

ANALYSIS & PDE

Volume 9

No. 6

2016

CHUANQIANG CHEN, FEI HAN AND QIANZHONG OU

THE INTERIOR C^2 ESTIMATE FOR THE MONGE-AMPÈRE
EQUATION
IN DIMENSION $n = 2$

THE INTERIOR C^2 ESTIMATE FOR THE MONGE–AMPÈRE EQUATION IN DIMENSION $n = 2$

CHUANQIANG CHEN, FEI HAN AND QIANZHONG OU

We introduce a new auxiliary function, and establish the interior C^2 estimate for the Monge–Ampère equation in dimension $n = 2$, which was first proved by Heinz by a geometric method.

1. Introduction

We consider the convex solution of the Monge–Ampère equation

$$\det D^2u = f(x) \quad \text{in } B_R(0) \subset \mathbb{R}^2. \quad (1-1)$$

When the solution u is convex, (1-1) is elliptic. It is well known that the interior C^2 estimate is an important problem for elliptic equations. For the Monge–Ampère equation in dimension $n = 2$, the corresponding interior C^2 estimate was established by Heinz [1959], and for higher dimensions $n \geq 3$ Pogorelov [1978] constructed his famous counterexample, namely irregular solutions to Monge–Ampère equations.

Later, Urbas [1990] generalized the counterexample for σ_k Hessian equations with $k \geq 3$. So the interior C^2 estimate of the σ_2 Hessian equation

$$\sigma_2(D^2u) = f \quad \text{in } B_R(0) \subset \mathbb{R}^n \quad (1-2)$$

is an interesting problem, where $\sigma_2(D^2u) = \sigma_2(\lambda(D^2u)) = \sum_{1 \leq i_1 < i_2 \leq n} \lambda_{i_1} \lambda_{i_2}$, the eigenvalues of D^2u are $\lambda(D^2u) = (\lambda_1, \dots, \lambda_n)$, and $f > 0$. For $n = 2$, (1-2) is the Monge–Ampère equation (1-1). For $n = 3$ and $f \equiv 1$, (1-2) can be viewed as a special Lagrangian equation, and Warren and Yuan [2009] obtained the corresponding interior C^2 estimate in their celebrated paper. Moreover, the problem is still open for general f with $n \geq 4$ and nonconstant f with $n = 3$.

Moreover, Pogorelov-type estimates for the Monge–Ampère equations and the σ_k Hessian equation ($k \geq 2$) were derived by Pogorelov [1978] and Chou and Wang [2001], respectively, and see [Guan et al. 2015; Li et al. 2016] for some generalizations.

In this paper, we introduce a new auxiliary function, and establish the interior C^2 estimate as follows:

Research of Chen is supported by the National Natural Science Foundation of China (No. 11301497 and No. 11471188). Research of Han is supported by the National Natural Science Foundation of China (No. 11161048). Research of Ou is supported by NSFC No. 11061013 and by Guangxi Science Foundation (2014GXNSFAA118028) and Guangxi Colleges and Universities Key Laboratory of Symbolic Computation and Engineering Data Processing.

MSC2010: 35B45, 35B65, 35J96.

Keywords: interior C^2 a priori estimate, Monge–Ampère equation, σ_2 Hessian equation, optimal concavity.

Theorem 1.1. *Let $u \in C^4(B_R(0))$ be a convex solution of the Monge–Ampère equation (1-1) in dimension $n = 2$, where $0 < m \leq f \leq M$ in $B_R(0)$. Then*

$$|D^2u(0)| \leq C_1 e^{C_2 \sup |Du|^2/R^2}, \tag{1-3}$$

where C_1 is a positive constant depending only on $m, M, R \sup |\nabla f|$ and $R^2 \sup |\nabla^2 f|$, and C_2 is a positive constant depending only on m and M .

Remark 1.2. By Trudinger’s gradient estimates [1997], we can bound $|D^2u(0)|$ in terms of u . In fact, we get from the convexity of u that

$$\sup_{B_{R/2}(0)} |Du| \leq \frac{\text{osc}_{B_R(0)} u}{R/2} \leq \frac{4 \sup_{B_R(0)} |u|}{R}$$

and

$$|D^2u(0)| \leq C_1 e^{C_2 \sup_{B_{R/2}(0)} |Du|^2/(R/2)^2} \leq C_1 e^{16C_2 \sup_{B_R(0)} |u|^2/R^4}. \tag{1-4}$$

Remark 1.3. The result was first proved by Heinz [1959]. In fact, Heinz’s proof depends on the strict convexity of solutions and the geometry of convex hypersurfaces in dimension two. Our proof, which is based on a suitable choice of auxiliary functions, is elementary and avoids geometric computations on the graph of solutions.

Remark 1.4. The interior C^2 estimate of the σ_2 Hessian equation (1-2) in higher dimensions is a longstanding problem. As we all know, it is hard to find a corresponding geometry in higher dimensions, so we cannot generalize Heinz’s proof or Warren and Yuan’s proof to higher dimensions. But the method in this paper and the optimal concavity in [Chen 2013] is helpful for this problem.

The rest of the paper is organized as follows. In Section 2, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. In Section 3, we introduce a new auxiliary function, and prove Theorem 1.1.

2. Derivatives of eigenvalues and eigenvectors

In this section, we give the calculations of the derivatives of eigenvalues and eigenvectors with respect to the matrix. We expect the following result is known to many people; for example see [Andrews 2007] for a similar result. For completeness, we give the result and a detailed proof.

Proposition 2.1. *Let $W = \{W_{ij}\}$ be an $n \times n$ symmetric matrix with eigenvalues $\lambda(W) = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and with corresponding continuous eigenvector field $\tau^i = (\tau_1^i, \dots, \tau_n^i) \in \mathbb{S}^{n-1}$. Suppose that $W = \{W_{ij}\}$ is diagonal with $\lambda_i = W_{ii}$ and corresponding eigenvector $\tau^i = (0, \dots, 0, 1, 0, \dots, 0) \in \mathbb{S}^{n-1}$ with the 1 in the i -th slot. If λ_k is distinct from the other eigenvalues, then we have, at the diagonal matrix W ,*

$$\frac{\partial \tau_i^k}{\partial W_{pq}} = \begin{cases} 0 & \text{if } i = k \text{ for all } p, q, \\ \frac{1}{\lambda_k - \lambda_i} & \text{if } i \neq k, p = i, q = k, \\ 0 & \text{otherwise;} \end{cases} \tag{2-1}$$

$$\frac{\partial^2 \tau_k^k}{\partial W_{pk} \partial W_{pk}} = -\frac{1}{(\lambda_k - \lambda_p)^2} \quad \text{if } p \neq k; \tag{2-2}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik} \partial W_{ii}} = \frac{1}{(\lambda_k - \lambda_i)^2} \quad \text{if } i \neq k; \quad \frac{\partial^2 \tau_i^k}{\partial W_{ik} \partial W_{kk}} = -\frac{1}{(\lambda_k - \lambda_i)^2} \quad \text{if } i \neq k; \tag{2-3}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{iq} \partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_q} \quad \text{if } i \neq k, i \neq q, q \neq k; \tag{2-4}$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{pq} \partial W_{rs}} = 0 \quad \text{otherwise.} \tag{2-5}$$

Proof. From the definitions of eigenvalue and eigenvector of a matrix W , we have

$$(W - \lambda_k I)\tau^k \equiv 0,$$

where τ^k is the eigenvector of W corresponding to the eigenvalue λ_k . That is, for $i = 1, \dots, n$,

$$(W_{ii} - \lambda_k)\tau_i^k + \sum_{j \neq i} W_{ij}\tau_j^k = 0. \tag{2-6}$$

When $W = \{W_{ij}\}$ is diagonal and λ_k is distinct from the other eigenvalues, λ_k and τ^k are C^2 at the matrix W . In fact,

$$\tau_k^k = 1 \quad \text{and} \quad \tau_i^k = 0 \quad \text{for } i \neq k \quad \text{at } W. \tag{2-7}$$

