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FOR HYPERBOLIC CORNER PROBLEMS, I:  
SELF-INTERACTION PHENOMENON**

# GEOMETRIC OPTICS EXPANSIONS FOR HYPERBOLIC CORNER PROBLEMS, I: SELF-INTERACTION PHENOMENON

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In this article we are interested in the rigorous construction of geometric optics expansions for hyperbolic corner problems. More precisely we focus on the case where self-interacting phases occur. Those phases are proper to the high frequency asymptotics for the corner problem and correspond to rays that can display a homothetic pattern after a suitable number of reflections on the boundary. To construct the geometric optics expansions in that framework, it is necessary to solve a new amplitude equation in view of initializing the resolution of the WKB cascade.

1. Introduction	1359
2. An overview of well-posedness for half-space and corner problems	1361
3. The phase generation process and examples	1366
4. Geometric optics expansions for self-interacted trapped rays	1376
5. Conclusion	1416
Acknowledgement	1417
References	1417

## 1. Introduction

The aim of this article is to give rigorous methods to construct geometric optics expansions for linear hyperbolic initial boundary value problems in the quarter-space. Such problems will be called corner problems and are of the form

$$\begin{cases} L(\partial)u^\varepsilon := \partial_t u^\varepsilon + A_1 \partial_1 u^\varepsilon + A_2 \partial_2 u^\varepsilon = 0, \\ B_1 u^\varepsilon|_{x_1=0} = g^\varepsilon, \quad B_2 u^\varepsilon|_{x_2=0} = 0, \quad u^\varepsilon|_{t \leq 0} = 0, \end{cases} \quad (x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}_+, \quad t \geq 0, \quad (1)$$

where the matrices  $A_1, A_2$  are in  $M_N(\mathbb{R})$  and where the boundary matrices  $B_1, B_2$  are elements of  $M_{p_1, N}(\mathbb{R})$  and  $M_{p_2, N}(\mathbb{R})$  respectively (the values of the integers  $p_1$  and  $p_2$  will be made precise in [Assumption 2.2](#)).

We have, in this article, chosen to work with only two space dimensions in order to save some notations. However, all the following results can be generalized if one looks at problem (1) with extra space variables  $x' \in \mathbb{R}^{d-2}$  (with, of course, the suitable modifications on the operator  $L(\partial)$  to preserve hyperbolicity).

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This article can be seen, in some sense, as a complement to the paper by Sarason and Smoller [1974] in which the authors give intuitions and elements of proof about how to construct geometric optics expansions but where the construction is not performed rigorously. To our knowledge it is the only paper about this subject in the literature for general first-order systems and we shall rely on some of the deep ideas of this seminal work. In particular, the links between the phase generation by reflections and the geometry of the characteristic variety will be the foundation of the proofs in this article (see Section 3 and [Sarason and Smoller 1974, Section 6] for more details).

Indeed, in [Sarason and Smoller 1974] the authors give examples of corner problems whose characteristic variety is such that, according to their argumentation, the associated ansatz of the geometric optics expansion has to contain more phases than the analogous ansatz for each problem in the half-spaces  $\{x_1 > 0\}$  and  $\{x_2 > 0\}$ . They also show that a new phenomenon, specific to corner problems, may happen for some characteristic variety configurations: the existence of “self-interacting phases”. By self-interacting phases we mean that some phases can regenerate themselves after a suitable number of reflections on both sides of the corner. Such spectral configurations trap part of the solution in a periodically repeating pattern of reflections from one side to the other (see Definition 4.8 and Figure 6 for more details).

Our aim is to give a *rigorous* construction of the geometric optics expansion when self-interacting phases occur. This result is achieved in Theorem 4.27. The most interesting thing during this construction is the appearance of a new amplitude equation whose resolution is needed to initialize the resolution of the whole cascade of equations. More precisely, the resolution of the new amplitude equation requires the invertibility of an operator acting on the trace of one of the self-interacting amplitudes. This operator arises under the form  $(I - \mathbb{T})$  and is reminiscent of Osher’s invertibility assumption [1973] for proving an a priori estimate for (1). We show in Theorems 4.28 and 4.29 that a sufficient and necessary (in many meaningful cases) condition for the new amplitude equation to be solvable in  $L^2(\mathbb{R}_+)$  is that the energy associated with the trapped information does not increase. Such a formulation matches with the naive (but intuitive) idea that if part of the information is trapped and increases after running through one cycle, then the associated geometric optics expansion will blow up after repeated cycles.

Inverting an operator of the form  $(I - \mathbb{T})$  in view of constructing the geometric optics expansion is not surprising. Indeed, if we make the analogy with the analysis of the initial boundary value problem in the half-space, the necessary and sufficient condition to ensure strong well-posedness is the so-called uniform Kreiss–Lopatinskii condition (see [Kreiss 1970] and Assumption 2.11). When one wants to construct geometric optics expansions for such problems in a half-space, a “microlocalized” version of this condition arises [Williams 1996]. So one should expect that an analogous situation takes place for the corner problem and that the solvability condition we exhibit here is a microlocalized version of a stronger condition ensuring well-posedness of (1).

The full characterization of strong well-posedness for the corner problem has not been achieved yet. Some partial results are known, for example for symmetric corner problems with strictly dissipative boundary conditions (in that framework, the strong well-posedness can easily be obtained with few modifications of the proofs of [Lax and Phillips 1960; Benzoni-Gavage and Serre 2007] for half-space problems). However, there are, to our knowledge, few results concerning the general framework, that

is to say, corner problems only satisfying the uniform Kreiss–Lopatinskii condition on each side. A fundamental contribution to this study is the article by Osher [1973]. In this paper, the author uses the invertibility of an operator of the form  $(I - \mathbb{T}_\zeta)$  — here  $\zeta$  denotes a time frequency — to establish a priori energy estimates. More precisely, he uses such an invertibility property to construct a “Kreiss-type symmetrizer” providing a priori energy estimates with a loss of regularity from the source terms to the solution. Unfortunately the number of losses in the estimates is not even explicit. However, some new results about the possibility to obtain energy estimates without loss can be found in [Benoit 2015].

We believe that, as for the half-space problems, the invertibility condition on  $(I - \mathbb{T})$  is a microlocalized version of Osher’s condition. It is also interesting to remark that the example given in Section 3E shows that the invertibility condition on  $(I - \mathbb{T})$  may not be satisfied if we only impose the uniform Kreiss–Lopatinskii condition on either side of the corner. But, looking still at the example of Section 3E, we observe that the invertibility condition on  $(I - \mathbb{T})$  is automatically satisfied if the boundary conditions are strictly dissipative.

The paper is organized as follows: In Section 2 we define some objects and introduce notations for dealing with geometric optics expansions for initial boundary value problems. We also give some known results about the well-posedness theory for the corner problem (1). In Section 3, we explain, and make complete, the phase generation process by reflection as studied in [Sarason and Smoller 1974]. We also briefly give an example of a  $2 \times 2$  corner problem for which geometric optics expansions contain infinitely many phases.

Section 4 is devoted to the proof of our main result. Firstly, we give a rigorous framework for the description of the phases obtained by successive reflections. This framework has to be general enough to take into account self-interacting phases. Then we construct the geometric optics expansion. To do that, it is, in a first time, necessary to exhibit a global “tree” structure on the set of phases, then to find a way to initialize the resolution. As already mentioned, the initialization requires solving a new amplitude equation for the trace of a self-interacting amplitude. The derivation of this equation is performed in Section 4B2. Then we show that, once we have organized the set of phases and we have constructed one of the self-interacting amplitudes, we can construct all amplitudes associated with phases “close to” the self-interacting ones. A more precise study of the structure of the phase set then permits to determine all the phases in the geometric optics expansion.

The end of Section 4 aims at justifying the geometric optics expansion and then at giving a necessary and sufficient condition to ensure that the operator  $(I - \mathbb{T})$  is invertible. We also give examples of corner problems with one loop and revisit some of the conclusions of [Sarason and Smoller 1974]. Finally, we make some comments on our results and give some prospects in Section 5.

## 2. An overview of well-posedness for half-space and corner problems

**2A. Notations and definitions.** Let

$$\Omega := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \geq 0, x_2 \geq 0\}, \quad \partial\Omega_1 := \Omega \cap \{x_1 = 0\}, \quad \text{and} \quad \partial\Omega_2 := \Omega \cap \{x_2 = 0\}$$

be the quarter-space and both its edges. For  $T > 0$ , we will define

$$\Omega_T := ]-\infty, T] \times \Omega, \quad \partial\Omega_{1,T} := ]-\infty, T] \times \partial\Omega_1, \quad \text{and} \quad \partial\Omega_{2,T} := ]-\infty, T] \times \partial\Omega_2.$$

The used function spaces will be the usual Sobolev spaces  $H^n(X)$ , with the notations  $L^2(X) = H^0(X)$  and  $H^\infty(X) := \bigcap_n H^n(X)$ , where  $X$  is some Banach space. But we will also need the weighted Sobolev spaces defined by  $H_\gamma^n(X) := \{u \in \mathcal{D}'(X) \mid e^{-\gamma t} u \in H^n(X)\}$  for  $\gamma > 0$ .

At last, during the construction of the WKB expansion, to make sure that amplitudes are smooth enough, we shall need the source term in (1) to be flat at the corner. The associated space of profiles is thus defined as

$$H_f^n := \{g \in H^n(\mathbb{R} \times \mathbb{R}_+) \mid \forall k \leq n, \partial_x^k g(t, x)|_{x=0} = 0\} \quad \forall n \in \mathbb{N} \cup \{\infty\}. \tag{2}$$

The flat-at-the-corner weighted Sobolev spaces  $H_{f,\gamma}^n$  are defined in a similar way.

Throughout  $\mathcal{L}$  will be the symbol of the differential operator  $L(\partial)$ ; i.e., for  $\tau \in \mathbb{R}$  and  $\xi \in \mathbb{R}^2$ ,

$$\mathcal{L}(\tau, \xi) := \tau I + \sum_{j=1}^2 \xi_j A_j.$$

The characteristic variety  $V$  of  $L(\partial)$  is given by

$$V := \{(\tau, \xi) \in \mathbb{R} \times \mathbb{R}^2 \mid \det \mathcal{L}(\tau, \xi) = 0\}.$$

In this article we choose to work with constantly hyperbolic operators. However, it has to be mentioned that the analysis of Section 4 is slightly easier in the particular framework of strictly hyperbolic operators. We thus assume the following property on  $L(\partial)$ :

**Assumption 2.1.** *There exists an integer  $q \geq 1$ , real-valued  $\lambda_1, \dots, \lambda_q$  analytic on  $\mathbb{R}^2 \setminus \{0\}$  and positive integers  $\mu_1, \dots, \mu_q$  such that*

$$\det \mathcal{L}(\tau, \xi) = \prod_{j=1}^q (\tau + \lambda_j(\xi))^{\mu_j} \quad \forall \xi \in \mathbb{S}^1,$$

where the semisimple eigenvalues  $\lambda_j(\xi)$  satisfy  $\lambda_1(\xi) < \dots < \lambda_q(\xi)$ .

Let us also assume that the boundary of  $\Omega$  is noncharacteristic, and that the matrices  $B_1$  and  $B_2$  induce the good number of boundary conditions, that is to say:

**Assumption 2.2.** *We assume that the matrices  $A_1, A_2$  are invertible. Then  $p_1$  (resp.  $p_2$ ), the number of lines of  $B_1$  (resp.  $B_2$ ), equals the number of positive eigenvalues of  $A_1$  (resp.  $A_2$ ).*

Moreover we also assume that  $B_1$  and  $B_2$  are of maximal rank.

Under Assumptions 2.1 and 2.2, we can define the resolvent matrices

$$\mathcal{A}_1(\zeta) := -A_1^{-1}(\sigma I + i\eta A_2) \quad \text{and} \quad \mathcal{A}_2(\zeta) := -A_2^{-1}(\sigma I + i\eta A_1),$$

where  $\zeta$  denotes an element of the frequency space

$$\Xi := \{\zeta := (\sigma = \gamma + i\tau, \eta) \in \mathbb{C} \times \mathbb{R}, \gamma \geq 0\} \setminus \{(0, 0)\}.$$

For convenience, we also introduce  $\Xi_0$  the boundary of  $\Xi$ :

$$\Xi_0 := \Xi \cap \{\gamma = 0\}.$$

For  $j = 1, 2$  and  $\zeta \in (\Xi \setminus \Xi_0)$ , we denote by  $E_j^s(\zeta)$  the stable subspace of  $\mathcal{A}_j(\zeta)$  and by  $E_j^u(\zeta)$  its unstable subspace. These spaces are well-defined according to [Hersh 1963]. The stable subspace  $E_j^s(\zeta)$  has dimension  $p_j$ , whereas  $E_j^u(\zeta)$  has dimension  $N - p_j$ . Let us recall the following theorem due to Kreiss [1970] and generalized by Métivier [2000] for constantly hyperbolic operators:

**Theorem 2.3** (block structure). *Under Assumptions 2.1–2.2, for all  $\underline{\zeta} \in \Xi$ , there exists a neighborhood  $\mathcal{V}$  of  $\underline{\zeta}$  in  $\Xi$ , integers  $L_1, L_2 \geq 1$ , two partitions  $N = v_{1,1} + \dots + v_{1,L_1} = v_{2,1} + \dots + v_{2,L_2}$  with  $v_{j,l}, v_{2,l} \geq 1$ , and two invertible matrices  $T_1, T_2$ , regular on  $\mathcal{V}$  such that for all  $\zeta \in \mathcal{V}$ ,*

$$\begin{aligned} T_1(\zeta)^{-1} \mathcal{A}_1(\zeta) T_1(\zeta) &= \text{diag}(\mathcal{A}_{1,1}(\zeta), \dots, \mathcal{A}_{1,L_1}(\zeta)), \\ T_2(\zeta)^{-1} \mathcal{A}_2(\zeta) T_2(\zeta) &= \text{diag}(\mathcal{A}_{2,1}(\zeta), \dots, \mathcal{A}_{2,L_2}(\zeta)), \end{aligned}$$

where the blocks  $\mathcal{A}_{j,l}(\zeta)$  have size  $v_{j,l}$  and satisfy one of the following alternatives:

- (i) All the elements in the spectrum of  $\mathcal{A}_{j,l}(\underline{\zeta})$  have positive real part.
- (ii) All the elements in the spectrum of  $\mathcal{A}_{j,l}(\underline{\zeta})$  have negative real part.
- (iii)  $v_{j,l} = 1$ ,  $\mathcal{A}_{j,l}(\underline{\zeta}) \in i\mathbb{R}$ ,  $\partial_{\mathcal{V}} \mathcal{A}_{j,l}(\underline{\zeta}) \in \mathbb{R} \setminus \{0\}$ , and  $\mathcal{A}_{j,l}(\zeta) \in i\mathbb{R}$  for all  $\zeta \in \mathcal{V} \cap \Xi_0$ .
- (iv)  $v_{j,l} > 1$ , and there exists  $k_{j,l} \in i\mathbb{R}$  such that

$$\mathcal{A}_{j,l}(\underline{\zeta}) = \begin{bmatrix} k_{j,l} & i & 0 \\ & \ddots & i \\ 0 & & k_{j,l} \end{bmatrix},$$

the coefficient in the lower left corner of  $\partial_{\mathcal{V}} \mathcal{A}_{j,l}(\underline{\zeta})$  is real and nonzero, and moreover,  $\mathcal{A}_{j,l}(\zeta) \in iM_{v_{j,l}}(\mathbb{R})$  for all  $\zeta \in \mathcal{V} \cap \Xi_0$ .

Thanks to this theorem it is possible to describe the four kinds of frequencies, one for each part of the boundary  $\partial\Omega$ :

**Definition 2.4.** For  $j = 1, 2$ , we denote by

- (1)  $\mathcal{E}_j$  the set of elliptic frequencies, that is to say, the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.3 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with one block of type (i) and one block of type (ii) only;
- (2)  $\mathcal{H}_j$  the set of hyperbolic frequencies, that is to say, the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.3 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with blocks of type (iii) only;
- (3)  $\mathcal{E}\mathcal{H}_j$  the set of mixed frequencies, that is to say, the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.3 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with one block of type (i), one of type (ii) and at least one of type (iii), but without a block of type (iv);
- (4)  $\mathcal{G}_j$  the set of glancing frequencies, that is to say, the set of  $\underline{\zeta} \in \Xi_0$  such that Theorem 2.3 for the matrix  $\mathcal{A}_j(\underline{\zeta})$  is satisfied with at least one block of type (iv).

Thus, by definition,  $\Xi_0$  admits the decomposition

$$\Xi_0 = \mathcal{E}_j \cup \mathcal{E}\mathcal{H}_j \cup \mathcal{H}_j \cup \mathcal{G}_j.$$

The study made in [Kreiss 1970; Métivier 2000] shows that the subspaces  $E_1^s(\underline{\zeta})$  and  $E_2^s(\underline{\zeta})$  admit a continuous extension up to  $\Xi_0$ . Moreover, if  $\underline{\zeta} \in \Xi_0 \setminus (\mathcal{G}_1 \cup \mathcal{G}_2)$ , one has the decomposition

$$\mathbb{C}^N = E_1^s(\underline{\zeta}) \oplus E_1^u(\underline{\zeta}) = E_2^s(\underline{\zeta}) \oplus E_2^u(\underline{\zeta}), \tag{3}$$

and for  $j \in \{1, 2\}$ ,

$$\begin{aligned} E_j^s(\underline{\zeta}) &= E_j^{s,e}(\underline{\zeta}) \oplus E_j^{s,h}(\underline{\zeta}), \\ E_j^u(\underline{\zeta}) &= E_j^{u,e}(\underline{\zeta}) \oplus E_j^{u,h}(\underline{\zeta}). \end{aligned}$$

where  $E_j^{s,e}(\underline{\zeta})$  (resp.  $E_j^{u,e}(\underline{\zeta})$ ) is the generalized eigenspace associated with eigenvalues of  $\mathcal{A}_j(\underline{\zeta})$  with negative (resp. positive) real part, and where the spaces  $E_j^{s,h}(\underline{\zeta})$  and  $E_j^{u,h}(\underline{\zeta})$  are sums of eigenspaces of  $\mathcal{A}_j(\underline{\zeta})$  associated with some purely imaginary eigenvalues of  $\mathcal{A}_j(\underline{\zeta})$ . From Assumption 2.2 we also have

$$\mathbb{C}^N = A_1 E_1^s(\underline{\zeta}) \oplus A_1 E_1^u(\underline{\zeta}) = A_2 E_2^s(\underline{\zeta}) \oplus A_2 E_2^u(\underline{\zeta}). \tag{4}$$

In fact, it is possible to give a more precise decomposition of the spaces  $E_j^{s,h}(\underline{\zeta})$  and  $E_j^{u,h}(\underline{\zeta})$ . Indeed, let  $\underline{\omega}_{m,j}$  be a purely imaginary eigenvalue of  $\mathcal{A}_j(\underline{\zeta})$ , that is,

$$\det(\underline{\tau} + \eta A_1 + \underline{\omega}_{m,2} A_2) = \det(\underline{\tau} + \underline{\omega}_{m,1} A_1 + \eta A_2) = 0.$$

Then, using Assumption 2.1, there exists an index  $k_{m,j}$  such that

$$\underline{\tau} + \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) = \underline{\tau} + \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) = 0,$$

where  $\lambda_{k_{m,j}}$  is smooth in both variables. Let us then introduce the following classification:

**Definition 2.5.** The set of incoming (resp. outgoing) phases for the side  $\partial\Omega_1$ , denoted by  $\mathfrak{I}_1$  (resp.  $\mathfrak{D}_1$ ), is the set of indices  $m$  such that the group velocity  $v_m := \nabla \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta})$  satisfies  $\partial_1 \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) > 0$  (resp.  $\partial_1 \lambda_{k_{m,1}}(\underline{\omega}_{m,1}, \underline{\eta}) < 0$ ).

Similarly, the set of incoming (resp. outgoing) phases for the side  $\partial\Omega_2$ , denoted by  $\mathfrak{I}_2$  (resp.  $\mathfrak{D}_2$ ), is the set of indices  $m$  such that the group velocity  $v_m := \nabla \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2})$  satisfies  $\partial_2 \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) > 0$  (resp.  $\partial_2 \lambda_{k_{m,2}}(\underline{\eta}, \underline{\omega}_{m,2}) < 0$ ).

Thanks to this definition, we can write the following decomposition of the stable and unstable components  $E_j^{s,h}(\underline{\zeta})$  and  $E_j^{u,h}(\underline{\zeta})$ :

**Lemma 2.6.** For all  $\underline{\zeta} \in \mathcal{H}_j \cup \mathcal{E}\mathcal{H}_j$ ,  $j = 1, 2$ , we have

$$E_1^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_1} \ker \mathcal{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \quad E_1^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{D}_1} \ker \mathcal{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \tag{5}$$

$$E_2^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{I}_2} \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}), \quad E_2^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathfrak{D}_2} \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}). \tag{6}$$

From [Assumption 2.2](#) we can also write

$$A_1 E_1^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathcal{D}_1} A_1 \ker \mathcal{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \quad A_1 E_1^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathcal{D}_1} A_1 \ker \mathcal{L}(\underline{\tau}, \underline{\omega}_{m,1}, \underline{\eta}), \quad (7)$$

$$A_2 E_2^{s,h}(\underline{\zeta}) = \bigoplus_{m \in \mathcal{D}_2} A_2 \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}), \quad A_2 E_2^{u,h}(\underline{\zeta}) = \bigoplus_{m \in \mathcal{D}_2} A_2 \ker \mathcal{L}(\underline{\tau}, \underline{\eta}, \underline{\omega}_{m,2}). \quad (8)$$

We refer, for example, to [\[Coulombel and Guès 2010\]](#) for a proof of this lemma.

**2B. Known results about strong well-posedness.** We consider the corner problem with source terms in the interior of  $\Omega_T$  and on either side of the boundary  $\partial\Omega_T$  given by

$$\begin{cases} L(\partial)u = f & \text{on } \Omega_T, \\ B_1 u|_{x_1=0} = g_1 & \text{on } \partial\Omega_{1,T}, \\ B_2 u|_{x_2=0} = g_2 & \text{on } \partial\Omega_{2,T}, \\ u|_{t \leq 0} = 0. \end{cases} \quad (9)$$

By strong well-posedness for the corner problem (9) we mean the following:

**Definition 2.7.** The corner problem (9) is said to be strongly well-posed if for  $T > 0$  and for all  $f \in L^2(\Omega_T)$  and  $g_j \in L^2(\partial\Omega_{j,T})$ , the corner problem (9) admits a unique solution  $u \in L^2(\Omega_T)$  with traces in  $L^2(\partial\Omega_{1,T})$  and  $L^2(\Omega_{2,T})$  satisfying the energy estimate

$$\begin{aligned} \|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \\ \leq C_T (\|f\|_{L^2(\Omega_T)}^2 + \|g_1\|_{L^2(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2) \end{aligned} \quad (10)$$

for some constant  $C_T$  depending on  $T$ .

As we have already mentioned in the [Introduction](#), the full characterization of strong well-posedness for the corner problem (9) has not been achieved yet. However, we have some partial results.

First of all, the strong well-posedness is proved in the particular framework of symmetric operators with strictly dissipative boundary conditions, that is, boundary conditions defined as follows:

**Definition 2.8.** For  $j = 1, 2$ , the boundary condition  $B_j u|_{x_j=0} = g_j$  is said to be strictly dissipative if the inequality

$$\langle A_j v, v \rangle < 0 \quad \forall v \in \ker B_j \setminus \{0\}$$

holds and  $\ker B_j$  is maximal (in the sense of inclusion) for this property.

We thus have the following result:

**Theorem 2.9** [\[Benoit 2015, chapitre 4\]](#). *Under [Assumption 2.2](#), if the matrices  $A_1$  and  $A_2$  are symmetric and if the boundary conditions of the corner problem (9) are strictly dissipative, then under a certain algebraic condition on the matrix  $A_1^{-1}A_2$ , the corner problem (9) is strongly well-posed in the sense of [Definition 2.7](#).*



We refer to [Benoit 2015, chapitre 4] for a proof of this result and for more details about the mentioned algebraic condition (see hypothèse 4.1.2 of [Benoit 2015]).<sup>1</sup> For more details about the algebraic condition we also refer to [Métivier and Rauch 2016] in which the necessity of imposing this condition is shown.

It is also easy to show (see [Benoit 2015, paragraphe 5.3.1]) that a necessary condition for (9) to be strongly well-posed is that each initial boundary value problem

$$\begin{cases} L(\partial)u = f, \\ B_j u|_{x_j=0} = g_j, \quad u|_{t \leq 0} = 0, \end{cases} \quad \text{on } \{x_j > 0, x_{3-j} \in \mathbb{R}\} \text{ for } j = 1, 2, \quad (11)$$

is strongly well-posed in the usual sense for initial boundary value problems in the half-space (see, for example, [Benzoni-Gavage and Serre 2007]).

This implies that Theorem 2.9 is not sharp (except for  $N=2$ , thanks to [Strang 1969]) because there exist nonstrictly dissipative boundary conditions leading to a strongly well-posed initial boundary value problem (11) (see, for example, [Benoit 2014, paragraphe 5.3]).

However, the set of the boundary conditions making (11) strongly well-posed has been characterized by [Kreiss 1970] and is composed of the boundary condition satisfying the so-called uniform Kreiss–Lopatinskii condition:

**Definition 2.10.** The initial boundary value problem (11) is said to satisfy the uniform Kreiss–Lopatinskii condition if for all  $\zeta \in \Xi$ , we have

$$\ker B_j \cap E_j^s(\zeta) = \{0\}.$$

So for the corner problem (9) to be strongly well-posed it is necessary that, for  $j = 1, 2$ , the initial boundary value problem (11) satisfies the “uniform” Kreiss–Lopatinskii condition. We thus make the assumption:

**Assumption 2.11.** For all  $\zeta \in \Xi$ , we have

$$\ker B_1 \cap E_1^s(\zeta) = \ker B_2 \cap E_2^s(\zeta) = \{0\}.$$

In particular, the restriction of  $B_1$  (resp.  $B_2$ ) to the stable subspace  $E_1^s(\zeta)$  (resp.  $E_2^s(\zeta)$ ) is invertible, and its inverse is denoted by  $\phi_1(\zeta)$  (resp.  $\phi_2(\zeta)$ ).

Unsurprisingly, the counterexample [Osher 1974a] shows that imposing the uniform Kreiss–Lopatinskii condition on each side of the boundary is not sufficient to ensure that the corner problem (9) is strongly well-posed.

### 3. The phase generation process and examples

Before constructing the geometric optics expansions, it is necessary to describe the expected phases in these expansions. Since the boundary of the domain  $\Omega$  is not flat, we expect that it is possible to generate more phases than for half-space problems. Indeed, at the very first glance, we can think that a ray of

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<sup>1</sup>We do not want to give more details about this condition because it is not used to construct the WKB expansion. Moreover, this condition will be satisfied by all our examples.

geometric optics can be reflected several times on the boundary of the domain, with different new phases generated at each reflection.

It is thus very important in order to postulate an ansatz to be able to describe all the phases that can be obtained by successive reflections on each side of the boundary.

Here, we shall go back to the discussion by Sarason and Smoller [1974] explaining this phenomenon and establishing a very strong link between the geometry of the characteristic variety of  $L(\partial)$  and the phase generation process.

As already mentioned in the [Introduction](#), we are interested here in corner problems which are homogeneous in the interior and on one side of the boundary. The only nonzero source term, which arises in the boundary condition on  $\partial\Omega_1$ , will be highly oscillating, and we want to understand which phases can be induced by this source term. We will here describe the phase generation process when the source term is taken on  $\partial\Omega_1$ ; the arguments are the same for a source term on  $\partial\Omega_2$ .

**3A. Source term induced phases.** Our problem of study is

$$\begin{cases} L(\partial)u^\varepsilon = 0 & \text{on } \Omega_T, \\ B_1u^\varepsilon|_{x_1=0} = g^\varepsilon & \text{on } \partial\Omega_{1,T}, \\ B_2u^\varepsilon|_{x_2=0} = 0 & \text{on } \partial\Omega_{2,T}, \\ u^\varepsilon|_{t \leq 0} = 0, \end{cases} \tag{12}$$

where the source term on  $\partial\Omega_{1,T}$  is given by

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}\varphi(t, x_2)}g(t, x_2), \tag{13}$$

where the amplitude  $g$  belongs to  $H_f^\infty$  and is zero for negative times. The planar phase  $\varphi$  is defined by

$$\varphi(t, x_2) := \underline{\tau}t + \underline{\xi}_2x_2$$

for two fixed real numbers  $\underline{\tau} > 0$  and  $\underline{\xi}_2$ .

The fact that  $g$  belongs to  $H_f^\infty$  implies that  $g^\varepsilon$  is zero at the corner. Assume that  $g$  identically vanishes in a neighborhood of the corner. Then by finite speed of propagation for the half-space problem, we can, at least during a small time interval, see the corner problem (12) as a boundary value problem in the half-space  $\{x_1 \geq 0\}$ .

