BOUNDARY $C^{1,\alpha}$ REGULARITY OF POTENTIAL FUNCTIONS IN OPTIMAL TRANSPORTATION WITH QUADRATIC COST

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We provide a different proof for the global $C^{1,\alpha}$ regularity of potential functions in the optimal transport problem, which was originally proved by Caffarelli. Our method applies to a more general class of domains.

1. Introduction

We study the global $C^{1,\alpha}$ regularity of potential functions in optimal transportation with quadratic cost. Let $\Omega$ and $\Omega^*$ be the source and target domains associated with densities $1/C < f < C$ and $1/C < g < C$, respectively, where $C$ is a positive constant. The optimal transport problem with quadratic cost is about finding a map $T : \Omega \to \Omega^*$ among all measure-preserving maps minimizing the transportation cost

$$\int_\Omega |x - T x|^2 \, dx.$$ 

Here the term “measure-preserving” means that $\int_{T^{-1}(B)} f = \int_B g$ for any Borel set $B \subset \Omega^*$. Brenier [1991] proved that one can find a convex function $u$ such that $T(x) = Du(x)$ for a.e. $x \in \Omega$.

Indeed, the convex function $u$ satisfies $\int_{(\partial u)^{-1}B} f = \int_B g$ for any Borel set $B \subset \Omega$, where $\partial u$ is the standard subgradient map of the convex function $u$. We call $u$ a Brenier solution of the optimal transport problem if it satisfies the property above. When the target domain $\Omega^*$ is convex, Caffarelli proved that $\partial u(\Omega) = \Omega^*$ and that $u$ is an Alexandrov solution, namely $u$ satisfies $(1/C) |A \cap \Omega| \leq |\partial u(A)| \leq C |A \cap \Omega|$ for any Borel set $A \subset \Omega$. Moreover, if we extend $u$ to $\mathbb{R}^n$ via

$$\tilde{u} := \sup \{ L \mid L \text{ is linear, } L|_{\Omega} \leq u, L(z) = u(z) \text{ for some } z \in \Omega \},$$

then $\tilde{u}$ is a globally Lipschitz convex solution of

$$C^{-1} \chi_\Omega \leq \det \tilde{u}_{ij} \leq C \chi_\Omega.$$ 

We will still use $u$ to denote this extended function. Caffarelli [1992b] proved interior $C^{1,\alpha}$ regularity by using his techniques for studying the standard Monge–Ampère-type equation; see [Caffarelli 1990a; 1990b; 1991].

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Then, Caffarelli [1992a] proved the boundary $C^{1,\alpha}$ regularity result under the condition that both $\Omega$ and $\Omega^*$ are convex. Below we will briefly discuss the main ideas involved in his proof. First, Caffarelli established a fundamental property of convex functions, namely the existence of sections centred at a given point (see the statement of Lemma 2.5). Then, he proved that such sections are decaying geometrically, namely there exists a constant $\delta$ such that

$$S_{\delta h}(y) \subset \frac{1}{2} S_h(x)$$

for any $y \in \frac{1}{2} S_h(x)$. (1-1)

Here $S_h(x)$ denotes the section of $u$ centred at $x$ with height $h$. From (1-1) we obtain the quantitative strict convexity estimate

$$u(z) \geq u(x) + Du(x) \cdot (z - x) + C|z - x|^\beta$$

for any $x, z \in \Omega$, (1-2)

for some $\beta > 1$. From (1-2), it is easy to check that $u^*$, the standard Legendre transform of $u$, is $C^{1,\alpha}$ on $\Omega^*$. Recall the well-known fact that $u^*$ is indeed the potential function of the optimal transport problem from $\Omega^*$ to $\Omega$. Therefore, by switching the role of $u$ and $u^*$ one can show the global $C^{1,\alpha}$ regularity of $u$.

The convexity of domains is crucial in Caffarelli’s approach. Indeed, the convexity of $\Omega^*$ ensures that $u^*$ is an Alexandrov solution, while the convexity of $\Omega^*$ ensures that the sections of $u^*$, centred at some point in $\Omega^*$, have some doubling property. Here we provide a different proof of the global $C^{1,\alpha}$ result. Instead of deducing the $C^{1,\alpha}$ regularity of $u$ from the strict convexity of $u^*$, we prove the $C^{1,\alpha}$ regularity of $u$ directly. Moreover, our method works for a slightly more general class of domains, namely we allow the source to be a domain obtained by removing finitely many disjoint convex subsets from a convex domain.

We would like to mention that in recent years the regularity of optimal transport maps has attracted much interest and there are many important works related to it; to cite a few, see [Figalli and Loeper 2009; Liu 2009; Trudinger and Wang 2009b; 2009a; Figalli and Rifford 2009; Loeper 2011; Loeper and Villani 2010; Liu et al. 2010; Kim and McCann 2010; Figalli et al. 2010; 2011; 2012; 2013a; 2013b].

The rest of the paper is organized as follows. In Section 2 we introduce some notations and preliminaries, and state the main results. Section 3 is devoted to the proof of global $C^1$ regularity. In the last section we complete the proof of the main results.

2. Preliminaries and main result

The main result of this paper is the following theorem:

**Theorem 2.1.** Let $\Omega$ and $\Omega^*$ be two bounded domains in $\mathbb{R}^n$, $n \geq 2$, and $f$ and $g$ be densities of two positive probability measures defined in $\Omega$ and $\Omega^*$, respectively, satisfying $C^{-1} \leq f, g \leq C$ for a positive constant $C$. Assume that $\Omega^*$ is convex and $\Omega$ is Lipschitz.