Taking the first derivative of (2-6), we have

$$\left(\frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}} \right) \tau_i^k + (W_{ii} - \lambda_k) \frac{\partial \tau_i^k}{\partial W_{pq}} + \sum_{j \neq i} \left(\frac{\partial W_{ij}}{\partial W_{pq}} \tau_j^k + W_{ij} \frac{\partial \tau_j^k}{\partial W_{pq}} \right) = 0.$$

Hence, for $i = k$, we get, from (2-7),

$$\frac{\partial \lambda_k}{\partial W_{pq}} = \frac{\partial W_{kk}}{\partial W_{pq}} = \begin{cases} 1 & \text{if } p = k, q = k, \\ 0 & \text{otherwise,} \end{cases} \tag{2-8}$$

and, for $i \neq k$,

$$(W_{ii} - \lambda_k) \frac{\partial \tau_i^k}{\partial W_{pq}} + \sum_{j \neq i} \frac{\partial W_{ij}}{\partial W_{pq}} \tau_j^k = 0,$$

then

$$\frac{\partial \tau_i^k}{\partial W_{pq}} = \frac{1}{\lambda_k - \lambda_i} \frac{\partial W_{ik}}{\partial W_{pq}} = \begin{cases} \frac{1}{\lambda_k - \lambda_i} & \text{if } p = i, q = k, \\ 0 & \text{otherwise.} \end{cases} \tag{2-9}$$

Since $\tau^k \in \mathbb{S}^{n-1}$, we have

$$1 = |\tau^k|^2 = (\tau_1^k)^2 + \dots + (\tau_k^k)^2 + \dots + (\tau_n^k)^2. \tag{2-10}$$

Taking the first derivative of (2-10), and using (2-7),

$$\frac{\partial \tau_k^k}{\partial W_{pq}} = 0 \quad \text{for all } (p, q). \quad (2-11)$$

For $i = k$, taking the second derivative of (2-6), and using (2-7),

$$\left(\frac{\partial^2 W_{kk}}{\partial W_{pq} \partial W_{rs}} - \frac{\partial^2 \lambda_k}{\partial W_{pq} \partial W_{rs}} \right) \tau_k^k + \sum_{j \neq k} \left(\frac{\partial W_{kj}}{\partial W_{pq}} \frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}} \frac{\partial \tau_j^k}{\partial W_{pq}} \right) = 0,$$

hence

$$\frac{\partial^2 \lambda_k}{\partial W_{pq} \partial W_{rs}} = \sum_{j \neq k} \left(\frac{\partial W_{kj}}{\partial W_{pq}} \frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{kj}}{\partial W_{rs}} \frac{\partial \tau_j^k}{\partial W_{pq}} \right) = \begin{cases} \frac{1}{\lambda_k - \lambda_q} & \text{if } p = k, q \neq k, r = q, s = k, \\ \frac{1}{\lambda_k - \lambda_s} & \text{if } r = k, s \neq k, p = s, q = k, \\ 0 & \text{if otherwise.} \end{cases} \quad (2-12)$$

For $i \neq k$,

$$\begin{aligned} \left(\frac{\partial W_{ii}}{\partial W_{pq}} - \frac{\partial \lambda_k}{\partial W_{pq}} \right) \frac{\partial \tau_i^k}{\partial W_{rs}} + \left(\frac{\partial W_{ii}}{\partial W_{rs}} - \frac{\partial \lambda_k}{\partial W_{rs}} \right) \frac{\partial \tau_i^k}{\partial W_{pq}} \\ + (W_{ii} - \lambda_k) \frac{\partial^2 \tau_i^k}{\partial W_{pq} \partial W_{rs}} + \sum_{j \neq i} \left(\frac{\partial W_{ij}}{\partial W_{pq}} \frac{\partial \tau_j^k}{\partial W_{rs}} + \frac{\partial W_{ij}}{\partial W_{rs}} \frac{\partial \tau_j^k}{\partial W_{pq}} \right) = 0, \end{aligned}$$

then

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik} \partial W_{ii}} = \frac{1}{\lambda_k - \lambda_i} \frac{\partial \tau_i^k}{\partial W_{ik}} = \frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_i} \quad \text{if } i \neq k; \quad (2-13)$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{iq} \partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i} \frac{\partial W_{iq}}{\partial W_{iq}} \frac{\partial \tau_q^k}{\partial W_{qk}} = \frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_q} \quad \text{if } i \neq k, i \neq q, q \neq k; \quad (2-14)$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{ik} \partial W_{kk}} = \frac{1}{\lambda_k - \lambda_i} \left(-\frac{\partial \lambda_k}{\partial W_{kk}} \frac{\partial \tau_i^k}{\partial W_{ik}} \right) = -\frac{1}{\lambda_k - \lambda_i} \frac{1}{\lambda_k - \lambda_i} \quad \text{if } i \neq k; \quad (2-15)$$

$$\frac{\partial^2 \tau_i^k}{\partial W_{pq} \partial W_{rs}} = 0 \quad \text{otherwise.} \quad (2-16)$$

From (2-10), we have

$$2\tau_k^k \frac{\partial^2 \tau_k^k}{\partial W_{pq} \partial W_{rs}} + 2 \sum_{i \neq k} \frac{\partial \tau_i^k}{\partial W_{pq}} \frac{\partial \tau_i^k}{\partial W_{rs}} = 0,$$

then

$$\frac{\partial^2 \tau_k^k}{\partial W_{pq} \partial W_{rs}} = - \sum_{i \neq k} \frac{\partial \tau_i^k}{\partial W_{pq}} \frac{\partial \tau_i^k}{\partial W_{rs}} = \begin{cases} -\frac{1}{\lambda_k - \lambda_p} \frac{1}{\lambda_k - \lambda_p} & \text{if } p \neq k, q = k, r = p, s = q, \\ 0 & \text{otherwise.} \end{cases}$$

The proof of Proposition 2.1 is finished. \square

Example 2.2. When $n = 2$, the matrix $\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix}$ has two eigenvalues

$$\lambda_1 = \frac{(u_{11} + u_{22}) + \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2} \quad \text{and} \quad \lambda_2 = \frac{(u_{11} + u_{22}) - \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2}$$

with $\lambda_1 \geq \lambda_2$. If $\lambda_1 > \lambda_2$,

$$\left(\begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} - \lambda_1 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} = 0,$$

we get

$$\xi_1 = \frac{(u_{22} - u_{11}) - \sqrt{(u_{11} - u_{22})^2 + 4u_{12}u_{21}}}{2} \quad \text{and} \quad \xi_2 = -u_{21}.$$

Then the eigenvector τ corresponding to λ_1 is

$$\tau = -\frac{(\xi_1, \xi_2)}{\sqrt{\xi_1^2 + \xi_2^2}}.$$

By direct calculations, we can get the derivatives of eigenvalues and the eigenvector τ with respect to the matrix, and this verifies Proposition 2.1.

3. Proof of Theorem 1.1

Now we start to prove Theorem 1.1.

Let $\tau(x) = \tau(D^2u(x)) = (\tau_1, \tau_2) \in \mathbb{S}^1$ be the continuous eigenvector field of $D^2u(x)$ corresponding to the largest eigenvalue. Let

$$\Sigma := \{x \in B_R(0) : r^2 - |x|^2 + \langle x, \tau(x) \rangle^2 > 0, r^2 - \langle x, \tau(x) \rangle^2 > 0\}, \tag{3-1}$$

where $r = R/\sqrt{2}$. It is easy to show that Σ is an open set and $B_r(0) \subset \Sigma \subset B_R(0)$. We introduce a new auxiliary function in Σ as follows:

$$\phi(x) = \eta(x)^\beta g\left(\frac{1}{2}|Du|^2\right)u_{\tau\tau}, \tag{3-2}$$

where $\eta(x) = (r^2 - |x|^2 + \langle x, \tau(x) \rangle^2)(r^2 - \langle x, \tau(x) \rangle^2)$ with $\beta = 4$ and $g(t) = e^{c_0 t/r^2}$ with $c_0 = 32/m$. In fact, $\langle x, \tau(x) \rangle$ is invariant under rotations of the coordinates, and so is $\eta(x)$.