Geometric optics expansions for boundary value problems in the half-space have already been studied (see, for example, [Williams 1996]) and, going back over the existing analysis, we expect that the source term  $g^\varepsilon$  on the side  $\partial\Omega_1$  induces in the interior of the domain several rays associated with the planar phases

$$\varphi^{0,k}(t, x) := \varphi(t, x_2) + \underline{\xi}_1^{0,k}x_1,$$

where the  $(\underline{\xi}_1^{0,k})_k$  are the roots in the  $\xi_1$ -variable of the dispersion relation

$$\det \mathcal{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0. \tag{14}$$

An important remark to understand the phase generation process is that the  $(\xi_1^{0,k})_k$  are the intersection points (with the convention that complex roots are viewed as intersection points at infinity) between the line  $\{(\tau, \xi_1, \xi_2), \xi_1 \in \mathbb{R}\}$  and the section of the characteristic variety  $V$  at  $\tau = \tau$ .

Let us denote by  $p_r$  the number of real roots of (14) and by  $2p_c$  the number of complex roots (which occur in conjugate pairs). We also assume that  $(\tau, \xi_2)$  is not a glancing frequency for the matrix  $\mathcal{A}_1$ ; hence  $p_r$  can be decomposed as  $p_r^i + p_r^o$ , where  $p_r^i$  (resp.  $p_r^o$ ) is the number of real roots inducing an incoming (resp. outgoing) group velocity (see Definition 2.5). We thus have  $p_1 = p_r^i + p_c$  and  $N - p_1 = p_r^o + p_c$  by Lemma 2.6. Firstly we shall consider  $\varphi_i^0$ , one of the  $p_r^i$  phases with an incoming group velocity, and  $\varphi_o^0$ , one of the  $p_r^o$  phases with an outgoing group velocity. We also denote by  $v_i^0$  and  $v_o^0$  the associated group velocities. Phases associated with complex roots will be dealt with separately. The following discussion should be performed for each such real phase.

We shall study separately the influence of the phases  $\varphi_i^0$  and  $\varphi_o^0$  upon the generation of phases.

**The phase  $\varphi_o^0$ :** The phase  $\varphi_o^0$ , associated with an outgoing group velocity, describes the “past” of the information reflected on the side  $\partial\Omega_1$  at the initial time. In other words, to know the origin of a point on the side  $\partial\Omega_1$ , it is sufficient to travel along the characteristic with group velocity  $v_o^0$  by rewinding time back to  $-\infty$ .

This leads us to separate two cases, making, thus, more precise the Definition 2.5:

**Definition 3.1.** An outgoing group velocity  $v = (v_1, v_2)$  for the side  $\partial\Omega_1$  (i.e.,  $v_1 < 0$ ) is said to be

- outgoing-incoming if  $v_2 > 0$ ,
- outgoing-outgoing if  $v_2 < 0$ .

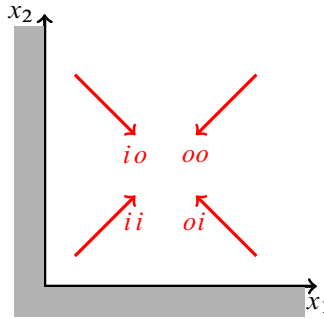
First subcase:  $v_o^0$  outgoing-outgoing. Let us fix a point on the side  $\partial\Omega_1$  and we draw the characteristic line with group velocity  $v_o^0$  passing through this point. Since  $v_o^0$  is outgoing for each side of the boundary, the information at the considered point of  $\partial\Omega_1$  can only come from information in the interior of the domain, which has been transported towards the side  $\partial\Omega_1$ ; see Figure 1. But, without a source term in the interior of  $\Omega$ , such information is zero. As a consequence, the amplitude  $u_o^0$  associated with the phase  $\varphi_o^0$  is zero, since according to Lax’s lemma [1957] it satisfies the transport equation

$$\begin{cases} \partial_t u_o^0 + v_o^0 \cdot \nabla_x u_o^0 = 0, \\ u_o^0|_{t \leq 0} = 0. \end{cases}$$

Outgoing-outgoing phases do not have any influence on the WKB expansion or on the phase generation process and are therefore ignored from now on.

Second subcase:  $v_o^0$  outgoing-incoming. Once again, we fix a point on the side  $\partial\Omega_1$  and we draw the characteristic line with group velocity  $v_o^0$  passing through this point. As in the subcase of an outgoing-outgoing, the information at the considered point of  $\partial\Omega_1$  cannot come from the interior of the domain.

However, the characteristic associated with the group velocity  $v_o^0$  hits the side  $\partial\Omega_2$  when we rewind the time back to  $-\infty$ , so the information at the point of the side  $\partial\Omega_1$  could a priori come from some



**Figure 1.** The four different kinds of phases.

information on the side  $\partial\Omega_2$ , which would have been transported towards the side  $\partial\Omega_1$ . But this is not possible at time  $t = 0$  since the boundary condition on  $\partial\Omega_2$  is homogeneous for negative times.

So, the amplitude associated with the outgoing-incoming phase  $\varphi_o^0$  is zero at time  $t = 0$  and even on a small time interval if  $g^\varepsilon$  identically vanishes near the corner. That is why we do not take into account the phase  $\varphi_o^0$  initially in the phase generation process.

Let us stress here that the phase  $\varphi_o^0$  is moved apart a priori only at time  $t = 0$ . Indeed, for some configuration of the characteristic variety, this phase can be generated at a future reflection on the side  $\partial\Omega_2$ , and will finally be included in the ansatz. We will make more comments on this point in [Section 3C](#), after having precisely described which reflections are taken into account.

**The phase  $\varphi_i^0$ :** The phase  $\varphi_i^0$  is associated with an incoming group velocity for the side  $\partial\Omega_1$ . Opposite to the phase  $\varphi_o^0$ , it describes the “future” of the source term  $g^\varepsilon$ . That is to say, when time goes to  $+\infty$ , part of the oscillations in  $g^\varepsilon$  is transported along the characteristic with group velocity  $v_i^0$ . So, the phase  $\varphi_i^0$  carries a nonzero information and has to be taken into account in the phase generation process.

However, once again, we have to separate two subcases, according to the following refinement of the [Definition 2.5](#):

**Definition 3.2.** An incoming group velocity  $v = (v_1, v_2)$  for the side  $\partial\Omega_1$  (i.e.,  $v_1 > 0$ ) is said to be

- incoming-incoming if  $v_2 > 0$ ,
- incoming-outgoing if  $v_2 < 0$ .

The four kinds of (nonglancing) oscillating phases used in this analysis are drawn in [Figure 1](#).

First subcase:  $v_i^0$  incoming-incoming. We choose a point  $(0, \underline{x}_2)$ , on  $\partial\Omega_1$  such that  $g^\varepsilon(0, \underline{x}_2)$  is nonzero and we draw the characteristic with velocity  $v_i^0$  passing through this point. When  $t$  goes to  $+\infty$ , the information transported along this ray will never hit the side  $\partial\Omega_2$  and will be unable to generate new phases by reflection. So, when the group velocity  $v_i^0$  is incoming-incoming, the phase generation process for the phase  $\varphi_i^0$  stops.

Second subcase:  $v_i^0$  incoming-outgoing. We fix a point  $(0, \underline{x}_2) \in \partial\Omega_1$  with  $g^\varepsilon(0, \underline{x}_2) \neq 0$ , and we draw the characteristic with velocity  $v_i^0$  passing through this point. As  $v_{i,2}^0$  is negative, this ray will hit, after

a while, the side  $\partial\Omega_2$ . We thus expect that this ray will give rise to reflected oscillations and that this reflection will create new phases. This reflection phenomenon and more specifically the new expected phases will be described in the next section. But before that, we will conclude the discussion on the phases induced by the source term  $g^\varepsilon$  by considering the possible complex-valued phases.

$\varphi_k^0$  **complex-valued:** We now deal with phases corresponding to roots in the  $\xi_1$ -variable to the dispersion relation (14) with nonzero imaginary part. Let us first introduce some vocabulary.

**Definition 3.3.** A phase  $\varphi_k^0$  with  $\underline{\xi}_1^{0,k} \in \mathbb{C} \setminus \mathbb{R}$  is said to be

- evanescent for the side  $\partial\Omega_1$  if  $\text{Im } \underline{\xi}_1^{0,k} > 0$ ,
- explosive for the side  $\partial\Omega_1$  if  $\text{Im } \underline{\xi}_1^{0,k} < 0$ .

Thanks to the construction of geometric optics expansion for complex-valued phases made, for example, in [Marcou 2010; Lescarret 2007], the expected behavior of the amplitudes associated with these phases is a propagation of information in the normal direction to the side  $\partial\Omega_1$ . However, this propagation is exponentially decreasing (resp. increasing) according to the variable  $x_1$  for the amplitudes linked with evanescent (resp. explosive) phases. In all that follows, as we are looking for amplitudes in  $L^2(\Omega)$  so as in [Lescarret 2007; Marcou 2010; Williams 1996] we do not take into account explosive phases.

Thus, we only keep the evanescent phases. Since, for regularity considerations on the oscillating amplitudes, we are working with a source term in  $H_f^\infty$ , this source term satisfies, in particular,  $g(t, 0) = 0$ . Consequently, the information carried by evanescent phases will never hit the side  $\partial\Omega_2$  and the evanescent phases for the side  $\partial\Omega_1$  are, as well as the incoming-incoming ones, stopping conditions in the phase generation process.

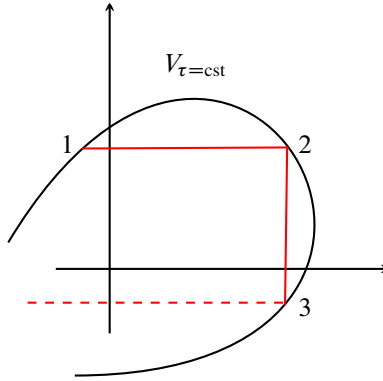
To summarize, the phases induced directly by the source term  $g^\varepsilon$  are the incoming (for the side  $\partial\Omega_1$ ) phases and the evanescent phases for the side  $\partial\Omega_1$ . Incoming-incoming and evanescent phases will not be reflected; thus we only have to study the reflections on  $\partial\Omega_2$  associated with incoming-outgoing phases.

**3B. The first reflection.** We assume that the dispersion relation (14) has at least one solution in the  $\xi_1$ -variable generating an incoming-outgoing group velocity. We shall describe the reflection of one of these phases. Of course, to determine all the expected phases in the WKB expansion, the following discussion has to be repeated for each of these phases.

Let  $\underline{\xi}_1^0$  be a fixed root in the  $\xi_1$ -variable to (14). We denote by  $v_i^0$  the associated incoming-outgoing group velocity which corresponds to rays emanating from  $\partial\Omega_1$  and hitting  $\partial\Omega_2$  in finite time. Let us also assume that time is large enough so that the ray associated with  $v_i^0$  and emanating from the support of  $g^\varepsilon$  has hit  $\partial\Omega_2$ . Once again, by (formal) finite speed of propagation arguments, the reflection of the ray can not hit immediately the side  $\partial\Omega_1$ . Thus during a small time, we can represent our situation as an initial boundary value problem in the half-space  $\{x_2 \geq 0\}$  whose boundary source term has been turned on by the amplitude for the outgoing (for the side  $\partial\Omega_2$ ) phase  $\varphi_i^0$ .

We thus have to determine the roots  $(\underline{\xi}_2^{1,k})_k$  in the  $\xi_2$ -variable of the dispersion relation

$$\det \mathcal{L}(\underline{\tau}, \underline{\xi}_1^0, \underline{\xi}_2) = 0. \tag{15}$$



**Figure 2.** The geometry of the characteristic variety and the phase generation.

Let us stress that we already know one of them, that is,  $\underline{\xi}_2$ . For each of the new roots, we associate to it the phase

$$\varphi^{1,k}(t, x) := \underline{\tau}t + \underline{\xi}_1^0 x_1 + \underline{\xi}_2^{1,k} x_2.$$

It is interesting to note that the  $(\underline{\xi}_2^{1,k})_k$  are the intersection points between  $V \cap \{\tau = \underline{\tau}\}$  and the line  $\{(\underline{\tau}, \underline{\xi}_1^0, \underline{\xi}_2), \underline{\xi}_2 \in \mathbb{R}\}$ . Thus to determine the phases generated by the source term, it is necessary to consider the intersection of  $V \cap \{\tau = \underline{\tau}\}$  with a horizontal line, and to determine the phases generated by the first reflection, we have to consider the intersection of  $V \cap \{\tau = \underline{\tau}\}$  with a vertical line (see Figure 2). To determine the phases generated by the second reflection, we will have to consider the intersection with a horizontal line and so on. We see that this process strongly depends on the geometry of the characteristic variety  $V$ .

Repeating exactly the same arguments as those used for the phases induced by the source term, we claim that outgoing-outgoing and incoming-outgoing phases can be neglected (at least initially for incoming-outgoing phases). Consequently, for real roots of (15), we just have to consider those associated with an incoming-incoming or outgoing-incoming group velocity. Let  $\varphi_i^1$  denote one of these phases and  $v_i^1$  its group velocity.

**$v_i^1$  incoming-incoming:** In that case, as when the group velocity  $v_i^0$  was incoming-incoming, the considered ray will never hit the side  $\partial\Omega_1$ , and it will never be reflected. The phase generation process for the phase  $\varphi_i^1$  stops, and we are free to study the reflection(s) of another root of (15).

**$v_i^1$  outgoing-incoming:** The reflected ray travels towards  $\partial\Omega_1$ , it will hit  $\partial\Omega_1$  after a while, and we will have to determine how it is reflected back. So the phase generation process for the phase  $\varphi_i^1$  continues.

Concerning complex roots of (15) (if such roots exist), we only add in the WKB expansion those associated with evanescent phases for the side  $\partial\Omega_2$  (that is to say, those satisfying  $\text{Im } \underline{\xi}_2^{1,k} > 0$ ). As for the complex-valued phases induced by the source term, they will never be reflected back, and the phase generation process for these phases stops.

**3C. Summary.** To summarize, the phase generation process is the following: We start from a source term on  $\partial\Omega_1$  and we only study the reflections for the incoming phases that it induces. If all of the

phases are incoming-incoming (or evanescent), then the process stops. Otherwise, we determine the reflections on  $\partial\Omega_2$  of all the incoming-outgoing phases and we shall consider them into the ansatz. If one of these reflected phases is outgoing-incoming, we will determine its reflection on  $\partial\Omega_1$ , otherwise the phase generation process stops. This leads us to consider sequences of phases which are alternatively incoming-outgoing and outgoing-incoming until we find an incoming-incoming or evanescent phase during a reflection, which ends the sequence.

There are, of course, two possibilities: either each of these sequences of phases generated by successive reflections is finite, and then the number of phases in the ansatz will be finite (see the example of [Section 3E](#)), or at least one of these sequences is infinite, and then the number of phases in the ansatz is infinite (see [Section 3D](#)).

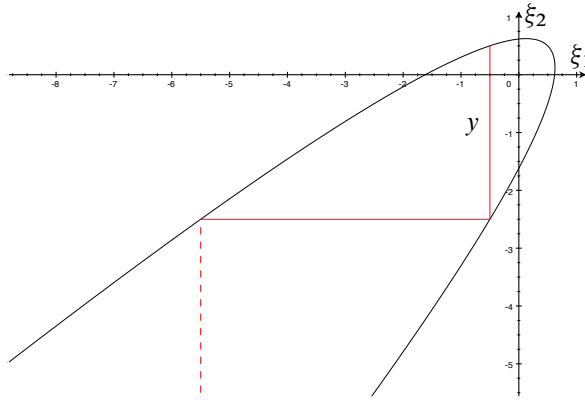
In all the preceding discussion, we used the tacit assumption that we never meet glancing phases. This assumption is satisfied in all our examples and it will be clearly stated in [Theorem 4.27](#). Formally glancing phases should be stopping criterion as well as incoming-incoming and evanescent phases. However, how to include rigorously glancing modes in the WKB expansion is left for future studies.

Let us also stress that during a reflection on the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ), the fact that outgoing-incoming (resp. incoming-outgoing) phases are not considered does not prevent these phases from appearing in the WKB expansion.

Indeed, let  $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$  be an incoming-outgoing phase generated by the source term  $g^\varepsilon$  and  $(\underline{\tau}, \underline{\xi}_1, \tilde{\xi}_2)$  be an outgoing-incoming phase also generated by the source term. This phase is a priori not taken into account in the WKB expansion at the first step of the phase generation process described above. Let us assume that the intersection between the characteristic variety  $V \cap \{\tau = \underline{\tau}\}$  and the line  $\{(\underline{\tau}, \underline{\xi}_1, \xi_2), \xi_2 \in \mathbb{R}\}$  contains a value of  $\xi_2$ , say  $\tilde{\xi}_2$ , such that the associated oscillating phase is outgoing-incoming and that the intersection between  $V \cap \{\tau = \underline{\tau}\}$  and the line  $\{(\underline{\tau}, \xi_1, \tilde{\xi}_2), \xi_1 \in \mathbb{R}\}$  contains the frequency  $(\underline{\tau}, \tilde{\xi}_1, \tilde{\xi}_2)$  (in other words, it is equivalent to say that there exists a rectangle with sides parallel to the  $x$ - and  $y$ -axes whose corners are four points of  $V \cap \{\tau = \underline{\tau}\}$ ). If the frequency  $(\underline{\tau}, \tilde{\xi}_1, \tilde{\xi}_2)$  is associated with an incoming-outgoing group velocity, we remark that by applying the phase generation process (more precisely during the third reflection), we have to consider the frequency  $(\underline{\tau}, \underline{\xi}_1, \tilde{\xi}_2)$ , which has been initially excluded.

Moreover, when we study the reflections of the phase associated with the frequency  $(\underline{\tau}, \tilde{\xi}_1, \tilde{\xi}_2)$  on the side  $\partial\Omega_1$ , we are led to consider one more time the phase with frequency  $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$ . So, the phase associated with the frequency  $(\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$  is “self-generating” or “self-interacting” because it is in the set of the phases that it generates. Such a configuration in the characteristic variety will be called a “loop”. An explicit example of a corner problem with a loop will be given in [Section 3E](#).

The fact that at each reflection there is more than one generated phase and this self-interaction phenomenon between the phases imply that there is no natural order on the set of phases as in the  $N = 2$  framework. Indeed, when  $N > 2$  we have to deal with a tree matching the phase generation at each reflection. Thus, constructing the WKB expansion when  $N > 2$  will be less intuitive than when  $N = 2$ , a framework in which it is sufficient to use the order induced by the phase generation process. In [Sections 4B1](#) and [4B4](#), we show how to overcome this lack of natural order in view of constructing the WKB expansion.



**Figure 3.** Phase generation for the corner problem of Section 3D.

**3D. An example with infinitely many phases.** The aim of this section is to illustrate the phase generation process and to give an explicit example of a corner problem whose geometric optics expansion contains an infinite number of phases. Moreover, this example will also stress the fact that the phase generation process is much simpler when  $N = 2$ , since it gives a natural order of construction of the WKB expansion.

Let us consider the corner problem (12) with

$$A_1 := \begin{bmatrix} 0 & -1 \\ -1 & -1 \end{bmatrix}, \quad A_2 := \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}.$$

It is thus clear that  $p_1 = p_2 = 1$ ; then we have to choose  $B_1, B_2 \in M_{1,2}(\mathbb{R})$  in such a way that the boundary conditions on  $\partial\Omega_1$  and  $\partial\Omega_2$  are strictly dissipative [Strang 1969]. Moreover, one can easily check that this corner problem satisfies Assumptions 2.1 and 2.2.

We choose

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(t + \frac{1}{2}x_2)} g(t, x)$$

for the source term on  $\partial\Omega_1$  in the corner problem (12). Then the phase generation process for this problem is precisely described in [Benoit 2015, paragraphe 6.6.1] and is illustrated in Figure 3. The phases that we have to consider form a “stairway” in a parabola (see Figure 3). The points of this stairway are labeled by two sequences  $(\xi_{1,p})_{p \in \mathbb{N}}$  and  $(\xi_{2,p})_{p \in \mathbb{N}}$  in such a way that points  $(\xi_{1,p}, \xi_{2,p})_{p \in \mathbb{N}}$  match with points in the “top of the parabola”, whereas points  $(\xi_{1,p}, \xi_{2,p+1})_{p \in \mathbb{N}}$  match with points in the “bottom of the parabola”. Finally we initialize at  $\xi_{1,0} = -\frac{1}{2}$  and  $\xi_{2,0} = \frac{1}{2}$ . A simple computation shows that we have

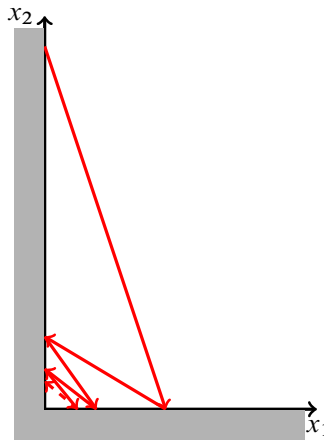
$$\xi_{1,p} = -2p^2 - 3p - \frac{1}{2}, \quad \xi_{2,p} = -2p^2 - p + \frac{1}{2},$$

and

$$v_p = \frac{1}{4p^2 + 4p + 2} \begin{bmatrix} 4p + 1 \\ -(4p + 3) \end{bmatrix}, \quad w_p = \frac{1}{4p^2 + 8p + 5} \begin{bmatrix} -(4p + 5) \\ 4p + 3 \end{bmatrix}.$$

So all the points of the “top” are associated with incoming-outgoing group velocities, while points of the “bottom” are associated with outgoing-incoming group velocities. Thus according to the phase generation process described above, the number of phases in the expansion will be infinite.





**Figure 4.** Appearance of the characteristics for problem of Section 3D.

We refer to [Benoit 2015, paragraphs 6.6.2 and 6.6.3] for a rigorous construction of the geometric optics expansion and a justification of its convergence towards the exact solution. The difficult part of this analysis does not come from the construction because we are in the simpler case  $N = 2$  but it comes from the justification. Indeed, when infinitely many phases occur, to ensure that the WKB expansion (at a finite order in terms of powers of  $\varepsilon$ ) makes sense, we have to ensure that a series converges.

Finally, we address the following phenomenon. If we fix a point  $(0, y_p) \in \partial\Omega_1$  and follow the characteristic with group velocity  $v_p$ , then we will hit  $\partial\Omega_2$  at time  $t_p^v$  in a point  $(x_p, 0)$ . Then if we start from  $(x_p, 0) \in \partial\Omega_2$  and follow the characteristic line with group velocity  $w_p$ , we will hit  $\partial\Omega_1$  at time  $t_p^w$  in a point  $(y_{p+1}, 0)$ . A simple computation shows that the considered sequences are given by

$$y_p = \frac{1}{4p+1}y_0, \quad x_p = \frac{1}{4p+3}y_0, \quad t_p^v = \frac{1}{v_{p,1}}y_p, \quad t_p^w = \frac{1}{v_{p,2}}x_p,$$

from which we deduce that from the starting point  $(0, y_0)$ ,  $y_0 > 0$ , we will get closer and closer to the corner at each reflection and will reach the corner in an infinite time. A scheme illustrating the characteristic lines for this corner problem is given in Figure 4.

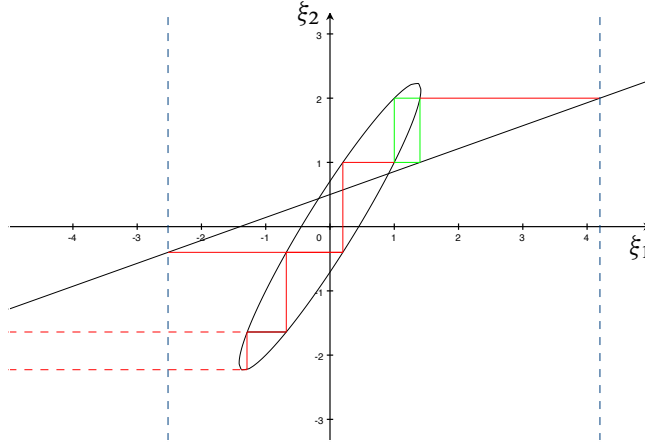
**3E. An example with a loop.** We consider the corner problem

$$\begin{cases} \partial_t u^\varepsilon + A_1 \partial_1 u^\varepsilon + A_2 \partial_2 u^\varepsilon = 0, \\ B_1 u^\varepsilon|_{x_1=0} = 0, \quad B_2 u^\varepsilon|_{x_2=0} = g^\varepsilon, \quad u^\varepsilon|_{t \leq 0} = 0, \end{cases} \quad (x_1, x_2) \in \Omega, \quad (16)$$

with

$$A_1 := \begin{bmatrix} 3/\sqrt{2} & 1/\sqrt{2} & 0 \\ 1/\sqrt{2} & -3/\sqrt{2} & 0 \\ 0 & 0 & 5/7 \end{bmatrix}, \quad A_2 := \begin{bmatrix} -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

This system does not have any physical meaning and is composed of a “wave type” equation and a scalar transport equation. It is clear that the corner problem (16) satisfies Assumption 2.2 with  $p_1 = 2$



**Figure 5.** Section of the characteristic variety and the phase generation for corner problem (16).

and  $p_2 = 1$ . The corner problem (16) does not satisfy Assumption 2.1, but it is hyperbolic in the sense of geometrically regular hyperbolic systems (see [Métivier and Zumbrun 2005, Definition 2.2]). This hyperbolicity assumption is sufficient for our discussion as long as we do not have to consider the ansatz frequencies corresponding to intersection points<sup>2</sup> of the different sheets of the characteristic variety.

For the corner problem (16), the equation of the section of the characteristic variety with the plane  $\{\tau = 1\}$  is given by

$$V_{\tau=1} : (5\xi_1^2 + 2\xi_2^2 - 6\xi_1\xi_2 + 1)(1 + \frac{5}{7}\xi_1 - 2\xi_2) = 0,$$

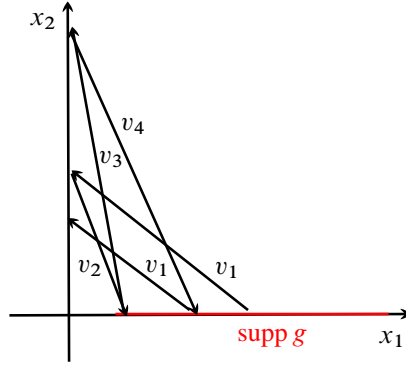
and is composed of an ellipse and a crossing line; see Figure 5. If we choose

$$g^\varepsilon(t, x_1) = e^{\frac{i}{\varepsilon}(t + \frac{7}{5}x_1)} g(t, x_1)$$

for the source term  $g^\varepsilon$  on  $\partial\Omega_2$  in (16) then after the application of the phase generation process (see Figure 5), we obtain a loop as introduced in Section 3C. The four self-interacting phases and the associated group velocities are given by

$$\begin{aligned} \varphi_1(t, x) &:= t + \frac{7}{5}x_1 + 2x_2, & v_1 &= \begin{bmatrix} -1 \\ 1/5 \end{bmatrix}, & \varphi_2(t, x) &:= t + \frac{7}{5}x_1 + x_2, & v_2 &= \begin{bmatrix} 5/7 \\ -2 \end{bmatrix}, \\ \varphi_3(t, x) &:= t + x_1 + x_2, & v_3 &= \begin{bmatrix} -2 \\ 1 \end{bmatrix}, & \varphi_4(t, x) &:= t + x_1 + 2x_2, & v_4 &= \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \end{aligned} \tag{17}$$

<sup>2</sup>We have to stress that these intersection points, specific to geometrically regular hyperbolic systems, can, generically, induce an infinite number of phases in the WKB expansion. Indeed, let us assume that in a given intersection point, one of the sheets of the variety is associated with an incoming-outgoing group velocity, whereas the other sheet is associated with an outgoing-incoming group velocity. Then, using the fact that the group velocities are regular, one can find a neighborhood on each sheet such that the group velocity does not change type on this neighborhood. It immediately follows that if a ray of the geometric optics expansion contains a frequency in these neighborhoods, it is automatically attracted toward the intersection point by forming a “stairway”, like in Section 3D. The fact that this phenomenon does not occur for the corner problem (16), and that the number of generated phases in finite, is somewhat very special.



**Figure 6.** The loop.

where  $v_1$  and  $v_3$  are outgoing-incoming, whereas  $v_2$  and  $v_4$  are incoming-outgoing. The precise values of the *fifteen other expected phases* in the WKB expansion can be explicitly computable but they are not really important for our actual discussion. Let us just stress that, as it can be seen in [Figure 5](#), there are four evanescent phases for the side  $\partial\Omega_2$  but there are no oscillating phases of the form  $(1, \xi_1, 0)$ . Thus, in particular, the technical [Assumption 4.9](#), used in the construction of the WKB expansion will be satisfied.

As in [Section 3D](#) we are not interested in the construction of the geometric optics expansion but we want to study the behavior of the rays associated with the phases  $(\varphi_j)_{j=1,\dots,4}$  when  $T$  is large.

