\in[0,1)$ and that

$$
M := \sup_{r\in[0,1)} \|\varphi(r\boldsymbol{S})\| < \infty.
$$

Note that $\varphi(r\boldsymbol{S})$ is a k-multi-Toeplitz operator and

$$
\varphi(r\boldsymbol{S})(h\otimes e_{\gamma_1}^1\otimes\cdots\otimes e_{\gamma_k}^k) = \sum_{\substack{\boldsymbol{\omega}=(\omega_1,\ldots,\omega_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ \boldsymbol{\omega}\sim_{\mathrm{rc}}\boldsymbol{\gamma}}} r^{\sum_{i=1}^k(|c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma})|+|c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma})|)} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h\otimes e_{\omega_1}^1\otimes\cdots\otimes e_{\omega_k}^k
$$

is a vector in $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. Hence, we deduce that, for each $\boldsymbol{\gamma}=(\gamma_1,\ldots,\gamma_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$,

$$
\left\langle r^{\sum_{i=1}^k(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma})+c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} \sum_{\substack{\boldsymbol{\omega}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+\\ \boldsymbol{\omega}\sim_{\mathrm{rc}}\boldsymbol{\gamma}}} A^*_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))} h, h\right\rangle \le \|\varphi(r\boldsymbol{S})\|^2\|h\|^2 \le M\|h\|^2
$$

for any $r\in[0,1)$ and $h\in\mathcal{E}$. Taking $r\to 1$, we get condition (i) of Theorem 1.5. Applying Theorem 1.5, we deduce that $\varphi(\boldsymbol{S})$ is the Fourier series of a k-multi-Toeplitz operator. $\square$

We remark that, due to Theorem 1.6, the order of the series in the definition of $\varphi_A(r\boldsymbol{S}_1, \ldots, r\boldsymbol{S}_k)$ (see Theorem 1.5(a)) is irrelevant.

Theorem 1.7. *Let $\boldsymbol{n} = (n_1, \ldots, n_k) \in \mathbb{N}^k$ and let $\mathcal{T}_{\boldsymbol{n}}$ be the set of all k-multi-Toeplitz operators on $\mathcal{E} \otimes \bigotimes_{i=1}^k F^2(H_{n_i})$. Then*

$$\mathcal{T}_{\boldsymbol{n}} = \overline{\operatorname{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}}^{\text{-SOT}} = \overline{\operatorname{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}}^{\text{-WOT}},$$

where $\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}$ is the polyball algebra.

Proof. Let

$$\mathcal{G} := \overline{\operatorname{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}}^{\|\cdot\|}.$$

According to Theorem 1.5, if $A \in \mathcal{T}_{\boldsymbol{n}}$ and $\varphi_A(\boldsymbol{S})$ is its Fourier series, then $\varphi_A(r\boldsymbol{S}) \in \mathcal{G}$ for any $r \in [0, 1)$ and $A = \text{SOT-lim}\, \varphi_A(r\boldsymbol{S})$. Consequently, $\mathcal{T}_{\boldsymbol{n}} \subseteq \bar{\mathcal{G}}^{\text{SOT}}$. Conversely, note that each monomial $\boldsymbol{S}_{\boldsymbol{\alpha}}^* \boldsymbol{S}_{\boldsymbol{\beta}}$, $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, is a k-multi-Toeplitz operator. This shows that, for each $Y \in \mathcal{G}$,

$$(I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,s}^*) Y (I_{\mathcal{E}} \otimes \boldsymbol{R}_{i,t}) = \delta_{st} Y, \quad s, t \in \{1, \ldots, n_i\},$$

for every $i \in \{1, \ldots, k\}$. Consequently, taking SOT-limits, we deduce that $\bar{\mathcal{G}}^{\text{SOT}} \subseteq \mathcal{T}_{\boldsymbol{n}}$, which proves that $\bar{\mathcal{G}}^{\text{SOT}} = \mathcal{T}_{\boldsymbol{n}}$.

Now, if $T \in \bar{\mathcal{G}}^{\text{WOT}}$, an argument as above shows that $T \in \mathcal{T}_{\boldsymbol{n}} = \bar{\mathcal{G}}^{\text{SOT}}$. Since $\bar{\mathcal{G}}^{\text{SOT}} \subseteq \bar{\mathcal{G}}^{\text{WOT}}$, we conclude that $\mathcal{T}_{\boldsymbol{n}} = \bar{\mathcal{G}}^{\text{SOT}} = \bar{\mathcal{G}}^{\text{WOT}}$. $\square$

Corollary 1.8. *The set of all k-multi-Toeplitz operators on $\bigotimes_{i=1}^k F^2(H_{n_i})$ coincides with*

$$\overline{\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}}^{\text{-SOT}} = \overline{\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}}^{\text{-WOT}},$$

where $\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}$ is the polyball algebra.

2. Bounded free k-pluriharmonic functions and the Dirichlet extension problem

In this section, we show that the bounded free k-pluriharmonic functions on $\boldsymbol{B}_{\boldsymbol{n}}$ are precisely the noncommutative Berezin transforms of k-multi-Toeplitz operators and solve the Dirichlet extension problem for the regular polyball $\boldsymbol{B}_{\boldsymbol{n}}$.

Definition 2.1. A function F with operator-valued coefficients in $B(\mathcal{E})$ is called free k-pluriharmonic on the polyball $\boldsymbol{B}_{\boldsymbol{n}}$ if it has the form

$$F(X) = \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1, \ldots, k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} A_{(\alpha_1, \ldots, \alpha_k; \beta_1, \ldots, \beta_k)} \otimes \boldsymbol{X}_{1,\alpha_1} \cdots \boldsymbol{X}_{k,\alpha_k} \boldsymbol{X}_{1,\beta_1}^* \cdots \boldsymbol{X}_{k,\beta_k}^*,$$

where the series converge in the operator norm topology for any $\boldsymbol{X} = (X_1, \ldots, X_k) \in \boldsymbol{B}_{\boldsymbol{n}}(\mathcal{H})$, with $X_i := (X_{i,1}, \ldots, X_{i,n_i})$, and any Hilbert space $\mathcal{H}$.

Due to the remark following Theorem 1.6, one can prove that the order of the series in the definition above is irrelevant. Note that any free holomorphic function on $\boldsymbol{B_n}$ is k-pluriharmonic. Indeed, according to [Popescu 2015b], any free holomorphic function on the polyball $\boldsymbol{B_n}$ has the form

$$f(\boldsymbol{X})=\sum_{m_1\in\mathbb{N}}\cdots\sum_{m_k\in\mathbb{N}}\sum_{\substack{\alpha_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i}} A_{(\alpha_1,\dots,\alpha_k)}\otimes \boldsymbol{X}_{1,\alpha_1}\cdots \boldsymbol{X}_{k,\alpha_k},\qquad \boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H}),$$

where the series converge in the operator norm topology. A function $F:\boldsymbol{B_n}(\mathcal{H})\to B(\mathcal{E}\otimes\mathcal{H})$ is called bounded if

$$\|F\|:=\sup_{\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})}\|F(\boldsymbol{X})\|<\infty.$$

A free k-pluriharmonic function is bounded if its representation on any Hilbert space is bounded. Denote by $\mathbf{PH}^\infty_{\mathcal{E}}(\boldsymbol{B_n})$ the set of all bounded free k-pluriharmonic functions on the polyball $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$. For each $m=1,2,\dots$, we define the norms $\|\cdot\|_m : M_m(\mathbf{PH}^\infty_{\mathcal{E}}(\boldsymbol{B_n}))\to[0,\infty)$ by setting

$$\|[F_{ij}]_m\|_m:=\sup\|[F_{ij}(\boldsymbol{X})]_m\|,$$

where the supremum is taken over all n-tuples $\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})$ and any Hilbert space $\mathcal{H}$. It is easy to see that the norms $\|\cdot\|_m$, $m=1,2,\dots$, determine an operator space structure on $\mathbf{PH}^\infty_{\mathcal{E}}(\boldsymbol{B_n})$, in the sense of Ruan (see, e.g., [Effros and Ruan 2000]).

Let $\boldsymbol{\mathcal{T}_n}$ be the set of all k-multi-Toeplitz operators on $\mathcal{E}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})$. According to Theorem 1.7, we have

$$\boldsymbol{\mathcal{T}_n}=\operatorname{span}\{f^*g : f,g\in B(\mathcal{E})\otimes_{\min}\boldsymbol{\mathcal{A}_n}\}^{\text{-SOT}},$$

where $\boldsymbol{\mathcal{A}_n}$ is the polyball algebra. The main result of this section is the following characterization of bounded free k-pluriharmonic functions:

Theorem 2.2. *If $F:\boldsymbol{B_n}(\mathcal{H})\to B(\mathcal{E})\otimes_{\min}B(\mathcal{H})$, then the following statements are equivalent*:

(i) *F is a bounded free k-pluriharmonic function*;

(ii) *there exists $A\in\boldsymbol{\mathcal{T}_n}$ such that*

$$F(\boldsymbol{X})=\boldsymbol{\mathcal{B}}^{\text{ext}}_{\boldsymbol{X}}[A]:=(I_{\mathcal{E}}\otimes\boldsymbol{K}^*_{\boldsymbol{X}})(A\otimes I_{\mathcal{H}})(I_{\mathcal{E}}\otimes\boldsymbol{K}_{\boldsymbol{X}}),\qquad \boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H}).$$

In this case, $A=\text{SOT-}\lim_{r\to1}F(r\boldsymbol{S})$. *Moreover*, *the map*

$$\Phi:\mathbf{PH}^\infty_{\mathcal{E}}(\boldsymbol{B_n})\to\boldsymbol{\mathcal{T}_n},\qquad \Phi(F):=A,$$

is a completely isometric isomorphism of operator spaces.

Proof. Assume that F is a bounded free k-pluriharmonic function on $\boldsymbol{B_n}$ and has the representation from Definition 2.1. Then, for any $r\in[0,1)$,

$$F(r\boldsymbol{S})\in\operatorname{span}\{f^*g : f,g\in B(\mathcal{E})\otimes_{\min}\boldsymbol{\mathcal{A}_n}\}^{\text{-}\|\cdot\|}$$

and, due to the noncommutative von Neumann inequality [Popescu 1999], we have $\sup_{r\in[0,1)}\|F(r\boldsymbol{S})\|=\|F\|_\infty<\infty$. According to Theorem 1.6, $F(\boldsymbol{S})$ is the formal Fourier series of a k-multi-Toeplitz operator

$A \in B(\mathcal{E} \otimes \bigotimes_{i=1} F^2(H_{n_i}))$ and $A = \text{SOT-lim}_{r\to 1} F(r\boldsymbol{S}) \in \boldsymbol{\mathcal{T}_n}$. Using the properties of the noncommutative Berezin kernel on polyballs, we have

$$F(r\boldsymbol{X}) = (I_\mathcal{E} \otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[F(r\boldsymbol{S}) \otimes I_\mathcal{H}](I_\mathcal{E} \otimes \boldsymbol{K}_{\boldsymbol{X}}), \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

Since the map $Y \mapsto Y \otimes I_\mathcal{H}$ is SOT-continuous on bounded subsets of $B(\mathcal{E} \otimes \bigotimes_{i=1} F^2(H_{n_i}))$, we deduce that

$$\text{SOT-}\lim_{r\to 1} F(r\boldsymbol{X}) = (I_\mathcal{E} \otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[A \otimes I_\mathcal{H}](I_\mathcal{E} \otimes \boldsymbol{K}_{\boldsymbol{X}}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A].$$

Since F is continuous in the norm topology on $\boldsymbol{B_n}(\mathcal{H})$, we have $F(r\boldsymbol{X}) \to F(\boldsymbol{X})$ as $r \to 1$. Consequently, the relation above implies $F(\boldsymbol{X}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$, which completes the proof that (i) implies (ii).

To prove that (ii) implies (i), let $A \in \mathcal{T}_{\boldsymbol{n}}$ and $F(\boldsymbol{X}) := \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$ for $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since A is a k-multi-Toeplitz operator, Theorem 1.5 shows that it has a formal Fourier series

$$\varphi(\boldsymbol{S}) := \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\} \\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*$$

with the property that the series $\varphi(r\boldsymbol{S})$ is convergent in the operator norm topology to an operator in $\text{span}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{-\|\cdot\|}$. Moreover, we have $A = \text{SOT-lim}_{r\to 1} \varphi(r\boldsymbol{S})$ and

$$\|A\| = \sup_{r\in[0,1)} \|\varphi(r\boldsymbol{S})\|.$$

Hence, the map $\boldsymbol{X} \mapsto \varphi(\boldsymbol{X})$ is a k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$. On the other hand, due to Theorem 1.6, we have $\varphi(r\boldsymbol{S}) = \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A]$, where

$$\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[u] := (I_\mathcal{E} \otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(u \otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_\mathcal{E} \otimes \boldsymbol{K}_{r\boldsymbol{S}}), \quad u \in B(\mathcal{E}_k),$$

and $\boldsymbol{K}_{r\boldsymbol{S}}$ is the noncommutative Berezin kernel associated with $r\boldsymbol{S} \in \boldsymbol{B_n}(\bigotimes_{i=1}^k F^2(H_{n_i}))$. Note that

$$\varphi(r\boldsymbol{X}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[\varphi(r\boldsymbol{S})] = (I_\mathcal{E} \otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[\varphi(r\boldsymbol{S}) \otimes I_\mathcal{H}](I_\mathcal{E} \otimes \boldsymbol{K}_{\boldsymbol{X}}).$$

Now, using continuity of φ on $\boldsymbol{B_n}(\mathcal{H})$ and the fact that $A = \text{SOT-lim}_{r\to 1} \varphi(r\boldsymbol{S})$, we deduce that

$$\varphi(\boldsymbol{X}) = \text{SOT-}\lim_{r\to 1} \varphi(r\boldsymbol{X}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A] = F(\boldsymbol{X}), \quad \boldsymbol{X} \in \boldsymbol{B}_n(\mathcal{H}).$$

To prove the last part of the theorem, let $[F_{ij}]_m \in M_m(\mathbf{PH}_\mathcal{E}^\infty(\boldsymbol{B_n}))$ and use the noncommutative von Neumann inequality to obtain

$$\|[F_{ij}]_m\| = \sup_{\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})} \|[F_{ij}(\boldsymbol{X})]_m\| = \sup_{r\in[0,1)} \|[F_{ij}(r\boldsymbol{S})]_m\|.$$

On the other hand, $A_{ij} := \text{SOT-lim}_{r\to 1} F_{ij}(rS)$ is a k-multi-Toeplitz operator and

$$F_{ij}(rS) = (I_\mathcal{E} \otimes \boldsymbol{K}_{r\boldsymbol{S}}^*)(A_{ij} \otimes I_{\bigotimes_{i=1}^k F^2(H_{n_i})})(I_\mathcal{E} \otimes \boldsymbol{K}_{r\boldsymbol{S}}).$$

Hence, we obtain

$$\sup_{r\in[0,1)} \|[F_{ij}(r\boldsymbol{S})]_m\| \le \|[A_{ij}]_m\|.$$

Since $[A_{ij}]_m := \text{SOT-}\lim_{r\to 1}[F_{ij}(rS)]_m$, we deduce that the inequality above is in fact an equality. This shows that Φ is a completely isometric isomorphisms of operator spaces. $\square$

As a consequence, we can obtain the following Fatou-type result concerning the boundary behaviour of bounded k-pluriharmonic functions.

Corollary 2.3. *If $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ is a bounded free k-pluriharmonic function and $\boldsymbol{X}$ is a pure element in $\boldsymbol{B_n}(\mathcal{H})^-$, then the limit*

$$\text{SOT-}\lim_{r\to 1} F(r\boldsymbol{X})$$

exists.

Proof. If $\boldsymbol{X}$ is a pure element in $\boldsymbol{B_n}(\mathcal{H})^-$, then the noncommutative Berezin kernel $\boldsymbol{K_X}$ is an isometry (see [Popescu 2016]). Since F is free k-pluriharmonic function on $\boldsymbol{B_n}$, we have

$$F(r\boldsymbol{S}) \in \overline{\text{span}}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{\|\cdot\|}$$

and $F(r\boldsymbol{S})$ converges in the operator norm topology. Consequently,

$$F(r\boldsymbol{X}) = (I_{\mathcal{E}} \otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[F(r\boldsymbol{S}) \otimes I_{\mathcal{H}}](I_{\mathcal{E}} \otimes \boldsymbol{K_X}).$$

Since F is bounded, Theorem 2.2 implies $\text{SOT-}\lim_{r\to 1} F(r\boldsymbol{S}) = A \in \boldsymbol{\mathcal{T}_n}$ and $\sup_{0\le r<1} \|F(r\boldsymbol{S})\| < \infty$. Using these facts in the relation above, we conclude that $\text{SOT-}\lim_{r\to 1} F(r\boldsymbol{X})$ exists. $\square$

We denote by $\mathbf{PH}^c_{\mathcal{E}}(\boldsymbol{B_n})$ the set of all free k-pluriharmonic functions on $\boldsymbol{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$, which have continuous extensions (in the operator norm topology) to the closed polyball $\boldsymbol{B_n}(\mathcal{H})^-$ for any Hilbert space $\mathcal{H}$. Throughout this section, we assume that $\mathcal{H}$ is an infinite-dimensional Hilbert space. In what follows we solve the Dirichlet extension problem for the regular polyballs.

Theorem 2.4. *If $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$, then the following statements are equivalent*:

(i) *F is a free k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$ such that $F(r\boldsymbol{S})$ converges in the operator norm topology as $r \to 1$.*

(ii) *There exists $A \in \boldsymbol{\mathcal{P}} := \overline{\text{span}}\{f^*g : f, g \in B(\mathcal{E}) \otimes_{\min} \boldsymbol{\mathcal{A}_n}\}^{\|\cdot\|}$ such that*

$$F(\boldsymbol{X}) = \boldsymbol{\mathcal{B}}^{\text{ext}}_{\boldsymbol{X}}[A], \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

(iii) *F is a free k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$ which has a continuous extension* (*in the operator norm topology*) *to the closed ball $\boldsymbol{B_n}(\mathcal{H})^-$.*

In this case, $A = \lim_{r\to 1} F(r\boldsymbol{S})$, where the convergence is in the operator norm. Moreover, the map

$$\Phi : \mathbf{PH}^c_{\mathcal{E}}(\boldsymbol{B_n}) \to \boldsymbol{\mathcal{P}}, \quad \Phi(F) := A,$$

is a completely isometric isomorphism of operator spaces.

Proof. Assume that (i) holds. Then F has a representation

$$F(\boldsymbol{X}) = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)} \otimes \boldsymbol{X}_{1,\alpha_1}\cdots \boldsymbol{X}_{k,\alpha_k}\boldsymbol{X}_{1,\beta_1}^*\cdots \boldsymbol{X}_{k,\beta_k}^*,$$

where the series converge in the operator norm topology for any $\boldsymbol{X} = (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})$. Since the series defining $F(r\boldsymbol{S})$ converges in the operator topology, we deduce that

$$A := \lim_{r\to 1} F(r\boldsymbol{S}) \in \boldsymbol{\mathcal{P}}. \tag{2-1}$$

On the other hand, we have

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[F(r\boldsymbol{S})] = (I_{\mathcal{E}} \otimes \boldsymbol{K}_{\boldsymbol{X}}^*)[F(r\boldsymbol{S}) \otimes I_{\mathcal{H}}](I_{\mathcal{E}} \otimes \boldsymbol{K}_{\boldsymbol{X}}) = F(r\boldsymbol{X})$$

for any $r \in [0, 1)$ and $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Hence, and using (2-1), we deduce that

$$\boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A] = \lim_{r\to 1} F(r\boldsymbol{X}) = F(\boldsymbol{X}),$$

which proves (ii). Now we show that (ii) implies (i). Assuming (ii) and taking into account Theorem 1.7, one can see that A is a k-multi-Toeplitz operator. As in the proof of Theorem 2.2, the map defined by $F(\boldsymbol{X}) := \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$, $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, is a bounded free k-pluriharmonic function. Moreover, we proved that

$$F(r\boldsymbol{S}) = \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A], \quad r \in [0, 1), \tag{2-2}$$

$F(r\boldsymbol{S}) \in \boldsymbol{\mathcal{P}}$ and also that $A = \text{SOT-}\lim_{r\to 1} F(r\boldsymbol{S})$ and $\|A\| = \sup_{r\in[0,1)} \|F(r\boldsymbol{S})\|$. Since $A \in \boldsymbol{\mathcal{P}}$, there is a sequence of polynomials q_m in $\boldsymbol{S}_{\boldsymbol{\alpha}}^*\boldsymbol{S}_{\boldsymbol{\beta}}$ such that $q_m \to A$ in norm as $m \to \infty$. For any $\epsilon > 0$, let $N \in \mathbb{N}$ be such that $\|A - q_m\| < \frac{1}{3}\epsilon$ for any $m \geq N$. Choose $\delta \in (0, 1)$ such that $\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[q_N] - q_N\| < \frac{1}{3}\epsilon$ for any $r \in (\delta, 1)$. Note that

$$\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A] - A\| \leq \|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A - q_N]\| + \|\boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[q_N] - q_N\| + \|q_N - A\| \leq \|A - q_N\| + \tfrac{2}{3}\epsilon < \epsilon$$

for any $r \in (\delta, 1)$. Therefore, $\lim_{r\to 1} \boldsymbol{\mathcal{B}}_{r\boldsymbol{S}}^{\text{ext}}[A] = A$ in the norm topology. Hence, and due to (2-2), we deduce that $\lim_{r\to 1} F(r\boldsymbol{S}) = A$ in the norm topology, which shows that (i) holds. Since $\mathcal{H}$ is infinite-dimensional, that (iii) implies (i) is clear.

It remains to prove that (ii) implies (iii). We assume that (ii) holds. Then there exists $A \in \boldsymbol{\mathcal{P}}$ such that $F(\boldsymbol{X}) = \boldsymbol{\mathcal{B}}_{\boldsymbol{X}}^{\text{ext}}[A]$ for all $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Due to Theorem 2.2, F is a bounded free k-pluriharmonic function on $\boldsymbol{B_n}(\mathcal{H})$. For any $\boldsymbol{Y} \in \boldsymbol{B_n}(\mathcal{H})^-$, one can show, as in the proof that (ii) implies (i), that $\widetilde{F}(\boldsymbol{Y}) := \lim_{r\to 1} \boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A]$ exists in the operator norm topology. Since $\|\boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A]\| \leq \|A\|$ for any $r \in [0, 1)$, we deduce that $\|\widetilde{F}(\boldsymbol{Y})\| \leq \|A\|$ for any $\boldsymbol{Y} \in \boldsymbol{B_n}(\mathcal{H})^-$. Note also that $\widetilde{F}$ is an extension of F. Lastly, we show that $\widetilde{F}$ is continuous on $\boldsymbol{B_n}(\mathcal{H})^-$. To this end, let $\epsilon > 0$ and, due to the equivalence of (ii) and (i), we can choose $r_0 \in [0, 1)$ such that $\|A - F(r_0\boldsymbol{S})\| < \frac{1}{3}\epsilon$. Since $A - F(r_0\boldsymbol{S}) \in \boldsymbol{\mathcal{P}}$, we deduce that

$$\|\widetilde{F}(\boldsymbol{Y}) - F(r_0\boldsymbol{Y})\| = \|\lim_{r\to 1} \boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A] - F(r_0\boldsymbol{Y})\| \leq \limsup_{r\to 1} \|\boldsymbol{\mathcal{B}}_{r\boldsymbol{Y}}^{\text{ext}}[A] - F(r_0\boldsymbol{Y})\| \leq \|A - F(r_0\boldsymbol{Y})\| < \tfrac{1}{3}\epsilon$$

for any $\boldsymbol{Y} \in \boldsymbol{B_n}(\mathcal{H})^-$. Since F is continuous on $\boldsymbol{B_n}(\mathcal{H})$, there is $\delta > 0$ such that $\|F(r_0\boldsymbol{Y}) - F(r_0\boldsymbol{W})\| < \frac{1}{3}\epsilon$ for any $\boldsymbol{W} \in \boldsymbol{B_n}(\mathcal{H})^-$ with $\|\boldsymbol{W} - \boldsymbol{Y}\| < \delta$. Now note that

$$\|\widetilde{F}(\boldsymbol{Y}) - \widetilde{F}(\boldsymbol{W})\| \le \|\widetilde{F}(\boldsymbol{Y}) - F(r_0\boldsymbol{Y})\| + \|F(r_0\boldsymbol{Y}) - F(r_0\boldsymbol{W})\| + \|F(r_0\boldsymbol{W}) - \widetilde{F}(\boldsymbol{W})\| < \epsilon$$

for any $\boldsymbol{W} \in \boldsymbol{B_n}(\mathcal{H})^-$ with $\|\boldsymbol{W} - \boldsymbol{Y}\| < \delta$. □

3. Naimark-type dilation theorem for direct products of free semigroups

In this section, we provide a Naimark-type dilation theorem for direct products $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ of unital free semigroups, and use it to obtain a structure theorem which characterizes the positive free k-pluriharmonic functions on the regular polyball with operator-valued coefficients.

Consider the unital semigroup $\boldsymbol{F_n^+} := \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ with neutral element $\boldsymbol{g} := (g_0^1, \ldots, g_0^k)$. Let $\boldsymbol{\omega} = (\omega_1, \ldots, \omega_k)$ and $\boldsymbol{\gamma} = (\gamma_1, \ldots, \gamma_k)$ be in $\mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$. We say that $\boldsymbol{\omega}$ and $\boldsymbol{\gamma}$ are left comparable, and write $\boldsymbol{\omega} \sim_{\mathrm{lc}} \boldsymbol{\gamma}$, if for each $i \in \{1, \ldots, k\}$, one of the conditions $\omega_i <_l \gamma_i$, $\gamma_i <_l \omega_i$ or $\omega_i = \gamma_i$ holds (see the definitions preceding Lemma 1.2). In this case, we define

$$c_l^+(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c_l^+(\omega_1, \gamma_1), \ldots, c_l^+(\omega_k, \gamma_k)) \quad \text{and} \quad c_l^-(\boldsymbol{\omega}, \boldsymbol{\gamma}) := (c_l^-(\omega_1, \gamma_1), \ldots, c_l^-(\omega_k, \gamma_k)),$$

where

$$c_l^+(\omega, \gamma) := \begin{cases} \omega \backslash_l \gamma & \text{if } \gamma <_l \omega, \\ g_0 & \text{if } \omega <_l \gamma \text{ or } \omega = \gamma, \end{cases} \quad \text{and} \quad c_l^-(\omega, \gamma) := \begin{cases} \gamma \backslash_r \omega & \text{if } \omega <_l \gamma, \\ g_0 & \text{if } \gamma <_l \omega \text{ or } \omega = \gamma. \end{cases}$$

We say that $K : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{E})$ is a *left k-multi-Toeplitz kernel* if $K(\boldsymbol{g}, \boldsymbol{g}) = I_{\mathcal{E}}$ and

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \begin{cases} K(c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega}); c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega})) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise.} \end{cases}$$

The kernel K is positive semidefinite if, for each $m \in \mathbb{N}$, any choice of $h_1, \ldots h_m \in \mathcal{E}$, and any $\boldsymbol{\sigma}^{(i)} := (\sigma_1^{(i)}, \ldots, \sigma_k^{(i)}) \in \boldsymbol{F_n^+}$, it satisfies the inequality

$$\sum_{i,j=1}^m \langle K(\boldsymbol{\sigma}^{(i)}, \boldsymbol{\sigma}^{(j)})h_j, h_i \rangle \ge 0.$$

Definition 3.1. A map $K : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{E})$ has a Naimark dilation if there exists a k-tuple of commuting row isometries $\boldsymbol{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, on a Hilbert space $\mathcal{K} \supset \mathcal{E}$, i.e., the nonselfadjoint algebra $\mathrm{Alg}(V_i)$ commutes with $\mathrm{Alg}(V_s)$ for any $i, s \in \{1, \ldots, k\}$ with $i \neq s$, such that

$$K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = P_{\mathcal{E}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}\big|_{\mathcal{E}}, \quad \boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F_n^+}.$$

The dilation is called minimal if $\mathcal{K} = \bigvee_{\boldsymbol{\omega} \in \boldsymbol{F_n^+}} \boldsymbol{V}_{\boldsymbol{\omega}} \mathcal{E}$.

Theorem 3.2. *A map $K : \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} \to B(\mathcal{H})$ is a positive semidefinite left k-multi-Toeplitz kernel on the direct product $\boldsymbol{F_n^+}$ of free semigroups if and only if it admits a Naimark dilation.*

Proof. Let $\mathcal{K}_0$ be the vector space of all sums of tensor monomials $\sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\boldsymbol{\sigma}}\otimes h_{\boldsymbol{\sigma}}$, where $\{h_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+}$ is a finitely supported sequence of vectors in $\mathcal{H}$. Define the sesquilinear form $\langle \cdot , \cdot \rangle_{\mathcal{K}_0}$ on $\mathcal{K}_0$ by setting

$$\left\langle \sum_{\boldsymbol{\omega}\in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\boldsymbol{\omega}}\otimes h_{\boldsymbol{\omega}}, \sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\boldsymbol{\sigma}}\otimes h'_{\boldsymbol{\sigma}} \right\rangle_{\mathcal{K}_0} := \sum_{\boldsymbol{\omega},\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle K(\boldsymbol{\sigma},\boldsymbol{\omega}) h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\rangle_{\mathcal{H}}, \quad h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\in \mathcal{H}.$$

Since K is positive semidefinite, so is $\langle \cdot , \cdot \rangle_{\mathcal{K}_0}$. Set $\mathcal{N}:=\{f\in \mathcal{K}_0 : \langle f, f\rangle = 0\}$ and define the Hilbert space obtained by completing $\mathcal{K}_0/\mathcal{N}$ with the induced inner product. For each $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, define the operator $V_{i,j}$ on $\mathcal{K}_0$ by setting

$$V_{i,j}\Bigl(\sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\boldsymbol{\sigma}}\otimes h_{\boldsymbol{\sigma}}\Bigr) := \sum_{\boldsymbol{\sigma}=(\sigma_1,\dots,\sigma_k)\in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\sigma_1}\otimes\cdots\otimes e_{\sigma_{i-1}}\otimes e_{g_j\sigma_i}\otimes e_{\sigma_{i+1}}\otimes\cdots\otimes e_{\sigma_k}\otimes h_{\boldsymbol{\sigma}}.$$

Note that if $p\in\{1,\dots,n_i\}$ then

$$\begin{aligned}
&\left\langle V_{i,j}\Bigl(\sum_{\boldsymbol{\omega}\in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\boldsymbol{\omega}}\otimes h_{\boldsymbol{\omega}}\Bigr), V_{i,p}\Bigl(\sum_{\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\boldsymbol{\sigma}}\otimes h'_{\boldsymbol{\sigma}}\Bigr)\right\rangle_{\mathcal{K}_0}\\
&\qquad = \sum_{\boldsymbol{\omega},\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle K(\sigma_1,\dots,\sigma_{i-1}, g_j\sigma_i,\sigma_{i+1},\dots,\sigma_k;\omega_1,\dots,\omega_{i-1},g_p\omega_i,\omega_{i+1},\dots,\omega_k)h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\rangle_{\mathcal{H}}\\
&\qquad = \begin{cases} \sum_{\boldsymbol{\omega},\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle K(\boldsymbol{\sigma},\boldsymbol{\omega}) h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}}\rangle_{\mathcal{H}} & \text{if } j=p,\\ 0 & \text{otherwise.}\end{cases}
\end{aligned}$$

Hence and using the definition of $\langle \cdot , \cdot \rangle_{\mathcal{K}_0}$, we deduce that, for each $i\in\{1,\dots,k\}$, the operators $V_{i,1},\dots,V_{i,n_i}$ can be extended by continuity to isometries on $\mathcal{K}$ with orthogonal ranges. Note also that, if $i, s\in\{1,\dots,k\}$, $i\neq s$, $j\in\{1,\dots,n_i\}$ and $t\in\{1,\dots,n_s\}$, then

$$V_{i,j}V_{st}(e_{\sigma_1}\otimes\cdots\otimes e_{\sigma_k}\otimes h) = e_{\sigma_1}\otimes\cdots\otimes e_{\sigma_{i-1}}\otimes e_{g_j\sigma_i}\otimes e_{\sigma_{i+1}}\cdots\otimes e_{\sigma_{s-1}}\otimes e_{\sigma_{g_t\sigma_s}}\otimes e_{\sigma_{s+1}}\otimes\cdots\otimes e_{\sigma_k}\otimes h$$

when $i<s$. This shows that $V_{ij}V_{st}=V_{st}V_{ij}$. Since

$$\langle e_{\boldsymbol{g}}\otimes h, e_{\boldsymbol{g}}\otimes h'\rangle_{\mathcal{K}} = \langle K(\boldsymbol{g},\boldsymbol{g})h, h'\rangle_{\mathcal{H}} = \langle h, h'\rangle_{\mathcal{H}}, \quad h, h'\in\mathcal{H},$$

we can embed $\mathcal{H}$ into $\mathcal{K}$ by setting $h=e_{\boldsymbol{g}}\otimes h$. Note that, for any $\boldsymbol{\omega},\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+$ and $h, h'\in\mathcal{H}$, we have

$$\langle V_{\boldsymbol{\sigma}}^* V_{\boldsymbol{\omega}} h, h'\rangle_{\mathcal{K}} = \langle V_{\boldsymbol{\omega}}h, V_{\boldsymbol{\sigma}}h'\rangle_{\mathcal{K}} = \langle e_{\boldsymbol{\omega}}\otimes h, e_{\boldsymbol{\sigma}}\otimes h'\rangle_{\mathcal{K}} = \langle K(\boldsymbol{\sigma},\boldsymbol{\omega})h, h'\rangle_{\mathcal{H}}.$$

Therefore, $K(\boldsymbol{\sigma},\boldsymbol{\omega}) = P_{\mathcal{H}}\boldsymbol{V}_{\boldsymbol{\sigma}}^*\boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{H}}$ for any $\boldsymbol{\sigma},\boldsymbol{\omega}\in \boldsymbol{F}_{\boldsymbol{n}}^+$. Since any element in $\mathcal{K}_0$ is a linear combination of vectors $\boldsymbol{V}_{\boldsymbol{\sigma}}h$, where $\boldsymbol{\sigma}\in \boldsymbol{F}_{\boldsymbol{n}}^+$ and $h\in\mathcal{H}$, we deduce that $\mathcal{K}=\bigvee_{\boldsymbol{\omega}\in \boldsymbol{F}_{\boldsymbol{n}}^+}\boldsymbol{V}_{\boldsymbol{\omega}}\mathcal{H}$, which proves the minimality of the Naimark dilation.

Now we prove the converse. Let $\boldsymbol{V}=(V_1,\dots,V_n)$ and $V_i=(V_{i,1},\dots,V_{i,n_i})$ be k-tuples of commuting row isometries on a Hilbert space $\mathcal{K}\supset\mathcal{H}$. Define $K:\boldsymbol{F}_{\boldsymbol{n}}^+\times\boldsymbol{F}_{\boldsymbol{n}}^+\to B(\mathcal{H})$ by setting $K(\boldsymbol{\sigma},\boldsymbol{\omega}) = P_{\mathcal{H}}\boldsymbol{V}_{\boldsymbol{\sigma}}^*\boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{H}}$ for any $\boldsymbol{\sigma},\boldsymbol{\omega}\in\boldsymbol{F}_{\boldsymbol{n}}^+$. Assume that $\boldsymbol{\sigma},\boldsymbol{\omega}\in\boldsymbol{F}_{\boldsymbol{n}}^+$ and $\boldsymbol{\sigma}\sim_{\rm lc}\boldsymbol{\omega}$. Using the commutativity of the row isometries $V_1,\dots,V_k$, we can assume without loss of generality that there is $p\in\{1,\dots,k\}$ such that

$\omega_1 \le_l \sigma_1, \ldots, \omega_p \le_l \sigma_p, \sigma_{p+1} \le_l \omega_{p+1}, \ldots, \sigma_k \le_l \omega$. Since each $V_i = (V_{i,1}, \ldots, V_{i,n_i})$ is an isometry, we have $V_{i,t}^* V_{i,s} = \delta_{ts} I$. Consequently, and using the commutativity of the row isometries, we deduce that

$$\begin{aligned}
&\langle V_{1,\omega_1} \cdots V_{k,\omega_k} h, V_{1,\sigma_1} \cdots V_{k,\sigma_k} h' \rangle \\
&\qquad = \langle V_{2,\omega_2} \cdots V_{k,\omega_k} h, V_{1,\sigma_1 \backslash_l \omega_1} V_{2,\sigma_2} \cdots V_{k,\sigma_k} h' \rangle \\
&\qquad = \langle V_{2,\omega_2} \cdots V_{k,\omega_k} h, V_{2,\sigma_2} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} h' \rangle \\
&\qquad \ \vdots \\
&\qquad = \langle V_{p+1,\omega_{p+1}} \cdots V_{k,\omega_k} h, V_{p+1,\sigma_{p+1}} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad = \langle V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} V_{p+2,\omega_{p+2}} \cdots V_{k,\omega_k} h, V_{p+2,\sigma_{p+2}} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad = \langle V_{p+2,\omega_{p+2}} \cdots V_{k,\omega_k} V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} h, V_{p+2,\sigma_{p+2}} \cdots V_{k,\sigma_k} V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad \ \vdots \\
&\qquad = \langle V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} \cdots V_{k,\omega_k \backslash_l \sigma_k} h, V_{1,\sigma_1 \backslash_l \omega_1} \cdots V_{p,\sigma_p \backslash_l \omega_p} h' \rangle \\
&\qquad = \langle V_{1,\sigma_1 \backslash_l \omega_1}^* \cdots V_{p,\sigma_p \backslash_l \omega_p}^* V_{p+1,\omega_{p+1} \backslash_l \sigma_{p+1}} \cdots V_{k,\omega_k \backslash_l \sigma_k} h, h' \rangle
\end{aligned}$$

for any $h, h' \in \mathcal{H}$. Therefore, for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$, we have

$$\begin{aligned}
K(\boldsymbol{\sigma}, \boldsymbol{\omega}) = P_{\mathcal{H}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}\big|_{\mathcal{H}} &= \begin{cases} P_{\mathcal{H}} \boldsymbol{V}_{c_l^+(\boldsymbol{\sigma},\boldsymbol{\omega})}^* \boldsymbol{V}_{c_l^-(\boldsymbol{\sigma},\boldsymbol{\omega})}\big|_{\mathcal{H}} & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise}, \end{cases} \\
&= \begin{cases} K(c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega}); c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega})) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise}, \end{cases}
\end{aligned}$$

and $K(\boldsymbol{g}, \boldsymbol{g}) = I_{\mathcal{H}}$. This shows that K is a left k-multi-Toeplitz kernel on $\boldsymbol{F}_{\boldsymbol{n}}^+$. On the other hand, for any finitely supported sequence $\{h_{\boldsymbol{\omega}}\}_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+}$ of elements in $\mathcal{H}$, we have

$$\sum_{\boldsymbol{\omega}, \boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle K(\boldsymbol{\sigma}, \boldsymbol{\omega}) h_{\boldsymbol{\omega}}, h_{\boldsymbol{\sigma}} \rangle = \sum_{\boldsymbol{\omega}, \boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle P_{\mathcal{H}} \boldsymbol{V}_{\boldsymbol{\sigma}}^* \boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{H}} h_{\boldsymbol{\omega}}, h_{\boldsymbol{\sigma}} \rangle = \Bigg\| \sum_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\omega}} h_{\boldsymbol{\omega}} \Bigg\|^2 \ge 0.$$

Therefore, K is a positive semidefinite left k-multi-Toeplitz kernel on $\boldsymbol{F}_{\boldsymbol{n}}^+$. $\square$

We remark that the Naimark dilation provided in Theorem 3.2 is minimal. To prove the uniqueness of the minimal Naimark dilation, let $\boldsymbol{V}' = (V_1', \ldots, V_n')$, $V_i' = (V_{i,1}', \ldots, V_{i,n_i}')$, be a k-tuple of commuting row isometries on a Hilbert space $\mathcal{K}' \supset \mathcal{H}$ such that $K(\sigma, \omega) = P_{\mathcal{H}}^{\mathcal{K}'} (\boldsymbol{V}_{\boldsymbol{\sigma}}')^* \boldsymbol{V}_{\boldsymbol{\omega}}'\big|_{\mathcal{H}}$ for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$ and with the property that $\mathcal{K} = \bigvee_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\omega}}' \mathcal{H}$. For any $x, y \in \mathcal{H}$, we have

$$\langle \boldsymbol{V}_{\boldsymbol{\omega}} x, \boldsymbol{V}_{\boldsymbol{\sigma}} y \rangle_{\mathcal{K}} = \langle K(\boldsymbol{\sigma}, \boldsymbol{\omega}) x, y \rangle_{\mathcal{H}} = \langle P_{\mathcal{H}}^{\mathcal{K}'} (\boldsymbol{V}_{\boldsymbol{\sigma}}')^* \boldsymbol{V}_{\boldsymbol{\omega}}' x, y \rangle_{\mathcal{K}'} = \langle \boldsymbol{V}_{\boldsymbol{\omega}}' x, \boldsymbol{V}_{\boldsymbol{\sigma}}' y \rangle_{\mathcal{K}'}.$$

Consequently, the map

$$W\Bigg(\sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\sigma}} h_{\boldsymbol{\sigma}} \Bigg) := \sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \boldsymbol{V}_{\boldsymbol{\sigma}}' h_{\boldsymbol{\sigma}},$$

where $\{h_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+}$ is any finitely supported sequence of vectors in $\mathcal{H}$, is well-defined. Due to the minimality of the spaces $\mathcal{K}$ and $\mathcal{K}'$, the map extends to a unitary operator W from $\mathcal{K}$ onto $\mathcal{K}'$. Note also that

$WV_{i,j} = V'_{i,j}W$ for any $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, which completes the proof of the uniqueness of the minimal Naimark dilation.

