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THE FINAL-STATE PROBLEM FOR THE CUBIC-QUINTIC NLS WITH NONVANISHING BOUNDARY CONDITIONS

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We construct solutions with prescribed scattering state to the cubic-quintic NLS

$$(i\partial_t + \Delta)\psi = \alpha_1\psi - \alpha_3|\psi|^2\psi + \alpha_5|\psi|^4\psi$$

in three spatial dimensions in the class of solutions with $|\psi(x)| \to c > 0$ as $|x| \to \infty$. This models disturbances in an infinite expanse of (quantum) fluid in its quiescent state — the limiting modulus c corresponds to a local minimum in the energy density.

Our arguments build on work of Gustafson, Nakanishi, and Tsai on the (defocusing) Gross–Pitaevskii equation. The presence of an energy-critical nonlinearity and changes in the geometry of the energy functional add several new complexities. One new ingredient in our argument is a demonstration that solutions of such (perturbed) energy-critical equations exhibit continuous dependence on the initial data with respect to the *weak* topology on H^1_x.

1. Introduction

We study the cubic-quintic nonlinear Schrödinger equation (NLS) with nonvanishing boundary conditions in three space dimensions:

$$\begin{cases} (i\partial_t + \Delta)\psi = \alpha_1\psi - \alpha_3|\psi|^2\psi + \alpha_5|\psi|^4\psi, & (t,x) \in \mathbb{R}\times\mathbb{R}^3, \\ \psi(0) = \psi_0. \end{cases} \tag{1-1}$$

We consider parameters $\alpha_1, \alpha_3, \alpha_5 > 0$ so that $\alpha_3^2 - 4\alpha_1\alpha_5 > 0$, which guarantees that the polynomial $\alpha_1 - \alpha_3 x + \alpha_5 x^2$ has two distinct positive roots $r_0^2 > r_1^2 > 0$. The boundary condition is given by

$$\lim_{|x|\to\infty} |\psi(t,x)| = r_0. \tag{1-2}$$

The choice of the larger root guarantees the energetic stability of the constant solution; it constitutes a local minimum of the energy functional (1-7).

Equation (1-1) appears in a great variety of physical problems. It is a model in superfluidity [Ginzburg and Pitaevskiĭ 1958; Ginzburg and Sobyanin 1976], descriptions of bosons [Barashenkov et al. 1989] and of defectons [Pushkarov and Kojnov 1978], the theory of ferromagnetic and molecular chains [Pushkarov and Primatarova 1984; 1986], and in nuclear hydrodynamics [Kartavenko 1984]. The popularity of this model can be explained by its simplicity combined with the fact that it captures an important phenomenology: the constituents of most fluids experience an attractive interaction at low densities and a repulsion at high

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densities. The recent paper [Killip et al. 2014] focuses on the analogous problem with data decaying at infinity, which constitutes a model for the dynamics of a finite body of fluid; the model (1-1) describes the behavior of a localized disturbance in an infinite expanse of fluid that is otherwise quiescent.

By rescaling both space-time and the values of ψ, it suffices to consider the case $r_0^2 = 1$ and $\alpha_5 = 1$. This leaves one free parameter

$$\gamma := 1 - r_1^2 \in (0,1), \tag{1-3}$$

in terms of which equation (1-1) becomes

$$\begin{cases} (i\partial_t + \Delta)\psi = (|\psi|^2 - 1)(|\psi|^2 - 1 + \gamma)\psi, \\ \psi(0) = \psi_0, \end{cases} \tag{1-4}$$

with the boundary condition

$$\lim_{|x|\to\infty} \psi(t,x) = 1. \tag{1-5}$$

As discussed in [Gérard 2006] (albeit in the context of the Gross–Pitaevskii equation), finite energy functions obeying (1-2) have a unique limiting phase as $|x| \to \infty$, which we can normalize to be zero, yielding (1-5). Furthermore, the dynamics of (1-1) preserve the value of this phase, so that the boundary condition is independent of time, as well. This breaks the gauge invariance of (1-1) and prohibits using a phase factor to remove the linear term in this equation. The presence of the linear term leads to weaker dispersion at low frequencies, which presents a key challenge in understanding the long-time behavior of solutions.

We are interested in perturbations of the constant solution $\psi \equiv 1$, and thus it is natural to introduce the function $u = u_1 + iu_2$ defined via $\psi = 1 + u$. Using (1-4), we arrive at the following equation for u:

$$\begin{cases} (i\partial_t + \Delta)u - 2\gamma u_1 = N(u), \\ u(0) = u_0, \end{cases} \tag{1-6}$$

where $N(u) = \sum_{j=2}^5 N_j(u)$, with

$$\begin{aligned} N_2(u) &= (3\gamma + 4)u_1^2 + \gamma u_2^2 + 2i\gamma u_1 u_2, \\ N_3(u) &= (\gamma + 8)u_1^3 + (\gamma + 4)u_1 u_2^2 + i[(\gamma + 4)u_1^2 u_2 + \gamma u_2^3], \\ N_4(u) &= 5u_1^4 + 6u_1^2 u_2^2 + u_2^4 + i[4u_1^3 u_2 + 4u_1 u_2^3], \\ N_5(u) &= |u|^4 u = u_1^5 + 2u_1^3 u_2^2 + u_2^4 u_1 + i[u_1^4 u_2 + 2u_1^2 u_2^3 + u_2^5]. \end{aligned}$$

The Hamiltonian for (1-4) is given by

$$E(\psi) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla\psi|^2\,dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} (|\psi|^2 - 1)^2\,dx + \frac{1}{6}\int_{\mathbb{R}^3} (|\psi|^2 - 1)^3\,dx. \tag{1-7}$$

Introducing the notation

$$q(u) := |\psi|^2 - 1 = 2u_1 + |u|^2,$$

we may write

$$2\gamma u_1 + N(u) = [\gamma q(u) + q(u)^2](1 + u)$$

and

$$E(1+u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{\gamma}{4}\int_{\mathbb{R}^3} q(u)^2\,dx + \frac{1}{6}\int_{\mathbb{R}^3} q(u)^3\,dx. \tag{1-8}$$

In the sequel we will write $E(u)$ for $E(1+u)$; when there is no risk of confusion we will simply write $q(u) = q$. Note that q represents density fluctuations relative to the constant background. The quantity $\int q(t,x)\,dx$, which represents the total surplus/deficit of matter relative to the constant background, is conserved in time; in this work we do not rely on this conservation law.

Well-posedness in the energy space. We define the energy space for (1-6) to be

$$\mathcal{E} := \bigl\{u \in \dot{H}^1_x(\mathbb{R}^3) : q(u) \in L^2_x(\mathbb{R}^3)\bigr\}, \tag{1-9}$$

with associated metric

$$[d_{\mathcal{E}}(u,v)]^2 := \|u-v\|^2_{\dot{H}^1_x} + \|q(u)-q(v)\|^2_{L^2_x},$$

and we let $\|u\|_{\mathcal{E}} := d_{\mathcal{E}}(u,0)$ denote the energy-norm.

To justify our choice of energy space, we first note that functions with finite energy-norm have finite energy. Indeed, using Sobolev embedding and the fact that $(L^3_x + L^6_x) \cap L^2_x \subset L^3_x$, it is not hard to see that if $u \in \mathcal{E}$ then $q(u) \in L^3_x$, and so $|E(u)| < \infty$. In fact,

$$|E(u)| \lesssim \|u\|^2_{\mathcal{E}} + \|u\|^3_{\mathcal{E}}.$$

On the other hand, in Lemma 3.1 we will show that for $\gamma \in \bigl[\frac{2}{3}, 1\bigr)$, functions with finite energy have finite energy-norm. When $\gamma \in \bigl(0, \frac{2}{3}\bigr)$, the energy is not coercive unless we impose an additional smallness assumption (see Lemma 3.2).

When the energy is not coercive, there is no unique candidate for the name "energy space". The authors of [Killip et al. 2012] worked with the following notion of energy space:

$$\mathcal{E}_{\text{KOPV}} := \bigl\{u \in \dot{H}^1_x(\mathbb{R}^3) \cap L^4_x(\mathbb{R}^3) : \operatorname{Re} u \in L^2_x(\mathbb{R}^3)\bigr\}.$$

Note that $\mathcal{E}_{\text{KOPV}} \subset \mathcal{E}$. In the same work, they also proved that (1-6) is globally well-posed for data $u_0 \in \mathcal{E}_{\text{KOPV}}$; in particular, solutions are unconditionally unique in $C(\mathbb{R}; \mathcal{E}_{\text{KOPV}})$.

In Section 3, we prove global well-posedness and unconditional uniqueness for (1-6) in the energy space $\mathcal{E}$ (see Theorem 3.3). As in [Killip et al. 2012; Tao et al. 2007; Zhang 2006], our approach is to regard the equation as a perturbation of the defocusing energy-critical NLS

$$(i\partial_t + \Delta)u = |u|^4 u, \tag{1-10}$$

which was proven to be globally well-posed, first in the radial case and then for general data in the celebrated papers [Bourgain 1999; Colliander et al. 2008]. Proving well-posedness for a Schrödinger equation in three dimensions that contains a quintic nonlinearity requires control over the $\dot{H}^1_x$-norm of the solution. As the energy (1-8) is not necessarily coercive for $\gamma \in \bigl(0, \frac{2}{3}\bigr)$, conservation of the Hamiltonian does not supply the requisite a priori bound. To resolve this issue we will require that both the energy and the kinetic energy of the data are small when $\gamma \in \bigl(0, \frac{2}{3}\bigr)$.

Statement of the main result. The stability of the equilibrium solution $\psi \equiv 1$ to (1-4) is equivalent to the small-data problem for (1-6). In this direction, there are two natural problems to consider, namely, the initial-value and the final-state problems for (1-6). For the former, the question is whether small and localized initial data lead to solutions that are global and decay as $|t|\to\infty$. For the latter, the question is whether one can construct a solution that scatters to a prescribed asymptotic state. In this paper we prove two results related to the final-state problem. We will address the initial-value problem in a forthcoming work.

To fit (1-6) into the standard framework of dispersive equations it is convenient to diagonalize the equation. Setting

$$U=|\nabla|\langle\nabla\rangle^{-1} \quad\text{and}\quad H=|\nabla|\langle\nabla\rangle, \quad \text{with } \langle\nabla\rangle:=\sqrt{2\gamma-\Delta} \text{ and } |\nabla|=(-\Delta)^{\frac12},$$

we arrive at the following equation for $v:=Vu:=u_1+iUu_2$:

$$\begin{cases}(i\partial_t-H)v=N_v(u):=U\,\mathrm{Re}[N(u)]+i\,\mathrm{Im}[N(u)],\\ v(0)=Vu_0.\end{cases} \tag{1-11}$$

Note that $u_{\mathrm{lin}}(t):=V^{-1}e^{-itH}Vu_+$ solves the equation

$$(i\partial_t+\Delta)u_{\mathrm{lin}}-2\gamma\,\mathrm{Re}\,u_{\mathrm{lin}}=0 \quad\text{with }\; u_{\mathrm{lin}}(0)=u_+; \tag{1-12}$$

this is the linearization of (1-6) about $u=0$.