If we start from a point  $(x_0, 0) \in \partial\Omega_2$  and make it travel along the characteristics with group velocity  $v_1, v_2, v_3$  and  $v_4$ , then after one cycle the ray will hit  $\partial\Omega_2$  after a time of travel  $t_0$  in a point  $(x_2, 0)$ . Some computations, like those made in [Section 3D](#), show that, for  $x_0 > 0$ , we have

$$x_{2p} = \beta^{-p} x_0, \quad t_p = \tilde{\alpha} \beta^{-p} x_0,$$

with

$$\beta^{-1} := \frac{1}{28} = \frac{v_{1,2}}{v_{1,1}} \frac{v_{2,1}}{v_{2,2}} \frac{v_{3,2}}{v_{3,1}} \frac{v_{4,1}}{v_{4,2}},$$

and  $\tilde{\alpha}$  a nonrelevant parameter for our purpose. Since  $\beta > 1$ , the ray concentrates at the corner. Moreover the total time of travel towards the corner  $\sum_{p \geq 0} t_p$  is the sum of a finite geometric sum so the ray reaches the corner in finite time.

We will come back in [Section 4D](#) to this example, and more precisely to the resolution of the new amplitude equation needed to construct the geometric optics expansion. Let us conclude this section by noting that [Figure 6](#) depicts the characteristics associated with the group velocities  $(v_j)_{j=1,\dots,4}$ .

#### 4. Geometric optics expansions for self-interacted trapped rays

Until the end of this paper, we will study the following hyperbolic corner problem with  $N$  equations:

$$\begin{cases} \partial_t u^\varepsilon + A_1 \partial_1 u^\varepsilon + A_2 \partial_2 u^\varepsilon = 0, \\ B_1 u^\varepsilon|_{x_1=0} = g^\varepsilon, \quad B_2 u^\varepsilon|_{x_2=0} = 0, \quad u^\varepsilon|_{t \leq 0} = 0, \end{cases} \quad (t, x_1, x_2) \in \Omega_T, \quad (18)$$

where we recall that  $A_1, A_2 \in M_N(\mathbb{R})$  with  $N \geq 2$ ,  $B_1 \in M_{p_1, N}(\mathbb{R})$  and  $B_2 \in M_{p_2, N}(\mathbb{R})$ . Our goal is to construct the WKB approximation to the solution  $u^\varepsilon$  to (18) when self-interacting phases occur. But before starting the construction of the geometric optics expansion, we shall give a precise and rigorous meaning of the phase generation process described in Section 3. This is the object of the following section.

**4A. General framework.** In this section we define a general framework wherein we can construct rigorously geometric optics expansions for corner problems. As already mentioned, the geometry of the characteristic variety influences the phase generation process and consequently it also influences the geometric optics expansion. Though not the most general, our framework will be general enough to take into account one-loop and self-interacting phases. Possible extensions are indicated at the end of this article.

**4A1. Definition of the frequency set and first properties.** Let us start with the definition of what we mean by a frequency set:

**Definition 4.1.** Let  $\mathcal{I}$  be a subset of  $\mathbb{N}$  and  $\underline{\tau} \in \mathbb{R}, \underline{\tau} \neq 0$ . A set indexed by  $\mathcal{I}$ ,

$$\mathcal{F} := \{f_i := (\underline{\tau}, \xi_1^i, \xi_2^i), i \in \mathcal{I}\},$$

will be a set of frequencies for the corner problem (18) if for all  $i \in \mathcal{I}$ , the frequency  $f_i$  satisfies

$$\det \mathcal{L}(f_i) = 0$$

and one of the following alternatives:

- (i)  $\xi_1^i, \xi_2^i \in \mathbb{R}$ .
- (ii)  $\xi_1^i \in (\mathbb{C} \setminus \mathbb{R}), \xi_2^i \in \mathbb{R}$  and  $\text{Im } \xi_1^i > 0$ .
- (iii)  $\xi_2^i \in (\mathbb{C} \setminus \mathbb{R}), \xi_1^i \in \mathbb{R}$  and  $\text{Im } \xi_2^i > 0$ .

In all that follows, if  $\mathcal{F}$  is a frequency set for the corner problem (18), we will define

$$\begin{aligned} \mathcal{F}_{\text{os}} &:= \{f_i \in \mathcal{F} \text{ satisfying (i)}\}, \\ \mathcal{F}_{\text{ev1}} &:= \{f_i \in \mathcal{F} \text{ satisfying (ii)}\}, \\ \mathcal{F}_{\text{ev2}} &:= \{f_i \in \mathcal{F} \text{ satisfying (iii)}\}. \end{aligned}$$

It is clear that the sets  $\mathcal{F}_{\text{os}}, \mathcal{F}_{\text{ev1}}$  and  $\mathcal{F}_{\text{ev2}}$  give a partition of  $\mathcal{F}$ . Moreover, to each  $f_i \in \mathcal{F}_{\text{os}}$ , we can associate a group velocity  $v_i := (v_{i,1}, v_{i,2})$ . Let us recall that the group velocity  $v_i$  is defined in Definition 2.5. The set  $\mathcal{F}_{\text{os}}$  can be decomposed as

$$\begin{aligned} \mathcal{F}_{\text{ii}} &:= \{f_i \in \mathcal{F}_{\text{os}} \mid v_{i,1}, v_{i,2} > 0\}, & \mathcal{F}_{\text{io}} &:= \{f_i \in \mathcal{F}_{\text{os}} \mid v_{i,1} > 0, v_{i,2} < 0\}, \\ \mathcal{F}_{\text{oi}} &:= \{f_i \in \mathcal{F}_{\text{os}} \mid v_{i,1} < 0, v_{i,2} > 0\}, & \mathcal{F}_{\text{oo}} &:= \{f_i \in \mathcal{F}_{\text{os}} \mid v_{i,1} < 0, v_{i,2} < 0\}, \\ \mathcal{F}_{\text{g}} &:= \{f_i \in \mathcal{F}_{\text{os}} \mid v_{i,1} = 0 \text{ or } v_{i,2} = 0\}, \end{aligned}$$

The partition of  $\mathcal{F}$  induces the following partition of  $\mathcal{I}$ :

$$\mathcal{I} = \mathcal{I}_{\text{g}} \cup \mathcal{I}_{\text{oo}} \cup \mathcal{I}_{\text{io}} \cup \mathcal{I}_{\text{oi}} \cup \mathcal{I}_{\text{ii}} \cup \mathcal{I}_{\text{ev1}} \cup \mathcal{I}_{\text{ev2}},$$

where we have denoted by  $\mathcal{I}_{\text{io}}$  (resp.  $\text{g, oo, oi, ii, ev1, ev2}$ ) the set of indices  $i \in \mathcal{I}$  such that the corresponding frequency  $f_i \in \mathcal{F}_{\text{io}}$  (resp.  $\text{g, oo, oi, ii, ev1, ev2}$ ).

From now on, the source term  $g^\varepsilon$  on the boundary in (18) is given by

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(\underline{\tau}t + \underline{\xi}_2 x_2)} g(t, x_2), \tag{19}$$

where the amplitude  $g$  belongs to  $H_f^\infty$  and is zero for negative times.

The following definition gives a precise framework for the phase generation process described in Section 3. More precisely, this definition qualifies the frequency set that contains all (and only) the frequencies linked with the expected nonzero amplitudes in the WKB expansion of the solution to the corner problem (18).

**Definition 4.2.** The corner problem (18) is said to be complete for reflections if there exists a set of frequencies  $\mathcal{F}$  satisfying the following properties:

- (i)  $\mathcal{F}$  contains the real roots (in the variable  $\xi_1$ ) associated with incoming-outgoing or incoming-incoming group velocities and the complex roots with positive imaginary part to the dispersion equation

$$\det \mathcal{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0.$$

- (ii)  $\mathcal{F}_g = \emptyset$ .<sup>3</sup>

- (iii) If  $(\underline{\tau}, \xi_1^i, \xi_2^i) \in \mathcal{F}_{io}$ , then  $\mathcal{F}$  contains all the roots (in the variable  $\xi_2$ ), denoted by  $\xi_2^P$ , of the dispersion relation  $\det \mathcal{L}(\underline{\tau}, \xi_1^i, \xi_2) = 0$  that satisfy one of the following two alternatives:

- (iii')  $\xi_2^P \in \mathbb{R}$  and the frequency  $(\underline{\tau}, \xi_1^i, \xi_2^P)$  is associated with an outgoing-incoming group velocity or an incoming-incoming group velocity.

- (iii'')  $\text{Im } \xi_2^P > 0$ .

- (iv) If  $(\underline{\tau}, \xi_1^i, \xi_2^i) \in \mathcal{F}_{oi}$ , then  $\mathcal{F}$  contains all the roots (in the variable  $\xi_1$ ), denoted by  $\xi_1^P$ , of the dispersion relation  $\det \mathcal{L}(\underline{\tau}, \xi_1, \xi_2^i) = 0$  that satisfy one of the following two alternatives:

- (iv')  $\xi_1^P \in \mathbb{R}$  and the frequency  $(\underline{\tau}, \xi_1^P, \xi_2^i)$  is associated with an incoming-outgoing or an incoming-incoming group velocity.

- (iv'')  $\text{Im } \xi_1^P > 0$ .

- (v)  $\mathcal{F}$  is minimal (for the inclusion) for the four preceding properties.

**Remark.** Property (i) establishes that the frequency set  $\mathcal{F}$  contains all the incoming phases for  $\partial\Omega_1$  that are induced by the source term  $g^\varepsilon$ .

Property (iii) (resp. (iv)) explains the generation by reflection on the side  $\partial\Omega_2$  (resp.  $\partial\Omega_1$ ) of a wave packet that emanates from the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ).

An immediate consequence of the minimality of  $\mathcal{F}$  is that  $\mathcal{F}_{oo}$  is empty. In all that follows, we will assume that the dispersion relation  $\det \mathcal{L}(\underline{\tau}, \xi_1, \underline{\xi}_2) = 0$  has at least one real solution  $\underline{\xi}_1$  such that the group velocity for the frequency  $\underline{f} := (\underline{\tau}, \underline{\xi}_1, \underline{\xi}_2)$  is incoming-outgoing. This assumption is, of course, not necessary. However, without this assumption, it is easy to see that the phase generation for the corner

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<sup>3</sup> This restriction is probably not necessary. However, for a first work on this subject we did not want to add the technicality induced by the determination of amplitudes associated with glancing frequencies (see [Williams 2000] for such a construction). Incorporating glancing modes in the WKB expansion is left for future studies.

problem (18) is not richer than the phase generation for the standard boundary value problem in the half-space  $\{x_1 \geq 0\}$ . Indeed, the minimality of the frequency set  $\mathcal{F}$  would imply, in this case,

$$\mathcal{F} = \mathcal{F}_{\text{ii}} \cup \mathcal{F}_{\text{ev1}} \quad \text{and} \quad \forall f_i \in \mathcal{F}, \quad \xi_2^i = \xi_2.$$

For a corner problem that is complete for reflections, one can define the following functions. These functions are defined on the index set  $\mathcal{I}$  and give, in the output, the indices “in the direct vicinity” of the input index:

$$\Phi, \Psi : \mathcal{I} \rightarrow \mathcal{P}_N(\mathcal{I}),$$

where  $\mathcal{P}_N(\mathcal{I})$  denotes the power set of  $\mathcal{I}$  with at most  $N$  elements. More precisely, for  $i \in \mathcal{I}$  and  $f_i = (\underline{\tau}, \xi_1^i, \xi_2^i)$ ,

$$\Phi(i) := \{j \in \mathcal{I} \mid \xi_2^j = \xi_2^i\} \quad \text{and} \quad \Psi(i) := \{j \in \mathcal{I} \mid \xi_1^j = \xi_1^i\}.$$

Thanks to these functions, the index set  $\mathcal{I}$  can be seen as a graph. This graph structure will be more abstract than the description of  $\mathcal{I}$  based on the wave packet reflections, but it will be easier to work with when we will construct the WKB expansion. This graph structure is defined by the following relation: two points  $i, j \in \mathcal{I}$  are linked by an edge if and only if  $i \in \Phi(j)$  or  $i \in \Psi(j)$ .

In terms of wave packet reflection, the set  $\Phi(i)$  (resp.  $\Psi(i)$ ) is the set of all indices of the phases that are considered in the reflection of the wave packet with phase associated to  $f_i$  on  $\partial\Omega_2$ . Let us stress that the index  $i$  is not necessarily the index of an incident ray but can be the index of one of the reflected rays.

It is easy to see that functions  $\Phi$  and  $\Psi$  have the following properties. One can also check that these properties are independent of the concept of “loop” that will be introduced in the following section.

**Proposition 4.3.** *If the corner problem (18) is complete for reflections, then  $\Phi$  and  $\Psi$  satisfy the following properties:*

- (i)  $\forall i \in \mathcal{I}$ , we have  $i \in \Psi(i)$  and  $i \in \Phi(i)$ .
- (ii)  $\forall i \in \mathcal{I}, \forall j \in \Psi(i), \forall k \in \Phi(i)$ , we have  $\Psi(i) = \Psi(j)$  and  $\Phi(i) = \Phi(k)$ .
- (iii)  $\forall i \in \mathcal{I}$ , we have  $\Phi(i) \cap \mathcal{I}_{\text{ev2}} = \emptyset$  and  $\Psi(i) \cap \mathcal{I}_{\text{ev1}} = \emptyset$ , and,  $\forall i \in \mathcal{I}_{\text{ev1}}, \forall j \in \mathcal{I}_{\text{ev2}}$ , we have  $\Psi(i) \subset \mathcal{I}_{\text{ev1}}$  and  $\Phi(i) \subset \mathcal{I}_{\text{ev2}}$ .
- (iv)  $\forall i \in \mathcal{I}_{\text{os}}$ , we have  $\#\{\Phi(i) \cap \mathcal{I}_{\text{ev1}} \cap \mathcal{I}_{\text{io}} \cap \mathcal{I}_{\text{ii}}\} \leq p_1$  and  $\#\{\Psi(i) \cap \mathcal{I}_{\text{ev2}} \cap \mathcal{I}_{\text{oi}} \cap \mathcal{I}_{\text{ii}}\} \leq p_2$ .
- (v)  $\forall i \in \mathcal{I}$ , we have, on one hand,  $\forall i_1, i_2 \in \Phi(i), i_1 \neq i_2$ ,

$$\Phi(i) \cap \Psi(i_1) = \{i_1\} \quad \text{and} \quad \Psi(i_1) \cap \Psi(i_2) = \emptyset,$$

and on the other hand,  $\forall j_1, j_2 \in \Psi(i), j_1 \neq j_2$ ,

$$\Psi(i) \cap \Phi(j_1) = \{j_1\} \quad \text{and} \quad \Phi(j_1) \cap \Phi(j_2) = \emptyset.$$

*Proof.* Properties (i), (ii) and (v) are direct consequences of the definition of the functions  $\Phi$  and  $\Psi$ . Property (iii) arises from the definition of the frequency set. Finally property (iv) is a consequence of the block structure theorem (see Theorem 2.3). □

Thanks to functions  $\Phi$  and  $\Psi$  it is easy to define the notion of two linked indices in the graph structure of  $\mathcal{I}$ . In terms of wave packet reflections, this notion means that the index  $i$  will be linked with the index  $j$  if and only if  $j$  is obtained from the wave packet associated with  $i$  after several reflections. In other terms, we can say that the index  $i$  generates the index  $j$ , or that  $i$  is the “father” of  $j$ . The following definition makes this notion more precise:

**Definition 4.4.** If  $\underline{i} \in \mathcal{I}_{io}$ , we say that the index  $j \in \mathcal{I}_{io} \cup \mathcal{I}_{ev1}$  (resp.  $j \in \mathcal{I}_{oi} \cup \mathcal{I}_{ev2}$ ) is linked with the index  $\underline{i}$  if there exists  $p \in 2\mathbb{N} + 1$  (resp.  $p \in 2\mathbb{N}$ ) and a sequence of indices  $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathcal{I}^p$  such that

$$\ell_1 \in \Psi(\underline{i}) \cap \mathcal{I}_{oi}, \quad \ell_2 \in \Phi(\ell_1) \in \mathcal{I}_{io}, \quad \dots, \quad j \in \Phi(\ell_p) \quad (\text{resp. } j \in \Psi(\ell_p)). \tag{\alpha'}$$

We say that the index  $j \in \mathcal{I}_{ii}$  is linked with the index  $\underline{i}$ , if there is a sequence of indices  $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathcal{I}^p$  such that

$$\ell_1 \in \Psi(\underline{i}) \cap \mathcal{I}_{oi}, \quad \ell_2 \in \Phi(\ell_1) \cap \mathcal{I}_{io}, \quad \dots, \quad j \in \begin{cases} \Phi(\ell_p) & \text{if } p \text{ is odd,} \\ \Psi(\ell_p) & \text{if } p \text{ is even.} \end{cases} \tag{\beta'}$$

If  $\underline{i} \in \mathcal{I}_{oi}$ , we say that the index  $j \in \mathcal{I}_{io} \cup \mathcal{I}_{ev1}$  (resp.  $j \in \mathcal{I}_{oi} \cup \mathcal{I}_{ev2}$ ) is linked with the index  $\underline{i}$ , if there exists  $p \in 2\mathbb{N}$  (resp.  $p \in 2\mathbb{N} + 1$ ) and a sequence of indices  $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathcal{I}^p$  such that

$$\ell_1 \in \Phi(\underline{i}) \cap \mathcal{I}_{io}, \quad \ell_2 \in \Psi(\ell_1) \in \mathcal{I}_{oi}, \quad \dots, \quad j \in \Phi(\ell_p) \quad (\text{resp. } j \in \Psi(\ell_p)). \tag{\alpha''}$$

We say that the index  $j \in \mathcal{I}_{ii}$  is linked with the index  $\underline{i}$ , if there exists a sequence of indices  $\ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathcal{I}^p$  such that

$$\ell_1 \in \Phi(\underline{i}) \cap \mathcal{I}_{io}, \quad \ell_2 \in \Psi(\ell_1) \cap \mathcal{I}_{oi}, \quad \dots, \quad j \in \begin{cases} \Psi(\ell_p) & \text{if } p \text{ is odd,} \\ \Phi(\ell_p) & \text{if } p \text{ is even.} \end{cases} \tag{\beta''}$$

Finally, if  $\underline{i} \in \mathcal{I}_{ii} \cup \mathcal{I}_{ev1} \cup \mathcal{I}_{ev2}$ , there is no element of  $\mathcal{I}$  linked with  $\underline{i}$ .

Moreover, we will say that an index  $j \in \mathcal{I}$  is linked with the index  $\underline{i}$  by a sequence of type  $H$  (for “horizontal”) (resp.  $V$  (for “vertical”)) and we will use the notation  $i \xrightarrow{H} j$  (resp.  $i \xrightarrow{V} j$ ) if the sequence  $(\underline{i}, \ell_1, \ell_2, \dots, \ell_p, j)$  satisfies  $(\alpha'')$  or  $(\beta'')$  (resp.  $(\alpha')$  or  $(\beta')$ ).

Let us comment a bit on this definition. In terms of wave packet reflections, if one fixes an index  $\underline{i} \in \mathcal{I}_{io}$ , an index  $j$  is linked with the index  $\underline{i}$  if  $j$  comes from  $\underline{i}$  after several reflections. More precisely, the incoming-outgoing ray associated with  $\underline{i}$  hits the side  $\partial\Omega_2$  and is reflected in the outgoing-incoming ray associated with the index  $\ell_1$ . Then the ray of index  $\ell_1$  hits the side  $\partial\Omega_1$ , and generates the incoming-outgoing ray associated with the index  $\ell_2$ . This ray hits the side  $\partial\Omega_2$  and so on until the ray associated with the index  $\ell_p$  generates by reflection the index  $j$ .

The distinction of cases based on the group velocity of the index  $j$  in the subcase  $(\alpha')$  considers the fact that a ray associated with an index in  $\mathcal{I}_{io} \cup \mathcal{I}_{ev1}$  (resp.  $\mathcal{I}_{oi} \cup \mathcal{I}_{ev2}$ ) can be generated by a ray associated with  $\ell_p$  only during a reflection on the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ), or equivalently after an even (resp. odd) number of reflections, whereas a ray with an incoming-incoming group velocity can be generated by the ray  $\ell_p$  during a reflection on the side  $\partial\Omega_1$  or one the side  $\partial\Omega_2$ . That is the reason why the subcase  $(\beta')$  differs from the subcase  $(\alpha')$ .

If one rather sees the index set  $\mathcal{I}$  with a graph structure, saying that  $j$  is linked with  $\underline{i}$  is no more than saying that starting from  $\underline{i}$  one can reach the index  $j$  by passing through the indices  $\ell_i$ , with the following rule of travel: if one reaches  $\ell_l$  by following a vertical (resp. horizontal) edge of the graph, then  $\ell_{l+1}$  will be reached by following a horizontal (resp. vertical) edge. A sequence of type  $H$  (resp.  $V$ ) just means that when we start from  $\underline{i}$ , the first edge is a horizontal (resp. vertical) one.

The following proposition is an immediate consequence of Definitions 4.2 and 4.4.

**Proposition 4.5.** *Let  $\mathcal{F}$  be a complete-for-reflection frequency set indexed by  $\mathcal{I}$ . Let  $\mathcal{I}_0$  be the set of indices in  $\mathcal{I}$  generated by the source term  $g^\varepsilon$ ; that is to say,*

$$\mathcal{I}_0 := \{i \in \mathcal{I}_{i0} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev1} \mid \det \mathcal{L}(\underline{\tau}, \xi_1^i, \xi_2) = 0\}.$$

Let  $\mathcal{I}_{\mathcal{R}}$  be the set of indices in  $\mathcal{I}$  linked with one of the elements of  $\mathcal{I}_0$ . Then

$$\mathcal{I}_{\mathcal{R}} = \mathcal{I}.$$

*Proof.* Let  $\mathcal{F}_{\mathcal{R}}$  be the set of frequencies indexed by  $\mathcal{I}_{\mathcal{R}}$ . It is clear that the set  $\mathcal{F}_{\mathcal{R}}$  satisfies properties (i)–(iv) of Definition 4.2. Let us describe the verification of property (iii) to be more convincing.

We fix  $\underline{i} \in \mathcal{I}_{\mathcal{R}}$ , an incoming-outgoing index. Let  $\underline{\ell}$  be a sequence that links  $\underline{i}$  to one of the indices of  $\mathcal{I}_0$ . Then indices in  $\mathcal{I}_{oi} \cap \Psi(\underline{i})$ ,  $\mathcal{I}_{ii} \cap \Psi(\underline{i})$  and  $\mathcal{I}_{ev2} \cap \Psi(\underline{i})$  are linked with an element in  $\mathcal{I}_0$  by the sequence  $(\underline{\ell}, \underline{i})$ . As a consequence, these indices are in  $\mathcal{I}_{\mathcal{R}}$ . We just showed that  $\mathcal{I}_{\mathcal{R}}$  satisfies property (iii) of Definition 4.2.

We now want to show that  $\mathcal{I}_{\mathcal{R}} = \mathcal{I}$ . By contradiction, we assume that there exists  $j \in (\mathcal{I} \setminus \mathcal{I}_{\mathcal{R}})$ . Firstly, if  $j \in \mathcal{I}_{ev1} \cup \mathcal{I}_{ev2} \cup \mathcal{I}_{ii}$ , then the frequency set indexed by  $\mathcal{I} \setminus \{j\}$  still satisfies properties (i)–(iv) in Definition 4.2. This fact contradicts the minimality of  $\mathcal{F}$ .

Then, if  $j \in \mathcal{I}_{i0} \cup \mathcal{I}_{oi}$ , we construct the set of indices linked with  $j$ , and we denote this set by  $\tilde{\mathcal{I}}$ . Let  $\tilde{\mathcal{F}}$  be the frequency set indexed by  $\tilde{\mathcal{I}}$ . The set  $(\tilde{\mathcal{F}} \cup \mathcal{F}_{\mathcal{R}}) \setminus (\tilde{\mathcal{F}} \cap \mathcal{F}_{\mathcal{R}})$  satisfies properties (i)–(iv) in Definition 4.2 and is strictly included in  $\mathcal{F}$  because  $j \in (\tilde{\mathcal{F}} \cap \mathcal{F}_{\mathcal{R}})$ . Once more, this fact is incompatible with the minimality of the frequency set  $\mathcal{F}$ . □

Proposition 4.5 concludes the description of our formal framework for frequency sets. Let us stress that in this framework we do not assume that the number of phases in the WKB expansion is finite. The assumption “ $\#\mathcal{F} < +\infty$ ” will only be used to make sure that the formal geometric optics expansion constructed in the following sections is relevant, in the sense that the expansion is well-defined and that it does indeed approximate the exact solution. But it will not be used to construct the WKB expansion, at least, at a formal level.

**4A2. Frequency sets with loops.** As mentioned in the beginning of this section, the aim of all that follows is to construct rigorous geometric optics expansions for corner problem where some amplitudes in the expansion display a self-interacting phenomenon. To do that, we will need to consider corner problems whose characteristic variety contains a “loop”. By loop, we mean that it is possible to find at least four points on the section of the characteristic variety  $V \cap \{\tau = \underline{\tau}\}$  such that if we draw the segments linking these points, we obtain a rectangle or a finite “stairway” (see Section 3C and [Sarason and Smoller 1974, Figure 8]).



Many kinds of loops are possible and few of them lead to a self-interaction phenomenon. That is why, in all that follows, we will assume that there is a unique loop and that this loop induces a self-interaction phenomenon. The uniqueness of the loop is probably not a necessary assumption, but it permits us to simplify many steps of the proof and to save a lot of combinatorial arguments. We refer to [Benoit 2015, paragraphe 6.10] for more details. The different kinds of loops are defined as follows:

**Definition 4.6.** Let  $i \in \mathcal{I}$ ,  $p \in 2\mathbb{N} + 1$  and  $\ell = (\ell_1, \dots, \ell_p) \in \mathcal{I}^p$  (we stress that elements of  $\ell$  are not necessarily distinct).

- We say that the index  $i \in \mathcal{I}$  admits a loop if there exists a sequence  $\ell$  satisfying

$$\ell_1 \in \Phi(i), \quad \ell_2 \in \Psi(\ell_1), \quad \dots, \quad i \in \Psi(\ell_p).$$

- A loop for an index  $i$  is said to be simple if the sequence  $\ell$  does not contain a periodically repeated subsequence.
- An index  $i \in \mathcal{I}_{io}$  (resp.  $i \in \mathcal{I}_{oi}$ ) admits a self-interaction loop if  $i$  admits a simple loop and if the sequence  $(i, \ell, i)$  is of type  $V$  (resp.  $H$ ) according to Definition 4.4.

From now on, let us assume the following:

**Assumption 4.7.** Let (18) be complete for the reflections. We assume that the frequency set  $\mathcal{F}$  contains a unique loop of size 3 and that this loop is a self-interaction loop. More precisely, we want the following properties to be satisfied:

- (vi)  $\exists (n_1, n_3) \in \mathcal{I}_{io}^2, (n_2, n_4) \in \mathcal{I}_{oi}^2$  such that

$$n_4 \in \Psi(n_1), \quad n_3 \in \Phi(n_4), \quad n_2 \in \Psi(n_3), \quad n_1 \in \Phi(n_2).$$

- (vii) Let  $i \in \mathcal{I}$  be an index with a loop  $\ell = (\ell_1, \dots, \ell_p)$ . Then  $p = 3$  and  $\{i, \ell_1, \ell_2, \ell_3\} = \{n_1, n_2, n_3, n_4\}$ .

The fact that we restrict our attention to a loop of size 3 is just to simplify as much as possible the redaction of the proof. However, all of the following construction can be generalized to loops with more than three elements.

One of the main difficulties induced by the presence of a loop is that the definition of linked indices does not permit us anymore to define a partial order on the frequency set, as can be done in the case  $N = 2$ . Indeed, if one considers indices  $n_1$  and  $n_3$  defined in Assumption 4.7 then we have  $n_1 \succ_{\nabla} n_3$  and  $n_3 \succ_{\nabla} n_1$  but  $n_1 \neq n_3$ . We will see in Section 4B1 how this new difficulty can be overcome.

We conclude this section by defining what we mean by “trapped” and “self-interacting” rays.

**Definition 4.8.** A ray of the geometric optics expansion is said to be trapped if when we follow its characteristics, we never escape from a compact set.

A ray of the geometric optics expansion is said to be self-interacting if when we follow its characteristics, we can find a repeating sequence of group velocities.

So a trapped ray is a ray which will never escape to “infinity”. The ray obtained by following the characteristic lines for the indices  $(n_1, n_2, n_3, n_4)$  is a self-interacting trapped ray, whereas, the ray described in Section 3D is non-self-interacting trapped ray.

To conclude this section let us give the following assumption about an extra special property which applies to the frequency set  $\mathcal{F}$ . This assumption will be useful in the beginning of Section 4B to derive the WKB cascade. Moreover, some extra comments about this assumption can be found in Section 5.