(i) If, for any given $x \in \Omega$, there exists a small ball $B_{r_x}(x)$ such that, for any convex set $\omega \subset B_{r_x}(x)$ centred in $\Omega$, we have $\int_{\omega} f \leq C \int_{\omega/2} f$ for some constant $C$ independent of $\omega$, then the potential function $u$ is $C^1(\Omega)$. (Here $f$ is defined to be 0 outside $\Omega$.)

(ii) If $\Omega$ is a domain obtained by removing finitely many disjoint convex subsets from a convex set, then the potential function $u$ is $C^{1,\alpha}(\Omega)$ for some $\alpha \in (0, 1)$. 
Remark 2.2. (a) It is easy to see that in Theorem 2.1(i) we allow \( \Omega \) to be any polytope (not necessarily convex). We also note that the \( C^1 \) regularity always holds in dimension two without any condition on \( \Omega \). This is a classical result of Alexandrov; see also [Figalli and Loeper 2009].

(b) One may want to prove higher regularity when the densities are smooth; however, in view of the following simple example we see that this is impossible. Let the dimension be \( n = 2 \). Let \( \Omega := B_2 - B_1 \), with uniform probability density, and let \( \Omega^* := B_{\sqrt{3}} \), with uniform probability density. Then by symmetry it is easy to compute that the optimal transport map is \( T(x) = \sqrt{|x|^2 - 1} x/|x| \), which is only \( C^{1/2} \) on \( \partial B_1 \subset \partial \Omega \).

In the following we will use \( S_h(x_0) \) to denote a section of \( u \) with height \( h \), namely

\[
S_h(x_0) := \{ x \mid u < p \cdot (x - x_0) + h \},
\]

where \( p \) is chosen so that \( x_0 \) is the centre of mass of \( S_h(x_0) \). We say a point \( x_0 \in \overline{\Omega} \) is localized (with respect to \( u \)) if, for any sequences \( h_k \to 0 \) and \( x_k \to x_0 \) satisfying \( x_0 \in S_{h_k}(x_k) \), we have that \( S_{h_k}(x_k) \) shrinks to the point \( x_0 \in \overline{\Omega} \).

Now we record a fundamental property of convex sets.

**Lemma 2.3** (John’s lemma). Let \( U \subset \mathbb{R}^n \) be a bounded, convex domain with its centre of mass at the origin. There exists an ellipsoid \( E \), also centred at the origin, such that

\[
E \subset U \subset n^{3/2} E.
\]

The original John’s lemma does not require that the ellipsoid is centered at the origin, and the constant \( n^{3/2} \) can be replaced by \( n \). We refer the reader to [Liu and Wang 2015] for a simple proof of the existence and uniqueness of such an ellipsoid.

By John’s lemma we can show the following property of convex functions:

**Lemma 2.4.** Let \( u : \mathbb{R}^n \to \mathbb{R} \) be a convex function. Let \( L \) be a supporting function of \( u \). Then any extreme point of \( \{ u = L \} \) is localized.

**Proof.** Suppose to the contrary that there exists an extreme point \( x_0 \) of \( \{ u = L \} \) which is not localized. Then there exist sequences \( x_k \to x_0 \) and \( h_k \to 0 \) such that \( x_0 \in S_{h_k}(x_k) \), and that \( S_{h_k}(x_k) \) contains a segment of length greater than or equal to some positive constant \( \delta \). Since \( S_{h_k}(x_k) \) is convex and centred at \( x_k \), by John’s lemma there exists a unit vector \( \xi_k \) such that \( I_k \), the segment connecting \( x_k - \delta/(2n^{3/2}) \xi_k \) and \( x_k + \delta/(2n^{3/2}) \xi_k \), is contained in \( S_{h_k}(x_k) \). Denote by \( L_k \) the defining function of \( S_{h_k}(x_k) \), namely \( S_{h_k}(x_k) = \{ u \leq L_k \} \). Then it is easy to see that \( DL_k \) is bounded; hence, by passing to a subsequence, \( L_k \to L_\infty \) for some linear function \( L_\infty \). Also by passing to a subsequence we may assume \( \xi_k \to \xi_\infty \) for some unit vector \( \xi_\infty \). Then \( u \) is linear on \( I_\infty \), which is the segment connecting \( x_0 - \delta/(2n^{3/2}) \xi_\infty \) and \( x_0 + \delta/(2n^{3/2}) \xi_\infty \). Hence \( I_\infty \subset \{ u = L \} \), which contradicts the assumption that \( x_0 \) is an extreme point of \( \{ u = L \} \). \( \square \)

The following property of sections of convex functions was proved by Caffarelli [1992a]. Here we provide a different proof by using a well-known fact that if a continuous map from a ball to itself fixes the boundary then it must be surjective. We learned this method from Wang; see [Sheng et al. 2004, Section 4].
Lemma 2.5. Let \( u : \mathbb{R}^n \to [0, \infty) \) be a convex function. Assume that:

1. \( u(0) = 0, \ u \geq 0. \)
2. \( u \) is finite in a neighbourhood of 0.
3. The graph of \( u \) contains no complete lines.

Then for \( h > 0 \) there exists a slope \( p \) such that the centre of mass of the section

\[
S_{h, p} := \{ x \mid u \leq x \cdot p + h \}
\]

is defined and equal to 0.

Proof. Let

\[
\begin{cases}
  u_k(x) = u(x) & \text{in } B_k, \\
  u_k = \infty & \text{in } \mathbb{R}^n - B_k.
\end{cases}
\]  

We only need to show the existence of sections \( S_k := \{ x \mid u_k \leq x \cdot p_k + h \} \) centred at 0 with bounded \( p_k \). Then \( S_{h, p} = \lim_{k \to \infty} S_k \) is the desired section in the lemma.