From the definition of Σ , we know $\eta(x) > 0$ in Σ , and $\eta = 0$ on $\partial\Sigma$. Assume the maximum of $\phi(x)$ in Σ is attained at $x_0 \in \Sigma$. By rotating the coordinates, we can assume $D^2u(x_0)$ is diagonal. In the following, we let $\lambda_i = u_{ii}(x_0)$, $\lambda = (\lambda_1, \lambda_2)$. Without loss of generality, we can assume $\lambda_1 \geq \lambda_2$ and $\tau(x_0) = (1, 0)$.

If

$$\eta\lambda_1 \leq 10^3 \left(1 + M + r \sup |\nabla f| + \frac{M}{m} \frac{\sup |Du|}{r} \right) r^4 =: \Theta,$$

then we easily get

$$\begin{aligned} u_{\tau(0)\tau(0)}(0) &\leq \frac{1}{r^{4\beta}} \phi(0) \leq \frac{1}{r^{4\beta}} \phi(x_0) \\ &\leq \Theta e^{c_0 \sup |Du|^2/r^2} \\ &\leq 10^3 (1 + M + r \sup |\nabla f|) e^{(c_0 + 2M/m) \sup |Du|^2/r^2}. \end{aligned}$$

Hence we get

$$|u_{\xi\xi}(0)| \leq u_{\tau(0)\tau(0)}(0) \leq 10^3(1 + M + r \sup |\nabla f|)e^{(c_0+2M/m) \sup |Du|^2/r^2} \quad \text{for all } \xi \in \mathbb{S}^1.$$

This completes the proof of Theorem 1.1 under the condition $\eta\lambda_1 \leq \Theta$.

Now, we assume $\eta\lambda_1 \geq \Theta$. Then we have

$$\lambda_1 = \frac{\eta\lambda_1}{\eta} \geq \frac{\Theta}{r^4} = 10^3 \left(1 + M + r \sup |\nabla f| + \frac{M \sup |Du|}{m r} \right). \tag{3-3}$$

From (1-1), we have

$$\lambda_2 = \frac{f}{\lambda_1} \leq \frac{M}{\lambda_1} < \lambda_1.$$

Hence λ_1 is distinct from the other eigenvalue, and $\tau(x)$ is C^2 at x_0 . Moreover, the test function

$$\varphi = \beta \log \eta + \log g \left(\frac{1}{2} |Du|^2 \right) + \log u_{11} \tag{3-4}$$

attains the local maximum at x_0 . In the following, all the calculations are at x_0 .

Then, we get

$$0 = \varphi_i = \beta \frac{\eta_i}{\eta} + \frac{g'}{g} \sum_k u_k u_{ki} + \frac{u_{11i}}{u_{11}},$$

so we have

$$\frac{u_{11i}}{u_{11}} = -\beta \frac{\eta_i}{\eta} - \frac{g'}{g} u_i u_{ii} \quad \text{if } i = 1, 2. \tag{3-5}$$

At x_0 , we also have

$$\begin{aligned} 0 \geq \varphi_{ii} &= \beta \left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2} \right) + \frac{g''g - g'^2}{g^2} \sum_k u_k u_{ki} \sum_l u_l u_{li} + \frac{g'}{g} \sum_k (u_{ki} u_{ki} + u_k u_{kii}) + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2} \\ &= \beta \left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2} \right) + \frac{g'}{g} \left(u_{ii}^2 + \sum_k u_k u_{kii} \right) + \frac{u_{11ii}}{u_{11}} - \frac{u_{11i}^2}{u_{11}^2}, \end{aligned}$$

since $g''g - g'^2 = 0$. Let

$$\begin{aligned} F^{11} &= \frac{\partial \det D^2 u}{\partial u_{11}} = \lambda_2, & F^{22} &= \frac{\partial \det D^2 u}{\partial u_{22}} = \lambda_1, \\ F^{12} &= \frac{\partial \det D^2 u}{\partial u_{12}} = 0, & F^{21} &= \frac{\partial \det D^2 u}{\partial u_{21}} = 0. \end{aligned}$$

Then from (1-1) we get

$$\lambda_2 = \frac{f}{\lambda_1}. \tag{3-6}$$

Differentiating (1-1) once, we get

$$F^{11} u_{11i} + F^{22} u_{22i} = f_i,$$

then

$$u_{22i} = \frac{1}{F^{22}} (f_i - F^{11} u_{11i}) = \frac{f_i}{\lambda_1} - \frac{f}{\lambda_1} \frac{u_{11i}}{u_{11}}. \tag{3-7}$$

Differentiating (1-1) twice, we get

$$\begin{aligned}
 F^{11}u_{1111} + F^{22}u_{2211} &= f_{11} - 2\frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111}u_{221} - 2\frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{112}^2 \\
 &= f_{11} - 2u_{111}u_{221} + 2u_{112}^2 \\
 &= f_{11} + 2u_{112}^2 - 2u_{111}\left(\frac{f_1}{\lambda_1} - \frac{f}{\lambda_1} \frac{u_{111}}{u_{11}}\right) \\
 &= f_{11} + 2u_{112}^2 - 2f_1\frac{u_{111}}{u_{11}} + 2f\left(\frac{u_{111}}{u_{11}}\right)^2
 \end{aligned} \tag{3-8}$$

and

$$\begin{aligned}
 F^{11}u_{1112} + F^{22}u_{2212} &= f_{12} - \frac{\partial^2 \det D^2u}{\partial u_{11}\partial u_{22}}u_{111}u_{222} - \frac{\partial^2 \det D^2u}{\partial u_{22}\partial u_{11}}u_{221}u_{112} - 2\frac{\partial^2 \det D^2u}{\partial u_{12}\partial u_{21}}u_{121}u_{212} \\
 &= f_{12} - u_{111}u_{222} - u_{112}u_{221} + 2u_{112}u_{221} \\
 &= f_{12} - u_{111}\left(\frac{f_2}{\lambda_1} - \frac{f}{\lambda_1} \frac{u_{112}}{u_{11}}\right) + u_{112}\left(\frac{f_1}{\lambda_1} - \frac{f}{\lambda_1} \frac{u_{111}}{u_{11}}\right) \\
 &= f_{12} + f_1\frac{u_{112}}{u_{11}} - f_2\frac{u_{111}}{u_{11}}.
 \end{aligned} \tag{3-9}$$