**Assumption 4.9.** *If  $\mathcal{F}_{\text{ev}1}$  (resp.  $\mathcal{F}_{\text{ev}1}$ ) is not empty then  $\mathcal{F}_{\text{os}}$  does not contain any element of the form  $f_k = (\underline{\tau}, 0, \xi_2^k)$  (resp.  $f_k = (\underline{\tau}, \xi_1^k, 0)$ ).*

**4A3. Some definitions and notation.** For  $j \in \mathcal{I}_{\text{ev}1} \cup \mathcal{I}_{\text{ii}} \cup \mathcal{I}_{\text{io}}$  (resp.  $i \in \mathcal{I}_{\text{ev}2} \cup \mathcal{I}_{\text{ii}} \cup \mathcal{I}_{\text{oi}}$ ), we denote by  $f^j := (\underline{\tau}, \xi_1^j, \xi_2^j)$  the associated frequency. Let us recall that thanks to the uniform Kreiss–Lopatinskii condition, it is possible to define  $\phi_1^j$  (resp.  $\phi_2^j$ ), the inverse of  $B_1$  (resp.  $B_2$ ) restricted to the stable subspace  $E_1^s(i \underline{\tau}, \xi_2^j)$  (resp.  $E_2^s(i \underline{\tau}, \xi_1^j)$ ).

To construct the amplitudes in the WKB expansion, we will need the following projectors and partial inverses:

**Definition 4.10.** For  $j \in \{1, 2\}$  and  $f_k = (\underline{\tau}, \xi_1^k, \xi_2^k) \in \mathcal{F}$ , let us denote by  $P_{1,s,j}^k$  (resp.  $P_{2,s,j}^k$ ), the projector on  $E_j^{s,e}(i \underline{\tau}, \xi_{3-j}^k)$  (resp.  $E_i^{u,e}(i \underline{\tau}, \xi_{3-j}^k)$ ) associated with the direct sum (3), and  $P_1^k$  (resp.  $P_2^k$ ) the projector on  $\ker \mathcal{L}(f_k)$  associated with the sums (5) (resp. (6)).

Let us denote by  $Q_{1,s,j}^k$  (resp.  $Q_{2,s,j}^k$ ), the projector on  $E_j^{s,e}(i \underline{\tau}, \xi_{3-j}^k)$  (resp.  $E_j^{u,e}(i \underline{\tau}, \xi_{3-j}^k)$ ) associated with the direct sum (4), and  $Q_j^k$  (resp.  $Q_2^k$ ) the projector on  $A_1 \ker \mathcal{L}(f_k)$  (resp.  $A_2 \ker \mathcal{L}(f_k)$ ) associated with the sums (7) (resp. (8)).

Let  $R_j^k$  be the partial inverse of  $\mathcal{L}(f_k)$ , defined by the two relations

$$R_j^k \mathcal{L}(f_k) = I - P_j^k, \quad P_j^k R_j^k = R_j^k Q_j^k = 0. \tag{20}$$

Finally, to simplify the notation as much as possible, set

$$S_1^k := P_1^k \phi_1^k, \quad S_2^k := P_2^k \phi_2^k, \quad S_{s,1}^k := P_{s,i}^k \phi_1^k, \quad S_{s,2}^k := P_{s,2}^k \phi_2^k.$$

An important remark is that, for  $k \in \mathcal{F}_{\text{os}}$ , if  $f_k$  is the associated frequency then  $\text{Ran } \mathcal{L}(f_k) = \ker Q_1^k = \ker Q_2^k$ , and that, for  $j \in \{1, 2\}$ , the projector  $Q_j^k$  induces an isomorphism from  $\text{Ran } P_j^k$  to  $\text{Ran } Q_j^k$ .

We will have to solve transport equations, so the following variables will be convenient:

$$\forall j \in \mathcal{I}_{\text{io}}, \quad t_{\text{io}}^j(t, x_1) := t - \frac{1}{v_{j,1}} x_1, \quad x_{\text{io}}^j(x_1, x_2) := x_2 - \frac{v_{j,2}}{v_{j,1}} x_1, \tag{21}$$

$$\forall j \in \mathcal{I}_{\text{oi}}, \quad t_{\text{oi}}^j(t, x_2) := t - \frac{1}{v_{j,2}} x_2, \quad x_{\text{oi}}^j(x_1, x_2) := x_1 - \frac{v_{j,1}}{v_{j,2}} x_2. \tag{22}$$

**4B. Construction of the WKB expansion.** During all the construction, we will have to consider three kinds of phases, namely oscillating phases, evanescent phases for the side  $\partial\Omega_1$  and evanescent phases for the side  $\partial\Omega_2$ . These will be denoted by

$$\begin{aligned} \varphi_k(t, x) &:= \langle (t, x), f_k \rangle, & f_k &\in \mathcal{F}_{\text{os}}, \\ \psi_{k,1}(t, x_2) &:= \langle (t, 0, x_2), f_k \rangle, & f_k &\in \mathcal{F}_{\text{ev}1} \cup \mathcal{F}_{\text{os}}, \\ \psi_{k,2}(t, x_1) &:= \langle (t, x_1, 0), f_k \rangle, & f_k &\in \mathcal{F}_{\text{ev}2} \cup \mathcal{F}_{\text{os}}. \end{aligned}$$

For a given amplitude  $g \in H_f^\infty$ , zero for negative times, we will work with a source term on the side  $\partial\Omega_1$  of the form

$$g^\varepsilon(t, x_2) := e^{\frac{i}{\varepsilon}(\tau t + x_2 \xi_2^{n_1})} g(t, x_2),$$

that is to say, a source term that “turns on” the index  $n_1$  on the loop, and that has an incoming group velocity for the side  $\partial\Omega_1$ . So, we expect that the source term  $g^\varepsilon$  will generate a wave packet propagating towards the side  $\partial\Omega_2$ .

As in [Lescarret 2007], evanescent modes will be treated in a “monoblock” way, that is to say, that for an index  $\underline{i} \in \mathcal{I}_{\text{ev}1}$  (resp.  $\underline{i} \in \mathcal{I}_{\text{ev}2}$ ), all the indices  $j \in \mathcal{I}_{\text{ev}1} \cap \Phi(\underline{i})$  (resp.  $j \in \mathcal{I}_{\text{ev}2} \cap \Psi(\underline{i})$ ) will contribute to a single vector-valued amplitude. To write off the ansatz and to describe with enough precision the boundary conditions, it is useful to introduce the two equivalence relations  $\underset{\Phi}{\sim}$  and  $\underset{\Psi}{\sim}$  defined by

$$\begin{aligned} i \underset{\Phi}{\sim} j &\iff j \in \Phi(i), \\ i \underset{\Psi}{\sim} j &\iff j \in \Psi(i). \end{aligned}$$

The fact that these relations are effectively equivalence relations is a direct consequence of Proposition 4.3.

Let  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ) be the set of equivalence classes for the relation  $\underset{\Phi}{\sim}$  (resp.  $\underset{\Psi}{\sim}$ ), and  $\mathcal{R}_1$  (resp.  $\mathcal{R}_2$ ) be a set of class representative for  $\mathcal{C}_1$  (resp.  $\mathcal{C}_2$ ). So  $\mathcal{R}_1$  (resp.  $\mathcal{R}_2$ ) is a set of indices which includes all the possible values for  $\xi_2$  (resp.  $\xi_1$ ) of the different frequencies. Let us define  $\mathfrak{R}_1$  and  $\mathfrak{R}_2$  by

$$\mathfrak{R}_1 := \{i \in \mathcal{R}_1 \mid \Phi(i) \cap \mathcal{I}_{\text{ev}1} \neq \emptyset\}, \tag{23}$$

$$\mathfrak{R}_2 := \{i \in \mathcal{R}_2 \mid \Psi(i) \cap \mathcal{I}_{\text{ev}2} \neq \emptyset\}. \tag{24}$$

$\mathfrak{R}_1$  (resp.  $\mathfrak{R}_2$ ) is a set of class representative of the values in  $\xi_2$  (resp.  $\xi_1$ ) for which there is an evanescent mode for the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ). At last, without loss of generality, we can always assume that  $n_1 \in \mathfrak{R}_2$ ; in other words, we choose  $n_1$  as a class representative of its equivalence class.

We take for the ansatz

$$\begin{aligned} u^\varepsilon(t, x) &\sim \sum_{k \in \mathcal{I}_{\text{os}}} e^{\frac{i}{\varepsilon} \varphi_k(t, x)} \sum_{n \geq 0} \varepsilon^n u_{n, k}(t, x) \\ &+ \sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon} \psi_{k, 1}(t, x_2)} \sum_{n \geq 0} \varepsilon^n U_{n, k, 1}\left(t, x, \frac{x_1}{\varepsilon}\right) + \sum_{k \in \mathfrak{R}_2} e^{\frac{i}{\varepsilon} \psi_{k, 2}(t, x_1)} \sum_{n \geq 0} \varepsilon^n U_{n, k, 2}\left(t, x, \frac{x_2}{\varepsilon}\right). \end{aligned} \tag{25}$$

And we now want to determine the profiles  $u_{n, k}$  and  $U_{n, k, i}$ . We are looking for oscillating profiles  $u_{n, k}$  in the space  $H^\infty(\Omega_T)$ , whereas, the space for the evanescent profiles is (see [Lescarret 2007]):

**Definition 4.11.** For  $i = 1, 2$ , the set  $P_{\text{ev}, i}$  of evanescent profiles for the side  $\partial\Omega_i$  is defined as functions  $U(t, x, X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$  for which there exists a positive  $\delta$  such that  $e^{\delta X_i} U(t, x, X_i) \in H^\infty(\Omega_T \times \mathbb{R}_+)$ .

Plugging the ansatz (25) in the evolution equation of the corner problem (18) and identifying in terms of powers of  $\varepsilon$  leads us to solve the cascade of equations

$$\left\{ \begin{array}{l} \mathcal{L}(d\varphi_k)u_{0,k} = 0 \quad \forall k \in \mathcal{I}_{os}, \\ i\mathcal{L}(d\varphi_k)u_{n+1,k} + L(\partial)u_{n,k} = 0 \quad \forall n \in \mathbb{N}, \forall k \in \mathcal{I}_{os}, \\ L_k(\partial_{X_1})U_{0,k,1} = 0 \quad \forall k \in \mathfrak{R}_1, \\ L_k(\partial_{X_1})U_{n+1,k,1} + L(\partial)U_{n,k,1} = 0 \quad \forall n \in \mathbb{N}, \forall k \in \mathfrak{R}_1, \\ L_k(\partial_{X_2})U_{0,k,2} = 0 \quad \forall k \in \mathfrak{R}_2, \\ L_k(\partial_{X_2})U_{n+1,k,2} + L(\partial)U_{n,k,2} = 0 \quad \forall n \in \mathbb{N}, \forall k \in \mathfrak{R}_2, \end{array} \right. \quad (26)$$

where the “fast” differentiation operators  $L_k(\partial_{X_1})$  and  $L_k(\partial_{X_2})$  are given by

$$\begin{aligned} L_k(\partial_{X_1}) &:= A_1(\partial_{X_1} - \mathcal{A}_1(\underline{\tau}, \xi_2^k)) \quad \text{for } k \in \mathfrak{R}_1, \\ L_k(\partial_{X_2}) &:= A_2(\partial_{X_2} - \mathcal{A}_2(\underline{\tau}, \xi_1^k)) \quad \text{for } k \in \mathfrak{R}_2. \end{aligned}$$

Then, plugging the ansatz (25) in the boundary conditions on the sides  $\partial\Omega_1$  and  $\partial\Omega_2$  gives

$$B_1 \left[ \sum_{k \in \mathcal{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,1}} u_{n,k}(t, 0, x_2) + \sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} U_{n,k,1}(t, 0, x_2, 0) + \sum_{k \in \mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,2} \left( t, 0, x_2, \frac{x_2}{\varepsilon} \right) \right] = \delta_{n,0} e^{\frac{i}{\varepsilon}\psi_{n_1,1}} g, \quad (27)$$

and

$$B_2 \left[ \sum_{k \in \mathcal{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,2}} u_{n,k}(t, x_1, 0) + \sum_{k \in \mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}} U_{n,k,2}(t, x_1, 0, 0) + \sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,1} \left( t, x_1, 0, \frac{x_1}{\varepsilon} \right) \right] = 0. \quad (28)$$

Let us study the first boundary condition. If there are no evanescent phases for the side  $\partial\Omega_2$  then it simply reads

$$B_1 \left[ \sum_{k \in \mathcal{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,1}} u_{n,k}(t, 0, x_2) + \sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} U_{n,k,1}(t, 0, x_2, 0) \right] = \delta_{n,0} e^{\frac{i}{\varepsilon}\psi_{n_1,1}} g,$$

whereas if there are evanescent phases for the side  $\partial\Omega_2$  we can use Assumption 4.9 to decouple (27) into

$$\left\{ \begin{array}{l} B_1 \left[ \sum_{k \in \mathcal{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,1}} u_{n,k}(t, 0, x_2) + \sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon}\psi_{k,1}} U_{n,k,1}(t, 0, x_2, 0) \right] = \delta_{n,0} e^{\frac{i}{\varepsilon}\psi_{n_1,1}} g, \\ B_1 \sum_{k \in \mathfrak{R}_2} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,2} \left( t, 0, x_2, \frac{x_2}{\varepsilon} \right) = 0. \end{array} \right. \quad (29)$$

Indeed Assumption 4.9 implies the linear independence of the phases  $\psi_{k,1}$  and  $\underline{\tau}t$ . The same reasoning for the boundary condition gives

$$\left\{ \begin{array}{l} B_2 \left[ \sum_{k \in \mathcal{I}_{os}} e^{\frac{i}{\varepsilon}\psi_{k,2}} u_{n,k}(t, x_1, 0) + \sum_{k \in \mathfrak{R}_2} e^{\frac{i}{\varepsilon}\psi_{k,2}} U_{n,k,2}(t, x_1, 0, 0) \right] = 0, \\ B_2 \sum_{k \in \mathfrak{R}_1} e^{\frac{i}{\varepsilon}\underline{\tau}t} U_{n,k,1} \left( t, x_1, 0, \frac{x_1}{\varepsilon} \right) = 0. \end{array} \right. \quad (30)$$

Now, using again the linear independence of the phases, the boundary conditions (29) and (30) can be decomposed as the following cascades of equations:

$$\left\{ \begin{array}{l} B_1 \left[ \sum_{j \in \Phi(n_1) \cap \mathcal{I}_{os}} u_{n,j} + U_{n,n_1,1|_{x_1=0}} \right]_{|_{x_1=0}} = \delta_{n,0} g \quad \forall n \in \mathbb{N}, \text{ if } n_1 \in \mathfrak{R}_1, \\ B_1 \left[ \sum_{j \in \Phi(n_1)} u_{n,j} \right]_{|_{x_1=0}} = \delta_{n,0} g \quad \forall n \in \mathbb{N}, \text{ if } n_1 \notin \mathfrak{R}_1, \\ B_1 \left[ \sum_{j \in \Phi(k) \cap \mathcal{I}_{os}} u_{n,j} + U_{n,k,1|_{x_1=0}} \right]_{|_{x_1=0}} = 0 \quad \forall n \in \mathbb{N}, \forall k \in \mathfrak{R}_1 \setminus \{n_1\}, \\ B_1 \left[ \sum_{j \in \Phi(k)} u_{n,j} \right]_{|_{x_1=0}} = 0 \quad \forall n \in \mathbb{N}, \forall k \notin \mathfrak{R}_1 \setminus \{n_1\}, \\ B_2 \left[ \sum_{j \in \Psi(k) \cap \mathcal{I}_{os}} u_{n,j} + U_{n,k,2|_{x_2=0}} \right]_{|_{x_2=0}} = 0 \quad \forall n \in \mathbb{N}, \forall k \in \mathfrak{R}_2, \\ B_2 \left[ \sum_{j \in \Psi(k)} u_{n,j} \right]_{|_{x_2=0}} = 0 \quad \forall n \in \mathbb{N}, \forall k \notin \mathfrak{R}_2. \end{array} \right. \quad (31)$$

and

$$\left\{ \begin{array}{l} B_1 \sum_{k \in \mathfrak{R}_2} U_{n,k,2} \left( t, 0, x_2, \frac{x_2}{\varepsilon} \right) = 0 \quad \forall n \in \mathbb{N}, \\ B_2 \sum_{k \in \mathfrak{R}_2} U_{n,k,1} \left( t, x_1, 0, \frac{x_1}{\varepsilon} \right) = 0 \quad \forall n \in \mathbb{N}. \end{array} \right. \quad (32)$$

At last, plugging the ansatz (25) in the initial condition of the corner problem (18) leads us to solve

$$\forall n \in \mathbb{N}, \quad \left\{ \begin{array}{l} u_{n,k}|_{t=0} = 0 \quad \forall k \in \mathcal{I}_{os}, \\ U_{n,k,1}|_{t=0} = 0 \quad \forall k \in \mathfrak{R}_1, \\ U_{n,k,2}|_{t=0} = 0 \quad \forall k \in \mathfrak{R}_2. \end{array} \right. \quad (33)$$

The main steps in the construction of the geometric optics expansion are the following. First, before solving the WKB cascade, we will describe a global structure on the set of indices  $\mathcal{I}$ . More precisely, this structure is based on a partition which takes into account the different relations that an index can have with the elements of the loop. We will thus be able to express  $\mathcal{I}$  as a union of nonintersecting “trees” (or ordered sets by the relations  $\xrightarrow{H}$  and  $\xrightarrow{V}$ , see Definition 4.4). Then, we will construct the amplitudes for the indices of the loop. To do this, we will need a new invertibility condition, which will be studied in Section 4D.

Thanks to the knowledge of the amplitudes associated with the loop, we will be able to construct the amplitudes in a direct neighborhood of the indices of the loop. In other terms, the new invertibility condition will be used to start the construction of the geometric optics expansion.

Then, to construct the remaining amplitudes, we will first make a more precise study of the structure of the trees that form  $\mathcal{I}$ . Using this more precise analysis, we will see that the construction of the amplitudes in these trees is rather easy because one can define a partial order on these trees.

The scheme of proof, and more precisely the order of construction of the amplitudes will be exactly the same for higher-order terms.

**4B1. Global structure of the set of indices  $\mathcal{I}$ .** In this section we will construct a partition of  $\mathcal{I}$  based upon the position of the indices compared to the loop index  $n_1$  and no more on the different kinds of elements in  $\mathcal{I}$ . More precisely, the partition will be based upon the different kinds of sequences that can link an index  $i$  to the loop index  $n_1$ .

The idea of the construction is the following; firstly, thanks to [Proposition 4.5](#), we know that every index  $i$  in  $\mathcal{I}$  is linked by a sequence of type  $V$  to one of the indices of  $\mathcal{I}_0$  (cf. [Definition 4.4](#)). Without loss of generality, one can always assume that for all indices  $i$  the sequence linking  $i$  to the index in  $\mathcal{I}_0$  does not start by the subsequence  $(n_4, n_2, n_3, n_1)$ .

The following lemma is also immediate:

**Lemma 4.12.** *For all  $i \in \mathcal{I}$ , there exists at least one sequence of type  $V$  linking  $i$  to  $n_1$ . Equivalently, for all  $i \in \mathcal{I}$ ,*

$$n_1 \succ_V i,$$

where the notation  $\succ_V$  has been introduced in [Definition 4.4](#).

*Proof.* It is sufficient to treat the case of indices  $i$  linked with  $i_0$  for  $i_0 \in \mathcal{I}_0 \setminus \{n_1\}$ . For such indices, there exists a sequence, denoted by  $\tilde{\ell}$ , of type  $V$  linking  $i$  to  $i_0$ . By definition,  $i_0 \in \Phi(n_1)$ . So  $i$  is linked with  $n_1$  by the type- $V$  sequence defined by  $\ell = (n_4, n_3, n_2, i_0, \tilde{\ell})$ . □

Now, let  $i \in \mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}$ , and let  $\ell^i = (\ell_1, \ell_2, \dots, \ell_p)$  be a type- $V$  sequence linking  $i$  to  $n_1$ . The way to construct the sets, denoted  $A_{a_l}, B_{b_m}, C_{c_q}, D_{d_r}$ , of the sought partition is based on the following algorithm:

Let  $\mathcal{C}_1 := \#\Psi(n_1) - 2$ , and

$$\Psi(n_1) \setminus \{n_1, n_4\} := \{a_1, a_2, \dots, a_{\mathcal{C}_1}\}.$$

Let  $l \in \{1, \dots, \mathcal{C}_1\}$ . We will say that  $i \in A_{a_l}$  if and only if the sequence  $\ell^i$  can be chosen such that  $\ell_1 = a_l$ .

At this stage, we have treated all the sequences that do not start with  $n_4$ . To treat the sequences that start with  $n_4$ , let  $\mathcal{C}_4 := \#\Phi(n_4) - 2$ , and

$$\Phi(n_4) \setminus \{n_3, n_4\} := \{b_1, b_2, \dots, b_{\mathcal{C}_4}\}.$$

Then for  $m \in \{1, \dots, \mathcal{C}_2\}$ , we will say that  $i \in B_{b_m}$  if and only if the sequence  $\ell^i$  can be chosen such that  $\ell_1 = n_4$  and  $\ell_2 = b_m$ .

Consequently we have treated all the sequences  $\ell^i$  except those starting with  $(n_4, n_3)$ .

Finally let  $\mathcal{C}_3 := \#\Psi(n_3) - 2$ ,  $\mathcal{C}_2 := \#\Phi(n_2) - 2$  and

$$\Psi(n_3) \setminus \{n_2, n_3\} := \{c_1, c_2, \dots, c_{\mathcal{C}_3}\}, \quad \Phi(n_2) \setminus \{n_1, n_2\} := \{d_1, d_2, \dots, d_{\mathcal{C}_2}\}.$$

We define the sets  $C_{c_q}$  and  $D_{d_r}$  by the relations:

- For  $q \in \{1, \dots, \mathfrak{C}_3\}$ , we have  $i \in C_{c_q}$  if and only if the sequence  $\ell^i$  can be chosen such that  $\ell_1 = n_4$ ,  $\ell_2 = n_3$  and  $\ell_3 = c_q$ .
- For  $r \in \{1, \dots, \mathfrak{C}_2\}$ , we have  $i \in D_{d_r}$  if and only if the sequence  $\ell^i$  can be chosen such that  $\ell_1 = n_4$ ,  $\ell_2 = n_3$ ,  $\ell_3 = n_2$  and  $\ell_4 = d_r$ .

This algorithm permits to consider all the possible sequences because no sequence starts with the subsequence  $(n_4, n_3, n_2, n_1)$ . Then, we repeat this construction for all the potential sequences linking  $i$  to  $n_1$ .

It is thus clear that

$$(\mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}) = \left( \bigcup_{l \leq \mathfrak{C}_1} A_{a_l} \right) \cup \left( \bigcup_{m \leq \mathfrak{C}_2} B_{b_m} \right) \cup \left( \bigcup_{q \leq \mathfrak{C}_3} C_{c_q} \right) \cup \left( \bigcup_{r \leq \mathfrak{C}_4} D_{d_r} \right). \tag{34}$$

The sets  $A_{a_l}$  and  $B_{b_m}$  can be characterized as follows:  $A_{a_l}$  is the set of indices  $i \in \mathcal{I}$  such that  $a_l \xrightarrow{H} i$ , whereas  $B_{b_m}$  is the set of indices  $i \in \mathcal{I}$  such that  $b_m \xrightarrow{V} i$ . In terms of wave packet reflection, the set  $A_{a_l}$  gathers the indices obtained by reflection of the phase associated with the index  $a_l$ , this phase being obtained by reflection of the wave packet associated with  $n_1$  on the side  $\partial\Omega_2$ . In a similar way,  $B_{b_m}$  gathers the indices obtained by reflection of the phase associated with the index  $b_j$ . The phase associated with  $b_j$  being obtained by reflection of the phase associated with  $n_4$  on the side  $\partial\Omega_1$ . An analogous characterization stands for the sets  $C_{c_q}$  and  $D_{d_r}$ .

**Lemma 4.13.** *The decomposition*

$$(\mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}) = \left( \bigcup_{l \leq \mathfrak{C}_1} A_{a_l} \right) \cup \left( \bigcup_{m \leq \mathfrak{C}_2} B_{b_m} \right) \cup \left( \bigcup_{n \leq \mathfrak{C}_3} C_{c_n} \right) \cup \left( \bigcup_{q \leq \mathfrak{C}_4} D_{d_q} \right)$$

is a partition of  $\mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}$ .

*Proof.* Let us first define the “mirror” sequence of a sequence by the relation

$$\forall \ell = (\ell_1, \ell_2, \dots, \ell_p) \in \mathcal{I}^p, \quad \bar{\ell} := (\ell_p, \ell_{p-1}, \dots, \ell_1) \in \mathcal{I}^p.$$

Let  $l, l' \in \{1, \dots, \mathfrak{C}_1\}$ ,  $l \neq l'$ .

**Proof of  $A_{a_l} \cap A_{a_{l'}} = \emptyset$ :** We argue by contradiction. Let us assume that there exists  $i \in A_{a_l} \cap A_{a_{l'}}$ . Then by definition, there exists a type- $H$  sequence  $\ell = (\ell_1, \dots, \ell_p)$  linking  $i$  to  $a_l$  and a type- $H$  sequence  $\ell' = (\ell'_1, \dots, \ell'_{p'})$  linking  $i$  to  $a_{l'}$ . We now have to consider several cases depending on the oddness/evenness of  $p$  and  $p'$ .

$p, p' \in 2\mathbb{N}$ : By the definition of type- $H$  sequences, we have  $i \in \Psi(\ell_p)$  and  $i \in \Psi(\ell'_{p'})$ . Thanks to property (ii) of Proposition 4.3,  $\ell'_{p'} \in \Psi(\ell_p)$ . The sequence  $(\ell, \bar{\ell}')$  is consequently a type- $H$  sequence linking  $a_l$  to  $a_{l'}$ . But  $a_l \in \Phi(a_{l'})$ , so the sequence  $(\ell, \bar{\ell}', a_{l'})$  is a loop for the index  $a_l$  with exactly  $p + p' + 1$  elements. This contradicts Assumption 4.7.

$p \in 2\mathbb{N}, p' \in 2\mathbb{N} + 1$ : Now  $i \in \Psi(\ell_p)$  and  $i \in \Phi(\ell'_{p'})$  or equivalently  $\ell'_{p'} \in \Phi(i)$ . The sequence  $(\ell, i, \bar{\ell}')$  is a type- $H$  sequence linking  $a_l$  to  $a_{l'}$ . Then  $(\ell, i, \bar{\ell}', a_{l'})$  is a loop for the index  $a_l$  with  $p + p' + 2$  elements. Once again, it contradicts [Assumption 4.7](#).

The case  $p, p' \in 2\mathbb{N} + 1$  is quite similar to the case  $p, p' \in 2\mathbb{N}$ , so we omit the proof.

We now deal with the proof of the property  $A_{a_l} \cap B_{b_m} = \emptyset$ , the other proofs showing that the other kinds of intersections are empty are analogous and consequently they will not be treated here.

**Proof of  $A_{a_l} \cap B_{b_m} = \emptyset$ :** Once again, we argue by contradiction. Let  $i \in A_{a_l} \cap B_{b_m}$ . Then by the definitions of the sets  $A_{a_l}$  and  $B_{b_m}$ , we have  $a_l \xrightarrow{H} i$  and  $b_m \xrightarrow{V} i$ . That is to say, there exists  $\ell = (\ell_1, \dots, \ell_p)$  a type- $H$  sequence linking  $i$  to  $a_l$  and a type- $V$  sequence  $\ell' = (\ell'_1, \dots, \ell'_{p'})$  linking  $i$  to  $b_m$ . We have to consider the following cases:

$p, p' \in 2\mathbb{N}$ : We have  $i \in \Psi(\ell_p)$  and  $i \in \Phi(\ell'_{p'})$ . So it is possible to show exactly as in the proof of one of the above subcases that the sequence  $(\ell, i, \bar{\ell}')$  links  $a_l$  to  $b_m$ . It follows from  $a_l \in \Psi(n_2)$  and  $n_4 \in \Phi(b_m)$  that the sequence  $(\ell, i, \bar{\ell}', b_m, n_4)$  is a loop for the index  $a_l$  with an odd number of elements.

$p \in 2\mathbb{N}, p' \in 2\mathbb{N} + 1$ : We can show that the sequence  $(\ell, \bar{\ell}')$  links  $a_l$  to  $b_m$ . So  $(\ell, \bar{\ell}', b_m, n_4)$  is a loop for  $a_l$  with an odd number of elements, which is again a contradiction with [Assumption 4.7](#).  $\square$

We have just shown that

$$\left( \bigcup_{i \leq c_1} A_{a_i} \right) \cup \left( \bigcup_{i \leq c_2} B_{b_i} \right) \cup \left( \bigcup_{i \leq c_3} C_{c_i} \right) \cup \left( \bigcup_{i \leq c_4} D_{d_i} \right) \tag{35}$$

is a partition of  $\mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}$ . A consequence is that to determine all the amplitudes in the WKB expansion, it will be sufficient to construct the amplitude for the indices on the loop and then the amplitudes in each set of the partition (35).

Moreover, the construction of the amplitudes in each set of the partition (35) can be made intrinsically in this set. Indeed, the fact that (35) is a partition implies that an index  $i$  in one set of (35) is only linked, by the boundary conditions (31), with other indices in the same set.