Our main result in this paper is the following theorem:

Theorem 1.1. *Suppose* $\gamma\in\big[\frac23,1\big)$. *For any* $u_+\in H^1_{\mathrm{real}}+i\dot H^1_{\mathrm{real}}$, *there exists a global solution* $u\in C(\mathbb{R};\mathcal{E})$ *to* (1-6) *such that*

$$\lim_{t\to\infty}\|u(t)-u_{\mathrm{lin}}(t)\|_{\dot H^1_x}=0, \tag{1-13}$$

where $u_{\mathrm{lin}}(t):=V^{-1}e^{-itH}Vu_+$. *Moreover, we have modified asymptotics in the energy space, in the sense that this same solution u obeys*

$$\lim_{t\to\infty}d_{\mathcal{E}}\big(u(t),\,u_{\mathrm{lin}}(t)-\gamma\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big)=0. \tag{1-14}$$

In the case $\gamma\in\big(0,\frac23\big)$, *both conclusions still hold if additionally* $\|u_+\|_{H^1_{\mathrm{real}}+i\dot H^1_{\mathrm{real}}}$ *is sufficiently small.*

Remark 1.2. The hypotheses on u_+ are not sufficient to guarantee that $u_{\mathrm{lin}}(t)\in\mathcal{E}$ at any time t; correspondingly, one cannot hope to say that u is close to u_{lin} in the energy space. Nonetheless, (1-13) does show that the modification in (1-14) only plays a role at very low frequencies. Indeed, simple computations show that the modification can be omitted, for example, when u_+ is a Schwartz function.

We do not guarantee uniqueness of the solution u in Theorem 1.1. Later, we will show uniqueness within a restricted class of solutions u for suitable scattering states u_+; see Theorem 1.4 and Corollary 1.7 below.

Discussion of relevant past results. To give proper context to our work, we need to discuss prior work of Gustafson, Nakanishi, and Tsai [Gustafson et al. 2006; 2007; 2009] on the Gross–Pitaevskii equation

$$\begin{cases}(i\partial_t+\Delta)\psi=(|\psi|^2-1)\psi,\\ \psi(0)=\psi_0,\\ \lim\limits_{|x|\to\infty}\psi(t,x)=1.\end{cases} \tag{1-15}$$

Note that unlike in (1-4), the cubic nonlinearity here is defocusing. Writing $\psi = 1 + u$, this equation preserves the energy

$$E_{\mathrm{GP}}(u) = \frac{1}{2}\int_{\mathbb{R}^3} |\nabla u|^2\,dx + \frac{1}{4}\int_{\mathbb{R}^3} q(u)^2\,dx. \tag{1-16}$$

In contrast to (1-8), this energy density is lacking the sign-indefinite $q(u)^3$-term. Correspondingly, the energy is coercive and the nonlinearity is energy-subcritical.

The final-state problem for the Gross–Pitaevskii equation was addressed by Gustafson et al. [2007; 2009] in two and three dimensions and in [Gustafson et al. 2006] in higher dimensions. They also considered the initial-value problem in dimensions $d \geq 3$ in [Gustafson et al. 2006; 2009].

The jumping-off point for Theorem 1.1 is an analogous result appearing in [Gustafson et al. 2009] for the Gross–Pitaevskii equation, which in turn builds on earlier work of Nakanishi [2001] on the (gauge-invariant) NLS. As our strategy is modeled closely on his, it is worth discussing in detail the following result:

Theorem 1.3 [Nakanishi 2001]. *Given $u_+ \in H^1_x(\mathbb{R}^3)$ and $\frac{2}{3} < p < \frac{4}{3}$, there is a solution to*

$$(i\partial_t + \Delta)u = |u|^p u \tag{1-17}$$

that obeys $e^{-it\Delta}u(t) \to u_+$ in $H^1_x(\mathbb{R}^3)$.

Sketch of proof. Nakanishi first defines solutions u^T to (1-17) with $u^T(T) = e^{iT\Delta}u_+$. As the problem is L^2_x-subcritical, these solutions are easily seen to be global with uniformly bounded H^1_x-norm (even in the focusing case).

By writing (1-17) in Duhamel form and exploiting the dispersive estimate (2-2), it is not difficult to show that for each $\phi \in C_c^\infty(\mathbb{R}^3)$, the collection of functions

$$\bigl\{t \mapsto \langle \phi, e^{-it\Delta}u^T(t)\rangle : T \in \mathbb{R}\bigr\} \tag{1-18}$$

forms an equicontinuous family on a compactification $\mathbb{R} \cup \{\pm\infty\}$ of the real line. In particular, each such function has limiting values as $t \to \pm\infty$. Applying Arzelà–Ascoli and the Cantor diagonal argument (H^1_x is separable), one can find a sequence $T_n \to \infty$ and a function $u^\infty \in L^\infty_t H^1_x$ so that

$$e^{-it\Delta}u^{T_n}(t) \rightharpoonup e^{-it\Delta}u^\infty(t) \quad \text{weakly in } H^1_x \text{ for each } t \in \mathbb{R}.$$

This construction guarantees that u^∞ has two further properties. First, the function $t \mapsto e^{-it\Delta}u^\infty(t)$ is weakly H^1_x-continuous on $\mathbb{R} \cup \{\pm\infty\}$, that is, when H^1_x is endowed with the weak topology. Secondly, for any $\phi \in C_c^\infty(\mathbb{R}^3)$,

$$\langle \phi, e^{-it\Delta}u^{T_n}(t)\rangle \to \langle \phi, e^{-it\Delta}u^\infty(t)\rangle \quad \text{as } n \to \infty, \text{ uniformly in } t \in \mathbb{R}.$$

Using these properties it is elementary to verify that $e^{-it\Delta}u^\infty(t) \rightharpoonup u_+$ as $t \to \infty$. This leaves two obligations: firstly, one must show that u^∞ is actually a solution to (1-17) and secondly, one must upgrade weak convergence to norm convergence.

Due to the H^1_x-subcriticality of the nonlinearity, the Rellich–Kondrashov theorem allows one to show that u^∞ is a weak solution to (1-17). For this problem, weak solutions with values in H^1_x are necessarily strong solutions and so we may conclude that u^∞ is a solution to (1-17).

Lastly, to upgrade weak convergence to strong convergence, one exploits conservation of mass and energy and the Radon–Riesz theorem. For example, one may argue as follows: The quantity

$$F(u) := \int_{\mathbb{R}^3} |\nabla u|^2 + \frac{2}{p+2}|u|^{p+2} + |u|^2 \, dx \tag{1-19}$$

is conserved under the flow (1-17). Exploiting this, dispersion of the linear flow, and weak lower-semicontinuity of norms, we deduce that

$$\varlimsup_{t\to\infty} \|e^{-it\Delta}u^\infty(t)\|^2_{H^1_x} \le F(u^\infty) \le \varliminf_{n\to\infty} F(u^{T_n}(0)) = \varliminf_{n\to\infty} F(u^{T_n}(T_n)) = \|u_+\|^2_{H^1_x}.$$

Given that $e^{-it\Delta}u^\infty(t) \rightharpoonup u_+$, we deduce that $e^{-it\Delta}u^\infty(t) \to u_+$ in H^1_x. □

In order to adapt this beautiful argument to the Gross–Pitaevskii setting, the authors of [Gustafson et al. 2009] had to overcome two significant obstacles: (i) One needs to make the (conserved) energy (1-16) associated to (1-15) play the role of F in the argument above. It is far from obvious that this has the requisite convexity. (ii) The simple arguments used to prove equicontinuity of the family (1-18) no longer work. This failure stems from lower-power terms in the nonlinearity combined with the fact that energy conservation gives poor a priori spatial decay of solutions; while it guarantees $q(u) \in L^2_x$, it only yields $u_1 \in L^3_x$ and no better than $u_2 \in L^6_x$. This is not sufficient decay to allow direct access to any of the integrable-in-time dispersive estimates obeyed by the propagator.

The key to obtaining equicontinuity of the analogue of the family (1-18) in the Gross–Pitaevskii setting is to exploit certain nonresonant structures in the nonlinearity that allow one to integrate by parts in time. In implementing this approach, one sees that it is necessary to exhibit such nonresonance in both the quadratic and cubic terms of the nonlinearity. Such a brute force attack is rather messy. The burden can be significantly reduced by using test functions whose Fourier support excludes the origin. We will demonstrate this (primarily expository) improvement over the arguments from [Gustafson et al. 2009] in the proof of Proposition 6.2 below. One particular virtue of this approach is that it makes clear from the start that the argument is inherently completely immune to the poor dispersion manifested by the propagator (2-4) at low frequencies.

In [Gustafson et al. 2009], the authors exploit the quadratic nonresonant structure in a more elegant way through the use of a normal form transformation

$$z = [u_1 + (2-\Delta)^{-1}|u|^2] + i\sqrt{-\Delta/(2-\Delta)}\, u_2. \tag{1-20}$$

In this work they also observe (and then utilize) the further nonresonant structure at the cubic level (akin to (6-30)). There is some flexibility in the choice of normal form that witnesses the requisite nonresonance; however, the particular one employed in [Gustafson et al. 2009] has the dramatic additional benefit of overcoming obstacle (i) described above. The necessary convexity of the energy functional becomes clearer when written in their new variables: with u and z related by (1-20),

$$E_{\mathrm{GP}}(u) = \tfrac{1}{2}\big\| \sqrt{2-\Delta}\, z \big\|_{L_x^2}^2 + \tfrac{1}{4}\big\| \sqrt{-\Delta/(2-\Delta)}\, |u|^2 \big\|_{L_x^2}^2. \tag{1-21}$$

The virtue of this identity is best understood in the context of (6-8). Because the right-most term in (1-21) is nonnegative, combining (1-21) with (6-8) yields

$$\overline{\lim_{t\to\infty}}\, \tfrac{1}{2}\big\| \sqrt{2-\Delta}\, z(t) \big\|_{L_x^2}^2 = \tfrac{1}{2}\big\| \sqrt{2-\Delta}\, z_+ \big\|_{L_x^2}^2,$$

where $z(t)$ and z_+ represent a particular solution and its putative scattering state, both in terms of the normal form variable. This is just what is needed as input for the Radon–Riesz theorem.

Discussion of the main result. In order to prove Theorem 1.1 we will need to capitalize on all of the ideas introduced in [Gustafson et al. 2009] to prove the analogous result for the Gross–Pitaevskii equation. In particular, we will exploit a normal form transformation modeled closely on (1-20), namely,

$$z = M(u) := \big[u_1 + \gamma(2\gamma-\Delta)^{-1}|u|^2\big] + i\sqrt{-\Delta/(2\gamma-\Delta)}\, u_2. \tag{1-22}$$

However, several new difficulties arise above and beyond those overcome in [Gustafson et al. 2009].
(i) The first group of new difficulties is associated to the presence of energy-critical terms in the nonlinearity.
(ii) The second group of difficulties stems from the shape of the energy functional.

(i) We begin by discussing the difficulties that arise from the energy-critical terms. As discussed earlier in the introduction, we already need to give consideration to the energy-critical terms in the proof of Theorem 3.3, which states that (1-6) admits global solutions for initial data in the energy space $\mathcal{E}$. A more significant challenge involves establishing a form of well-posedness with respect to the weak $\dot{H}_x^1$ topology (see Theorem 4.1), as we will now explain.

In the argument of Nakanishi described above, it was used that weak limits (in the H_x^1 topology pointwise in time) of strong solutions to (1-17) are themselves strong solutions. In the subcritical case, one sees relatively easily that such limits are weak solutions (via Rellich–Kondrashov) and can then exploit earlier work (see [Cazenave 2003, Chapters 3–4]) showing that weak solutions with values in H_x^1 are strong solutions. In particular, solutions converging weakly to zero (in H_x^1) by concentrating will actually converge to zero in the space-time norms used to construct such solutions. In a similar way, we see that increasingly concentrated parts of a solution (which will drop out under taking a weak limit) do not affect parts of the solution living at unit scale.

These arguments break down in the presence of the quintic nonlinearity, which is energy-critical. In particular, initial data that converge weakly to zero in H_x^1 by concentrating at a point lead to solutions that do not go to zero in the space-time norms needed for well-posedness. Correspondingly, highly concentrated parts of a solution may have large norm and so, naively at least, have a nontrivial effect on the remainder of the solution. Thus, it is not clear that weak limits of solutions should even be continuous in time! The key to escaping this nightmare is to show that two parts of a solution have little effect on one another if they live at widely separated scales. We will achieve this by employing concentration compactness techniques.