Hence

$$\begin{aligned}
 0 &\geq \sum_{i=1}^2 F^{ii}\varphi_{ii} \\
 &= \beta \sum_i F^{ii}\left(\frac{\eta_{ii}}{\eta} - \frac{\eta_i^2}{\eta^2}\right) + \frac{g'}{g} \sum_i F^{ii}u_{ii}^2 + \frac{g'}{g} \sum_k u_k f_k + \frac{1}{u_{11}} \sum_i F^{ii}u_{11ii} - \sum_i F^{ii}\left(\frac{u_{11i}}{u_{11}}\right)^2 \\
 &= \beta\lambda_2\left(\frac{\eta_{11}}{\eta} - \frac{\eta_1^2}{\eta^2}\right) + \beta\lambda_1\left(\frac{\eta_{22}}{\eta} - \frac{\eta_2^2}{\eta^2}\right) + \frac{g'}{g}(\lambda_1 + \lambda_2)f + \frac{g'}{g}(u_1f_1 + u_2f_2) \\
 &\quad + \frac{1}{u_{11}}\left(f_{11} + 2u_{112}^2 - 2f_1\frac{u_{111}}{u_{11}} + 2f\left(\frac{u_{111}}{u_{11}}\right)^2\right) - \lambda_2\left(\frac{u_{111}}{u_{11}}\right)^2 - \lambda_1\left(\frac{u_{112}}{u_{11}}\right)^2 \\
 &\geq \beta\left(\frac{f}{\lambda_1} \frac{\eta_{11}}{\eta} + \lambda_1\frac{\eta_{22}}{\eta}\right) - \beta\frac{f}{\lambda_1} \frac{\eta_1^2}{\eta^2} - \beta\lambda_1\frac{\eta_2^2}{\eta^2} + \frac{f}{\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2 - 2\frac{f_1}{u_{11}} \frac{u_{111}}{u_{11}} \\
 &\quad + \frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2 + \frac{\lambda_1}{2}\left(\beta\frac{\eta_2}{\eta} + \frac{g'}{g}u_2u_{22}\right)^2 + \frac{g'}{g}f\lambda_1 - \frac{g'}{g}|\nabla u||\nabla f| - \frac{|f_{11}|}{\lambda_1} \\
 &\geq \beta\left(\frac{f}{\lambda_1} \frac{\eta_{11}}{\eta} + \lambda_1\frac{\eta_{22}}{\eta}\right) - \beta\frac{f}{\lambda_1} \frac{\eta_1^2}{\eta^2} - \beta\lambda_1\frac{\eta_2^2}{\eta^2} + \frac{f}{2\lambda_1}\left(\frac{u_{111}}{u_{11}}\right)^2 \\
 &\quad + \frac{\lambda_1}{2}\left(\frac{u_{112}}{u_{11}}\right)^2 + \frac{\beta^2}{2}\lambda_1\frac{\eta_2^2}{\eta^2} + \beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2 + \frac{g'}{g}f\lambda_1 - \frac{g'}{g}|\nabla u||\nabla f| - \frac{|f_{11}|}{\lambda_1} - 2\frac{f_1^2}{f\lambda_1}.
 \end{aligned} \tag{3-10}$$

Lemma 3.1. Under the condition $\eta\lambda_1 \geq \Theta$ we have, at x_0 ,

$$\beta\frac{f}{\lambda_1} \frac{\eta_1^2}{\eta^2} \leq \frac{8\beta fr^6}{\eta^2\lambda_1} + \frac{\lambda_1}{4}\left(\frac{u_{112}}{u_{11}}\right)^2 \quad \text{and} \quad \beta f\frac{g'}{g}\frac{\eta_2}{\eta}u_2 \geq -4\beta f\frac{g'}{g}\frac{r^4}{\eta} \frac{|u_2|}{r}, \tag{3-11}$$

and

$$\begin{aligned} \beta \left(\frac{f}{\lambda_1} \frac{\eta_{11}}{\eta} + \lambda_1 \frac{\eta_{22}}{\eta} \right) &\geq -\frac{1}{2} \frac{g'}{g} f \lambda_1 - \beta \lambda_1 \left(\frac{\eta_2}{\eta} \right)^2 - \frac{f}{2\lambda_1} \left(\frac{u_{111}}{u_{11}} \right)^2 - \frac{\lambda_1}{4} \left(\frac{u_{112}}{u_{11}} \right)^2 - 2\beta f \frac{r^2}{\eta \lambda_1} \\ &\quad - 4\beta |f_{12}| \frac{r^4}{\eta \lambda_1} - 32\beta \frac{f_1^2 r^4}{\eta \lambda_1^3} - 8\beta \frac{|f_1 f_2| r^4}{\eta \lambda_1^3} - 24\beta \frac{|f_1| r^3}{\eta \lambda_1} - \left(\frac{6\beta |f_1|^2 r^4}{\eta \lambda_1} + \frac{12\beta f r^2}{\eta \lambda_1} \right) \\ &\quad - \left(\frac{8\beta f r^4 |f_2|^2}{\eta \lambda_1^3} + \frac{(48\beta)^2 f r^6}{\eta^2 \lambda_1} + \frac{32\beta^2 |f_2|^2 r^8}{\eta^2 \lambda_1 f} \right). \end{aligned} \tag{3-12}$$

Proof. At x_0 , $\tau = (\tau_1, \tau_2) = (1, 0)$. Then from Proposition 2.1 we get

$$\langle x, \partial_i \tau \rangle = \sum_{m=1}^2 x_m \frac{\partial \tau_m}{\partial x_i} = \sum_{m=1}^2 x_m \frac{\partial \tau_m}{\partial u_{pq}} u_{pqi} = x_2 \frac{\partial \tau_2}{\partial u_{pq}} u_{pqi} = x_2 \frac{u_{12i}}{\lambda_1 - \lambda_2} \quad \text{if } i = 1, 2. \tag{3-13}$$

From the definition of η , then we have, at x_0 ,

$$\eta = (r^2 - |x|^2 + \langle x, \tau \rangle^2)(r^2 - \langle x, \tau \rangle^2) = (r^2 - x_2^2)(r^2 - x_1^2). \tag{3-14}$$

Taking the first derivative of η , we get

$$\begin{aligned} \eta_i &= (-2x_i + 2\langle x, \tau \rangle \langle x, \tau \rangle_i)(r^2 - \langle x, \tau \rangle^2) + (r^2 - |x|^2 + \langle x, \tau \rangle^2)(-2\langle x, \tau \rangle \langle x, \tau \rangle_i) \\ &= (-2x_i + 2x_1(\delta_{i1} + \langle x, \partial_i \tau \rangle))(r^2 - x_1^2) + (r^2 - x_2^2)(-2x_1(\delta_{i1} + \langle x, \partial_i \tau \rangle)) \\ &= \begin{cases} -2x_1(r^2 - x_2^2) + 2x_1 \langle x, \partial_1 \tau \rangle (x_2^2 - x_1^2) & \text{if } i = 1, \\ -2x_2(r^2 - x_1^2) + 2x_1 \langle x, \partial_2 \tau \rangle (x_2^2 - x_1^2) & \text{if } i = 2. \end{cases} \end{aligned}$$

Hence

$$\begin{aligned} \beta \frac{f}{\lambda_1} \frac{\eta_1^2}{\eta^2} &= \beta \frac{f}{\lambda_1} \left(\frac{-2x_1(r^2 - x_2^2)}{\eta} + 2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \\ &\leq \beta \frac{f}{\lambda_1} \left(\frac{8r^6}{\eta^2} + \frac{8r^8}{\eta^2} \left(\frac{u_{112}}{u_{11}} \right)^2 \right) \leq \frac{8\beta f r^6}{\eta^2 \lambda_1} + \frac{\lambda_1}{4} \left(\frac{u_{112}}{u_{11}} \right)^2. \end{aligned} \tag{3-15}$$

Also we have

$$\begin{aligned} \frac{\eta_2}{\eta} &= \frac{-2x_2(r^2 - x_1^2)}{\eta} + 2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{u_{221}}{\lambda_1 - \lambda_2} \\ &= \frac{-2x_2(r^2 - x_1^2)}{\eta} + 2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{1}{\lambda_1 - \lambda_2} \left(\frac{f_1}{\lambda_1} - \frac{f}{\lambda_1} \frac{u_{111}}{u_{11}} \right) \\ &= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left(1 - x_1 \frac{(x_2^2 - x_1^2)(r^2 - x_2^2)}{\eta} \frac{1}{\lambda_1 - \lambda_2} \frac{f_1}{\lambda_1} \right) + 2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{1}{\lambda_1 - \lambda_2} \frac{f}{\lambda_1} \left(\beta \frac{\eta_1}{\eta} + \frac{g'}{g} u_{11} u_{11} \right) \\ &= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left(1 - x_1 \frac{(x_2^2 - x_1^2)(r^2 - x_2^2)}{\eta} \frac{1}{\lambda_1 - \lambda_2} \left(\frac{f_1}{\lambda_1} + \frac{g'}{g} u_{11} f \right) \right) \\ &\quad + 2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{1}{\lambda_1 - \lambda_2} \frac{f}{\lambda_1} \beta \left(\frac{-2x_1(r^2 - x_2^2)}{\eta} + 2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \frac{u_{112}}{\lambda_1 - \lambda_2} \right) \end{aligned}$$