We should mention that there is a dual Naimark-type dilation for positive semidefinite right k-multi-Toeplitz kernels. A kernel $\Gamma : \boldsymbol{F}_{\boldsymbol{n}}^+ \times \boldsymbol{F}_{\boldsymbol{n}}^+ \to B(\mathcal{E})$ is called *right k-multi-Toeplitz* if $\Gamma(\boldsymbol{g}, \boldsymbol{g}) = I_{\mathcal{E}}$ and

$$\Gamma(\boldsymbol{\sigma}, \boldsymbol{\omega}) = \begin{cases} \Gamma(c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega}); c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{rc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise,} \end{cases}$$

where $c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega})$, $c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})$ are defined by (1-1). We say that Γ has a Naimark dilation if there exists a k-tuple $\boldsymbol{W} = (W_1, \dots, W_n)$, $W_i = (W_{i,1}, \dots, W_{i,n_i})$, of commuting row isometries on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ such that $\Gamma(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) = P_{\mathcal{E}} \boldsymbol{W}_{\boldsymbol{\sigma}}^* \boldsymbol{W}_{\boldsymbol{\omega}}|_{\mathcal{E}}$ for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$.

Theorem 3.3. *A map $\Gamma : \boldsymbol{F}_{\boldsymbol{n}}^+ \times \boldsymbol{F}_{\boldsymbol{n}}^+ \to B(\mathcal{H})$ is a positive semidefinite right k-multi-Toeplitz kernel on $\boldsymbol{F}_{\boldsymbol{n}}^+$ if and only if it admits a Naimark dilation. In this case, there is a minimal dilation which is uniquely determined up to isomorphism.*

Proof. We only sketch the proof, which is very similar to that of Theorem 3.2, pointing out the differences. First, $\mathcal{K}_0$ is the vector space of all sums of tensor monomials $\sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\sigma}}} \otimes h_{\boldsymbol{\sigma}}$, where $\{h_{\boldsymbol{\sigma}}\}_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+}$ is a finitely supported sequence of vectors in $\mathcal{H}$, while the sesquilinear form $\langle \cdot, \cdot \rangle_{\mathcal{K}_0}$ on $\mathcal{K}_0$ is defined by setting

$$\left\langle \sum_{\boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\omega}}} \otimes h_{\boldsymbol{\omega}}, \sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\sigma}}} \otimes h'_{\boldsymbol{\sigma}} \right\rangle_{\mathcal{K}_0} := \sum_{\boldsymbol{\omega}, \boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} \langle \Gamma(\boldsymbol{\sigma}, \boldsymbol{\omega}) h_{\boldsymbol{\omega}}, h'_{\boldsymbol{\sigma}} \rangle_{\mathcal{H}}.$$

For each $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$, we define the operator $W_{i,j}$ on $\mathcal{K}_0$ by setting

$$W_{i,j}\left(\sum_{\boldsymbol{\sigma} \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\boldsymbol{\sigma}}} \otimes h_{\boldsymbol{\sigma}} \right) := \sum_{\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_k) \in \boldsymbol{F}_{\boldsymbol{n}}^+} e_{\tilde{\sigma}_1} \otimes \cdots \otimes e_{\tilde{\sigma}_{i-1}} \otimes e_{g_j \tilde{\sigma}_i} \otimes e_{\tilde{\sigma}_{i+1}} \otimes \cdots \otimes e_{\tilde{\sigma}_k} \otimes h_{\boldsymbol{\sigma}}.$$

Taking into account the relations

$$c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim = c_l^+(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) \quad \text{and} \quad c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim = c_l^-(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}),$$

we deduce that

$$\begin{aligned} P_{\mathcal{H}} \boldsymbol{W}_{\boldsymbol{\sigma}}^* \boldsymbol{W}_{\boldsymbol{\omega}}|_{\mathcal{H}} &= \begin{cases} P_{\mathcal{H}} \boldsymbol{W}_{c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega})}^* \boldsymbol{W}_{c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega})}|_{\mathcal{H}} & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \Gamma(c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim; c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega})^\sim) & \text{if } \boldsymbol{\sigma} \sim_{\mathrm{lc}} \boldsymbol{\omega}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \begin{cases} \Gamma(c_r^+(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}); c_r^-(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}})) & \text{if } \tilde{\boldsymbol{\sigma}} \sim_{\mathrm{rc}} \tilde{\boldsymbol{\omega}}, \\ 0 & \text{otherwise,} \end{cases} \\ &= \Gamma(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) \end{aligned}$$

for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}_{\boldsymbol{n}}^+$. The rest of the proof is similar to that of Theorem 3.2. We leave it to the reader. □

Let F be a free k-pluriharmonic on the polyball $\boldsymbol{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$ with representation

$$F(\boldsymbol{X})=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\ i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\ |\beta_i|=m_i^+}} A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes \boldsymbol{X}_{1,\alpha_1}\cdots \boldsymbol{X}_{k,\alpha_k}\boldsymbol{X}_{1,\beta_1}^*\cdots \boldsymbol{X}_{k,\beta_k}^*, \tag{3-1}$$

where the series converge in the operator norm topology for any $\boldsymbol{X}=(X_1,\dots,X_k)\in\boldsymbol{B_n}(\mathcal{H})$, with $X_i:=(X_{i,1},\dots,X_{i,n_i})$, and any Hilbert space $\mathcal{H}$. We associate to F the kernel $\Gamma_F:\boldsymbol{F_n^+}\times\boldsymbol{F_n^+}\to B(\mathcal{E})$ given by

$$\Gamma_F(\boldsymbol{\sigma},\boldsymbol{\omega}):=\begin{cases}A_{(c_r^+(\boldsymbol{\sigma},\boldsymbol{\omega});c_r^-(\boldsymbol{\sigma},\boldsymbol{\omega}))} & \text{if } \boldsymbol{\sigma}\sim_{\text{rc}}\boldsymbol{\omega},\\ 0 & \text{otherwise.}\end{cases} \tag{3-2}$$

One can easily see that Γ_F is a right k-multi-Toeplitz kernel on $\boldsymbol{F_n^+}$. In what follows, we prove a Schur-type result for positive k-pluriharmonic functions in polyballs.

Theorem 3.4. *Let F be a k-pluriharmonic function on the regular polyball $\boldsymbol{B_n}$, with coefficients in $B(\mathcal{E})$. Then F is positive on $\boldsymbol{B_n}$ if and only if the kernel Γ_{F_r} is positive semidefinite for any $r\in[0,1)$, where F_r stands for the mapping $\boldsymbol{X}\mapsto F(r\boldsymbol{X})$.*

Proof. For every $\boldsymbol{\gamma}:=(\gamma_1,\dots,\gamma_k)\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, we set $e_{\boldsymbol{\gamma}}:=e^1_{\gamma_1}\otimes\cdots\otimes e^k_{\gamma_k}$ and $\boldsymbol{S_\gamma}:=\boldsymbol{S}_{1,\gamma_1}\cdots\boldsymbol{S}_{k,\gamma_k}$. Let F be a k-pluriharmonic function with representation (3-1). Taking into account Lemma 1.2, for each $\boldsymbol{\gamma},\boldsymbol{\omega}\in\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, $r\in[0,1)$, and $h,h'\in\mathcal{E}$, we have

$$\begin{aligned}\langle F(r\boldsymbol{S})(h\otimes e_{\boldsymbol{\gamma}}),h'\otimes e_{\boldsymbol{\omega}}\rangle &=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\\ i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,|\beta_i|=m_i^+}}\langle A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}h,h'\rangle r^{\sum_{i=1}^k(|\alpha_i|+|\beta_i|)}\langle \boldsymbol{S_\alpha S_\beta^*}e_{\boldsymbol{\gamma}},e_{\boldsymbol{\omega}}\rangle\\ &=\begin{cases}r^{\sum_{i=1}^k(|\alpha_i|+|\beta_i|)}\langle A_{(c_r^+(\boldsymbol{\omega},\boldsymbol{\gamma});c_r^-(\boldsymbol{\omega},\boldsymbol{\gamma}))}h,h'\rangle & \text{if } \boldsymbol{\omega}\sim_{\text{rc}}\boldsymbol{\gamma},\\ 0 & \text{otherwise,}\end{cases}\\ &=\langle\Gamma_{F_r}(\boldsymbol{\omega},\boldsymbol{\gamma})h,h'\rangle.\end{aligned}$$

Hence, we deduce that the kernel Γ_{F_r} is positive semidefinite for any $r\in[0,1)$ if and only if $F(r\boldsymbol{S})\geq 0$ for any $r\in[0,1)$. Now, let $\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})$ and let $r\in(0,1)$ be such that $(1/r)\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm and completely positive, so is $\mathrm{id}\otimes\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, we obtain

$$F(\boldsymbol{X})=(\mathrm{id}\otimes\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}})[F(r\boldsymbol{S})]\geq 0,\qquad \boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H}).$$

Note that if F is positive on $\boldsymbol{B_n}$ then $F(r\boldsymbol{S})\geq 0$ for any $r\in[0,1)$. □

Corollary 3.5. *Let $f:\boldsymbol{B_n}(\mathcal{H})\to B(\mathcal{E})\otimes_{\min}B(\mathcal{H})$ be a free holomorphic function. Then the following statements are equivalent*:

(i) $\Re f\geq 0$ *on the polyball* $\boldsymbol{B_n}$.

(ii) $\Re f(r\boldsymbol{S}) \geq 0$ *for any* $r \in [0, 1)$.

(iii) *The right k-multi Toeplitz kernel* $\Gamma_{\Re f_r}$ *is positive semidefinite for any* $r \in [0, 1)$.

Let us define the *free k-pluriharmonic Poisson kernel* by setting

$$\boldsymbol{\mathcal{P}}(\boldsymbol{Y}, \boldsymbol{X}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}^+_{n_i},\, i \in \{1, \dots, k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} Y^*_{1,\tilde{\alpha}_1} \cdots Y^*_{k,\tilde{\alpha}_k} Y_{1,\tilde{\beta}_1} \cdots Y_{k,\tilde{\beta}_k} \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X^*_{1,\beta_1} \cdots X^*_{k,\beta_k}$$

for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and any $\boldsymbol{Y} = (Y_1, \dots, Y_k)$ with $Y_i = (Y_{i,1}, \dots, Y_{i,n_i}) \in B(\mathcal{K})^{n_i}$ such that the series above is convergent in the operator norm topology. Let $\Omega \subset \boldsymbol{F}^+_{\boldsymbol{n}} \times \boldsymbol{F}^+_{\boldsymbol{n}}$ be the set of all $(\boldsymbol{\alpha}, \boldsymbol{\beta})$ where $\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_k)$, $\boldsymbol{\beta} = (\beta_1, \dots, \beta_k) \in \boldsymbol{F}^+_{\boldsymbol{n}}$ are such that $\alpha_i, \beta_i \in \mathbb{F}^+_{n_i}$, $|\alpha_i| = m_i^-$, $|\beta_i| = m_i^+$ for some $m_i \in \mathbb{Z}$.

Theorem 3.6. *A map* $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ *with* $F(0) = I$ *is a positive free k-pluriharmonic function on the regular polyball if and only if it has the form*

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}^*_{\tilde{\boldsymbol{\alpha}}} \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}^*_{\boldsymbol{\beta}},$$

where $\boldsymbol{V} = (V_1, \dots, V_k)$ *is a k-tuple of commuting row isometries on a space* $\mathcal{K} \supset \mathcal{E}$ *such that*

$$\sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}^*_{\tilde{\boldsymbol{\alpha}}} \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes r^{|\boldsymbol{\alpha}| + |\boldsymbol{\beta}|} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{S}^*_{\boldsymbol{\beta}} \geq 0, \quad r \in [0, 1),$$

and the series is convergent in the operator topology.

Proof. Assume that F is a positive free k-pluriharmonic function which has the representation (3-1) and $F(0) = I$. Due to Theorem 3.4, $F(r\boldsymbol{S}) \geq 0$ and the right k-multi-Toeplitz kernel Γ_{F_r} is positive semidefinite for any $r \in [0, 1)$. Taking limits as $r \to \infty$, we deduce that Γ_F is positive semidefinite as well. According to Theorem 3.3, Γ_F has a Naimark-type dilation. Therefore, there is a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$ of commuting row isometries on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ such that $\Gamma(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) = P_{\mathcal{E}} \boldsymbol{V}^*_{\boldsymbol{\sigma}} \boldsymbol{V}_{\boldsymbol{\omega}}|_{\mathcal{E}}$ for any $\boldsymbol{\sigma}, \boldsymbol{\omega} \in \boldsymbol{F}^+_{\boldsymbol{n}}$. Using (3-1) and (3-2), we deduce that

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}^*_{\tilde{\boldsymbol{\alpha}}} \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}^*_{\boldsymbol{\beta}},$$

where the convergence is in the norm topology. This shows, in particular, that $F(r\boldsymbol{S})$ is convergent.

To prove the converse, assume that $\boldsymbol{V} = (V_1, \dots, V_k)$ is a k-tuple of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ such that

$$\sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}^*_{\tilde{\boldsymbol{\alpha}}} \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes r^{|\boldsymbol{\alpha}| + |\boldsymbol{\beta}|} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{S}^*_{\boldsymbol{\beta}} \geq 0, \quad r \in [0, 1), \tag{3-3}$$

and the convergence is in the operator norm topology. Let $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm and completely positive, so is $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, we obtain

$$F(\boldsymbol{X}) := (\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}) \Bigg(\sum_{(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Omega} P_{\mathcal{E}} \boldsymbol{V}^*_{\tilde{\boldsymbol{\alpha}}} \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes r^{|\boldsymbol{\alpha}| + |\boldsymbol{\beta}|} \boldsymbol{S}_{\boldsymbol{\alpha}} \boldsymbol{S}^*_{\boldsymbol{\beta}} \Bigg) \geq 0, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}). \qquad \square$$

We remark that the condition (3-3) is equivalent to the condition that the kernel defined by the relation $\Gamma_{rV}(\tilde{\sigma}, \tilde{\omega}) := r^{|\sigma|+|\omega|} P_{\mathcal{E}} V_{\sigma}^* V_{\omega}|_{\mathcal{E}}$ for any $\sigma, \omega \in F_n^+$ is positive semidefinite. We should also mention that one can find a version of the theorem above when the condition $F(0) = I$ is dropped. In this case, $F(0) = A \otimes I$ with $A \geq 0$ and we set

$$G_\epsilon := [(A + \epsilon I_{\mathcal{E}})^{-1/2} \otimes I](F + \epsilon I_{\mathcal{E}} \otimes I)[(A + \epsilon I_{\mathcal{E}})^{-1/2} \otimes I], \quad \epsilon > 0.$$

Since G_ϵ is a positive k-pluriharmonic function with $G_\epsilon(0) = I$, we can apply Theorem 3.6 to get a family $V(\epsilon) = (V_1(\epsilon), \dots, V_k(\epsilon))$ of k-tuples of commuting row isometries on a space $\mathcal{K}_\epsilon \supset \mathcal{E}$ such that

$$F(X) = \lim_{\epsilon \to 0} \sum_{(\alpha,\beta)\in\Omega} (A + \epsilon I_{\mathcal{E}})^{1/2} \big[P_{\mathcal{E}} V_{\tilde{\alpha}}^*(\epsilon) V_{\tilde{\beta}}(\epsilon) \big|_{\mathcal{E}} \big] (A + \epsilon I_{\mathcal{E}})^{1/2} \otimes X_\alpha X_\beta^*,$$

where the convergence is in the operator norm topology.

Definition 3.7. A k-tuple $V = (V_1, \dots, V_k)$ of commuting row isometries $V_i = (V_{i,1}, \dots, V_{i,n_i})$ is called *pluriharmonic* if the free k-pluriharmonic Poisson kernel $\mathcal{P}(V, rS)$ is a positive operator for any $r \in [0, 1)$.

Proposition 3.8. *Let $V = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, be a k-tuple of commuting row isometries. Then V is pluriharmonic in each of the following cases*:

(i) $k = 1$ *and* $n_1 \in \mathbb{N}$.

(ii) V *is doubly commuting*, *i.e.*, *the C^*-algebra $C^*(V_i)$ commutes with $C^*(V_s)$ if $i, s \in \{1, \dots, k\}$ with $i \neq s$.*

(iii) $n_1 = \cdots = n_k = 1$.

Proof. It is easy to see that V is pluriharmonic if the condition in (i) is satisfied. Under the condition (ii), the proof that V is pluriharmonic is similar to the proof of Theorem 4.2(i), when we replace the universal operator R with V. Now, we assume that $n_1 = \cdots = n_k = 1$. Then $V = (V_1, \dots, V_k)$, where $V_1, \dots, V_k$ are commuting isometries on a Hilbert space $\mathcal{K}$. It is well known [Sz.-Nagy et al. 2010] that there is a k-tuple $U = (U_1, \dots, U_k)$ of commuting unitaries on a Hilbert space $\mathcal{G} \supset \mathcal{K}$ such that $U_i|_{\mathcal{K}} = V_i$ for $i \in \{1, \dots, k\}$. Due to Fuglede's theorem (see [Douglas 1998]), the unitaries are doubly commuting. Due to (ii), $\mathcal{P}(U, rS)$ is a well-defined positive operator for any $r \in [0, 1)$, where the convergence defining the free k-pluriharmonic Poisson kernel $\mathcal{P}(U, rS)$ is in the operator norm topology. On the other hand, we have

$$\mathcal{P}(V, rS) = (P_{\mathcal{K}} \otimes I)\mathcal{P}(U, rS)|_{\mathcal{K}\otimes\bigotimes_{i=1}^k F^2(H_{n_i})} \geq 0,$$

which completes the proof. ☐

Proposition 3.9. *Let $V = (V_1, \dots, V_k)$ be a pluriharmonic tuple of commuting row isometries on a Hilbert space $\mathcal{K}$ and let $\mathcal{E} \subset \mathcal{K}$ be a subspace. Then the map*

$$F(X) := (P_{\mathcal{E}} \otimes I)\mathcal{P}(V, X)|_{\mathcal{E}\otimes\mathcal{H}}, \quad X \in B_n(\mathcal{H}),$$

is a positive free k-pluriharmonic function on the polyball B_n with operator-valued coefficients in $B(\mathcal{E})$ and $F(0) = I$.

Moreover, *in the particular cases* (i) *and* (iii) *of Proposition 3.8*, *each positive free k-pluriharmonic function F with $F(0) = I$ has the form above.*

Proof. Since $\boldsymbol{V}$ is a tuple of commuting row isometries, the free k-pluriharmonic Poisson kernel $\boldsymbol{\mathcal{P}}(\boldsymbol{V}, r\boldsymbol{S})$ is a positive operator for any $r \in [0, 1)$ and so is the compression $(P_{\mathcal{E}} \otimes I)\boldsymbol{\mathcal{P}}(\boldsymbol{V}, r\boldsymbol{S})|_{\mathcal{E}\otimes\bigotimes_{i=1}^{k} F^2(H_{n_i})}$. Let $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)X}$ is continuous in the operator norm and completely positive, we deduce that

$$F(\boldsymbol{X}) := (P_{\mathcal{E}} \otimes I)\boldsymbol{\mathcal{P}}(\boldsymbol{V}, \boldsymbol{X})|_{\mathcal{E}\otimes\mathcal{H}} \geq 0, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}),$$

where the convergence of $\mathcal{P}(\boldsymbol{V}, \boldsymbol{X})$ is in the operator norm topology. Therefore, F is a positive free k-pluriharmonic function on the polyball $\boldsymbol{B_n}$ with operator-valued coefficients in $B(\mathcal{E})$ and $F(0) = I$. To prove the second part of this proposition, assume that F is a positive free k-pluriharmonic function with $F(0) = I$. According to Theorem 3.6, F has the form

$$F(\boldsymbol{X}) = \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Omega} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^{*} \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^{*},$$

where $\boldsymbol{V} = (V_1, \ldots, V_k)$ is a k-tuple of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ and the convergence of the series is in the operator norm topology. Since in the particular cases (i) and (ii) of Proposition 3.8 $\boldsymbol{V}$ is pluriharmonic, one can easily complete the proof. $\square$

We remark that the theorem above contains, in particular, a structure theorem for positive k-harmonic functions on the regular polydisk included in $[B(\mathcal{H})]_1 \times_c \cdots \times_c [B(\mathcal{H})]_1$, which extends the corresponding classical result on scalar polydisks [Rudin 1969]. In the general case of the polyball it is unknown if all positive free k-pluriharmonic functions F with $F(0) = I$ have the form of Proposition 3.9.

4. Berezin transforms of completely bounded maps in regular polyballs

We define a class of noncommutative Berezin transforms of completely bounded linear maps and give necessary and sufficient conditions for a function to be the Poisson transform of a completely bounded or completely positive map.

Let $\mathcal{H}$ be a Hilbert space and identify the set $M_m(B(\mathcal{H}))$ of $m \times m$ matrices with entries from $B(\mathcal{H})$ with $B(\mathcal{H}^{(m)})$, where $\mathcal{H}^{(m)}$ is the direct sum of m copies of $\mathcal{H}$. Thus we have a natural C^*-norm on $M_m(B(\mathcal{H}))$. If $\mathcal{X}$ is an operator space, i.e., a closed subspace of $B(\mathcal{H})$, we consider $M_m(\mathcal{X})$ as a subspace of $M_m(B(\mathcal{H}))$ with the induced norm. Let $\mathcal{X}$, $\mathcal{Y}$ be operator spaces and let $u : \mathcal{X} \to \mathcal{Y}$ be a linear map. Define the map $u_m : M_m(\mathcal{X}) \to M_m(\mathcal{Y})$ by $u_m([x_{ij}]) := [u(x_{ij})]$. We say that u is completely bounded if $\|u\|_{\mathrm{cb}} := \sup_{m\geq 1} \|u_m\| < \infty$. If $\|u\|_{\mathrm{cb}} \leq 1$ then u is completely contractive; if u_m is an isometry for any $m \geq 1$ then u is completely isometric; and if u_m is positive for all m then u is called completely positive. For basic results concerning completely bounded maps and operator spaces we refer to [Paulsen 1986; Pisier 2001; Effros and Ruan 2000].

Let $\mathcal{K}$ be a Hilbert space and let $\mu : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{K})$ be a completely bounded map. It is well known (see, e.g., [Paulsen 1986]) that there exists a completely bounded linear map

$$\hat{\mu} := \mu \otimes \mathrm{id} : B\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg) \otimes_{\min} B(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

such that $\hat{\mu}(f \otimes Y) := \mu(f) \otimes Y$ for $f \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ and $Y \in B(\mathcal{H})$. Moreover, $\|\hat{\mu}\|_{\mathrm{cb}} = \|\mu\|_{\mathrm{cb}}$ and, if μ is completely positive, then so is $\hat{\mu}$. We introduce the *noncommutative Berezin transform* associated with μ as the map

$$\boldsymbol{\mathcal{B}}_\mu : B\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg) \times \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

defined by

$$\boldsymbol{\mathcal{B}}_\mu(A, \boldsymbol{X}) := \hat{\mu}[C_{\boldsymbol{X}}^*(A \otimes I_{\mathcal{H}})C_{\boldsymbol{X}}], \quad A \in B\bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\bigg),\ \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}),$$

where the operator $C_{\boldsymbol{X}} \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i}) \otimes \mathcal{H}\big)$ is defined by

$$C_{\boldsymbol{X}} := (I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1}$$

and the *defect mapping* $\boldsymbol{\Delta}_{\boldsymbol{X}} : B(\mathcal{H}) \to B(\mathcal{H})$ is given by

$$\boldsymbol{\Delta}_{\boldsymbol{X}} := (\mathrm{id} - \Phi_{X_1}) \circ \cdots \circ (\mathrm{id} - \Phi_{X_k}),$$

where $\Phi_{X_i} : B(\mathcal{H}) \to B(\mathcal{H})$ is the completely positive linear map defined by

$$\Phi_{X_i}(Y) := \sum_{j=1}^{n_i} X_{i,j} Y X_{i,j}^*, \quad Y \in B(\mathcal{H}).$$

We need to show that the operator $I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*$ is invertible. Let $\boldsymbol{Y} = (Y_1, \ldots, Y_k)$ with $Y_i := (Y_{i,1}, \ldots, Y_{i,n_i}) \in B(\mathcal{H})^{n_i}$. We introduce the spectral radius of $\boldsymbol{Y}$ by setting

$$r(\boldsymbol{Y}) := \limsup_{(p_1,\ldots,p_k)\in\mathbb{Z}_+^k} \Bigg\| \sum_{\substack{\alpha_i \in \mathbb{F}_{n_i}^+,\ |\alpha_i| = p_i \\ i \in \{1,\ldots,k\}}} Y_{\boldsymbol{\alpha}} Y_{\boldsymbol{\alpha}}^* \Bigg\|^{\frac{1}{2(p_1+\cdots+p_k)}},$$

where $Y_{\boldsymbol{\alpha}} := Y_{1,\alpha_1} \cdots Y_{k,\alpha_k}$ for $\boldsymbol{\alpha} := (\alpha_1, \ldots, \alpha_k) \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$ and $Y_{i,\alpha_i} := Y_{i,j_1} \cdots Y_{i,j_p}$ for $\alpha_i = g_{j_1}^i \cdots g_{j_p}^i \in \mathbb{F}_{n_i}^+$. We remark that, when $k = 1$, we recover the spectral radius of an n_i-tuple of operators, i.e., $r(Y_i) = \lim_{p\to\infty} \big\| \sum_{\beta_i \in \mathbb{F}_{n_i}^+, |\beta_i| = p} Y_{i,\beta_i} Y_{i,\beta_i}^* \big\|^{1/2p}$. Note also that

$$r(Y_i) = r(\boldsymbol{R}_{i,1} \otimes Y_{i,1}^* + \cdots + \boldsymbol{R}_{i,n_i} \otimes Y_{i,n_i}^*)$$

and $r(Y_i) \le r(\boldsymbol{Y})$ for any $i \in \{1, \ldots, k\}$. Consequently, if $r(\boldsymbol{Y}) < 1$ then $r(Y_i) < 1$ and the spectrum of $\boldsymbol{R}_{i,1} \otimes Y_{i,1}^* + \cdots + \boldsymbol{R}_{i,n_i} \otimes Y_{i,n_i}^*$ is included in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. In particular, when $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, the noncommutative von Neumann inequality [Popescu 1999] implies $r(\boldsymbol{X}) \le r(t\boldsymbol{S}) = t$ for some $t \in (0, 1)$, which proves our assertion.

Proposition 4.1. *Let $\mathcal{B}_\mu$ be the noncommutative Berezin transform associated with a completely bounded linear map $\mu : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{K})$.*

(i) *If $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ is fixed, then*

$$\mathcal{B}_\mu(\,\cdot\,, \boldsymbol{X}) : B\Bigg(\bigotimes_{i=1}^k F^2(H_{n_i})\Bigg) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

is a completely bounded linear map with $\|\mathcal{B}_\mu(\,\cdot\,, \boldsymbol{X})\|_{\mathrm{cb}} \le \|\mu\|_{\mathrm{cb}} \|C_{\boldsymbol{X}}\|^2$.

(ii) *If μ is selfadjoint, then $\mathcal{B}_\mu(A^*, \boldsymbol{X}) = \mathcal{B}_\mu(A, \boldsymbol{X})^*$. Moreover, if μ is completely positive then so is the map $\mathcal{B}_\mu(\,\cdot\,, \boldsymbol{X})$.*

(iii) *If $A \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is fixed, then the map*

$$\mathcal{B}_\mu(A, \cdot\,) : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{K}) \otimes_{\min} B(\mathcal{H})$$

is continuous and $\|\mathcal{B}_\mu(A, \boldsymbol{X})\| \le \|\mu\|_{\mathrm{cb}} \|A\| \|C_{\boldsymbol{X}}\|^2$ for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$.

Proof. Parts (i) and (ii) follow easily from the definition of the noncommutative Berezin transform associated with μ. To prove (iii), let $\boldsymbol{X}, \boldsymbol{Y} \in \boldsymbol{B_n}(\mathcal{H})$ and note that

$$\begin{aligned}\|\mathcal{B}_\mu(A, \boldsymbol{X}) - \mathcal{B}_\mu(A, \boldsymbol{Y})\| &\le \|\mu\| \|C_{\boldsymbol{X}}^*(A \otimes I_{\mathcal{H}})(C_{\boldsymbol{X}} - C_{\boldsymbol{Y}})\| + \|\mu\| \|(C_{\boldsymbol{X}}^* - C_{\boldsymbol{Y}}^*)(A \otimes I_{\mathcal{H}})C_{\boldsymbol{Y}}\| \\ &\le \|\mu\| \|A\| \|C_{\boldsymbol{X}} - C_{\boldsymbol{Y}}\| (\|C_{\boldsymbol{X}}\| + \|C_{\boldsymbol{Y}}\|).\end{aligned}$$

The continuity of the map $\boldsymbol{X} \mapsto \mathcal{B}_\mu(A, \boldsymbol{X})$ will follow once we prove that $\boldsymbol{X} \mapsto C_{\boldsymbol{X}}$ is a continuous map on $\boldsymbol{B_n}(\mathcal{H})$. Note that

$$\|C_{\boldsymbol{X}} - C_{\boldsymbol{Y}}\| \le \|\boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}\| \left\| \prod_{i=1}^k (I - \boldsymbol{R}_{X_i^*})^{-1} - \prod_{i=1}^k (I - \boldsymbol{R}_{Y_i^*})^{-1} \right\| + \|\boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2} - \boldsymbol{\Delta}_{\boldsymbol{Y}}(I)^{1/2}\| \left\| \prod_{i=1}^k (I - \boldsymbol{R}_{X_i^*})^{-1} \right\|,$$

where $\boldsymbol{R}_{X_i^*} := I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*$. Since the maps $\boldsymbol{X} \mapsto \prod_{i=1}^k (I - \boldsymbol{R}_{X_i^*})^{-1}$ and $\boldsymbol{X} \mapsto \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}$ are continuous on $\boldsymbol{B_n}(\mathcal{H})$ in the operator norm topology, our assertion follows. The inequality in (iii) is obvious. □

We remark that the noncommutative Poisson transform introduced in [Popescu 1999] is in fact a particular case of the noncommutative Berezin transform associated with a linear functional. Indeed, let τ be the linear functional on $B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ defined by $\tau(A) := \langle A(1), 1\rangle$. If $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ is fixed, then $\mathcal{B}_\tau(\,\cdot\,, \boldsymbol{X}) : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to B(\mathcal{H})$ is a completely contractive linear map and

$$\langle \mathcal{B}_\tau(A, \boldsymbol{X})x, y\rangle = \langle C_{\boldsymbol{X}}^*(A \otimes I_{\mathcal{H}})C_{\boldsymbol{X}}(1 \otimes x), 1 \otimes y\rangle, \quad x, y \in \mathcal{H}.$$

Hence, we have

$$\mathcal{B}_\tau(A, \boldsymbol{X}) = \boldsymbol{K}_{\boldsymbol{X}}^*(A \otimes I)\boldsymbol{K}_{\boldsymbol{X}},$$

where $\boldsymbol{K_X}$ is the noncommutative Berezin kernel at $\boldsymbol{X}$. Note also that if $A \in B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$ is fixed, then $\boldsymbol{\mathcal{B}}_\tau(A, \cdot): \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{H})$ is a bounded continuous map and $\|\boldsymbol{\mathcal{B}}_\tau(A, \boldsymbol{X})\| \le \|A\|$ for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$.

We mention that, if $n_1 = \cdots = n_k = 1$, $\mathcal{H} = \mathbb{C}$ and $\boldsymbol{X} = \boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_k) \in \mathbb{D}^k$, then we recover the Berezin transform of a bounded linear operator on the Hardy space $H^2(\mathbb{D}^k)$, i.e.,

$$\boldsymbol{\mathcal{B}}_\tau(A, \lambda) = \prod_{i=1}^k (1 - |\lambda_i|^2)\langle A k_{\boldsymbol{\lambda}}, k_{\boldsymbol{\lambda}}\rangle, \quad A \in B(H^2(\mathbb{D}^k)),$$

where $k_\lambda(\boldsymbol{z}) := \prod_{i=1}^k (1 - \bar{\lambda}_i z_i)^{-1}$ and $\boldsymbol{z} = (z_1, \dots, z_k) \in \mathbb{D}^k$.

Define the set

$$\Lambda := \big\{(\boldsymbol{\sigma}, \boldsymbol{\omega}) \in \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} : \boldsymbol{\sigma} \sim_{\text{lc}} \boldsymbol{\omega} \text{ and } (\boldsymbol{\sigma}, \boldsymbol{\omega}) = (c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega}), c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega}))\big\}. \tag{4-1}$$

Set $\tilde{\Lambda} := \{(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) : (\boldsymbol{\sigma}, \boldsymbol{\omega}) \in \Lambda\}$ and note that

$$\tilde{\Lambda} := \big\{(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) \in \boldsymbol{F_n^+} \times \boldsymbol{F_n^+} : \tilde{\boldsymbol{\sigma}} \sim_{\text{rc}} \tilde{\boldsymbol{\omega}} \text{ and } (\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}) = (c_r^+(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}), c_r^-(\tilde{\boldsymbol{\sigma}}, \tilde{\boldsymbol{\omega}}))\big\}.$$

Moreover, we have $\Lambda = \tilde{\Lambda}$. In the case $(\boldsymbol{\sigma}, \boldsymbol{\omega}) \in \Lambda$, one can easily see that $c_l^+(\boldsymbol{\sigma}, \boldsymbol{\omega}) = c_r^+(\boldsymbol{\sigma}, \boldsymbol{\omega})$ and $c_l^-(\boldsymbol{\sigma}, \boldsymbol{\omega}) = c_r^-(\boldsymbol{\sigma}, \boldsymbol{\omega})$.

In what follows, we introduce the *noncommutative Poisson transform* of a completely positive linear map on the operator system

$$\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} := \operatorname{span}\{\boldsymbol{R_\alpha^*}\boldsymbol{R_\beta} : \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+\},$$

where $\boldsymbol{R} := (\boldsymbol{R}_1, \dots, \boldsymbol{R}_k)$ and $\boldsymbol{R}_i := (\boldsymbol{R}_{i,1}, \dots, \boldsymbol{R}_{i,n_i})$ is the n_i-tuple of right creation operators (see Section 1). Regard $M_m(\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n})$ as a subspace of $M_m\big(B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)\big)$. Let $M_m(\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n})$ have the norm structure that it inherits from the (unique) norm structure on the C^*-algebra $M_m\big(B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)\big)$. We remark that

$$\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} = \operatorname{span}\{\boldsymbol{R_\alpha^*}\boldsymbol{R_\beta} : (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda\} = \operatorname{span}\{\boldsymbol{R_{\tilde{\alpha}}^*}\boldsymbol{R_{\tilde{\beta}}} : (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda\},$$

where $\Lambda = \tilde{\Lambda}$ is given by (4-1). If $\mu : \boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n} \to B(\mathcal{E})$ is a completely bounded linear map, then there exists a unique completely bounded linear map

$$\hat{\mu} := \mu \otimes \mathrm{id} : \overline{\boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n}}^{\|\cdot\|} \otimes_{\min} B(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$$

such that

$$\hat{\mu}(A \otimes Y) = \mu(A) \otimes Y, \quad A \in \boldsymbol{\mathcal{R}_n^*}\boldsymbol{\mathcal{R}_n},\ Y \in B(\mathcal{H}).$$

Moreover, $\|\hat{\mu}\|_{\text{cb}} = \|\mu\|_{\text{cb}}$ and, if μ is completely positive, then so is $\hat{\mu}$.