Before tackling the full equation (1-6), one should first ensure that one can prove that weak limits of solutions are themselves solutions in the case of the energy-critical NLS equation (1-10). Questions of this type appear to have been studied before only in the case of the energy-critical wave equation [Bahouri and Gérard 1999]. As there, we proceed by harnessing the full power of the associated concentration compactness ideas. Specifically, one starts with a nonlinear profile decomposition and then further exploits some of the decoupling ideas used in its proof. In this paper, we will implement this strategy in the setting of (1-6); this is ample guidance for anyone seeking to reconstruct the argument for (1-10).

As a precursor to the nonlinear profile decomposition needed to prove that weak limits of solutions to (1-6) are themselves solutions, we must first develop a linear profile decomposition adapted to (1-6); see Proposition 4.3. Despite the fact that the linear equation underlying (1-6) differs from that underlying (1-10), we are able to adapt the profile decomposition for the linear Schrödinger equation to our setting, rather than proceeding ab initio. To develop the *nonlinear* profile decomposition, we need to construct solutions to (1-6) associated to each linear profile. For profiles living at unit scale, existence of these solutions (and all requisite bounds) follows from Theorem 3.3. Profiles whose characteristic length scale diverges can be approximated by linear solutions on bounded time intervals and so require no special attention. However, highly concentrated profiles require independent treatment; this is the content of Proposition 4.5. There are two subtle points here: (a) Such profiles are merely $\dot{H}^1_x$ and so do not have finite energy. (b) The characteristic time scale associated to such profiles is very short; thus, understanding such solutions even on a bounded interval essentially requires an understanding of their infinite time behavior.

The nonlinear profile decomposition posits that the nonlinear evolution of the initial data can be approximated by the sum of the nonlinear evolutions of its constituent profiles. This is verified by demonstrating decoupling of the profiles inside the nonlinearity (see Lemma 4.7) and exploiting a suitable stability theory for the equation (see Proposition 3.5). The latter requires certain a priori bounds, which are shown to hold in Lemma 4.6. Once it is known that the nonlinear profile decomposition faithfully represents the true solution, it is relatively elementary to complete the proof of well-posedness in the weak topology, that is, the proof of Theorem 4.1.

This completes our discussion of the new difficulties (relative to [Gustafson et al. 2009]) associated to the presence of energy-critical nonlinear terms.

(ii) We turn to the second main group of difficulties mentioned above, which stem from the shape of the energy functional. First, the lack of coercivity when $\gamma \in \big(0, \frac{2}{3}\big)$ was discussed already as an obstacle to proving global well-posedness. In this case, we restore coercivity by imposing a smallness condition on the initial data.

As also discussed above, convexity of the energy functional plays a key role in upgrading weak convergence to strong convergence in the argument of Nakanishi, via an argument of Radon–Riesz type. The analogue of (1-21) for our equation is as follows: For $z = M(u)$ as in (1-22),

$$E(u) = \tfrac{1}{2}\|\langle\nabla\rangle z\|^2_{L^2_x} + \tfrac{1}{4}\gamma\|U|u|^2\|^2_{L^2_x} + \int \tfrac{1}{6}q(u)^3\,dx. \tag{1-23}$$

Unlike its analogue (1-21), this does not yield an inequality between the energy and the H^1_x-norm of z. Indeed, the leading-order correction is the sign-indefinite term $\frac{4}{3}\int(u_1)^3\,dx$. Correspondingly, we will need to be concerned with the structure of our solution $u^\infty(t)$ as $t \to \infty$ to ensure that it does not contain surplus energy beyond that needed for its (putative) scattering state. Recall that $u^{\infty}(t)$ is merely constructed as a weak limit of solutions $u^{T_n}(t)$ defined by their values at $t = T_n$, which gives very little a priori information on its structure.

The resolution of this dilemma is to prove a form of energy decoupling between the part of the solution matching the scattering state and any residual part; see Lemma 6.3. Ultimately, this energy decoupling shows that any residual part of the solution must converge to zero in norm, which in fact obviates any explicit implementation of the Radon–Riesz-style argument described above.

Existence of wave operators. Recall that in Theorem 1.1, we cannot guarantee uniqueness of the nonlinear solution with prescribed scattering state. However, we are able to guarantee uniqueness under stronger hypotheses. Specifically, for scattering states with good linear decay, we can guarantee that there is only one nonlinear solution scattering to it with comparable decay. The decay of such solutions will be measured in the norm

$$\|u\|_{X_T} := \sup_{t\geq T} t^{\frac{1}{2}} \|u(t)\|_{H_x^{1,3}(\mathbb{R}^3)}.$$

Theorem 1.4. *Fix $\gamma \in (0,1)$. There exists $\eta > 0$ so that if $u_+ \in H^1_{\mathrm{real}} + i\dot{H}^1_{\mathrm{real}}$ satisfies*

$$\|V^{-1}e^{-itH}Vu_+\|_{X_1} \leq \eta, \tag{1-24}$$

then there exists a global solution $u \in C(\mathbb{R};\mathcal{E})$ to (1-6) *such that*

$$\lim_{t\to\infty} \|u(t) - V^{-1}e^{-itH}Vu_+\|_{H^1_{\mathrm{real}}+i\dot{H}^1_{\mathrm{real}}} = 0. \tag{1-25}$$

Moreover u is unique in the class of solutions with $\|u\|_{X_T} \leq 4\eta$ for some $T \geq 1$.

Remark 1.5. The proof of this theorem gives a quantitative rate in (1-25), namely,

$$\|u(t) - V^{-1}e^{-itH}Vu_+\|_{H^1_{\mathrm{real}}+i\dot{H}^1_{\mathrm{real}}} \lesssim t^{-\frac{1}{4}}. \tag{1-26}$$

Remark 1.6. Writing $u_{\mathrm{lin}}(t) = V^{-1}e^{-itH}Vu_+$, we note that $u_+ \in H^1_{\mathrm{real}} + i\dot{H}^1_{\mathrm{real}}$ and $\|u_{\mathrm{lin}}\|_{X_1} < \infty$ guarantee that u_{lin} is uniformly bounded in the energy space $\mathcal{E}$ for $t \geq 1$.

Finally, we observe that we can guarantee the smallness condition (1-24) by assuming control over weighted norms.

Corollary 1.7. *Let $\gamma \in (0,1)$ and $u_+ \in H^1_{\mathrm{real}} + i\dot{H}^1_{\mathrm{real}}$. If*

$$\|\langle x\rangle^{\frac{1}{2}+}\langle\nabla\rangle u_+\|_{L^2_x} + \|\langle x\rangle^{\frac{4}{3}+}\langle\nabla\rangle^{\frac{5}{6}}\operatorname{Re} u_+\|_{L^2_x}$$

is sufficiently small, then there exists a global solution $u \in C(\mathbb{R};\mathcal{E})$ to (1-6) *such that* (1-25) *holds.*

We prove Theorem 1.4 and Corollary 1.7 in Section 7. The proof, which relies primarily on dispersive and Strichartz estimates, consists of a contraction mapping argument that simultaneously solves the requisite PDE for $z = M(u)$ and inverts the normal form transformation. The argument differs little from that used to prove Theorem 1.1 in [Gustafson et al. 2007].

Outline of the paper. In Section 2 we set some notation and collect several useful lemmas.

Section 3 concerns the well-posedness of (1-6) in the energy space. We prove Theorem 3.3, giving global well-posedness and unconditional uniqueness in the energy space for (1-6). We also prove a stability result, Proposition 3.5.

The proof of the main result, Theorem 1.1, is ultimately carried out in Section 6. The strategy is modeled on the proof of Theorem 1.3 sketched above. Recalling that proof, we can broadly describe the three main steps as follows: (a) weak convergence uniformly in time, (b) well-posedness in the weak topology, and (c) strong convergence. As discussed above, new difficulties in our setting prevent a naive implementation of Nakanishi's strategy. Thus, we need to establish some preliminary results before launching into the proof of Theorem 1.1.

In Section 4, we consider step (b) and prove Theorem 4.1; briefly, this theorem states that if $u_n(0) \rightharpoonup u_0$ in $\dot{H}_x^1$, then $u_n(t) \rightharpoonup u(t)$ in $\dot{H}_x^1$ for all t, where u_n and u are solutions to (1-6) with initial data $u_n(0)$ and u_0, respectively. As described above, ingredients include (i) a linear profile decomposition adapted to (1-6) and (ii) a way to construct nonlinear solutions associated to the linear profiles. We prove the linear profile decomposition Proposition 4.3 by adapting the energy-critical linear profile decomposition for the Schrödinger propagator. For linear profiles living at unit length scales, we use Theorem 3.3 to construct the corresponding nonlinear profiles. The construction of nonlinear profiles in the case of highly concentrated linear profiles is more delicate and relies on the main result of [Colliander et al. 2008]. Specifically, we approximate such solutions to (1-6) by solutions to the energy-critical NLS and invoke the stability result, Proposition 3.5. The details are carried out in Proposition 4.5.

In Section 5, we discuss the normal form transformation, which is needed for steps (a) and (c). As discussed in the subsection on page 1526, low powers in the nonlinearity and poor spatial decay are problematic for establishing the equicontinuity needed to prove weak convergence. To remedy this, we exploit non-resonant structure in the equation via the normal form transformation M defined in (1-22). We prove some continuity and invertibility properties of this transformation in Proposition 5.1. We also prove Lemma 5.3 relating the energy and the inverse of the normal form transformation, which plays a role in step (c).

With the results of Section 4 and Section 5 in place, we are in a position to prove Theorem 1.1 in Section 6. Following the strategy of Nakanishi and using the normal form transformation and Theorem 4.1, we first construct the putative scattering solution u^∞. Working with the variables $z^\infty = M(u^\infty)$, we then prove a weak convergence result, Proposition 6.2. Having removed the worst quadratic terms via normal form transformation, establishing the requisite equicontinuity is a more feasible prospect; as in the work of [Gustafson et al. 2009], however, we still need to exhibit additional nonresonance at the cubic level.

We next upgrade to strong convergence, still at the level of z^∞. This relies largely on an energy decoupling lemma, Lemma 6.3. Finally, to complete the proof of Theorem 1.1, we show that strong convergence for z^∞ implies the desired convergence properties for u^∞. For this, we make use of results proved in Section 5 concerning the inverse of the normal form transformation (e.g., Lemma 5.3).

Finally, in Section 7 we prove Theorem 1.4 and Corollary 1.7. These results are much simpler than Theorem 1.1; they follow from a contraction mapping argument and rely primarily on Strichartz/dispersive estimates.

2. Notation and useful lemmas

Some notation. We write $A \lesssim B$ or $A = O(B)$ to indicate that $A \le CB$ for some constant $C > 0$. Dependence of implicit constants on various parameters will be indicated with subscripts. For example, $A \lesssim_\varphi B$ means that $A \le CB$ for some $C = C(\varphi)$. The dependence of implicit constants on the parameter γ defined in (1-3) will not be explicitly indicated. We write $A \sim B$ if $A \lesssim B$ and $B \lesssim A$. We write $A \ll B$ if $A \le cB$ for some small $c > 0$.

We write a complex-valued function u as $u = u_1 + iu_2$. When X is a monomial, we use the notation $\emptyset(X)$ to denote a finite linear combination of products of the factors of X, where Mikhlin multipliers (e.g., Littlewood–Paley projections) and/or complex conjugation may be additionally applied in each factor. We extend $\emptyset$ to polynomials via $\emptyset(X + Y) = \emptyset(X) + \emptyset(Y)$.