A last consequence of the fact that (35) is a partition of the frequency set is the following refinement of [Proposition 4.3](#):

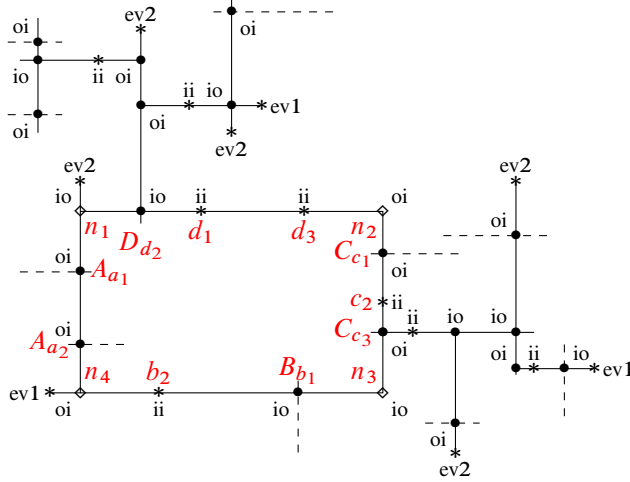
**Proposition 4.14.** *Let (18) be complete for the reflections, under [Assumption 4.7](#). Let  $\mathcal{I}$  be the index set; then  $\Phi$  and  $\Psi$  satisfy, in addition to the properties of [Proposition 4.3](#), the two extra properties:*

$$\begin{aligned} \text{(viii)} \quad & \Phi(n_1) \setminus \{n_2\} \subset \mathcal{I}_{i_0} \cup \mathcal{I}_{i_{ii}} \cup \mathcal{I}_{ev1}, & \Psi(n_1) \setminus \{n_1\} \subset \mathcal{I}_{o_i} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev2}, \\ & \Phi(n_4) \setminus \{n_4\} \subset \mathcal{I}_{i_0} \cup \mathcal{I}_{i_{ii}} \cup \mathcal{I}_{ev1}, & \Psi(n_3) \setminus \{n_3\} \subset \mathcal{I}_{o_i} \cup \mathcal{I}_{ii} \cup \mathcal{I}_{ev2}. \end{aligned}$$

(ix) *Let  $i \in \mathcal{I}_{ii} \cup \mathcal{I}_{ev1}$  and  $j \in \mathcal{I}_{ii} \cup \mathcal{I}_{ev2}$ . Then*

$$\begin{aligned} i \in \Phi(n_1) &\implies \Psi(i) = \{i\}, & j \in \Psi(n_1) &\implies \Phi(j) = \{j\}, \\ i \in \Phi(n_4) &\implies \Psi(i) = \{i\}, & j \in \Psi(n_3) &\implies \Phi(j) = \{j\}. \end{aligned}$$





**Figure 7.** “Tree structure” of the frequency set  $\mathcal{F}$ .

*Proof.* (viii) We will here just show the first assertion, that is to say,  $\Phi(n_1) \setminus \{n_2\} \subset \mathcal{S}_{io} \cup \mathcal{S}_{ii} \cup \mathcal{S}_{ev1}$ . By contradiction, let  $i \in \Phi(n_1) \cap \mathcal{S}_{oi}$ ,  $i \neq n_2$ . Then, there exists  $j \in \Psi(i) \cap \mathcal{S}_{io}$ ; otherwise, the frequency set indexed by  $\mathcal{S} \setminus \{i\}$  is strictly included in  $\mathcal{F}$  and satisfies (i)–(iv) of Definition 4.2.

Thanks to Lemma 4.12, we know that there exists a type- $V$  sequence,  $\ell = (\ell_1, \ell_2, \dots, \ell_p)$ , with necessarily  $p \in 2\mathbb{N} + 1$  (because  $n_1, j \in \mathcal{S}_{io}$ , see Definition 4.4), such that  $n_1 \succ_{\mathcal{V}} j$ . The sequence  $(\ell, j, i)$  is a self-interacting loop for  $n_1$  with an odd number of elements, but it is not the same loop as  $\{n_1, n_2, n_3, n_4\}$ . This is a contradiction of Assumption 4.7.

(ix) This uses exactly the same reasoning as for (viii) and we will omit it here. The only difference is that we cannot conclude that the loop is a self-interacting one because it may contain indices in  $\mathcal{S}_{ii}$ . Then we need the uniqueness assumption of a loop and not only the uniqueness assumption of a self-interacting loop.  $\square$

Property  $\Phi(n_1) \setminus \{n_2\} \subset \mathcal{S}_{io} \cup \mathcal{S}_{ii} \cup \mathcal{S}_{ev1}$  of (viii) in Proposition 4.14 means that even if the characteristic variety contains a loop, all the frequencies (but  $n_2$ ) associated with outgoing-incoming group velocity are initially discarded. We already justified this observation in the phase generation process described in Section 3. Property (ix) means that, thanks to the uniqueness assumption of a loop, an incoming-incoming phase in the direct neighborhood of the loop can only be generated by reflection on one side of  $\partial\Omega$  and not on both sides.

Thanks to Proposition 4.14, partition (35) can be rewritten as

$$\begin{aligned}
 (\mathcal{S} \setminus \{n_1, n_2, n_3, n_4\}) = & \left( \bigcup_{a_l \in \Psi(n_1) \cap \mathcal{S}_{io}} A_{a_l} \quad \bigcup_{a_l \in \Psi(n_1) \cap (\mathcal{S}_{ev1} \cup \mathcal{S}_{ii})} \{a_l\} \right) \cup \left( \bigcup_{b_m \in \Phi(n_4) \cap \mathcal{S}_{oi}} B_{b_m} \quad \bigcup_{b_m \in \Phi(n_4) \cap (\mathcal{S}_{ev2} \cup \mathcal{S}_{ii})} \{b_m\} \right) \\
 & \cup \left( \bigcup_{c_q \in \Psi(n_3) \cap \mathcal{S}_{io}} C_{c_q} \quad \bigcup_{c_q \in \Psi(n_3) \cap (\mathcal{S}_{ev1} \cup \mathcal{S}_{ii})} \{c_q\} \right) \cup \left( \bigcup_{d_r \in \Phi(n_2) \cap \mathcal{S}_{io}} D_{d_r} \quad \bigcup_{d_r \in \Phi(n_2) \cap (\mathcal{S}_{ev2} \cup \mathcal{S}_{ii})} \{d_r\} \right). \quad (36)
 \end{aligned}$$

Let us conclude this section by noting that Figure 7 illustrates the “tree structure” of the frequency set  $\mathcal{F}$ .

**4B2. Determination of the amplitudes on the loop and invertibility condition.** Now that the global structure of the frequency set is described, and thanks to the new properties of functions  $\Phi$  and  $\Psi$ , it is time to start the construction of the amplitudes in the WKB expansion. A good (and natural) choice to initialize this construction is to determine first the amplitudes associated with the loop indices. To do that, a new amplitude equation will be derived (see [Section 3E](#)).

The cascades of equations (26), (31) and (33) written for  $n = 0$  and  $k = n_1$  tell us that the amplitude  $u_{0,n_1}$  satisfies

$$\begin{cases} \mathcal{L}(d\varphi_{n_1})u_{0,n_1} = 0, \\ i\mathcal{L}(d\varphi_{n_1})u_{1,n_1} + L(\partial)u_{0,n_1} = 0, \end{cases} \tag{37}$$

in the interior, the boundary conditions

$$\begin{cases} B_1 \left[ \sum_{j \in \Phi(n_1) \cap \mathcal{I}_{os}} u_{0,j} + U_{0,n_1,1|_{X_1=0}} \right]_{|_{x_1=0}} = g, & \text{if } n_1 \in \mathfrak{R}_1, \\ B_1 \left[ \sum_{j \in \Phi(n_1)} u_{0,j} \right]_{|_{x_1=0}} = g, & \text{if } n_1 \in \mathfrak{R}_2 \setminus \mathfrak{R}_1, \end{cases} \tag{38}$$

and

$$\begin{cases} B_2 \left[ \sum_{j \in \Psi(n_1) \cap \mathcal{I}_{os}} u_{0,j} + U_{0,n_1,1|_{X_1=0}} \right]_{|_{x_2=0}} = 0, & \text{if } n_1 \in \mathfrak{R}_2, \\ B_2 \left[ \sum_{j \in \Psi(n_1)} u_{0,j} \right]_{|_{x_2=0}} = 0, & \text{if } n_1 \in \mathfrak{R}_1 \setminus \mathfrak{R}_2, \end{cases} \tag{39}$$

and finally the initial condition

$$u_{0,n_1}|_{t \leq 0} = 0. \tag{40}$$

We will now explain the method of resolution of equations (37)–(40). The ideas described below are classical; they explain why the amplitudes associated with oscillating phases satisfy transport equations in the example of [Section 3E](#) and they will be applied to all the oscillating amplitudes.

Firstly, let us remark that the first equation of (37) tells us that the amplitude  $u_{0,n_1}$  belongs to  $\ker \mathcal{L}(d\varphi_{n_1})$ . In other words, we have the so-called polarization condition

$$P_1^{n_1} u_{0,n_1} = u_{0,n_1},$$

where  $P_1^{n_1}$  is the projector defined in [Definition 4.10](#). Now, composing the second equation of (37) with the projector  $Q_1^{n_1}$  defined in [Definition 4.10](#) and using the polarization condition give us

$$Q_1^{n_1} L(\partial) P_1^{n_1} u_{0,n_1} = 0.$$

But Lax’s lemma [[1957](#)] tells us that if the corner problem (18) is constantly hyperbolic then we have the relation

$$Q_1^{n_1} L(\partial) P_1^{n_1} = (\partial_t + v_{n_1} \cdot \nabla_x) Q_1^{n_1} P_1^{n_1},$$

where  $v_{n_1}$  is the group velocity associated with the phase  $\varphi_{n_1}$ . So, the amplitude  $u_{0,n_1}$  satisfies the transport equation

$$(\partial_t + v_{n_1} \cdot \nabla_x) Q_1^{n_1} u_{0,n_1} = 0.$$

We are now interested in the boundary conditions. As mentioned in [Section 3](#), the boundary conditions needed to solve a transport equation in a quarter-space are linked with the nature of the transport velocity. Let us recall the four possible alternatives:

- The transport velocity is outgoing-outgoing; then no boundary condition has to be imposed.
- The transport velocity is incoming-outgoing; then the transport equation needs a boundary condition on  $\partial\Omega_1$  only.
- The transport velocity is outgoing-incoming; then the transport equation needs a boundary condition on  $\partial\Omega_2$  only.
- The transport velocity is incoming-incoming; then the transport equation needs a boundary condition on  $\partial\Omega_1$  and on  $\partial\Omega_2$ .

Here, by assumption we have  $n_1 \in \mathcal{S}_{i_0}$ , so no boundary condition on  $\partial\Omega_2$  has to be imposed and we only keep the boundary condition on  $\partial\Omega_2$ . Consequently, the amplitude  $u_{0,n_1}$  satisfies the transport equation

$$\begin{cases} (\partial_t + v_{n_1} \cdot \nabla_x) Q_1^{n_1} u_{0,n_1} = 0, \\ B_1 \left[ \sum_{k \in \Phi(n_1)} u_{0,k} \right]_{|x_1=0} = g, & \text{if } n_1 \notin \mathfrak{R}_1, \\ u_{0,n_1}|_{t \leq 0} = 0, \end{cases} \tag{41}$$

and

$$\begin{cases} (\partial_t + v_{n_1} \cdot \nabla_x) Q_1^{n_1} u_{0,n_1} = 0, \\ B_1 \left[ \sum_{k \in \Phi(n_1)} u_{0,k} + U_{0,n_1,1}|_{x_1=0} \right]_{|x_1=0} = g, & \text{if } n_1 \in \mathfrak{R}_1. \\ u_{0,n_1}|_{t \leq 0} = 0, \end{cases} \tag{42}$$

In both cases, using the fact that  $\Phi(n_1) \cap \mathcal{S}_{oi} = \{n_2\}$  thanks to (vi) of [Assumption 4.7](#) and (viii) of [Proposition 4.14](#), the boundary condition of (41) reads

$$u_{0,n_1}|_{x_1=0} + \sum_{k \in (\Phi(n_1) \cap (\mathcal{S}_{i_0} \cup \mathcal{S}_{ii})) \setminus \{n_1\}} u_{0,k}|_{x_1=0} = \phi_1^{n_1} [g - B_1 u_{0,n_2}|_{x_1=0}]$$

when  $n_1 \notin \mathfrak{R}_1$ , and

$$u_{0,n_1}|_{x_1=0} + \sum_{k \in (\Phi(n_1) \cap (\mathcal{S}_{i_0} \cup \mathcal{S}_{ii})) \setminus \{n_1\}} u_{0,k}|_{x_1=0} + U_{0,n_1,1}|_{x_1=x_1=0} = \phi_1^{n_1} [g - B_1 u_{0,n_2}|_{x_1=0}]$$

when  $n_1 \in \mathfrak{R}_1$ . Multiplying these conditions by the projector  $P_1^{n_1}$ , using the fact that the  $u_{0,k}$  are polarized on  $\ker \mathcal{L}(d\varphi_k)$ , we obtain, in both cases, that the trace  $u_{0,n_1}$  on  $\partial\Omega_1$  is given by

$$u_{0,n_1}|_{x_1=0} = S_1^{n_1} [g - B_1 u_{0,n_2}|_{x_1=0}],$$

where we recall that the matrix  $S_1^{n_1}$  has been introduced in [Definition 4.10](#).

It is now easy to integrate (41) along the characteristics. We obtain the expression of  $u_{0,n_1}$  according to its trace on  $\partial\Omega_1$ ; more precisely,

$$u_{0,n_1}(t, x) = S_1^{n_1} [g - B_1 u_{0,n_2}|_{x_1=0}] (t_{io}^{n_1}(t, x_1), x_{io}^{n_1}(x_1, x_2)),$$

where the new variables  $t_{io}^{n_1}$  and  $x_{io}^{n_1}$  are defined in (21). As a consequence, the trace of  $u_{0,n_1}$  on  $\partial\Omega_2$  is given by

$$u_{0,n_1}(t, x_1, 0) = S_1^{n_1} [g - B_1 u_{0,n_2}|_{x_1=0}] \left( t_{io}^{n_1}(t, x_1), -\frac{v_{n_1,2}}{v_{n_1,1}} x_1 \right). \tag{43}$$

Then we can repeat exactly the same reasoning for the second element  $n_2$  of the loop. Indeed, using the fact that  $n_2 \in \mathcal{S}_{oi}$ ,  $u_{0,n_2}$  will be determined by integration along the characteristics from its trace on  $\partial\Omega_2$ . Thanks to Assumption 4.7 and property (viii) of Proposition 4.14, the trace  $u_{0,n_2}|_{x_2=0}$  will depend only of the trace  $u_{0,n_3}|_{x_2=0}$ . The trace  $u_{0,n_2}|_{x_1=0}$ , which appears in (43), is consequently given by

$$u_{0,n_2}(t, 0, x_2) = -S_2^{n_2} B_2 u_{0,n_3}|_{x_2=0} \left( t_{oi}^{n_2}(t, x_2), -\frac{v_{n_2,1}}{v_{n_2,2}} x_2 \right). \tag{44}$$

At last, repeating the same method, we obtain the traces of the two remaining amplitudes for the indices of the loop:

$$u_{0,n_3}(t, x_1, 0) = -S_1^{n_3} B_1 u_{0,n_4}|_{x_1=0} \left( t_{io}^{n_3}(t, x_1), -\frac{v_{n_3,2}}{v_{n_3,1}} x_1 \right). \tag{45}$$

and

$$u_{0,n_4}(t, 0, x_2) = -S_2^{n_4} B_2 u_{0,n_1}|_{x_2=0} \left( t_{oi}^{n_4}(t, x_2), -\frac{v_{n_4,1}}{v_{n_4,2}} x_2 \right). \tag{46}$$

An important point in this analysis is that at each step of the computation, there is one and only one outgoing phase coupled with the incoming phases in the equivalence classes, for the relations  $\tilde{\mathfrak{F}}$  and  $\tilde{\mathfrak{V}}$ , of the indices  $n_j$ . This fact will, a priori, not be true if one considers a frequency set containing several loops.

Thus, combining equations (43)–(46) we obtain, after some computations, the functional equation determining the trace  $u_{0,n_1}|_{x_2=0}$ :

$$(I - \mathbb{T})u_{0,n_1}|_{x_2=0} = S_1^{n_1} g \left( t - \frac{1}{v_{n_1,1}} x_1, -\frac{v_{n_1,2}}{v_{n_1,1}} x_1 \right), \tag{47}$$

where  $\mathbb{T}$  is the operator defined by

$$(\mathbb{T}w)(t, x_1) := Sw(t + \alpha x_1, \beta x_1), \tag{48}$$

with

$$\begin{aligned} S &:= S_1^{n_1} B_1 S_2^{n_2} B_2 S_1^{n_3} B_1 S_2^{n_4} B_2, \\ \alpha &:= \frac{1}{v_{n_1,1}} \left( -1 + \frac{v_{n_1,2}}{v_{n_2,2}} - \frac{v_{n_1,2} v_{n_2,1}}{v_{n_2,2} v_{n_3,1}} + \frac{v_{n_1,2} v_{n_2,1} v_{n_3,2}}{v_{n_2,2} v_{n_3,1} v_{n_4,2}} \right) < 0, \\ \beta &:= \frac{v_{n_4,1}}{v_{n_4,2}} \frac{v_{n_3,2}}{v_{n_3,1}} \frac{v_{n_2,1}}{v_{n_2,2}} \frac{v_{n_1,2}}{v_{n_1,1}} > 0. \end{aligned} \tag{49}$$

Given (47), we make the following assumption:

**Assumption 4.15.** For all  $\gamma > 0$ , the operator  $(I - \mathbb{T})$  defined in (48) is invertible from  $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$  to  $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$  uniformly with respect with the parameter  $\gamma > 0$ .

However, for  $T > 0$  and for a source term in  $L^2([-\infty, T] \times \mathbb{R}_+)$  that is zero for negative times, this assumption will only give us amplitudes for indices of the loop which are  $L^2(\Omega_T)$ . This is not sufficient to construct the amplitudes of high order in the WKB expansion nor to make sure that the amplitudes linked with an incoming-incoming group velocity are  $H^1(\Omega_T)$ . We thus need to reinforce Assumption 4.15 in the following way:

**Assumption 4.16.** Let  $2 \leq K \leq \infty$ . For all  $\gamma > 0$ , the operator  $(I - \mathbb{T})$  defined in (48) is invertible from  $H_{f,\gamma}^K$  to  $H_{f,\gamma}^K$  uniformly with respect with  $\gamma > 0$ .

Let us stress that Assumption 4.16 is (at this stage of the analysis) purely formal and is introduced to construct the WKB expansion. We will show in Section 4D the following proposition:

**Proposition 4.17.** If  $|S| < \sqrt{\beta}$  (where  $S$  and  $\beta$  are defined in (49)), for all  $\gamma > 0$ , the operator  $(I - \mathbb{T})$  is uniformly invertible from  $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$  to  $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$ . In particular, for all  $T > 0$ , equation (47) admits a unique solution  $u \in L^2([-\infty, T] \times \mathbb{R}_+)$ , zero for negative times, if the source term  $G$  is in  $L^2(\partial\Omega_{1,T})$  and is zero for negative times.

If  $\beta \leq 1$  and  $G \in H_f^\infty$ , under the assumption  $|S| < \sqrt{\beta}$ , the solution  $u$  of the equation  $(I - \mathbb{T})u = G$  is in  $H_f^\infty$ .

If  $\beta > 1$  let  $K \in \mathbb{N}$  and  $G \in H_f^K$ ; then under the assumption  $|S|\beta^{K-\frac{1}{2}} < 1$ , the solution  $u$  of  $(I - \mathbb{T})u = G$  is in  $H_f^K$ .

Assumption  $|S| < \sqrt{\beta}$ , or  $|S|\beta^{K-\frac{1}{2}} < 1$ , gives us a framework in which we can, firstly ensure enough regularity to construct (at least up to a finite order) the amplitudes in the WKB expansion, and secondly construct the incoming-incoming amplitudes. More details and comments about the condition  $|S| < \sqrt{\beta}$  will be given in Section 4D.

From now on we denote by  $K \in \mathbb{N} \cup \{+\infty\}$  the largest integer such that the solution  $u$  of the equation  $(I - \mathbb{T})u = G$  is  $H_f^K$  for  $G \in H_f^K$ . In view of constructing the first corrector term and to ensure that the WKB expansion is a good approximation to the exact solution, we need  $K \geq 3$ .

From all these considerations about (47), it follows that the trace  $u_{0,n_1}|_{x_2=0}$  is uniquely determined in  $H_f^K$  by the formula

$$u_{0,n_1}(t, x_1, 0) = (I - \mathbb{T})^{-1} S_1^{n_1} g \left( t - \frac{1}{v_{n_1,1}} x_1, -\frac{v_{n_1,2}}{v_{n_1,1}} x_1 \right), \tag{50}$$

an equation which enables us to construct the amplitudes  $u_{0,n_j}$ ,  $j = 1, \dots, 4$ , by using (46), (45) and (44) and integrating along the corresponding characteristics.

We summarize this construction of the amplitudes associated with loop indices by the following proposition:

**Proposition 4.18.** *Under Assumptions 2.1–2.2 on the complete-for-reflections corner problem (18) and under Assumptions 4.7 and 4.16, for  $j = 1, \dots, 4$  and for all  $T > 0$ , there exist functions  $u_{0,n_j} \in H^K(\Omega_T)$ , with traces in  $H_f^K$ , satisfying the cascades of equations (26), (31), (32) and (33) written for  $n = 0$  and  $k = n_j$ .*

**4B3.** *Determination of the amplitudes in the direct neighborhood of the loop.* In this section we will show that the knowledge of the amplitudes on the loop and the global structure of the index set  $\mathcal{I}$  described in Section 4B1 are sufficient to construct the amplitudes in the direct neighborhood of the indices of the loop.

We have chosen to separate this construction from the construction of the amplitudes linked with indices in the different sets of the partition (36). This choice is motivated by the following two reasons. Firstly we think that it is important to make the computations explicit at least once (mainly because we have not yet described the construction for evanescent phases). Secondly, the construction in the close neighborhood of the loop remains unchanged, instead of the determination of the amplitudes associated with indices in the different sets of the partition (36), under a weaker uniqueness assumption of the loop (i.e., an assumption imposing the uniqueness of a self-interaction loop, but which allows other types of loops in the frequency set).

We will here describe the determination of the amplitudes in  $\Phi(n_4)$ ; the construction for amplitudes in  $\Psi(n_4)$ ,  $\Phi(n_2)$  or  $\Psi(n_2)$  is exactly the same. Using property (viii) of Proposition 4.14, we know that  $\Phi(n_4) \cap \mathcal{I}_{oi} = \{n_4\}$ . The boundary condition (31) written for  $k = n_4$  (if we choose  $n_4$  as a class representative of its own equivalence class for the relation  $\sim_{\mathfrak{F}}$ ) is given by

$$B_1 \left[ \sum_{j \in \Phi(n_4) \cap (\mathcal{I}_{io} \cup \mathcal{I}_{ii})} u_{0,j} + U_{0,n_4,1|_{X_1=0}} \right]_{|_{X_1=0}} = -B_1 u_{0,n_4|_{X_1=0}} \quad \text{if } n_4 \in \mathfrak{R}_1, \tag{51}$$

$$B_1 \left[ \sum_{j \in \Phi(n_4) \cap (\mathcal{I}_{io} \cup \mathcal{I}_{ii})} u_{0,j} \right]_{|_{X_1=0}} = -B_1 u_{0,n_4|_{X_1=0}} \quad \text{if } n_4 \in \mathfrak{B}_1 \setminus \mathfrak{R}_1, \tag{52}$$

where in both cases, the source term is a known element of  $H_f^K$ .

Applying the uniform Kreiss–Lopatinskii condition to equations (51) and (52), and composing by the projectors  $P_1^j$  for  $j \in \Phi(n_4) \cap (\mathcal{I}_{io} \cup \mathcal{I}_{ii})$ , and/or by  $P_{s,1}^{n_4}$ , leads us to solve the uncoupled boundary conditions

$$\forall j \in \Phi(n_4) \cap (\mathcal{I}_{io} \cup \mathcal{I}_{ii}), \quad u_{0,j|_{X_1=0}} = -S_1^j B_1 u_{0,n_4|_{X_1=0}}, \tag{53}$$

and if, moreover,  $n_4 \in \mathfrak{R}_1$ ,

$$U_{0,n_4,1|_{X_1=x_1=0}} = -S_{s,1}^j B_1 u_{0,n_4|_{X_1=0}}. \tag{54}$$

Thus, the construction of the possible evanescent amplitude for the side  $\Omega_1$  can be made independently of the construction of the amplitudes for oscillating phases.

Let us first briefly recall how to determine amplitudes for oscillating phases. We have several cases to take into account.

$j \in \mathcal{I}_{\text{io}}$ : Lax’s lemma and the polarization condition enable us to show that the amplitude  $u_{0,j}$  satisfies a transport equation with an incoming-outgoing velocity  $v_j$ . That is why to construct this amplitude we just need to know its trace on  $\partial\Omega_1$ . This trace is determined by (53), so integrating along the characteristics,  $u_{0,j}$  is given by

$$u_{0,j}(t, x) = S_1^j B_1 u_{0,n_4|x_1=0}(t_{\text{io}}^j(t, x_1), x_{\text{io}}^j(x_1, x_2)). \tag{55}$$

An important point for the end of the proof (more specifically for the construction of incoming-incoming amplitudes in the set  $B_{b_j}$ ) is that  $u_{0,j} \in H^K(\Omega_T)$  for all  $T > 0$  and that its trace on the side  $\partial\Omega_2$  is  $H_f^K$ . We can easily see this fact in the formula (55).

In other words, the flatness at the corner of the source term  $g^\varepsilon$  is transmitted to the amplitudes close to the loop.

$j \in \mathcal{I}_{\text{ii}}$ : In this case,  $u_{0,j}$  is solution to a transport equation with incoming-incoming velocity, so its determination needs the traces on both sides  $\partial\Omega_1$  and  $\partial\Omega_2$ . The boundary condition (53) gives the trace on  $\partial\Omega_1$ . Concerning the trace on  $\partial\Omega_2$ , property (ix) of Proposition 4.14 shows that  $j$  is the only element in its equivalence class for the relation  $\approx_\Psi$ ; in particular,  $j \notin \mathfrak{R}_2$ . So the boundary condition (31) written for  $k = j$  reads

$$B_2 u_{0,j|x_2=0} = 0.$$

Using the uniform Kreiss–Lopatinskii condition, it follows that  $u_{0,j|x_2=0} = 0$ . So the amplitude  $u_{0,j}$  satisfies the transport equation

$$u_{0,j} = P_1^j u_{0,j} = P_2^j u_{0,j}, \quad \begin{cases} (\partial_t + v_j \cdot \nabla_x) Q_1^j u_{0,j} = 0, \\ u_{0,j|x_1=0} = -S_1^j B_1 u_{0,n_4|x_1=0}, \quad u_{0,j|x_2=0} = 0, \quad u_{0,j|t \leq 0} = 0. \end{cases}$$

To solve this transport equation, we use the flatness at the corner of  $u_{0,n_4|x_1=0}$  to extend the problem in the half-space  $\{x_1 \geq 0\}$  by extending  $u_{0,j}$  by zero to  $\{x_2 < 0\}$ , we integrate along the characteristics, and then we restrict the constructed solution to the quarter-space. The obtained solution  $u_{0,j}$  is in  $H^K(\Omega_T)$ , thanks to the fact that  $u_{0,n_4|x_1=0}$  is flat at the corner.

One can also easily check that the obtained solution  $u_{0,j}$  satisfies the property: if  $x_1 \geq 0$ , then  $u_{0,j|x_1=x_1} \in H_f^K$ . This extra regularity of  $u_{0,j}$  will also be needed during the construction of higher-order terms.

$n_4 \in \mathfrak{R}_1$ : The determination of the amplitude associated with an evanescent index for the side  $\partial\Omega_1$  (or even  $\partial\Omega_2$ ) follows (in some sense) the same kind of ideas as the determination of amplitudes linked with oscillating indices. Indeed, it will be easy to construct the amplitude linked with an evanescent index if we know its trace (on  $\partial\Omega_1$  for elements of  $\mathcal{I}_{\text{ev}1}$  and the trace on  $\partial\Omega_2$  for indices of  $\mathcal{I}_{\text{ev}2}$ ).

However, we will in this proof treat the evanescent modes in only one block, as in [Lescarret 2007]; that is why the associated amplitudes will not satisfy transport equations as in the oscillating case. Thus, we first recall the evolution equations and the boundary conditions satisfied by such amplitudes and then we will give a method to solve these equations.