Take a large ball \( B_r \). For any \( p \in B_r \), let \( z_p \) be the centre of mass of the section \( S_p := \{ x \mid u_k(x) \leq x \cdot p + h \} \). Then we obtain a mapping \( M_1 : p \to z_p \) from \( B_r \) to \( \mathbb{R}^n \). If \( p \in \partial B_r \), it is easy to see that \( p \cdot z_p > 0 \) provided \( r \) is sufficiently large.

If there is no \( p \in B_r \) such that \( z_p = 0 \), then we can define a mapping \( M_2 : z_p \to t_p z_p \), where \( t_p > 0 \) is a constant such that \( t_p z_p \in \partial B_r \). We then obtain a continuous mapping \( M = M_2 \circ M_1 \) from \( B_r \) to \( \partial B_r \) with the property that

\[
p \cdot M(p) > 0 \quad \text{on } \partial B_r. \tag{2-2}
\]

To get a contradiction, we extend the mapping \( M \) to \( B_{2r} \) as follows. For any point \( p \in \partial B_{2r} \), let \( p_1 = p, \ p_0 = \frac{1}{2} p \in \partial B_r \) and \( p_t = (1 - t)p_0 + p_1 \). We extend the mapping \( M \) to \( B_{2r} \) by letting \( M(p_t) = (1 - t)M(p_0) + t p_1 \). Then, by (2-2), \( M(p) \neq 0 \) on \( B_{2r} \), and \( M \) is the identity mapping on \( \partial B_{2r} \). This is a contradiction.

Hence, for each \( k > 0 \), there exists a \( p_k \in \mathbb{R}^n \) such that \( S_k := \{ x \mid u_k \leq x \cdot p_k + h \} \) is centred at 0. Moreover, \( |p_k| \leq C \) for some constant independent of \( k \). Indeed, we can argue as follows: By rotating the coordinates we may assume \( p_k = (a, 0, \ldots, 0) \) with \( a > 0 \). Let \( \alpha^+ = \sup \{ x_1 \mid (x_1, 0, \ldots, 0) \in S_k \} \) and \( \alpha^- = -\inf \{ x_1 \mid (x_1, 0, \ldots, 0) \in S_k \} \). Then \( \alpha^+/\alpha^- \to \infty \) as \( a \to \infty \). Since \( S_k \) is centred at 0, \( a \) cannot be too large. \( \square \)

The following Alexandrov-type estimates were proved by Caffarelli [1996]:

Lemma 2.6. Let \( u \) be a convex solution of

\[
det D^2 u = d\mu
\]

in the convex domain \( S \) with \( u = 0 \) on \( \partial S \). Assume \( S \) is normalized, namely \( B_1 \subset S \subset n^{3/2} B_1 \). Assume \( d\mu(S) \leq \theta d\mu \left( \frac{1}{2} S \right) \) for some constant \( \theta \), where \( \frac{1}{2} S \) is a dilation of \( S \) with respect to the origin.

(a) \( \frac{1}{C} \inf_S u^n \leq d\mu(S) \leq C \inf_S u^n \), where \( C \) is a constant depending only on \( \theta \).

(b) \( |u(x)|^n \leq C d\mu(S) d(x, \partial S) \).
3. Global $C^1$ regularity

In this section, we prove Theorem 2.1(i).

**Lemma 3.1.** Suppose $u$ is a globally Lipschitz convex function. Assume that $u$ is $C^1$ at all of the extreme points of a convex set $K = \{u = L\}$, where $L$ is a linear function satisfying $u \geq L$ and $u(y) = L(y)$ for some $y \in \mathbb{R}^n$. Then $u$ is $C^1$ on $K$.

**Proof.** By subtracting $L$ we may assume $K = \{u = 0\}$. If $K$ is a bounded convex set, then for any $x \in K$ we have

$$ x = \sum_{i=1}^{k} \lambda_i x_i, $$

where $x_i, i = 1, \ldots, k$, are extreme points of $K$, $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i = 1$. Since $u$ is $C^1$ at $x_i, i = 1, \ldots, k$, we have $0 \leq u(z) = o(z - x_i)$, $i = 1, \ldots, k$. Now, by convexity we have

$$ 0 \leq u(z) = u \left( \sum_{i=1}^{k} \lambda_i (z - x + x_i) \right) \leq \sum_{i=1}^{k} \lambda_i u(z - x + x_i) = \sum_{i=1}^{k} \lambda_i o(z - x) = o(z - x). $$

Hence, $u$ is $C^1$ at $x$.

If $K$ is unbounded, it is well-known that $K = \text{covext}[K] + \text{rc}[K]$, where covext$[K]$ is the convex hull of the extreme points of $K$, and rc$[K] := \lim_{t \to 0} t K$ is the recession cone of $K$. Hence we need only to show that $u$ is $C^1$ at points represented by $x = x_0 + q$, where $x_0$ is an extreme point of $K$ and $q \in \text{rc}[K]$. For any $M \geq 0$, by using the facts that $u$ is Lipschitz and $x_1 := x_0 + M q \in K$ we have that $u(z - x + x_1) \leq C |z - x|$. By convexity we have

$$ u(z) = u \left( \frac{M-1}{M} (z - x + x_0) + \frac{1}{M} (z - x + x_1) \right) \leq \frac{M-1}{M} o(|z - x|) + \frac{C}{M} |z - x|. $$

By letting $M \to \infty$ we have $0 \leq u(z) \leq o(|z - x|)$. Hence $u$ is $C^1$ at $x$. \hfill $\Box$

Since $u$ is convex, for any unit vector $\gamma$ the lateral derivatives

$$ \partial_+^\gamma u(x) =: \lim_{t \searrow 0} t^{-1} (u(x + t \gamma) - u(x)) \quad \text{and} \quad \partial_-^\gamma u(x) =: \lim_{t \searrow 0} t^{-1} (u(x) - u(x - t \gamma)) $$

exist. To prove that $u \in C^1(\overline{\Omega})$, it suffices to prove that

$$ \partial_+^\gamma u(x_0) = \partial_-^\gamma u(x_0) \quad (3.1) $$

at any point $x_0 \in \partial \Omega$ for any unit vector $\gamma$. By convexity, it suffices to prove this for $\xi = \xi_k$ for all $k = 1, 2, \ldots, n$, where $\xi_k, k = 1, \ldots, n$, are any $n$ linearly independent unit vectors.