$$= \frac{-2x_2(r^2 - x_1^2)}{\eta} \left(1 - x_1 \frac{(x_2^2 - x_1^2)(r^2 - x_2^2)}{\eta} \frac{1}{\lambda_1 - \lambda_2} \left(\frac{f_1}{\lambda_1} + \frac{g'}{g} u_1 f - \frac{f}{\lambda_1} \beta \frac{2x_1(r^2 - x_2^2)}{\eta} \right) \right) \\ + \left(2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \right)^2 \frac{f}{(\lambda_1 - \lambda_2)^2} \beta \left(-\beta \frac{\eta_2}{\eta} - \frac{g'}{g} u_2 u_{22} \right),$$

then we get

$$\left(1 + \beta^2 \left(2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \right)^2 \frac{f}{(\lambda_1 - \lambda_2)^2} \right) \frac{\eta_2}{\eta} \\ = \frac{-2x_2(r^2 - x_1^2)}{\eta} \left(1 - x_1 \frac{(x_2^2 - x_1^2)(r^2 - x_2^2)}{\eta} \frac{1}{\lambda_1 - \lambda_2} \left(\frac{f_1}{\lambda_1} + \frac{g'}{g} u_1 f - \frac{f}{\lambda_1} \beta \frac{2x_1(r^2 - x_2^2)}{\eta} \right) \right) \\ + \frac{-2x_2(r^2 - x_1^2)}{\eta} \frac{2x_1^2 x_2 (x_2^2 - x_1^2)^2 (r^2 - x_2^2)}{\eta^2} \frac{f}{(\lambda_1 - \lambda_2)^2} \beta \frac{g'}{g} u_2 \frac{f}{\lambda_1}. \quad (3-16)$$

Hence

$$\beta f \frac{g'}{g} \frac{\eta_2}{\eta} u_2 \geq -\beta f \frac{g'}{g} \frac{2r^3}{\eta} \left(1 + \frac{2r^5}{\eta \lambda_1} \left(\frac{|f_1|}{\lambda_1} + \frac{g'}{g} |u_1| f + \frac{2\beta f r^3}{\eta \lambda_1} \right) + \frac{16\beta r^9}{\eta^2} \frac{f^2}{\lambda_1^3} \frac{g'}{g} |u_2| \right) |u_2| \\ \geq -\beta f \frac{g'}{g} \frac{2r^3}{\eta} \left(1 + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \right) |u_2| \\ = -4\beta f \frac{g'}{g} \frac{r^4}{\eta} \frac{|u_2|}{r}. \quad (3-17)$$

In fact, $\eta_2/\eta \approx -2x_2(r^2 - x_1^2)/\eta$ if $\eta\lambda_1$ is big enough, and we get from (3-16)

$$\frac{\eta_2^2}{\eta^2} \geq \left(\left(1 + \beta^2 \left(2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \right)^2 \frac{f}{(\lambda_1 - \lambda_2)^2} \right) \frac{\eta_2}{\eta} \right)^2 \left(1 - \beta^2 \left(2x_1 x_2 \frac{x_2^2 - x_1^2}{\eta} \right)^2 \frac{f}{(\lambda_1 - \lambda_2)^2} \right)^2 \\ \geq \left(\frac{-2x_2(r^2 - x_1^2)}{\eta} \right)^2 \left(1 - \frac{2r^5}{\eta \lambda_1} \left(\frac{|f_1|}{\lambda_1} + \frac{g'}{g} |u_1| f + \frac{2\beta f r^3}{\eta \lambda_1} \right) - \frac{8\beta r^9}{\eta^2 \lambda_1^2} \frac{g'}{g} |u_2| \frac{f^2}{\lambda_1} \right)^2 \left(1 - \beta^2 \frac{16r^8}{\eta^2} \frac{f}{\lambda_1^2} \right)^2 \\ \geq \left(\frac{-2x_2(r^2 - x_1^2)}{\eta} \right)^2 \left(1 - \frac{1}{10^3} - \frac{64}{10^3} - \frac{1}{10} - \frac{1}{10^3} \right)^2 \left(1 - \frac{1}{10^3} \right)^2 \\ \geq \frac{1}{2} \left(\frac{-2x_2(r^2 - x_1^2)}{\eta} \right)^2.$$

Taking second derivatives of η , we get

$$\eta_{ii} = (-2 + 2\langle x, \tau \rangle \langle x, \tau \rangle_{ii} + 2\langle x, \tau \rangle_i \langle x, \tau \rangle_i) (r^2 - \langle x, \tau \rangle^2) \\ + 2(-2x_i + 2\langle x, \tau \rangle \langle x, \tau \rangle_i) (-2\langle x, \tau \rangle \langle x, \tau \rangle_i) \\ + (r^2 - |x|^2 + \langle x, \tau \rangle^2) (-2\langle x, \tau \rangle \langle x, \tau \rangle_{ii} - 2\langle x, \tau \rangle_i \langle x, \tau \rangle_i) \\ = (-2 + 2x_1 \langle x, \tau \rangle_{ii} + 2(\delta_{i1} + \langle x, \partial_i \tau \rangle)) (r^2 - x_1^2) \\ + 2(-2x_i + 2x_1(\delta_{i1} + \langle x, \partial_i \tau \rangle)) (-2x_1(\delta_{i1} + \langle x, \partial_i \tau \rangle)) \\ + (r^2 - x_2^2) (-2x_1 \langle x, \tau \rangle_{ii} - 2(\delta_{i1} + \langle x, \partial_i \tau \rangle)^2),$$

so

$$\eta_{11} = -2(r^2 - x_2^2) - 2x_1(x_1^2 - x_2^2)\langle x, \tau \rangle_{11} + (4x_2^2 - 12x_1^2)\langle x, \partial_1 \tau \rangle + (2x_2^2 - 10x_1^2)\langle x, \partial_1 \tau \rangle^2, \quad (3-18)$$

$$\eta_{22} = -2(r^2 - x_1^2) - 2x_1(x_1^2 - x_2^2)\langle x, \tau \rangle_{22} + 8x_1x_2\langle x, \partial_2 \tau \rangle + (2x_2^2 - 10x_1^2)\langle x, \partial_2 \tau \rangle^2. \quad (3-19)$$

Hence

$$\begin{aligned} \beta \left(\frac{f}{\lambda_1} \frac{\eta_{11}}{\eta} + \lambda_1 \frac{\eta_{22}}{\eta} \right) &= -2\beta \left(\frac{f}{\lambda_1} \frac{r^2 - x_2^2}{\eta} + \lambda_1 \frac{r^2 - x_1^2}{\eta} \right) - 2\beta \frac{x_1(x_1^2 - x_2^2)}{\eta} \left(\frac{f}{\lambda_1} \langle x, \tau \rangle_{11} + \lambda_1 \langle x, \tau \rangle_{22} \right) \\ &\quad + \beta \frac{f}{\lambda_1} \left(\frac{x_2(4x_2^2 - 12x_1^2)}{\eta} \frac{u_{112}}{\lambda_1 - \lambda_2} + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta} \left(\frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \right) \\ &\quad + \beta \lambda_1 \left(\frac{8x_1x_2^2}{\eta} \frac{u_{221}}{\lambda_1 - \lambda_2} + \frac{x_2^2(2x_2^2 - 10x_1^2)}{\eta} \left(\frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 \right). \end{aligned} \quad (3-20)$$