Plugging the ansatz (25) in the evolution equation of the corner problem (18) we have seen that the amplitude  $U_{n,n_4,1}$  has to satisfy the cascade of equations

$$\begin{cases} L_{n_4}(\partial_{X_1})U_{0,n_4,1} = 0, \\ L_{n_4}(\partial_{X_1})U_{n,n_4,1} + L(\partial)U_{n-1,n_4,1} = 0, \quad \forall n \geq 1, \end{cases} \tag{56}$$

where

$$L_{n_4}(\partial_{X_1}) := A_1(\partial_{X_1} - \mathcal{A}_1(\underline{\tau}, \xi_2^{n_4})).$$

The boundary condition has also already been studied in the case  $n_4 \in \mathfrak{R}_1$  and is given by (51). So  $U_{0,n_4,1}$  has to satisfy the system

$$\begin{cases} L_{n_4}(\partial_{X_1})U_{0,n_4,1} = 0, \\ B_1 \left[ \sum_{j \in \Phi(n_4) \cap (\mathcal{J}_{i0} \cup \mathcal{J}_{ii})} u_{0,j} + U_{0,n_4,1}|_{X_1=0} \right]_{|X_1=0} = -B_1 u_{0,n_4}|_{X_1=0}, \\ U_{0,n_4,1}|_{t \leq 0} = 0, \end{cases} \tag{57}$$

and one has also to keep in mind that the boundary condition (32) will also have to be satisfied. First let us solve (57). Let us recall the following lemma from [Lescarret 2007], which permits us to solve (56) in the profile space  $P_{ev,1}$ .

**Lemma 4.19.** *For  $i = 1, 2$ , and  $\underline{k} \in \mathfrak{R}_i$ , let*

$$\mathbb{P}_{ev,i}^{\underline{k}} U(X_i) := e^{X_i \mathcal{A}_i(\underline{\tau}, \xi_{3-i}^{\underline{k}})} P_{s,i}^{\underline{k}} U(0), \tag{58}$$

$$\mathbb{Q}_{ev,i}^{\underline{k}} F(X_i) := \int_0^{X_i} e^{(X_i-s) \mathcal{A}_i(\underline{\tau}, \xi_{3-i}^{\underline{k}})} P_{s,i}^{\underline{k}} A_i^{-1} F(s) ds - \int_{X_i}^{+\infty} e^{(X_i-s) \mathcal{A}_i(\underline{\tau}, \xi_{3-i}^{\underline{k}})} P_{u,i}^{\underline{k}} A_i^{-1} F(s) ds. \tag{59}$$

Then, for all  $F \in P_{ev,i}$ , the equation

$$L_{\underline{k}}(\partial_{X_i})U = F$$

admits a solution in  $P_{ev,i}$ . Moreover, this solution is given by

$$U = \mathbb{P}_{ev,i}^{\underline{k}} U + \mathbb{Q}_{ev,i}^{\underline{k}} F.$$

This lemma tells us that the evanescent amplitude of leading order  $U_{0,n_4,1}$  satisfies  $\mathbb{P}_{ev,1}^{n_4} U_{0,n_4,1} = U_{0,n_4,1}$ . This relation is analogous to the polarization condition for oscillating phases and thanks to the definition of  $\mathbb{P}_{ev,1}^{n_4}$ , enables us to determine  $U_{0,n_4,1}$  if we know its trace on  $\{X_1 = 0\}$ .

Unfortunately the system (57) does not give any information about this trace but only on the “double” trace on  $\{x_1 = X_1 = 0\}$ . This is determined by

$$U_{0,n_4,1}|_{X_1=x_1=0} = -S_{s,1}^j B_1 u_{0,n_4}|_{x_1=0}.$$

It is then sufficient to lift the “double” trace in a “single” one. As in [Lescarret 2007], for example, choose

$$U_{0,n_4,1}(t, x, 0) := -\chi(x_1) S_{s,1}^j B_1 u_{0,n_4}|_{x_1=0},$$

where  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  satisfies  $\chi(0) = 1$ .



Now that the trace of  $U_{0,n_4,1}$  on  $\{X_1 = 0\}$  is determined, we can apply the operator  $\mathbb{P}_{\text{ev},1}^{n_4}$ . Thus by construction, the amplitude

$$U_{0,n_4,1}(t, x, X_1) = -\chi(x_1)e^{X_1 \mathcal{A}_1(\underline{x}, \xi_2^{n_4})} S_{s,1}^{n_4} B_1 u_{0,n_4|x_1=0}(t, x_2) \tag{60}$$

is a solution to the system of equations (57).

Now, we consider the contribution of  $U_{0,n_4,1}(t, x_1, 0, x_1/\varepsilon)$  in (32). From (60), this trace is explicitly given by

$$U_{0,n_4,1}\left(t, x_1, 0, \frac{x_1}{\varepsilon}\right) = -\chi(x_1)e^{\frac{x_1}{\varepsilon} \mathcal{A}_1(\underline{x}, \xi_2^{n_4})} S_{s,1}^{n_4} B_1 u_{0,n_4|x_1=x_2=0},$$

but from the flatness of  $u_{0,n_4|x_1=0}$  at the corner, it follows that  $U_{0,n_4,1}(t, x_1, 0, x_1/\varepsilon)$  is zero and that it does not contribute<sup>4</sup> to (32).

The determination of evanescent amplitudes for the side  $\partial\Omega_2$  that can appear when we construct the amplitudes associated with indices in  $\Psi(n_3)$  or  $\Psi(n_1)$  is totally similar. For example, for the indices in  $\Psi(n_3)$ , we will start by determining  $U_{0,n_3,2}$  on  $\{x_2 = X_2 = 0\}$  by using the boundary condition; then we lift this double trace in a single one on  $\{X_2 = 0\}$ . This trace is finally propagated in the interior of  $\Omega$  by the operator  $\mathbb{P}_{\text{ev},2}^{n_3}$  defined in (58). We also obtain that  $U_{0,n_3,2}(t, 0, x_2, x_2/\varepsilon)$  is zero and as a consequence,  $U_{0,n_3,2}(t, 0, x_2, x_2/\varepsilon)$  does not contribute in (32).

Then we repeat this construction for all the indices in the direct neighborhood of the loop, so the indices whose amplitudes have still to be determined in partition (36) are

$$\left(\bigcup_{a_l} A_{a_l} \setminus \{a_l\}\right) \cup \left(\bigcup_{b_m} B_{b_m} \setminus \{b_m\}\right) \cup \left(\bigcup_{c_q} C_{c_q} \setminus \{c_q\}\right) \cup \left(\bigcup_{d_r} D_{d_r} \setminus \{d_r\}\right);$$

that is to say, it only remains to determine the amplitudes linked with indices in the trees of the partition (36). Before constructing these amplitudes, we will need to have a more precise description of the structure of those trees. It is the subject of the following section.

**4B4. Local structure of the trees.** Let us concentrate on the internal structure of the trees  $A_{a_l}$  appearing in the partition (36) of  $\mathcal{S}$ . The description for the trees  $B_{b_m}$ ,  $C_{c_q}$  and  $D_{d_r}$  is, up to a few modifications, analogous and will not be given in detail here. Let us recall that a tree  $A_{a_l}$  has for its root an index  $a_l \in (\Psi(n_1) \cap \mathcal{S}_{\text{oi}}) \setminus \{n_4\}$  and is the set of indices  $j$  linked with  $a_l$  by a sequence of type  $H$  (see Definition 4.4). To simplify the future notations, we will define  $A_{a_l} := A_a$ .

The following proposition has already been mentioned in Section 4B1, and is the main proposition needed to understand the structure of  $A_a$ .

**Proposition 4.20.** *Let  $j \in A_a$ . Then there exists a unique sequence  $\ell$  of type  $H$  linking  $j$  to  $a$ .*

*Proof.* By contradiction, let  $\ell = (\ell_1, \ell_2, \dots, \ell_p)$  and  $\ell' = (\ell'_1, \ell'_2, \dots, \ell'_{p'})$ ,  $\ell \neq \ell'$ , be two sequences of type  $H$  which link  $j$  to  $a$ . We will separate several cases depending on the oddness/evenness of the lengths  $p$  and  $p'$ , and without loss of generality we assume that  $p \leq p'$ .

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<sup>4</sup>We will in fact show in the following that all the contributions of the evanescent phases for the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ) in (32) are zero. As a consequence, the boundary condition (32) is in fact trivially satisfied as soon as the oscillating amplitudes remain flat at the corner.

$p, p' \in 2\mathbb{N}$ : We have to distinguish two different subcases:

- If  $\ell' = (\ell, \ell'_{p+1}, \dots, \ell_{p'})$ , then  $\ell'_{p+1} \in \Phi(\ell'_{p'})$ . This allows us to show that the sequence  $(\ell'_{p+2}, \dots, \ell_{p'})$  is a loop for the index  $\ell'_{p+1}$  of length  $p' - p - 1$ . Thanks to [Assumption 4.7](#), it is impossible.
- If  $\ell' \neq (\ell, \ell'_{p+1}, \dots, \ell_{p'})$ , let  $m$  be the first integer such that  $\ell_m \neq \ell'_m$ . From the preceding subcase, we can assume that  $1 \leq m < p$ . We will here deal with the case  $m \in 2\mathbb{N} + 1$  (the case  $m \in 2\mathbb{N}$  can be treated in a similar way, up to modification of the type of sequence). We have,  $\ell_m \in \Phi(\ell'_m)$ .

We have, once again two different possibilities:

- There exists  $l$ , where  $m + 1 < l \leq p$ , such that  $k_l = k'_l$ . Then let  $\underline{l}$  be the first integer  $l$ , where  $m + 1 < l \leq p$ , such that  $k_l = k'_l$ . Then if  $\underline{l} \in 2\mathbb{N}$  (resp.  $\underline{l} \in 2\mathbb{N} + 1$ ), we have  $\ell_{\underline{l}-1} \in \Psi(\ell'_{\underline{l}-1})$  (resp.  $\ell_{\underline{l}-1} \in \Phi(\ell'_{\underline{l}-1})$ ). Consequently the sequence  $(\ell'_m, \dots, \ell'_{\underline{l}-1})$  is a sequence of type  $H$  linking  $\ell_{\underline{l}-1}$  to  $\ell_m$ , and the sequence  $(\ell_{m+1}, \dots, \ell_{\underline{l}-2})$  is a sequence of type  $V$  linking  $\ell_{\underline{l}-1}$  to  $\ell_m$ . From this observation, we deduce that the sequence  $(\ell'_m, \dots, \ell'_{\underline{l}-1}, \ell_{\underline{l}-1}, \ell_{\underline{l}-2}, \dots, \ell_{m+1})$  is a loop for the index  $\ell_m$ . Once again, this fact contradicts [Assumption 4.7](#).
- If, for all  $q \in \{m+2, \dots, p\}$ , the indices  $\ell_q$  and  $\ell'_q$  are distinct, we easily see that  $(\ell'_m, \dots, \ell'_p, \ell_p, \dots, \ell_{m-1})$  is a loop for  $\ell_m$ .

We now consider the second subcase, that is to say:

$p \in 2\mathbb{N}, p' \in 2\mathbb{N} + 1$ : If  $\ell' = (\ell, \ell'_{p+1}, \dots, \ell'_{p'})$ , we can show that  $(\ell'_{p+1}, \dots, \ell'_{p'})$  is a loop for  $j$ , whereas, if  $\ell' \neq (\ell, \ell'_{p+1}, \dots, \ell'_{p'})$ , we can repeat the analysis made in the subcase  $p, p' \in 2\mathbb{N}$  to treat the subcase, “There exists  $l$ , where  $m + 1 < l \leq p$ , such that  $\ell_l = \ell'_l$ ”. If, for all  $q \in \{m + 2, \dots, p\}$ , the indices  $\ell_q$  and  $\ell'_q$  are distinct, we can easily show that  $(\ell'_m, \dots, \ell'_p, j, \ell_p, \dots, \ell_{m-1})$  is a loop for  $\ell_m$ .

The other case,  $p, p' \in 2\mathbb{N} + 1$ , is analogous, up to the inversion of the role played by the functions  $\Phi$  and  $\Psi$ , to the case  $p, p' \in 2\mathbb{N}$ . This case, is left to the reader. □

**Remark.** As indicated in [Section 4B1](#), the uniqueness of the sequence linking  $\underline{j} \in A_a$  to the root  $\underline{a}$  depends, in a nontrivial way, on [Assumption 4.7](#).

Thanks to [Proposition 4.20](#), it is now possible to give a more precise (and final) version of the properties satisfied by functions  $\Phi$  and  $\Psi$ :

**Proposition 4.21.** *Let  $\underline{j} \in A_a \setminus \{a\}$ . We denote by  $\ell = (\ell_1, \dots, \ell_p)$  the sequence of type  $H$  linking  $\underline{j}$  to  $\underline{a}$ . Then, according to the parity of  $p$ , we have:*

- (i) *If  $p \in 2\mathbb{N}$ , then  $\underline{j} \notin \mathcal{S}_{oi}$ . Moreover, if  $\underline{j} \in \mathcal{S}_{ev1} \cup \mathcal{S}_{ii}$  then  $\Psi(\underline{j}) = \{\underline{j}\}$ .*
- (ii) *If  $p \in 2\mathbb{N} + 1$ , then  $\underline{j} \notin \mathcal{S}_{io}$ . Moreover, if  $\underline{j} \in \mathcal{S}_{ev2} \cup \mathcal{S}_{ii}$  then  $\Phi(\underline{j}) = \{\underline{j}\}$ .*

*Proof.* We will consider the case  $p \in 2\mathbb{N}$ . Let us first show that  $\underline{j} \notin \mathcal{S}_{oi}$ . By contradiction, we assume that  $\underline{j} \in \mathcal{S}_{oi}$ , but using the fact that the frequency set  $\mathcal{F}$  is minimal, we can assume that  $\Psi(\underline{j}) \cap \mathcal{S}_{io} \neq \emptyset$ .

Let  $\underline{i} \in \Psi(\underline{j}) \cap \mathcal{S}_{io}$ . According to the analysis made in [Section 4B1](#), we have  $\underline{i} \in A_a$ . Let  $\ell' = (\ell'_1, \dots, \ell'_{p'})$  be the sequence linking  $\underline{i}$  to the root  $\underline{a}$ . Reiterating the arguments used in the proof of [Proposition 4.20](#), it is sufficient to study the case  $\ell_i \neq \ell'_i$  for all  $i$ .

If  $p' \in 2\mathbb{N}$ , we can then show that  $(\ell', i, j, \ell_p, \dots, \ell_2)$  is a loop with an odd number of elements for the index  $\ell_1$ , whereas if  $p' \in 2\mathbb{N} + 1$ , the sequence  $(\ell', i, \ell_p, \dots, \ell_2)$  is a loop for  $\ell_1$ . Both cases contradict [Assumption 4.7](#).

The proof of the assertion “If  $j \in \mathcal{S}_{ii} \cup \mathcal{S}_{ev1}$ , then  $\Psi(j) = \{j\}$ ” follows the same reasoning. □

The same proposition, up to some adaptations on the oddness/evenness according to the considered tree, is true for all trees in partition [\(36\)](#).

In terms of wave packet reflection, [Proposition 4.21](#) states that, on one hand, during a reflection on the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ), an outgoing-incoming phase (resp. incoming-outgoing) cannot generate (resp. incoming-outgoing) outgoing-incoming phases. This “natural” idea used in [\[Sarason and Smoller 1974\]](#) is now rigorously justified.

**4B5. Determination of the amplitudes for indices in the trees.** Thanks to the precise description of the internal structure of the different trees in the partition [\(36\)](#), it is easy to determine all the remaining amplitudes in the WKB expansion, and to conclude the construction of the leading-order terms. Once again, we will here only deal with a tree  $A_{\underline{a}}$ . The construction is analogous for the other trees.

Let  $\underline{j}$  be any index of  $A_{\underline{a}}$ . We will show that it is always possible to determine the amplitude  $u_{0,\underline{j}}$ . Thanks to [Proposition 4.20](#), there exists a unique sequence of type  $H$ , denoted by  $\ell^{\underline{j}} = \ell = (\ell_1, \dots, \ell_p)$ , linking  $\underline{j}$  to the root  $\underline{a}$ . The first step in the construction of the amplitude  $u_{0,\underline{j}}$  is to remark that independently of the determination of  $u_{0,\underline{j}}$ , we can always first determine the amplitudes  $u_{0,\ell_i}$ ,  $i = 1, \dots, p$ .

Indeed, by the definition of sequences of type  $H$  (see [Definition 4.4](#)),  $\ell_1 \in \Phi(\underline{a}) \cap \mathcal{S}_{io}$ . So, the amplitude  $u_{0,\ell_1}$  satisfies a transport equation given by

$$\left\{ \begin{aligned} &(\partial_t + v_{\ell_1} \cdot \nabla_x) Q_1^{\ell_1} u_{0,\ell_1} = 0, \\ &B_1 \left[ \sum_{i \in \Phi(\ell_1) \cap (\mathcal{S}_{io} \cup \mathcal{S}_{ii})} u_{0,i|_{x_1=0}} \right] = -B_1 u_{0,\underline{a}|_{x_1=0}}, \quad \text{if } \ell_1 \in \mathcal{B}_1 \setminus \mathcal{R}_1, \\ &u_{0,\ell_1|_{t \leq 0}} = 0, \end{aligned} \right. \tag{61}$$

$$\left\{ \begin{aligned} &(\partial_t + v_{\ell_1} \cdot \nabla_x) Q_1^{\ell_1} u_{0,\ell_1} = 0, \\ &B_1 \left[ \sum_{i \in \Phi(\ell_1) \cap (\mathcal{S}_{io} \cup \mathcal{S}_{ii})} u_{0,i} + U_{0,\ell_1,1|_{x_1=0}} \right]_{|_{x_1=0}} = -B_1 u_{0,\underline{a}|_{x_1=0}}, \quad \text{if } \ell_1 \in \mathcal{R}_1, \\ &u_{0,\ell_1|_{t \leq 0}} = 0, \end{aligned} \right.$$

because all the elements of  $\Phi(\ell_1)$  are linked with  $\underline{a}$  by a sequence of length zero. Thanks to property (i) of [Proposition 4.21](#), it follows that  $\Phi(\ell_1) \cap \mathcal{S}_{oi} = \{\underline{a}\}$ . Consequently, multiplying [\(61\)](#) by  $S_1^{\ell_1}$  (see [Definition 4.10](#)), we can write

$$u_{0,\ell_1|_{x_1=0}} = -S_1^{\ell_1} B_1 u_{0,\underline{a}|_{x_1=0}}.$$

This equation determines the trace of  $u_{0,\ell_1}$  on the side  $\partial\Omega_1$  because the amplitude  $u_{0,\underline{a}}$  and its trace have already been determined in [Section 4B3](#). Integrating [\(61\)](#) along the characteristics, we determine  $u_{0,\ell_1} \in H^K(\Omega_T)$  and the trace  $u_{0,\ell_1|_{x_2=0}} \in H_f^K$ .

Then we are interested in the construction of the amplitude  $u_{0,\ell_2}$ , by the definition of type- $H$  sequences,  $\ell_2 \in \Psi(\ell_1) \cap \mathcal{S}_{oi}$ . Once again, we can apply [Proposition 4.21](#) to show that

$$\Psi(\ell_2) \cap \mathcal{S}_{io} = \{\ell_1\}.$$

This allows us to rewrite the transport equation on  $u_{0,\ell_2}$  in the form

$$\begin{cases} (\partial_t + v_{\ell_2} \cdot \nabla_x) Q_2^{\ell_2} u_{0,\ell_2} = 0, \\ u_{0,\ell_2}|_{x_2=0} = -S_2^{\ell_2} B_2 u_{0,\ell_1}|_{x_2=0}, \quad u_{0,\ell_2}|_{t \leq 0} = 0, \end{cases} \tag{62}$$

and we solve this equation by integration along the characteristics.

We can reiterate the same kind of resolutions of transport equations for all the indices of the sequence  $\ell$ . This operation permits us to construct all the amplitudes  $u_{0,\ell_l}$ ,  $l = 1, \dots, p$ . The important point in these recursive resolutions is that since the first amplitude  $u_{0,\underline{a}}$  has its trace on  $\partial\Omega_1$  in  $H_f^K$ , this flatness at the corner is transmitted to all the amplitudes indexed by the sequence  $\ell$ .

Indeed, integration along the characteristics gives an explicit formula and it is easy to see on this formula that the traces of the  $u_{0,\ell_l}$  are in  $H_f^K$  for all  $1 \leq l \leq p$ .

Once we have constructed all the amplitudes associated to the indices of  $\ell$ , it is easy to determine the amplitude  $u_{0,\underline{j}}$ . We distinguish the following cases depending of the nature of the index  $\underline{j}$ .

- $\underline{j} \in \mathcal{S}_{io}$  (resp.  $\underline{j} \in \mathcal{S}_{oi}$ ): [Proposition 4.21](#) tells us that an index in  $\mathcal{S}_{io} \cap A_{\underline{a}}$  (resp.  $\mathcal{S}_{oi}$ ) can appear only after an even (resp. odd) number of reflections, in other terms, the length of the sequence  $\ell$ ,  $p \in 2\mathbb{N}$  (resp.  $p \in 2\mathbb{N} + 1$ ). Using the fact that  $\underline{j} \in \mathcal{S}_{io}$  (resp.  $\underline{j} \in \mathcal{S}_{oi}$ ), to construct the amplitude  $u_{0,\underline{j}}$  it is sufficient to know  $u_{0,\underline{j}}|_{x_1=0}$  (resp.  $u_{0,\underline{j}}|_{x_2=0}$ ). But, [Proposition 4.21](#) implies that  $\ell_p$  is the only element of  $\mathcal{S}_{oi}$  (resp.  $\mathcal{S}_{io}$ ) in  $\Phi(\underline{j})$  (resp.  $\Psi(\underline{j})$ ); multiplying by  $S_1^{\underline{j}}$  (resp.  $S_2^{\underline{j}}$ ), we can determine the trace  $u_{0,\underline{j}}|_{x_1=0}$  (resp.  $u_{0,\underline{j}}|_{x_2=0}$ ) as a function of the trace  $u_{0,\ell_p}|_{x_1=0}$  (resp.  $u_{0,\ell_p}|_{x_2=0}$ ). Consequently the amplitude  $u_{0,\underline{j}}$  is constructed.

Moreover, we can show that  $u_{0,\underline{j}} \in H^K(\Omega)$ , and that its traces on the sides  $\partial\Omega_1$  and  $\partial\Omega_2$  are in  $H_f^K$ . This fact will be crucial to construct the incoming-incoming phases as well as the evanescent phases that may appear in the WKB expansion.

- $\underline{j} \in \mathcal{S}_{ii}$ : An incoming-incoming index may appear after an even number of reflections as well as after an odd number of reflections. We will here deal with the case  $p \in 2\mathbb{N}$ ; the case  $p \in 2\mathbb{N} + 1$  is totally similar. [Proposition 4.21](#) implies, on the one hand, that  $\ell_p$  is the only index in  $\Phi(\underline{j}) \cap \mathcal{S}_{oi}$  and, on the other hand, that  $\Psi(\underline{j}) = \{\underline{j}\}$ . So the boundary conditions for the amplitude  $u_{0,\underline{j}}$  can be written in the form

$$u_{0,\underline{j}}|_{x_1=0} = -S_1^{\underline{j}} B_1 u_{0,\ell_p}|_{x_1}, \quad u_{0,\underline{j}}|_{x_2=0} = 0.$$

It follows that the amplitude  $u_{0,\underline{j}}$  satisfies the incoming-incoming transport equation

$$\begin{cases} (\partial_t + v_{\underline{j}} \cdot \nabla_x) Q_1^{\underline{j}} u_{0,\underline{j}} = 0, \\ u_{0,\underline{j}}|_{x_1=0} = -S_1^{\underline{j}} B_1 u_{0,\ell_p}|_{x_1=0}, \quad u_{0,\underline{j}}|_{x_2=0} = 0, \quad u_{0,\underline{j}}|_{t \leq 0} = 0. \end{cases} \tag{63}$$

To solve this equation, we extend the source term  $-S_1^j B_1 u_{0,\ell_p|_{x_1=0}}$  by zero on  $\{x_2 < 0\}$  (this extension gives a regular function because  $u_{0,\ell_p|_{x_1=0}} \in H_f^K$ ); then we restrict to  $\{x_2 \geq 0\}$  the solution to the transport equation in the half-space  $\{x_1 \geq 0, x_2 \in \mathbb{R}\}$ .

Consequently we have constructed  $u_{0,\underline{j}} \in H^K(\Omega_T)$ , such that for all  $x_1 \geq 0$ , we have  $u_{0,\underline{j}|_{x_1=x_1}} \in H_f^K$ .

- $\underline{j} \in \mathfrak{R}_1$  (resp.  $\underline{j} \in \mathfrak{R}_2$ ): As in the case  $\underline{j} \in \mathcal{S}_{i_0}$  (resp.  $\underline{j} \in \mathcal{S}_{oi}$ ), [Proposition 4.21](#) tells us that an evanescent index for the side  $\partial\Omega_1$  (resp.  $\partial\Omega_2$ ) can only appear after an even (resp. odd) number of reflections. Moreover, [Proposition 4.21](#) also implies that the only index of  $\mathcal{S}_{oi}$  (resp.  $\mathcal{S}_{i_0}$ ) and  $\Phi(\underline{j})$  (resp.  $\Psi(\underline{j})$ ) is  $\ell_p$ . We are now interested in the construction of the evanescent amplitude  $U_{0,\underline{j},1}$  (resp.  $U_{0,\underline{j},2}$ ).

Repeating the construction described in [Section 4B3](#), to determine the amplitude  $U_{0,\underline{j},1}$  (resp.  $U_{0,\underline{j},2}$ ), it is sufficient to start by determining the double trace on  $\{X_1 = x_1 = 0\}$  (resp.  $\{X_2 = x_2 = 0\}$ ). Using the fact that  $\ell_p$  is the only element of  $\mathcal{S}_{oi}$  (resp.  $\mathcal{S}_{i_0}$ ) in  $\Phi(\underline{j})$  (resp.  $\Psi(\underline{j})$ ) allows us to show that this double trace is given by

$$U_{0,\underline{j},1}(t, 0, x', 0) = -S_{s,1}^j B_1 u_{0,\ell_p|_{x_1=0}}$$

$$\text{(resp. } U_{0,\underline{j},2}(t, x', 0, 0) = -S_{s,2}^j B_2 u_{0,\ell_p|_{x_2=0}}).$$

We then lift this double trace to a single one by setting

$$U_{0,\underline{j},1}(t, x, 0) := -\chi(x_1) S_{s,1}^j B_1 u_{0,\ell_p|_{x_1=0}}$$

$$\text{(resp. } U_{0,\underline{j},2}(t, x, 0) := -\chi(x_2) S_{s,2}^j B_2 u_{0,\ell_p|_{x_2=0}}),$$

where  $\chi \in \mathcal{C}_c^\infty(\mathbb{R})$  is such that  $\chi(0) = 1$ . Finally, [Lemma 4.19](#) shows that the function  $U_{0,\underline{j},1}$  (resp.  $U_{0,\underline{j},2}$ ) defined by

$$U_{0,\underline{j},1}(t, x, X_1) = -\chi(x_1) e^{X_1 \mathcal{A}_1} S_{s,1}^j B_1 u_{0,\ell_p|_{x_1=0}} \tag{64}$$

$$\text{(resp. } U_{0,\underline{j},2}(t, x, X_2) = -\chi(x_2) e^{X_2 \mathcal{A}_2} S_{s,2}^j B_2 u_{0,\ell_p|_{x_2=0}})$$

is a solution to the cascades of equations (26), (31) and (33) written for  $n = 0$  and  $k = \underline{j}$ .

We are now interested in the influence of the evanescent phases previously constructed in the extra boundary condition (32). For example, let us study the trace on  $\{x_2 = 0\}$  of an evanescent phase for the side  $\partial\Omega_1$ . From (64) this trace is explicitly given by

$$U_{0,\underline{j},1}\left(t, x_1, 0, \frac{x_1}{\varepsilon}\right) = -\chi(x_1) e^{X_1 \mathcal{A}_1} S_{s,1}^j B_1 u_{0,\ell_p|_{x_1=0}}(t, 0)$$

because the only term depending on  $x_2$  in (64) is  $u_{0,\ell_p|_{x_1=0}}$ . The flatness at the corner of  $u_{0,\ell_p|_{x_1=0}}$  shows that  $U_{0,\underline{j},1}(t, x_1, 0, x_1/\varepsilon)$  is in fact zero and that it does not contribute in (32). The same result is also true for all the other evanescent amplitudes (for both sides) in the tree  $A_a$ .