For a time interval I we write $L_t^q L_x^r(I \times \mathbb{R}^3)$ for the Banach space of functions $u : I \times \mathbb{R}^3 \to \mathbb{C}$ equipped with the norm

$$\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} = \left(\int_I \|u(t)\|_{L_x^r(\mathbb{R}^3)}^q \, dt\right)^{\frac{1}{q}},$$

with the usual adjustments when q or r is infinity. If $q = r$ we write $L_t^q L_x^q = L_{t,x}^q$. We often abbreviate $\|u\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} = \|u\|_{L_t^q L_x^r}$ and $\|u\|_{L_x^r(\mathbb{R}^3)} = \|u\|_{L_x^r}$. We also write $C(I; X)$ to denote the space of continuous functions on I taking values in X.

We use the following convention for the Fourier transform on $\mathbb{R}^3$:

$$\hat{f}(\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{-ix\cdot\xi} f(x)\, dx \quad \text{so that} \quad f(x) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{ix\cdot\xi} \hat{f}(\xi)\, d\xi.$$

The fractional differential operator $|\nabla|^s$ is defined by $\widehat{|\nabla|^s f}(\xi) = |\xi|^s \hat{f}(\xi)$. We will also make use of the following Fourier multiplier operators (and powers thereof):

$$\begin{aligned}
\langle\xi\rangle &= \sqrt{2\gamma + |\xi|^2}, & \langle\nabla\rangle &= \sqrt{2\gamma - \Delta},\\
U(\xi) &= \sqrt{|\xi|^2(2\gamma + |\xi|^2)^{-1}}, & U &= \sqrt{(-\Delta)(2\gamma - \Delta)^{-1}},\\
H(\xi) &= \sqrt{|\xi|^2(2\gamma + |\xi|^2)}, & H &= \sqrt{(-\Delta)(2\gamma - \Delta)}.
\end{aligned}$$

Fix $\gamma \in (0, 1)$ as in (1-3). We define homogeneous and inhomogeneous Sobolev norms $\dot{H}_x^{s,r}$ and $H_x^{s,r}$ as the completion of Schwartz functions under the norms

$$\|f\|_{\dot{H}_x^{s,r}} := \|(-\Delta)^{\frac{s}{2}} f\|_{L_x^r} \quad \text{and} \quad \|f\|_{H_x^{s,r}} := \|(2\gamma - \Delta)^{\frac{s}{2}} f\|_{L_x^r},$$

respectively. When $r = 2$ we abbreviate $\dot{H}_x^{s,2} = \dot{H}_x^s$ and $H_x^{s,2} = H_x^s$. Note that this definition of the H_x^s-norm is equivalent (up to constants depending on γ) to the standard one, which uses the operator $(1 - \Delta)^{\frac{s}{2}}$.

Basic harmonic analysis. We employ the standard Littlewood–Paley theory. Let ϕ be a radial bump function supported in $\left\{|\xi| \le \frac{11}{10}\right\}$ and equal to 1 on the unit ball. For $N \in 2^{\mathbb{Z}}$ we define the Littlewood–Paley projections

$$\widehat{P_{\leq N}u}(\xi) = \phi\big(\tfrac{1}{N}\xi\big)\hat{u}(\xi), \quad \widehat{P_N u}(\xi) = \big[\phi\big(\tfrac{1}{N}\xi\big) - \phi\big(\tfrac{1}{2N}\xi\big)\big]\hat{u}(\xi), \quad \text{and} \quad P_{>N} = \mathrm{Id} - P_{\leq N}.$$

These operators commute with all other Fourier multiplier operators. They are self-adjoint and bounded on every L_x^p and H_x^s space for $1 \leq p \leq \infty$ and $s \geq 0$. We write $P_{\mathrm{lo}} = P_{\leq 1}$ and $P_{\mathrm{hi}} = P_{>1}$.

The Littlewood–Paley projections obey the following standard estimates.

Lemma 2.1 (Bernstein estimates). *For $1 \leq r \leq q \leq \infty$ and $s \geq 0$ we have*

$$\big\| |\nabla|^s P_{\leq N} u \big\|_{L_x^r(\mathbb{R}^3)} \lesssim N^s \| P_{\leq N} u\|_{L_x^r(\mathbb{R}^3)},$$
$$\| P_{>N} u\|_{L_x^r(\mathbb{R}^3)} \lesssim N^{-s} \big\| |\nabla|^s P_{>N} u \big\|_{L_x^r(\mathbb{R}^3)},$$
$$\| P_{\leq N} u\|_{L_x^q(\mathbb{R}^3)} \lesssim N^{\frac{3}{r}-\frac{3}{q}} \| P_{\leq N} u\|_{L_x^r(\mathbb{R}^3)}.$$

We will need the following:

Lemma 2.2 (fractional chain rule, [Christ and Weinstein 1991]). *Suppose $G \in C^1(\mathbb{C})$ and $s \in (0, 1]$. Let $1 < r, r_2 < \infty$ and $1 < r_1 \leq \infty$ satisfy $1/r_1 + 1/r_2 = 1/r$. Then*

$$\big\| |\nabla|^s G(u) \big\|_{L_x^r} \lesssim \| G'(u)\|_{L_x^{r_1}} \big\| |\nabla|^s u \big\|_{L_x^{r_2}}.$$

We will also need the following result concerning bilinear Fourier multipliers. For a real-valued function $B(\xi_1, \xi_2)$ we define the operator $B[f, g]$ via

$$\widehat{B[f,g]}(\xi) := (2\pi)^{\frac{3}{2}} \int_{\mathbb{R}^3} B(\eta, \xi - \eta) \hat{f}(\eta) \hat{g}(\xi - \eta)\, d\eta. \tag{2-1}$$

Lemma 2.3 (Coifman–Meyer bilinear estimate, [Coifman and Meyer 1978; Meyer and Coifman 1991]). *If the symbol $B(\xi_1, \xi_2)$ satisfies*

$$|\partial_{\xi_1}^{\alpha} \partial_{\xi_2}^{\beta} B(\xi_1, \xi_2)| \lesssim_{\alpha,\beta} (|\xi_1| + |\xi_2|)^{-(|\alpha|+|\beta|)}$$

for all multi-indices α, β up to sufficiently high order, then

$$\| B[f, g]\|_{L_x^r} \lesssim \| f\|_{L_x^{r_1}} \| g\|_{L_x^{r_2}}$$

for all $1 < r < \infty$ and $1 < r_1, r_2 < \infty$ satisfying $1/r = 1/r_1 + 1/r_2$.

Linear estimates. We record here the dispersive and Strichartz estimates for the propagators $e^{it\Delta}$ and e^{-itH}.

As is well known, the linear Schrödinger propagator in three space dimensions can be written as

$$[e^{it\Delta} f](x) = (4\pi i t)^{-\frac{3}{2}} \int_{\mathbb{R}^3} e^{\frac{i|x-y|^2}{4t}} f(y)\, dy$$

for $t \neq 0$. This yields the dispersive estimates

$$\| e^{it\Delta} f\|_{L_x^r(\mathbb{R}^3)} \lesssim |t|^{-(\frac{3}{2}-\frac{3}{r})} \| f\|_{L_x^{r'}(\mathbb{R}^3)} \tag{2-2}$$

for $t \neq 0$, where $2 \leq r \leq \infty$ and $1/r + 1/r' = 1$. This estimate can be used to prove the standard Strichartz estimates for $e^{it\Delta}$. We state the result we need in three space dimensions.

Proposition 2.4 (Strichartz estimates for $e^{it\Delta}$, [Ginibre and Velo 1992; Keel and Tao 1998; Strichartz 1977]). *For a space-time slab $I \times \mathbb{R}^3$ and $2 \le q, \tilde{q} \le \infty$ with $2/q + 3/r = 2/\tilde{q} + 3/\tilde{r} = \frac{3}{2}$, we have*

$$\left\| e^{it\Delta}\varphi + \int_0^t e^{i(t-s)\Delta} F(s)\, ds \right\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \lesssim \|\varphi\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)}.$$

Using stationary phase, one can prove a similar dispersive estimate for e^{-itH} (see [Gustafson et al. 2006; 2009]). In fact, there is a small gain at low frequencies compared to the estimates for the linear Schrödinger propagator; while the dispersion relation for this propagator has less curvature in the radial direction than that for Schrödinger, this is more than compensated for by the increased curvature in the angular directions.

Proposition 2.5 (estimates for e^{-itH}, [Gustafson et al. 2006, 2009]). *For $2 \le r \le \infty$ we have*

$$\|e^{-itH} f\|_{L_x^r(\mathbb{R}^3)} \lesssim |t|^{-(\frac{3}{2}-\frac{3}{r})} \|U^{\frac{1}{2}-\frac{1}{r}} f\|_{L_x^{r'}(\mathbb{R}^3)} \tag{2-3}$$

for $t \neq 0$. In particular, for a space-time slab $I \times \mathbb{R}^3$ and $2 \le q, \tilde{q} \le \infty$ with $2/q + 3/r = 2/\tilde{q} + 3/\tilde{r} = \frac{3}{2}$, we have

$$\left\| e^{-itH}\varphi + \int_0^t e^{-i(t-s)H} F(s)\, ds \right\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} \lesssim \|\varphi\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}(I\times\mathbb{R}^3)}.$$

For an interval I and $s \ge 0$ we define the Strichartz norm by

$$\|u\|_{\dot{S}^s(I)} = \sup\left\{ \left\| |\nabla|^s u \right\|_{L_t^q L_x^r(I\times\mathbb{R}^3)} : 2 \le q \le \infty,\ \frac{2}{q} + \frac{3}{r} = \frac{3}{2} \right\}.$$

The Strichartz space $\dot{S}^s(I)$ is then defined to be the closure of test functions under this norm. We let $\dot{N}^s(I)$ denote the corresponding dual Strichartz space.

In several places it will be more convenient to work with (1-6) rather than the diagonalized (1-11). The linear propagator associated with (1-6) takes the form

$$V^{-1} e^{-itH} V \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \cos(tH) & U\sin(tH) \\ -U^{-1}\sin(tH) & \cos(tH) \end{bmatrix} \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \tag{2-4}$$

for any function $f = f_1 + i f_2$. We will make use of the following Strichartz estimates for this propagator:

Lemma 2.6. *Fix $T > 0$. Given $2 < q, \tilde{q} \le \infty$ with $2/q + 3/r = 2/\tilde{q} + 3/\tilde{r} = \frac{3}{2}$, we have*

$$\left\| V^{-1} e^{-itH} V\varphi + \int_0^t V^{-1} e^{-i(t-s)H} V F(s)\, ds \right\|_{L_t^q L_x^r} \lesssim_T \|\varphi\|_{L_x^2} + \|F\|_{L_t^{\tilde{q}'} L_x^{\tilde{r}'}}, \tag{2-5}$$

where all space-time norms are over $[-T, T] \times \mathbb{R}^3$.

Proof. As we are excluding the endpoint, it suffices (via a TT^* argument) to prove the result when $F \equiv 0$; moreover, it clearly suffices to consider each entry in the matrix (2-4) separately. In view of the boundedness of U, three out of four of these matrix elements obey the same Strichartz estimates as e^{-itH}; see Proposition 2.5. As $P_{\mathrm{hi}} U^{-1}$ is also bounded, we need only prove Strichartz estimates for

$P_{\mathrm{lo}}U^{-1}\sin(tH)$. However, this is easily done via Hölder and Bernstein's inequality:

$$\begin{aligned}\|P_{\mathrm{lo}}U^{-1}\sin(tH)\varphi\|_{L_t^qL_x^r([-T,T]\times\mathbb{R}^3)} &\lesssim T^{\frac{1}{q}}\|P_{\mathrm{lo}}U^{-1}\sin(tH)\varphi\|_{L_t^\infty L_x^2([-T,T]\times\mathbb{R}^3)}\\ &\lesssim T^{1+\frac{1}{q}}\|\varphi\|_{L_x^2(\mathbb{R}^3)}.\end{aligned}\tag{2-6}$$

This completes the proof of the lemma. □

At high frequencies, the operator e^{-itH} closely resembles the Schrödinger propagator (on bounded time intervals); specifically, we have

$$\sqrt{|\xi|^2(2\gamma+|\xi|^2)}=|\xi|^2+\gamma+m(\xi)\quad\text{with } |m(\xi)|\lesssim\langle\xi\rangle^{-2}.\tag{2-7}$$

Indeed, it is not difficult to verify that $m(\xi)$ defines a Mikhlin multiplier. This observation will play a key role in our treatment of highly concentrated profiles in Section 4. For the moment, however, we simply use it to obtain a crude local smoothing estimate.