**Proof of Theorem 2.1(i).** By Lemmas 3.1 and 2.4 we only need to show that $u$ is $C^1$ at localized points. Assume to the contrary that $u$ is not $C^1$ at $x_0 \in \partial \Omega$. Let us assume that $x_0 = 0$, $u(0) = 0$, $u \geq 0$ and $\partial_+^1 u(0) > \partial_-^1 u(0) = 0$. Since $\partial \Omega$ is Lipschitz, we may also assume that $-t e_1 \in \Omega$ for $t \in (0, 1)$, where $e_1$ is the first coordinate direction.
Now we consider a section $S_h(x')$, where $x' = (-a', 0, \ldots, 0)$ for some small constant $0 < a' < \frac{1}{2}r_0$, where $r_0 := r_{x_0}$ is the radius in the condition of Theorem 2.1(i). Note that by John’s lemma there exists an ellipsoid $E$ with centre $x'$ such that $E \subset S_h(x') \subset n^{3/2}E$. Since $u$ is Lipschitz and $\partial^{+}u(0) > 0$, we have that $C^{-1} \leq u(\varepsilon e_1) \leq C \varepsilon$ for any small positive $\varepsilon$, where $C$ is a positive constant. Since $\partial^{+}u(0) = 0$, we have $u(-Ma'e_1) = o(a')$, where $M = 2n^{3/2}$. Hence, we can choose small $\varepsilon$ and $a'$ so that the following properties hold:

1. $o(a') = u(-Ma'e_1) \leq C^{-1} \varepsilon \ll a'$,
2. $\varepsilon e_1$ is on the boundary of some section $S_h(x')$, and
3. $S_h(x') \subset B_{r_0}(0)$.

The existence of such a section $S_h(x')$ in (2) follows from the property that a centred section, say $S_h(x)$, various continuously with respect to the height $h$; see [Caffarelli and McCann 2010, Lemma A.8], and (3) follows from the assumption that $x_0 = 0$ is localized.

Let $L$ be the defining linear function of $S_h(x')$; by (1) it is easy to see that $L$ is increasing in the $e_1$ direction (see Figure 1); hence,

$$ (L - u)(0) \geq (L - u)(x') = h. \quad (3-2) $$

Since $\int_{S_h(x')} f \leq C \int_{\frac{1}{2}S_h(x')} f$, we have that

$$ (L - u)(0) \leq C \left(\frac{\varepsilon}{a'}\right)^{\frac{1}{2}} h, \quad (3-3) $$

contradicting (3-2), since $a' \gg \varepsilon$. Here we have followed the argument of [Caffarelli 1996]. Indeed, let $A$ be an affine transform normalizing $S_h(x')$; then $v := (u - L)(A^{-1}x)/h$ satisfies $\det D^2v = f(A^{-1}x)/h^n$ in $A(S_h(x'))$ and $v = 0$ on $\partial S_h(x')$. Hence, by applying Lemma 2.6 to $v$ and translating back to $u$ we get (3-3).

Hence, $u$ must be $C^1$ at any localized point $x_0$. Therefore $u \in C^1(\mathbb{R}^n)$. \qed
Remark 3.2. The proof of Theorem 2.1(i) shares some similarities with the proof of $C^1$ regularity for the obstacle problem in [Savin 2005] (see Proposition 2.8 in that paper).

4. Global $C^{1,\alpha}$ regularity

In this section, we prove Theorem 2.1(ii). First we point out that to prove $u \in C^{1,\alpha}(\bar{\Omega})$, it suffices to prove that there exist constants $C > 0$, $\alpha \in (0, 1)$ and $r > 0$ such that, for any point $x_0 \in \bar{\Omega}$,

$$u(x) - \ell_{x_0}(x) \leq C|x - x_0|^{1+\alpha}$$  \hfill (4-1)

for every $x \in B_r(x_0) \cap \Omega$. From (4-1) one can prove that $u \in C^{1,\alpha}(\bar{\Omega})$, using the convexity of $u$. In the following we will show that a relaxed version of (4-1) is enough to show $u \in C^{1,\alpha}(\bar{\Omega})$, and it has the advantage of avoiding some annoying limiting picture.

By the assumption of Theorem 2.1(ii) we write $\Omega = U - \sum_{i=1}^{k} C_i$, where $U$ is an open convex set, and $C_i$, $i = 1, \ldots, k$, are closed disjoint convex subsets of $U$; see Figure 2. Given any $x \in \bar{\Omega}$, we introduce the function

$$\rho_x(t) := \sup \left\{ u(z) - u(x) - Du(x) \cdot (z-x) \mid |z-x| = t, x + s \frac{z-x}{|z-x|} \in \bar{\Omega} \text{ for any } s \in [0, r_0] \right\},$$  \hfill (4-2)

where $r_0$ is a fixed small positive constant depending on $\Omega$, and its smallness will be clear in the proof of Lemma 4.1. Indeed, we need to take $r_0$ small enough that $B_{r_0}(x) \cap \partial U$ can be represented as the graph of some Lipschitz function for any $x \in \partial U$ with the Lipschitz constant independent of $x$, and that

$$r_0 \ll \min \{ \text{dist}(\partial U, \partial C_i), \text{dist}(\partial C_j, \partial C_l) \mid i = 1, \ldots, k, 1 \leq j \neq l \leq k \}.$$  

Lemma 4.1. Suppose that there exist $r > 0$ and $\delta \in (0, 1)$ such that for any $x \in \bar{\Omega}$ we have

$$\rho_x \left( \frac{1}{2} t \right) \leq \frac{1}{2} (1 - \delta) \rho_x(t)$$  \hfill (4-3)

whenever $t \leq r$. Then $u \in C^{1,\alpha}(\bar{\Omega})$ for some $\alpha \in (0, 1)$.