We define the *free pluriharmonic Poisson kernel* by setting

$$\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X}) := \sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\ i \in \{1, \dots, k\} \\ |\alpha_i| = m_i^-,\ |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*$$

for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology. We need to show that the latter convergence holds. Indeed, note that, for each $i \in \{1, \dots, k\}$ and $r \in [0, 1)$, we have

$$\begin{aligned} W_i &:= \sum_{m_i \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} \boldsymbol{R}_{i,\tilde{\alpha}_i}^* \boldsymbol{R}_{i,\tilde{\beta}_i} \otimes r^{|\alpha_i| + |\beta_i|} \boldsymbol{S}_{i,\alpha_i} \boldsymbol{S}_{i,\beta_i}^* \\ &= \lim_{p_i \to \infty} \Bigg(\sum_{\substack{\alpha_i \in \mathbb{F}_{n_i} \\ 0 < |\alpha_i| \le p_i}} \boldsymbol{R}_{i,\tilde{\alpha}_i}^* \otimes r^{|\alpha_i|} \boldsymbol{S}_{i,\alpha_i} + \sum_{\substack{\beta_i \in \mathbb{F}_{n_i} \\ 0 \le |\beta_i| \le p_i}} \boldsymbol{R}_{i,\tilde{\beta}_i} \otimes r^{|\beta_i|} \boldsymbol{S}_{i,\beta_i}^* \Bigg), \end{aligned}$$

where the limit is in the operator norm topology. One can easily see that

$$\begin{aligned} W_1 \cdots W_k &= \boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) \\ &:= \lim_{p_1 \to \infty} \cdots \lim_{p_k \to \infty} \sum_{\substack{m_1 \in \mathbb{Z} \\ |m_1| \le p_1}} \cdots \sum_{\substack{m_k \in \mathbb{Z} \\ |m_k| \le p_k}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \\ &\qquad \otimes r^{\sum_{i=1}^k (|\alpha_i| + |\beta_i|)} \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \\ &\qquad \times \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*. \end{aligned}$$

Therefore, the series defining $\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S})$, i.e.,

$$\sum_{m_1 \in \mathbb{Z}} \cdots \sum_{m_k \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \otimes r^{\sum_{i=1}^k (|\alpha_i| + |\beta_i|)} \boldsymbol{S}_{1,\alpha_1} \cdots \boldsymbol{S}_{k,\alpha_k} \boldsymbol{S}_{1,\beta_1}^* \cdots \boldsymbol{S}_{k,\beta_k}^*$$

are convergent in the operator norm topology. We remark that, due to the fact that the operators $W_1, \dots, W_k$ commute, the order of the limits above is irrelevant. Fix $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X}$ is in $\boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm, so is $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, applying $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ to the relation above, we deduce that

$$\begin{aligned} &(\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}})[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S})] \\ &\quad = \lim_{p_1 \to \infty} \cdots \lim_{p_k \to \infty} \sum_{\substack{m_1 \in \mathbb{Z} \\ |m_1| \le p_1}} \cdots \sum_{\substack{m_k \in \mathbb{Z} \\ |m_k| \le p_k}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+,\, i \in \{1,\dots,k\} \\ |\alpha_i| = m_i^-,\, |\beta_i| = m_i^+}} \boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k} \\ &\qquad \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*, \end{aligned}$$

where the limits are in the operator norm topology. This proves our assertion. Now, we introduce the *noncommutative Poisson transform* of a completely bounded linear map $\mu : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ to be the map $\boldsymbol{\mathcal{P}}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X}) := \hat{\mu}[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})], \qquad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

The next result contains some of the basic properties of the noncommutative Poisson kernel and the noncommutative Poisson transform.

Theorem 4.2. *Let $\mu : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ be a completely bounded linear map.*

(i) *The map* $\boldsymbol{X} \mapsto \boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})$ *is a positive k-pluriharmonic function on the polyball* $\boldsymbol{B_n}$, *with coefficients in* $B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big)$, *and has the factorization* $\mathcal{P}(\boldsymbol{R}, \boldsymbol{X}) = C_{\boldsymbol{X}}^* C_{\boldsymbol{X}}$, *where*

$$C_{\boldsymbol{X}} := (I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

(ii) *The noncommutative Poisson transform* $\boldsymbol{\mathcal{P}}\mu$ *is a free k-pluriharmonic function on the regular polyball* $\boldsymbol{B_n}$ *that coincides with the Berezin transform* $\boldsymbol{\mathcal{B}}_\mu(I, \cdot)$.

(iii) *If* μ *is a completely positive linear map*, *then* $\boldsymbol{\mathcal{P}}\mu$ *is a positive free k-pluriharmonic function on* $\boldsymbol{B_n}$.

Proof. The fact that $\boldsymbol{X} \mapsto \boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X})$ is a free k-pluriharmonic function on the polyball $\boldsymbol{B_n}$ with coefficients in $\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}$ was proved in the remarks preceding the theorem. Setting $\Lambda_i := \boldsymbol{R}_{i,1} \otimes r \boldsymbol{S}_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes r \boldsymbol{S}_{i,n_i}^*$ for each $i \in \{1, \dots, k\}$, we have

$$\begin{aligned}
W_i &:= \sum_{m_i \in \mathbb{Z}} \sum_{\substack{\alpha_i, \beta_i \in \mathbb{F}_{n_i}^+ \\ |\alpha_i| = m_i^-, \, |\beta_i| = m_i^+}} \boldsymbol{R}_{i,\tilde{\alpha}_i}^* \boldsymbol{R}_{i,\tilde{\beta}_i} \otimes r^{|\alpha_i| + |\beta_i|} \boldsymbol{S}_{i,\alpha_i} \boldsymbol{S}_{i,\beta_i}^* \\
&= (I - \Lambda_i)^{-1} - I + (I - \Lambda_i^*)^{-1} \\
&= (I - \Lambda_i^*)^{-1} [(I - \Lambda_i) - (I - \Lambda_i^*)(I - \Lambda_i) + (I - \Lambda_i^*)](I - \Lambda_i)^{-1} \\
&= (I - \Lambda_i^*)^{-1} \left[I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \left(I_{\bigotimes_{i=1}^k F^2(H_{n_i})} - \sum_{j=1}^{n_i} r^2 \boldsymbol{S}_{i,j} \boldsymbol{S}_{i,j}^* \right) \right] (I - \Lambda_i)^{-1}.
\end{aligned}$$

Recall that $\boldsymbol{R}_{i,s} \boldsymbol{R}_{j,t} = \boldsymbol{R}_{j,t} \boldsymbol{R}_{i,s}$ and $\boldsymbol{R}_{i,s} \boldsymbol{R}_{j,t}^* = \boldsymbol{R}_{j,t}^* \boldsymbol{R}_{i,s}$ for any $i, j \in \{1, \dots, k\}$ with $i \neq j$ and for any $s \in \{1, \dots, n_i\}$ and $t \in \{1, \dots, n_j\}$. Similar commutation relations hold for the universal model $\boldsymbol{S}$. Since $\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) = W_1 \cdots W_k$ and $W_1, \dots, W_k$ are commuting positive operators, we deduce that

$$\begin{aligned}
\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) = \left(\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes r \boldsymbol{S}_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes r \boldsymbol{S}_{i,n_i})^{-1} \right) & \\
\times (I \otimes \boldsymbol{\Delta}_{r\boldsymbol{S}}(I)) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes r \boldsymbol{S}_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes r \boldsymbol{S}_{i,n_i}^*)^{-1} &
\end{aligned}$$

for any $r \in [0, 1)$, and $\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S}) = C_{r\boldsymbol{S}}^* C_{r\boldsymbol{S}} \geq 0$. Now, let $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$ and let $r \in (0, 1)$ be such that $(1/r)\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. Since the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ is continuous in the operator norm and completely positive, so is $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$. Consequently, applying $\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}}$ to the relations above, we deduce that

$$\begin{aligned}
\boldsymbol{\mathcal{P}}(\boldsymbol{R}, \boldsymbol{X}) &= (\mathrm{id} \otimes \boldsymbol{\mathcal{B}}_{(1/r)\boldsymbol{X}})[\boldsymbol{\mathcal{P}}(\boldsymbol{R}, r\boldsymbol{S})] \\
&= \prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} (I \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1} \\
&= C_{\boldsymbol{X}}^* C_{\boldsymbol{X}},
\end{aligned}$$

which completes the proof of (i).

Using the results above and the continuity of $\hat{\mu}$ in the operator norm, we deduce that the noncommutative Berezin transform $\boldsymbol{\mathcal{B}}_\mu(I,\cdot)$ associated with μ coincides with the Poisson transform $\boldsymbol{\mathcal{P}}\mu$. Indeed, we have

$$\begin{aligned}
\boldsymbol{\mathcal{B}}_\mu(I,\boldsymbol{X}) &= \hat{\mu}(C_{\boldsymbol{X}}^* C_{\boldsymbol{X}}) \\
&= \sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\dots,k\}\\ |\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}} \mu(\boldsymbol{R}_{1,\tilde{\alpha}_1}^*\cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1}\cdots \boldsymbol{R}_{k,\tilde{\beta}_k})\otimes X_{1,\alpha_1}\cdots X_{k,\alpha_k} X_{1,\beta_1}^*\cdots X_{k,\beta_k}^* \\
&= \hat{\mu}(\boldsymbol{\mathcal{P}}(\boldsymbol{R},\boldsymbol{X})) \\
&= (\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X})
\end{aligned}$$

for any $\boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology of $B(\mathcal{K}\otimes\mathcal{H})$. This proves (ii). Note also that the Poisson transform $\boldsymbol{\mathcal{P}}\mu$ is a free k-pluriharmonic function on $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$. If μ is completely positive, then so is $\hat{\mu}$. Using the fact that $\hat{\mu}(C_{\boldsymbol{X}}^*C_{\boldsymbol{X}}) = (\boldsymbol{\mathcal{P}}\mu)(\boldsymbol{X})$, we deduce (iii). □

Consider the particular case when $n_1=\cdots=n_k=1$, $\mathcal{H}=\mathcal{K}=\mathbb{C}$, $\boldsymbol{X}=(X_1,\dots,X_k)$ with $X_j=r_je^{i\theta_j}\in\mathbb{D}$, and μ is a complex Borel measure on $\mathbb{T}^k$. Note that μ can be seen as a bounded linear functional on $C(\mathbb{T}^k)$. Consequently, there is a unique bounded linear functional $\hat{\mu}$ on the operator system generated by the monomials $S_1^{m_1^-}\cdots S_k^{m_k^-} S_1^{*\,m_1^+}\cdots S_k^{*\,m_k^+}$, where $m_1,\dots,m_k\in\mathbb{Z}$, and $S_1,\dots,S_k$ are the unilateral shifts acting on the Hardy space $H^2(\mathbb{T}^k)$, such that

$$\hat{\mu}(S_1^{m_1^-}\cdots S_k^{m_k^-} S_1^{*\,m_1^+}\cdots S_k^{*\,m_k^+}) = \mu(e^{im_1^-\varphi_1}\cdots e^{im_k^-\varphi_k}e^{-im_1^+\varphi_1}\cdots e^{-im_k^+\varphi_k}),\quad m_1,\dots,m_k\in\mathbb{Z}.$$

Indeed, if p is any polynomial function of the form

$$p(z_1,\dots,z_k,\bar{z}_1,\dots,\bar{z}_k) = \sum a_{(m_1,\dots,m_k)} z_1^{m_1^-}\cdots z_k^{m_k^-}\bar{z}_1^{m_1^+}\cdots\bar{z}_k^{m_k^+},\quad (z_1,\dots,z_k)\in\mathbb{D}^k,$$

where $a_{(m_1,\dots,m_k)}\in\mathbb{C}$, then, due to the noncommutative von Neumann inequality [Popescu 1999], we have

$$\begin{aligned}
|\hat{\mu}(p(S_1,\dots,S_k,S_1^*,\dots,S_k^*))| &= |\mu(p(e^{i\varphi_1},\dots,e^{i\varphi_k},e^{-i\varphi_1},\dots,e^{-i\varphi_k}))| \\
&\le \|\mu\|\,\|p(S_1,\dots,S_k,S_1^*,\dots,S_k^*)\|.
\end{aligned}$$

Therefore, $\hat{\mu}$ is a bounded linear functional on the operator system $\operatorname{span}\{\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{A}}_{\boldsymbol{n}}\}^{-\|\cdot\|}$. Note that the noncommutative Poisson transform of $\hat{\mu}$, i.e., $B_{\hat{\mu}}(I,\cdot)$, coincides with the classical Poisson transform of μ. Indeed, for any $z=(r_1e^{i\theta_1},\dots,r_ke^{i\theta_k})\in\mathbb{D}^k$, we have

$$\begin{aligned}
\boldsymbol{\mathcal{B}}_{\hat{\mu}}(I,z) &= \sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k} \hat{\mu}(S_1^{p_1^-}\cdots p_k^{p_k^-} S_1^{*\,p_1^+}\cdots S_k^{*\,p_k^+}) z_1^{p_1}\cdots z_k^{p_k} \\
&= \sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k} \mu(\bar{\zeta}_1^{p_1}\cdots\bar{\zeta}^{p_k}) z_1^{p_1}\cdots z_k^{p_k} \\
&= \sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k} \left(\int_{\mathbb{T}^k}\bar{\zeta}_1^{p_1}\cdots\bar{\zeta}^{p_k}\,d\mu(\zeta)\right) z_1^{p_1}\cdots z_k^{p_k} \\
&= \int_{\mathbb{T}^k}\left(\sum_{(p_1,\dots,p_k)\in\mathbb{Z}^k} r_1^{|p_1|}\cdots r_k^{|p_k|}e^{ip_1(\theta_1-\varphi_1)}\cdots e^{ip_k(\theta_k-\varphi_k)}\right)d\mu(\zeta) = \int_{\mathbb{T}^k} P(z,\zeta)\,d\mu(\zeta),
\end{aligned}$$

where

$$P(z,\zeta)=P_{r_1}(\theta_1-\varphi_1)\cdots P_{r_k}(\theta_k-\varphi_k),\qquad \zeta=(e^{i\varphi_1},\dots,e^{i\varphi_k})\in\mathbb{T}^k,$$

and $P_r(\theta-\varphi)=(1-r^2)/(1-2r\cos(\theta-\varphi)+r^2)$ is the Poisson kernel of the unit disc (see [Rudin 1969]).

We recall that Λ denotes the set of all pairs $(\boldsymbol{\alpha},\boldsymbol{\beta})\in \boldsymbol{F}_{\boldsymbol{n}}^+\times\boldsymbol{F}_{\boldsymbol{n}}^+$, where $\boldsymbol{F}_{\boldsymbol{n}}^+:=\mathbb{F}_{n_1}^+\times\cdots\times\mathbb{F}_{n_k}^+$, with the property that $\boldsymbol{\alpha}\sim_{\mathrm{lc}}\boldsymbol{\beta}$ and $(\boldsymbol{\alpha},\boldsymbol{\beta})=(c_l^+(\boldsymbol{\alpha},\boldsymbol{\beta}),c_l^-(\boldsymbol{\alpha},\boldsymbol{\beta}))$. We remark that $(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda$ if and only if $(\tilde{\boldsymbol{\alpha}},\tilde{\boldsymbol{\beta}})\in\Lambda$. As before, we use the notation $\tilde{\boldsymbol{\alpha}}=(\tilde{\boldsymbol{\alpha}}_1,\dots,\tilde{\boldsymbol{\alpha}}_k)$ if $\boldsymbol{\alpha}=(\boldsymbol{\alpha}_1,\dots,\boldsymbol{\alpha}_k)\in\mathbb{F}_{\boldsymbol{n}}^+$.

Throughout the rest of this section, we assume that $\mathcal{E}$ is a separable Hilbert space.

Lemma 4.3. *Let $\mu:\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}\to B(\mathcal{E})$ be a completely bounded linear map. For each $r\in[0,1)$, define the linear map $\mu_r:\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}\to B(\mathcal{E})$ by*

$$\mu_r(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}):=r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|}\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}),\qquad(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda,$$

where $|\boldsymbol{\alpha}|:=|\alpha_1|+\cdots+|\alpha_k|$ if $\boldsymbol{\alpha}=(\alpha_1,\dots,\alpha_k)\in\mathbb{F}_{\boldsymbol{n}}^+$. Then

(i) *μ_r is a completely bounded linear map*;

(ii) $\|\mu\|_{\mathrm{cb}}=\sup_{0\le r<1}\|\mu_r\|_{\mathrm{cb}}=\lim_{r\to1}\|\mu_r\|_{\mathrm{cb}}$;

(iii) *$\mu_r(A)\to\mu(A)$ in the operator norm topology as $r\to1$ for any $A\in\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}$*;

(iv) *if μ is completely positive, then so is μ_r for any $r\in[0,1)$.*

Proof. Let

$$p(\boldsymbol{R}^*,\boldsymbol{R}):=\sum_{\substack{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda'\subset\Lambda\\ \mathrm{card}(\Lambda')<\aleph_0}}a_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}},\qquad a_{(\boldsymbol{\alpha},\boldsymbol{\beta})}\in\mathbb{C},$$

and $0\le r_1<r_2\le 1$. Using the noncommutative von Neumann inequality [Popescu 1999], we deduce that

$$\|\mu_{r_1}(p(\boldsymbol{R}^*,\boldsymbol{R}))\|=\|\mu(p(r_1\boldsymbol{R}^*,r_1\boldsymbol{R}))\|=\left\|\mu_{r_2}\left(p\left(\frac{r_1}{r_2}\boldsymbol{R}^*,\frac{r_1}{r_2}\boldsymbol{R}\right)\right)\right\|\le\|\mu_{r_2}\|\,\|p(\boldsymbol{R}^*,\boldsymbol{R})\|.$$

In particular, we have $\|\mu_r\|\le\|\mu\|$ for any $r\in[0,1)$. Similarly, passing to matrices over $\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}$, one can show that $\|\mu_{r_1}\|_{\mathrm{cb}}\le\|\mu_{r_2}\|_{\mathrm{cb}}$ if $0\le r_1<r_2\le1$, and $\|\mu_r\|_{\mathrm{cb}}\le\|\mu\|_{\mathrm{cb}}$ for any $r\in[0,1)$. Now, one can easily see that $\mu_r(A)\to\mu(A)$ in the operator norm topology as $r\to1$ for any $A\in\mathcal{R}_{\boldsymbol{n}}^*\mathcal{R}_{\boldsymbol{n}}$, and $\|\mu\|_{\mathrm{cb}}=\sup_{0\le r<1}\|\mu_r\|_{\mathrm{cb}}$. Hence, and using the fact that the function $r\mapsto\|\mu_r\|_{\mathrm{cb}}$ is increasing for $r\in[0,1)$, we deduce that $\lim_{r\to1}\|\mu_r\|_{\mathrm{cb}}$ exists and it is equal to $\|\mu\|_{\mathrm{cb}}$.

To prove (iv), note that $\mu_r(p(\boldsymbol{R}^*,\boldsymbol{R}))=\mu\big(\mathcal{B}_{r\boldsymbol{R}}[p(\boldsymbol{S}^*,\boldsymbol{S})]\big)$. Since the noncommutative Berezin transform $\mathcal{B}_{r\boldsymbol{R}}$ and μ are completely positive linear maps and $p(\boldsymbol{R}^*,\boldsymbol{R})$ is unitarily equivalent to $p(\boldsymbol{S}^*,\boldsymbol{S})$, we deduce that μ_r is a completely positive linear map for each $r\in[0,1)$. □

Let F be a free k-pluriharmonic function on the polyball $\boldsymbol{B}_{\boldsymbol{n}}$, with operator-valued coefficients in $B(\mathcal{E})$, with representation

$$F(\boldsymbol{X})=\sum_{m_1\in\mathbb{Z}}\cdots\sum_{m_k\in\mathbb{Z}}\sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\,i\in\{1,\dots,k\}\\|\alpha_i|=m_i^-,\,|\beta_i|=m_i^+}}A_{(\alpha_1,\dots,\alpha_k;\beta_1,\dots,\beta_k)}\otimes\boldsymbol{X}_{1,\alpha_1}\cdots\boldsymbol{X}_{k,\alpha_k}\boldsymbol{X}_{1,\beta_1}^*\cdots\boldsymbol{X}_{k,\beta_k}^*.$$

We associate to F and each $r \in [0, 1)$ the linear map $\nu_{F_r} : \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}} \to B(\mathcal{E})$ by setting

$$\nu_{F_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) := r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} A_{(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_k)}, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda. \tag{4-2}$$

We remark that ν_{F_r} is uniquely determined by the radial function $r \mapsto F(r\boldsymbol{S})$. Indeed, note that, if $x := x_1 \otimes \cdots \otimes x_k$, $y = y_1 \otimes \cdots \otimes y_k$ satisfy (1-3) and $h, \ell \in \mathcal{E}$, we have

$$\langle F(r\boldsymbol{S})(h \otimes x), \ell \otimes y\rangle = \langle r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} A_{(\alpha_1,\ldots,\alpha_k;\beta_1,\ldots,\beta_k)} h, \ell\rangle = \langle \nu_{F_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) h, \ell\rangle, \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda.$$

In what follows, we denote by $C^*(\boldsymbol{R})$ the C^*-algebra generated by the right creation operators $\boldsymbol{R}_{i,j}$, where $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$.

Theorem 4.4. *Let $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a free k-pluriharmonic function. Then the following statements are equivalent*:

(i) *There exists a completely bounded linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ such that $F = \boldsymbol{\mathcal{P}}\mu$.*

(ii) *The linear maps $\{\nu_{F_r}\}_{r\in[0,1)}$ associated with F are completely bounded and* $\sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}} < \infty$.

(iii) *There exist a k-tuple $\boldsymbol{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, of doubly commuting row isometries acting on a Hilbert space $\mathcal{K}$ and bounded linear operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ such that*

$$F(\boldsymbol{X}) = (W_1^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2 \otimes I),$$

where

$$C_{\boldsymbol{X}}(\boldsymbol{V}) := (I \otimes \boldsymbol{\Delta}_{\boldsymbol{X}}(I)^{1/2}) \prod_{i=1}^{k} (I - V_{i,1} \otimes X_{i,1}^* - \cdots - V_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

Moreover, in this case we can choose μ such that $\|\mu\|_{\mathrm{cb}} = \sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}}$.

Proof. Assume that (i) holds. Then

$$F(\boldsymbol{X}) = \sum_{m_1\in\mathbb{Z}} \cdots \sum_{m_k\in\mathbb{Z}} \sum_{\substack{\alpha_i,\beta_i\in\mathbb{F}_{n_i}^+,\, i\in\{1,\ldots,k\} \\ |\alpha_i|=m_i^-,\, |\beta_i|=m_i^+}} \mu(\boldsymbol{R}_{1,\tilde{\alpha}_1}^* \cdots \boldsymbol{R}_{k,\tilde{\alpha}_k}^* \boldsymbol{R}_{1,\tilde{\beta}_1} \cdots \boldsymbol{R}_{k,\tilde{\beta}_k}) \otimes X_{1,\alpha_1} \cdots X_{k,\alpha_k} X_{1,\beta_1}^* \cdots X_{k,\beta_k}^*$$

for any $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$, where the convergence is in the operator norm topology. Set $A_{(\boldsymbol{\alpha};\boldsymbol{\beta})} := \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}})$ for any $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda$. Consequently, for each $r \in [0, 1)$, we have

$$\nu_{F_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) := r^{|\boldsymbol{\alpha}|+|\boldsymbol{\beta}|} \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}), \quad (\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda.$$

We recall that $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda$ if and only if $(\tilde{\boldsymbol{\alpha}}, \tilde{\boldsymbol{\beta}}) \in \Lambda$. Applying Lemma 4.3, we deduce that $\{\nu_{F_r}\}$ is a completely bounded map and

$$\|\mu|_{\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}}\|_{\mathrm{cb}} = \sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}} < \infty,$$

which proves that (i) implies (ii).

Now, we prove that (ii) implies (i). Assume that F is a free pluriharmonic function on $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$ and such that condition (ii) holds. Let $\{q_j\}$ be a countable dense subset of $\boldsymbol{\mathcal{R}_n^*\mathcal{R}_n}$. For instance, we can consider all the operators of the form

$$p(\boldsymbol{R}^*, \boldsymbol{R}) := \sum_{(\boldsymbol{\alpha},\boldsymbol{\beta})\in\Lambda:\, |\boldsymbol{\alpha}|\le m, |\boldsymbol{\beta}|\le m} a_{(\boldsymbol{\alpha},\boldsymbol{\beta})} \boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}},$$

where $m \in \mathbb{N}$ and the coefficients $a_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$ lie in some countable dense subset of the complex plane. For each j, we have $\|\nu_{F_r}(q_j)\| \le M\|q_j\|$ for any $r \in [0, 1)$, where $M := \sup_{0\le r<1} \|\nu_{F_r}\|_{\mathrm{cb}}$.

Due to the Banach–Alaoglu theorem [Douglas 1998], the ball $[B(\mathcal{E})]_M^-$ is compact in the w^*-topology. Since $\mathcal{E}$ is a separable Hilbert space, $[B(\mathcal{E})]_M^-$ is a metric space in the w^*-topology which coincides with the weak operator topology on $[B(\mathcal{E})]_M^-$. Consequently, the diagonal process guarantees the existence of a sequence $\{r_m\}_{m=1}^\infty$ such that $r_m \to 1$ and WOT-$\lim_{m\to 1} \nu_{F_{r_m}}(q_j)$ exists for each q_j. Fix $A \in \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n}$ and $x, y \in \mathcal{E}$ and let us prove that $\{\langle \nu_{F_{r_m}}(A)x, y\rangle\}_{m=1}^\infty$ is a Cauchy sequence. Let $\epsilon > 0$ and choose q_j so that $\|q_j - A\| < \epsilon/(3M\|x\|\|y\|)$. Now we choose N so that $\left|\langle(\nu_{F_{r_m}}(q_j) - \nu_{F_{r_k}}(q_j))x, y\rangle\right| < \frac{1}{3}\epsilon$ for any $m, k > N$. Due to the fact that

$$\begin{aligned}\left|\langle(\nu_{F_{r_m}}(A)-\nu_{F_{r_k}}(A))x, y\rangle\right| &\le \left|\langle \nu_{F_{r_m}}(A-q_j)x, y\rangle\right| + \left|\langle(\nu_{F_{r_m}}(q_j)-\nu_{F_{r_k}}(q_j))x, y\rangle\right| + \left|\langle \nu_{F_{r_k}}(q_j-A)x, y\rangle\right|\\ &\le 2M\|x\|\|y\|\|A-q_j\| + \left|\langle(\nu_{F_{r_m}}(q_j)-\nu_{F_{r_k}}(q_j))x, y\rangle\right|,\end{aligned}$$

we deduce that $\left|\langle(\nu_{F_{r_m}}(A) - \nu_{F_{r_k}}(A))x, y\rangle\right| < \epsilon$ for $m, k > N$. Therefore,

$$\Phi(x, y) := \lim_{m\to\infty} \langle \nu_{F_{r_m}}(A)x, y\rangle$$

exists for any $x, y \in \mathcal{E}$ and defines a functional $\Phi : \mathcal{E}\times\mathcal{E} \to \mathbb{C}$ which is linear in the first variable and conjugate linear in the second. Moreover, we have $|\Phi(x, y)| \le M\|A\|\|x\|\|y\|$ for any $x, y \in \mathcal{E}$. Due to the Riesz representation theorem, there exists a unique bounded linear operator $B(\mathcal{E})$, which we denote by $\nu(A)$, such that $\Phi(x, y) = \langle \nu(A)x, y\rangle$ for $x, y \in \mathcal{E}$. Therefore,

$$\nu(A) = \text{WOT-}\lim_{r_m\to 1} \nu_{F_{r_m}}(A), \quad A \in \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n},$$

and $\|\nu(A)\| \le M\|A\|$. Note that $\nu : \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n} \to B(\mathcal{E})$ is a completely bounded map. Indeed, if $[A_{ij}]_m$ is an $m \times m$ matrix over $\boldsymbol{\mathcal{R}_n^*\mathcal{R}_n}$, then $[\nu(A_{ij})]_m =$ WOT-$\lim_{r_k\to 1}[\nu_{F_{r_k}}(A_{ij})]_m$. Hence, $\|[\nu(A_{ij})]_m\| \le M\|[A_{ij}]_m\|$ for all m, and so $\|\nu\|_{\mathrm{cb}} \le M$. Note also that $\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = A_{(\boldsymbol{\alpha};\boldsymbol{\beta})}$ for any $(\boldsymbol{\alpha}, \boldsymbol{\beta}) \in \Lambda$, where $A_{(\boldsymbol{\alpha};\boldsymbol{\beta})}$ are the coefficients of F. By Wittstock's extension theorem [1981; 1984], there exists a completely bounded linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ which extends ν such that $\|\mu\|_{\mathrm{cb}} = \|\nu\|_{\mathrm{cb}}$. Since $F = \boldsymbol{\mathcal{P}}\mu$, the proof of (i) is complete.

Now, we prove the equivalence of (i) with (iii). If (i) holds, then according to Theorem 8.4 from [Paulsen 1986] there exist a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{K})$, and bounded operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ with $\|\mu\| = \|W_1\|\|W_2\|$ such that

$$\mu(A) = W_1^*\pi(A)W_2, \quad A \in C^*(\boldsymbol{R}). \tag{4-3}$$

Set $V_{i,j} := \pi(\boldsymbol{R}_{i,j})$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$ and note that $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, is a k-tuple of doubly commuting row isometries. Using Theorem 4.2, one can easily see that the equality $F = \boldsymbol{\mathcal{P}}\mu$ implies the one from (iii). Now, we prove that (iii) implies (i). Since the k-tuple $\boldsymbol{V} = (V_1, \dots, V_n)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, consists of doubly commuting row isometries on a Hilbert space $\mathcal{K}$, the noncommutative von Neumann inequality [Popescu 1999] implies that the map $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ defined by

$$\pi(\boldsymbol{R}_{\boldsymbol{\alpha}} \boldsymbol{R}_{\boldsymbol{\beta}}^*) := \boldsymbol{V}_{\boldsymbol{\alpha}} \boldsymbol{V}_{\boldsymbol{\beta}}^*, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{F}_{\boldsymbol{n}}^+,$$

is a $*$-representation of $C^*(\boldsymbol{R})$. Define $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ by setting $\mu(A) := W_1^* \pi(A) W_2$, $A \in C^*(\boldsymbol{R})$, and note that μ is a completely bounded linear map. Using the relation

$$F(\boldsymbol{X}) = (W_1^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2 \otimes I)$$

and the factorization $P(\boldsymbol{V}, \boldsymbol{X}) = C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})$ (see also Theorem 4.2), we deduce that $F(\boldsymbol{X}) = \boldsymbol{\mathcal{P}}\mu(\boldsymbol{X})$ for $\boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H})$. □

We introduce the space $\mathbf{PH}^1(\boldsymbol{B_n})$ of all free k-pluriharmonic functions F on $\boldsymbol{B_n}$ such that the linear maps $\{\nu_{F_r}\}_{r\in[0,1)}$ associated with F are completely bounded and set $\|F\|_1 := \sup_{0\le r<1} \|\nu_{F_r}\|_{\text{cb}} < \infty$. As a consequence of Theorem 4.4, one can see that $\|\cdot\|_1$ is a norm on $\mathbf{PH}^1(\boldsymbol{B_n})$ and $(\mathbf{PH}^1(\boldsymbol{B_n}), \|\cdot\|_1)$ is a Banach space that can be identified with the Banach space $\text{CB}(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}, B(\mathcal{E}))$ of all completely bounded linear maps from $\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}$ to $B(\mathcal{E})$.

Corollary 4.5. *Let $F : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a free k-pluriharmonic function. Then the following statements are equivalent*:

(i) *There exists a completely positive linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ such that $F = \boldsymbol{\mathcal{P}}\mu$.*

(ii) *The linear maps $\{\nu_{F_r}\}_{r\in[0,1)}$ associated with F are completely positive.*

(iii) *There exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of doubly commuting row isometries acting on a Hilbert space $\mathcal{K} \supset \mathcal{E}$ and a bounded operator $W : \mathcal{E} \to \mathcal{K}$ such that*

$$F(\boldsymbol{X}) = (W^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W \otimes I).$$

Proof. The proof is similar to that of Theorem 4.4. Note that for (i) implies (ii) we have to use Lemma 4.3(iv). For the converse, note that if ν_{F_r}, $r \in [0, 1)$, are completely positive linear maps then

$$\|\nu_{F_r}\|_{\text{cb}} = \|\nu_{F_r}(1)\| = \|\nu_{F_r}\| = \|A_{(\boldsymbol{g};\boldsymbol{g})}\|,$$

where $\boldsymbol{g} = (g_0^1, \dots, g_0^k)$ is the identity element in $\boldsymbol{F}_{\boldsymbol{n}}^+$. As in the proof of Theorem 4.4, we find a completely bounded map $\nu : \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}} \to B(\mathcal{E})$ such that

$$\nu(A) = \text{WOT-}\lim_{r_m \to 1} \nu_{F_{r_m}}(A), \quad A \in \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}.$$

Since ν_{F_r}, $r \in [0, 1)$, are completely positive linear maps, one can easily see that ν is completely positive. Using Arveson's extension theorem [1969], we find a completely positive map $\mu : C^*(\boldsymbol{R}) \to \mathbb{C}$ which extends ν and such that $\|\mu\|_{\text{cb}} = \|\nu\|_{\text{cb}}$. We also have that $F = \boldsymbol{\mathcal{P}}\mu$. Now, the proof that (iii) is equivalent to (i) uses Stinespring's representation theorem [1955] and is similar to the same equivalence from Theorem 4.4. We leave it to the reader. □

An open question remains. Is any positive free k-pluriharmonic function on the regular polyball $\boldsymbol{B_n}$ the Poisson transform of a completely positive linear map? The answer is positive if $k = 1$ (see [Popescu 2009]) and also when $n_1 = \cdots = n_k$ (see Section 3).

5. Herglotz–Riesz representations for free holomorphic functions with positive real parts

In this section, we introduce the noncommutative Herglotz–Riesz transform of a completely positive linear map $\mu : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ and obtain Herglotz–Riesz representation theorems for free holomorphic functions with positive real parts in regular polyballs.

Define the space

$$\mathbf{RH}(\boldsymbol{B_n}) := \operatorname{span}\{\Re f : f \in \mathrm{Hol}_{\mathcal{E}}(\boldsymbol{B_n})\},$$

where $\mathrm{Hol}_{\mathcal{E}}(\boldsymbol{B_n})$ is the set of all free holomorphic functions in the polyball $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$. Let $\tau : B\big(\bigotimes_{i=1}^k F^2(H_{n_i})\big) \to \mathbb{C}$ be the bounded linear functional defined by $\tau(A) = \langle A1, 1\rangle$. We remark that the radial function associated with $\varphi \in \mathbf{RH}(\boldsymbol{B_n})$, i.e., $r \mapsto \varphi(r\boldsymbol{R})$ for $r \in [0, 1)$, uniquely determines the family $\{\nu_{\varphi_r}\}_{r\in[0,1)}$ of linear maps $\nu_{\varphi_r} : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ defined by (4-2). Indeed, note that

$$\nu_{\varphi_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) := (\mathrm{id} \otimes \tau)[(I \otimes \boldsymbol{R}_{\boldsymbol{\alpha}}^*)\varphi(r\boldsymbol{R})],$$
$$\nu_{\varphi_r}(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}) := (\mathrm{id} \otimes \tau)[\varphi(r\boldsymbol{R})(I \otimes \boldsymbol{R}_{\boldsymbol{\alpha}})],$$

for any $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_k) \in \boldsymbol{F}_n^+ := \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, where $\tilde{\boldsymbol{\alpha}} = (\tilde{\alpha}_1, \ldots, \tilde{\alpha}_k)$ and $\boldsymbol{R}_{\boldsymbol{\alpha}} := \boldsymbol{R}_{1,\alpha_1} \cdots \boldsymbol{R}_{k,\alpha_k}$, and $\nu_{\varphi_r}(\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is different from $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_n^+$. Consider the space

$$\mathbf{RH}^1(\boldsymbol{B_n}) := \big\{\varphi \in \mathbf{RH}(\boldsymbol{B_n}) : \nu_{\varphi_r} \text{ is completely bounded and } \sup_{0\le r<1} \|\nu_{\varphi_r}\|_{\mathrm{cb}} < \infty\big\}.$$

If $\varphi \in \mathbf{RH}^1(\boldsymbol{B_n})$, we define $\|\varphi\|_1 := \sup_{0\le r<1} \|\nu_{\varphi_r}\|_{\mathrm{cb}}$. Denote by $\mathrm{CB}_0(\boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n, B(\mathcal{E}))$ the space of all completely bounded linear maps $\lambda : \boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n \to B(\mathcal{E})$ such that $\lambda(\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_n^+$.

Theorem 5.1. $(\mathbf{RH}^1(\boldsymbol{B_n}), \|\cdot\|_1)$ *is a Banach space which can be identified with the Banach space* $\mathrm{CB}_0(\boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n, B(\mathcal{E}))$. *Moreover, if* $\varphi : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ *is a function, then the following statements are equivalent*:

(i) φ *is in* $\mathbf{RH}^1(\boldsymbol{B_n})$.

(ii) *There is a unique completely bounded linear map* $\mu_\varphi \in \mathrm{CB}_0(\boldsymbol{\mathcal{R}}_n^* \boldsymbol{\mathcal{R}}_n, B(\mathcal{E}))$ *such that* $\varphi = \boldsymbol{\mathcal{P}}\mu_\varphi$.

(iii) *There exist a k-tuple* $\boldsymbol{V} = (V_1, \ldots, V_k)$, $V_i = (V_{i,1}, \ldots, V_{i,n_i})$, *of doubly commuting row isometries on a Hilbert space* $\mathcal{K}$ *and bounded linear operators* $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ *such that*

$$\varphi(\boldsymbol{X}) = (W_1^* \otimes I)[C_{\boldsymbol{X}}(\boldsymbol{V})^* C_{\boldsymbol{X}}(\boldsymbol{V})](W_2 \otimes I)$$

and $W_1^* \boldsymbol{V}_{\boldsymbol{\alpha}}^* \boldsymbol{V}_{\boldsymbol{\beta}} W_2 = 0$ *if* $\boldsymbol{R}_{\boldsymbol{\alpha}}^* \boldsymbol{R}_{\boldsymbol{\beta}}$ *is not equal to* $\boldsymbol{R}_{\boldsymbol{\gamma}}$ *or* $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ *for some* $\boldsymbol{\gamma} \in \boldsymbol{F}_n^+$.