Lemma 2.7 (local smoothing). *Given $T>0$ and $R>0$,*

$$\big\||\nabla|^{\frac{1}{2}}V^{-1}e^{-itH}V\varphi\big\|_{L^2_{t,x}(\{|t|\le T\}\times\{|x|\le R\})}\lesssim_{R,T}\|\varphi\|_{L_x^2}.\tag{2-8}$$

Proof. We treat high and low frequencies separately. In the low-frequency regime, we exploit (2-4) and argue as in (2-6) to deduce that

$$\big\||\nabla|^{\frac{1}{2}}P_{\mathrm{lo}}V^{-1}e^{-itH}V\varphi\big\|_{L^2_{t,x}(\{|t|\le T\}\times\{|x|\le R\})}\lesssim T^{\frac{1}{2}}(1+T)\|\varphi\|_{L_x^2}.$$

In the high-frequency regime, we can use the usual local smoothing estimate for the Schrödinger equation together with

$$\big\||\nabla|^{\frac{1}{2}}P_{\mathrm{hi}}V^{-1}[e^{-itH}-e^{-it(\gamma-\Delta)}]V\varphi\big\|_{L_t^\infty L_x^2(\{|t|\le T\}\times\mathbb{R}^3)}\lesssim T\|\varphi\|_{L_x^2},$$

which follows from (2-7). □

In practice, we will use the following corollary.

Corollary 2.8. *Let K be a compact subset of $I\times\mathbb{R}^3$ for some interval $I\subset\mathbb{R}$. Then the following estimates hold*:

$$\|\nabla e^{it\Delta}f\|_{L^2_{t,x}(K)}\lesssim_K\|e^{it\Delta}f\|^{\frac{1}{3}}_{L^{10}_{t,x}(I\times\mathbb{R}^3)}\|f\|^{\frac{2}{3}}_{\dot H^1_x},$$

$$\|\nabla V^{-1}e^{-itH}Vf\|_{L^2_{t,x}(K)}\lesssim_K\|V^{-1}e^{-itH}Vf\|^{\frac{1}{3}}_{L^{10}_{t,x}(I\times\mathbb{R}^3)}\|f\|^{\frac{2}{3}}_{\dot H^1_x}.$$

Proof. Fix $N>0$. By the Bernstein and Hölder inequalities,

$$\|\nabla P_{\le N}e^{it\Delta}f\|_{L^2_{t,x}(K)}\lesssim_K N\|e^{it\Delta}f\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)}.$$

By the local smoothing estimate for $e^{it\Delta}$ and Bernstein, we also have

$$\|\nabla P_{>N}e^{it\Delta}f\|_{L^2_{t,x}(K)}\lesssim_K\big\||\nabla|^{\frac{1}{2}}P_{>N}f\big\|_{L_x^2}\lesssim_K N^{-\frac{1}{2}}\|\nabla f\|_{L_x^2}.$$

Optimizing in the choice of N yields the first estimate.

To obtain the second estimate one argues in exactly the same way, making use of Lemma 2.7. □

3. Global well-posedness in the energy space

In this section we discuss the well-posedness of (1-6) in the energy space. We begin by justifying the name "energy space" given to the set $\mathcal{E}$ defined in (1-9). Recall from the Introduction that if $u \in \mathcal{E}$, then $|E(u)| < \infty$. The following two lemmas prove that if the energy of u is finite, then $u \in \mathcal{E}$; when $\gamma \in \left(0, \frac{2}{3}\right)$, this requires an additional smallness condition.

Lemma 3.1. *If $\gamma \in \left(\frac{2}{3}, 1\right)$ and $E(u) < \infty$, then $u \in \mathcal{E}$ with $\|u\|_{\mathcal{E}}^2 \lesssim E(u)$. If $\gamma = \frac{2}{3}$ and $E(u) < \infty$, then $u \in \mathcal{E}$ with*

$$\|\nabla u\|_{L_x^2}^2 \lesssim E(u) \quad \textit{and} \quad \|q\|_{L_x^2}^2 \lesssim E(u) + [E(u)]^3.$$

Proof. When $\gamma > \frac{2}{3}$ we use the fact that $q \geq -1$ in (1-8) to write

$$E(u) \geq \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{\gamma}{4} \int \left(1 - \frac{2}{3\gamma}\right) q^2 \, dx,$$

which immediately implies the result.

We now turn to the case when $\gamma = \frac{2}{3}$. In this case, the energy takes the form

$$E(u) = \frac{1}{2} \int |\nabla u|^2 \, dx + \frac{1}{6} \int q^2 (q+1) \, dx.$$

As $q \geq -1$, we have $q^2(q+1) \geq 0$. Thus $u \in \dot{H}_x^1(\mathbb{R}^3)$ and $\|\nabla u\|_{L_x^2}^2 \lesssim E(u)$.

To estimate the L_x^2-norm of q, we note that

$$\int_{\{q \geq -\frac{1}{2}\}} q^2 \, dx \leq 2 \int q^2 (q+1) \, dx \lesssim E(u).$$

On the other hand, if $q < -\frac{1}{2}$ then $|u_1| > \frac{1}{4}$; thus, by Chebyshev's inequality and Sobolev embedding,

$$\int_{\{q < -\frac{1}{2}\}} q^2 \, dx \leq 4^6 \|u_1\|_{L_x^6}^6 \lesssim \|\nabla u\|_{L_x^2}^6 \lesssim [E(u)]^3. \qquad \square$$

We next consider the full range $\gamma \in (0, 1)$. In this case, we can guarantee coercivity of the energy under an appropriate smallness assumption.

Lemma 3.2. *For any $\gamma \in (0, 1)$ there exists $\delta_\gamma > 0$ so that the following hold*:

(i) *If $E(u) < \infty$ and $\|\nabla u_1\|_{L_x^2}^2 \leq \delta_\gamma$, then $u \in \mathcal{E}$ with $\|u\|_{\mathcal{E}}^2 \lesssim E(u)$.*

(ii) *For any ball B,*

$$\|\nabla u_1\|_{L_x^2(\mathbb{R}^3)}^2 \leq \delta_\gamma \implies \int_{B^c} \tfrac{1}{2} |\nabla u|^2 + \tfrac{1}{4} \gamma q^2 + \tfrac{1}{6} q^3 \, dx \geq 0. \tag{3-1}$$

(iii) *If $u : I \times \mathbb{R}^3 \to \mathbb{C}$ is a solution to (1-6) with $E(u) \leq \frac{1}{4} \delta_\gamma$ and $\|\nabla \operatorname{Re} u(t_0)\|_{L_x^2}^2 \leq \delta_\gamma$ for some $t_0 \in I$, then*

$$\|\nabla \operatorname{Re} u\|_{L_t^\infty L_x^2(I \times \mathbb{R}^3)}^2 \leq \delta_\gamma \quad \textit{and} \quad \|u\|_{L^\infty(I;\mathcal{E})}^2 \lesssim E(u).$$

Proof. We begin by writing

$$E(u) = \int \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{8}\gamma q^2 + \tfrac{1}{6}q^2\big(q + \tfrac{3}{4}\gamma\big)\,dx$$
$$\geq \int \tfrac{1}{2}|\nabla u|^2 + \tfrac{1}{8}\gamma q^2\,dx + \int_{\{q<-\frac{3}{4}\gamma\}} \tfrac{1}{6}q^2\big(q + \tfrac{3}{4}\gamma\big)\,dx.$$

For $q < -\frac{3}{4}\gamma$ we have $|u_1| > \frac{3}{8}\gamma$. Thus, by Chebyshev's inequality and Sobolev embedding, we have

$$\big|\{q < -\tfrac{3}{4}\gamma\}\big| \leq \Big(\frac{8}{3\gamma}\Big)^6 \|u_1\|_{L_x^6}^6 \lesssim \gamma^{-6}\|\nabla u_1\|_{L_x^2}^6.$$

Recalling that $q \geq -1$, we find that for $\|\nabla u_1\|_{L_x^2}^2 \ll \gamma^{\frac{3}{2}}$ we have

$$\bigg|\int_{\{q<-\frac{3}{4}\gamma\}} \tfrac{1}{6}q^2\big(q + \tfrac{3}{4}\gamma\big)\,dx\bigg| \lesssim \gamma^{-6}\|\nabla u_1\|_{L_x^2}^6 \leq \tfrac{1}{4}\|\nabla u_1\|_{L_x^2}^2.$$

Thus

$$E(u) \geq \int \tfrac{1}{4}|\nabla u|^2 + \tfrac{1}{8}\gamma q^2\,dx,$$

which yields conclusion (i) of the lemma. Claim (iii) also follows from this and a continuity argument.

To obtain (ii), we repeat the argument above, using the fact that Sobolev embedding holds in the exterior of any ball B. □

We next turn to the question of global well-posedness for (1-6) with initial data $u_0 \in \mathcal{E}$. From the lemmas above we see that $u(t) \in \mathcal{E}$ and $\|\nabla u(t)\|_{L_x^2}^2 \lesssim E(u_0)$ for all times of existence, whenever (1) $\gamma \in \big[\frac{2}{3}, 1\big)$ or (2) $\gamma \in \big(0, \frac{2}{3}\big)$ and $E(u_0)$ and $\|\nabla \operatorname{Re} u_0\|_{L_x^2}$ are sufficiently small. This a priori bound on $\|\nabla u(t)\|_{L_x^2}$ allows us to treat (1-6) as a perturbation of the defocusing energy-critical NLS, which was proven to be globally well-posed with finite space-time bounds in [Colliander et al. 2008]. See also [Killip et al. 2012; Tao et al. 2007] for similar perturbative arguments.

Theorem 3.3 (global well-posedness and unconditional uniqueness). *For $\gamma \in \big[\frac{2}{3}, 1\big)$ and $u_0 \in \mathcal{E}$, there exists a unique global solution $u \in C(\mathbb{R}; \mathcal{E})$ to* (1-6).

For $\gamma \in \big(0, \frac{2}{3}\big)$, if $u_0 \in \mathcal{E}$ satisfies $\|\nabla \operatorname{Re} u_0\|_{L_x^2}^2 \leq \delta_\gamma$ and $E(u_0) \leq \frac{1}{4}\delta_\gamma$, then there exists a unique global solution $u \in C(\mathbb{R}; \mathcal{E})$ to (1-6). *Here δ_γ is as in Lemma 3.2.*

In both cases the solution remains uniformly bounded in $\mathcal{E}$ and for any $T > 0$,

$$\|u\|_{\dot{S}^1([-T,T])} \lesssim_T 1.$$

Remark 3.4. When $\gamma \in \big(0, \frac{2}{3}\big)$, smallness of the initial data is only exploited to prove global existence; the proof we present below guarantees uniqueness of any solution in $C(I; \mathcal{E})$ on any time interval $I \subseteq \mathbb{R}$.