Direct calculations yield

$$\begin{aligned} \langle x, \tau \rangle_{11} &= \frac{\partial^2}{\partial x_1^2} \left(\sum_{m=1}^2 x_m \tau_m \right) = 2 \frac{\partial \tau_1}{\partial x_1} + \sum_{m=1}^2 x_m \frac{\partial^2 \tau_m}{\partial x_1^2} \\ &= 2 \frac{\partial \tau_1}{\partial u_{pq}} u_{pq1} + \sum_{m=1}^2 x_m \left(\frac{\partial \tau_m}{\partial u_{pq}} u_{pq11} + \frac{\partial^2 \tau_m}{\partial u_{pq} \partial u_{rs}} u_{pq1} u_{rs1} \right) \\ &= 0 + x_1 \frac{\partial^2 \tau_1}{\partial u_{pq} \partial u_{rs}} u_{pq1} u_{rs1} + x_2 \left(\frac{\partial \tau_2}{\partial u_{pq}} u_{pq11} + \frac{\partial^2 \tau_2}{\partial u_{pq} \partial u_{rs}} u_{pq1} u_{rs1} \right) \\ &= -x_1 \left(\frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 + x_2 \left(\frac{1}{\lambda_1 - \lambda_2} \right) u_{1211} + 2x_2 \left(-\frac{u_{112}u_{111}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{112}u_{221}}{(\lambda_1 - \lambda_2)^2} \right). \end{aligned}$$

Similarly, we have

$$\begin{aligned} \langle x, \tau \rangle_{22} &= \frac{\partial^2}{\partial x_2^2} \left(\sum_{m=1}^2 x_m \tau_m \right) = 2 \frac{\partial \tau_2}{\partial x_2} + \sum_{m=1}^2 x_m \frac{\partial^2 \tau_m}{\partial x_2^2} \\ &= 2 \frac{\partial \tau_2}{\partial u_{pq}} u_{pq2} + \sum_{m=1}^2 x_m \left(\frac{\partial \tau_m}{\partial u_{pq}} u_{pq22} + \frac{\partial^2 \tau_m}{\partial u_{pq} \partial u_{rs}} u_{pq2} u_{rs2} \right) \\ &= 2 \left(\frac{1}{\lambda_1 - \lambda_2} \right) u_{221} - x_1 \left(\frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 + x_2 \left(\frac{1}{\lambda_1 - \lambda_2} \right) u_{1222} + 2x_2 \left(-\frac{u_{112}u_{221}}{(\lambda_1 - \lambda_2)^2} + \frac{u_{222}u_{221}}{(\lambda_1 - \lambda_2)^2} \right), \end{aligned}$$

then

$$\begin{aligned} \frac{f}{\lambda_1} \langle x, \tau \rangle_{11} + \lambda_1 \langle x, \tau \rangle_{22} &= -x_1 \frac{f}{\lambda_1} \left(\frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 + 2\lambda_1 \left(\frac{u_{221}}{\lambda_1 - \lambda_2} \right) - x_1 \lambda_1 \left(\frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 \\ &\quad + x_2 \left(\frac{1}{\lambda_1 - \lambda_2} \right) \left(f_{12} + f_1 \frac{u_{112}}{u_{11}} - f_2 \left(\frac{u_{111}}{u_{11}} \right) \right) \\ &\quad + 2x_2 \left(-\frac{u_{112}}{(\lambda_1 - \lambda_2)^2} f_1 + \frac{u_{221}}{(\lambda_1 - \lambda_2)^2} f_2 \right). \end{aligned} \quad (3-21)$$

From (3-20) and (3-21), we get

$$\begin{aligned}
& \beta \left(\frac{f}{\lambda_1} \frac{\eta_{11}}{\eta} + \lambda_1 \frac{\eta_{22}}{\eta} \right) \\
&= -2\beta \left(\frac{f}{\lambda_1} \frac{r^2 - x_2^2}{\eta} + \lambda_1 \frac{r^2 - x_1^2}{\eta} \right) - 2\beta \frac{x_1 x_2 (x_1^2 - x_2^2)}{\eta} \left(\frac{1}{\lambda_1 - \lambda_2} \right) \left(f_{12} - f_2 \left(\frac{u_{111}}{u_{11}} \right) \right) \\
&\quad + \left(\frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \left(2\beta \frac{f}{\lambda_1} \frac{x_1^2 (x_1^2 - x_2^2)}{\eta} + \beta \frac{f}{\lambda_1} \frac{x_2^2 (2x_2^2 - 10x_1^2)}{\eta} \right) \\
&\quad + \frac{u_{112}}{\lambda_1 - \lambda_2} \left(-2\beta \frac{x_1 (x_1^2 - x_2^2)}{\eta} \left(x_2 \frac{f_1}{\lambda_1} - 2x_2 \frac{f_1}{\lambda_1 - \lambda_2} \right) + \beta \frac{f}{\lambda_1} \frac{x_2 (4x_2^2 - 12x_1^2)}{\eta} \right) \\
&\quad + \left(\frac{u_{221}}{\lambda_1 - \lambda_2} \right)^2 \left(2\beta \lambda_1 \frac{x_1^2 (x_1^2 - x_2^2)}{\eta} + \beta \lambda_1 \frac{x_2^2 (2x_2^2 - 10x_1^2)}{\eta} \right) \\
&\quad + \frac{u_{221}}{\lambda_1 - \lambda_2} \left(-2\beta \frac{x_1 (x_1^2 - x_2^2)}{\eta} \left(2\lambda_1 + 2x_2 \frac{f_2}{\lambda_1 - \lambda_2} \right) + \beta \lambda_1 \frac{8x_1 x_2^2}{\eta} \right) \\
&\geq -2\beta \lambda_1 \frac{r^2 - x_1^2}{\eta} - 2\beta f \frac{r^2}{\eta \lambda_1} - 2\beta |f_{12}| \frac{r^4}{\eta (\lambda_1 - \lambda_2)} - 2\beta |f_2| \frac{r^4}{\eta (\lambda_1 - \lambda_2)} \left| \frac{u_{111}}{u_{11}} \right| \\
&\quad - \left(\frac{u_{112}}{\lambda_1 - \lambda_2} \right)^2 \beta \frac{f}{\lambda_1} \frac{8r^4}{\eta} - \left| \frac{u_{112}}{\lambda_1 - \lambda_2} \right| \left(6\beta \frac{|f_1|}{\lambda_1 - \lambda_2} \frac{r^4}{\eta} + \beta \frac{f}{\lambda_1} \frac{12r^3}{\eta} \right) \\
&\quad - \frac{1}{(\lambda_1 - \lambda_2)^2} (u_{221})^2 \beta \lambda_1 \frac{8r^4}{\eta} - \frac{1}{\lambda_1 - \lambda_2} |u_{221}| \left(4\beta \frac{r^4 |f_2|}{\eta \lambda_1} + \beta \lambda_1 \frac{12r^3}{\eta} \right) \\
&\geq -2\beta \lambda_1 \frac{r^2 - x_1^2}{\eta} - 2\beta f \frac{r^2}{\eta \lambda_1} - 4\beta |f_{12}| \frac{r^4}{\eta \lambda_1} - 4\beta |f_2| \frac{r^4}{\eta \lambda_1} \left| \frac{u_{111}}{u_{11}} \right| \\
&\quad - \left(\frac{u_{112}}{u_{11}} \right)^2 \beta \frac{f}{\lambda_1} \frac{16r^4}{\eta} - \left| \frac{u_{112}}{u_{11}} \right| \left(12\beta \frac{|f_1|}{\lambda_1} \frac{r^4}{\eta} + \beta \frac{f}{\lambda_1} \frac{24r^3}{\eta} \right) \\
&\quad - 2 \left(\frac{f_1^2}{\lambda_1^4} + \frac{f^2}{\lambda_1^4} \left(\frac{u_{111}}{u_{11}} \right)^2 \right) \beta \lambda_1 \frac{16r^4}{\eta} - \left(\frac{|f_1|}{\lambda_1^2} + \frac{f}{\lambda_1^2} \left| \frac{u_{111}}{u_{11}} \right| \right) \left(8\beta \frac{r^4 |f_2|}{\eta \lambda_1} + \beta \lambda_1 \frac{24r^3}{\eta} \right) \\
&\geq -2\beta \lambda_1 \frac{r^2 - x_1^2}{\eta} - 2\beta f \frac{r^2}{\eta \lambda_1} - 4\beta |f_{12}| \frac{r^4}{\eta \lambda_1} - 32\beta \frac{f_1^2 r^4}{\eta \lambda_1^3} - 8\beta \frac{|f_1 f_2| r^4}{\eta \lambda_1^3} - 24\beta \frac{|f_1| r^3}{\eta \lambda_1} \\
&\quad - \frac{\lambda_1}{2} \left(\frac{u_{112}}{u_{11}} \right)^2 \left(\frac{32\beta f r^4}{\eta \lambda_1^2} + \frac{12\beta r^4}{\eta \lambda_1^2} + \frac{24\beta f r^4}{\eta \lambda_1^2} \right) - \frac{\lambda_1}{2} \left(\frac{12\beta |f_1|^2 r^4}{\eta \lambda_1^2} + \frac{24\beta f r^2}{\eta \lambda_1^2} \right) \\
&\quad - \frac{f}{2\lambda_1} \left(\frac{u_{111}}{u_{11}} \right)^2 \left(\frac{64\beta f r^4}{\eta \lambda_1^2} + \frac{16\beta r^4}{\eta \lambda_1^2} + \frac{1}{4} + \frac{1}{4} \right) \\
&\quad - \frac{f}{2\lambda_1} \left(\frac{16\beta r^4 |f_2|^2}{\eta \lambda_1^2} + \left(\frac{48\beta r^3}{\eta} \right)^2 + \left(\frac{8\beta r^4 |f_2|}{\eta} \frac{1}{f} \right)^2 \right)
\end{aligned}$$