Proof. Define the map $\Psi : \mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}, B(\mathcal{E})) \to \mathbf{RH}^1(\boldsymbol{B_n})$ by $\Psi(\mu) := \boldsymbol{\mathcal{P}}\mu$. To prove the injectivity of Ψ, let μ_1, μ_2 be in $\mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}, B(\mathcal{E}))$ such that $\Psi(\mu_1) = \Psi(\mu_2)$. Due to the uniqueness of the representation of a free k-pluriharmonic function and the definition of the noncommutative Poisson transform of a completely bounded map on $\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}$, we deduce that $\mu_1(\boldsymbol{R_\alpha}) = \mu_2(\boldsymbol{R_\alpha})$ and $\mu_1(\boldsymbol{R}_{\boldsymbol{\alpha}}^*) = \mu_2(\boldsymbol{R}_{\boldsymbol{\alpha}}^*)$ for $\boldsymbol{\alpha} \in \mathbb{F}_{n_1}^+ \times \cdots \times \mathbb{F}_{n_k}^+$, and $\mu_1(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R_\beta}) = \mu_2(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R_\beta}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R_\beta}$ is not equal to $\boldsymbol{R_\gamma}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_{\boldsymbol{n}}^+$. Hence, we deduce that $\mu_1 = \mu_2$.

By Theorem 4.4, for any $\varphi \in \mathbf{RH}^1(\boldsymbol{B_n})$ there is a completely bounded linear map $\mu_\varphi \in \mathrm{CB}(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}, B(\mathcal{E}))$ such that $\varphi = \boldsymbol{\mathcal{P}}\mu_\varphi$ and $\|\varphi\|_1 = \|\mu_\varphi\|_{\mathrm{cb}}$. This proves that the map Ψ is surjective and $\|\boldsymbol{\mathcal{P}}\mu_\varphi\|_1 = \|\mu_\varphi\|_{\mathrm{cb}}$. Therefore, (i) is equivalent to (ii).

Now, the latter equivalence and Theorem 4.2 imply

$$\varphi(\boldsymbol{X}) = (\boldsymbol{\mathcal{P}}\mu_\varphi)(\boldsymbol{X}) = \hat{\mu}_\varphi(C_{\boldsymbol{X}}^* C_{\boldsymbol{X}}), \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}), \tag{5-1}$$

where

$$C_{\boldsymbol{X}} := (I_{\bigotimes_{i=1}^k F^2(H_{n_i})} \otimes \boldsymbol{\Delta_X}(I)^{1/2}) \prod_{i=1}^k (I - \boldsymbol{R}_{i,1} \otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i} \otimes X_{i,n_i}^*)^{-1}.$$

Due to Wittstock's extension theorem [1984], there exists a completely bounded map $\Phi : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ that extends μ_φ with $\|\mu_\varphi\|_{\mathrm{cb}} = \|\Phi\|_{\mathrm{cb}}$. According to Theorem 8.4 from [Paulsen 1986], there exist a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{K})$, and bounded operators $W_1, W_2 : \mathcal{E} \to \mathcal{K}$ with $\|\Phi\| = \|W_1\|\|W_2\|$ such that

$$\Phi(A) = W_1^*\pi(A)W_2, \quad A \in C^*(\boldsymbol{R}). \tag{5-2}$$

Set $V_{i,j} := \pi(\boldsymbol{R}_{i,j})$ for $i \in \{1, \ldots, k\}$ and $j \in \{1, \ldots, n_i\}$ and note that $\boldsymbol{V} = (V_1, \ldots, V_n)$ is a k-tuple of doubly commuting row isometries $V_i = (V_{i,1}, \ldots, V_{i,n_i})$. Using (5-1) and (5-2), one can deduce (iii). The proof that (iii) implies (i) is similar to the proof of the same implication from Theorem 4.4. □

Consider now the subspace of free holomorphic functions $\boldsymbol{H}^1(\boldsymbol{B_n}) := \mathrm{Hol}(\boldsymbol{B_n}) \bigcap \mathbf{PH}^1(\boldsymbol{B_n})$ together with the norm $\|\cdot\|_1$. Using Theorem 5.1, we can obtain the following weak analogue of the F. and M. Riesz theorem [Hoffman 1962] in our setting.

Corollary 5.2. $(\boldsymbol{H}^1(\boldsymbol{B_n}), \|\cdot\|_1)$ *is a Banach space which can be identified with the annihilator of* $\boldsymbol{\mathcal{R}}_n$ *in* $\mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}, B(\mathcal{E}))$, *i.e.*,

$$(\boldsymbol{\mathcal{R}}_n)^\perp := \{\mu \in \mathrm{CB}_0\,(\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}}, B(\mathcal{E})) : \mu(\boldsymbol{R_\alpha}) = 0 \text{ for all } \boldsymbol{\alpha} \in \boldsymbol{F}_{\boldsymbol{n}}^+,\ |\boldsymbol{\alpha}| \geq 1\}.$$

Moreover, for each $f \in \boldsymbol{H}^1(\boldsymbol{B_n})$, *there is a unique completely bounded linear map* $\mu_f \in (\boldsymbol{\mathcal{R}}_n)^\perp$ *such that* $f = \boldsymbol{\mathcal{P}}\mu_f$.

Given a completely bounded linear map $\mu : \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^*\boldsymbol{\mathcal{R}}_{\boldsymbol{n}} \to B(\mathcal{E})$, we introduce the *noncommutative Fantappiè transform* of μ to be the map $\boldsymbol{\mathcal{F}}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{\mathcal{F}}\mu)(\boldsymbol{X}) := \hat{\mu}\left(\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} \right)$$

for $\boldsymbol{X} := (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})$. We remark that the noncommutative Fantappiè transform is a linear map and $\boldsymbol{\mathcal{F}}\mu$ is a free holomorphic function in the open polyball $\boldsymbol{B_n}$ with coefficients in $B(\mathcal{E})$.

Let $\mu : \boldsymbol{\mathcal{R}_n^*\mathcal{R}_n} \to B(\mathcal{E})$ be a completely positive linear map. We introduce the *noncommutative Herglotz–Riesz transform* of μ on the regular polyball to be the map $\boldsymbol{H}\mu : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ defined by

$$(\boldsymbol{H}\mu)(\boldsymbol{X}) := \hat{\mu}\left(2\prod_{i=1}^{k}(I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)$$

for $\boldsymbol{X} := (X_1, \dots, X_k) \in \boldsymbol{B_n}(\mathcal{H})$. Note that $(\boldsymbol{H}\mu)(\boldsymbol{X}) = 2(\boldsymbol{\mathcal{F}}\mu)(\boldsymbol{X}) - \mu(I) \otimes I$.

Theorem 5.3. *Let f be a function from the polyball $\boldsymbol{B_n}(\mathcal{H})$ to $B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$. Then the following statements are equivalent*:

(i) *f is a free holomorphic function with $\Re f \geq 0$ and the linear maps $\{\nu_{\Re f_r}\}_{r\in[0,1)}$ associated with $\Re f$ are completely positive.*

(ii) *The function f admits a Herglotz–Riesz representation*

$$f(\boldsymbol{X}) = (\boldsymbol{H}\mu)(\boldsymbol{X}) + i\Im f(0),$$

where $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ is a completely positive linear map with the property that $\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_{\boldsymbol{n}}^+$.

(iii) *There exist a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$, $V_i = (V_{i,1}, \dots, V_{i,n_i})$, of doubly commuting row isometries on a Hilbert space $\mathcal{K}$ and a bounded linear operator $W : \mathcal{E} \to \mathcal{K}$ such that*

$$f(\boldsymbol{X}) = (W^* \otimes I)\left(2\prod_{i=1}^{k}(I - V_{i,1}^* \otimes X_{i,1} - \cdots - V_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)(W \otimes I) + i\Im f(0)$$

and $W^*\boldsymbol{V}_{\boldsymbol{\alpha}}^*\boldsymbol{V}_{\boldsymbol{\beta}}W = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_{\boldsymbol{n}}^+$.

Proof. We prove that (i) implies (ii). Let f have the representation $f(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha}\in\boldsymbol{F}_{\boldsymbol{n}}^+} A_{(\boldsymbol{\alpha})} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}}$. Due to Corollary 4.5, there exists a completely positive linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ such that $\Re f = \boldsymbol{\mathcal{P}}\mu$. Consequently, we have

$$\mu(I) = \tfrac{1}{2}(A_{(\boldsymbol{g})} + A_{(\boldsymbol{g})}^*), \quad \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}) = \tfrac{1}{2}A_{(\boldsymbol{\alpha})}^*, \quad \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = \tfrac{1}{2}A_{(\boldsymbol{\alpha})} \quad \text{for all } \boldsymbol{\alpha} \in \boldsymbol{F}_{\boldsymbol{n}}^+,\ |\boldsymbol{\alpha}| \geq 1,$$

and $\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma} \in \boldsymbol{F}_{\boldsymbol{n}}^+$. Using the definition of the Herglotz–Riesz transform, we obtain

$$\begin{aligned}(\boldsymbol{H}\mu)(\boldsymbol{X}) &:= \hat{\mu}\left(2\prod_{i=1}^{k}(I - \boldsymbol{R}_{i,1}^* \otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^* \otimes X_{i,n_i})^{-1} - I\right)\\ &= \sum_{\boldsymbol{\alpha}\in\boldsymbol{F}_{\boldsymbol{n}}^+} A_{(\boldsymbol{\alpha})} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} + A_{(\boldsymbol{g})} \otimes I - \tfrac{1}{2}(A_{(\boldsymbol{g})} + A_{(\boldsymbol{g})}^*) \otimes I\\ &= f(\boldsymbol{X}) - \tfrac{1}{2}(A_{(\boldsymbol{g})}^* - A_{(\boldsymbol{g})}) \otimes I\\ &= f(\boldsymbol{X}) - i\Im f(0),\end{aligned}$$

which proves (ii). Now we prove that (ii) implies (i). Assume that (ii) holds. Then

$$f(\boldsymbol{X}) = 2(\mathcal{F}\mu)(\boldsymbol{X}) - \mu(I)\otimes I - i\Im f(0)$$

is a free holomorphic function on the polyball $\boldsymbol{B_n}$. Taking into account that $\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = 0$ if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma}\in \boldsymbol{F}_{\boldsymbol{n}}^+$, and using Theorem 4.2, we deduce that

$$\begin{aligned}
&\tfrac{1}{2}(f(\boldsymbol{X}) + f(\boldsymbol{X})^*)\\
&\quad = \tfrac{1}{2}((\boldsymbol{H}\mu)(\boldsymbol{X}) + (\boldsymbol{H}\mu)(\boldsymbol{X})^*)\\
&\quad = \hat{\mu}\biggl(\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^*\otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^*\otimes X_{i,n_i})^{-1} - I + \prod_{i=1}^k (I - \boldsymbol{R}_{i,1}\otimes X_{i,1}^* - \cdots - \boldsymbol{R}_{i,n_i}\otimes X_{i,n_i}^*)^{-1}\biggr)\\
&\quad = \hat{\mu}(\boldsymbol{\mathcal{P}}(\boldsymbol{R},\boldsymbol{X})) \geq 0.
\end{aligned}$$

Therefore, $\Re f$ is a free k-pluriharmonic function such that $\Re f = \boldsymbol{\mathcal{P}}\mu$. Due to Corollary 4.5, we deduce that the linear maps $\{\nu_{\Re f_r}\}_{r\in[0,1)}$ associated with $\Re f$ are completely positive.

Now, we prove that (ii) implies (iii). Assume that (ii) holds. According to Stinespring's representation theorem [1955], there is a Hilbert space $\mathcal{K}$, a $*$-representation $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{K})$ and a bounded $W:\mathcal{E}\to\mathcal{K}$ with $\|\mu(I)\| = \|W\|^2$ such that $\mu(A) = W^*\pi(A)W$ for all $A\in C^*(\boldsymbol{R})$. Setting $V_{i,j} := \pi(\boldsymbol{R}_{ij})$ for all $i\in\{1,\dots,k\}$ and $j\in\{1,\dots,n_i\}$, it is clear that the k-tuple $\boldsymbol{V} = (V_1,\dots,V_k)$, $V_i = (V_{i,1},\dots,V_{i,n_i})$, consists of doubly commuting row isometries. Note that, if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma}\in \boldsymbol{F}_{\boldsymbol{n}}^+$, then

$$W^*\boldsymbol{V}_{\boldsymbol{\alpha}}^*\boldsymbol{V}_{\boldsymbol{\beta}}W = W^*\pi(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}})W = \mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = 0.$$

Now, one can easily see that the relation $f(\boldsymbol{X}) = (\boldsymbol{H}\mu)(\boldsymbol{X}) + i\Im f(0)$ leads to the representation in (iii), which completes the proof.

It remains to prove that (iii) implies (ii). To this end, assume that (iii) holds. Since the k-tuple $\boldsymbol{V} = (V_1,\dots,V_k)$, $V_i = (V_{i,1},\dots,V_{i,n_i})$, consists of doubly commuting row isometries on a Hilbert space $\mathcal{K}$, the noncommutative von Neumann inequality [Popescu 1999] implies that the map $\pi : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ defined by

$$\pi(\boldsymbol{R}_{\boldsymbol{\alpha}}\boldsymbol{R}_{\boldsymbol{\beta}}^*) := \boldsymbol{V}_{\boldsymbol{\alpha}}\boldsymbol{V}_{\boldsymbol{\beta}}^*, \quad \boldsymbol{\alpha},\boldsymbol{\beta}\in \boldsymbol{F}_{\boldsymbol{n}}^+,$$

is a $*$-representation of $C^*(\boldsymbol{R})$. Define $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ by setting $\mu(A) := W^*\pi(A)W$. It is clear that μ is a completely positive linear map and (iii) implies

$$f(\boldsymbol{X}) = \hat{\mu}\biggl(2\prod_{i=1}^k (I - \boldsymbol{R}_{i,1}^*\otimes X_{i,1} - \cdots - \boldsymbol{R}_{i,n_i}^*\otimes X_{i,n_i})^{-1} - I\biggr) + i\Im f(0).$$

Note also that

$$\mu(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}) = W^*\pi(\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}})W = W^*\boldsymbol{V}_{\boldsymbol{\alpha}}^*\boldsymbol{V}_{\boldsymbol{\beta}}W = 0$$

if $\boldsymbol{R}_{\boldsymbol{\alpha}}^*\boldsymbol{R}_{\boldsymbol{\beta}}$ is not equal to $\boldsymbol{R}_{\boldsymbol{\gamma}}$ or $\boldsymbol{R}_{\boldsymbol{\gamma}}^*$ for some $\boldsymbol{\gamma}\in \boldsymbol{F}_{\boldsymbol{n}}^+$. This shows that (ii) holds. $\square$

In the particular case when $n_1 = \cdots = n_k = 1$, we obtain an operator-valued extension of Korányi–Pukánszky integral representation for holomorphic functions with positive real part on polydisks [Korányi and Pukánszky 1963].

In what follows, we say that f has a Herglotz–Riesz representation if Theorem 5.3(ii) is satisfied.

Theorem 5.4. *Let $f : \boldsymbol{B_n}(\mathcal{H}) \to B(\mathcal{E}) \otimes_{\min} B(\mathcal{H})$ be a function, where $\boldsymbol{n} = (n_1, \dots, n_k) \in \mathbb{N}^k$. If f admits a Herglotz–Riesz representation, then f is a free holomorphic function with $\Re f \geq 0$.*

Conversely, if f is a free holomorphic function such that $\Re f \geq 0$, then there is a unique completely positive linear map $\mu : \boldsymbol{\mathcal{R}}_{\boldsymbol{n}}^* + \boldsymbol{\mathcal{R}}_{\boldsymbol{n}} \to B(\mathcal{E})$ such that

$$f(\boldsymbol{Y}) = (\boldsymbol{H}\mu)(k\boldsymbol{Y}) + i\Im f(0), \quad \boldsymbol{Y} \in \frac{1}{k}\boldsymbol{B_n}(\mathcal{H}).$$

Moreover,

$$f(\boldsymbol{X}) = 2\sum_{\boldsymbol{\alpha} \in \boldsymbol{F}_{\boldsymbol{n}}^+} k^{|\boldsymbol{\alpha}|}\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} - \mu(I) \otimes I, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

Proof. The forward implication was proved in Theorem 5.3. We prove the converse. Assume that f has the representation

$$f(\boldsymbol{X}) = \sum_{\boldsymbol{\alpha} \in \boldsymbol{F}_{\boldsymbol{n}}^+} A_{(\boldsymbol{\alpha})} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}}, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}). \tag{5-3}$$

First we consider the case when $\frac{1}{2}(A_{(\boldsymbol{g})} + A_{(\boldsymbol{g})}^*) = I_{\mathcal{E}}$. Since $\Re f \geq 0$ and $\Re f(0) = I$, Theorem 3.6 shows that there is a k-tuple $\boldsymbol{V} = (V_1, \dots, V_k)$ of commuting row isometries on a space $\mathcal{K} \supset \mathcal{E}$ such that

$$\Re f(\boldsymbol{X}) = \sum_{(\boldsymbol{\sigma},\boldsymbol{\omega}) \in \boldsymbol{F}_{\boldsymbol{n}}^+ \times \boldsymbol{F}_{\boldsymbol{n}}^+} P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} \boldsymbol{X}_{\boldsymbol{\beta}}^*.$$

The uniqueness of the representation of free k-pluriharmonic functions on $\boldsymbol{B_n}$ implies

$$P_{\mathcal{E}} \boldsymbol{V}_{\tilde{\boldsymbol{\alpha}}}^* \boldsymbol{V}_{\tilde{\boldsymbol{\beta}}}\big|_{\mathcal{E}} = \begin{cases} \frac{1}{2}A_{(\boldsymbol{\alpha})} & \text{if } \boldsymbol{\alpha} \in \boldsymbol{F}_{\boldsymbol{n}}^+, \boldsymbol{\beta} = \boldsymbol{g}, \\ \frac{1}{2}A_{(\boldsymbol{\beta})}^* & \text{if } \boldsymbol{\beta} \in \boldsymbol{F}_{\boldsymbol{n}}^+, \boldsymbol{\alpha} = \boldsymbol{g}, \\ 0 & \text{otherwise.} \end{cases} \tag{5-4}$$

Set $T_{i,j} := (1/k)V_{i,j}$ for $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. According to Proposition 1.9 from [Popescu 2016], the k-tuple $\boldsymbol{T} := (T_1, \dots, T_k)$, $T_i := (T_{i,1}, \dots, T_{i,n_i})$, is in the closed polyball $\boldsymbol{B_n}(\mathcal{K})$. Using Theorem 7.2 from [Popescu 2016], we find a k-tuple $\boldsymbol{W} := (W_1, \dots, W_k)$ of doubly commuting row isometries on a Hilbert space $\mathcal{G} \supset \mathcal{K}$ such that $W_{i,j}^*|_{\mathcal{K}} = T_{i,j}^*$ for all $i \in \{1, \dots, k\}$ and $j \in \{1, \dots, n_i\}$. Define the linear map $\mu : C^*(\boldsymbol{R}) \to B(\mathcal{E})$ by setting

$$\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}} \boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = P_{\mathcal{E}}\big[P_{\mathcal{K}} \boldsymbol{W}_{\tilde{\boldsymbol{\beta}}} \boldsymbol{W}_{\tilde{\boldsymbol{\alpha}}}^*|_{\mathcal{K}}\big]\big|_{\mathcal{E}}, \quad \boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{F}_{\boldsymbol{n}}^+.$$

Note that μ is a completely positive linear map with the property that $\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = (1/2k^{|\boldsymbol{\beta}|})A_{(\boldsymbol{\beta})}^*$ and $\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = (1/2k^{|\boldsymbol{\alpha}|})A_{(\boldsymbol{\alpha})}$ if $\boldsymbol{\alpha}, \boldsymbol{\beta} \in \boldsymbol{F}_{\boldsymbol{n}}^+$ with $\boldsymbol{\alpha} \neq \boldsymbol{g}$ and $\boldsymbol{\beta} \neq \boldsymbol{g}$, and $\mu(I) = I_{\mathcal{E}}$. Consequently, (5-3) and (5-4) imply

$$f(\boldsymbol{X}) = 2\sum_{\boldsymbol{\alpha} \in \boldsymbol{F}_{\boldsymbol{n}}^+} k^{|\boldsymbol{\alpha}|}\mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) \otimes \boldsymbol{X}_{\boldsymbol{\alpha}} - \mu(I) \otimes I, \quad \boldsymbol{X} \in \boldsymbol{B_n}(\mathcal{H}).$$

Setting $\boldsymbol{Y} := (1/k)\boldsymbol{X}$, we deduce that

$$\begin{aligned} f(\boldsymbol{Y}) &= 2\sum_{\boldsymbol{\alpha}\in\boldsymbol{F}_n^+} \mu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*)\otimes k^{|\boldsymbol{\alpha}|}\boldsymbol{Y}_{\boldsymbol{\alpha}} - \mu(I)\otimes I \\ &= \hat{\mu}\biggl(2\prod_{i=1}^{k}(I-\boldsymbol{R}_{i,1}^*\otimes kY_{i,1}-\cdots-\boldsymbol{R}_{i,n_i}^*\otimes kY_{i,n_i})^{-1}-I\biggr) \\ &= (\boldsymbol{H}\mu)(k\boldsymbol{Y}) \end{aligned}$$

for any $\boldsymbol{Y}\in(1/k)\boldsymbol{B_n}(\mathcal{H})$, which completes the proof when $A_{(\boldsymbol{g})}=I_{\mathcal{E}}$. Now, we consider the case when $C_{(\boldsymbol{g})} := \frac{1}{2}(A_{(\boldsymbol{g})}+A_{(\boldsymbol{g})}^*)\ge 0$. For each $\epsilon>0$, define the free holomorphic function

$$g_\epsilon := (C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}[f+\epsilon I_{\mathcal{E}}\otimes I_{\mathcal{H}}](C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}$$

and note that $\Re g_\epsilon(0)=I$. Applying the first part of the proof to g_ϵ, we find a completely positive linear map $\mu_\epsilon : C^*(\boldsymbol{R})\to B(\mathcal{E})$ with the property that

$$\mu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = \frac{1}{2k^{|\boldsymbol{\beta}|}}(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}A_{(\boldsymbol{\beta})}^*(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}$$

and

$$\mu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = \frac{1}{2k^{|\boldsymbol{\alpha}|}}(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}A_{(\boldsymbol{\alpha})}(C_{(\boldsymbol{g})}+\epsilon I)^{-1/2}$$

if $\boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{F}_n^+$ with $\boldsymbol{\alpha}\ne\boldsymbol{g}$ and $\boldsymbol{\beta}\ne\boldsymbol{g}$, and $\mu_\epsilon(I)=I_{\mathcal{E}}$. Setting $\nu_\epsilon(\xi) := (C_{(\boldsymbol{g})}+\epsilon I)^{1/2}\mu_\epsilon(\xi)(C_{(\boldsymbol{g})}+\epsilon I)^{1/2}$ for all $\xi\in C^*(\boldsymbol{R})$, one can easily see that ν_ϵ is a completely positive linear map with the property that $\nu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = (1/2k^{|\boldsymbol{\beta}|})A_{(\boldsymbol{\beta})}^*$ and $\nu_\epsilon(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = (1/2k^{|\boldsymbol{\alpha}|})A_{(\boldsymbol{\alpha})}$ if $\boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{F}_n^+$ with $\boldsymbol{\alpha}\ne\boldsymbol{g}$ and $\boldsymbol{\beta}\ne\boldsymbol{g}$, and $\nu_\epsilon(I)=C_{(\boldsymbol{g})}+\epsilon I_{\mathcal{E}}$. Following the proofs of Theorem 4.4 and Corollary 4.5, we find a completely positive linear map $\nu : C^*(\boldsymbol{R})\to B(\mathcal{E})$ such that $\nu(\xi) = \text{WOT-}\lim_{\epsilon_k\to 0}\nu_{\epsilon_k}(\xi)$ for $\xi\in C^*(\boldsymbol{R})$, where $\{\epsilon_k\}$ is a sequence of positive numbers converging to zero. Consequently, we have $\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\beta}}}) = (1/2k^{|\boldsymbol{\beta}|})A_{(\boldsymbol{\beta})}^*$ and $\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*) = (1/2k^{|\boldsymbol{\alpha}|})A_{(\boldsymbol{\alpha})}$ if $\boldsymbol{\alpha},\boldsymbol{\beta}\in\boldsymbol{F}_n^+$ with $\boldsymbol{\alpha}\ne\boldsymbol{g}$ and $\boldsymbol{\beta}\ne\boldsymbol{g}$, and $\nu(I)=C_{(\boldsymbol{g})}$. As in the first part of this proof, one can easily see that

$$f(\boldsymbol{Y}) = (\boldsymbol{H}\nu)(k\boldsymbol{Y})+i\Im f(0), \quad \boldsymbol{Y}\in\frac{1}{k}\boldsymbol{B_n}(\mathcal{H}),$$

and

$$f(\boldsymbol{X}) = 2\sum_{\boldsymbol{\alpha}\in\boldsymbol{F}_n^+} k^{|\boldsymbol{\alpha}|}\nu(\boldsymbol{R}_{\tilde{\boldsymbol{\alpha}}}^*)\otimes\boldsymbol{X}_{\boldsymbol{\alpha}} - \nu(I)\otimes I, \quad \boldsymbol{X}\in\boldsymbol{B_n}(\mathcal{H}). \qquad \square$$

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Received 3 Dec 2015. Revised 29 Feb 2016. Accepted 12 Apr 2016.

GELU POPESCU: gelu.popescu@utsa.edu
Department of Mathematics, The University of Texas at San Antonio, One UTSA Circle, San Antonio, TX 78249, United States

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1235

BOHNENBLUST–HILLE INEQUALITIES FOR LORENTZ SPACES VIA INTERPOLATION

ANDREAS DEFANT AND MIECZYSŁAW MASTYŁO

We prove that the Lorentz sequence space $\ell_{2m/(m+1),1}$ is, in a precise sense, optimal among all symmetric Banach sequence spaces satisfying a Bohnenblust–Hille-type inequality for m-linear forms or m-homogeneous polynomials on $\mathbb{C}^n$. Motivated by this result we develop methods for dealing with subtle Bohnenblust–Hille-type inequalities in the setting of Lorentz spaces. Based on an interpolation approach and the Blei–Fournier inequalities involving mixed-type spaces, we prove multilinear and polynomial Bohnenblust–Hille-type inequalities in Lorentz spaces with subpolynomial and subexponential constants. An application to the theory of Dirichlet series improves a deep result of Balasubramanian, Calado and Queffélec.

1. Introduction and classical results

In seminal work, Bohnenblust and Hille [1931] proved that there exists a positive function f on $\mathbb{N}$ such that, for each n and every m-homogeneous polynomial on $\mathbb{C}^n$, the ℓ_p-norm with $p = 2m/(m+1)$ of the set of its coefficients is bounded above by the constant $f(m)$ times the supremum norm of the polynomial on the unit polydisc $\mathbb{D}^n$. The primary interest of this result is that $f(m)$ is independent of the dimension n and, moreover, the exponent $2m/(m+1)$ is optimal. This result was a key point in the celebrated solution by Bohnenblust and Hille of Bohr's absolute convergence problem for Dirichlet series (see, e.g., [Bohnenblust and Hille 1931; Bohr 1913; Defant et al. 2016; Defant and Sevilla-Peris 2014]).

Recently, more sophisticated results were obtained and successfully applied to verify several long-standing conjectures in the convergence theory for Dirichlet series (and intimately related complex analysis in high dimensions). A striking improvement was given in [Defant et al. 2011], proving that $f(m)$ in fact grows at most exponentially in m, and a recent result even states that $f(m)$ is subexponential, in the sense that for every $\varepsilon > 0$ there is a constant $C(\varepsilon)$ such that $f(m) \le C(\varepsilon)(1+\varepsilon)^m$ for each $m \in \mathbb{N}$ [Bayart et al. 2014b]. Estimates of this type proved to be useful in many different areas of analysis, for example the modern $\mathcal{H}_p$-theory of Dirichlet series and (the intimately connected) infinite-dimensional holomorphy (see, e.g., [Bayart et al. 2014a; Defant and Sevilla-Peris 2014]), the study of summing polynomials in Banach spaces (see [Albuquerque et al. 2014; Defant et al. 2012; Dimant and Sevilla-Peris 2013], for example), and even in quantum information theory (see [Montanaro 2012]) and more generally in Fourier analysis of Boolean functions. A good general reference in this area is the recent book of O'Donnell [2014].

Mastyło was supported by the Foundation for Polish Science (FNP).
MSC2010: 46B70, 47A53.
Keywords: Bohnenblust–Hille inequality, Dirichlet polynomials, Dirichlet series, homogeneous polynomials, interpolation spaces, Lorentz spaces.

Our aim is to prove multilinear and polynomial Bohnenblust–Hille inequalities in the setting of Lorentz spaces. In the remainder of this introduction we give more precise details on the state of the art of BH inequalities (multilinear and polynomial) and isolate the two natural problems that mainly concern us.

We will consider Banach sequence spaces $(X(I), \|\cdot\|_X)$ of $\mathbb{C}$-valued sequences $(x_i)_{i\in I}$, which are defined over arbitrary given (index) sets I. In what follows, Lorentz spaces will play an important role. Given $1 \le p < \infty$ and $1 \le q \le \infty$, the Lorentz space $\ell_{p,q}(I)$ ($\ell_{p,q}$ for short) on a nonempty set I consists of all $x = (x_i)_{i\in I}$ for which the expression

$$\|x\|_{\ell_{p,q}} = \begin{cases} \left(\sum_{k\in J} x_k^{*q}(k^{q/p} - (k-1)^{q/p})^q\right)^{1/q} & \text{if } q < \infty, \\ \sup_{k\in J} k^{1/p} x_k^* & \text{if } q = \infty, \end{cases} \tag{1}$$

is finite. Here, as usual, for a given $x = (x_i)_{i\in I} \in \ell_\infty(I)$, we denote by $x^* = (x_j^*)_{j\in J}$ the nonincreasing rearrangement of x, defined by

$$x_j^* = \inf\bigl\{\lambda > 0 : \operatorname{card}\{i \in I : |x_i| > \lambda\} \le j\bigr\}, \quad j \in J,$$

where $J = \{1, \dots, n\}$ whenever $\operatorname{card} I = n$, and $J = \mathbb{N}$ whenever I is infinite. The expression (1) is a norm if $q \le p$ and a quasinorm if $q > p$. In the second case, $\|\cdot\|_{\ell_{p,q}}$ is equivalent to a norm. Of course, $\ell_{p,p}$ is the Minkowski space ℓ_p, since the map $x \mapsto x^*$ is an isometry.

The following two finite index sets will be of special interest: for each $m, n \in \mathbb{N}$,

$$\mathcal{M}(m,n) = \{\boldsymbol{i} = (i_1, \dots, i_m) : i_k \in \mathbb{N},\ 1 \le i_k \le n\} \quad \text{and} \quad \mathcal{J}(m,n) = \{\boldsymbol{j} \in \mathcal{M}(m,n) : j_1 \le j_2 \le \cdots \le j_m\}.$$

Below we explain the two inequalities we are interested in, the so-called multilinear and polynomial Bohnenblust–Hille inequalities, and we motivate the two problems we intend to handle.

The multilinear BH inequality. Given a Banach sequence space X (defined over arbitrary index sets) and $m \in \mathbb{N}$, we denote by

$$\mathrm{BH}_X^{\mathrm{mult}}(m) \in [1, \infty]$$

the best constant $C \ge 1$ such that for each n and every complex matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ we have

$$\|(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}\|_X \le C\|a\|_\infty, \tag{2}$$

where

$$\|a\|_\infty = \sup_{\substack{\|(x_i^k)_{i=1}^n\|_\infty \le 1 \\ 1\le k\le m}} \left| \sum_{\boldsymbol{i}=(i_1,\dots,i_m)\in\mathcal{M}(m,n)} a_{\boldsymbol{i}} x_{i_1}^1 \cdots x_{i_m}^m \right|.$$

For the sake of completeness we give a short review of the history of the inequalities of the form (2), emphasizing those results, old and very recent, which are of relevance to this article. (For more on that we once again refer to [Defant and Sevilla-Peris 2014].) The case $m = 2$ reflects a famous result of Littlewood [1930]:

$$\mathrm{BH}_{\ell_{4/3}}^{\mathrm{mult}}(2) < \infty.$$

Solving Bohr's so-called absolute convergence problem on Dirichlet series, Bohnenblust and Hille [1931] studied the case of arbitrary m and proved that

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1)}}(m) < \infty. \tag{3}$$

This result was improved by [Blei and Fournier 1989; Fournier 1987] showing that, even,

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) < \infty. \tag{4}$$

In Section 4 we give a modified version of their proof from [Blei and Fournier 1989].

Finally, Bayart, Pellegrino and Seoane-Sepúlveda [Bayart et al. 2014b] showed that the constants in (3) are subpolynomial in the following sense: there is a constant $\kappa > 1$ such that for all m we have

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1)}}(m) \le \kappa m^{(1-\gamma)/2}, \tag{5}$$

where γ is the Euler–Masceroni constant. Note that there exists a uniform constant $C > 0$ such that, for any finite index set I,

$$\|\ell_p(I) \hookrightarrow \ell_{p,1}(I)\| \le C \log(\operatorname{card} I); \tag{6}$$

hence, by (5), there exists $\delta > 1$ such that, for each m, n and every matrix $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$,

$$\|(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}\|_{2m/(m+1),1} \le m^{\delta}(\log n)\|a\|_\infty.$$

In view of this, and comparing with (4) and (5), the following natural question appears:

Problem 1. *Does there exist a constant $\delta > 0$ such that for each m we have*

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) \le m^{\delta}?$$

We provide far-reaching partial solutions extending all results mentioned before. The main contributions are given in Theorems 6 and 12.

The polynomial BH inequality. Every m-homogenous polynomial

$$P(z) = \sum_{\substack{\alpha\in\mathbb{N}_0^n\\ |\alpha|=m}} c_\alpha z^\alpha$$

in n complex variables $z = (z_1, \ldots, z_n) \in \mathbb{C}^n$ can be uniquely rewritten in the form

$$P(z) = \sum_{\boldsymbol{j}\in\mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1}\cdots z_{j_m}, \tag{7}$$

and we denote its supremum norm by

$$\|P\|_\infty = \sup_{\|(z_i)_{i=1}^n\|_\infty\le 1} \Bigg| \sum_{\boldsymbol{j}=(i_1,\ldots,i_n)\in\mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1}\cdots z_{j_m}\Bigg|.$$

Given a Banach sequence space X (defined over an arbitrary index set) and $m \in \mathbb{N}$, we denote by

$$\mathrm{BH}^{\mathrm{pol}}_X(m) \in [1, \infty]$$

the best constant $C \geq 1$ such that, for each n and every m-homogeneous polynomial P as in (7), we have

$$\|(c_{\boldsymbol{j}}(P))_{\boldsymbol{j}\in\mathcal{J}(m,n)}\|_X \leq C\|P\|_\infty. \tag{8}$$

Let us again give a short review of the most important results on such inequalities (for more information, again see [Defant and Sevilla-Peris 2014]).

By inventing polarization, Bohnenblust and Hille [1931] deduced from (3) that

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1)}}(m) < \infty. \tag{9}$$

The fact that $p = 2m/(m+1)$ is optimal here was a crucial step in the solution of Bohr's so-called absolute convergence problem. Again, mainly motivated by problems on the general theory of Dirichlet series and holomorphic functions in high dimensions, the first qualitative improvement of the constants was done in [Defant et al. 2011]: for every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that, for all m,

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1)}}(m) \leq C(\varepsilon)(\sqrt{2}+\varepsilon)^m. \tag{10}$$

Bayart et al. [2014b] proved that these constants are even subexponential in the following sense:

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1)}}(m) \leq C(\varepsilon)(1+\varepsilon)^m. \tag{11}$$

We are going to see that a standard polarization argument extends (9) to Lorentz spaces:

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1),1}}(m) < \infty; \tag{12}$$

but the following problem will turn out to be much more challenging:

Problem 2. *To what extent do* (10) *and* (11) *hold when we replace* $\ell_{2m/(m+1)}$ *by the Lorentz sequence space* $\ell_{2m/(m+1),1}$?

Concerning the extension of (10), our main result is given in Theorem 14.

Why do Lorentz spaces play an essential role within the context of Bohnenblust–Hille inequalities? We prove (see Theorem 1) that, among all symmetric Banach sequence spaces X satisfying a multilinear or polynomial Bohnenblust–Hille inequality as in (2) or (8), the sequence space $X = \ell_{2m/(m+1),1}$ is the smallest one (and in this sense the "best").