Proof. As mentioned above, Lemmas 3.1 and 3.2 imply that under the hypotheses of Theorem 3.3 we have $\|\nabla u(t)\|_{L_x^2}^2 \lesssim E(u_0)$ for all times t of existence. This allows us to treat (1-6) as a perturbation of the defocusing energy-critical NLS. Indeed, we may rewrite (1-6) as

$$(i\partial_t + \Delta)u = |u|^4 u + \mathcal{R}(u),$$

where $\mathcal{R}(u) = 2\gamma \operatorname{Re} u + \sum_{j=2}^{4} N_j(u)$. Noting that the "error" $\mathcal{R}(u)$ is energy-subcritical, one may argue as in [Killip et al. 2012, Section 4.2] to construct a global solution $u \in C(\mathbb{R}; \mathcal{E}) \cap L_t^{10} \dot{H}_x^{1,\frac{30}{13}}(\mathbb{R} \times \mathbb{R}^3)$ to (1-6). A key ingredient in this argument is the main result in [Colliander et al. 2008], which guarantees that the defocusing energy-critical NLS is globally well-posed with finite $L_t^{10} \dot{H}_x^{1,\frac{30}{13}}(\mathbb{R} \times \mathbb{R}^3)$ norm. We omit the details of this argument. Instead, we present the proof of uniqueness of solutions in the energy space, because the choice of energy space in this paper does not allow for a direct implementation of the methods in [Killip et al. 2012, Section 4.3].

Fix a compact time interval $I = [0, \tau]$ with $\tau > 0$ small. Let $u \in C(\mathbb{R}; \mathcal{E}) \cap L_t^{10} \dot{H}_x^{1,\frac{30}{13}}(\mathbb{R} \times \mathbb{R}^3)$ be the solution to (1-6) constructed via the perturbative argument described above. Suppose $\tilde{u} \in C(I; \mathcal{E})$ is another solution such that $\tilde{u}(0) = u(0)$. We wish to show that $u = \tilde{u}$ almost everywhere on $I \times \mathbb{R}^3$.

To this end, we define $w = \tilde{u} - u$ and let $0 < \eta < 1$ be a small parameter to be determined below. As $w(0) = 0$ and $w \in C(I; \dot{H}_x^1)$, we can choose τ small enough so that

$$\|w\|_{L_t^\infty \dot{H}_x^1(I \times \mathbb{R}^3)} \le \eta. \tag{3-2}$$

As $\nabla u \in L_t^{10} L_x^{\frac{30}{13}}(I \times \mathbb{R}^3)$, we may also use Sobolev embedding and choose τ possibly even smaller to guarantee that

$$\|u\|_{L_{t,x}^{10}(I \times \mathbb{R}^3)} \le \eta. \tag{3-3}$$

We also note that as u and $\tilde{u}$ are bounded in $\mathcal{E}$, we have that $q(u)$, $q(\tilde{u})$ are bounded in L_x^2; u, $\tilde{u}$ are bounded in L_x^6; and u_1, $\tilde{u}_1$ are bounded in $L_x^3 \cap L_x^6$.

We will first show that w is bounded in Strichartz spaces on $I \times \mathbb{R}^3$. To see this, we write

$$(i\partial_t + \Delta) w = 2\gamma \tilde{u}_1 + N(\tilde{u}) - [2\gamma u_1 + N(u)],$$

where $N(u)$ is as in (1-6). We make use of $q(u)$ and $q(\tilde{u})$ to rewrite

$$\begin{aligned}(i\partial_t + \Delta) w = O(|\tilde{u}|^5 + |u|^5) + O(|\tilde{u}|^4 + |u|^4) + O(|\tilde{u}|^3 + |u|^3) \\ + \gamma q(\tilde{u}) + (2\gamma+4)\tilde{u}_1^2 + 2i\gamma \tilde{u}_1 \tilde{u}_2 - [\gamma q(u) + (2\gamma+4)u_1^2 + 2i\gamma u_1 u_2].\end{aligned}$$

As $w(0) = 0$, we can use Strichartz to estimate

$$\begin{aligned}\|w\|_{L_t^2 L_x^6} + \|w\|_{L_t^4 L_x^3} + \|w\|_{L_t^\infty L_x^2} \lesssim \|\tilde{u}^5\|_{L_t^2 L_x^{6/5}} + \|u^5\|_{L_t^2 L_x^{6/5}} + \|\tilde{u}^4\|_{L_t^{4/3} L_x^{3/2}} \\ + \|u^4\|_{L_t^{4/3} L_x^{3/2}} + \|\tilde{u}^3\|_{L_t^1 L_x^2} + \|u^3\|_{L_t^1 L_x^2} \\ + \|q(\tilde{u})\|_{L_t^1 L_x^2} + \|q(u)\|_{L_t^1 L_x^2} + \|\tilde{u}\tilde{u}_1\|_{L_t^1 L_x^2} + \|u u_1\|_{L_t^1 L_x^2},\end{aligned}$$

where all space-time norms are over $I \times \mathbb{R}^3$. Using Hölder, we find

$$\|u^5\|_{L_t^2 L_x^{6/5}} \lesssim \tau^{1/2} \|u\|_{L_t^\infty L_x^6}^5, \quad \|u^4\|_{L_t^{4/3} L_x^{3/2}} \lesssim \tau^{3/4} \|u\|_{L_t^\infty L_x^6}^4, \quad \|u^3\|_{L_t^1 L_x^2} \lesssim \tau \|u\|_{L_t^\infty L_x^6},$$
$$\|q(u)\|_{L_t^1 L_x^2} \lesssim \tau \|q(u)\|_{L_t^\infty L_x^2}, \quad \|u u_1\|_{L_t^1 L_x^2} \lesssim \tau \|u\|_{L_t^\infty L_x^6} \|u_1\|_{L_t^\infty L_x^3},$$

and we can estimate similarly for $\tilde{u}$. Thus we conclude

$$\|w\|_{L_t^2 L_x^6} + \|w\|_{L_t^4 L_x^3} + \|w\|_{L_t^\infty L_x^2} < \infty. \tag{3-4}$$

We will show that, in fact,

$$\|w\|_{L_t^2 L_x^6} + \|w\|_{L_t^4 L_x^3} + \|w\|_{L_t^\infty L_x^2} = 0, \tag{3-5}$$

which implies $w = 0$ almost everywhere, as desired. To this end, we again rewrite the equation for w, using z to indicate that either w or u may appear. We have

$$(i\partial_t + \Delta)w = O\big(|w||u|^4 + |w|^5 + |w||z|^3 + |w||z|^2 + |w||z| + |w|\big).$$

We now use Strichartz, (3-2) and (3-3) to estimate

$$\begin{aligned}
&\|w\|_{L_t^2 L_x^6} + \|w\|_{L_t^4 L_x^3} + \|w\|_{L_t^\infty L_x^2} \\
&\quad\lesssim \|wu^4\|_{L_t^{10/9} L_x^{30/17}} + \|w^5\|_{L_t^2 L_x^{6/5}} + \|wz^3\|_{L_t^2 L_x^{6/5}} + \|wz^2\|_{L_t^{4/3} L_x^{3/2}} + \|wz\|_{L_t^1 L_x^2} + \|w\|_{L_t^1 L_x^2} \\
&\quad\lesssim \|u\|_{L_{t,x}^{10}}^4 \|w\|_{L_t^2 L_x^6} + \|w\|_{L_t^\infty L_x^6}^4 \|w\|_{L_t^2 L_x^6} + \tau^{\frac14} \|z\|_{L_t^\infty L_x^6}^3 \|w\|_{L_t^4 L_x^3} \\
&\qquad\qquad + \tau^{\frac12} \|z\|_{L_t^\infty L_x^6}^2 \|w\|_{L_t^4 L_x^3} + \tau^{\frac34} \|z\|_{L_t^\infty L_x^6} \|w\|_{L_t^4 L_x^3} + \tau \|w\|_{L_t^\infty L_x^2} \\
&\quad\lesssim \eta^4 \|w\|_{L_t^2 L_x^6} + \tau^{\frac14} \|w\|_{L_t^4 L_x^3} + \tau \|w\|_{L_t^\infty L_x^2}.
\end{aligned}$$

Choosing η, τ sufficiently small and using (3-4), we conclude that (3-5) holds and so $u = \tilde u$ almost everywhere on $I \times \mathbb{R}^3$. As uniqueness is a local property, this yields uniqueness in the energy space for solutions to (1-6). □

Next we develop a stability theory for (1-6), which we will need in Section 4.

Proposition 3.5 (stability theory). *Fix $T > 0$ and let $\tilde u : [-T, T] \times \mathbb{R}^3 \to \mathbb{C}$ be a solution to the perturbed equation*

$$(i\partial_t + \Delta - 2\gamma\,\mathrm{Re})\tilde u = N(\tilde u) + e$$

for some function e. Suppose that

$$\|\tilde u\|_{L_t^\infty \dot H_x^1([-T,T]\times\mathbb{R}^3)} + \|\nabla \tilde u\|_{L_t^{10} L_x^{30/13}([-T,T]\times\mathbb{R}^3)} \le L \tag{3-6}$$

for some constant $L > 0$. Let $u_0 \in \dot H_x^1(\mathbb{R}^3)$ and assume that

$$\|\tilde u(0) - u_0\|_{\dot H_x^1} + \left\| \int_0^t e^{i(t-s)\Delta} \nabla e(s)\, ds \right\|_{L_t^\infty L_x^2 \cap L_{t,x}^{10/3}([-T,T]\times\mathbb{R}^3)} \le \varepsilon \tag{3-7}$$

for some $\varepsilon \le \varepsilon_0(L, T)$. Then for $\varepsilon_0(L, T)$ sufficiently small there exists a solution $u : [-T, T] \times \mathbb{R}^3 \to \mathbb{C}$ to (1-6) *with data $u(0) = u_0$ and*

$$\|\nabla(\tilde u - u)\|_{L_t^\infty L_x^2 \cap L_{t,x}^{10/3}([-T,T]\times\mathbb{R}^3)} \le C(L, T)\varepsilon, \tag{3-8}$$

$$\|u\|_{\dot S^1([-T,T])} \le C(L, T). \tag{3-9}$$

Proof. The existence of the solution u on a small neighborhood of $t = 0$ follows from the arguments described in Theorem 3.3. In that setting, the solution could be extended globally due to energy control. That argument does not apply here as $u_0 \in \dot H_x^1$ by itself does not guarantee finiteness of the energy;

furthermore, we permit here large data even when $\gamma < \frac{2}{3}$, in which case the energy need not be coercive. However, these earlier arguments do show that if a solution should blow up in finite time, then the $\dot{S}^1$-norm must diverge. Consequently, we can prove that the solution exists and obeys (3-8) and (3-9) on the whole interval $[-T, T]$ by showing that it obeys (3-8) and (3-9) on any subinterval $0 \ni I \subseteq [-T, T]$ on which it does exist. This is what we do.