In this section we have shown that an arbitrary amplitude in the tree  $A_a$  can always be constructed. As a consequence, all the amplitudes in the tree  $A_a$  can be determined. Then it is sufficient to repeat the method of construction for each tree in the partition (36). So we have constructed all the amplitudes

associated with indices in  $\mathcal{I} \setminus \{n_1, n_2, n_3, n_4\}$ , and we have thus finished the construction of the leading term in the WKB expansion. We summarize the analysis with the following proposition:

**Proposition 4.22.** *Under Assumptions 2.1–2.2 on the complete-for-reflections corner problem (18) and under Assumptions 4.7–4.9 and 4.16, there exist functions  $(u_{0,i})_{i \in \mathcal{I}_{os}}$ ,  $(U_{0,i,1})_{i \in \mathfrak{R}_1}$  and  $(U_{0,i,2})_{i \in \mathfrak{R}_2}$  satisfying the cascades of equations (26), (31), (32) and (33) written for  $n = 0$ .*

*Moreover, the functions  $(u_{0,i})_{i \in \mathcal{I}_{os}}$  admit the following regularity. For all  $T > 0$ :*

- *If  $i \in \mathcal{I}_{io} \cup \mathcal{I}_{oi}$  then  $u_{0,i} \in H^K(\Omega_T)$  and the traces  $u_{0,i}|_{x_1=0}$  and  $u_{0,i}|_{x_2=0}$  are in  $H_f^K$ .*
- *If  $i \in \mathcal{I}_{ii}$  then  $u_{0,i} \in H^K(\Omega_T)$ . Moreover, if  $\Psi(i) = \{i\}$  (resp.  $\Phi(i) = \{i\}$ ) then for all  $\underline{x}_1 > 0$  (resp.  $\underline{x}_2 > 0$ ), the trace  $u_{0,i}|_{x_1=\underline{x}_1}$  (resp.  $u_{0,i}|_{x_2=\underline{x}_2}$ ) is  $H_f^K$ .*

*The functions  $(U_{0,i,1})_{i \in \mathfrak{R}_1}$  (resp.  $(U_{0,i,2})_{i \in \mathfrak{R}_2}$ ) are in  $P_{ev1}$  (resp.  $P_{ev2}$ ). Moreover, all the functions  $(U_{0,i,1})_{i \in \mathfrak{R}_1}$  (resp.  $(U_{0,i,2})_{i \in \mathfrak{R}_2}$ ) satisfy  $U_{0,i,1}(t, x_1, 0, X_1) = 0$  (resp.  $U_{0,i,2}(t, 0, x_2, X_2) = 0$ ).*

**4B6.** *Construction of the higher-order terms in the WKB expansion.* The construction for the higher-order terms in the WKB expansion looks like the construction of the leading-order term. In particular, the order of resolution will be the same: we start with the amplitudes on the loop, and then we show that the knowledge of these amplitudes is sufficient to construct any amplitude in the trees of the partition (36). In this section we only give the main steps of the construction of the term of order  $\varepsilon$ , without all details. Let us begin with the oscillating amplitudes.

For  $\underline{k} \in \mathcal{I}_{os}$ , the amplitude  $u_{1,\underline{k}}$  satisfies the equations

$$\begin{cases} i\mathcal{L}(d\varphi_{\underline{k}})u_{1,\underline{k}} + L(\partial)u_{0,\underline{k}} = 0, \\ u_{1,\underline{k}}|_{l \leq 0} = 0, \end{cases} \tag{65}$$

with the two boundary conditions

$$\begin{aligned} B_1 \left[ \sum_{\underline{k} \in \Phi(\underline{k}) \cap \mathcal{I}_{os}} u_{1,\underline{k}} \right]_{|x_1=0} &= 0 \quad \text{if } \underline{k} \notin \mathfrak{R}_1, \\ B_1 \left[ \sum_{\underline{k} \in \Phi(\underline{k}) \cap \mathcal{I}_{os}} u_{1,\underline{k}} + U_{1,\underline{k},1}|_{x_1=0} \right]_{|x_1=0} &= 0 \quad \text{if } \underline{k} \in \mathfrak{R}_1, \end{aligned} \tag{66}$$

and

$$\begin{aligned} B_2 \left[ \sum_{\underline{k} \in \Psi(\underline{k}) \cap \mathcal{I}_{os}} u_{1,\underline{k}} \right]_{|x_2=0} &= 0 \quad \text{if } \underline{k} \notin \mathfrak{R}_2, \\ B_2 \left[ \sum_{\underline{k} \in \Psi(\underline{k}) \cap \mathcal{I}_{os}} u_{1,\underline{k}} + U_{1,\underline{k},2}|_{x_2=0} \right]_{|x_2=0} &= 0 \quad \text{if } \underline{k} \in \mathfrak{R}_2. \end{aligned} \tag{67}$$

In a classical way, we compose the first equation of (65) by the partial inverse  $R_1^{\underline{k}}$  if  $\underline{k} \in \mathcal{I}_{io}$ ,  $R_2^{\underline{k}}$  if  $\underline{k} \in \mathcal{I}_{oi}$  and by  $R_1^{\underline{k}}$  or  $R_2^{\underline{k}}$  if  $\underline{k} \in \mathcal{I}_{ii}$ . Let us recall that this partial inverse satisfies: for  $i = 1, 2$ ,

$$R_i^{\underline{k}} \mathcal{L}(d\varphi_{\underline{k}}) = I - P_i^{\underline{k}}, \quad P_i^{\underline{k}} R_i^{\underline{k}} = R_i^{\underline{k}} Q_i^{\underline{k}} = 0.$$

The first equation of (65), after this composition, reads

$$(I - P_i^k)u_{1,k} = iR_i^k L(\partial)u_{0,k} \tag{68}$$

and determines in a unique way the unpolarized part of  $u_{1,k}$ . Indeed, at this stage of the analysis, the term of the right-hand side of (68) has already been constructed. As  $u_{0,k} \in H^K(\Omega_T)$  and its traces on the sides  $\partial\Omega_1$  and  $\partial\Omega_2$  are in  $H_f^K$ , the unpolarized part of  $u_{1,k}$  belongs to  $H^{K-1}(\Omega_T)$  with traces in  $H_f^{K-1}$ . To complete the construction of the oscillating amplitude  $u_{1,k}$ , we just have to construct its polarized part, that is to say,  $P_1^k u_{1,k}$  (or equivalently  $P_2^k u_{1,k}$ ).

To determine the polarized part, we will repeat with some modifications, the method described for the leading order. First, we remark that the evolution equation for the amplitude  $u_{2,k}$  is

$$i\mathcal{L}(d\varphi_k)u_{2,k} + L(\partial)u_{1,k} = 0,$$

and, after composition by  $Q_1^k$  for  $k \in \mathcal{S}_{io}$ , by  $Q_2^k$  for  $k \in \mathcal{S}_{oi}$  and by  $Q_1^k$  or  $Q_2^k$  for  $k \in \mathcal{S}_{ii}$ , is given by

$$\begin{aligned} Q_i^k L(\partial)P_i^k u_{1,k} &= -Q_i^k L(\partial)(I - P_i^k)u_{1,k} \\ &= -iQ_i^k L(\partial)R_i^k L(\partial)u_{0,k}. \end{aligned}$$

Thanks to Lax’s lemma [1957], this equation is a transport equation with speed  $v_k$  on the polarized part  $P_i^k u_{1,k}$ . As a consequence,  $Q_i^k P_i^k u_{1,k}$  satisfies the same transport equation (with a nonzero source term in the interior of  $\Omega$ ) as the transport equation satisfied by  $u_{0,k}$ . This observation leads us to apply the same method of construction as in Sections 4B3 and 4B5.

More precisely, we start with the indices on the loop; to fix the ideas, we will describe the construction of  $u_{1,n_4}$ . We have already seen that its unpolarized part is known. To construct the polarized part of  $u_{1,n_4}$ , since it travels with an outgoing-incoming velocity, we need to know its trace on  $\partial\Omega_2$ . Repeating the computation made in Section 4B2, we obtain an invertibility condition which reads

$$(I - \mathbb{T})P_1^{n_1}u_{1,n_1|x_2=0} = G_1,$$

where  $G_1 \in H_f^{K-1}$  only depends on the unpolarized traces of the amplitudes associated with the elements of the loop. Assumption 4.15 implies that  $P^{n_4}u_{1,n_4}$  is solution to the transport equation

$$\begin{cases} (\partial_t + v_{n_4} \cdot \nabla_x)Q_2^{n_4}P_2^{n_4}u_{1,n_4} = -iQ_2^{n_4}L(\partial)R_2^{n_4}L(\partial)u_{0,n_4}, \\ P_2^{n_4}u_{1,n_4|x_2=0} = -S_2^{n_4}B_2[(I - \mathbb{T})^{-1}G_1 + (I - P_2^{n_4})u_{1,n_4|x_2=0} + (I - P_1^{n_1})u_{1,n_1|x_2=0}], \quad P_2^{n_4}u_{1,n_4|l \leq 0} = 0. \end{cases}$$

All the source terms in this equation are known, so we can integrate along the characteristics to determine  $P_2^{n_4}u_{1,n_4}$ . The source term in the interior is  $H^{K-2}(\Omega_T)$  and the source term on the boundary is  $H_f^{K-1}$ , so the solution  $P_2^{n_4}u_{1,n_4}$  is in  $H^{K-2}(\Omega_T)$  with traces on  $\partial\Omega_1$  and  $\partial\Omega_2$  in  $H_f^{K-2}$ . The fact that the construction of the term of order one in  $\varepsilon$  needs two derivatives is classical, and more generally, the construction of the term of order  $N_0$  needs  $2N_0$  derivatives on the  $u_{0,j}$ .

When the amplitudes associated with indices on the loop are determined, the construction of the polarized parts of the other oscillating amplitudes follows exactly the same method. In particular the

order of resolution is the same order as the order described in Section 4B5. That is why we will not give more details about this construction.

We are now interested in the construction of the evanescent amplitudes of order  $\varepsilon$ . Although these amplitudes do not satisfy transport equations, the method of construction is based on the same ideas as the method for oscillating amplitudes. Indeed, we remark that the amplitudes  $U_{1,k,i}$  can be decomposed in a polarized part (whose construction will use the techniques of the construction of  $U_{0,k,i}$ ) and an unpolarized part only depending of the known amplitude  $U_{0,k,i}$ .

In this section we will only consider evanescent amplitudes for the side  $\partial\Omega_1$ , so we let  $\underline{k} \in \mathfrak{R}_1$ . The amplitude  $U_{1,\underline{k},1}$  satisfies the system of equations

$$\begin{cases} L_{\underline{k}}(\partial_{X_1})U_{1,\underline{k},1} + L(\partial)U_{0,\underline{k},1} = 0, \\ B_1 \left[ \sum_{k \in \Phi(\underline{k}) \cap \mathcal{J}_{os}} u_{1,k} + U_{1,\underline{k},1}|_{X_1=0} \right]_{|x_1=0} = 0, \\ U_{1,\underline{k},1}|_{t \leq 0} = 0, \end{cases} \tag{69}$$

but thanks to Lemma 4.19, we know that the first equation of this system has a solution reading

$$U_{1,\underline{k},1} = \mathbb{P}_{ev1}^{\underline{k}} U_{1,\underline{k},1} - \mathbb{Q}_{ev1}^{\underline{k}} L(\partial)U_{0,\underline{k},1}, \tag{70}$$

where we recall that the projectors  $\mathbb{P}_{ev1}^{\underline{k}}$  and  $\mathbb{Q}_{ev1}^{\underline{k}}$  are defined in (58) and (59). Using the fact that the amplitude  $U_{0,\underline{k},1}$  has already been constructed, the unpolarized part of  $U_{1,\underline{k},1}$ , namely  $\mathbb{Q}_{ev1}^{\underline{k}} L(\partial)U_{0,\underline{k},1}$ , is known. It is thus sufficient to construct the polarized part of  $U_{1,\underline{k},1}$ , namely  $\mathbb{P}_{ev1}^{\underline{k}} U_{1,\underline{k},1}$ . To do that, we repeat the construction used for  $U_{0,\underline{k},1}$ . By the definition of  $\mathbb{P}_{ev1}^{\underline{k}}$ , we know  $\mathbb{P}_{ev1}^{\underline{k}} U_{1,\underline{k},1}$  will be determined if we can construct the trace of  $U_{1,\underline{k},1}$  on  $\{X_1 = 0\}$ .

Firstly, the boundary condition (69) and Proposition 4.21 give the double trace on  $\{x_1 = X_1 = 0\}$ . More precisely, this double trace is given by

$$U_{1,\underline{k},1}|_{x_1=X_1=0} = -S_{s,1}^{\underline{k}} u_{1,\underline{k}}|_{x_1=0}, \tag{71}$$

where the source term is known because we have already constructed the oscillating amplitudes of order  $\varepsilon$ . To conclude we lift this double trace on  $\{x_1 = X_1 = 0\}$  in a single trace  $\{X_1 = 0\}$  exactly as has been done for  $U_{0,\underline{k},1}$ , and then we apply the operator  $\mathbb{P}_{ev1}^{\underline{k}}$ .

Finally we have to study the contribution of  $U_{1,\underline{k},1}|_{x_2=0}$  in the boundary condition (32). From the decomposition (70), it is in fact sufficient to study the traces on  $\{x_2 = 0\}$  of  $\mathbb{P}_{ev1}^{\underline{k}} U_{1,\underline{k},1}$  and of  $\mathbb{Q}_{ev1}^{\underline{k}} L(\partial)U_{0,\underline{k},1}$ . Concerning the first term, since it has been constructed exactly as lower-order evanescent amplitudes, its trace on  $\{x_2 = 0\}$  will be zero if and only if (see (71)) the trace of  $u_{1,\underline{k}}|_{x_1=0}$  on  $\{x_2 = 0\}$  is zero. This point is a consequence of the fact that oscillating amplitudes of order one are, as the amplitudes of order zero, flat at the corner.

Concerning the value of the trace on  $\{x_2 = 0\}$  of  $\mathbb{Q}_{ev1}^{\underline{k}} L(\partial)U_{0,\underline{k},1}$ , let us first remark that from (59), the operator  $\mathbb{Q}_{ev1}^{\underline{k}}$  only acts on the fast variable  $X_1$ . As a consequence it does not influence the value of a trace for the slow variable  $x_2$ . We thus have to study  $L(\partial)U_{0,\underline{k},1}(t, x_1, 0, x_1/\varepsilon)$ . From (64), we can compute



$$L(\partial)U_{0,k,1}\left(t, x_1, x_2, \frac{x_1}{\varepsilon}\right) = \left(\chi'(x_1) + \frac{1}{\varepsilon}\right)M_1\left(\frac{x_1}{\varepsilon}\right)u_{1,k|_{x_1=0}}(t, x_2) + M_0\left(\frac{x_1}{\varepsilon}\right)\partial_t u_{1,k|_{x_1=0}}(t, x_2) \\ + M_2\left(\frac{x_1}{\varepsilon}\right)\partial_2 u_{1,k|_{x_1=0}}(t, x_2),$$

where  $M_i(x_1/\varepsilon) := -A_i e^{\frac{x_1}{\varepsilon} \mathcal{A}_1} S_{s,1}^k B_1$  (with the convention  $A_0 := I$ ). Evaluating the previous formula at  $x_2 = 0$ , we obtain from the flatness of  $u_{1,k|_{x_1=0}}$  at the corner that the trace on  $\{x_2 = 0\}$  of  $\mathbb{Q}_{\text{ev}1}^k L(\partial)U_{0,k,1}$  is zero. So the construction of the WKB expansion for the corner problem (18) is complete. To summarize we give the following theorem:

**Theorem 4.23.** *Under Assumptions 2.1–2.2 on the complete-for-reflections corner problem (18) and under Assumptions 4.7 and 4.16, if  $[\cdot]$  denotes the integer-part function, then there exist functions  $(u_{n,k})_{n \leq [\frac{K}{2}], k \in \mathcal{I}_{\text{os}}}$ ,  $(U_{n,k,1})_{n \leq [\frac{K}{2}], k \in \mathfrak{R}_1}$  and  $(U_{n,k,2})_{n \leq [\frac{K}{2}], k \in \mathfrak{R}_2}$  satisfying the cascades of equations (26), (31), (32) and (33).*

Moreover, the functions  $u_{n,k}$  admit the following regularity. For all  $T > 0$ :

- If  $k \in \mathcal{I}_{\text{io}} \cup \mathcal{I}_{\text{oi}}$  then  $u_{n,k} \in H^{K-2n}(\Omega_T)$  and the traces  $u_{n,k|_{x_1=0}}$  and  $u_{n,k|_{x_2=0}}$  are in  $H_f^{K-2n}$ .
- If  $n \in \mathcal{I}_{\text{ii}}$ , then  $u_{n,k} \in H^{K-2n}(\Omega_T)$ . Moreover, if  $\Psi(k) = \{k\}$  (resp.  $\Phi(k) = \{k\}$ ) then for all  $\underline{x}_1 > 0$  (resp.  $\underline{x}_2 > 0$ ), the trace  $u_{n,k|_{x_1=\underline{x}_1}}$  (resp.  $u_{n,k|_{x_2=\underline{x}_2}}$ ) is  $H_f^{K-2n}$ .

The  $U_{n,k,1}$  (resp.  $U_{n,k,2}$ ) are in  $P_{\text{ev}1}$  (resp.  $P_{\text{ev}2}$ ). Moreover, the  $U_{n,k,1}$ ,  $U_{n,k,2}$  satisfy for all  $n, k$ ,  $U_{n,k,1}(t, x_1, 0, X_1) = U_{n,k,2}(t, 0, x_2, X_2) = 0$ .

**4C. Justification of the WKB expansion.** In this section we show that, if the corner problem (18) is strongly well-posed, the truncated WKB expansion constructed in the preceding section is a good approximation to the exact solution  $u^\varepsilon$  of the corner problem (18). Let us recall what we mean by strong well-posedness:

**Definition 4.24.** The corner problem is said to be strongly well-posed if for all  $f \in L^2(\Omega_T)$ ,  $g_1 \in L^2(\partial\Omega_{1,T})$  and  $g_2 \in L^2(\partial\Omega_{2,T})$  zero for negative times, the system

$$\begin{cases} \partial_t u + A_1 \partial_1 u + A_2 \partial_2 u = f, \\ B_1 u|_{x_1=0} = g_1, \quad B_2 u|_{x_2=0} = g_2, \quad u|_{t \leq 0} = 0, \end{cases}$$

admits a unique solution  $u \in L^2(\Omega_T)$ , with traces in  $L^2(\partial\Omega_{1,T})$  and  $L^2(\partial\Omega_{2,T})$ , satisfying the energy estimate

$$\|u\|_{L^2(\Omega_T)}^2 + \|u|_{x_1=0}\|_{L^2(\partial\Omega_{1,T})}^2 + \|u|_{x_2=0}\|_{L^2(\partial\Omega_{2,T})}^2 \leq C_T (\|f\|_{L^2(\Omega_T)}^2 + \|g_1\|_{L^2(\partial\Omega_{1,T})}^2 + \|g_2\|_{L^2(\partial\Omega_{2,T})}^2). \quad (72)$$

Let us recall that the strong well-posedness of the corner problem is demonstrated for the particular class of symmetric corner problems with strictly dissipative boundary conditions.

To justify the convergence of the WKB expansion, we need to be sure that the amplitudes are regular enough. The regularity has already been studied for the oscillating amplitudes. Concerning the evanescent's

amplitudes, the following proposition shows that they are regular and also gives their size according to the small parameter  $\varepsilon$ .

**Proposition 4.25.** *Let  $U$  be an element of  $P_{ev,1}$  (resp.  $P_{ev,2}$ ). Then the functions  $U(t, x, x_1/\varepsilon)$  and  $(L(\partial)U(t, x, X_1))|_{X_1=\frac{x_1}{\varepsilon}}$  (resp.  $U(t, x, x_2/\varepsilon)$  and  $(L(\partial)U(t, x, X_2))|_{X_2=\frac{x_2}{\varepsilon}}$ ) are  $O(\varepsilon^{\frac{1}{2}})$  in  $L^2(\Omega_T)$ .*

We refer to [Benoit 2014] for a proof of this result.

Before showing that the truncated WKB expansion is a good approximation to the exact solution to the corner problem (18), we have to make sure that the truncated WKB expansion makes sense. Indeed, we saw in Section 3D that when there was an infinite number of phases generated by successive reflections, it was not clear that the sum of all amplitudes defining the WKB expansion converges. That is why, to avoid this difficulty we will restrict ourselves to a finite number of phases:

**Assumption 4.26.** *We assume that the number of phases generated by successive reflections on the sides  $\partial\Omega_1$  and  $\partial\Omega_2$  is finite. That is to say,  $\#\mathcal{F} < +\infty$ .*

With this extra assumption, it is clear that the truncated WKB expansion makes sense. The main theorem of this article is:

**Theorem 4.27.** *Suppose Assumptions 2.1–2.2 for the complete-for-reflection corner problem (18) and Assumptions 4.7, 4.9, 4.16 and 4.26 hold. Then, for  $N_0 \in \mathbb{N}$ , with  $N_0 \leq [\frac{K}{2} - \frac{3}{2}]$ , we denote by  $u_{app,N_0}^\varepsilon$  the geometric optics expansion truncated at order  $N_0$  defined by*

$$u_{app,N_0}^\varepsilon := \sum_{k \in \mathcal{J}_{os}} e^{\frac{i}{\varepsilon} \varphi_k(t,x)} \sum_{n=0}^{N_0} \varepsilon^n u_{n,k}(t, x) + \sum_{k \in \mathcal{R}_1} e^{\frac{i}{\varepsilon} \psi_{k,1}(t,x_2)} \sum_{n=0}^{N_0} \varepsilon^n U_{n,k,1}\left(t, x, \frac{x_1}{\varepsilon}\right) + \sum_{k \in \mathcal{R}_2} e^{\frac{i}{\varepsilon} \psi_{k,2}(t,x_1)} \sum_{n=0}^{N_0} \varepsilon^n U_{n,k,2}\left(t, x, \frac{x_2}{\varepsilon}\right),$$

where functions  $u_{n,k}$ ,  $U_{n,k,1}$  and  $U_{n,k,2}$  are given by Proposition 4.22. Then, if the corner problem (18) is strongly well-posed, let  $u^\varepsilon$  be its exact solution, and the error  $u^\varepsilon - u_{app,N_0}^\varepsilon$  is  $O(\varepsilon^{N_0+1})$  in  $L^2(\Omega_T)$ .

*Proof.* Since we assumed that  $N_0 \leq [\frac{K}{2} - \frac{3}{2}]$ , the term of order  $\varepsilon^{N_0+1}$  of the WKB expansion makes sense and is at least in  $H^1(\Omega_T)$ . By construction of the  $u_{n,k}$ ,  $U_{n,k,1}$  and  $U_{n,k,2}$ , for  $n \leq N_0 + 1$ , the remainder  $u^\varepsilon - u_{app,N_0+1}^\varepsilon$  satisfies the corner problem

$$\begin{cases} L(\partial)(u^\varepsilon - u_{app,N_0+1}^\varepsilon) = f_{N_0+1}^\varepsilon, \\ B_1(u^\varepsilon - u_{app,N_0+1}^\varepsilon)|_{x_1=0} = 0, \quad B_2(u^\varepsilon - u_{app,N_0+1}^\varepsilon)|_{x_2=0} = 0, \quad (u^\varepsilon - u_{app,N_0+1}^\varepsilon)|_{t \leq 0} = 0, \end{cases} \tag{73}$$

with

$$f_{N_0+1}^\varepsilon := \varepsilon^{N_0+1} \left[ \sum_{k \in \mathcal{J}_{os}} e^{\frac{i}{\varepsilon} \varphi_k} L(\partial)u_{N_0+1,k} + \sum_{k \in \mathcal{R}_1} e^{\frac{i}{\varepsilon} \psi_{k,1}} (L(\partial)U_{N_0+1,k,1})|_{X_1=\frac{x_1}{\varepsilon}} + \sum_{k \in \mathcal{R}_2} e^{\frac{i}{\varepsilon} \psi_{k,2}} (L(\partial)U_{N_0+1,k,2})|_{X_2=\frac{x_2}{\varepsilon}} \right].$$

But the corner problem (18) is supposed to be strongly well-posed, so we can use the energy estimate (72), to obtain

$$\|u^\varepsilon - u_{\text{app},N_0+1}^\varepsilon\|_{L^2(\Omega_T)} \leq C_T \|f_{N_0+1}^\varepsilon\|_{L^2(\Omega_T)}.$$

The right-hand side can be estimated by

$$\begin{aligned} \|f_{N_0+1}^\varepsilon\|_{L^2(\Omega_T)} &\leq \varepsilon^{N_0+1} \left[ \sum_{k \in \mathcal{I}_{\text{os}}} \|L(\partial)u_{N_0+1,k}\|_{L^2(\Omega_T)} + \sum_{k \in \mathcal{I}_{\text{ev}1}} \|L(\partial)U_{N_0+1,k,1}(\cdot, \cdot, X_1)|_{X_1=\frac{x_1}{\varepsilon}}\|_{L^2(\Omega_T)} \right. \\ &\quad \left. + \sum_{k \in \mathcal{I}_{\text{ev}2}} \|L(\partial)U_{N_0+1,k,2}(\cdot, \cdot, X_2)|_{X_2=\frac{x_2}{\varepsilon}}\|_{L^2(\Omega_T)} \right] \\ &\leq C \varepsilon^{N_0+1} \end{aligned}$$

because, according to Proposition 4.25,  $(L(\partial)U_{N_0,k,1})|_{X_1=\frac{x_1}{\varepsilon}}$  and  $(L(\partial)U_{N_0+1,k,2})|_{X_2=\frac{x_2}{\varepsilon}}$  are  $O(\varepsilon^{\frac{1}{2}})$  in  $L^2(\Omega_T)$ , whereas  $L(\partial)u_{N_0+1,k}$  are  $O(1)$  in  $L^2(\Omega_T)$ , because  $u_{N_0+1,k}$  is at least in  $H^{\frac{1}{4}}(\Omega_T)$ .

We thus have shown that

$$\|u^\varepsilon - u_{\text{app},N_0+1}^\varepsilon\|_{L^2(\Omega_T)} \leq C_T \varepsilon^{N_0+1},$$

and we conclude by the triangle inequality. □

**4D. Study of the invertibility condition (47).** In this section we will give a sufficient (and also necessary in several relevant cases) condition ensuring that the invertibility condition (47) is satisfied. Let us recall that this condition reads

$$u_{0,n_1|x_2=0}(t, x_1) - S u_{0,n_1|x_2=0}(t - \alpha x_1, \beta x_1) = S_1^{n_1} g(t + \delta x_1, \kappa x_1), \tag{74}$$

with  $\alpha, \beta > 0$ , and  $\delta < 0, \kappa > 0$ . The exact expressions of these parameters are given by

$$\begin{aligned} S &:= S_1^{n_1} B_1 S_2^{n_2} B_2 S_1^{n_3} B_1 S_2^{n_4} B_2, \\ \alpha &:= -\frac{1}{v_{n_1,1}} \left( -1 + \frac{v_{n_1,2}}{v_{n_2,2}} - \frac{v_{n_1,2}v_{n_2,1}}{v_{n_2,2}v_{n_3,1}} + \frac{v_{n_1,2}v_{n_2,1}v_{n_3,2}}{v_{n_2,2}v_{n_3,1}v_{n_4,2}} \right), \\ \beta &:= \frac{v_{n_4,1}}{v_{n_4,2}} \frac{v_{n_3,2}}{v_{n_3,1}} \frac{v_{n_2,1}}{v_{n_2,2}} \frac{v_{n_1,2}}{v_{n_1,1}}. \end{aligned}$$

If we assume that  $\dim \ker \mathcal{L}(i\tau, \xi_1^{n_1}, \xi_2^{n_1}) = 1$  (an assumption which is automatically satisfied in the strictly hyperbolic framework), then (74) is in fact a scalar equation:

$$u(t, x_1) - \tilde{S}u(t - \alpha x_1, \beta x_1) = G(t, x_1), \tag{75}$$

where, thanks to the polarization condition, we write  $u_{0,n_1|x_2=0}(t, x_1) = u(t, x_1)e_{n_1}$ , with  $e_{n_1}$  chosen such that  $\ker(\mathcal{L}(i\tau, \xi_1^{n_1}, \xi_2^{n_1})) = \text{Span } e_{n_1}$ . The scalar  $\tilde{S}$  is defined by the equality

$$S e_{n_1} = \tilde{S} e_{n_1},$$

and without loss of generality we can assume that  $\tilde{S} \neq 0$ .

In all that follows, it will be more convenient to rewrite (75) as

$$(I - \mathbb{T})u = G, \tag{76}$$

where

$$(\mathbb{T}u)(t, x_1) := \tilde{S}u(t - \alpha x_1, \beta x_1). \tag{77}$$

A sufficient condition for (74) (and thus also (75)) to have a unique solution in the profiles space  $L^2(]-\infty, T] \times \mathbb{R}_+)$  is given by the following theorem:

**Theorem 4.28.** *If*

$$|S| < \sqrt{\beta},$$

*then for all  $\gamma > 0$ , for all  $G \in L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$ , the functional equation (74) admits a unique solution  $u \in L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$ , polarized on  $\ker \mathcal{L}(i \underline{\tau}, \xi_1^{n_1}, \xi_2^{n_1})$ , and satisfying*

$$\|u\|_{L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})} \leq C \|G\|_{L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})},$$

*where  $C$  does not depend on the parameter  $\gamma$ .*

*In particular, for all  $T > 0$ , if  $G$  is in  $L^2(\partial\Omega_{2,T})$  and is zero for negative times, then (74) has a unique solution  $u \in L^2(\partial\Omega_{2,T})$ , polarized on  $\ker(\mathcal{L}(i \underline{\tau}, \xi_1^{n_1}, \xi_2^{n_1}))$ , and satisfying*

$$\|u\|_{L^2(\partial\Omega_{2,T})} \leq C_T \|G\|_{L^2(\partial\Omega_{2,T})}.$$

*Proof.* To solve (76) in a unique way, it is sufficient that  $\mathbb{T}$  is a contraction on  $L^2_\gamma(\mathbb{R}_+ \times \mathbb{R})$  (or equivalently on  $L^2(]-\infty, T] \times \mathbb{R}_+)$ ). A simple computation shows that is it effectively the case under the assumption  $|S| < \sqrt{\beta}$ . □

**Remark.** The fact that we are interested in uniform bounds (compared with the parameter  $\gamma$ ) of the operator  $\mathbb{T}$  is motivated by the following fact. In the analysis of the initial boundary value problem in the half-space, one starts to deal with global problems in time. Then from the uniformity of the energy estimate compared to  $\gamma$  follows a principle of causality which allows for the restriction to problems where the time variable lies in  $]-\infty, T]$ . We refer to [Benzoni-Gavage and Serre 2007; Chazarain and Piriou 1981] for more details about this proof.