Figure 2. Domain $\Omega$. 

Proof. For $t = r/2^k$, we have
\[
\rho_x(t) \leq \frac{(1 - \delta)k}{2^k} \rho_x(r) \leq \frac{t}{r} (1 - \delta) \log(rt)/\log^2 \rho_x(r) = C t^{1+\alpha},
\]
where $C$ depends on $r$, $\delta$ and $\rho_x(r)$, and $\alpha = -\log(1 - \delta)/\log 2$.

Suppose $x, y \in \overline{\Omega}$ and $|x - y| \ll r \ll r_0$. We need to consider two cases:

(a) $x, y$ are close to $\partial U$.

(b) $x, y$ are close to $\partial C_i$ for some $1 \leq i \leq k$.

We will deal with case (a) first; case (b) follows from a similar argument. Without loss of generality we may assume that $B_{3r_1} \subset U$ for some small fixed $r_1$, that $r_0 \ll r_1$, and that $\text{dist}(\partial B_{3r_1}, \partial U) \gg r_1$. Denote by $\mathcal{C}_{x, r_1}$ the convex hull of $x$ and $B_{r_1}$. By convexity, $\mathcal{C}_{x, 3r_1} \subset U$. Then we prove the following claim:

Claim 1. For any $z \in B_{r/2}(x) \cap \mathcal{C}_{x, 2r_1}$, we have $|Du(x) - Du(z)| \leq C|z - x|^\alpha$.

Proof of Claim 1. Observe that $\text{dist}(z, \partial \mathcal{C}_{x, 3r_1}) \geq (1/C)|z - x|$ for some large constant $C$. Hence, $B_{1/C}|z-x|(z) \subset B_{3r_1} \cap \mathcal{C}_{x, 3r_1}$. Now, for any $\tilde{z} \in \partial B_{1/C}|z-x|(z)$, by (4-4) we have that
\[
u(\tilde{z}) \leq u(x) + Du(x) \cdot (\tilde{z} - x) + C|\tilde{z} - x|^{1+\alpha}.
\]
By convexity we also have
\[
u(\tilde{z}) \geq u(z) + Du(z) \cdot (\tilde{z} - z)
\]
and
\[
u(z) \geq u(x) + Du(x) \cdot (z - x).
\]
By (4-5), (4-6) and (4-7) we have
\[
(Du(z) - Du(x)) \cdot (\tilde{z} - z) \leq C|\tilde{z} - x|^{1+\alpha}.
\]
Note that $|\tilde{z} - z| \approx |\tilde{z} - x| \approx |z - x|$ provided $\tilde{z} \in \partial B_{1/C}|z-x|(z)$ and $C$ is sufficiently large. Since (4-8) holds for any $\tilde{z} \in \partial B_{1/C}|z-x|(z)$, it follows that $|Du(x) - Du(z)| \leq C|z - x|^\alpha$.\]

Now suppose $|x - y| \ll r$. If either $y \in \mathcal{C}_{x, 2r_1}$ or $x \in \mathcal{C}_{y, 2r_1}$ holds, then by Claim 1 we have $|Du(x) - Du(y)| \leq C|x - y|^\alpha$. Otherwise one may find a point $z \in \mathcal{C}_{x, r_1} \cap \mathcal{C}_{y, r_1}$ such that $|z - x| \approx |z - y| \approx |z - y|$. Then by applying the estimate in Claim 1 we have
\[
|Du(x) - Du(y)| \leq |Du(x) - Du(z)| + |Du(y) - Du(z)| \leq C(|x - z|^\alpha + |y - z|^\alpha) \leq C|x - y|^\alpha.
\]
We can prove case (b) by a similar argument. Indeed, $\partial C_i \cap B_r(x)$ can be represented as the graph of some Lipschitz function for any fixed $x \in \partial C_1$ provided $r \ll r_0$. Then, by the assumption that the $C_i$ are disjoint, it is easy to find a small ball $B_{3r_1} \subset \Omega$ such that $\mathcal{C}_{x, 3r_1} \subset \Omega$ for any $z \in B_{r}(x) \cap \overline{\Omega}$. Then, by a similar argument to the proof of case (a), we can show that $|Du(x) - Du(y)| \leq C|x - y|^\alpha$ provided $|x - y| \ll r$.

The following lemma shows that the centred sections are well-localized provided the heights are sufficiently small.
Lemma 4.2. There exists a height \( h_0 > 0 \) such that, for any \( x \in \Omega \), the section \( S_h(x) \) intersects at most one of \( \partial U, \partial C_i, i = 1, \ldots, m \), provided \( h < h_0 \).