For brevity, we define the following norm: given a time interval $[a, b] \subset \mathbb{R}$,

$$\|u\|_{Y([a,b])} := \|\nabla u\|_{L_t^\infty L_x^2 \cap L_{t,x}^{10/3}([a,b]\times\mathbb{R}^3)}.$$

Given $0 < \eta < 1$ to be chosen later, we divide I into intervals J where

$$|J| \le \eta \quad \text{and} \quad \|\nabla \tilde{u}\|_{L_t^{10} L_x^{30/13}(J\times\mathbb{R}^3)} \le \eta. \tag{3-10}$$

The number K of such intervals depends only on L, T, and η. Below we will show that for η sufficiently small,

$$\inf_{t_0\in J} \|\tilde{u}(t_0) - u(t_0)\|_{\dot{H}_x^1} \le \eta \implies \|\tilde{u} - u\|_{Y(J)} \le A \inf_{t_0\in J} \|\tilde{u}(t_0) - u(t_0)\|_{\dot{H}_x^1} \tag{3-11}$$

for some absolute constant A on such intervals J. Iterating this completes the proof of (3-8) and yields constants

$$\varepsilon_0 = A^{-K(L,T,\eta)}\eta \quad \text{and} \quad C(L, T) = K(L, T, \eta) A^{K(L,T,\eta)}.$$

We now verify (3-11). Writing $u = \tilde{u} + v$, we use Strichartz and (3-7) to estimate

$$\|v\|_{Y(J)} \lesssim \inf_{t_0\in J} \|v(t_0)\|_{\dot{H}_x^1} + \big\|\nabla[N(\tilde{u}+v) - N(\tilde{u})]\big\|_{\dot{N}^0(J)} + |J|\,\|v\|_{L_t^\infty \dot{H}_x^1} + \varepsilon,$$

where $N(\cdot)$ denotes the nonlinearity, as in (1-6). Moreover,

$$\begin{aligned}\big\|\nabla[N(\tilde{u}+v) - N(\tilde{u})]\big\|_{\dot{N}^0(J)} &\lesssim \|\nabla\tilde{u}\|_{L_t^{10}L_x^{30/13}} \|v\|_{L_{t,x}^{10}} \sum_{k=2}^{5} |J|^{\frac{5-k}{4}} \Big(\|\tilde{u}\|_{L_{t,x}^{10}}^{k-2} + \|v\|_{L_{t,x}^{10}}^{k-2}\Big) \\ &\quad + \|\nabla v\|_{L_t^{10}L_x^{30/13}} \sum_{k=2}^{5} |J|^{\frac{5-k}{4}} \Big(\|\tilde{u}\|_{L_{t,x}^{10}}^{k-1} + \|v\|_{L_{t,x}^{10}}^{k-1}\Big),\end{aligned}$$

where all space-time norms are over $J \times \mathbb{R}^3$. Using Sobolev embedding and (3-10), we therefore obtain

$$\|v\|_{Y(J)} \lesssim \inf_{t_0\in J} \|v(t_0)\|_{\dot{H}_x^1} + \sum_{k=1}^{5} \eta^{\frac{5-k}{4}} \|v\|_{Y(J)}^k + \varepsilon.$$

Choosing η sufficiently small, a simple bootstrap argument yields (3-11).

Using the fact that u is a solution to (1-6), a further application of the Strichartz inequality gives (3-9). □

We also record the following corollary.

Corollary 3.6 (small-data space-time bounds). *Given $T > 0$ there exists $\eta(T) > 0$ such that*

$$\|u_0\|_{\dot{H}_x^1} \le \eta(T) \implies \|u\|_{\dot{S}^1([-T,T])} \lesssim_T \|u_0\|_{\dot{H}_x^1},$$

where u denotes the solution to (1-6) *with data $u(0) = u_0$.*

Proof. We apply Proposition 3.5 with $\tilde{u} = e^{it\Delta}u_0$. By the Strichartz inequality,

$$\|\tilde{u}\|_{L_t^\infty \dot{H}_x^1([-T,T]\times\mathbb{R}^3)} + \|\nabla\tilde{u}\|_{L_t^{10} L_x^{30/13}([-T,T]\times\mathbb{R}^3)} \lesssim \|u_0\|_{\dot{H}_x^1},$$

while a little computation yields

$$\left\| \int_0^t e^{i(t-s)\Delta} e(s)\, ds \right\|_{\dot{S}^1([-T,T])} \lesssim \sum_{k=1}^{5} T^{\frac{5-k}{4}} \|u_0\|_{\dot{H}_x^1}^k.$$

Proposition 3.5 now gives the claim, provided $\eta(T)$ is taken sufficiently small. □

4. Well-posedness in the weak topology

In this section we prove the following well-posedness result in the weak $\dot{H}_x^1$ topology. As described in the Introduction, this theorem will play a key role in the proof of Theorem 1.1 in Section 6.

Theorem 4.1 (weak topology well-posedness). *Let $\gamma \in (0, 1)$ and let $\{u_n(0)\}_{n\geq 1}$ be a bounded sequence in $\mathcal{E}$. Assume that $u_n(0) \rightharpoonup u_0$ weakly in $\dot{H}_x^1(\mathbb{R}^3)$. If $\gamma \in \left(0, \frac{2}{3}\right)$ we assume additionally that*

$$\|\nabla \operatorname{Re} u_n(0)\|_{L_x^2} \leq \delta_\gamma \quad \text{and} \quad E(u_n(0)) \leq \tfrac{1}{4}\delta_\gamma,$$

where δ_γ is as in Theorem 3.3. Then there exists a unique solution $u \in C(\mathbb{R};\mathcal{E})$ to (1-6) *with $u(0) = u_0$, and for all $t \in \mathbb{R}$ we have*

$$u_n(t) \rightharpoonup u(t) \quad \text{weakly in } \dot{H}_x^1(\mathbb{R}^3), \tag{4-1}$$

where $u_n \in C(\mathbb{R};\mathcal{E})$ denotes the solution to (1-6) *with initial data $u_n(0)$, whose existence is guaranteed by Theorem 3.3.*

We begin with the following lemma, which guarantees that the limit u_0 belongs to the energy space and obeys the necessary smallness conditions when $\gamma \in \left(0, \frac{2}{3}\right)$, so that the existence and uniqueness of the solution $u \in C(\mathbb{R};\mathcal{E})$ follow from Theorem 3.3.

Lemma 4.2. *Fix $\gamma \in (0, 1)$ and suppose $\{u_n\}_{n\geq 1}$ is a bounded sequence in $\mathcal{E}$ that satisfies $u_n(x - x_n) \rightharpoonup u_0(x)$ weakly in $\dot{H}_x^1(\mathbb{R}^3)$ for some sequence $\{x_n\}_{n\geq 1} \subseteq \mathbb{R}^3$. Then $u_0 \in \mathcal{E}$. Moreover, if $\gamma \geq \frac{2}{3}$, then*

$$E(u_0) \leq \liminf_{n\to\infty} E(u_n). \tag{4-2}$$

If $\gamma \in \left(0, \frac{2}{3}\right)$ and $\|\nabla \operatorname{Re} u_n\|_{L_x^2}^2 \leq \delta_\gamma$, then $\|\nabla \operatorname{Re} u_0\|_{L_x^2}^2 \leq \delta_\gamma$ and (4-2) *holds. Here δ_γ is as in Theorem 3.3.*

Proof. Without loss of generality, we may assume that $x_n \equiv 0$.

To prove that $u_0 \in \mathcal{E}$, it suffices to show that $q(u_0) \in L_x^2$. As $u_n \rightharpoonup u_0$ weakly in $\dot{H}_x^1(\mathbb{R}^3)$, invoking Rellich–Kondrashov and passing to a subsequence, we deduce that $u_n \to u_0$ in $L_x^p(K)$ for any $2 \leq p < 6$ and any compact set $K \subset \mathbb{R}^3$. Therefore, for any ball $B \subset \mathbb{R}^3$,

$$\int_B |q(u_0(x))|^2\, dx = \lim_{n\to\infty} \int_B |q(u_n(x))|^2\, dx \leq \liminf_{n\to\infty} \int_{\mathbb{R}^3} |q(u_n(x))|^2\, dx < \infty.$$

As the bound does not depend on B, this proves $q(u_0) \in L_x^2$.

Proceeding similarly and using (weak) lower semicontinuity of the $\dot H^1_x$- and L^6_x-norms, we obtain

$$\int_B \tfrac{1}{2}|\nabla u_0|^2 + \tfrac{1}{4}\gamma q(u_0)^2 + \tfrac{1}{6}q(u_0)^3\,dx \le \liminf_{n\to\infty}\int_B \tfrac{1}{2}|\nabla u_n|^2 + \tfrac{1}{4}\gamma q(u_n)^2 + \tfrac{1}{6}q(u_n)^3\,dx$$

for any ball B. It is crucial here that the sextic term in the energy appears with a positive coefficient.

When $\gamma \in \left[\frac{2}{3}, 1\right)$, the energy density is positive and so the right-hand side above is majorized by $\liminf E(u_n)$. When $\gamma \in \left(0, \frac{2}{3}\right)$, we use instead (3-1) to reach the same conclusion. As $u_0 \in \mathcal{E}$, the dominated convergence theorem yields (4-2). □

We next prove a linear profile decomposition adapted to (1-12) for $\dot H^1_x$-bounded sequences. Beginning with the profile decomposition for the linear Schrödinger equation, we group the profiles according to the behavior of their associated parameters. We also show that the error term vanishes in the limit under propagation by $V^{-1}e^{-itH}V$ (in addition to propagation by $e^{it\Delta}$).

Proposition 4.3 (linear profile decomposition). *Suppose $\{f_n\}_{n\ge1}$ is a bounded sequence in $\dot H^1_x(\mathbb{R}^3)$ and let $T>0$. Passing to a subsequence, there exists $J^* \in \{0,1,2,\dots\}\cup\{\infty\}$ and for each finite $1\le j\le J^*$ there exist a nonzero profile $\phi^j \in \dot H^1_x(\mathbb{R}^3)$, scales $\{\lambda^j_n\}_{n\ge1}\subset(0,\infty)$, and positions $\{(t^j_n, x^j_n)\}_{n\ge1}\subset\mathbb{R}\times\mathbb{R}^3$ conforming to one of the following two scenarios*:

- $\lambda^j_n \equiv 1$ *and* $t^j_n \equiv 0$,
- $\lambda^j_n \to 0$ *as* $n\to\infty$ *and either* $t^j_n\equiv 0$ *or* $t^j_n(\lambda^j_n)^{-2}\to\pm\infty$ *as* $n\to\infty$,

so that for any finite $0\le J\le J^*$ we have the decomposition

$$f_n(x) = \sum_{j=1}^{J} e^{-it^j_n\Delta}\left[(\lambda^j_n)^{-\frac{1}{2}}\phi^j\left(\frac{x-x^j_n}{\lambda^j_n}\right)\right] + w^J_n(x)$$

satisfying the following properties:

$$(\lambda^j_n)^{\frac{1}{2}}\big(e^{it^j_n\Delta}f_n\big)(\lambda^j_n x + x^j_n) \rightharpoonup \phi^j \quad \textit{weakly in } \dot H^1_x, \tag{4-3}$$

$$\lim_{J\to J^*}\limsup_{n\to\infty}\big[\|V^{-1}e^{-itH}Vw^J_n\|_{L^{10}_{t,x}([-T,T]\times\mathbb{R}^3)} + \|e^{it\Delta}w^J_n\|_{L^{10}_{t,x}([-T,T]\times\mathbb{R}^3)}\big] = 0, \tag{4-4}$$

$$\sup_J \limsup_{n\to\infty}\left[\|f_n\|^2_{\dot H^1_x} - \sum_{j=1}^{J}\|\phi^j\|^2_{\dot H^1_x} - \|w^J_n\|^2_{\dot H^1_x}\right] = 0, \tag{4-5}$$

$$(\lambda^j_n)^{\frac{1}{2}}(e^{it^j_n\Delta}w^J_n)(\lambda^j_n x + x^j_n) \rightharpoonup 0 \quad \textit{weakly in } \dot H^1_x \textit{ for all } 1\le j\le J, \tag{4-6}$$

$$\lim_{n\to\infty}\frac{\lambda^j_n}{\lambda^l_n} + \frac{\lambda^l_n}{\lambda^j_n} + \frac{|x^j_n - x^l_n|^2}{\lambda^j_n\lambda^l_n} + \frac{|t^j_n - t^l_n|}{\lambda^j_n\lambda^l_n} = \infty \quad \textit{for all } j\ne l. \tag{4-7}$$

Proof. Using the linear profile decomposition for the Schrödinger propagator for bounded sequences in $\dot H^1_x$ (see, for example, [Keraani 2001] or [Visan 2014, Theorem 4.1]), we obtain a decomposition

$$f_n(x) = \sum_{j=1}^{J} e^{-it^j_n\Delta}\left[(\lambda^j_n)^{-\frac{1}{2}}\phi^j\left(\frac{x-x^j_n}{\lambda^j_n}\right)\right] + r^J_n(x) \tag{4-8}$$

satisfying (4-3), (4-5), (4-6), and (4-7) (with w_n^J replaced by r_n^J), as well as

$$\lim_{J\to J^*}\limsup_{n\to\infty}\|e^{it\Delta}r_n^J\|_{L^{10}_{t,x}([-T,T]\times\mathbb{R}^3)}=0. \tag{4-9}$$

We will first show that we may assume the parameters conform to the two scenarios described above; in particular, we will show that we may absorb any other bubbles of concentration into the error r_n^J, while maintaining condition (4-9). To complete the proof of the proposition, we will show that condition (4-9) (for the new error term) suffices to prove (4-4). Note that it is essential in what follows that we work on a compact time interval.