2. Preliminaries

Throughout the paper, for a given finite set $\{X_i\}_{i\in I}$ of Banach spaces which are all contained in some linear space $\mathscr{X}$, we denote by $\bigoplus_{i\in I} X_i$ the Banach space of all $x \in \bigcap_{i\in I} X_i$ equipped with the norm

$$\|x\|_{\bigoplus_{i\in I} X_i} = \sum_{i\in I} \|x\|_{X_i}.$$

For each $m \in \mathbb{N}$ we denote by $\mathcal{M}(m)$ and $\mathcal{J}(m)$ the union of all $\mathcal{M}(m,n)$ and $\mathcal{J}(m,n)$, $n \in \mathbb{N}$, respectively. We define an equivalence relation in $\mathcal{M}(m,n)$ in the following way: $\boldsymbol{i} \sim \boldsymbol{j}$ if there is a permutation σ of $\{1,\dots,m\}$ such that $(i_1,\dots,i_m) = (j_{\sigma(1)},\dots,j_{\sigma(m)})$, and denote by $[\boldsymbol{i}]$ the equivalence class of $\boldsymbol{i} \in \mathcal{M}(m,n)$. The following disjoint partition of $\mathcal{M}(m,n)$ will be very useful:

$$\mathcal{M}(m,n) = \bigcup_{\boldsymbol{j} \in \mathcal{J}(m,n)} [\boldsymbol{j}].$$

For $1 \le k \le m$, let $\mathcal{P}_k(m)$ denote the set of all subsets of $\{1,\dots,m\}$ with cardinality k. We denote the complement of $S \in \mathcal{P}_k(m)$ in $\{1,\dots,m\}$ by $\widehat{S}$. If $S \in \mathcal{P}_k(m)$, then let $\mathcal{M}(S,n)$ be the set of all indices $\boldsymbol{i}: S \to \{1,\dots,n\}$, so in the special case $S = \{1,\dots,k\}$ we clearly have that $\mathcal{M}(k,n) = \mathcal{M}(S,n)$. Finally, for $\boldsymbol{i} \in \mathcal{M}(S,n)$ and $\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)$ we define $\boldsymbol{i} \oplus \boldsymbol{j} \in \mathcal{M}(m,n)$ through

$$\boldsymbol{i} \oplus \boldsymbol{j} = \begin{cases} \boldsymbol{i} & \text{on } S, \\ \boldsymbol{j} & \text{on } \widehat{S}. \end{cases}$$

Given m, n, $k \in \mathbb{N}$ with $1 \le k < m$ and $1 \le p$, $q \le \infty$, we define the norm $\|\cdot\|_{(m,n,k,p,q)}$ on the space $\mathbb{C}^{\mathcal{M}(m,n)}$ of all matrices $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ by

$$\|a\|_{(m,n,k,p,q)} = \sum_{S \in \mathcal{P}_k(m)} \Biggl(\sum_{\boldsymbol{i} \in \mathcal{M}(S,n)} \Biggl(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)} |a_{\boldsymbol{i} \oplus \boldsymbol{j}}|^q \Biggr)^{p/q} \Biggr)^{1/p},$$

and denote the corresponding Banach space by

$$\bigoplus_{S \in \mathcal{P}_k(m)} \ell_p(S)[\ell_q(\widehat{S})].$$

Clearly, this is the ℓ_1-sum of all Banach spaces $\ell_p(S)[\ell_q(\widehat{S})]$, where $\ell_p(S)[\ell_q(\widehat{S})]$ is, by definition, $\mathbb{C}^{\mathcal{M}(m,n)}$ normed by

$$\|a\|_{\ell_p(S)[\ell_q(\widehat{S})]} = \Biggl(\sum_{\boldsymbol{i} \in \mathcal{M}(S,n)} \Biggl(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)} |a_{\boldsymbol{i} \oplus \boldsymbol{j}}|^q \Biggr)^{p/q} \Biggr)^{1/p}.$$

We will consider (classes of) Banach lattices. Of particular importance are *symmetric* spaces. We recall that a Banach lattice E on a measure space (Ω, Σ, μ) is said to be symmetric if $g \in E$ and $\|f\|_E = \|g\|_E$ whenever $\mu_f = \mu_g$ and $f \in E$. Here μ_f denotes the distribution function of f, defined by $\mu_f(\lambda) = \mu\{t \in \Omega : |f(t)| > \lambda\}$ for $\lambda \ge 0$. Throughout the paper, by a Banach sequence lattice on a finite or countable set I we mean a real or complex Banach lattice E on the measure space $(I, 2^I, \mu)$ (on I, for short), where μ is the counting measure. In the case when E is symmetric, E is said to be a symmetric Banach (sequence) space.

A symmetric space E is called *fully symmetric* whenever it is an exact interpolation space between $L_1(\mu)$ and $L_\infty(\mu)$; that is, for any linear operator $T: L_1(\mu) + L_\infty(\mu) \to L_1(\mu) + L_\infty(\mu)$ such that $\|T\|_{L_1(\mu) \to L_1(\mu)} \le 1$ and $\|T\|_{L_\infty(\mu) \to L_\infty}(\mu) \le 1$ we have that T maps E into E and $\|T\|_{E \to E} \le 1$. It is well known that symmetric spaces that have the Fatou property or have order continuous norm are fully symmetric (see [Bennett and Sharpley 1988; Kreĭn et al. 1982], for example).

We will need the concept of discretization of a Banach lattice. Let (Ω, Σ, μ) be a measure space and let $d = \{\Omega_k\}_{k=1}^N \subset \Sigma$ be a measurable partition of Ω, i.e., $\Omega = \bigcup_{k=1}^N \Omega_k$, where $\Omega_i \cap \Omega_j = \varnothing$ for each $i, j \in \{1, \dots, N\}$ with $i \neq j$. Then, given a Banach lattice X on (Ω, Σ, μ), the discretization X^d is the Banach space of all simple functions $f \in X$ of the form $f = \sum_{k=1}^N \xi_k \chi_{\Omega_k} \in X$, equipped with the induced norm from X.

The notion of Lorentz spaces over arbitrary measure spaces will be essential in what follows. Given a measure space (Ω, Σ, μ) and $0 < p < \infty,\ 0 < q \le \infty$, the Lorentz space $L_{p,q}(\Omega, \mu)$ ($L_{p,q}(\Omega)$ or $L_{p,q}$, for short) is defined to be the space of all (equivalence classes of) measurable functions f on Ω, equipped with the quasinorm

$$\|f\|_{L_{p,q}} = \begin{cases} \left((q/p) \int_0^\infty f^*(t)^q t^{q/p-1}\, dt\right)^{1/q} & \text{if } q < \infty, \\ \sup_{t>0} t^{1/p} f^*(t) & \text{if } q = \infty, \end{cases}$$

where f^* is the decreasing rearrangement of f, defined on $[0, \infty)$ by

$$f^*(t) = \inf\{s > 0 : \mu_f(s) \le t\}.$$

(We adopt the convention $\inf \varnothing = \infty$.) In the case when $\Omega = I$ is a nonempty set with counting measure μ, the space $L_{p,q}(\Omega, \mu)$ in fact coincides with the Lorentz sequence space $\ell_{p,q}(I)$ already defined in (1). Indeed, in this case, given a function $f = x$ on $\Omega = I$ we have $x_k^* = f^*(t)$ for every $t \in [k-1, k),\ k \in J$, where $J = \{1, \dots, \operatorname{card} I\}$ if I is finite and $J = \mathbb{N}$ if I is infinite. Thus $\|f\|_{L_{p,q}} = \|x\|_{\ell_{p,q}}$, where the latter norm is as defined by the formula (1).

We recall that the Köthe dual space $(\ell_{p,1})'$ of the Lorentz space $\ell_{p,1} = \ell_{p,1}(I)$ coincides with the Marcinkiewicz space m_p, which consists of all complex sequences $x = (x_i)_{i \in I}$ such that

$$\|x\|_{m_p} = \sup_{k \in J} \frac{1}{k^{1/p}} \sum_{j=1}^k x_j^* < \infty,$$

and which, with this norm, forms a Banach space. Moreover, we note that by standard comparison with the integral of t^α on $[1, N]$, we have for each $N \in \mathbb{N}$ and every $\alpha \in (0, 1)$,

$$\sum_{k=1}^N \frac{1}{k^\alpha} < \frac{1}{1-\alpha} N^{1-\alpha}. \tag{13}$$

Combining this inequality (for $\alpha = 1/p$) with $x_k^* \le k^{-1/p} \|x\|_{\ell_{p,\infty}}$ for $k \in J$ yields

$$m_p = \ell_{p,\infty}$$

up to equivalent norms:

$$\frac{1}{p'} \|x\|_{m_p} \le \|x\|_{\ell_{p,\infty}} \le \|x\|_{m_p}, \quad x \in \ell_{p,\infty}.$$

(As usual we write $1/p' := 1 - 1/p$.) Many of our arguments will be based on interpolation theory. Here we recall some of its basic concepts and provide some special facts we are going to use. Recall that if

$\vec{A} = (A_0, A_1)$ is a quasinormed couple then, for any $a \in A_0 + A_1$, we define the K-functional

$$K(t, a; \vec{A}) = \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a_0 + a_1 = a\}, \quad t > 0.$$

For $0 < \theta < 1$ and $0 < q < \infty$, the real interpolation space $(A_0, A_1)_{\theta,q}$ is the space of all $a \in A_0 + A_1$, equipped with the quasinorm

$$\|a\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(t, a; \vec{A}))^q \, \frac{dt}{t}\right)^{1/q},$$

with an obvious modification for $q = \infty$.

The following well-known and easily verified interpolation property holds: if (A_0, A_1) and (B_0, B_1) are two quasinormed couples, T is a map from (A_0, A_1) to (B_0, B_1) (i.e., $T\colon A_0 + A_1 \to B_0 + B_1$ and the restrictions of T to A_j are bounded from A_j to B_j for each $j \in \{0, 1\}$) with the quasinorms $M_j = \|T : A_j \to B_j\|$, then $T\colon (A_0, A_1)_{\theta,q} \to (B_0, B_1)_{\theta,q}$ is also bounded and, for its quasinorm M, we have

$$M \le M_0^{1-\theta} M_1^{\theta}.$$

Lorentz spaces arise naturally in the real interpolation method since most of their important properties can be derived from real interpolation theorems. We briefly review some basic definitions. The pair (L_1, L_∞) is especially important for the understanding of the space $L_{p,q}$. It is well known that, for every $f \in L_1 + L_\infty$,

$$K(t, f; L_1, L_\infty) = \int_0^t f^*(s)\, ds = t f^{**}(t), \quad t > 0.$$

Hence, for each $\theta \in (0, 1)$,

$$\|f\|_{\theta,q} = \left(\int_0^\infty [t^{1-\theta} f^{**}(t)]^q \, \frac{dt}{t}\right)^{1/q}.$$

An immediate consequence of Hardy's inequality is the following well-known formula, which states that, for $1 < p < \infty$, $1 \le q \le \infty$ and $\theta = 1 - 1/p$,

$$(L_1, L_\infty)_{\theta,q} = L_{p,q},$$

and, moreover,

$$\frac{1}{p'}\|f\|_{(L_1,L_\infty)_{\theta,q}} \le \|f\|_{L_{p,q}} \le \|f\|_{(L_1,L_\infty)_{\theta,p}}.$$

The following result will be used (which follows from the more general Theorem 4.3 of [Holmstedt 1970]): Let $1/p = (1-\theta)/p_0 + \theta/p_1$, $0 < p_0$, $p_1 < \infty$, $p_0 \ne p_1$ and $0 < q \le \infty$. Then, up to equivalent norms, we have

$$(L_{p_0}, L_{p_1})_{\theta,q} = L_{p,q}.$$

More precisely,

$$\begin{aligned} C^{-1}\theta^{-\min(1/q,1/p_0)}(1-\theta)^{-\min(1/q,1/p_1)}\Big(\frac{p}{q}\Big)^{1/q}\|f\|_{L_{p,q}} \\ \le \|f\|_{(L_{p_0},L_{p_1})_{\theta,q}} \\ \le C\theta^{-\max(1/q,1/p_0)}(1-\theta)^{-\max(1/q,1/p_1)}\Big(\frac{p}{q}\Big)^{1/q}\|f\|_{L_{p,q}}, \end{aligned} \tag{14}$$

where $C>0$ is a universal constant.

We will also make intensive use of complex interpolation, and denote by $[A_0,A_1]_\theta$ the complex interpolation spaces as defined, for example, in [Calderón 1964]. We recall that if X_0 and X_1 are two complex Banach lattices on a measure space (Ω,Σ,μ) then

$$[X_0,X_1]_\theta = X_0^{1-\theta}X_1^\theta, \tag{15}$$

with equality of norms provided one of the spaces has order continuous norm; here, following Calderón, we denote by $X_0^{1-\theta}X_1^\theta$ the Calderón space of all $x\in L^0(\mu)$ such that $|x|\le\lambda|x_0|^{1-\theta}|x_1|^\theta$ μ-a.e. on Ω for some constant $\lambda>0$ and some $x_i\in X_i$ with $\|x_i\|_{X_i}\le 1$ for $i=0,1$. We put

$$\|x\|_{X_0^{1-\theta}X_1^\theta} = \inf\lambda.$$

3. The optimality of Lorentz spaces

The following theorem motivates our study; we show that, in the context of multilinear and polynomial Bohnenblust–Hille inequalities, Lorentz spaces are in a certain sense optimal. Before we state and prove these results we recall that, if X is a symmetric Banach sequence space on I and χ_A denotes the indicator function of a set $A\subset I$, clearly $\|\chi_A\|_X$ depends only on $\mathrm{card}(A)$. The function $\phi_X(k)=\|\chi_A\|_X$, where $A\subset I$ with $\mathrm{card}(A)=k$, is called the *fundamental function* of X. It is well known (see, e.g., [Kreĭn et al. 1982, Theorem 2.5.2]) that, if $1\le p<\infty$ and X is a symmetric Banach sequence space on I such that $\|\chi_A\|_X=\mathrm{card}(A)^{1/p}$ for every indicator function χ_A (that is, $\phi_X(k)=k^{1/p}$ for every $A\subset I$ with $\mathrm{card}(A)=k$), then $\ell_{p,1}\hookrightarrow X$ with

$$\|x\|_X\le\|x\|_{\ell_{p,1}},\quad x\in\ell_{p,1}.$$

Thus $\ell_{p,1}$ is the smallest symmetric Banach sequence space on I whose norm coincides with the ℓ_p-norm on indicator functions.

Theorem 1. *Fix a positive integer m. The Lorentz space $\ell_{2m/(m+1),1}$ is the smallest symmetric Banach sequence space X such that $\mathrm{BH}_X^{\mathrm{mult}}(m)<\infty$. Also, the Lorentz space $\ell_{2m/(m+1),1}$ is the smallest symmetric Banach sequence space X such that $\mathrm{BH}_X^{\mathrm{pol}}(m)<\infty$.*

Proof. We follow an argument inspired by [Bohnenblust and Hille 1931]. Assume that X is a symmetric Banach sequence space such that $\mathrm{BH}_X^{\mathrm{mult}}(m)<\infty$, i.e., for each $n\in\mathbb{N}$ and every complex matrix $a=(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ we have

$$\|a\|_X\le\mathrm{BH}_X^{\mathrm{mult}}(m)\|a\|_\infty. \tag{16}$$

It suffices to show that the fundamental function

$$\phi(n) := \Big\| \sum_{i=1}^{n} e_i \Big\|_X, \quad n \in \mathbb{N}, \tag{17}$$

satisfies

$$\phi(n) \le C(m) n^{(m+1)/(2m)} \tag{18}$$

for each $n \in \mathbb{N}$. For fixed N, choose some $N \times N$ matrix (a_{rs}) such for every r, s we have $|a_{rs}| = 1$ and $\sum_{k=1}^{N} a_{rk}\bar{a}_{sk} = N\delta_{rs}$ (e.g., $a_{rs} = e^{2\pi i rs/N}$ with $1 \le r, s \le N$), and define the matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ by

$$a_{i_1 \dots i_m} = a_{i_1 i_2} \cdots a_{i_{m-1} i_m}.$$

Since $|a_{i_1 \dots i_m}| = 1$, we have $\phi(N^m) = \|a\|_X$. We now estimate the norm $\|a\|_\infty$. We first do the trilinear case $m = 3$, where the argument becomes more transparent. We take $x, y, z \in \mathbb{C}^N$ with supremum norm at most 1; then, using the Cauchy–Schwarz inequality and the properties of the matrix, we have

$$\begin{aligned}
\Big| \sum_{i,j,k} a_{ij} a_{jk} x_i y_j z_k \Big| &\le \sum_k \Big| \sum_{i,j} a_{ij} a_{jk} x_i y_j \Big| |z_k| \\
&\le N^{1/2} \bigg(\sum_k \Big| \sum_{i,j} a_{ij} a_{jk} x_i y_j \Big|^2 \bigg)^{1/2} \\
&= N^{1/2} \bigg(\sum_{\substack{i_1, i_2 \\ j_1, j_2}} a_{i_1 j_1} \bar{a}_{i_2 j_2} x_{i_1} \bar{x}_{i_2} y_{j_1} \bar{y}_{j_2} \sum_k a_{j_1 k} \bar{a}_{j_2 k} \bigg)^{1/2} \\
&= N^{1/2} N^{1/2} \bigg(\sum_{\substack{i_1, i_2 \\ j}} a_{i_1 j} \bar{a}_{i_2 j} x_{i_1} \bar{x}_{i_2} y_j \bar{y}_j \bigg)^{1/2} = N \bigg(\sum_j \Big| \sum_i a_{ij} x_i \Big|^2 |y_j|^2 \bigg)^{1/2} \\
&\le N \bigg(\sum_{i_1 i_2} \sum_j a_{i_1 j} \bar{a}_{i_2 j} x_{i_1} \bar{x}_{i_2} \bigg)^{1/2} = N^{3/2} \bigg(\sum_i |x_i|^2 \bigg)^{1/2} \le N^{4/2}.
\end{aligned}$$

In the general case we take $z^{(1)}, \dots, z^{(m)} \in \mathbb{C}^N$, each with supremum norm at most 1, and repeat this procedure to get

$$\Big| \sum_{i_1, \dots, i_m = 1}^{N} a_{i_1 i_2} \cdots a_{i_{m-1} i_m} z^{(1)}_{i_1} \cdots z^{(m)}_{i_m} \Big| \le N^{m/2} \bigg(\sum_{i_1} |z^{(1)}_{i_1}|^2 \bigg)^{1/2} \le N^{m/2} N^{1/2}. \tag{19}$$

Hence $\|a\|_\infty \le N^{(m+1)/2}$ for each N, and by (16) we have $\phi(N^m) \le \mathrm{BH}^{\mathrm{mult}}_X(m)(N^m)^{(m+1)/(2m)}$. Since for each positive integer n there is N such that $N^m \le n < (N+1)^m$, we finally obtain (18).

To prove the second statement, we assume that X is a symmetric Banach sequence space such that, for each n and every m-homogeneous polynomial $P(z) = \sum_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} c_\alpha z^\alpha$, we have

$$\big\| (c_\alpha)_{\alpha \in \mathbb{N}_0^n, |\alpha| = m} \big\|_X \le \mathrm{BH}^{\mathrm{pol}}_X(m) \|P\|_\infty.$$

Following nontrivial ideas of Bohnenblust and Hille [1931] it is possible to modify the proof of the first statement, which leads to a sort of deterministic proof of the second statement. Here we give an alternative, probabilistic argument. As in (17) we consider the fundamental function $\phi(n)$, $n \in \mathbb{N}$, of X. Then, by the Kahane–Salem–Zygmund inequality (see [Kahane 1985], for example), there is a constant $C_{\mathrm{KSZ}} \geq 1$ such that for every choice of N there are signs $\varepsilon_\alpha = \pm 1$ for which

$$\sup_{z\in\mathbb{D}^N}\Big|\sum_{\substack{\alpha\in\mathbb{N}_0^N\\ |\alpha|=m}} \varepsilon_\alpha z^\alpha\Big| \leq C_{\mathrm{KSZ}}\Big(N\binom{m+N-1}{m}\log m\Big)^{1/2}.$$

Since the sequence $(\phi(N)/N)$ is nonincreasing and for each N we have

$$\frac{N^m}{m!} \leq \binom{N+m-1}{m} \leq N^m,$$

it follows that $\phi(N^m) \leq m!\,\phi\big(\binom{N+m-1}{m}\big)$ for each N. Combining the above estimates we conclude that, for each N,

$$\phi(N^m) \leq \mathrm{BH}_X^{\mathrm{pol}}(m) C_{\mathrm{KSZ}} m! \sqrt{\log m}\,(N^m)^{(m+1)/(2m)}.$$

This easily implies that there exists a constant $C(m) > 0$ such that

$$\phi(n) \leq C(m) n^{(m+1)/(2m)}, \quad n \in \mathbb{N},$$

and the conclusion again follows. □

4. Multilinear BH inequalities for Lorentz spaces revisited

In this section we present a slightly modified proof of (4), which was first given in [Blei and Fournier 1989]. We need to prove four preliminary lemmas.

Lemma 2. *For each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ and each $S \subset \mathcal{M}(m,n)$,*

$$\frac{1}{E(S)}\sum_{\boldsymbol{i}\in S}|a_{\boldsymbol{i}}| \leq m\|a\|_{\ell_{m/(m-1),\infty}},$$

where

$$E(S) := \max_{1\leq k\leq m} \operatorname{card}\{i_k : \boldsymbol{i} \in S\}.$$

Proof. Clearly

$$k^{(m-1)/m} a_k^* \leq \|a\|_{\ell_{m/(m-1),\infty}}, \quad 1 \leq k \leq n^m.$$

Now note that $\sum_{\boldsymbol{i}\in S}|a_{\boldsymbol{i}}|$ has not more that $E(S)^m$ summands and that $\sum_{k=1}^{E(S)^m} a^*(k)$ sums the first $E(S)^m$ many largest $|a_{\boldsymbol{i}}|$, $\boldsymbol{i} \in S$. As a consequence, we obtain by (13) (with $\alpha = 1 - 1/m$) that

$$\sum_{\boldsymbol{i}\in S}|a_{\boldsymbol{i}}| \leq \sum_{k=1}^{E(S)^m} a_k^* \leq \|a\|_{\ell_{m/(m-1),\infty}} \sum_{k=1}^{E(S)^m} k^{-(m-1)/m} \leq m\|a\|_{\ell_{m/(m-1),\infty}} E(S),$$

as desired. □

Lemma 3. *For each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ the index set $\mathcal{M}(m,n)$ splits into a union of m subsets S_k such that, for every $1 \le q < \infty$,*

$$\max_{1 \le k \le m} \|a^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\widehat{\{k\}})]} \le m^{1/q} \|a\|_{\ell_{qm/(m-1),\infty}},$$

where, for $S \subset \mathcal{M}(m,n)$, we put $a^S = a_{\boldsymbol{i}}$ for $\boldsymbol{i} \in S$ and $a^S = 0$ for $\boldsymbol{i} \notin S$.

Proof. It suffices to show the desired inequality for $q = 1$: for arbitrary $1 < q < \infty$ apply the case $q = 1$ to $|a|^{1/q}$ instead of to a. In view of Lemma 2 we show that there are appropriate sets S_k for which

$$\max_{1 \le k \le m} \|a^{S_k}\|_{\ell_\infty(\{k\})[\ell_1(\widehat{\{k\}})]} \le \sup_{S \subset \mathcal{M}(m,n)} \frac{1}{E(S)} \sum_{\boldsymbol{i} \in S} |a_{\boldsymbol{i}}|,$$

and without loss of generality we may assume that the supremum on the right side is at most 1. Given $1 \le k \le m$, observe that

$$\sum_{\ell=1}^{n} \sum_{\substack{\boldsymbol{i} \in \mathcal{M}(m,n) \\ i_k = \ell}} |a_{\boldsymbol{i}}| \le \sum_{\boldsymbol{i} \in \mathcal{M}(m,n)} |a_{\boldsymbol{i}}| \le E(\mathcal{M}(m,n)) = n.$$

Hence there is some $1 \le \ell(k) \le n$ such that for

$$T_k^1 = \{\boldsymbol{j} \in \mathcal{M}(m,n) : j_k = \ell(k)\}$$

we have

$$\sum_{\boldsymbol{i} \in T_k^1} |a_{\boldsymbol{i}}| \le 1.$$

Then, for

$$N_1 = \mathcal{M}(m,n) \setminus \bigcup_{k=1}^{m} T_k^1,$$

we obviously get $E(N_1) \le n - 1$. If we now repeat this procedure with N_1 instead of $\mathcal{M}(m,n)$, then we obtain m new index sets T_k^2, $1 \le k \le m$, in N_1, for which

$$\sum_{\boldsymbol{i} \in T_k^2} |a_{\boldsymbol{i}}| \le 1$$

and

$$E(N_2) \le n - 2 \quad \text{with } N_2 = \Big(\mathcal{M}(m,n) \setminus \bigcup_{k=1}^{m} T_k^1\Big) \setminus \Big(\bigcup_{k=1}^{m} T_k^2\Big).$$

Continuing for $j \in \{3, \dots, n\}$, we find index sets T_k^j, $1 \le j \le n$, $1 \le k \le m$, such that

$$\sum_{\boldsymbol{i} \in T_k^j} |a_{\boldsymbol{i}}| \le 1, \quad 1 \le k \le m,\ 1 \le j \le n \tag{20}$$

and

$$E(N_n) = 0 \quad \text{with } N_n = \mathcal{M}(m,n) \setminus \bigcup_{j=1}^{n} \bigcup_{k=1}^{m} T_k^j.$$

Define, for $1 \le k \le m$,

$$S_k = \bigcup_{j=1}^{n} T_k^j .$$

Obviously, we have that $N_n = \varnothing$ and hence

$$\mathcal{M}(m,n) = \bigcup_{k=1}^{m} S_k .$$

Finally, for any $1 \le k \le m$,

$$\|a^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\widehat{\{k\}})]} = \sup_{1\le j\le n} \sum_{\boldsymbol{i}\in\mathcal{M}(\widehat{\{k\}},n)} |a^{S_k}_{\boldsymbol{i}\oplus j}| \le \sup_{1\le j\le n} \sum_{\substack{\boldsymbol{i}\in\mathcal{M}(\widehat{\{k\}},n)\\ \boldsymbol{i}\oplus j\in\bigcup_{l=1}^{n} T_k^l}} |a_{\boldsymbol{i}\oplus j}| \le 1.$$

Let us comment on the argument for the last estimate: Assume without loss of generality that $n = 2$. Then, by construction, given $j = 1$ or $j = 2$ we have that either $\boldsymbol{i} \oplus j \in T_k^1$ for all $\boldsymbol{i} \in \mathcal{M}(\widehat{\{k\}}, n)$ or $\boldsymbol{i} \oplus j \in T_k^2$ for all $\boldsymbol{i} \in \mathcal{M}(\widehat{\{k\}}, n)$. The conclusion follows from (20). □

Lemma 4. *For each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ and every $1 \le q < \infty$,*

$$\|a\|_{\ell_{qm/((q-1)m+1),1}} \le m^{1/q} \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\widehat{\{k\}})]}.$$

Proof. Since for every $1 < r < \infty$ we have $m_r = \ell_{r,\infty}$ with $\|\cdot\|_{\ell_{r,\infty}} \le \|\cdot\|_{m_r}$ and $(\ell_{r,1})' = m_r$ isometrically, the required inequality follows by Lemma 3 and a simple duality argument. Indeed, take a matrix a and sets S_k according to Lemma 3. Then

$$\begin{aligned}
\sum_{\boldsymbol{i}\in\mathcal{M}(m,n)} |a_{\boldsymbol{i}} b_{\boldsymbol{i}}| &\le \sum_{1\le k\le m} \sum_{\boldsymbol{i}\in\mathcal{M}(m,n)} |a_{\boldsymbol{i}} b^{S_k}_{\boldsymbol{i}}| \\
&\le \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\widehat{\{k\}})]} \|b^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\widehat{\{k\}})]} \\
&\le \max_{1\le k\le m} \|b^{S_k}\|_{\ell_\infty(\{k\})[\ell_q(\widehat{\{k\}})]} \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\widehat{\{k\}})]} \\
&\le m^{1/q} \|b\|_{\ell_{qm/(m-1),\infty}} \sum_{1\le k\le m} \|a\|_{\ell_1(\{k\})[\ell_{q'}(\widehat{\{k\}})]},
\end{aligned}$$

the desired conclusion. □

The last lemma needed is the following so-called mixed BH inequality (this is a simple consequence of the multilinear Khinchine inequality; see, e.g., [Bayart et al. 2014b; Bohnenblust and Hille 1931; Defant et al. 2016]).

Lemma 5. *For each n and each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ we have*

$$\sum_{j=1}^{n} \Big(\sum_{\boldsymbol{i}\in\mathcal{M}(\widehat{\{k\}},n)} |a_{\boldsymbol{i}\oplus j}|^2 \Big)^{1/2} \le \sqrt{2}^{\,m-1} \|a\|_\infty, \quad 1 \le k \le m.$$

Combining Lemmas 4 (with $q = 2$) and 5 gives the proof of (4). As a byproduct we get the following estimate for the constant:

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) \le m^{1/2}\sqrt{2}^{\,m-1}.$$

We note a disadvantage of this proof: it does not give polynomial growth of $\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m)$ in m as we obtained for $\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1)}}(m)$ in (5).

4.1. ***Polynomial growth, part I.*** We are going to give a first improvement of the result from (5). Our estimate shows that the symmetric Banach sequence space

$$X = \ell_{2m/(m+1),2(m-1)/m}$$

satisfies the BH inequality from (2) with a constant growing subpolynomially in m. It is important to note that X is strictly larger than the Lorentz space $\ell_{2m/(m+1),1}$; however, X has the same fundamental function as $\ell_{2m/(m+1),1}$, which of course fits with Theorem 1.

Theorem 6. *There exists a constant $\delta > 0$ such that, for each m,*

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),2(m-1)/m}}(m) \le m^{\delta}.$$

The proof combines ideas and tools from [Blei and Fournier 1989; Bohnenblust and Hille 1931; Littlewood 1930] with some more recent ones from [Bayart et al. 2014b]. The following lemma, the proof of which is explicitly included in the proof of [Bayart et al. 2014b, Proposition 3.1], is crucial. For $1 \le p \le 2$ we write $A_p \ge 1$ for the best constant in the Khinchine–Steinhaus inequality: for each choice of finitely many $\alpha_1, \dots, \alpha_N \in \mathbb{C}$,

$$\|(\alpha_k)_{k=1}^N\|_{\ell_2} \le A_p \biggl(\int_{\mathbb{T}^N} \biggl| \sum_{k=1}^N \alpha_k z_k \biggr|^p dz \biggr)^{1/p},$$

where dz stands for the normalized Lebesgue measure on the N-dimensional torus $\mathbb{T}^N$. Recall that $A_p \le \sqrt{2}$ for all $1 \le p \le 2$.

Lemma 7. *For each n, each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ and each $1 \le k < m$, we have*

$$\|a\|_{(m,n,k,2k/(k+1),2)} \le A^{m-k}_{2k/(k+1)} \,\mathrm{BH}^{\mathrm{mult}}_{\ell_{2k/(k+1)}}(k) \|a\|_\infty.$$

The second lemma needed is an immediate consequence of [Blei and Fournier 1989, Theorem 7.2]:

Lemma 8. *For each $1 \le q < \infty$ there is a constant $C_q \ge 1$ such that, for each $1 \le t < q$ and each matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$,*

$$\|a\|_{\ell_{mqt/(mq+t-q),t}} \le C_q m \|a\|_{(m,n,m-1,t,q)}.$$

Proof of Theorem 6. For $q = 2$ and $t = 2(m-1)/m$ we have $mqt/(mq+t-q) = 2m/(m+1)$. Hence, given a matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$, Lemma 8 yields

$$\|a\|_{\ell_{2m/(m+1),2(m-1)/m}} \le C_2 m \|a\|_{(m,n,m-1,2(m-1)/m,2)}.$$

Moreover, by Lemma 7 we have

$$\|a\|_{(m,n,m-1,2(m-1)/m,2)} \le A_{2(m-1)/m}\, \mathrm{BH}^{\mathrm{mult}}_{\ell_{2(m-1)/m}}(m-1)\|a\|_\infty.$$

Combining with (5) we conclude (because $A_p \le \sqrt{2}$ for each $1 \le p \le 2$) that

$$\|a\|_{\ell_{2m/(m+1),2(m-1)/m}} \le C_2 m\sqrt{2}\kappa(m-1)^{(1-\gamma)/2}\|a\|_\infty,$$

as required. □

4.2. ***Polynomial growth, part II.*** In this section we use complex and real interpolation as well as results from [Fournier 1987] to improve Theorem 6 considerably (see Theorem 12). The starting point for what we intend to prove is the following result:

Lemma 9. *For each $m, n, k \in \mathbb{N}$ with $1 \le k \le m$ we have that*

$$\Big\| \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_\infty(\hat{S})] \hookrightarrow \ell_{m/k,1}(\mathcal{M}(m,n)) \Big\| \le \binom{m}{k}^{-1}.$$

Proof. A variant of this result is mentioned without proof in [Fournier 1987, p. 69] — the special case $k = 1$ is given in Fournier's Theorem 4.1; for the general case, analyze the proof of that theorem and use in particular his Theorem 3.3 instead of Theorem 3.1, in combination with Cauchy's inequality. □

We will need the following obvious technical result; since we here are interested in precise norm estimates, we prefer to include a proof.

Lemma 10. *Let J be a finite set and let Y and X_j, $j \in J$, be Banach lattices on a measure space (Ω, Σ, μ). Then $\bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta) = \big(\bigoplus_{j\in J} X_j\big)^{1-\theta}Y^\theta$ for every $\theta \in (0,1)$, with*

$$\Big\| \bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta) \hookrightarrow \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta \Big\| \le \operatorname{card} J,$$

$$\Big\| \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta \hookrightarrow \bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta) \Big\| \le \operatorname{card} J.$$

Proof. Choose $x \in \bigoplus_{j\in J}(X_j^{1-\theta}Y^\theta)$ with norm less than 1. Since $\|x\|_{X_j^{1-\theta}Y^\theta} < 1$ for each $j \in J$, there exist $y_j \in Y$ and $x_j \in X_j$ with $\|y_j\|_Y \le 1$ and $\|x_j\|_{X_j} \le 1$ for each $j \in J$ such that

$$|x| \le |x_j|^{1-\theta}|y_j|^\theta, \quad j \in J.$$

This implies

$$|x| \le \big(\min_{k\in J}|x_k|\big)^{1-\theta}\big(\max_{k\in J}|y_k|\big)^\theta.$$

Clearly, $\big\|\min_{k\in J}|x_k|\big\|_{\bigoplus_{j\in J}X_j} \le \sum_{j\in J}\|x_j\|_{X_j} \le \operatorname{card} J$ and $\big\|\max_{k\in J}|y_k|\big\|_Y \le \operatorname{card} J$ yield

$$x \in \Big(\bigoplus_{j\in J} X_j\Big)^{1-\theta} Y^\theta$$

with

$$\|x\|_{(\bigoplus_{j\in J} X_j)^{1-\theta} Y^\theta} \le \operatorname{card} J.$$

This shows the first estimate from our statement. The proof of the second statement is straightforward. □

Now we use real and complex interpolation to deduce, from Lemma 9, the following result:

Lemma 11. *For each $m, n, k \in \mathbb{N}$ with $1 \le k \le m$ we have*

$$\Big\| \bigoplus_{S\in\mathcal{P}_k(m)} \ell_{2k/(k+1)}(S)[\ell_2(\hat{S})] \hookrightarrow \ell_{2m/(m+1),2k/(k+1)}(\mathcal{M}(m,n)) \Big\| \le 2\binom{m}{k}^{3/2}.$$

Proof. We claim that the following norm estimate holds:

$$\Big\| \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\hat{S})] \hookrightarrow \ell_{2m/(m+k),1}(\mathcal{M}) \Big\| \le \sqrt{\binom{m}{k}}, \tag{21}$$

where $\mathcal{M} = \mathcal{M}(m,n)$. Indeed, combining complex interpolation first with Lemma 10 (with norm $\binom{m}{k}$) and then with Lemma 9 (with norm $\binom{m}{k}^{-1/2}$), we obtain

$$\begin{aligned}
\bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\hat{S})] &= \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)\big[[\ell_1(\hat{S}), \ell_\infty(\hat{S})]_{1/2}\big] \\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(S)[\ell_1(\hat{S})], \ell_1(S)[\ell_\infty(\hat{S})]\big]_{1/2} \\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(\mathcal{M}), \ell_1(S)[\ell_\infty(\hat{S})]\big]_{1/2} \\
&\hookrightarrow \Big[\ell_1(\mathcal{M}), \bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_\infty(\hat{S})]\Big]_{1/2} && \text{with norm } \le \binom{m}{k} \\
&\hookrightarrow [\ell_1(\mathcal{M}), \ell_{m/k,1}(\mathcal{M})]_{1/2} && \text{with norm } \le \binom{m}{k}^{-1/2} \\
&= \ell_{2m/(m+k),1}(\mathcal{M}).
\end{aligned}$$

Observe that the last equality here holds with equality of norms; to see this note that for every $1 < p < \infty$ and $0 < \theta < 1$ we have, by (15),

$$E := [\ell_1(\mathcal{M}), \ell_{p,1}(\mathcal{M})]_\theta = \ell_1(\mathcal{M})^{1-\theta} \ell_{p,1}(\mathcal{M})^\theta.$$

Taking Köthe duals we obtain $E' = \ell_\infty(\mathcal{M})^{1-\theta}(m_p(\mathcal{M}))^\theta = (m_p)^{1/\theta}$, which, for $\theta = \frac{1}{2}$ and $p = m/k$, gives $E' = m_{2m/(m-k)}(\mathcal{M})$, and by duality

$$E = \ell_{2m/(m+k),1}(\mathcal{M}).$$

This proves the claim from (21). Now, for $\theta_k = (k-1)/k$ we have

$$[\ell_1(S), \ell_2(S)]_{\theta_k} = \ell_{2k/(k+1)}(S).$$

Hence we deduce from (21) and, again, Lemma 10 that

$$
\begin{aligned}
\bigoplus_{S\in\mathcal{P}_k(m)} \ell_{2k/(k+1)}(S)[\ell_2(\widehat{S})] &= \bigoplus_{S\in\mathcal{P}_k(m)} [\ell_1(S),\ell_2(S)]_{\theta_k}[\ell_2(\widehat{S})] \\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(S)[\ell_2(\widehat{S})],\ell_2(S)[\ell_2(\widehat{S})]\big]_{\theta_k} \\
&= \bigoplus_{S\in\mathcal{P}_k(m)} \big[\ell_1(S)[\ell_2(\widehat{S})],\ell_2(\mathcal{M})\big]_{\theta_k} \\
&\hookrightarrow \Big[\bigoplus_{S\in\mathcal{P}_k(m)} \ell_1(S)[\ell_2(\widehat{S})],\ell_2(\mathcal{M})\Big]_{\theta_k} \qquad \text{with norm } \le \binom{m}{k} \\
&\hookrightarrow [\ell_{2m/(m+k),1}(\mathcal{M}),\ell_2(\mathcal{M})]_{\theta_k} \qquad \text{with norm } \le \binom{m}{k}^{(1-\theta_k)/2}
\end{aligned}
$$

and so the norm of the inclusion map is less than or equal to

$$\binom{m}{k}\binom{m}{k}^{(1-\theta_k)/2} = \binom{m}{k}^{1+1/(2k)} \le \binom{m}{k}^{3/2}.$$

We now need the equality

$$[\ell_{2m/(m+k),1}(\mathcal{M}),\ell_2(\mathcal{M})]_{\theta_k} = \ell_{2m/(m+1),2k/(k+1)}$$

with

$$\big\|[\ell_{2m/(m+k),1}(\mathcal{M}),\ell_2(\mathcal{M})]_{\theta_k} \hookrightarrow \ell_{2m/(m+1),2k/(k+1)}(\mathcal{M})\big\| \le 2.$$

In fact, from (15) it follows that for $1 \le q_j \le p_j < \infty$ with $j = 0, 1$ and $\theta \in (0, 1)$ we have

$$[\ell_{p_0,q_0},\ell_{p_1,q_1}]_\theta = (\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta.$$

And, further, for $1/p = (1-\theta)/p_0 + \theta/p_1$ and $1/q = (1-\theta)/q_0 + \theta/q_1$ it can be shown, similarly to in the nonatomic case in [Grafakos and Mastyło 2014, Lemma 4.1], that in the atomic case we have

$$(\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta = \ell_{p,q}$$

with

$$\|(\ell_{p_0,q_0})^{1-\theta}(\ell_{p_1,q_1})^\theta \hookrightarrow \ell_{p,q}\| \le 2^{1/p}.$$

Thus, taking $\theta = (k-1)/k,\ q_0 = 1,\ p_0 = 2m/(m+k)$ and $p_1 = q_1 = 2$, we obtain the required embedding. Combining all together, we finally arrive at

$$\Big\|\bigoplus_{S\in\mathcal{P}_k(m)} \ell_{2k/(k+1)}(S)[\ell_2(\widehat{S})] \hookrightarrow \ell_{2m/(m+1),2k/(k+1)}\Big\| \le 2\binom{m}{k}^{3/2},$$

which completes the proof. □

A combination of (5) and Lemmas 7 and 11 leads to the following substantial improvement of Theorem 6:

Theorem 12. *For each $m, k \in \mathbb{N}$ with $1 \le k \le m$ we have*

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),2k/(k+1)}}(m) \le 2\binom{m}{k}^{3/2} A^{m-k}_{2k/(k+1)} \, \mathrm{BH}^{\mathrm{mult}}_{\ell_{2k/(k+1)}}(k).$$

In particular, for each k there is some $\delta(k) > 0$ such that, for each $m > k$,

$$\mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),2(m-k)/(m-k+1)}}(m) \le m^{\delta(k)}.$$

5. The polynomial BH inequality for Lorentz spaces

Let us start with a standard polarization argument, showing how the multilinear BH inequality in Lorentz spaces from (4) transfers to a polynomial BH inequality in Lorentz spaces (as already stated in (12)).