Proof. Suppose to the contrary there exist sequences \( x_k \in \Omega \) and \( h_k \to 0 \), such that \( S_{h_k}(x_k) \) intersects at least two of \( \partial U, \partial C_i, i = 1, \ldots, m \). Passing to a subsequence we may assume \( x_k \to y \in \Omega \). Since \( u \) is strictly convex in the interior of \( \Omega \), we have either \( y \in \partial U \) or \( y \in \partial C_i \) for some \( i \). Denote by \( L_k \) the defining function of \( S_{h_k}(x_k) \), namely \( S_{h_k}(x_k) = \{ u \leq L_k \} \). Then, passing to a subsequence we may assume \( L_k \to L \) for some affine function \( L \), and \( S_{h_k}(x_k) \to S \subset \{ u \leq L \} \). It follows from the properties of \( S_{h_k}(x_k) \) that:

(i) \( S \) is centred at \( y \).

(ii) \( S \) intersects at least two of \( \partial U, \partial C_i, i = 1, \ldots, m \).

(iii) \( L(y) = \lim_{k \to \infty} L_k(x_k) = \lim_{k \to \infty} u(x_k) + h_k = u(y) \).

By (i) and (iii) we have that \( S \subset \{ u = L \} \). Then by (ii) we see that \( S \) passes through the interior of \( \Omega \), which contradicts the fact that \( u \) is strictly convex in the interior of \( \Omega \). \( \square \)

Proof of Theorem 2.1(ii). Step 1. The main observation in this step is that if (4-3) is violated for small \( \delta \), then \( u \) is close to a linear function on a segment connecting \( x \) and some point \( z \in \Omega \). Hence, if (4-3) is violated for arbitrary \( r, \delta \), then one can find a sequence of points \( x_k \) such that \( u \) is more and more linear around \( x_k \) in some direction as \( k \to \infty \). The “almost linearity” will be clear if we perform blow-up and an affine transform on \( u \) properly restricted to some carefully chosen section around \( x_k \), and a line segment will appear on the graph of the limiting function. The detailed argument goes as follows.

To prove \( \rho_x(t) \leq C t^{1+\alpha} \) for any \( x \in \Omega \) and any \( t \leq r \), by Lemma 4.1 we assume to the contrary that there exist sequences \( t_k \leq 1/k, \delta_k = 1/k \) and \( x_k \in \Omega \) such that

\[
\rho_{x_k}(\frac{1}{2}t_k) \geq \frac{1}{2}(1 - 1/k)\rho_{x_k}(t_k). \quad (4-9)
\]

Suppose the supremum in (4-2) (when \( x = x_k \) and \( t = \frac{1}{2}t_k \)) is attained at \( \frac{1}{2}(x_k + z_k) \in \Omega \); by the definition of \( \rho_x \) we see that \( \frac{z_k}{x_k} \subset \Omega \), where \( \frac{z_k}{x_k} \) denotes the segment connecting \( z_k \) and \( x_k \). By passing to a subsequence, we may assume \( x_k \to x_\infty \in \partial \Omega \).

Choosing sections. For each \( k \), let \( S_{h_k}(x_k) \) be a section of \( u \) with centre \( x_k \), where \( h_k \) is chosen so that \( z_k \in \partial S_{h_k}(x_k) \). Similar to the proof of Theorem 2.1(i), the existence of such a section follows from the property that a centred section, say \( S_h(x) \), varies continuously with respect to the height \( h \); see [Caffarelli and McCann 2010, Lemma A.8] for a proof. It is easy to see that \( h_k \to 0 \).

Normalization. Let \( L_k \) be the defining function of \( S_{h_k}(x_k) \). We normalize the section \( S_{h_k}(x_k) \) by a linear transformation \( T_k \), and let \( S_k = T_k(S_{h_k}(x_k)) \). Note that \( T_k(x_k) = 0 \) and \( B_1 \subset S_k \subset n^{3/2}B_1 \). Also we let \( u_k = (u - L_k)(T_k^{-1}x)/h_k \). Then \( u_k \) solves

\[
\begin{align*}
\det D^2 u_k &= f_k &\text{in } S_k, \\
u_k &= 0 &\text{on } \partial S_k,
\end{align*}
\]

(4-10)

where \( f_k = h_k^{-n}(\det T_k)^{-1} f(T_k^{-1}x)/g(Du(T_k^{-1}x)) \). After a rotation of coordinates, we may assume \( T_k(z_k) \) is on the \( x_1 \)-axis.
Linearity estimate. Let
\[ v_k(x) := u(x) - Du(x_k) \cdot (x - x_k) - u(x_k); \]
from (4-9) we have that
\[ v_k\left(\frac{1}{2}(x_k + z_k)\right) \geq \frac{1}{2}(1 - 1/k)v_k(z_k). \]
Let
\[ \tilde{L}_k(x) := L_k(x) - Du(x_k) \cdot (x - x_k) - u(x_k). \]

Then we have that \( S_{h_k}(x_k) = \{v_k \leq \tilde{L}_k\}. \) Since \( S_{h_k}(x_k) \) is centred at \( x_k, \) \( z_k \in \partial S_{h_k}(x_k), \) \( v_k \geq 0 \) and \( \tilde{L}_k(x_k) = h_k, \) by John’s lemma we have that \( 0 \leq \tilde{L}_k(z_k) \leq 2n^{3/2}h_k. \) Now,
\[
(v_k - \tilde{L}_k)\left(\frac{1}{2}(x_k + z_k)\right) - \frac{1}{2}\left(1 - \frac{1}{k}\right)((v_k - \tilde{L}_k)(x_k) + (v_k - \tilde{L}_k)(z_k)) \geq -\frac{1}{2k}(\tilde{L}_k(x_k) + \tilde{L}_k(z_k)) \geq -\frac{3n^{3/2}}{2k}h_k.
\]
Since \( v_k - \tilde{L}_k = u - L_k, \) from the above estimate and the definition of \( u_k \) we have
\[
u_k\left(\frac{1}{2}T_k z_k\right) \geq \frac{1}{2}\left(1 - \frac{1}{k}\right)(u_k(0) + u_k(T_k z_k)) - \frac{3n^{3/2}}{2k}.
\]  \hspace{1cm} (4-11)

Limiting problem. Now, by convexity we may take limits \( S_k \to S_\infty \) and \( u_k \to u_\infty. \) Let \( f_\infty \) be the weak limit of \( f_k. \) Then \( u_\infty \) satisfies \( \det D^2u_\infty = f_\infty \) in the Alexandrov sense. Let \( z_\infty := \lim_{k \to \infty} T_k(z_k). \) By (4-11) we have
\[
u_\infty = L \text{ on the segment connecting 0 and } z_\infty.
\]  \hspace{1cm} (4-12)
where \( L \) is a supporting function of \( u_\infty \) at 0.