$$\begin{aligned} &\geq -2\beta\lambda_1 \frac{r^2 - x_1^2}{\eta} - 2\beta f \frac{r^2}{\eta\lambda_1} - 4\beta|f_{12}| \frac{r^4}{\eta\lambda_1} - 32\beta \frac{f_1^2 r^4}{\eta\lambda_1^3} - 8\beta \frac{|f_1 f_2| r^4}{\eta\lambda_1^3} - 24\beta \frac{|f_1| r^3}{\eta\lambda_1} \\ &\quad - \frac{\lambda_1}{4} \left(\frac{u_{112}}{u_{11}} \right)^2 - \left(\frac{6\beta|f_1|^2 r^4}{\eta\lambda_1} + \frac{12\beta f r^2}{\eta\lambda_1} \right) \\ &\quad - \frac{f}{2\lambda_1} \left(\frac{u_{111}}{u_{11}} \right)^2 - \left(\frac{8\beta f r^4 |f_2|^2}{\eta\lambda_1^3} + \frac{(48\beta)^2 f r^6}{\eta^2 \lambda_1} + \frac{32\beta^2 |f_2|^2 r^8}{\eta^2 \lambda_1 f} \right). \end{aligned}$$

Now we just need to estimate $-2\beta\lambda_1(r^2 - x_1^2)/\eta$. If $x_2^2 \leq \frac{1}{2}r^2$, we get

$$-2\beta\lambda_1 \frac{r^2 - x_1^2}{\eta} = -\frac{8}{r^2 - x_2^2} \lambda_1 \geq -\frac{16}{r^2} \lambda_1 \geq -\frac{1}{2} \frac{c_0}{r^2} f \lambda_1 = -\frac{1}{2} \frac{g'}{g} f \lambda_1.$$

If $x_2^2 \geq \frac{1}{2}r^2$, we get

$$-2\beta\lambda_1 \frac{r^2 - x_1^2}{\eta} = -\frac{8}{r^2 - x_2^2} \lambda_1 \geq -\frac{x_2^2}{r^2 - x_2^2} \frac{8}{r^2 - x_2^2} \lambda_1 = -\beta\lambda_1 \frac{1}{2} \left(\frac{2x_2}{r^2 - x_2^2} \right)^2 \geq -\beta\lambda_1 \left(\frac{\eta_2}{\eta} \right)^2.$$

Hence

$$-2\beta\lambda_1 \frac{r^2 - x_1^2}{\eta} \geq -\frac{1}{2} \frac{g'}{g} f \lambda_1 - \beta\lambda_1 \left(\frac{\eta_2}{\eta} \right)^2 \tag{3-22}$$

and

$$\begin{aligned} \beta \left(\frac{f}{\lambda_1} \frac{\eta_{11}}{\eta} + \lambda_1 \frac{\eta_{22}}{\eta} \right) &\geq -\frac{1}{2} \frac{g'}{g} f \lambda_1 - \beta\lambda_1 \left(\frac{\eta_2}{\eta} \right)^2 - \frac{f}{2\lambda_1} \left(\frac{u_{111}}{u_{11}} \right)^2 - \frac{\lambda_1}{4} \left(\frac{u_{112}}{u_{11}} \right)^2 \\ &\quad - 2\beta f \frac{r^2}{\eta\lambda_1} - 4\beta|f_{12}| \frac{r^4}{\eta\lambda_1} - 32\beta \frac{f_1^2 r^4}{\eta\lambda_1^3} - 8\beta \frac{|f_1 f_2| r^4}{\eta\lambda_1^3} - 24\beta \frac{|f_1| r^3}{\eta\lambda_1} \\ &\quad - \left(\frac{6\beta|f_1|^2 r^4}{\eta\lambda_1} + \frac{12\beta f r^2}{\eta\lambda_1} \right) \\ &\quad - \left(\frac{8\beta f r^4 |f_2|^2}{\eta\lambda_1^3} + \frac{(48\beta)^2 f r^6}{\eta^2 \lambda_1} + \frac{32\beta^2 |f_2|^2 r^8}{\eta^2 \lambda_1 f} \right). \end{aligned} \tag{3-23}$$

This concludes the proof of Lemma 3.1. □

Now we continue to prove Theorem 1.1. From (3-10) and Lemma 3.1, we get

$$\begin{aligned} 0 &\geq \sum_{i=1}^2 F^{ii} \varphi_{ii} \\ &\geq \frac{1}{2} \frac{g'}{g} f \lambda_1 - 2\beta f \frac{r^2}{\eta\lambda_1} - 4\beta|f_{12}| \frac{r^4}{\eta\lambda_1} - 32\beta \frac{f_1^2 r^4}{\eta\lambda_1^3} - 8\beta \frac{|f_1 f_2| r^4}{\eta\lambda_1^3} - 24\beta \frac{|f_1| r^3}{\eta\lambda_1} \\ &\quad - \left(\frac{6\beta|f_1|^2 r^4}{\eta\lambda_1} + \frac{12\beta f r^2}{\eta\lambda_1} \right) - \left(\frac{8\beta f r^4 |f_2|^2}{\eta\lambda_1^3} + \frac{(48\beta)^2 f r^6}{\eta^2 \lambda_1} + \frac{32\beta^2 |f_2|^2 r^8}{\eta^2 \lambda_1 f} \right) \\ &\quad - 4\beta f \frac{g'}{g} \frac{r^4}{\eta} \frac{|u_2|}{r} - \frac{g'}{g} |\nabla u| |\nabla f| - \frac{|f_{11}|}{\lambda_1} - 2 \frac{f_1^2}{f \lambda_1} - \frac{8\beta f r^6}{\eta^2 \lambda_1} \end{aligned}$$

$$\begin{aligned} &\geq \frac{c_0 m}{2r^2} \lambda_1 - \frac{8r^2 \cdot M}{\eta \lambda_1} - \frac{32r^2 \cdot r^2 |f_{12}|}{\eta \lambda_1} - \frac{128r^2 \cdot r^2 f_1^2}{\eta \lambda_1^3} - \frac{32r^2 \cdot r^2 |f_1 f_2|}{\eta \lambda_1^3} - \frac{96r^2 \cdot r |f_1|}{\eta \lambda_1} \\ &\quad - \frac{24r^2 \cdot r^2 |f_1|^2}{\eta \lambda_1} - \frac{48r^2 \cdot M}{\eta \lambda_1} - \frac{32r^2 \cdot M \cdot r^2 |f_2|^2}{\eta \lambda_1^3} - \frac{192^2 \cdot M \cdot r^6}{\eta^2 \lambda_1} - \frac{512r^6 \cdot r^2 |f_2|^2 \cdot 1/m}{\eta^2 \lambda_1} \\ &\quad - \frac{16r^2 \cdot c_0 m |u_2|}{\eta} - \frac{c_0}{r^2} \cdot r |\nabla f| \cdot \frac{|\nabla u|}{r} - \frac{|f_{11}|}{\lambda_1} - 2 \frac{f_1^2}{m \lambda_1} - \frac{32Mr^6}{\eta^2 \lambda_1}. \end{aligned}$$