Theorem 13. *Given $m \in \mathbb{N}$, there is a constant $C > 0$ such that for every m-homogeneous polynomial $P = \sum_{\boldsymbol{j} \in \mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1} \cdots z_{j_m}$ in n complex variables we have*

$$\|(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{\ell_{2m/(m+1),1}} \le C \|P\|_\infty;$$

in other terms,

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1),1}}(m) < \infty.$$

Proof. Take some m-homogeneous polynomial P as above, and let $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ be the associated symmetric matrix. Then for every $\boldsymbol{j} \in \mathcal{J}(m,n)$ we have

$$c_{\boldsymbol{j}} = \mathrm{card}[\boldsymbol{j}] a_{\boldsymbol{j}}$$

and, by standard polarization,

$$\|a\|_\infty \le \frac{m^m}{m!} \|P\|_\infty.$$

Obviously,

$$\left\| \ell_{p,1}(\mathcal{M}(m,n)) \hookrightarrow \ell_{p,1}(\mathcal{J}(m,n)), (b_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)} \mapsto (b_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)} \right\| \le 1.$$

Combining all this we obtain

$$\begin{aligned} \|(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{2m/(m+1),1} &= \|(\mathrm{card}[\boldsymbol{j}] a_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{\ell_{2m/(m+1),1}} \\ &\le \|(\mathrm{card}[\boldsymbol{i}] a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{\ell_{2m/(m+1),1}} \\ &\le m! \|(a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{\ell_{2m/(m+1),1}} \\ &\le m! \, \mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) \|a\|_\infty \le m^m \, \mathrm{BH}^{\mathrm{mult}}_{\ell_{2m/(m+1),1}}(m) \|P\|_\infty, \end{aligned}$$

which is the estimate we aimed for. □

5.1. ***Hypercontractive growth.*** We now improve the preceding theorem by showing for $X = \ell_{2m/(m+1),1}$ that the constant $\mathrm{BH}^{\mathrm{pol}}_X(m)$ in fact has hypercontractive growth in m; this extends (10) from Minkowski spaces $\ell_{2m/(m+1)}$ to Lorentz spaces $\ell_{2m/(m+1),1}$.

Theorem 14. *For every $\varepsilon > 0$ there is a constant $C(\varepsilon) > 0$ such that, for each m,*

$$\mathrm{BH}^{\mathrm{pol}}_{\ell_{2m/(m+1),1}}(m) \le C(\varepsilon)(\sqrt{2} + \varepsilon)^m.$$

Our proof needs four preliminary lemmas. The understanding of the diagonal operator

$$D(m,n)\colon \mathbb{C}^{\mathcal{M}(m,n),s} \hookrightarrow \mathbb{C}^{\mathcal{J}(m,n)}, \quad (a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)} \mapsto (\mathrm{card}[\boldsymbol{j}]^{(m+1)/(2m)} a_{\boldsymbol{j}})_{\boldsymbol{j}\in\mathcal{J}(m,n)},$$

will turn out to be crucial; here $\mathbb{C}^{\mathcal{M}(m,n),s}$ stands for all symmetric matrices in $\mathbb{C}^{\mathcal{M}(m,n)}$, namely all matrices $(a_{\boldsymbol{i}})_{\boldsymbol{i}\in\mathcal{M}(m,n)}$ for which $a_{\boldsymbol{i}} = a_{\boldsymbol{j}}$ whenever $\boldsymbol{j} \in [\boldsymbol{i}]$. Moreover, for $1 < p < \infty$ denote by $\ell_{p,1}^s(\mathcal{M}(m,n))$ the subspace $\mathbb{C}^{\mathcal{M}(m,n),s}$ of $\ell_{p,1}(\mathcal{M}(m,n))$, and similarly define the subspace $\ell_p^s(\mathcal{M}(m,n))$ for $1 \le p < \infty$.

In Lemma 16 we will use interpolation in order to establish norm estimates for these diagonal operators in Lorentz sequence spaces. In order to do so, we need another technical lemma on real interpolation:

Lemma 15. *Let X_0 and X_1 be fully symmetric spaces on a measure space (Ω, Σ, μ). If X_0^d and X_1^d are discretizations of X_0 and X_1 generated by the same measurable partition of Ω, then for every $\theta \in (0,1)$ and $1 \le q \le \infty$ the inclusion map* $\mathrm{id}\colon (X_0^d, X_1^d)_{\theta,q} \to (X_0, X_1)_{\theta,q}$ *is an isometric isomorphism, i.e.,*

$$\|f\|_{(X_0^d,X_1^d)_{\theta,q}} = \|f\|_{(X_0,X_1)_{\theta,q}} \quad \textit{for } f \in (X_0^d, X_1^d)_{\theta,q}.$$

Proof. Let $\{\Omega_k\}_{k=1}^N \subset \Sigma$ be a given measurable partition of Ω. Define the linear map

$$P\colon L_1(\mu) + L_\infty(\mu) \to L_1(\mu) + L_\infty(\mu), \quad f \mapsto \sum_{k=1}^N \left(\frac{1}{\mu(\Omega_k)} \int_{\Omega_k} f \, d\mu \right) \chi_{\Omega_k}.$$

Since $P\colon (L_1(\mu), L_\infty(\mu)) \to (L_1(\mu), L_\infty(\mu))$ with $\|P\|_{L_1(\mu)\to L_1(\mu)} \le 1$ and $\|P\|_{L_\infty(\mu)\to L_\infty(\mu)} \le 1$, and X_0 and X_1 are fully symmetric, it follows that

$$P\colon (X_0, X_1) \to (X_0^d, X_1^d)$$

with $\|P\|_{X_j \to X_j^d} \le 1$ for $j \in \{0,1\}$. This implies that, for every $f \in X_0^d + X_1^d$, we have, since $P(f) = f$,

$$K(t, f; X_0^d, X_1^d) = K(t, Pf; X_0, X_1) \le K(t, f; X_0, X_1), \quad t > 0.$$

Since the opposite inequality is obvious, the required statement follows. ☐

The next result will be essential:

Lemma 16. *There is a uniform constant $L > 0$ such that, for each m and n,*

$$\left\| D(m,n)\colon \ell_{2m/(m+1),1}^s(\mathcal{M}(m,n)) \hookrightarrow \ell_{2m/(m+1),1}(\mathcal{J}(m,n)) \right\| \le Lm.$$

Proof. The proof is based on interpolation, and the abbreviations $\mathcal{M} = \mathcal{M}(m,n)$ and $\mathcal{J} = \mathcal{J}(m,n)$ will be used. We claim that

$$\|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\| \le 1, \quad \|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\| \le \sqrt{m}. \tag{22}$$

Indeed, for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$ we have

$$\begin{aligned} \|D(m,n)a\|_{\ell_1(\mathcal{J})} &= \sum_{\boldsymbol{j}\in\mathcal{J}} \mathrm{card}[\boldsymbol{j}]^{(m+1)/(2m)} |a_{\boldsymbol{j}}| = \sum_{\boldsymbol{j}\in\mathcal{J}} \mathrm{card}[\boldsymbol{j}]^{(m+1)/(2m)-1} \,\mathrm{card}[\boldsymbol{j}]\, |a_{\boldsymbol{j}}| \\ &\le \sum_{\boldsymbol{j}\in\mathcal{J}} \mathrm{card}[\boldsymbol{j}]\, |a_{\boldsymbol{j}}| = \sum_{\boldsymbol{i}\in\mathcal{M}} |a_{\boldsymbol{i}}| = \|a\|_{\ell_1^s(\mathcal{M})} \end{aligned}$$

and

$$\|D(m,n)a\|_{\ell_2(\mathcal{J})} = \Big(\sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]^{(m+1)/m}|a_{\boldsymbol{j}}|^2\Big)^{1/2} = \Big(\sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]^{(m+1)/m-1}\operatorname{card}[\boldsymbol{j}]|a_{\boldsymbol{j}}|^2\Big)^{1/2}$$
$$= (m!)^{1/(2m)}\Big(\sum_{\boldsymbol{j}\in\mathcal{J}} \operatorname{card}[\boldsymbol{j}]|a_{\boldsymbol{j}}|^2\Big)^{1/2} \le \sqrt{m}\Big(\sum_{\boldsymbol{i}\in\mathcal{M}} |a_{\boldsymbol{i}}|^2\Big)^{1/2} = \sqrt{m}\,\|a\|_{\ell_2^s(\mathcal{M})},$$

which proves (22). We now apply the two-sided norm estimate from (14). In the special case when $p_0 = q_0 = 1$, $p_1 = q_1 = 2$, $q = 1$ and $\theta = (m-1)/m$, we have $p = 2m/(m+1)$ and, in particular, $1 \le (p/q)^{1/q} = 2m/(m+1) < 2$. Then, for $I = \mathcal{M}(m,n)$ or $I = \mathcal{J}(m,n)$,

$$(\ell_1(I), \ell_2(I))_{(m-1)/m,1} = \ell_{2m/(m+1),1}(I),$$

and there is $C > 0$ such that, for all $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\frac{m^{3/2}}{C(m-1)}\|a\|_{\ell_{2m/(m+1),1}(I)} \le \|a\|_{(\ell_1(I),\ell_2(I))_{(m-1)/m,1}} \le \frac{Cm^2}{m-1}\|a\|_{\ell_{2m/(m+1),1}(I)}. \tag{23}$$

It follows from Lemma 15 that

$$\|a\|_{(\ell_1^s(\mathcal{M}),\ell_2^s(\mathcal{M}))_{(m-1)/m,1}} = \|a\|_{(\ell_1(\mathcal{M}),\ell_2(\mathcal{M}))_{(m-1)/m,1}} \quad \text{for } a \in \mathbb{C}^{\mathcal{M}(m,n),s}. \tag{24}$$

Now we interpolate; we recall that, for every operator T between interpolation pairs (A_0, A_1) and (B_0, B_1) and every $0 < \theta < 1$, we have

$$\|T\colon (A_0, A_1)_{\theta,1} \to (B_0, B_1)_{\theta,1}\| \le \|T\colon A_0 \to B_0\|^{1-\theta}\|T\colon A_1 \to B_1\|^{\theta}.$$

In particular,

$$\big\|D(m,n)\colon (\ell_1^s(\mathcal{M}), \ell_2^s(\mathcal{M}))_{(m-1)/m,1} \to (\ell_1(\mathcal{J}), \ell_2(\mathcal{J}))_{(m-1)/m,1}\big\|$$
$$\le \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}.$$

As a consequence we obtain that, for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\begin{aligned}
&\frac{m^{3/2}}{C(m-1)}\|D(m,n)a\|_{\ell_{2m/(m+1),1}(\mathcal{M})}\\
&\quad\overset{(23)}{\le} \|D(m,n)a\|_{(\ell_1(\mathcal{J}),\ell_2(\mathcal{J}))_{(m-1)/m,1}}\\
&\quad\le \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}\|a\|_{(\ell_1^s(\mathcal{M}),\ell_2^s(\mathcal{M}))_{(m-1)/m,1}}\\
&\quad\overset{(24)}{=} \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}\|a\|_{(\ell_1(\mathcal{M}),\ell_2(\mathcal{M}))_{(m-1)/m,1}}\\
&\quad\overset{(23)}{\le} \|D(m,n)\colon \ell_1^s(\mathcal{M}) \to \ell_1(\mathcal{J})\|^{1/m}\|D(m,n)\colon \ell_2^s(\mathcal{M}) \to \ell_2(\mathcal{J})\|^{(m-1)/m}\frac{Cm^2}{m-1}\|a\|_{\ell_{2m/(m+1),1}(\mathcal{J})}.
\end{aligned}$$

Combining the above estimates with (22), we conclude that, for every $a \in \mathbb{C}^{\mathcal{M}(m,n),s}$,

$$\|D(m,n)a\|_{\ell_{2m/(m+1),1}(\mathcal{M})} \le C^2\sqrt{m}\sqrt{m}^{(m-1)/m}\|a\|_{\ell_{2m/(m+1),1}(\mathcal{J})} \le C^2 m\|a\|_{\ell_{2m/(m+1),1}(\mathcal{J})},$$

and this completes the proof. □

In what follows we will need the Khinchine–Steinhaus inequality for homogeneous polynomials due to [Bayart 2002]: given $0 < p < q < \infty$, for every m-homogeneous polynomial P on $\mathbb{C}^n$ we have

$$\left(\int_{\mathbb{T}^n} |P(z)|^q \, dz\right)^{1/q} \le \sqrt{\frac{q}{p}}^{\,m} \left(\int_{\mathbb{T}^n} |P(z)|^p \, dz\right)^{1/p}; \tag{25}$$

note that it is shown in [Defant and Mastyło 2015, Theorem 2.1] that the constant $\sqrt{q/p}$ that appears is optimal. For the proof of Theorem 14, this fact will only be used for the case $p = 1$ and $q = 2$.

Next, we also require a lemma — which is (implicitly) in [Bayart et al. 2014b] and (explicitly) in [Defant et al. 2016, Section 9] — however only in the case $k = 1$.

Lemma 17. *Let $P = \sum_{\boldsymbol{j} \in \mathcal{J}(m,n)} c_{\boldsymbol{j}} z_{j_1} \cdots z_{j_m}$ be an m-homogeneous polynomial in n variables and let $a = (a_{\mathbf{i}})_{\mathbf{i} \in \mathcal{M}(m,n)}$ be its associated symmetric matrix. Then for every $S \in \mathcal{P}_k(m)$, $1 \le k \le m$, we have*

$$\left(\sum_{\boldsymbol{i} \in \mathcal{M}(S,n)} \left(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{S},n)} \operatorname{card}[\boldsymbol{j}]\, |a_{\boldsymbol{i} \oplus \boldsymbol{j}}|^2\right)^{\frac{1}{2}\frac{2k}{k+1}}\right)^{\frac{k+1}{2k}} \le \sqrt{\frac{k+1}{k}}^{\,m-k} \frac{(m-k)!\, m^m}{(m-k)^{m-k} m!} B^{\mathrm{mult}}_{\ell_{2k/(k+1)}}(k) \|P\|_\infty.$$

The fourth lemma is an immediate consequence of [Blei and Fournier 1989, Theorem 3.3]; here we will use only the case $q = 2$.

Lemma 18. *Given $1 \le q < \infty$, there is a constant $C_q \ge 1$ such that, for every matrix $a = (a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$,*

$$\|a\|_{\ell_{mq/(m+q-1),1}} \le C_q m \|a\|_{(m,n,1,1,q)}.$$

We are now ready to give the proof of Theorem 14.

Proof of Theorem 14. Assume that P is an m-homogeneous polynomial on $\mathbb{C}^n$ with coefficients $(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}$ and denote by $(a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}$ the coefficients of the associated symmetric m-linear form A. We have the simple fact that, for all $\boldsymbol{i} \in \mathcal{M}(\{1\}, n)$ and $\boldsymbol{j} \in \mathcal{M}(\widehat{\{1\}}, n)$,

$$\operatorname{card}[\boldsymbol{i} \oplus \boldsymbol{j}] \le m \operatorname{card}[\boldsymbol{j}].$$

Hence we deduce from Lemmas 16, 18 (with $q = 2$) and 17 (with $k = 1$) that, for each m and n,

$$\begin{aligned}
&\|(c_{\boldsymbol{j}})_{\boldsymbol{j} \in \mathcal{J}(m,n)}\|_{2m/(m+1),1} \\
&\qquad = \|(\operatorname{card}[\boldsymbol{i}]\, a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{J}(m,n)}\|_{2m/(m+1),1} \\
&\qquad \le Lm \|(\operatorname{card}[\boldsymbol{i}]^{1-(m+1)/2m} a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{2m/(m+1),1} \\
&\qquad \le LmC_2 m \|(\operatorname{card}[\boldsymbol{i}]^{1-(m+1)/(2m)} a_{\boldsymbol{i}})_{\boldsymbol{i} \in \mathcal{M}(m,n)}\|_{(m,n,1,1,2)} \\
&\qquad = LmC_2 m \max_{S \in \mathcal{P}_1(m)} \sum_{\boldsymbol{i} \in \mathcal{M}(\{1\},n)} \left(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{\{1\}},n)} \left|\operatorname{card}[\boldsymbol{i} \oplus \boldsymbol{j}]^{(m-1)/(2m)} a_{\boldsymbol{i} \oplus \boldsymbol{j}}\right|^2\right)^{1/2} \\
&\qquad \le LmC_2 m \max_{S \in \mathcal{P}_1(m)} \sum_{\boldsymbol{i} \in \mathcal{M}(\{1\},n)} \left(\sum_{\boldsymbol{j} \in \mathcal{M}(\widehat{\{1\}},n)} \left|(m \operatorname{card}[\boldsymbol{j}])^{(m-1)/(2m)} a_{\boldsymbol{i} \oplus \boldsymbol{j}}\right|^2\right)^{1/2}
\end{aligned}$$

$$\le LmC_2 m\, m^{(m-1)/(2m)} \max_{S\in\mathcal{P}_1(m)} \sum_{\boldsymbol{i}\in\mathcal{M}(\{1\},n)} \Bigg(\sum_{\boldsymbol{j}\in\mathcal{M}(\{\widehat{1}\},n)} \operatorname{card}[\boldsymbol{j}]^{(m-1)/m} |a_{\boldsymbol{j}}|^2 \Bigg)^{1/2}$$

$$\le LmC_2 m\, m^{(m-1)/(2m)} \max_{S\in\mathcal{P}_1(m)} \sum_{\boldsymbol{i}\in\mathcal{M}(\{1\},n)} \Bigg(\sum_{\boldsymbol{j}\in\mathcal{M}(\{\widehat{1}\},n)} \operatorname{card}[\boldsymbol{j}]\, |a_{\boldsymbol{j}}|^2 \Bigg)^{1/2}$$

$$\le LmC_2 m\, m^{(m-1)/(2m)} \sqrt{2}^{m-1} \times \frac{(m-1)!\,m^m}{(m-1)^{m-1} m!} \times B_{\ell_1}^{\mathrm{mult}}(1) \times \|P\|_\infty .$$

This completes the argument. □

We conclude with the following remark: The estimate (11) suggests that the constant $\sqrt{2}$ in Theorem 14 could be improved. Here $\sqrt{2}$ appears since our proof applies (25) for $p = 1$ and $q = 2$, which is an inequality on homogeneous polynomials of arbitrary degree m. We have already indicated that the constant $\sqrt{2}$ in the inequality (25) is optimal (note that, in contrast to this, the best constant in (25) for polynomials of degree only $m = 1$ equals $\sqrt{\pi}/2$; see [Sawa 1985; König 2014]).

5.2. ***The Balasubramanian–Calado–Queffélec result revisited.*** In this section we improve a remarkable result by Balasubramanian, Calado and Queffélec [Balasubramanian et al. 2006]. By $\mathcal{P}({}^m c_0)$ we denote the linear space of all m-homogeneous continuous polynomials on c_0, which, together with the supremum norm on the open unit ball in c_0, forms a Banach space. On the subspace c_{00} of all finite sequences in c_0, each such polynomial has a unique monomial series decomposition $P(z) = \sum_{|\alpha|=m} c_\alpha(P) z^\alpha$, $z \in c_{00}$, (or, in different notation, $P(z) = \sum_{\boldsymbol{j}\in\mathcal{J}(m)} c_{\boldsymbol{j}} z_{\boldsymbol{j}}$, $z \in c_{00}$). A Dirichlet series $D = \sum_n a_n n^{-s}$ is said to be m-homogeneous whenever $a_n \neq 0$ implies $n = \mathfrak{p}^\alpha$ and $|\alpha| = m$ (where $\mathfrak{p}$ is the sequence of primes). All m-homogeneous Dirichlet series $D = \sum_n a_n n^{-s}$ which converge on $\{s : \operatorname{Re} s > 0\}$ and are such that the holomorphic function $D(s) = \sum_{n=1}^\infty a_n n^{-s}$ for $\operatorname{Re} s > 0$ is bounded form (together with the supremum norm on $\{s : \operatorname{Re} s > 0\}$) the Banach space $\mathcal{H}_\infty^m$.

It is remarkable that there is a unique isometric isomorphism

$$\mathfrak{B}\colon \mathcal{P}({}^m c_0) \to \mathcal{H}_\infty^m, \quad P = \sum_{|\alpha|=m} c_\alpha(P) z^\alpha \mapsto D = \sum_n a_n n^{-s},$$

such that $c_\alpha = a_n$ whenever $n = \mathfrak{p}^\alpha$. (For more information see [Defant et al. 2016; Defant and Sevilla-Peris 2014; Queffélec and Queffélec 2013].) Then the following theorem is an immediate consequence of this identification and Theorem 14:

Theorem 19. *For every Dirichlet series $D = \sum_n a_n n^{-s} \in \mathcal{H}_\infty^m$ we have $(a_n^*) \in \ell_{2m/(m+1),1}$. More precisely, for every $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that, for every $D \in \mathcal{H}_\infty^m$,*

$$\sum_{n=1}^\infty a_n^* \frac{1}{n^{(m-1)/(2m)}} \le C(\varepsilon)(\sqrt{2}+\varepsilon)^m \|D\|_\infty. \tag{26}$$

At the end of the previous section we discuss in some detail why our proof of Theorem 14 and then also (26) leads to the constant $\sqrt{2}$.

Note that for every sequence $a = (a_n) \in \ell_{2m/(m+1),1}$ we have

$$\sum_{n=1}^{\infty} |a_n| \frac{1}{n^{(m-1)/(2m)}} \leq \sum_{n=1}^{\infty} a_n^* \frac{1}{n^{(m-1)/(2m)}} \asymp \|a\|_{\ell_{2m/(m+1),1}} < \infty.$$

Balasubramanian et al. [2006] proved that there is a constant $c(m) > 0$ such that, for every Dirichlet series $D = \sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}_{\infty}^m$,

$$\sum_{n=1}^{\infty} |a_n| \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} \leq c(m) \|D\|_{\infty}, \tag{27}$$

and in addition it is shown that the exponent in the log term is optimal. In contrast to (26), it is unknown whether the best constant in (27) has exponential growth.

A natural question appears: how is this result related to the estimate from Theorem 19? To see this, let $\ell_1(\omega)$ be the weighted ℓ_1-space with weight $\omega = (\omega_n)$ given by

$$\omega_n = \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}}, \quad n \in \mathbb{N}. \tag{28}$$

We observe that $\ell_1(\omega)$ is different from $\ell_{2m/(m+1),1}$; in fact, if we would have $\ell_1(\omega) \subset \ell_{2m/(m+1),1}$, or equivalently $\ell_1 \subset \ell_{2m/(m+1),1}(\omega^{-1})$, then by the closed graph theorem

$$\sup_{n \in \mathbb{N}} \|e_n\|_{\ell_{2m/(m+1),1}(\omega^{-1})} < \infty.$$

But since, for each $n \in \mathbb{N}$,

$$\|e_n\|_{\ell_{2m/(m+1),1}(\omega^{-1})} = \left\| \frac{e_n}{\omega_n} \right\|_{\ell_{2m/(m+1),1}} = \frac{n^{(m-1)/(2m)}}{(\log n)^{(m-1)/m}},$$

we get a contradiction. Similarly, if $\ell_{2m/(m+1),1} \subset \ell_1(\omega)$ then there would exist a constant $C > 0$ such that, for each $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} \frac{(\log n)^{(m-1)/2}}{n^{(m-1)/(2m)}} = \left\| \sum_{n=1}^{N} e_n \right\|_{\ell_1(\omega)} \leq C \left\| \sum_{n=1}^{N} e_n \right\|_{\ell_{2m/(m+1),1}} = C N^{(m+1)/(2m)},$$

which is again impossible. We conclude the paper with the following formal improvement of Theorem 19 and the Balasubramanian–Calado–Queffélec result (27):

Corollary 20. *For each $m \in \mathbb{N}$ and every Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s} \in \mathcal{H}_{\infty}^m$,*

$$(a_n)_n \in \ell_1(\omega) \cap \ell_{2m/(m+1),1},$$

where the weight ω is given by the formula (28).

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Received 14 Jan 2016. Revised 12 Feb 2016. Accepted 30 Mar 2016.

ANDREAS DEFANT: andreas.defant@uni-oldenburg.de
Institut für Mathematik, Carl von Ossietzky Universität, Postfach 2503, D-26111 Oldenburg, Germany

MIECZYSŁAW MASTYŁO: mastylo@amu.edu.pl
Faculty of Mathematics and Computer Science, Adam Mickiewicz University in Poznań, Umultowska 87, 61-614 Poznań, Poland

ANALYSIS AND PDE
Vol. 9, No. 5, 2016
dx.doi.org/10.2140/apde.2016.9.1259

ON THE NEGATIVE SPECTRUM OF THE ROBIN LAPLACIAN IN CORNER DOMAINS

VINCENT BRUNEAU AND NICOLAS POPOFF

Dedicated to Monique Dauge on the occasion of her 60th birthday

For a bounded corner domain Ω, we consider the attractive Robin Laplacian in Ω with large Robin parameter. Exploiting multiscale analysis and a recursive procedure, we have a precise description of the mechanism giving the bottom of the spectrum. It allows also the study of the bottom of the essential spectrum on the associated tangent structures given by cones. Then we obtain the asymptotic behavior of the principal eigenvalue for this singular limit in any dimension, with remainder estimates. The same method works for the Schrödinger operator in $\mathbb{R}^n$ with a strong attractive δ-interaction supported on $\partial\Omega$. Applications to some Ehrling-type estimates and the analysis of the critical temperature of some superconductors are also provided.

1. Introduction

1A. ***Context: Robin Laplacian with large parameter.*** Let M be a Riemannian manifold of dimension n without boundary and Ω an open domain of M (in practice one may think $M=\mathbb{R}^n$ or $M=\mathbb{S}^n$). We are interested in the eigenvalue problem

$$\begin{cases} -\Delta u=\lambda u & \text{on } \Omega, \\ \partial_\nu u-\alpha u=0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

Here $\alpha\in\mathbb{R}$ is the Robin parameter and ∂_ν denotes the outward normal to the boundary of Ω. We assume that Ω belongs to a general class of corner domains defined recursively, such as in [Dauge 1988]. This class of corner domains of M, precisely defined in Section 2, consists of open bounded sets $\Omega\subset M$ such that each point in $\partial\Omega$ can be associated with a *tangent cone*. We ask the sections of these tangent cones

MSC2010: 35J10, 35P15, 47F05, 81Q10.
Keywords: Robin Laplacian, eigenvalues estimates, corner domains.

to satisfy the same property, that is, as open sets of $\mathbb{S}^{n-1}$ to themselves be corner domains. The corner domains of $\mathbb{S}^0$ being its nonempty subsets, this leads to a natural recursive definition of corner domains; see [Dauge 1988; Bonnaillie-Noël et al. 2016a, Section 3] for a more complete description and examples. Note that these domains include various possible geometries, like regular domains, polyhedra and circular cones.

We denote by $\mathcal{Q}_\alpha[\Omega]$ the quadratic form of the Robin Laplacian on Ω with parameter α:

$$\mathcal{Q}_\alpha[\Omega](u) := \|\nabla u\|^2_{L^2(\Omega)} - \alpha \|u\|^2_{L^2(\partial\Omega)}, \quad u \in H^1(\Omega). \tag{2}$$

Since Ω is bounded and is the finite union of Lipschitz domains (see [Dauge 1988, Lemma AA.9]), the trace injection from $H^1(\Omega)$ into $L^2(\partial\Omega)$ is compact and the quadratic form $\mathcal{Q}_\alpha[\Omega]$ is lower semibounded. We define $L_\alpha[\Omega]$, its self-adjoint extension, whose spectrum is a sequence of eigenvalues, and we denote by $\lambda(\Omega, \alpha)$ the first one. It is the principal eigenvalue of the system (1).

The study of the spectrum of $L_\alpha[\Omega]$ has received some attention in the past years, in particular for the singular limit $\alpha \to +\infty$. This problem appeared first in a model of reaction diffusion for which the absorption mechanism competes with a boundary term [Lacey et al. 1998], and more recently it was established that the understanding of $\lambda(\Omega, \alpha)$ provides information on the critical temperature of surface superconductivity under zero magnetic field [Giorgi and Smits 2007]. Let us mention that such models are also used in the quantum Hall effect and topological insulators to justify the appearance of edge states (see [Asorey et al. 2015]).

In view of the quadratic form, it is not difficult to see that $\lambda(\Omega, \alpha) \to -\infty$ as $\alpha \to +\infty$ (while in the limit $\alpha \to -\infty$ they converge to those of the Dirichlet Laplacian). When $\Omega \subset \mathbb{R}^n$ is smooth, $\lambda(\Omega, \alpha) \le -\alpha^2$ for all $\alpha \ge 0$; see [Giorgi and Smits 2007, Theorem 2.1]. More precisely, it is known that $\lambda(\Omega, \alpha) \sim C_\Omega \alpha^2$ as $\alpha \to +\infty$ for some particular domains: for smooth domains, $C_\Omega = -1$ (see [Lacey et al. 1998; Lou and Zhu 2004] and [Daners and Kennedy 2010] for higher eigenvalues), and, for planar polygonal domains with corners of opening $(\theta_k)_{k=1,\dots,N}$,

$$C_\Omega = -\max_{0<\theta_k<\pi} \left(1, \sin^{-2} \tfrac{1}{2}\theta_k\right).$$

This last formula, conjectured in [Lacey et al. 1998], is proved in [Levitin and Parnovski 2008]. For general domains Ω having a piecewise smooth boundary it is natural to study the operator on tangent spaces and, from homogeneity reasons (see Lemma 3.2), one expects that $\lambda(\Omega, \alpha) \sim C_\Omega \alpha^2$ when $\alpha \to +\infty$, with some negative constant C_Ω. Levitin and Parnovski [2008] consider domains with corners satisfying the uniform interior cone condition. For each $x \in \partial\Omega$, they introduce $E(\Pi_x)$, the bottom of the spectrum of the Robin Laplacian on an infinite model cone Π_x (if x is a regular point, it is a half-space) and show

$$\lim_{\alpha \to +\infty} \frac{\lambda(\Omega, \alpha)}{\alpha^2} = \inf_{x \in \partial\Omega} E(\Pi_x). \tag{3}$$

But we have no guarantee concerning the finiteness of $E(\Pi_x)$ and, moreover, even if it is finite, we don't know if their infimum over $\partial\Omega$ is reached. Then an important question is to understand more precisely the influence of the geometry (of the boundary) of Ω in the asymptotic behavior of $\lambda(\Omega, \alpha)$ in order to give meaning to (3) (in particular proving that $\inf_{x \in \partial\Omega} E(\Pi_x)$ is finite) and, if possible, to obtain some remainder estimates.

1B. ***Local energies on admissible corner domains.*** In this article, our purpose is to develop a framework in the study of such asymptotics by introducing the *local energy* function $x \mapsto E(\Pi_x)$ on the recursive class of corner domains (see [Dauge 1988]). The natural tangent structures are given by dilation-invariant domains, more succinctly referred as *cones*. When the domain is a convenient cone Π, the quadratic form in (2) may still be defined on $H^1(\Pi)$. By immediate scaling, $\mathcal{Q}_\alpha[\Pi]$ is unitarily equivalent to $\alpha^2\mathcal{Q}_1[\Pi]$. Therefore the case where the parameter is equal to 1 plays an important role and we write $\mathcal{Q}[\Pi] = \mathcal{Q}_1[\Pi]$. For a general cone, we don't know whether $\mathcal{Q}[\Pi]$ is lower semibounded, and we define

$$E(\Pi) = \inf_{\substack{u \in H^1(\Pi) \\ u \neq 0}} \frac{\mathcal{Q}[\Pi](u)}{\|u\|^2},$$

the ground state energy of the Robin Laplacian on Π. For $x \in \overline{\Omega}$, denote by Π_x the tangent cone at x. When Π_x is the full space (corresponding to interior points), there is no boundary and $E(\Pi_x) = 0$, whereas, when Π_x is a half-space (corresponding to regular points of the boundary), it is easy to see that $E(\Pi_x) = E(\mathbb{R}_+) = -1$ (see [Daners and Kennedy 2010]). Moreover, when Π_x is an infinite planar sector S_θ of opening θ, $E(\Pi_x)$ is given by

$$E(S_\theta) = \begin{cases} -\sin^{-2}\frac{1}{2}\theta & \text{if } \theta \in (0, \pi), \\ -1 & \text{if } \theta \in (\pi, 2\pi); \end{cases} \tag{4}$$

see [Lacey et al. 1998; Levitin and Parnovski 2008]. No such explicit expressions are available for general cones in higher dimensions. In view of (3), we introduce the infimum of local energy $E(\Pi_x)$ for $x \in \overline{\Omega}$, which, from the above remarks, is also the infimum on the boundary:

$$\mathscr{E}(\Omega) := \inf_{x \in \partial\Omega} E(\Pi_x). \tag{5}$$

Our goal is to prove the finiteness of $\mathscr{E}(\Omega)$ (and firstly of $E(\Pi_x)$ for $x \in \overline{\Omega}$) for admissible corner domains and to give an estimate of $\lambda(\Omega, \alpha) - \alpha^2\mathscr{E}(\Omega)$ for α large. In view of the above particular cases, the local energy is clearly discontinuous (even for smooth domains it is piecewise constant with values in $\{0, -1\}$). We will use a recursive procedure in order to prove the finiteness and the lower semicontinuity of the local energy in the general case. It relies also on a multiscale analysis to get an estimate of the first eigenvalue, as developed in [Bonnaillie-Noël et al. 2016a] for the semiclassical magnetic Laplacian. Unlike [Bonnaillie-Noël et al. 2016a], where the complexity of model problems limits the study to dimension 3, for the Robin Laplacian we have a good understanding of the ground state energy on corner domains in any dimension. Moreover these techniques allow an analog spectral study of the Schrödinger operator with δ-interaction supported on closed corner hypersurfaces and on conical surfaces.

1C. ***Results for the Robin Laplacian.*** We define below generic notions associated with cones.

Definition 1.1. A cone Π is a domain of $\mathbb{R}^n$ which is dilation invariant:

$$\rho x \in \Pi \quad \text{for all } x \in \Pi,\ \rho > 0.$$

The section of a cone Π is $\Pi \cap \mathbb{S}^{n-1}$, generically denoted by ω. We say that two cones Π_1 and Π_2 are equivalent, and we write $\Pi_1 \equiv \Pi_2$, if they can be deduced one from another by a rotation. Given a cone Π, there exists $d \in \mathbb{N}$ with $0 \le d \le n$ such that

$$\Pi \equiv \mathbb{R}^{n-d} \times \Gamma, \quad \text{with } \Gamma \text{ a cone in } \mathbb{R}^d.$$

When d is minimal for such an equivalence, we say that Γ is the reduced cone of Π. When $d = n$, so that $\Pi = \Gamma$, we say that Π is irreducible.

In the following, $\mathfrak{P}_n$ denotes the class of admissible cones of $\mathbb{R}^n$ and $\mathfrak{D}(M)$ denotes the class of admissible corner domains on a given Riemannian manifold M without boundary. We refer to Section 2 for precise definitions of these classes of domains.

Theorem 1.2. *Let $\Pi \in \mathfrak{P}_n$ be an admissible cone.*

(1) *$E(\Pi) > -\infty$ and the Robin Laplacian $L[\Pi]$ is well defined as the Friedrichs extension of $Q[\Pi]$ with form domain $D(Q[\Pi]) = H^1(\Pi)$.*

(2) *Let Γ be the reduced cone of Π. Then the bottom of the essential spectrum of $L[\Gamma]$ is $\mathscr{E}(\omega)$, where ω is the section of Γ.*

This theorem generalizes to cones having no regular section the result of [Pankrashkin 2016], where the bottom of the essential spectrum is proved to be -1 for cones with regular section (as discussed at the end of Section 1A, in this case $\mathscr{E}(\omega) = -1$).

The crucial point of this theorem is to show that the Robin Laplacian on a cone, far from the origin, can be linked to the Robin Laplacian on the section of the cone, with a parameter related to the distance to the origin.

Notice that this theorem provides an effective procedure to compute the bottom of the essential spectrum for Laplacians on cones. In particular, as shown by Remark 6.4, we obtain that [Levitin and Parnovski 2008, Theorem 3.5] is incorrect in dimension $n \ge 3$; indeed, we construct a cone which contains an hyperplane passing through the origin for which the bottom of the essential spectrum (then of the spectrum) of the Robin Laplacian is below -1.

The next step is to minimize the local energy on a corner domain Ω and to prove that $\mathscr{E}(\Omega)$ is finite. Thanks to Theorem 1.2, we will be able to prove some monotonicity properties (on singular chains; see Section 2B for the definition), which, combined with continuity of the local energy (for the topology of singular chains), allow us to apply [Bonnaillie-Noël et al. 2016a, Section 3] and to obtain:

Theorem 1.3. *For any corner domain $\Omega \in \mathfrak{D}(M)$, the energy function $x \mapsto E(\Pi_x)$ is lower semicontinuous on $\overline{\Omega}$ and we have $\mathscr{E}(\Omega) > -\infty$.*

To get asymptotics of $\lambda(\Omega, \alpha)$ with control of the remainders, we need to control error terms when using change of variables and cut-off functions. However, the principal curvatures of the regular part of a corner domain may be unbounded in dimension $n \ge 3$ (think of a circular cone), so the standard estimates when using approximation of metrics may blow up. We use a multiscale analysis to overcome this difficulty and we get the following result:

Theorem 1.4. *Let $\Omega \in \mathfrak{D}(M)$ with $n \geq 2$ the dimension of M. Then there exists $\alpha_0 \in \mathbb{R}$, two constants $C^{\pm} > 0$ and two integers $0 \leq \overline{\nu} \leq \overline{\nu}_+ \leq n-2$ such that*

$$-C^{-}\alpha^{2-2/(2\overline{\nu}_++3)} \leq \lambda(\Omega, \alpha) - \alpha^2 \mathscr{E}(\Omega) \leq C^{+}\alpha^{2-2/(2\overline{\nu}+3)} \quad \textit{for all } \alpha \geq \alpha_0.$$

The constant $\overline{\nu}$ corresponds to the degree of degeneracy of the curvatures near the minimizers of the local energy; its precise definition can be found in (29). The constant $\overline{\nu}_+$ describes the degeneracy of the curvatures globally in $\overline{\Omega}$; see Lemma 4.1. In particular, when Ω is polyhedral (that is, a domain with bounded curvatures on the regular part), $\overline{\nu} = \overline{\nu}_+ = 0$.