To fully understand the condition  $|S| < \sqrt{\beta}$ , it is important to note the following fact: if one considers a point  $(0, L) \in \partial\Omega_1$  and follows the characteristic curves associated with the indices on the loop, then after three reflections, this traveling point goes back to  $\partial\Omega_1$  in a new position  $(0, L') \in \partial\Omega_1$ . Some basic computations show that

$$\beta = \frac{L}{L'}. \tag{78}$$

So, we have three possible behaviors depending of the value of  $\beta$ :

- If  $\beta > 1$ , then traveling along the bicharacteristics, the information approaches the corner.
- If  $\beta < 1$ , then traveling along the bicharacteristics, the information goes away from the corner.
- If  $\beta = 1$ , then the travel along the bicharacteristics is periodic.

The condition  $|S| < \sqrt{\beta}$  imposes that after one turn along the bicharacteristics associated with the loop, the  $L^2$ -norm of the trace has decreased.

In the scalar case, that is, when the matrix  $S$  can be replaced by the scalar  $\tilde{S}$ ,<sup>5</sup> that is to say, when the rank of the projector  $P_1^{n_1}$  is one, we can show that the condition  $|\tilde{S}| < \sqrt{\beta}$  is also necessary for the well-posedness of (75). The idea of the proof is to use Laplace transformation in the time variable to reduce to a situation already studied by Osher [1974b].

**Theorem 4.29.** *Let  $\alpha, \beta > 0$  and  $\tilde{S} \in \mathbb{R} \setminus \{0\}$  be such that  $\tilde{S} > \sqrt{\beta}$ . Then the equation*

$$u(t, x) - \tilde{S}u(t - \alpha x, \beta x) = G \tag{79}$$

satisfies one of the alternatives:

- (i) If  $\beta < 1$ , then (79) written for  $G = 0$  admits a nonzero solution in  $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$ , for all  $\gamma > 0$ .
- (ii) If  $\beta > 1$ , then there exists  $g \in L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$  such that (79) does not have any solution.

*Proof.* We begin with the proof of (i). We are looking for a nonzero solution  $u$  written in the form  $u(t, x) = H(t)v(t, x)$ , where  $H$  is the Heaviside function. Applying the Laplace transform in the time variable to (79) leads us to solve

$$\hat{v}(\sigma, x) - \tilde{S}e^{-\alpha\sigma x}\hat{v}(\sigma, \beta x) = 0, \tag{80}$$

where  $\sigma \in \mathbb{C}$ ,  $\text{Re } \sigma > 0$ , is the dual variable of  $t$ . Following [Osher 1974b], let

$$\hat{\hat{v}}(\sigma, x) = e^{\frac{\alpha\sigma x}{1-\beta} - \frac{\ln \tilde{S}}{\ln \beta} x}.$$

It is easy to check that this function is a solution to (80). But  $\text{Re}(\alpha\sigma/(1-\beta)) < 0$ , so using the assumption  $\tilde{S} > \sqrt{\beta}$ , it follows that  $\hat{\hat{v}} \in L^2_x(\mathbb{R}_+)$ .

However, in order to come back to the time variable, we want to apply the Paley–Wiener theorem to  $\hat{\hat{v}}$ . That is why we denote by  $\hat{v}(\sigma, x)$  the following modification of  $\hat{\hat{v}}$ :

$$\hat{v}(\sigma, x) = \frac{1}{(1+\sigma)}\hat{\hat{v}}(\sigma, x).$$

It is easy to see that  $\hat{v}$  is still a solution to (80). Moreover,

$$\sup_{\gamma > 0} \int_{\mathbb{R}} \int_{\mathbb{R}_+} |v(\gamma + i\eta, x)|^2 dx d\eta \leq \sup_{\gamma > 0} \left( \int_{\mathbb{R}_+} x^{-2\frac{\ln \tilde{S}}{\ln \beta}} e^{\frac{2x\alpha\gamma}{1-\beta}} dx \right) \int_{\mathbb{R}} \frac{1}{1+\eta^2} d\eta \leq C.$$

We can thus apply the Paley–Wiener theorem, so there exists  $v \in \bigcap_{\gamma > 0} L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$  such that  $\hat{v}$  is the Laplace transform. As a consequence we have constructed a nonzero solution to (79).

To show (ii), using the same proof as in [Osher 1974b], it is sufficient to remark that the adjoint of  $\mathbb{T}$  is given by

$$\mathbb{T}^*v = -\frac{\tilde{S}}{\beta}v\left(t - \frac{\alpha}{\beta}x, \frac{1}{\beta}x\right).$$

---

<sup>5</sup>When  $S$  is a matrix and not a number, it seems more difficult to show the analog of Theorem 4.29. That is why the restriction to strictly hyperbolic operators is an easy way to obtain sharp results.

So if  $\beta > 1$ , the operator  $\mathbb{T}^*$  is as in (i), so

$$\frac{\tilde{S}}{\beta} > \frac{1}{\sqrt{\beta}} \iff \tilde{S} > \sqrt{\beta},$$

and  $\mathbb{T}^*$  is not injective. As a consequence,  $\mathbb{T}$  is not surjective. □

During the construction of the geometric optics expansion we saw that the invertibility of the operator  $I - \mathbb{T}$  on the weighted space  $L^2_\gamma(\mathbb{R} \times \mathbb{R}_+)$  was not sufficient to construct the term of order  $\varepsilon$  which is, however, necessary if we want to show that the truncated expansion approximates the exact solution. More precisely, to construct the first corrector term, it is necessary that  $I - \mathbb{T}$  is (at least) invertible from  $H_f^3$  to  $H_f^3$ , and more generally if one wants the remainder  $u^\varepsilon - u^\varepsilon_{\text{app}, N_0}$  to be  $O(\varepsilon^{N_0+1})$ , it is needed that  $I - \mathbb{T}$  is invertible from  $H_f^{N_0+3}$  in  $H_f^{N_0+3}$  (to ensure that the term of order  $N_0 + 1$  is at least in  $H^1(\Omega_T)$ ). The following theorem shows that the solution to the functional equation (75) given by [Theorem 4.28](#) inherits (under some restrictions) the regularity  $H_f^K$  of the source term. There are two different cases to handle:

**Theorem 4.30.** (i) *If  $0 < \beta \leq 1$  and if  $|S| < \sqrt{\beta}$ , then  $I - \mathbb{T}$  is invertible from  $H_f^\infty$  to  $H_f^\infty$ .*

(ii) *Let  $K \in \mathbb{N}$ , if  $\beta > 1$  and if  $|S|\beta^{K-\frac{1}{2}} < 1$ , then  $I - \mathbb{T}$  is invertible from  $H_f^K$  to  $H_f^K$ .*

*Proof.* Let

$$u(t, x_1) = G(t, x_1) + \sum_{j=1}^{+\infty} S^j G(t + X_{\alpha, \beta}^j x_1, \beta^j x_1), \tag{81}$$

where

$$X_{\alpha, \beta}^j := \sum_{k=0}^{j-1} \alpha \beta^k + \beta^j \alpha.$$

It is easy to check that, under the assumption  $|S| < \sqrt{\beta}$ , we have  $u$  is a solution to (75) which belongs to  $L^2(\partial\Omega_{2,T})$ . According to [Theorem 4.28](#), it is unique.

Then, we show that the solution  $u$  defined by (81) inherits the regularity of  $G$ . Firstly, according to the particular form of (75), it is clear that independently of  $\beta$ , for all  $n \in \mathbb{N}$ ,

$$\|\partial_t^n u\|_{L^2(\partial\Omega_{2,T})} \leq C \|\partial_t^n g\|_{L^2(\partial\Omega_{2,T})}, \tag{82}$$

so we only have to deal with the derivatives in the spatial variable, because it will also permit us to deal with the cross-derivatives by using (82). For  $n \in \mathbb{N}$ , a simple computation gives

$$\partial_{x_1}^n u = \partial_{x_1}^n G + \sum_{j=1}^{+\infty} S^j \left[ \sum_{l=0}^n \binom{n}{l} (X_{\alpha, \beta}^j)^l (\beta^j \kappa)^{n-l} \partial_t^{n-l} \partial_{x_1}^l G \right] (t + X_{\alpha, \beta}^j x_1, \beta^j x_1) \tag{83}$$

and leads us to a distinction depending on the value of  $\beta$ .

If  $\beta \leq 1$ , then all the constants appearing during the derivation can be abruptly bounded from above and we obtain

$$\|\partial_{x_1}^n u\|_{L^2(\partial\Omega_{2,T})} \leq \|\partial_{x_1}^n G\|_{L^2(\partial\Omega_{2,T})} + C_{n,\alpha} \sum_{j=1}^{+\infty} \left(\frac{|S|}{\sqrt{\beta}}\right)^j \sum_{l=0}^n \|\partial_t^{n-l} \partial_{x_1}^l G\|_{L^2(\partial\Omega_{2,T})},$$

where we used the change of variable

$$\begin{bmatrix} s \\ y \end{bmatrix} = \begin{bmatrix} 1 & X_{\alpha,\beta}^j \\ 0 & \beta^j \end{bmatrix} \begin{bmatrix} t \\ x_1 \end{bmatrix}$$

to force the appearance of the factor  $\sqrt{\beta}$ . As a consequence, under the assumption  $|S| < \sqrt{\beta}$ , the solution  $u$  given by (81) is in  $H^\infty(\partial\Omega_{2,T})$  and we can check by (83) that its trace is also in  $H_f^\infty$ .

If  $\beta > 1$ , we have

$$\|\partial_{x_1}^n u\|_{L^2(\partial\Omega_{2,T})} \leq \|\partial_{x_1}^n G\|_{L^2(\partial\Omega_{2,T})} + C_{n,\alpha} \sum_{j=1}^{+\infty} (|S|\beta^{n-\frac{1}{2}})^j \sum_{l=0}^n \|\partial_t^{n-l} \partial_{x_1}^l G\|_{L^2(\partial\Omega_{2,T})}$$

for  $0 \leq n \leq K$ , this sum is finite thanks to the assumption  $|S|\beta^{K-\frac{1}{2}} < 1$ . We have thus shown that  $u \in H^K(\partial\Omega_{2,T})$ . Once again, the flatness at the corner is given by computing the trace in (83), so we have  $u \in H_f^K$ . □

**Remark.** As in the situation where an infinite number of phases was present in the WKB expansion (see the example in Section 3D), we can remark that when the source term  $g$  is in  $\mathcal{C}_c^\infty$  with its support away from the corner, if we restrict ourselves to the construction of the WKB expansion for a finite time  $T < +\infty$ , then the number of nonzero terms in the sum (81) is finite. Thus, in this framework the operator  $(I - \mathbb{T})$  is automatically invertible (independently of the parameters  $\beta$  and  $S$ ). Its inverse is given by (81).

Theorem 4.30 seems to indicate that a corner concentration phenomenon is more difficult to handle with than a separation from the corner phenomenon. Indeed, if  $\beta < 1$ , the error between the exact solution and the truncated WKB expansion is  $O(\varepsilon^N)$  with  $N$  arbitrarily large, whereas if  $\beta > 1$ , the norms of the derivatives of the solution to (75) seem to get larger and larger. To prevent this “blow up”, we have made the assumption  $|S|\beta^{K-\frac{1}{2}} < 1$  which “implies” that there exists a maximal  $N_0$  such that the error is  $O(\varepsilon^{N_0})$ . We do not claim that, for  $\beta > 1$ , Theorem 4.30 is sharp. But it is sufficient to treat the example of Section 3E.

**4E. Examples for which the invertibility condition (75) is satisfied.**

**4E1.** *The example of Section 3E.* We come back to the corner problem (16) and more precisely to the resolution of the amplitude equation (75) for this problem.

First of all we have to specify the chosen boundary conditions. We set for  $B_1$  and  $B_2$  in (16)

$$B_1 := \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \end{bmatrix}, \quad B_2 := [0 \quad -\delta \quad 1], \tag{84}$$

where  $\delta \in \mathbb{R}$ ,  $\delta \neq 0$ , is a fixed parameter.

It is easy to see that the boundary condition defined by  $B_1$  on the side  $\partial\Omega_1$  is strictly dissipative; in particular, it satisfies the uniform Kreiss–Lopatinskii condition (see [Benzoni-Gavage and Serre 2007, Proposition 4.4]). The real parameter  $\delta \neq 0$  encodes the dissipativity on the side  $\partial\Omega_2$  in the following way:

- If  $|\delta| > 2^{-\frac{1}{4}}$ , the boundary condition defined by  $B_2$  is strictly dissipative.
- If  $|\delta| = 2^{-\frac{1}{4}}$ , the boundary condition defined by  $B_2$  is maximal dissipative.
- If  $0 < |\delta| < 2^{-\frac{1}{4}}$ , the boundary condition defined by  $B_2$  is not dissipative but it satisfies the uniform Kreiss–Lopatinskii condition.

Reiterating the same kind of computations as those described in Section 4B2, using the fact that  $\dim \mathcal{L}(d\varphi_1) = 1$ , shows that the amplitude equation for the amplitude associated with the phase  $\varphi_1$  is scalar and is given by

$$u(t, 0, x_2) = -\frac{\frac{1}{5} - 6\sqrt{2}}{(\sqrt{2} + 1)(\frac{1}{5} - \sqrt{2})\delta^2}u(t - \alpha x_2, 0, 28x_2) + G(t, x_2), \tag{85}$$

where  $u$  is a real-valued function,  $G$  is an explicitly computable, but nonrelevant, source term and  $\alpha$  is also not relevant.

According to Theorem 4.28, if

$$|\delta| > \sqrt{\frac{\frac{1}{5} - 6\sqrt{2}}{(\sqrt{2} + 1)(\frac{1}{5} - \sqrt{2})\sqrt{28}}} := \delta_0 \approx 0.73$$

then the functional equation (85) admits a unique solution. We are thus able to construct the leading-order term of the geometric optics expansion for more parameters than those leading to strictly dissipative boundary conditions.

On the contrary, if

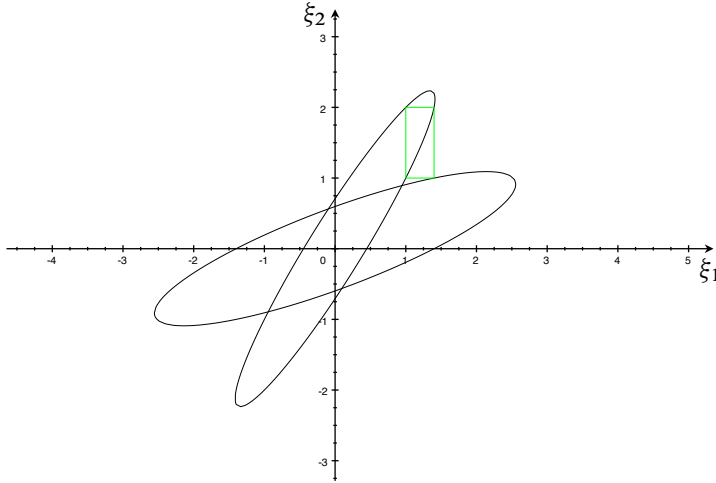
$$0 < |\delta| < \delta_0,$$

we are in a nondissipative framework, and according to Theorem 4.29, equation (85) admits a nonzero solution for  $G = 0$ , so the leading-order term in the WKB expansion is not determined in a unique way. This example shows that imposing the uniform Kreiss–Lopatinskii condition on each side of the boundary is not sufficient to construct the geometric optics expansion in a unique way. It seems to be a good argument in favor of the fact that the same situation is true for the strong well-posedness of the corner problem (18).

**4E2.** *The example of Sarason and Smoller [1974].* In [Sarason and Smoller 1974], the authors construct an example of  $4 \times 4$  strictly hyperbolic operator whose section of the characteristic variety contains a loop. This example, with the example of Section 3E, constitute, to our knowledge, the only two examples of corner problems with a loop in the literature.

The main idea of the construction is a perturbation argument: we first choose a centered ellipse and we fix three points  $P_2, P_3, P_4$  on this ellipse such that angle  $\widehat{P_2 P_3 P_4}$  is a perpendicular angle and the group velocities are incoming-outgoing in  $P_3$  and outgoing-incoming in  $P_2$  and  $P_4$ . This choice determines





**Figure 8.** The construction of Sarason and Smoller for  $p_1 := 5\xi_1^2 + 2\xi_2^2 - 6\xi_1\xi_2 - 1$  and  $p_2 := \frac{25}{49}\xi_1^2 + \frac{14}{5}\xi_2^2 - 2\xi_1\xi_2 - 1$ .

in a unique way a point  $P_1$  such that  $P_1P_2P_3P_4$  is a rectangle. Then we construct a second ellipse meeting  $P_1$  with an incoming-outgoing group velocity at this point (see Figure 8).

The variety constructed can be written in the form

$$p_1(\tau, \xi_1, \xi_2)p_2(\tau, \xi_1, \xi_2) = 0,$$

where the polynomials  $p_1$  and  $p_2$  are homogeneous of degree two. This variety contains the loop  $(P_1, P_2, P_3, P_4)$ , but it cannot represent the section at  $\tau = 1$  of the characteristic variety of a strictly hyperbolic operator. Indeed, the two ellipses constructed previously intersect in four points, namely  $Q_1, Q_2, Q_3$  and  $Q_4$ . However, it can be shown that it is the section at  $\tau = 1$  of the characteristic variety of a geometrically regular hyperbolic operator with  $A_1$  and  $A_2$  of block diagonal form

$$A_1 := \begin{bmatrix} -a_1 & a_2 & 0 & 0 \\ a_2 & a_1 & 0 & 0 \\ 0 & 0 & -\tilde{a}_1 & \tilde{a}_2 \\ 0 & 0 & \tilde{a}_2 & \tilde{a}_1 \end{bmatrix} \quad \text{and} \quad A_2 := \begin{bmatrix} -b & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & -\tilde{b} & 0 \\ 0 & 0 & 0 & \tilde{b} \end{bmatrix}$$

for suitable real parameters  $a_1, a_2, \tilde{a}_1, \tilde{a}_2, b$  and  $\tilde{b}$  (we refer to [Benoit 2015, paragraphe 6.9.6] or [Sarason and Smoller 1974] for more details about the construction of  $A_1$  and  $A_2$ ).

Once the operator  $L(\partial)$  is constructed, we add the boundary conditions

$$B_1u|_{x_1=0} := g^\varepsilon, \quad B_2u|_{x_2=0} := 0,$$

where  $B_1$  and  $B_2$  are defined by

$$B_1 := \begin{bmatrix} 1 & 0 & 0 & -\delta \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B_2 := \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\delta & 0 & 0 & 1 \end{bmatrix}. \tag{86}$$

It is easy to check, using the particular form of  $A_1$  and  $A_2$  and the fact that the boundary conditions (86) written for  $\delta = 0$  are strictly dissipative, that  $B_1$  and  $B_2$  satisfy the uniform Kreiss–Lopatinskii condition for all  $\delta$ . It is also easy to check that the boundary conditions  $B_1$  and  $B_2$  are strictly dissipative if  $\delta$  is sufficiently small.

Now that boundary conditions are fixed, we want to study the invertibility condition  $|S| < \sqrt{\beta}$ , which appears when we construct the WKB expansion of the corner problem<sup>6</sup>

$$\begin{cases} L(\partial)u^\varepsilon = 0, \\ B_1 u^\varepsilon|_{x_1=0} = g^\varepsilon, \quad B_2 u^\varepsilon|_{x_2=0} = 0, \quad u^\varepsilon|_{t \leq 0} = 0, \end{cases} \tag{87}$$

where the source term is

$$g^\varepsilon = e^{\frac{i}{\varepsilon}(t+P_{1,2}x_2)} g,$$

with  $g \in H_f^\infty$ , zero for negative times, and where  $P_1 := (P_{1,1}, P_{1,2})$ .

The factor  $\beta$  only depends of the coefficients of the operator  $L(\partial)$ . In particular, it does not depend on  $\delta$  and can be explicitly computed. The term  $S$  can be considered as a scalar (see Section 4D) and it is given by

$$S := S_1^{P_1} B_1 S_2^{P_2} B_2 S_1^{P_3} B_1 S_2^{P_4} B_2,$$

where we have made the amalgam between the indices of the loop and the associated frequencies.

Once again, using the fact that the operator  $L(\partial)$  defined two  $2 \times 2$  uncoupled systems, the stable subspace  $E_2^s(i, P_{4,1})$  is given by  $E_2^s(i, P_{4,1}) = \text{vect}\{(0, 0, p_4, q_4), (\tilde{p}, \tilde{q}, 0, 0)\}$ . Thus, we can easily compute

$$S_2^{P_4} B_2 \begin{bmatrix} p_1 \\ q_1 \\ 0 \\ 0 \end{bmatrix} = \mathbb{P}^{P_4} \left( \begin{bmatrix} \tilde{p} \\ \tilde{q} \\ q_1 \\ 0 \\ 0 \end{bmatrix} + \delta \begin{bmatrix} 0 \\ 0 \\ \frac{p_4}{q_4} (q_1 \frac{\tilde{p}}{\tilde{q}} - p_1) \\ q_1 \frac{\tilde{p}}{\tilde{q}} - p_1 \end{bmatrix} \right) := \tilde{c} \delta \begin{bmatrix} 0 \\ 0 \\ p_4 \\ q_4 \end{bmatrix},$$

where  $\tilde{c}$  is not zero and only depends of the projector  $\mathbb{P}^{P_4}$  on  $\ker(\mathcal{L}(\underline{\tau}, P_4))$  and we set  $\ker(\mathcal{L}(\underline{\tau}, P_1)) = \text{vect}\{(p_1, q_1, 0, 0)\}$ . Repeating exactly the same arguments for the other terms composing  $S$ , it is easy to show that the invertibility condition (75) is equivalent to

$$c\delta^2 < \sqrt{\beta}, \quad c > 0.$$

This condition is not satisfied for large values of  $\delta$ . Let us remark on the following points: firstly, the blow up phenomenon is in  $\delta^2$  and not in  $\delta^4$  as predicted in [Sarason and Smoller 1974]. Secondly, for  $|\delta|$  small enough, the invertibility condition is satisfied and we are in the strictly dissipative framework.

Moreover, the condition  $c\delta^2 < \sqrt{\beta}$  is more precise than the analogous condition of [Sarason and Smoller 1974]. Indeed it says that since we are working with  $L^2$ -norms, to prevent the signal from

<sup>6</sup>We do not try here to determine all the phases in the WKB expansion because, due to a concentration of the phases in the neighborhood of the intersection points of the ellipses (see Section 3E), the number of expected phases in the expansion will be infinite.

increasing in strength after a complete circuit, we must have that the amplification caused by the boundary condition is less than the contraction of the support of the source term after a complete circuit.

Finally, since we have  $\beta > 1$  and  $S$  “scalar”, [Theorem 4.29](#) tells us that if the condition  $c\delta^2 < \sqrt{\beta}$  is not satisfied then the WKB expansion cannot be constructed for any source term on the boundary. This conclusion is, in fact, worse than the conclusion of Sarason and Smoller, which says that in this case the corner problem is poorly posed (see [[Sarason and Smoller 1974](#), Definition 4.3]).

To construct a strictly hyperbolic corner problem whose characteristic variety contains a loop, Sarason and Smoller use a perturbation argument. More precisely, they introduce a small coupling between the two  $2 \times 2$  uncoupled systems defining  $L(\partial)$  constructed in such a way to “pull apart” the two intersecting ellipses of the characteristic variety; see Sections 7 and 9 of [[Sarason and Smoller 1974](#)]. They show that if the perturbation is small enough then the obtained corner problem is strictly hyperbolic and the boundary conditions defined by  $B_1$  and  $B_2$  satisfy the uniform Kreiss–Lopatinskii condition, and finally that the perturbed system admits a loop in the section of its characteristic variety.

## 5. Conclusion

In this article we have shown a theorem which gives a rigorous geometric optics expansion for a hyperbolic corner problem when the number of phases generated by reflections is finite, but our theorem is general enough to apply to problems involving self-interacting phases. For such problems, a sequence of propagation of, at least, four fixed group velocities is repeated ad vitam aeternam. The construction of the geometric optics expansion then needs the solvability of a new amplitude equation, which is an invertibility condition in the spirit of Osher’s condition [[1973](#)](see [Assumption 4.15](#)).

Of course, the construction given in this article can also be made if the section of characteristic variety does not contain any loop. Without any surprise, in that framework the construction is much simpler because the results in [Sections 4B1 and 4B4](#) are more or less immediate and we can construct the expansion as it has been done in [[Benoit 2015](#), paragraphs 6.4–6.6]. Moreover, the construction can also be adapted if the source term on the boundary does not turn on a self-interacting phase but if this phase appears after several reflections.

We also think that it should be possible to show a version of [Theorem 4.27](#) without the assumption of the nonappearance of glancing modes during the phase generation process. Indeed, if one starts with a hyperbolic frequency, then nothing ensures that after several reflections a glancing mode will not appear in the phase generation process. However, thanks to [[Williams 1996; 2000](#)], we think that, after the suitable modifications of the oscillating scales in the ansatz, glancing modes will, more or less, behave like evanescent modes in the sense that they will create boundary layers in the expansion and that they will not be reflected from one side to the other.

The proof of [Theorem 4.27](#) should also work when there are several loops. There are two cases to distinguish: first, if there is still a unique loop of interaction but other loops that are not interaction loops and, second, the case where there is more than one interaction loop. In the first case, the proof of [Theorem 4.27](#) just has to be a bit adapted in a more technical way. In the second case, we think that the

proof of [Theorem 4.27](#) can also be adapted as long as the interaction loops do not intersect themselves, but this is left for future work.

Finally, let us give some comments about [Assumption 4.9](#). On one hand, this restriction is really anecdotal because in the phase generation process described in [Section 3](#) there is no mathematical reason to systematically meet one of the axes  $\{\xi_1 = 0\}$  or  $\{\xi_2 = 0\}$ . The process depends on the geometry of the characteristic variety and on the phase of the source term. Moreover, it is easy to see that [Assumption 4.9](#) is clearly not sharp (but it has the advantage of simplicity). Indeed what we really need in our construction is the following sharper version of [Assumption 4.9](#): Let  $\underline{i} \in \mathcal{S}_{os}$  be the index associated to a frequency given by  $(\underline{\tau}, 0, \xi_2^k)$  (resp.  $(\underline{\tau}, \xi_1^k, 0)$ ) and let  $\Gamma$  be defined by

$$\Gamma := \{i \in \mathcal{S} \setminus \underline{i} \xrightarrow{V} i\},$$

(resp.  $\Gamma := \{i \in \mathcal{S} \setminus \underline{i} \xrightarrow{H} i\}$ ).

Then  $\Gamma \cap \mathcal{S}_{ev1}$  (resp.  $\Gamma \cap \mathcal{S}_{ev2}$ ) is empty. However, this new version seems not to be really less restrictive than [Assumption 4.9](#) as soon as we have some kinds of symmetries in the characteristic variety.

But on the other hand, and in the author’s opinion, an interesting question may be to consider WKB expansions without [Assumption 4.9](#). Indeed in such a framework one of the boundary conditions is given by, for example,

$$B_1 \left[ \sum_{k \in \mathcal{S}_{os}} u_{0,k} + \sum_{k \in \mathcal{N}_2} U_{0,k,2} \right]_{|x_1=0} = \mathcal{G},$$

and it seems to be not so easy to adapt the previous construction when such a boundary condition occurs because the resolution described in this paper is mainly based on the fact that we are able to determine oscillating phases before evanescent ones. That is why the construction without [Assumption 4.9](#) is left for future studies.

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# ANALYSIS & PDE

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A complete study of the lack of compactness and existence results of a fractional Nirenberg equation via a flatness hypothesis, I	1285
Wael Abdelhedi, Hichem Chtioui and Hichem Hajaiej	
On positive solutions of the $(p, A)$ -Laplacian with potential in Morrey space	1317
Yehuda Pinchover and Georgios Psaradakis	
Geometric optics expansions for hyperbolic corner problems, I: Self-interaction phenomenon	1359
Antoine Benoit	
The interior $C^2$ estimate for the Monge–Ampère equation in dimension $n = 2$	1419
Chuanqiang Chen, Fei Han and Qianzhong Ou	
Bounded solutions to the Allen–Cahn equation with level sets of any compact topology	1433
Alberto Enciso and Daniel Peralta-Salas	
Hölder estimates and large time behavior for a nonlocal doubly nonlinear evolution	1447
Ryan Hynd and Erik Lindgren	
Boundary $C^{1,\alpha}$ regularity of potential functions in optimal transportation with quadratic cost	1483
Elina Andriyanova and Shibing Chen	
Commutators with fractional differentiation and new characterizations of BMO-Sobolev spaces	1497
Yanping Chen, Yong Ding and Guixiang Hong	