So we get

$$\eta \lambda_1 \leq C \left(1 + \frac{|\nabla u|}{r} \right) r^4, \tag{3-24}$$

where C is a positive constant depending only on $c_0, m, M, r|\nabla f|$ and $r^2|\nabla^2 f|$. So we easily get

$$u_{\tau(0)\tau(0)}(0) \leq \frac{1}{r^{4\beta}} \phi(0) \leq \frac{1}{r^{4\beta}} \phi(x_0) \leq C \left(1 + \frac{\sup |Du|}{r} \right) e^{c_0 \sup |Du|^2/r^2} \leq C e^{(c_0+2) \sup |Du|^2/r^2}$$

and

$$|u_{\xi\xi}(0)| \leq u_{\tau(0)\tau(0)}(0) \leq C e^{(c_0+2) \sup |Du|^2/r^2} \quad \text{for all } \xi \in \mathbb{S}^1. \tag{3-25}$$

This completes the proof of Theorem 1.1 under the condition $\eta \lambda_1 \geq \Theta$. Hence Theorem 1.1 holds.

Remark 3.2. The eigenvector field τ is important. In fact, it is well-defined when the largest eigenvalue is distinct from the others, and τ depends only on the adjoint matrix. For the Monge–Ampère equation in dimension $n \geq 3$, we do not know whether the largest eigenvalue is distinct from the others, so our method is not suitable for this case.

Acknowledgements

The authors would like to express sincere gratitude to Prof. Xi-Nan Ma for the constant encouragement in this subject. Also, the authors are very grateful to the referee for the careful reading and valuable suggestions.

References

- [Andrews 2007] B. Andrews, “Pinching estimates and motion of hypersurfaces by curvature functions”, *J. Reine Angew. Math.* **608** (2007), 17–33. MR 2339467 Zbl 1129.53044
- [Chen 2013] C. Chen, “Optimal concavity of some Hessian operators and the prescribed σ_2 curvature measure problem”, *Sci. China Math.* **56**:3 (2013), 639–651. MR 3017945 Zbl 1276.35081
- [Chou and Wang 2001] K.-S. Chou and X.-J. Wang, “A variational theory of the Hessian equation”, *Comm. Pure Appl. Math.* **54**:9 (2001), 1029–1064. MR 1835381 Zbl 1035.35037
- [Guan et al. 2015] P. Guan, C. Ren, and Z. Wang, “Global C^2 -estimates for convex solutions of curvature equations”, *Comm. Pure Appl. Math.* **68**:8 (2015), 1287–1325. MR 3366747 Zbl 1327.53009
- [Heinz 1959] E. Heinz, “On elliptic Monge–Ampère equations and Weyl’s embedding problem”, *J. Analyse Math.* **7** (1959), 1–52. MR 0111943 Zbl 0152.30901
- [Li et al. 2016] M. Li, C. Ren, and Z. Wang, “An interior estimate for convex solutions and a rigidity theorem”, *J. Funct. Anal.* **270**:7 (2016), 2691–2714. MR 3464054 Zbl 06548316

- [Pogorelov 1978] A. V. Pogorelov, *The Minkowski multidimensional problem*, Wiley, New York, 1978. MR 0478079 Zbl 0387.53023
- [Trudinger 1997] N. S. Trudinger, “Weak solutions of Hessian equations”, *Comm. Partial Differential Equations* **22**:7-8 (1997), 1251–1261. MR 1466315 Zbl 0883.35035
- [Urbas 1990] J. I. E. Urbas, “On the existence of nonclassical solutions for two classes of fully nonlinear elliptic equations”, *Indiana Univ. Math. J.* **39**:2 (1990), 355–382. MR 1089043 Zbl 0724.35028
- [Warren and Yuan 2009] M. Warren and Y. Yuan, “Hessian estimates for the sigma-2 equation in dimension 3”, *Comm. Pure Appl. Math.* **62**:3 (2009), 305–321. MR 2487850 Zbl 1173.35388

Received 29 Sep 2015. Revised 28 Dec 2015. Accepted 12 May 2016.

CHUANQIANG CHEN: cqchen@mail.ustc.edu.cn

Department of Applied Mathematics, Zhejiang University of Technology, No. 288, Liuhe Road, Xihu District, Hangzhou, 310023, Zhejiang Province, China

FEI HAN: klhanqingfei@126.com

School of Mathematics Sciences, Xinjiang Normal University, No. 102, Xinyi Road, Shayibake District, Urumqi, 830054, Xinjiang Uygur Autonomous Region, China

QIANZHONG OU: ouqzh@163.com

School of Science, Hezhou University, No. 18, Xihuan Road, Hezhou, 542899, Guangxi Province, China

Analysis & PDE

msp.org/apde

EDITORS

EDITOR-IN-CHIEF

Patrick Gérard
patrick.gerard@math.u-psud.fr
Université Paris Sud XI
Orsay, France

BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France nicolas.burq@math.u-psud.fr	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Yuval Peres	University of California, Berkeley, USA peres@stat.berkeley.edu
Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Wilhelm Schlag	University of Chicago, USA schlag@math.uchicago.edu
Vaughan Jones	U.C. Berkeley & Vanderbilt University vaughan.f.jones@vanderbilt.edu	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbm@math.mit.edu	András Vasy	Stanford University, USA andras@math.stanford.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpms.cam.ac.uk	Maciej Zworski	University of California, Berkeley, USA zworski@math.berkeley.edu

PRODUCTION

production@msp.org
Silvio Levy, Scientific Editor


See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2016 is US \$235/year for the electronic version, and \$430/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 9 No. 6 2016

A complete study of the lack of compactness and existence results of a fractional Nirenberg equation via a flatness hypothesis, I	1285
WAEEL ABDELHEDI, HICHEM CHTIQUI and HICHEM HAJAJEJ	
On positive solutions of the (p, A) -Laplacian with potential in Morrey space	1317
YEHUDA PINCHOVER and GEORGIOS PSARADAKIS	
Geometric optics expansions for hyperbolic corner problems, I: Self-interaction phenomenon	1359
ANTOINE BENOIT	
The interior C^2 estimate for the Monge–Ampère equation in dimension $n = 2$	1419
CHUANQIANG CHEN, FEI HAN and QIANZHONG OU	
Bounded solutions to the Allen–Cahn equation with level sets of any compact topology	1433
ALBERTO ENCISO and DANIEL PERALTA-SALAS	
Hölder estimates and large time behavior for a nonlocal doubly nonlinear evolution	1447
RYAN HYND and ERIK LINDGREN	
Boundary $C^{1,\alpha}$ regularity of potential functions in optimal transportation with quadratic cost	1483
ELINA ANDRIYANOVA and SHIBING CHEN	
Commutators with fractional differentiation and new characterizations of BMO-Sobolev spaces	1497
YANPING CHEN, YONG DING and GUIXIANG HONG	



2157-5045(2016)9:6;1-6