The proof of the lower bound relies on a multiscale partition of the unity where the size of the balls optimizes the error terms. The upper bound is less classical: using the concept of *singular chain*, we isolate a tangent "subreduced cone" for which the bottom of the spectrum corresponds to an isolated eigenvalue (below the essential spectrum). Then we construct recursive quasimodes, coming from this tangent "subreduced cone".

Note finally that for regular domains more precise asymptotics involving the mean curvature can be found ([Pankrashkin 2013; Helffer and Kachmar 2014] in dimension 2 and [Pankrashkin and Popoff 2015; 2016] for higher dimensions). A precise analysis is also done for particular polygonal geometries: the tunneling effect in some symmetry cases [Helffer and Pankrashkin 2015], and reduction to the boundary when the domain is the exterior of a convex polygon [Pankrashkin 2015]. In all these cases, the local energy is piecewise constant, and new geometric criteria appear near the set of minimizers. In fact, the local energy can be seen as a potential in the standard theory of the harmonic approximation [Dimassi and Sjöstrand 1999] and, under additional hypotheses on the local energy, it is reasonable to expect more precise asymptotics in higher dimensions. For polygons (dimension 2), another approach would consist in comparing the limit problem to a problem on a graph, in the spirit of [Grieser 2008], the nodes (resp. edges) corresponding to the vertices (resp. sides) of the polygons. But it is not clear how such an approach could be generalized to any dimension.

1D. ***Applications of the method for the Schrödinger operator with δ-interaction.*** Let $\Omega \in \mathfrak{D}(M)$ be a corner domain and let $S = \partial\Omega$ be its boundary. We consider $L_\alpha^\delta[M, S]$, the self-adjoint extension associated with the quadratic form

$$\mathcal{Q}_\alpha^\delta[M, S](u) := \|\nabla u\|_{L^2(M)}^2 - \alpha \|u\|_{L^2(S)}^2, \quad u \in H^1(M).$$

The associated boundary problem is the Laplacian with the derivative jump condition across the closed hypersurface S: $[\partial_\nu u]_{\partial\Omega} = \alpha u$. It is well known (see, e.g., [Brasche et al. 1994]) that, since S is bounded, $L_\alpha^\delta[\mathbb{R}^n, S]$ is a relatively compact perturbation of $L_0 = -\Delta$ on $L^2(\mathbb{R}^n)$, and then

$$\sigma_{\text{ess}}(L_\alpha^\delta[\mathbb{R}^n, S]) = \sigma_{\text{ess}}(L_0) = [0, +\infty).$$

Moreover, $L_\alpha^\delta[\mathbb{R}^n, S]$ has a finite number of negative eigenvalues. If we denote by $\lambda^\delta(S, \alpha)$ the lowest one, by applying our techniques developed for the Robin Laplacian all the above results are still valid, replacing $\lambda(\Omega, \alpha)$ by $\lambda^\delta(S, \alpha)$. In particular, for $x \in S$, the tangent cone to Ω at x is Π_x and its boundary is denoted by S_x. We still define the tangent operator as $L_1^\delta[\mathbb{R}^n, S_x]$, and the associated local energy $E^\delta(S_x)$ at x, and their infimum $\mathscr{E}^\delta(S)$. Then:

Theorem 1.5. *Theorems 1.2–1.4 remain valid when replacing the Robin Laplacian $L_\alpha[\Omega]$ by the δ-interaction Laplacians $L_\alpha^\delta[M, S]$, $\lambda(\Omega, \alpha)$ by $\lambda^\delta(S, \alpha)$, $E(\Pi_x)$ by $E^\delta(S_x)$ and $\mathscr{E}(\Omega)$ by $\mathscr{E}^\delta(S)$.*

When x belongs to the regular part of S, S_x is an hyperplane and

$$E^\delta(\mathbb{R}^n, S_x) = E^\delta(\mathbb{R}, \{0\}) = -\tfrac{1}{4}; \tag{6}$$

see [Exner and Yoshitomi 2002]. Therefore $\mathscr{E}^\delta(S) = -\frac{1}{4}$ when S is regular, and we obtain the known main term of the asymptotic expansion of $\lambda^\delta(S, \alpha)$ proved in dimension 2 or 3 (see [Exner and Yoshitomi 2002; Exner and Pankrashkin 2014; Dittrich et al. 2016]).

To our best knowledge the only studies for δ-interactions supported on nonsmooth hypersurfaces are for broken lines and conical domains with circular section (see [Behrndt et al. 2014; Duchêne and Raymond 2014; Exner and Kondej 2015; Lotoreichik and Ourmières-Bonafos 2015]). In that case, we clearly have $\sigma(L_\alpha^\delta[\mathbb{R}^n, S]) = \alpha^2\sigma(L_1^\delta[\mathbb{R}^n, S])$ (see Lemma 3.2), and it is proved in the above references that the bottom of the essential spectrum of $L^\delta[\mathbb{R}^n, S]$ is $-\frac{1}{4}$. In view of our result, it remains true when the section of the conical surface is smooth. Moreover, our work seems to be the first result giving the main asymptotic behavior of $\lambda^\delta(S, \alpha)$ for interactions supported by general closed hypersurfaces with corners.

Remark 1.6. For the Robin Laplacian and the δ-interaction Laplacian, we can add a smooth positive weight function G in the boundary conditions. These conditions become, for the Robin condition, $\partial_\nu u = \alpha G(x)u$, and, for the δ-interaction case, $[\partial_\nu u] = \alpha G(x)u$. In our analysis, for $x \in \partial\Omega$ fixed, we have only to change α into $\alpha G(x)$ and, clearly, the results are still true by replacing $\mathscr{E}(\Omega)$ and $\mathscr{E}^\delta(S)$ by

$$\mathscr{E}_G(\Omega) := \inf_{x\in\partial\Omega} G(x)^2 E(\Pi_x), \quad \mathscr{E}_G^\delta(S) := \inf_{x\in S} G(x)^2 E^\delta(S_x).$$

For the Robin Laplacian, these cases were already considered in [Levitin and Parnovski 2008; Colorado and García-Melián 2011].

1E. ***Organization of the article.*** In Section 2, we recall the definitions of corner domains, in the spirit of [Dauge 1988; Maz′ja and Plamenevskiĭ 1977], and we give some properties proved in [Bonnaillie-Noël et al. 2016a]. Section 3 is devoted to the effects of perturbations on the quadratic form of the Robin Laplacian. It contains several technical lemmas used in the following sections.

Section 4 contains the proof of the lower bound of Theorem 1.4. It is based on a multiscale analysis in order to counterbalance the possible blow-up of curvatures in corner domains. In particular it involves the lower bound $\liminf_{\alpha\to+\infty} \lambda(\Omega, \alpha)/\alpha^2 \geq \mathscr{E}(\Omega)$ in any dimension, which is also used in Sections 5 and 6. Notice that in Section 4, at this stage of the analysis, the quantity $\mathscr{E}(\Omega)$ is still not known to be finite; its finiteness will be the recursive hypothesis of the next two sections.

Section 5 is a step in a recursive proof of Theorem 1.3 developed in Section 6. Then, when the finiteness of $\mathscr{E}(\Omega)$ is stated, Theorem 1.2 is a direct consequence of Lemmas 5.2 and 5.3 (see the end of Section 6A).

In Section 7, we prove the upper bound of Theorem 1.4. This is done by exploiting the results of Section 6 in order to find a tangent problem that admits an eigenfunction associated with $\mathscr{E}(\Omega)$. Then we construct recursive quasimodes, qualified either as *sitting* or *sliding*, from the language of [Bonnaillie-Noël et al. 2016a].

In Section 8 we give two possible applications of our results. A purely mathematical one concerns optimal estimates in compact injections of Sobolev spaces. In the second one we recall how, from the study of $\lambda(\Omega, \alpha)$, we derive properties on the critical temperature for zero fields for systems with enhanced surface superconductivity (where α^{-1} is related to the penetration depth).

2. Corner domains

Here we give some background of so-called admissible corner domains; see [Dauge 1988; Bonnaillie-Noël et al. 2016a].

2A. ***Tangent cones and recursive class of corner domains.*** Let M be a Riemannian manifold without boundary. We define recursively the class of admissible corner domains $\mathfrak{D}(M)$ and admissible cones $\mathfrak{P}_n$, in the spirit of [Dauge 1988]:

Initialization: $\mathfrak{P}_0$ has one element, $\{0\}$. $\mathfrak{D}(\mathbb{S}^0)$ is formed by all nonempty subsets of $\mathbb{S}^0$.

Recurrence: For $n \geq 1$,

(1) a cone Π (see Definition 1.1) belongs to $\mathfrak{P}_n$ if and only if the section of Π belongs to $\mathfrak{D}(\mathbb{S}^{n-1})$,

(2) $\Omega \in \mathfrak{D}(M)$ if and only if Ω is bounded and, for any $x \in \overline{\Omega}$, there exists a tangent cone $\Pi_x \in \mathfrak{P}_n$ to Ω at x.

By definition, Π_x is the tangent cone to Ω at $x \in \overline{\Omega}$ if there exists a local map $\psi_x : \mathcal{U}_x \mapsto \mathcal{V}_x$, where $\mathcal{U}_x$ and $\mathcal{V}_x$ are neighborhoods (called *map-neighborhoods*) of x in M and of 0 in $\mathbb{R}^n$, respectively, and ψ_x is a diffeomorphism such that

$$\psi_x(x) = 0, \quad (\mathrm{d}\psi_x)(x) = \mathbb{I}, \quad \psi_x(\mathcal{U}_x \cap \Omega) = \mathcal{V}_x \cap \Pi_x \quad \text{and} \quad \psi_x(\mathcal{U}_x \cap \partial\Omega) = \mathcal{V}_x \cap \partial \Pi_x. \tag{7}$$

In dimension 2, cones are half-planes, sectors and the full plane. The corner domains are (curvilinear) polygons on M with a finite number of vertices, each one of opening in $(0, \pi) \cup (\pi, 2\pi)$. This includes, of course, regular domains.

The key quantity in order to estimate errors when making a change of variables is

$$\kappa(x) = \|\mathrm{d}\psi\|_{W^{1,\infty}(\mathcal{U}_x)}. \tag{8}$$

It depends not only on x, but also on the choice of the local map. Note that, unlike for a regular domain, the curvature of the regular part of a corner domain may be unbounded (think of a circular cone). Therefore, $\kappa(x)$ is not bounded in general when picking an atlas of $\overline{\Omega}$. An important subclass of corner domains are those who are *polyhedral*: a cone is said to be polyhedral if its boundary is contained in a finite union of hyperplanes, and a domain is called polyhedral if all its tangent cones are polyhedral.

As proven in [Bonnaillie-Noël et al. 2016a], for a polyhedral domain it is possible to find an atlas such that κ is bounded. In the general case, we will have to control the possible blow-up of κ.

A list of examples can be found in [Bonnaillie-Noël et al. 2016a, Section 3.1]. Let us recall that, in dimension 2, all cones are polyhedral and therefore so are all corner domains, but this is not true anymore when $n \geq 3$: circular cones are typical examples of cones which are not polyhedral.

2B. ***Singular chains.*** For $x_0 \in \overline{\Omega}$, we denote by $\Gamma_{x_0} \in \mathfrak{P}_{d_0}$ the reduced cone of Π_{x_0} — see Definition 1.1 — and ω_{x_0} the section of Γ_{x_0}. A singular chain $\mathbb{X} = (x_0, \dots, x_p)$ is a sequence of points, with $x_0 \in \overline{\Omega}$, $x_1 \in \overline{\omega}_{x_0}$, and so on. We denote by $\mathfrak{C}(\Omega)$ the set of singular chains (in Ω), $\mathfrak{C}_{x_0}(\Omega)$ the set of chains initiated at x_0 and $\mathfrak{C}^*_{x_0}(\Omega)$ the set of $\mathbb{X} \in \mathfrak{C}_{x_0}(\Omega)$ such that $\mathbb{X} \neq (x_0)$. We denote by $l(\mathbb{X})$ the integer $p+1$ that is the length of the chain. Note that $1 \leq l(\mathbb{X}) \leq n+1$, and that $l(\mathbb{X}) \geq 2$ when $\mathbb{X} \in \mathfrak{C}^*_{x_0}(\Omega)$.

With a chain $\mathbb{X}$ is canonically associated a cone, denoted by $\Pi_{\mathbb{X}}$, called a tangent structure:

- If $\mathbb{X} = (x_0)$, then $\Pi_{\mathbb{X}} = \Pi_{x_0}$.
- If $\mathbb{X} = (x_0, x_1)$, write as above, in some adapted coordinates, $\Pi_{x_0} = \mathbb{R}^{n-d_0} \times \Gamma_{x_0}$. Let C_{x_1} be the tangent cone to ω_{x_0} at x_1. Then, in the adapted coordinates, $\Pi_{\mathbb{X}} = \mathbb{R}^{n-d_0} \times \langle x_1 \rangle \times C_{x_1}$, where $\langle x_1 \rangle$ is the vector space spanned by x_1 in Γ_{x_0}.
- And so on for longer chains.

We refer to [Bonnaillie-Noël et al. 2016a, Section 3.4] for complete definitions. Since singular chains are one of the tools of our analysis, we provide below some examples for a better understanding. In these examples, we assume for simplicity that Π_{x_0} is irreducible.

- If $x_1 \in \Pi_{x_0}$ (an interior point), then $\Pi_{(x_0,x_1)}$ is the full space.
- If x_1 is in the regular part of the boundary of ω_{x_0}, then C_{x_1} is a half-space of $\mathbb{R}^{n-1}$ and $\Pi_{(x_0,x_1)}$ is a half-space of $\mathbb{R}^n$. In particular, for a cone with regular section, all chains of length 2 are associated either with a half-space or the full space. The chains of length 3 are associated with the full space, and there are no longer chains.
- If $\Pi_{x_0} \subset \mathbb{R}^3$ is such that its section is a polygon and if x_1 is one of its vertices, then C_{x_1} is a two-dimensional sector, and $\Pi_{(0,x_1)}$ is a wedge. If x_2 is on the boundary of the sector C_{x_1}, then $\Pi_{(x_0,x_1,x_2)}$ is a half-space, but, if x_2 is on the interior of the sector, then $\Pi_{(0,x_1,x_2)} = \mathbb{R}^3$.

Given a cone $\Pi \in \mathfrak{P}_n$, we will also consider chains of Π, for example chains in $\mathfrak{C}_0(\Pi)$ are of the form $(0, x_1, \dots)$, where x_1 belongs to the closure of the section of the reduced cone of Π.

The main idea is to consider the local energy as a function not only defined on $\overline{\Omega}$, but also on singular chains: $\mathfrak{C}(\Omega) \ni \mathbb{X} \mapsto E(\Pi_{\mathbb{X}})$. In order to show regularity properties of this function, we define a partial order on singular chains: we say that $\mathbb{X} \leq \mathbb{X}'$ if $l(\mathbb{X}) \leq l(\mathbb{X}')$ and $x_k = x'_k$ for all $k \leq l(\mathbb{X})$. We also define a distance between cones through the action of isomorphisms:

$$\mathbb{D}(\Pi, \Pi') = \frac{1}{2}\left\{ \min_{\substack{L \in \mathsf{BGL}_n \\ L\Pi = \Pi'}} \| L - \mathbb{I}_n \| + \min_{\substack{L \in \mathsf{BGL}_n \\ L\Pi' = \Pi}} \| L - \mathbb{I}_n \| \right\}, \tag{9}$$

where BGL_n is the ring of linear isomorphisms L of $\mathbb{R}^n$ with norm $\|L\| \leq 1$. Note that by definition the distance between two cones is $+\infty$ if they do not belong to the same orbit for the action of BGL_n on $\mathfrak{P}_n$.

We then define the natural distance, inherited on $\mathfrak{C}(\Omega)$, by $\mathbb{D}(\mathbb{X}, \mathbb{X}') = \|x_0 - x_0'\| + \mathbb{D}(\Pi_{\mathbb{X}}, \Pi_{\mathbb{X}'})$; see [Bonnaillie-Noël et al. 2016a, Definition 3.22]. Then [Bonnaillie-Noël et al. 2016a, Theorem 3.25] states that any function $F : \mathfrak{C}(\Omega) \to \mathbb{R}$, monotonous and continuous with respect to $\mathbb{D}$, is lower semicontinuous when restricted to $\overline{\Omega}$ (which corresponds to chains of length 1). We will show these two criteria; see Corollaries 6.2 and 6.3.

3. Change of variables and perturbation of the metric

This section contains mainly technical lemmas, which are useful in the following sections. We define the operator with metric and we show the influence of a change of variables from a corner domains toward tangent cones on the quadratic form.

3A. ***Change of variables and operator with metrics.*** We need to know how a change of variables transforms the quadratic form of the Robin Laplacian. Indeed, we will consider diffeomorphisms $\psi : \mathcal{O} \to \mathcal{O}'$, where $\mathcal{O}$ and $\mathcal{O}'$ are open sets, in these two situations:

- $\mathcal{O}$ and $\mathcal{O}'$ will be cones in $\mathfrak{P}_n$ and ψ will be a linear map on $\mathbb{R}^n$, or
- $\mathcal{O}$ and $\mathcal{O}'$ will be map-neighborhoods, respectively of a point in a closure of a corner domain and of 0 in the associated tangent cone.

This change of variables will induce a regular metric $\mathrm{G} : \mathcal{O}' \to \mathrm{GL}_n$. In the case where ψ is linear, G will be constant.

Let $L^2_{\mathrm{G}}(\mathcal{O}')$ be the space of the square-integrable functions for the weight $|\mathrm{G}|^{-1/2}$, endowed with its natural norm $\|v\|_{L^2_G} := \int_{\mathcal{O}'} |v|^2 |\mathrm{G}|^{-1/2}$. Due to the previous hypotheses, $L^2_{\mathrm{G}}(\mathcal{O}') = L^2(\mathcal{O}')$. Let $g = \mathrm{G}|_{\partial\mathcal{O}'}$ be the restriction of the metric to the boundary. We introduce the quadratic form

$$\mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](v) = \int_{\mathcal{O}'} \langle \mathrm{G}\nabla v, \nabla v\rangle |\mathrm{G}|^{-1/2} - \alpha \int_{\partial\mathcal{O}'} |v|^2 |g|^{-1/2}.$$

Due to the above hypotheses on $\mathcal{O}'$ and G, we can define this quadratic form on $H^1(\mathcal{O}')$, endowed with the weighted norm $\|\cdot\|_{L^2_G}$.

Lemma 3.1. *Let $\mathcal{O}$ and $\mathcal{O}'$ be open sets and $\psi : \mathcal{O} \mapsto \mathcal{O}'$ a diffeomorphism as above. Let $\mathrm{J} := \mathrm{d}(\psi^{-1})$ be the Jacobian of ψ^{-1} and $\mathrm{G} := \mathrm{J}^{-1}(\mathrm{J}^{-1})^\top$ the associated metric. Then, for all $u \in H^1(\mathcal{O})$,*

$$\mathcal{Q}_\alpha[\mathcal{O}](u) = \mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](u \circ \psi^{-1}) \quad \textit{and} \quad \|u\|_{L^2(\mathcal{O})} = \|u \circ \psi^{-1}\|_{L^2_G(\mathcal{O}')}.$$

Said differently, if we define $\mathrm{U} : u \mapsto u \circ \psi^{-1}$, then U is an isometry from $L^2(\mathcal{O})$ onto $L^2_G(\mathcal{O}')$, and $\mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}]\mathrm{U} = \mathcal{Q}_\alpha[\mathcal{O}]$. We will also use scaling on cones:

Lemma 3.2. *Let Π be a cone and $u \in H^1(\Pi)$. For $\alpha > 0$, we define $u_\alpha(x) := \alpha^{-n/2} u(x/\alpha)$. Then*

$$\|u_\alpha\|_{L^2} = \|u\|_{L^2} \quad \textit{and} \quad \mathcal{Q}_\alpha[\Pi](u) = \alpha^2 \mathcal{Q}[\Pi](u_\alpha).$$

In particular, $\mathcal{Q}_\alpha[\Pi]$ and $\alpha^2\mathcal{Q}[\Pi]$ are unitarily equivalent.

3B. ***Approximation of metrics.*** We will be led to consider situations where $\mathrm{J}-\mathbb{I}$ is small (and so is $\mathrm{G}-\mathbb{I}$). Therefore, for $v \in H^1(\mathcal{O}')$, we compute

$$\mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](v)-\mathcal{Q}_\alpha[\mathcal{O}'](v)=\int_{\mathcal{O}'}\langle(\mathrm{G}-\mathbb{I})\nabla v, \nabla v\rangle|\mathrm{G}|^{-1/2}+\int_{\mathcal{O}'}|\nabla v|^2(|\mathrm{G}|^{-1/2}-1)+\alpha\int_{\partial\mathcal{O}'}|v|^2(|g|^{-1/2}-1)$$

and therefore

$$\begin{aligned}&\big|\mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](v)-\mathcal{Q}_\alpha[\mathcal{O}'](v)\big|\\ &\qquad\le \big(\|\mathrm{G}-\mathbb{I}\|_{L^\infty_v}(\||\mathrm{G}|^{-1/2}-1\|_{L^\infty_v}+1)+\||\mathrm{G}|^{-1/2}-\mathbb{I}\|_{L^\infty_v}\big)\|\nabla v\|^2_{L^2}+\alpha\||g|^{-1/2}-1\|_{L^\infty_v}\|v\|_{L^2(\partial\mathcal{O}')},\end{aligned}$$

where $\|\cdot\|_{L^\infty_v}$ denotes the L^∞ norm on $\operatorname{supp} v$. Assume now that $\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}\le 1$; then there exists a universal constant $C>0$ such that

$$\big|\mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](v)-\mathcal{Q}_\alpha[\mathcal{O}'](v)\big|\le C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}(\|\nabla v\|^2_{L^2}+\alpha\|v\|_{L^2(\partial\mathcal{O}')}). \tag{10}$$

This may be written as

$$\begin{aligned}&(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|\nabla v\|^2_{L^2}-\alpha(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2(\partial\mathcal{O}')}\\ &\qquad\le \mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](v)\le(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|\nabla v\|^2_{L^2}-\alpha(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2(\partial\mathcal{O}')}\end{aligned}$$

That is, for $\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}$ small enough:

$$\begin{aligned}&(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\bigg(\|\nabla v\|^2_{L^2}-\alpha\frac{1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}{1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}\|v\|_{L^2(\partial\mathcal{O}')}\bigg)\\ &\qquad\le \mathcal{Q}_\alpha[\mathcal{O}', \mathrm{G}](v)\le(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\bigg(\|\nabla v\|^2_{L^2}-\alpha\frac{1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}{1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}}\|v\|_{L^2(\partial\mathcal{O}')}\bigg).\end{aligned} \tag{11}$$

Similarly, we have a norm approximation: assuming that $\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v}\le 1$,

$$(1-C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2}\le\|v\|_{L^2_G}\le(1+C\|\mathrm{J}-\mathbb{I}\|_{L^\infty_v})\|v\|_{L^2}\quad\text{for all } v\in L^2(\mathcal{O}'). \tag{12}$$

By applying the previous inequality to tangent geometries with a constant metric, we will deduce the continuity of the local energy on strata in Section 6A.

3C. ***Functions with small support.*** The following lemma compares the quadratic form with a metric to the one without metric for functions concentrated near the origin of a tangent cone:

Lemma 3.3. *Let $\Omega\in\mathfrak{D}(M)$, let $x_0\in\overline{\Omega}$, and let $\psi_{x_0}:\mathcal{U}_{x_0}\to\mathcal{V}_{x_0}$ be a map-neighborhood of x_0. Let* G *be the associated metric, defined in Lemma 3.1. Then there exist universal positive constants c and C such that, for all $r\in(0,c/\kappa(x_0))$ with $\mathcal{B}(0,r)\subset\mathcal{V}_{x_0}$, and all $v\in H^1(\Pi_{x_0})$ compactly supported in $\mathcal{B}(0,r)$,*

$$(1-Cr\kappa(x_0))\mathcal{Q}_{\alpha^-}[\Pi_{x_0}](v)\le\mathcal{Q}_\alpha[\Pi_{x_0},\mathrm{G}](v)\le(1+Cr\kappa(x_0))\mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v), \tag{13}$$

where

$$\alpha^\pm(r,x_0)=\alpha\frac{1\mp Cr\kappa(x_0)}{1\pm Cr\kappa(x_0)} \tag{14}$$

and

$$\big|\|v\|_{L^2}-\|v\|_{L^2_G}\big|\le Cr\kappa(x_0)\|v\|_{L^2}.$$

Here $\kappa(x)$ is as defined in (8).

Proof. Let J be the Jacobian of $\psi_{x_0}^{-1}$. Since v is supported in a ball $\mathcal{B}(0,r)$ and $\mathrm{J}(0)=\mathbb{I}$, by the direct Taylor inequality we get $\|\mathrm{J}-\mathbb{I}\|_{L^\infty(\mathcal{B}(0,r))}\le r\|\mathrm{J}\|_{W^{1,\infty}(\mathcal{O})}=r\kappa(x_0)$. We use (10), and we follow the same steps leading to (11) and (12). □

Remark 3.4. When the quadratic forms are negative, the above inequality implies

$$\mathcal{Q}_{\alpha^-}[\Pi_{x_0}](v)\le\mathcal{Q}_\alpha[\Pi_{x_0},\mathrm{G}](v)\le\mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v). \tag{15}$$

The following lemma will be useful when studying the essential spectrum of tangent operators:

Lemma 3.5. *Let $\Omega\in\mathfrak{D}(M)$ and choose $x_0\in\overline{\Omega}$ such that $E(\Pi_{x_0})$ is finite. Let $\mathcal{U}_{x_0}$ be a map-neighborhood of x_0. Then*

$$\limsup_{\alpha\to+\infty}\inf_{\substack{u\in H^1(\Omega),\ \|u\|=1\\ \operatorname{supp}u\subset\mathcal{U}_{x_0}}}\alpha^{-2}\mathcal{Q}_\alpha[\Omega](u)\le E(\Pi_{x_0}).$$

This property is still true if $\Omega\in\mathfrak{P}_n$.

Proof. Obviously, $E(\Pi_{x_0})<0$. Let $\epsilon>0$ be such that $E(\Pi_{x_0})+\epsilon<0$. Note that

$$\frac{E(\Pi_{x_0})+\epsilon}{E(\Pi_{x_0})+\frac{1}{2}\epsilon}\in(0,1). \tag{16}$$

The functions in $H^1(\Pi_{x_0})$ with compact support are dense in $H^1(\Pi_{x_0})$, therefore there exists $v_\epsilon\in H^1(\Pi_{x_0})$ with compact support such that $\|v_\epsilon\|=1$ and $\mathcal{Q}[\Pi_{x_0}](v_\epsilon)<E(\Pi_{x_0})+\frac{1}{2}\epsilon$. Let $\mathcal{V}_{x_0}=\psi_{x_0}(\mathcal{U}_{x_0})$; we choose $r>0$ such that

$$\mathcal{B}(0,r)\subset\mathcal{V}_{x_0}\quad\text{and}\quad r\le\frac{c}{\kappa(x_0)}, \tag{17a}$$

$$\left(\frac{1-Cr\kappa(x_0)}{1+Cr\kappa(x_0)}\right)^2\left(E(\Pi_{x_0})+\frac{\epsilon}{2}\right)<E(\Pi_{x_0})+\epsilon. \tag{17b}$$

Conditions (17a) will allow us to apply Lemma 3.3. Note that (17b) is possible because of (16). The reason for this last condition will appear later. The value $\alpha^+=\alpha^+(x_0,r)$ is well defined in (14). The (normalized) test function

$$v_{\epsilon,\alpha^+}(x):=(\alpha^+)^{n/2}v_\epsilon(\alpha^+x)$$

satisfies

$$\mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v_{\epsilon,\alpha^+})=(\alpha^+)^2\mathcal{Q}[\Pi_{x_0}](v_\epsilon) \tag{18}$$

(see Lemma 3.2) and its support is

$$\operatorname{supp}v_{\epsilon,\alpha^+}=(\alpha^+)^{-1}\operatorname{supp}v_\epsilon.$$

Therefore there exists α large enough such that

$$\operatorname{supp}v_{\epsilon,\alpha^+}\subset\mathcal{B}(0,r), \tag{19}$$

so we can apply Lemma 3.3. Therefore, by combining (18) with estimates (13), we get

$$\begin{aligned}\mathcal{Q}_\alpha[\Pi_{x_0}, \mathrm{G}](v_{\epsilon,\alpha^+}) &\le (1+cr\kappa(x_0))\mathcal{Q}_{\alpha^+}[\Pi_{x_0}](v_{\epsilon,\alpha^+})\\ &= (1+cr\kappa(x_0))(\alpha^+)^2\mathcal{Q}[\Pi_{x_0}](v_\epsilon)\\ &\le (1+cr\kappa(x_0))(\alpha^+)^2\big(E(\Pi_{x_0})+\tfrac{1}{2}\epsilon\big).\end{aligned}$$

Due to (17a) and (19), we can define

$$u_{\epsilon,\alpha} := v_{\epsilon,\alpha^+}\circ\psi_{x_0}^{-1},$$

with $\operatorname{supp} u_{\epsilon,\alpha}\subset \mathcal{U}_{x_0}$, and Lemma 3.1 gives $\mathcal{Q}_\alpha[\Omega](u_{\epsilon,\alpha}) = \mathcal{Q}_\alpha[\Pi_{x_0}, G](v_{\epsilon,\alpha})$. Moreover, $\|u_{\epsilon,\alpha}\|^2 = \|v\|^2_{L^2_G}\le 1+Cr\kappa(x_0)$; therefore, keeping in mind that for ϵ small enough $E(\Pi_{x_0})+\frac{1}{2}\epsilon<0$, we get

$$\frac{\mathcal{Q}_\alpha[\Omega](u_{\epsilon,\alpha})}{\|u_{\epsilon,\alpha}\|^2}\le(\alpha^+)^2\Big(E(\Pi_{x_0})+\frac{\epsilon}{2}\Big)=\bigg(\frac{1-Cr\kappa(x_0)}{1+Cr\kappa(x_0)}\bigg)^2\alpha^2\Big(E(\Pi_{x_0})+\frac{\epsilon}{2}\Big).$$

Setting $u=u_{\epsilon,\alpha}/\|u_{\epsilon,\alpha}\|$ and using (17b), we have proved

$$\mathcal{Q}_\alpha[\Omega](u)\le E(\Pi_{x_0})+\epsilon$$

and we get the lemma. Since, locally, a cone of $\mathfrak{P}_n$ satisfies the same properties as a corner domain, the above proof works when Ω is a cone. □

Remark 3.6. As a direct consequence of the previous lemma, the min-max principle would provide a rough upper bound for $\limsup_{\alpha\to+\infty}\lambda(\alpha,\Omega)/\alpha^2$ by $\mathscr{E}(\Omega)$. But, at this stage, we still don't know whether $\mathscr{E}(\Omega)$ is finite or not when Ω is an n-dimensional corner domain.

4. Lower bound: multiscale partition of the unity

In this section, we prove the lower bound of Theorem 1.4 for any domain $\Omega\in\mathfrak{D}(M)$. We note at this point that this lower bound has interest only when $\mathscr{E}(\Omega)>-\infty$, which is not proved yet.

It relies on a multiscale partition of the unity of the domain by balls. Near each of these balls, we will perform a change of variables toward the tangent cone at the center of the ball, and we will estimate the remainder. However, the curvature of the boundary near each center of a ball may be large as this one is close to a conical point. We will counterbalance this effect by choosing balls of radius smaller with regard to the distances to conical points.

The following lemma is a consequence of [Bonnaillie-Noël et al. 2016a, Section 3.4.4 and Lemma B.1]:

Lemma 4.1. *Let $\Omega\in\mathfrak{D}(M)$ and let $\overline{\nu}_+$ be the smallest integer satisfying*

$$l(\mathbb{X})\ge\overline{\nu}_+ \implies \Pi_{\mathbb{X}} \textit{ is polyhedral} \qquad \textit{for all } \mathbb{X}\in\mathfrak{C}(\Omega).$$

For each sequence of scales $(\delta_k)_{0\le k\le\overline{\nu}_+}$ in $(0,+\infty)$ there exists $h_0>0$, an integer $L>0$ and a constant $c(\Omega)>0$ such that, for all $h\in(0,h_0)$, there exists an h-dependent finite set of points $\mathcal{P}\subset\overline{\Omega}$ such that, for all $p\in\mathcal{P}$, there exists $0\le k\le\overline{\nu}_+$ such that:

- *The ball* $\mathcal{B}(p, 2h^{\delta_0+\ldots+\delta_k})$ *is contained in a map-neighborhood of* p.
- *The curvature associated with this map-neighborhood* (*defined by* (8)) *satisfies*

$$\kappa(p) \le \frac{c(\Omega)}{h^{\delta_0+\ldots+\delta_{k-1}}}.$$

- $\overline{\Omega} \subset \bigcup_{p\in\mathcal{P}} \mathcal{B}(p, h^{\delta_0+\ldots+\delta_k})$, *and each point of* $\overline{\Omega}$ *belongs to at most* L *of these balls.*

We will need the standard IMS formula;[1] see for example [Simon 1983, Lemma 3.1]:

Lemma 4.2. *Let* $\chi_1, \ldots, \chi_N \in \mathcal{C}^\infty(\overline{\Omega})$ *be such that* $\sum_{l=1}^N \chi_l^2 = 1$. *Then*

$$\|\nabla u\|^2 = \sum_{l=1}^N \|\nabla(\chi_l u)\|^2 - \sum_{l=1}^N \|u\nabla\chi_l\|^2 \quad \textit{for all } u \in H^1(\Omega).$$

We set $h = \alpha^{-1}$ and we now choose a partition of unity $(\chi_p)_{p\in\mathcal{P}}$ associated with the balls provided by the previous lemma; each χ_p is C^∞ and is supported in the ball $\mathcal{B}(p, 2\alpha^{-(\delta_0+\ldots+\delta_k)})$, and

$$\begin{cases} \sum_{p\in\mathcal{P}} \chi_p^2 = 1 & \text{on } \overline{\Omega}, \\ \sum_{p\in\mathcal{P}} \|\nabla\chi_p\|_\infty^2 \le C(\Omega)\alpha^{2\delta} & \text{with } \delta = \delta_0 + \cdots + \delta_{\bar{\nu}_+}. \end{cases} \tag{20}$$

We apply Lemma 4.2 together with the uniform estimates of gradients (20):

$$\mathcal{Q}_\alpha[\Omega](u) = \sum_{p\in\mathcal{P}} \mathcal{Q}_\alpha[\Omega](\chi_p u) - \sum_{p\in\mathcal{P}} \|u\nabla\chi_p\|^2 \ge \sum_{p\in\mathcal{P}} \mathcal{Q}_\alpha[\Omega](\chi_p u) - C(\Omega)\alpha^{2\delta}\|u\|^2.$$

Therefore we are left with the task of estimating $\mathcal{Q}_\alpha[\Omega](\chi_p u)$ from below for each $p \in \mathcal{P}$. Let ψ_p be a local map on $\mathcal{B}(p, 2\alpha^{-(\delta_0+\ldots+\delta_k)})$ and $v_p := (\chi_p u)\circ\psi_p^{-1}$. Let G_p be the associated metric. Then we deduce from Lemmas 3.1 and 3.3 that (recall that the quadratic forms are negative)

$$\begin{aligned} \frac{\mathcal{Q}_\alpha[\Omega](\chi_p u)}{\|\chi_p u\|^2} &= \frac{\mathcal{Q}_\alpha[\Pi_p, \mathrm{G}_p](v_p)}{\|v_p\|_{\mathrm{G}_p}^2} \\ &\ge (1 + C\alpha^{-(\delta_0+\ldots+\delta_k)}\kappa(p))\frac{\mathcal{Q}_{\alpha^-}[\Pi_p](v_p)}{\|v_p\|^2} \\ &\ge (1 + C\alpha^{-(\delta_0+\ldots+\delta_k)}\kappa(p))(\alpha^-)^2 E(\Pi_c) \ge (1 + C'\alpha^{-(\delta_0+\ldots+\delta_k)}\kappa(p))\alpha^2\mathscr{E}(\Omega) \\ &= \alpha^2\mathscr{E}(\Omega) + O(\alpha^{2-\delta_k}), \end{aligned}$$

where we have used Lemma 4.1 to control $\kappa(p)$.

Lemma 4.2 provides

$$\mathcal{Q}_\alpha[\Omega](u) \ge \left(\alpha^2\mathscr{E}(\Omega) + \sum_{k=0}^{\bar{\nu}_+} O(\alpha^{2-\delta_k}) + O(\alpha^{2\delta})\right)\|u\|^2 \quad \text{for all } u \in H^1(\Omega).$$

Recall that $\delta = \sum_{k=0}^{\bar{\nu}_+}\delta_k$; these remainders are optimized by choosing $\delta_0 = \cdots = \delta_{\bar{\nu}_+}$ and $2 - \delta_0 = 2\delta = 2(\bar{\nu}_+ + 1)\delta_0$, that is, $\delta_0 = 2/(2\bar{\nu}_+ + 3)$. We deduce from the min-max principle that there exists $\alpha_0 \in \mathbb{R}$

[1] IMS stands for *Ismagilov, Morgan and Simon.*

and $C^- > 0$ such that

$$\lambda(\Omega,\alpha) \geq \alpha^2 \mathscr{E}(\Omega) - C^- \alpha^{2-2/(2\overline{\nu}_+ + 3)} \quad \text{for all } \alpha \geq \alpha_0, \tag{21}$$

which is the lower bound of Theorem 1.4.

5. Tangent operator

In this section we describe the Robin Laplacian on a cone Π, linking some parts of its spectrum with its section ω.