Step 2. In this step, we need to consider two situations:

(a) \( x_\infty \in \partial C_i \) for some \( 1 \leq i \leq k. \)
(b) \( x_\infty \in \partial U. \)

In each case, a contradiction is obtained at some carefully chosen extreme point (denoted by \( y) \) of \( \{u_\infty = L\}. \) Heuristically, we can choose a section of \( u_\infty \) (denoted by \( S \)) around \( y \) such that \( y \) is much closer to \( \partial S \) in one direction than in the opposite direction. Hence, on one hand the Alexandrov-type estimate Lemma 2.6(a) shows that \( h, \) the height of the section \( S, \) should not be too small. On the other hand, Lemma 2.6(b) shows that \( h \) is very small, which is a contradiction.

We deal with case (a) first.

Proof in case (a). Note that since \( x_\infty \in \partial C_i \) for some \( 1 \leq i \leq k \) and \( h_k \to 0 \) as \( k \to \infty, \) by Lemma 4.2 we have that the support of \( f_k \) can be represented by \( S_k - A_k \) when \( k \) is large, where \( A_k \) is an open convex subset of \( S_k. \) Let the convex set \( A_\infty \) be the limit of the \( A_k. \) Then \( S_\infty - A_\infty \) is the support of \( f. \) Since the centre of mass of \( S_\infty \) is 0 and \( 0 \in S_\infty - A_\infty, \) we have that the volume of \( S_\infty - A_\infty \) is positive. Hence, it is easy to see that there exists a constant \( C \) such that
\[
C^{-1} \chi_{S_\infty - A_\infty} \leq f_\infty \leq C \chi_{S_\infty - A_\infty}.
\]

Since \( \overline{S_k - A_k} \subset \Omega, \) we have \( 0 \subset S_\infty \cap A_\infty = \emptyset. \)

Subcase 1: \( \{u_\infty = L\} \) contains an interior point of \( S_\infty - A_\infty. \)

Subcase 2: \( \{u_\infty = L\} \cap S_\infty \subset A_\infty. \)
Figure 3. Two related sections.

For subcase 1, take $x_0 \in (S_\infty - \tilde{A}_\infty) \cap \{u = L\}$. Take $\delta$ sufficiently small that $B_\delta (x_0) \subset S_\infty - \tilde{A}_\infty$.

Choosing an extreme point. Let $y \in \{u = L\}$ be the point such that:

1. $u_\infty (y) = \inf_{\{u = L\}} u_\infty$.
2. $y$ is an extreme point of the convex set $\{u_\infty = L\} \cap \{u_\infty = u(y)\}$.

It is easy to see that $y$ is an extreme point of $\{u_\infty = L\}$.

Cutting a suitable section. By rotating the coordinates we may assume that $\{u_\infty = L\} \subset \{x_1 \leq b\}$ for some constant $b > 0$, and that $\{u_\infty = L\} \cap \{x_1 = b\} = \{y\}$. Then we consider the section $S = \{u_\infty < L + \varepsilon (x_1 - b + a)\}$ (see Figure 3), where we fix $a$ sufficiently small and then take $\varepsilon \ll a$, so that $S \subseteq S_\infty$ and $a \gg a := \max \{x_1 \mid (x_1, 0, \ldots, 0) \in S\} - b$.

Using Alexandrov estimates to obtain a contradiction. On one hand, by the Alexandrov estimate we have

$$|S|^2 > C \frac{a}{d} \varepsilon^n.$$  \hfill (4-13)

On the other hand, we consider another section $\tilde{S} = \{u_\infty < L + C \varepsilon\}$. Since $u$ is Lipschitz, it is easy to see that $S \subseteq \tilde{S}$ provided $C$ (independent of $\varepsilon$) is sufficiently large. By convexity we have $|B_\delta (x_0) \cap \tilde{S}| \geq C |	ilde{S}|$. 
for some constant $C$. We claim

$$|S|^2 \leq C\varepsilon^n,$$

(4-14)

where the constant $C$ is independent of $d$. The claim follows from the following argument. Let $v = u_\infty - L - C\varepsilon$. Let $G \equiv \tilde{S} \cap B_\delta(x_0)$. By John’s lemma, there exists an affine transformation $A$ with $\text{det} A = 1$ such that

$$B_{\tilde{r}} \subset A(G) \subset n^{3/2}B_{\tilde{r}}$$

for some $\tilde{r}$. Now $\tilde{v} = v(A^{-1}x)$ satisfies $\text{det} D^2 \tilde{v} = f_\infty(A^{-1}x) \geq C^{-1}$ in $A(G)$ and $|v| \leq C\varepsilon$ in $A(G)$. Then we have

$$C^{-1}|G| \leq \int_{G/2} f_\infty = |\partial \tilde{v}(A(\frac{1}{2}G))| \leq C\varepsilon^n.$$

(4-15)

Equation (4-14) follows from (4-15) and the fact that $|\tilde{S}| \approx |G| \approx \tilde{r}^n$. Since $d \ll a$, it is easy to see that (4-14) contradicts (4-13).