5A. *Semiboundedness of the operator on tangent cones.*

Lemma 5.1. *Let $\Pi \in \mathfrak{P}_n$ and let ω be its section. Let $R \geq 0$, and let $u \in H^1(\Pi)$ with support in $\mathcal{B}(0, R)^\complement$.*[2] *Then*

$$\mathcal{Q}[\Pi](u) \geq \left(\inf_{r>R} \frac{\lambda(\omega, r)}{r^2} \right) \|u\|^2_{L^2(\Pi)}.$$

Proof. Let $\varphi : (r,\theta) \mapsto r\theta$ be the change of variables from $\mathbb{R}_+ \times \omega$ into Π and denote by $v(r,\theta) := u \circ \varphi^{-1}$ the function associated with the change of variables. We have

$$\|\nabla u\|^2_{L^2(\Pi)} = \int_{r>R} \left(|\partial_r v|^2 + \frac{1}{r^2} \|\nabla_\theta v(r,\cdot)\|^2_{L^2(\omega)} \right) r^{n-1}\, \mathrm{d}r;$$

therefore,

$$\begin{aligned}
\mathcal{Q}[\Pi](u) &\geq \int_{r>R} \frac{1}{r^2} \|\nabla_\theta v(r,\cdot)\|^2_{L^2(\omega)} r^{n-1}\, \mathrm{d}r - \int_{r>R} \|v(r,\cdot)\|^2_{L^2(\partial\omega)} r^{n-2}\, \mathrm{d}r \\
&= \int_{r>R} \frac{1}{r^2} \mathcal{Q}_r[\omega](v(r,\cdot)) r^{n-1}\, \mathrm{d}r \geq \int_{r>R} \frac{1}{r^2} \lambda(\omega, r) \|v(r,\cdot)\|^2_{L^2(\omega)} r^{n-1}\, \mathrm{d}r \\
&\geq \inf_{r>R} \frac{\lambda(\omega,r)}{r^2} \int_{r>R} \|v(r,\cdot)\|^2_{L^2(\omega)} r^{n-1}\, \mathrm{d}r
\end{aligned}$$

and the lemma follows. □

We now prove the following:

Lemma 5.2. *Let $\Pi \in \mathfrak{P}_n$ be such that its section ω satisfies $\mathscr{E}(\omega) > -\infty$. Then $E(\Pi) > -\infty$ and the Robin Laplacian $L[\Pi]$ is well defined as the Friedrichs extension of $Q[\Pi]$ with form domain $D(Q[\Pi]) = H^1(\Pi)$.*

Proof. Since $\mathscr{E}(\omega)$ is supposed to be finite, (21) implies

$$\liminf_{r \to +\infty} \frac{\lambda(\omega, r)}{r^2} \geq \mathscr{E}(\omega). \tag{22}$$

Let χ_1 and χ_2 be two regular cut-off functions defined on $\mathbb{R}_+$ such that $\operatorname{supp} \chi_1 \subset [0, 2R)$, $\chi_1 = 1$ on $[0, R]$ and $\chi_1^2 + \chi_2^2 = 1$. Lemma 4.2 provides

$$\mathcal{Q}[\Pi](u) = \sum_{i=1,2} \mathcal{Q}[\Pi](\chi_i u) - \sum_{i=1,2} \|\nabla \chi_i u\|^2. \tag{23}$$

[2] $R = 0$ is included, with $\mathcal{B}(0,0) = \varnothing$.

Denote by D_0^R the set of functions in $H^1(\Pi \cap \mathcal{B}(0, 2R))$ supported in $\mathcal{B}(0, 2R)$. Since $\Pi \cap \mathcal{B}(0, 2R)$ is a corner domain, D_0^R has compact injection into $L^2(\partial\Pi \cap \mathcal{B}(0, 2R))$; see [Dauge 1988, Corollary AA.15]. We deduce the existence of a constant $C_1(R) \in \mathbb{R}$ such that

$$\mathcal{Q}[\Pi](\chi_1 u) \geq C_1(R)\|\chi_1 u\|^2_{L^2(\Pi\cap\mathcal{B}(0,2R))} = C_1(R)\|\chi_1 u\|^2_{L^2(\Pi)}.$$

Let $\epsilon > 0$; from (22) we deduce the existence of $R > 0$ such that

$$\frac{\lambda(\omega, r)}{r^2} \geq \mathscr{E}(\omega) - \epsilon \quad \text{for all } r > R$$

and therefore Lemma 5.1 gives

$$\mathcal{Q}(\chi_2 u) \geq (\mathscr{E}(\omega) - \epsilon)\|\chi_2 u\|^2_{L^2(\Pi)}.$$

There exists $C_2 > 0$ such that $\sum_i \|\nabla\chi_i\|^2 \leq C_2 R^{-2}$ for all $R > 0$. Therefore we deduce that there exists $C_3 = C_3(R, \epsilon, \omega) \in \mathbb{R}$ such that

$$\mathcal{Q}[\Pi](u) \geq C_3\|u\|^2_{L^2(\Pi)}.$$

We deduce that the quadratic form is lower semibounded and the operator $L[\Pi]$ is well defined as the self-adjoint extension of $\mathcal{Q}[\Pi]$, and its form domain is $H^1[\Pi]$. □

5B. ***Bottom of the essential spectrum for irreducible cones.*** Let $\Pi \in \mathfrak{P}_m$ with $m \geq n$, and let Γ be its reduced cone. In some suitable coordinates, we may write

$$\Pi = \mathbb{R}^{m-n} \times \Gamma$$

with $\Gamma \in \mathfrak{P}_n$ an irreducible cone. The associated Robin Laplacian admits the decomposition

$$L[\Pi] = -\Delta_{\mathbb{R}^{m-n}} \otimes \mathbb{I}_n + \mathbb{I}_{m-n} \otimes L[\Gamma]. \tag{24}$$

In particular,

$$\mathfrak{S}(L[\Pi]) = [E(\Gamma), +\infty).$$

Moreover, if $E(\Gamma)$ is a discrete eigenvalue for $L[\Gamma]$ and u is an associated eigenfunction (with exponential decay), then $\mathbb{I} \otimes u$ is called an L^∞-generalized eigenfunction for $L[\Pi]$ (this is linked to the notion of L^∞-spectral pair). Therefore we are led to investigate the bottom of the essential spectrum of $L[\Gamma]$. We prove:

Lemma 5.3. *Let $\Gamma \in \Pi_n$ be an irreducible cone of section ω such that $\mathscr{E}(\omega) > -\infty$. Then the bottom of the essential spectrum of $L[\Pi]$ is $\mathscr{E}(\omega)$.*

Proof. From Persson's lemma [1960], the bottom of the essential spectrum of $L[\Gamma]$ is the limit, as $R \to +\infty$, of

$$\Sigma(R) := \inf_{\substack{\Psi\in H^1(\Gamma),\ \Psi\neq 0\\ \operatorname{supp}(\Psi)\cap\mathcal{B}(0,R)=\varnothing}} \frac{\mathcal{Q}[\Gamma](\Psi)}{\|\Psi\|^2}.$$

Lower bound. From Lemma 5.1, we get directly

$$\liminf_{R\to+\infty} \Sigma(R) \geq \liminf_{R\to+\infty} \frac{\lambda(\omega, R)}{R^2}$$

and we deduce from (22) that

$$\liminf_{R\to+\infty} \Sigma(R) \geq \mathscr{E}(\omega).$$

Upper bound. By scaling — see Lemma 3.2 — we immediately have

$$\Sigma(R) = R^{-2} \inf_{\substack{\Psi\in H^1(\Gamma),\, \Psi\neq 0\\ \operatorname{supp}(\Psi)\cap\mathcal{B}(0,1)=\varnothing}} \frac{\mathcal{Q}_R[\Gamma](\Psi)}{\|\Psi\|^2}.$$

Each point x in $\overline{\Gamma}\setminus\overline{\mathcal{B}(0,1)}$ has a tangent cone Π_x. If we let $x_1 := x/|x| \in \overline{\omega}$, and let C_{x_1} be the tangent cone to ω at x_1, then $\Pi_x \equiv \mathbb{R}\times C_{x_1}$. Therefore, by tensor decomposition of the Robin Laplacian (see (24)), $E(C_{x_1}) = E(\Pi_x)$. Thus the finiteness of $\mathscr{E}(\omega)$ implies the finiteness of $E(\Pi_x)$, and from Lemma 3.5 we have

$$\limsup_{R\to+\infty} \Sigma(R) \leq E(\Pi_x) \quad \text{for all } x\in\overline{\Gamma}\setminus\overline{\mathcal{B}(0,1)}. \tag{25}$$

Using moreover that

$$\inf_{x\in\overline{\Gamma}\setminus\overline{\mathcal{B}(0,1)}} E(\Pi_x) = \inf_{x_1\in\partial\omega} E(C_{x_1}) = \mathscr{E}(\omega), \tag{26}$$

and taking the infimum in (25) over $x\in\overline{\Gamma}\setminus\overline{\mathcal{B}(0,1)}$, we deduce

$$\limsup_{R\to+\infty} \Sigma(R) \leq \mathscr{E}(\omega),$$

and the lemma follows. □

6. Infimum of the local energies in corner domains

6A. ***Finiteness of the infimum of the local energies.*** In this section, we prove the finiteness of $\mathscr{E}(\Omega)$ for any $\Omega\in\mathfrak{D}(M)$ and for any n-dimensional manifold M without boundary, by induction on the dimension n.

In dimension 1, bounded domains are intervals and the associated tangent cones are either half-lines or the full line whose associated energies are respectively -1 and 0 (by explicit computations), therefore the infimum of the local energies is finite.

Let $n\geq 2$ be fixed and let us assume that, for any corner domain ω of an $n-1$-dimensional Riemannian manifold without boundary, we have

$$\mathscr{E}(\omega) > -\infty.$$

We want to prove that the same holds in dimension n.

As a consequence of the recursive hypothesis, $E(\Pi)$ is finite for all $\Pi\in\mathfrak{P}_n$ — see Lemma 5.2 — and we can study the regularity of the local energy with respect to the geometry of a cone:

Proposition 6.1. *Assume the recursive hypothesis in dimension $n-1$. Then the map $\Pi\mapsto E(\Pi)$ is continuous on $\mathfrak{P}_n$ for the distance $\mathbb{D}$ defined in* (9).

Proof. Let $\Pi \in \mathfrak{P}_n$ and let $(\Pi_k)_{k\in\mathbb{N}}$ be a sequence of cones with $\mathbb{D}(\Pi_k, \Pi) \to 0$ as $k \to +\infty$. This means that there exists a sequence $(\mathrm{J}_k)_{k\in\mathbb{N}}$ in GL_n with $\mathrm{J}_k(\Pi_k) = \Pi$, $\|\mathrm{J}_k\| \le 1$ and $\|\mathrm{J}_k - \mathbb{I}\| \to 0$ as $k \to +\infty$. Then, as a direct consequence of (11) and (12), we deduce that

$$\lim_{k\to+\infty} \frac{\mathcal{Q}[\Pi, \mathrm{G}_k](v)}{\|v\|^2_{L^2_{\mathrm{G}_k}}} = \frac{\mathcal{Q}[\Pi](v)}{\|v\|^2} \quad \text{for all } v \in H^1(\Pi).$$

Recall that the form domain of $\mathcal{Q}[\Pi, \mathrm{G}_k]$ is $H^1(\Pi)$; see Section 5A. Since $\mathcal{Q}[\Pi_k]$ and $\mathcal{Q}[\Pi, \mathrm{G}_k]$ are unitarily equivalent (see Lemma 3.1), we deduce that $E(\Pi_k) \to E(\Pi)$ as $k \to +\infty$. □

By definition of the distance on singular chains (see Section 2B), we get:

Corollary 6.2. *Assume the recursive hypothesis in dimension $n-1$. Let M be an n-dimensional manifold as above, and let $\Omega \in \mathfrak{D}(M)$ be a corner domain. Then the map $\mathbb{X} \mapsto E(\Pi_{\mathbb{X}})$ is continuous on $\mathfrak{C}(\Omega)$ for the distance $\mathbb{D}$. In particular, $x \mapsto E(\Pi_x)$ is continuous on each stratum of $\overline{\Omega}$.*

Let M be an n-dimensional manifold as above, let $\Omega \in \mathfrak{D}(M)$ and let $x_0 \in \partial\Omega$; in what follows, Γ_{x_0} is the reduced cone of Π_{x_0} and $\omega_{x_0} \in \mathfrak{D}(\mathbb{S}^{d-1})$ is its section, with $d \le n$. We note that (26) may be written as

$$\mathscr{E}(\omega_{x_0}) = \inf_{x_1\in\partial\omega_{x_0}} E(\Pi_{(x_0,x_1)}).$$

Therefore, Lemmas 5.2 and 5.3 show that

$$E(\Pi_{x_0}) \le E(\Pi_{(x_0,x_1)}) \quad \text{for all } x_1 \in \bar{\omega}_{x_0}.$$

We deduce by immediate recursion:

Corollary 6.3. *Let $\mathbb{X}_1$ and $\mathbb{X}_2$ be two singular chains in $\mathfrak{C}(\Omega)$ satisfying $\mathbb{X}_1 \le \mathbb{X}_2$; we have*

$$E(\Pi_{\mathbb{X}_1}) \le E(\Pi_{\mathbb{X}_2}).$$

We combine this with Corollary 6.2 and we can apply [Bonnaillie-Noël et al. 2016a, Theorem 3.25] to get the lower semicontinuity of the local energy function $x \mapsto E(\Pi_x)$, and, from the compactness of $\overline{\Omega}$, we deduce that $\mathscr{E}(\Omega)$ is finite. This concludes the proof of Theorem 1.3 by induction.

As a consequence, Lemmas 5.2 and 5.3 imply Theorem 1.2.

6B. *Second energy level.* Note that for a cone which is not irreducible, the spectrum consists in essential spectrum, and Theorem 1.2 does not apply. However, there still exists a threshold in the spectrum: the second energy level of the tangent operator of a cone $\Pi \in \mathfrak{P}_n$ is defined as

$$\mathscr{E}^*(\Pi) := \inf_{\mathbb{X}\in\mathfrak{C}^*_0(\Pi)} E(\Pi_{\mathbb{X}}),$$

where we recall that $\mathfrak{C}^*_0(\Pi)$, defined in Section 2B, is the set of singular chains of Π of the form $\mathbb{X} = (0, \dots)$ and with $l(\mathbb{X}) \ge 2$, where $l(\mathbb{X})$ is the length of the chain.

Using Corollary 6.3 with $\mathbb{X}_1 = (0)$, then taking the infimum over the chain $\mathbb{X}_2 \geq \mathbb{X}_1$ with $l(\mathbb{X}_2) \geq 2$, we get $E(\Pi) \leq \mathscr{E}^*(\Pi)$. We also get $\mathscr{E}^*(\Pi) = \inf_{x_1 \in \partial\omega} E(\Pi_{(0,x_1)})$ and therefore, by (26),

$$\mathscr{E}(\omega) = \mathscr{E}^*(\Pi), \tag{27}$$

where ω is the section of the reduced cone of Π. The quantity $\mathscr{E}^*$ will be the discriminating value in the analysis carried out in Section 7.

6C. ***Examples.*** The inequality $E(\Pi) \leq \mathscr{E}^*(\Pi)$ is strict if and only if the operator on the reduced cone has eigenvalues below the essential spectrum. The presence (or absence) of a discrete spectrum is an interesting question in itself, and we describe here some examples for which this question has been studied. Due to the clear decomposition of the Robin Laplacian on a cone of the form $\mathbb{R}^{m-n} \times \Gamma$ — see (24) — we only treat the case of irreducible cones.

When Γ is the half-line, $E(\Gamma) = -1 < 0 = \mathscr{E}^*(\Gamma)$, and an associated eigenfunction is $x \mapsto e^{-x}$. The case of sectors is given by (4): the inequality is strict if and only if the sector is convex. In that case, an associated eigenfunction is $(x, y) \mapsto e^{-x/\sin\theta}$, where x denotes the variable associated with the axis of symmetry of the sector, and θ is the opening angle.

Pankrashkin [2016] provides geometrical conditions on three-dimensional cones with regular section. He shows that, when $\Gamma \in \mathfrak{P}_3$ is a cone such that $\mathbb{R}^3 \setminus \Gamma$ is convex, $E(\Gamma) = \mathscr{E}^*(\Gamma)$. On the other hand, if $\mathbb{R}^3 \setminus \Gamma$ is not convex, then $E(\Gamma)$ is a discrete eigenvalue below the essential spectrum.

Note finally that Levitin and Parnovski [2008] use a geometrical parameter to give a more explicit expression of $E(\Pi)$ when the section of Π is a polygonal domain that admits an inscribed circle.

Remark 6.4. In [Levitin and Parnovski 2008, Theorem 3.5], it is stated that the bottom of the spectrum of the Robin Laplacian on a cone which contains an hyperplane passing through the origin is -1. The following example shows that this statement is incorrect because the bottom of the essential spectrum could be below -1: Take a spherical polygon $\omega \subset \mathbb{S}^2$ such that

- ω is included in a hemisphere,
- ω has at least a vertex of opening $\theta \in (\pi, 2\pi)$.

Let $\Pi \subset \mathbb{R}^3$ be the cone of section ω, and let $\widetilde{\Pi}$ be its complement in $\mathbb{R}^3$. The cone $\widetilde{\Pi}$ contains a half-space, has an edge with opening angle $\tilde{\theta} = 2\pi - \theta \in (0, \pi)$. Then, from Theorem 1.2 and (4), we get that the bottom of the essential spectrum of $L[\Pi]$ is below $-\sin^{-2}\frac{1}{2}\tilde{\theta}$, and therefore $E(\widetilde{\Pi}) < -1$.

7. Upper bound: construction of quasimodes

In order to prove the upper bound of Theorem 1.4, we construct recursive quasimodes. The subsections below correspond to the following plan:

(A) Use the analysis of Section 6 to find a chain $\mathbb{X}_\nu = (x_0, \ldots, x_\nu) \in \mathfrak{C}(\Omega)$ such that $L(\Pi_{\mathbb{X}_\nu})$ admits a generalized eigenfunction associated with the value $\mathscr{E}(\Omega)$, then construct a quasimode for $L_\alpha[\Pi_{\mathbb{X}_\nu}]$. We do this by using scaling and cut-off functions in a standard way.

(B) Use a recursive procedure (together with a multiscale analysis) to construct a quasimode on Π_{x_0}.

(C) Use this quasimode to construct a final quasimode on Ω, and choose the scales to optimize the remainders.

7A. ***A quasimode on a tangent structure.*** The next proposition uses the quantity $\mathscr{E}^*$ to state that there always exist a tangent structure that admits an L^∞-generalized eigenfunction associated with the ground state energy.

Proposition 7.1. *Let $\Pi \in \mathfrak{P}_n$. Then there exists $\mathbb{X} \in \mathfrak{C}_0(\Pi)$ satisfying*

$$E(\Pi_{\mathbb{X}}) = E(\Pi) \quad \textit{and} \quad E(\Pi_{\mathbb{X}}) < \mathscr{E}^*(\Pi_{\mathbb{X}}). \tag{28}$$

Let $\Gamma_{\mathbb{X}} \in \mathfrak{P}_d$ be the irreducible cone of $\Pi_{\mathbb{X}}$. Then there exists an L^∞-generalized eigenfunction for $L[\Pi_{\mathbb{X}}]$ associated with $E(\Pi)$. Moreover it has the form $\mathbb{1} \otimes \Psi_{\mathbb{X}}$, in coordinates associated with the decomposition $\Pi_{\mathbb{X}} \equiv \mathbb{R}^{n-d} \times \Gamma_{\mathbb{X}}$, where $\Psi_{\mathbb{X}}$ has exponential decay.

Proof. The proof of the existence of $\mathbb{X}$ is recursive over the dimension d of the reduced cone of Π. The initialization is clear; indeed, when $d = 1$, we have that Π is a half-plane, $E(\Pi) = E(\mathbb{R}_+) = -1$ and $\mathscr{E}^*(\Pi) = E(\mathbb{R}) = 0$. Moreover, $\psi_{\mathbb{X}}(x) = e^{-x}$ provides an eigenfunction for $L[\mathbb{R}^+]$.

We now prove the heredity. First we find a chain $\mathbb{X}$ satisfying (28):

- If $E(\Pi) < \mathscr{E}^*(\Pi)$, then $\mathbb{X} = (0)$ and $\Pi_{\mathbb{X}} = \Pi$.

- If $E(\Pi) = \mathscr{E}^*(\Pi)$, we use Theorem 1.2: the function $x_1 \mapsto E(\Pi_{x_1})$ is lower semicontinuous on $\bar{\omega}$, where ω is the section of the reduced cone of Π. Therefore there exists $x_1 \in \partial\omega$ such that $\mathscr{E}^*(\Pi) = \mathscr{E}(\omega) = E(\Pi_{x_1}) = E(\Pi_{(0,x_1)})$. The dimension of the reduced cone of $\Pi_{(0,x_1)}$ is lower than that of Π; therefore, by the recursive hypothesis, there exists $\mathbb{X}' \in \mathfrak{C}_0(\Pi_{(0,x_1)})$ such that $E(\Pi_{\mathbb{X}'}) = E(\Pi_{(0,x_1)})$ and $E(\Pi_{\mathbb{X}'}) < \mathscr{E}^*(\Pi_{\mathbb{X}'})$. We write this chain in the form $\mathbb{X}' = (0, \mathbb{X}'')$, and the chain $\mathbb{X}'$ is pulled back into an element of $\mathfrak{C}_0(\Pi)$ by setting $\mathbb{X} = (0, x_1, \mathbb{X}'') \in \mathfrak{C}_0(\Pi)$. Note that $\Pi_{\mathbb{X}} = \Pi_{\mathbb{X}'}$, so that $E(\Pi_{(0,x_1)}) = E(\Pi_{\mathbb{X}}) = \mathscr{E}^*(\Pi) = E(\Pi)$ and $E(\Pi_{\mathbb{X}}) < \mathscr{E}(\Pi_{\mathbb{X}})$.

From Theorem 1.2 and (27), $E(\Pi_{\mathbb{X}}) < \mathscr{E}^*(\Pi_{\mathbb{X}})$ means that $E(\Pi_{\mathbb{X}})$ is an eigenvalue of $L(\Gamma_{\mathbb{X}})$ below the essential spectrum; therefore, there exists an associated eigenfunction $\Psi_{\mathbb{X}}$ with exponential decay, and $(y, z) \mapsto \Psi(z)$ for $(y, z) \in \mathbb{R}^{n-d} \times \Gamma_{\mathbb{X}}$ is clearly an L^∞-generalized eigenfunction for $L[\mathbb{R}^{n-d} \times \Gamma_{\mathbb{X}}]$. ☐

First, thanks to the lower semicontinuity of local energies, we choose $x_0 \in \partial\Omega$ such that $E(\Pi_{x_0}) = \mathscr{E}(\Omega)$. Then, using Proposition 7.1, we pick a singular chain $\mathbb{X}_\nu = (x_0, \dots, x_\nu)$ such that $L[\Pi_{\mathbb{X}_\nu}]$ has a generalized eigenfunction associated with $E(\Pi_{x_0})$. We let $\mathbb{X}_k = (x_0, \dots, x_k)$ for $0 \le k \le \nu$, and $\Pi_k := \Pi_{\mathbb{X}_k}$.

We define

$$\bar{\nu} := \inf\{k \ge 0 : \Pi_k \text{ is polyhedral}\}. \tag{29}$$

The index $\bar{\nu}$ provides the shortest chain such that $\Pi_{\bar{\nu}}$ is polyhedral, with $\bar{\nu} = +\infty$ when Π_ν is not polyhedral. Moreover, when $\bar{\nu}$ is finite the tangent structure Π_k is polyhedral for all $\bar{\nu} \le k \le \nu$, and $\bar{\nu} \le n-2$, since any chain of length strictly larger than $n-2$ is associated either with a half-space or with the full space.

The tangent structure Π_ν is (in some suitable coordinates) $\mathbb{R}^p \times \Gamma_\nu$ with Γ_ν irreducible. We denote by π_{Γ_ν} the projection onto Γ_ν associated with this decomposition. Then, by Proposition 7.1, there exists an eigenfunction u with exponential decay for $L[\Gamma_\nu]$ associated with $E(\Pi_\nu)$.

Let $\chi \in \mathcal{C}^\infty(\mathbb{R}^+)$ be a cut-off function with compact support satisfying

$$\chi(r) = 1 \quad \text{if } r \leq 1 \qquad \text{and} \qquad \chi(r) = 0 \quad \text{if } r \geq 2.$$

We define the scaled cut-off function

$$\chi_\alpha(r) = \chi(\alpha^\delta r),$$

where $\delta \in (0, 1)$ will be chosen later. The initial quasimode is

$$u_\nu(x) = \chi_\alpha(|x|)u(\pi_\Gamma(\alpha x)), \quad x \in \Pi_{\mathbb{X}_\nu}.$$

Standard computations show that

$$\frac{\mathcal{Q}_\alpha[\Pi_\nu](u_\nu)}{\|u_\nu\|^2} = \alpha^2\mathscr{E}(\Omega) + \frac{\|\nabla(\chi_\alpha)u_\nu\|^2}{\|u_\nu\|^2};$$

in particular,

$$\frac{\mathcal{Q}_\alpha[\Pi_\nu](u_\nu)}{\|u_\nu\|^2} = \alpha^2\mathscr{E}(\Omega) + O(\alpha^{2\delta}). \tag{30}$$

7B. ***Getting up along the chains.*** The previous section provides a quasimode u_ν for $L[\Pi_\nu]$. The aim of this section is a recursive decreasing procedure in order to get a quasimode for $L[\Pi_0]$. Therefore, this step is skipped if $\nu = 0$. This case happens when $E(\Pi_{x_0}) < \mathscr{E}^*(\Pi_{x_0})$, and the quasimode is called *sitting*, as was introduced in [Bonnaillie-Noël et al. 2016a]. Otherwise we suppose that $\nu \geq 1$, and we will construct quasimodes u_k defined on Π_k, for $0 \leq k \leq \nu$. These quasimodes are called *sliding*.

In what follows, $(d_k(\alpha))_{k=1,\dots,\nu}$ and $(r_k(\alpha))_{k=0,\dots,\nu}$ are positive sequences of shifts and radii (to be determined) going to 0 as $\alpha \to +\infty$.

Let $1 \leq k \leq \nu$ and assume that $u_k \in H^1(\Pi_k)$ is constructed and is supported in a ball $\mathcal{B}(0, r_k(\alpha))$. This is already done for $k = \nu$; see the last section. For $1 \leq k \leq \nu$, we define

$$v_k = d_k(\alpha)(0, x_k) \in \Pi_{k-1},$$

where $(0, x_k) \in \Pi_{k-1}$ are cylindrical coordinates associated with the decomposition $\Pi_{k-1} = \mathbb{R}^{p_k} \times \Gamma_{k-1}$. Intuitively, v_k is a point of Π_{k-1} satisfying $\|v_k\| = d_k(\alpha)$ and is collinear to $(0, x_k)$.

We construct u_{k-1} as follows:

- *Local map at v_k*: The tangent cone to Π_{k-1} at v_k is Π_k itself. Let $\psi_k : \mathcal{U}_{v_k} \mapsto \mathcal{V}_{v_k}$ be a local map. The map-neighborhoods $\mathcal{U}_{v_k}$ and $\mathcal{V}_{v_k}$ (of $v_k \in \Pi_{k-1}$ and $0 \in \Pi_k$, respectively) can be chosen of diameters smaller than $c_k d_k(\alpha)$, where c_k is the diameter of the map-neighborhood of x_k. Moreover, when $k \geq \overline{\nu}$, Π_k is polyhedral, so ψ_k is a translation. When this is not the case, by elementary scaling, $\kappa(v_k) \leq \kappa(x_k)/d_k(\alpha)$; see [Bonnaillie-Noël et al. 2016a, Section 3] for more details on this process. Since the $(x_k)_{0\leq k\leq\nu}$ are fixed, we can choose ν fixed map-neighborhoods associated with these points, and a constant $c(\Omega) > 0$ such that

$$\kappa(v_k) \leq \begin{cases} c(\Omega)/d_k(\alpha) & \text{if } k \leq \overline{\nu}, \\ c(\Omega) & \text{if } k \geq \overline{\nu} + 1. \end{cases} \tag{31}$$

We now add the constraint that

$$\frac{r_k(\alpha)}{d_k(\alpha)} \to 0 \quad \text{as } \alpha \to +\infty \qquad \text{if } k \leq \bar{\nu}, \tag{32}$$

so that $r_k \kappa(v_k) \to 0$ for all $1 \leq k \leq \nu$, and we can define, for α large enough,

$$\tau_k := \frac{1 - C r_k \kappa(v_k)}{1 + C r_k \kappa(v_k)}, \tag{33}$$

where C is the constant appearing in Lemma 3.3.

- *Change of variables*: First we rescale u_k (the reason for this will appear later): let

$$\tilde{u}_k(x) = \tau_k(\alpha)^{n/2} u(\tau_k(\alpha) x). \tag{34}$$

This function satisfies

$$\|\tilde{u}_k\| = \|u_k\| \quad \text{and} \quad \mathcal{Q}_{\alpha_k^+}[\Pi_k](\tilde{u}_k) = \tau_k(\alpha)^2 \mathcal{Q}_\alpha[\Pi_k](u_k), \tag{35}$$

where $\alpha_k^+ = \tau_k(\alpha)\alpha$. Recall that $\operatorname{supp} u_k \subset \mathcal{B}(0, r_k(\alpha))$ by the recursive hypothesis on u_k. Then, due to (32), we have $c_k d_k(\alpha) > r_k(\alpha)/\tau_k(\alpha)$ for α large enough, and therefore

$$\operatorname{supp} \tilde{u}_k \subset \mathcal{B}(0, r_k(\alpha)/\tau_k(\alpha)) \subset \mathcal{V}_k.$$

As a consequence, we can define on $\mathcal{U}_k \cap \Pi_{k-1}$ the function

$$u_{k-1} = \tilde{u}_k \circ \psi_k. \tag{36}$$

We can extend this function by 0 outside its support so that $u_{k-1} \in H^1(\Pi_{k-1})$. Its support is inside a ball centered at 0 and of size $d_k + \operatorname{diam}(\mathcal{U}_k) = (1 + c_k) d_k$, so we set

$$r_{k-1} := (1 + c_k) d_k. \tag{37}$$

We derive from this recursive procedure a quasimode u_0 on Π_0, localized in a ball $\mathcal{B}(0, r_0(\alpha))$.

7C. *Quasimode on the initial domain Ω and choice of the scales.* Now we set $v_0 := x_0$, and we still define τ_0 by (33), then $\tilde{u}_0$ by (34) and u_{-1} by (36). Note that $\kappa(v_0)$ is constant since $v_0 = x_0$ is fixed. We compare $\mathcal{Q}_\alpha[\Pi_{k-1}](u_{k-1})$ with $\mathcal{Q}_\alpha[\Pi_k](u_k)$ for $0 \leq k \leq \nu$. We have, from Lemma 3.1,

$$\mathcal{Q}_\alpha[\Pi_k, \mathrm{G}_k](\tilde{u}_k) = \mathcal{Q}_\alpha[\Pi_{k-1}](u_{k-1}), \tag{38}$$

where $\mathrm{G}_k := \mathrm{J}_k^{-1}(\mathrm{J}_k^{-1})^\top$ is the associated metric with $\mathrm{J}_k := \mathrm{d}\psi_k^{-1}$.

Since, by construction, $r_k \kappa(v_k) \to 0$, we can apply Lemma 3.3, in particular the inequality (15):

$$\mathcal{Q}_\alpha[\Pi_k, G_k](\tilde{u}_k) \leq \mathcal{Q}_{\alpha_k^+}[\Pi_k](\tilde{u}_k).$$

Combining this with (35) and (38) we get, for all $0 \leq k \leq \nu$,

$$\mathcal{Q}_\alpha[\Pi_{k-1}](u_{k-1}) \leq \tau_k(\alpha)^2 \mathcal{Q}_\alpha[\Pi_k](u_k),$$

and therefore

$$\mathcal{Q}_\alpha[\Omega](u_{-1}) \leq \prod_{k=0}^{\nu} \tau_k(\alpha)^2 \mathcal{Q}_\alpha[\Pi_\nu](u_\nu).$$

Recall that $\kappa(v_0)$ is fixed; we get, from (31),

$$\mathcal{Q}_\alpha[\Omega](u_{-1}) \leq \left(1 + C\left(r_0 + \frac{r_1}{d_1} + \cdots + \frac{r_{\bar{\nu}}}{d_{\bar{\nu}}} + r_{\bar{\nu}+1} + \cdots + r_\nu\right)\right) \mathcal{Q}_\alpha[\Pi_\nu](u_\nu).$$

We now choose $r_k(\alpha) = \alpha^{-\sum_{p=0}^{k} \delta_k}$ when $k \leq \bar{\nu}$ and $r_k = r_{\bar{\nu}}$ when $k > \bar{\nu}$, with $\delta_k > 0$. The shifts are set by (37), so that $r_k/d_k = O(\alpha^{-\delta_k})$ for all $1 \leq k \leq \bar{\nu}$. Moreover, the scale δ of Section 7A is related by $\delta = \sum_{k=0}^{\bar{\nu}} \delta_k$, and (30) provides

$$\mathcal{Q}_\alpha[\Omega](u_{-1}) \leq \left(1 + \sum_{k=0}^{\bar{\nu}} O(\alpha^{-\delta_k})\right)(\alpha^2 \mathscr{E}(\Omega) + O(\alpha^{2\delta})) = \alpha^2 \mathscr{E}(\Omega) + \sum_{k=0}^{\bar{\nu}} O(\alpha^{2-\delta_k}) + O(\alpha^{2\delta}).$$

The error terms are the same as in Section 7C; therefore, we make the same choice of scales $\delta_k = 2/(2\bar{\nu}+3)$ for all $0 \leq k \leq \bar{\nu}$. By construction, u_{-1} is normalized, therefore the min-max theorem implies the upper bound of Theorem 1.4.

8. Applications

In the applications below, one must keep in mind that the finiteness of $\mathscr{E}(\Omega)$ is one of our results, and that this quantity can be made more explicit for particular geometries; see [Levitin and Parnovski 2008]. Moreover, this quantity goes to $-\infty$ as the corners of a domain Ω gets sharper: this is clear in dimension 2 since the local energy at a corner of opening θ goes to $-\infty$ as $\theta \to 0$; see (4). In higher dimension, it could be possible to use the approach from [Bonnaillie-Noël et al. 2016b] in order to show that the local energy goes to $-\infty$ for sharp cones (see the definition of a sharp cone therein).

8A. ***On the optimal constant in relative bounds zero for the trace operator.*** The trace injection from $H^1(\Omega)$ into $L^2(\partial\Omega)$ being compact, the following relative 0-bound holds: for all $\epsilon > 0$, there exists $C(\epsilon) > 0$ such that

$$\|u\|^2_{L^2(\partial\Omega)} \leq \epsilon \|\nabla u\|^2_{L^2(\Omega)} + C(\epsilon)\|u\|^2_{L^2(\Omega)} \quad \text{for all } u \in H^1(\Omega). \tag{39}$$

This inequality is a particular case of Ehrling's lemma. It can be written as

$$\mathcal{Q}_{1/\epsilon}[\Omega](u) \geq -\frac{C(\epsilon)}{\epsilon}\|u\|^2_{L^2(\Omega)} \quad \text{for all } u \in H^1(\Omega).$$

Thus, by definition of $\lambda(\Omega, \alpha)$, for each $\epsilon > 0$ the best constant $C(\epsilon)$ in (39) is

$$C(\epsilon) = -\epsilon\lambda\left(\Omega, \frac{1}{\epsilon}\right).$$

From Theorem 1.4, we obtain that this constant is essentially $\epsilon^{-1}|\mathscr{E}(\Omega)|$. More precisely:

Proposition 8.1. *Let $\Omega \in \mathfrak{D}(M)$ be an admissible corner domain. Then there exist $\epsilon_0 > 0$ and $\gamma \in \left(0, \frac{2}{3}\right)$ such that, for all $\epsilon \in (0, \epsilon_0)$,*

$$\|u\|^2_{L^2(\partial\Omega)} \leq \epsilon \|\nabla u\|^2_{L^2(\Omega)} + \left(\frac{|\mathscr{E}(\Omega)|}{\epsilon} + O(\epsilon^{\gamma-1})\right)\|u\|^2_{L^2(\Omega)} \quad \textit{for all } u \in H^1(\Omega).$$

Let us recall that the finiteness of $\lambda(\Omega, \alpha)$ is closely related to the compactness of the injection of $H^1(\Omega)$ into $L^2(\partial\Omega)$ and, for some cusps, where $\lambda(\Omega, \alpha) = -\infty$, this injection is not compact (see [Nazarov and Taskinen 2011; Daners 2013]).

8B. ***Transition temperature of superconducting models.*** In the study of superconducting models, the physics literature has explored over the years the possibility of increasing the critical fields. Another more interesting and more recent idea is to increase the temperature below which the normal state (i.e., the critical point of the Ginzburg–Landau energy for which the material is nowhere in the superconducting state) is not stable. For zero fields associated to a superconducting body Ω, enhanced surface superconductivity is modeled via a negative penetration depth $b < 0$ and, following [Giorgi and Smits 2007], this critical temperature is given by

$$T^b_c(\Omega) = T_{c_0} - T_{c_0}\lambda\left(\Omega, \frac{\xi(0)}{|b|}\right), \tag{40}$$

where $\xi(0) > 0$ is the so-called coherence length at zero temperature, T_{c_0} is the vacuum zero field critical temperature for $b = \infty$ (corresponding to a superconductor surrounded by vacuum) and $\lambda(\Omega, \alpha)$ is the first eigenvalue of the Robin problem.

Thanks to Theorem 1.4, for $|b|$ small enough we have

$$T^b_c(\Omega) \geq T_{c_0} + T_{c_0}\frac{\xi(0)^2}{|b|^2}\left(|\mathscr{E}(\Omega)| + O(|b|^\gamma)\right)$$

for some $\gamma \in \left(0, \frac{2}{3}\right)$. Since $|\mathscr{E}(\Omega)| \geq 1$ and goes to $+\infty$ as the corners of $\partial\Omega$ become sharper, our results are consistent with the general physical principle of increase of $T^b_c(\Omega)$ due to confinement (see for instance [Montevecchi and Indekeu 2000, Section 4] and see [Yampolskii and Peeters 2000; Baelus et al. 2002] concerning superconducting properties of nanostructuring materials).

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Received 18 Jan 2016. Revised 30 Mar 2016. Accepted 29 Apr 2016.

VINCENT BRUNEAU: vbruneau@math.u-bordeaux1.fr
Institut de Mathematiques, Université de Bordeaux I, 351 Cours de La Libération, 33405 Talence, France

NICOLAS POPOFF: nicolas.popoff@math.u-bordeaux1.fr
Institut de Mathematiques, Université de Bordeaux I, 351 Cours de La Libération, 33405 Talence, France

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Volume 9 No. 5 2016

2157-5045(2016)9:5;1-7