For subcase 2, we need to choose the extreme point more carefully.

**Choosing an extreme point.** Let $\tilde{K} \subset \mathbb{R}^n$ be a supporting plane of the convex set $A_\infty$ at 0. If $A_\infty$ is not $C^1$ at 0 we choose $\tilde{K}$ to be the one containing $z_0$. Let $y'$ be the point where $u_\infty$ attains its minimum on $D := \{u = L\} \cap \tilde{K} \cap S_\infty$. It is easy to check that $D$ is a convex set, and the set $D \cap \{x \mid u(x) = u(y')\}$ is also convex. Let $y$ be an extreme point of $D \cap \{x \mid u(x) = u(y')\}$. We claim that $y$ is an extreme point of $\{u = L\}$. Indeed, suppose not; then there exist $y_1, y_2 \in \{u = L\} \cap S_\infty \subset \tilde{K} \subset \mathbb{R}^n$ such that $y = \frac{1}{2}(y_1 + y_2)$. Since $\tilde{K}$ is a supporting plane of $A_\infty$ and $y \in \tilde{K}$, we have that $y_1, y_2 \in D$. However, since $u(y) = \min\{u(x) \mid x \in D\}$, we have $y_1, y_2 \in D \cap \{x \mid u(x) = u(y')\}$, which contradicts the choice of $y$ as an extreme point of $D \cap \{x \mid u(x) = u(y')\}$.

**Cutting a suitable section.** By subtracting $L$ and translating the coordinates we may assume that $y = 0$, that $u_\infty \geq 0$, that $u_\infty(t e_1) = 0$ for $t \in (0, 1)$, and that $u_\infty(t e_1) > 0$ for $t < 0$. Let $0 < \varepsilon \ll a$ be small positive numbers. Let $S_h(\varepsilon e_1)$ be a section of $u_\infty$ with centre $ae_1$, where $h$ is chosen so that $-\varepsilon e_1 \in \partial S_h(\varepsilon e_1)$. Since $y$ is an extreme point of $\{u = L\}$, we have that $S_h(\varepsilon e_1) \subseteq S_\infty$ provided $h$ is sufficiently small. Note that $h \rightarrow 0$ as $\varepsilon \rightarrow 0$.

**Using Alexandrov estimates to obtain a contradiction.** Since $A_\infty$ is convex, it is easy to see that

$$\int_{S_h(ae_1)} f_\infty \leq C \int_{\frac{1}{2}S_h(ae_1)} f_\infty$$

for some constant $C$. Let $L_1$ be the defining function of the section $S_h(\varepsilon e_1)$, which is obviously decreasing in the $e_1$ direction. Hence $(L_1 - u_\infty)(0) \geq h$. Then by Lemma 2.6 we also have

$$(L_1 - u_\infty)(0) \leq C \left(\frac{\varepsilon}{a}\right)^{1/n} h,$$

which contradicts the previous estimate. \qed

**Proof in case (b).** The proof in case (b) follows from a similar argument to the proof of [Caffarelli 1992a, Lemma 4]; we sketch the argument here. Note that $f_k$ is now supported in a convex domain $D_k \subset \tilde{S}_k$. 
Let \( D_\infty \) := \( \lim_{k \to \infty} D_k \). We have \( z_\infty \in D_\infty \). Let \( L \) be the supporting function of \( u_\infty \) at 0 such that \( 0 \in S \subset \{ u_\infty = L \} \). Similarly to the proof of subcase 1 of case (a), let \( y \in \{ u_\infty = L \} \) be the point such that:

1. \( u_\infty(y) = \inf_{\{ u_\infty = L \}} u_\infty \).
2. \( y \) is an extreme point of the convex set \( \{ u_\infty = L \} \cap \{ u_\infty = u(y) \} \).

It is easy to see that \( y \) is an extreme point of \( \{ u_\infty = L \} \). Observe that \( y \in D_\infty \), since otherwise \( u_k \) has positive Monge–Ampère measure outside \( D_k \) for large \( k \). Let \( z = (1-\sigma) y + \sigma z_\infty \) for some small positive \( \sigma \); we may also find a section satisfying \( S_h(z) := \{ u_\infty < L \} \subseteq S_\infty \) and \( y + \varepsilon (y - z_\infty)/|y - z_\infty| \in \partial S_h(z) \) for small \( \varepsilon \ll \sigma \). Since \( y \in D_\infty \), there exists a sequence \( y_k \in D_k \) such that \( y_k \to y \) as \( k \to \infty \). Let

\[
\tilde{z}_k := (1-\sigma) y_k + \sigma T(z_k);
\]

it is easy to see that \( \tilde{z}_k \to z \) as \( k \to \infty \). Recall that \( z_\infty := \lim_{k \to \infty} T(z_k) \) with \( T(z_k) \in D_k \). Let \( \tilde{S}_k := \{ u_k \leq L_k \} \) be a section of \( u_k \) centred at \( \tilde{z}_k \) with height \( h \). Then, passing to a subsequence, \( \tilde{S}_k \to S_h(z) \) in Hausdorff distance. In particular, \( \tilde{S}_k \cap S_k \) provided \( k \) is sufficiently large. Then, by Lemma 2.6, we have that

\[
C h \leq (L_k - u_k)(y_k) \leq \left( \frac{\varepsilon}{\sigma} \right)^{1/n} h
\]

for large \( k \), which is a contradiction because \( \varepsilon \ll \sigma \).

Theorem 2.1(ii) follows from the above discussions.

\[
\square
\]

### References


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