We will use the notation

$$\phi_n^j(x):=e^{-it_n^j\Delta}\Big[(\lambda_n^j)^{-\frac12}\phi^j\Big(\frac{x-x_n^j}{\lambda_n^j}\Big)\Big].$$

We begin with the following lemma.

Lemma 4.4. *If* $|t_n^j|+\lambda_n^j\to\infty$ *as* $n\to\infty$, *then*

$$\lim_{n\to\infty}\|e^{it\Delta}\phi_n^j\|_{L^{10}_{t,x}([-T,T]\times\mathbb{R}^3)}=0.$$

Proof. A direct computation gives

$$\|e^{it\Delta}\phi_n^j\|_{L^{10}_{t,x}([-T,T]\times\mathbb{R}^3)}=\|e^{it\Delta}\phi^j\|_{L^{10}_{t,x}(I\times\mathbb{R}^3)},$$

where

$$I=\Big[\frac{-t_n^j-T}{(\lambda_n^j)^2},\frac{-t_n^j+T}{(\lambda_n^j)^2}\Big].$$

If $\lambda_n^j\to\infty$, then the lengths of the time intervals appearing on the right-hand side of the equality above shrink to zero; consequently, by the dominated convergence theorem combined with the Strichartz inequality, we deduce the claim.

Passing to a subsequence, we may henceforth assume that $\lambda_n^j\to\lambda^j\in[0,\infty)$. In this case, we have $|t_n^j|\to\infty$, and so the time intervals escape to infinity. Thus the claim follows once again from the dominated convergence theorem combined with the Strichartz inequality. □

Discarding the bubbles of concentration whose parameters satisfy the hypotheses of Lemma 4.4, we can now see that we may reduce attention to the two scenarios described in Proposition 4.3. Indeed, passing to a subsequence, we may assume that $\lambda_n^j\to\lambda^j\in[0,\infty)$ and $t_n^j\to t^j\in(-\infty,\infty)$. If $\lambda^j\neq0$, then we may assume that $\lambda_n^j\equiv1$ and $t_n^j\equiv0$ by redefining the corresponding profile to be $(\lambda^j)^{-\frac12}e^{-it^j\Delta}[\phi^j(\cdot/\lambda^j)]$. The error incurred by this modification can be absorbed into r_n^J; indeed, we have

$$\begin{aligned}\Big\|\phi_n^j-(\lambda^j)^{-\frac12}e^{-it^j\Delta}\Big[\phi^j\Big(\frac{x-x_n^j}{\lambda^j}\Big)\Big]\Big\|_{\dot H^1_x}\qquad\qquad&\\ \le\Big\|(\lambda_n^j)^{-\frac12}\phi^j\Big(\frac{x}{\lambda_n^j}\Big)-(\lambda^j)^{-\frac12}\phi^j\Big(\frac{x}{\lambda^j}\Big)\Big\|_{\dot H^1_x}+\Big\|(e^{-it_n^j\Delta}-e^{-it^j\Delta})\Big[(\lambda^j)^{-\frac12}\phi^j\Big(\frac{x}{\lambda^j}\Big)\Big]\Big\|_{\dot H^1_x},&\end{aligned}$$

which tends to zero as $n \to \infty$ by the strong convergence of the linear Schrödinger propagator. If instead $\lambda^j = 0$, then passing to a further subsequence we may assume that either $t_n^j \equiv 0$ or $t_n^j (\lambda_n^j)^{-2} \to \pm\infty$ as $n \to \infty$. Indeed, if there is a subsequence along which $t_n^j (\lambda_n^j)^{-2} \to \tau \in (-\infty, \infty)$, then we redefine the profile to be $e^{-i\tau\Delta}\phi^j$ and $t_n^j \equiv 0$. It is easy to see that the resulting error can be absorbed into r_n^J.

It remains to prove that the new error w_n^J (which consists of r_n^J plus the bubbles of concentration whose parameters satisfy the hypotheses of Lemma 4.4) obeys (4-4). This is a consequence of the following: if

$$\lim_{J \to J^*} \limsup_{n \to \infty} \|e^{it\Delta} w_n^J\|_{L_{t,x}^{10}([-T,T]\times\mathbb{R}^3)} = 0,$$

then

$$\lim_{J \to J^*} \limsup_{n \to \infty} \| V^{-1} e^{-itH} V w_n^J \|_{L_{t,x}^{10}([-T,T]\times\mathbb{R}^3)} = 0.$$

To prove this final implication, we argue as follows: In view of the representation (2-4) and the boundedness of U and $P_{\mathrm{hi}} U^{-1}$, it suffices to verify that $e^{\mp itH} e^{\mp it\Delta}$ and $P_{\mathrm{lo}} U^{-1} \sin(tH) e^{-it\Delta}$ are Mikhlin multipliers with bounds that are uniform for $t \in [-T, T]$. In the former case, this follows from (2-7); with regard to the latter, see (2-6).

This completes the proof of Proposition 4.3. □

In the proof of Theorem 4.1, we will construct solutions to (1-6) associated to each ϕ_n^j. For profiles conforming to the first scenario in Proposition 4.3, we can achieve this by an application of Lemma 4.2 and Theorem 3.3. For profiles conforming to the second scenario, this is a more difficult problem, which we address in the following proposition.

Proposition 4.5 (highly concentrated nonlinear profiles). *Let $\phi \in \dot{H}_x^1(\mathbb{R}^3)$ and $T > 0$. Assume $\{\lambda_n\}_{n\geq 1} \subset (0, \infty)$ and $\{(t_n, x_n)\}_{n\geq 1} \subset \mathbb{R} \times \mathbb{R}^3$ satisfy $\lambda_n \to 0$ and either $t_n \equiv 0$ or $t_n \lambda_n^{-2} \to \pm\infty$. Then for n sufficiently large, there exists a solution u_n to* (1-6) *with initial data*

$$u_n(0, x) = \phi_n(x) := e^{-it_n\Delta}\left[\lambda_n^{-\frac{1}{2}} \phi\left(\frac{x - x_n}{\lambda_n}\right)\right]$$

satisfying

$$\|u_n\|_{\dot{S}^1([-T,T])} \leq C(\|\phi\|_{\dot{H}_x^1}). \tag{4-10}$$

Moreover, for all $\varepsilon > 0$ there exist $\phi_\varepsilon, \psi_\varepsilon \in C_c^\infty([-T, T] \times \mathbb{R}^3)$ such that

$$\limsup_{n\to\infty} \left\| u_n(t, x) - e^{-i\gamma t} \lambda_n^{-\frac{1}{2}} \phi_\varepsilon\left(\frac{t - t_n}{\lambda_n^2}, \frac{x - x_n}{\lambda_n}\right) \right\|_{L_{t,x}^{10}([-T,T]\times\mathbb{R}^3)} \leq \varepsilon, \tag{4-11}$$

$$\limsup_{n\to\infty} \left\| \nabla u_n(t, x) - e^{-i\gamma t} \lambda_n^{-\frac{3}{2}} \psi_\varepsilon\left(\frac{t - t_n}{\lambda_n^2}, \frac{x - x_n}{\lambda_n}\right) \right\|_{L_{t,x}^{\frac{10}{3}}([-T,T]\times\mathbb{R}^3)} \leq \varepsilon. \tag{4-12}$$

Proof. As (1-6) is space-translation invariant, without loss of generality we may assume that $x_n \equiv 0$.

We proceed via a perturbative argument. Specifically, using a solution to the defocusing energy-critical NLS, we will construct an approximate solution $\tilde{u}_n$ to (1-6) with initial data asymptotically matching ϕ_n. This approximate solution will have good space-time bounds inherited from the solution to the defocusing energy-critical NLS. Using the stability result Proposition 3.5, we will then deduce that for n sufficiently large, there exist true solutions u_n to (1-6) with $u_n(0) = \phi_n$ that inherits the space-time bounds of $\tilde{u}_n$, thus proving (4-10). We turn to the details.

If $t_n \equiv 0$, let v be the solution to the defocusing energy-critical NLS

$$(i\partial_t + \Delta)v = |v|^4 v \tag{4-13}$$

with initial data $v(0) = \phi$. If $t_n\lambda_n^{-2} \to \pm\infty$, let v be the solution to (4-13) which scatters in $\dot{H}_x^1$ to $e^{it\Delta}\phi$ as $t \to \pm\infty$. By the main result in [Colliander et al. 2008], we have

$$\|v\|_{\dot{S}^1(\mathbb{R})} \le C(\|\phi\|_{\dot{H}_x^1}).$$

We are now in a position to introduce the approximate solutions $\tilde{u}_n$ to (1-6). For $n \ge 1$, we define

$$\tilde{u}_n(t, x) := e^{-i\gamma t}\lambda_n^{-\frac{1}{2}} v\left(\frac{t - t_n}{\lambda_n^2}, \frac{x}{\lambda_n}\right).$$

The phase factor $e^{-i\gamma t}$ is necessary. It replaces the linear factor in (1-6) by a nonresonant term; see (4-15).

Note that

$$\|\tilde{u}_n\|_{\dot{S}^1(\mathbb{R})} = \|v\|_{\dot{S}^1(\mathbb{R})} \le C(\|\phi\|_{\dot{H}_x^1}). \tag{4-14}$$

Moreover, $\tilde{u}_n(0)$ asymptotically matches the initial data $u_n(0) = \phi_n$; indeed, by construction, we have

$$\|\tilde{u}_n(0) - \phi_n\|_{\dot{H}_x^1} = \left\| v\left(-\frac{t_n}{\lambda_n^2}\right) - e^{-i(t_n/\lambda_n^2)\Delta}\phi \right\|_{\dot{H}_x^1} \to 0 \quad \text{as } n \to \infty.$$

To invoke the stability result Proposition 3.5 and deduce claim (4-10), it remains to show that $\tilde{u}_n$ is an approximate solution to (1-6) on the interval $[-T, T]$ as $n \to \infty$. A computation yields

$$e_n := (i\partial_t + \Delta - 2\gamma\,\mathrm{Re})\tilde{u}_n - N(\tilde{u}_n) = -\gamma\overline{\tilde{u}_n} - \sum_{j=2}^{4} N_j(\tilde{u}_n). \tag{4-15}$$

To establish (4-10), we have to verify that the error e_n satisfies the smallness condition in (3-7) for n sufficiently large.

Let $\delta > 0$ to be chosen later. There exist $T_1, T_2 > 0$ sufficiently large so that

$$\|v\|_{L_{t,x}^{10}(\{|t|>T_1\}\times\mathbb{R}^3)} < \delta, \tag{4-16}$$

$$\|v(t) - e^{it\Delta}v_\pm\|_{\dot{H}_x^1} < \delta \quad \text{for } \pm t > T_2, \tag{4-17}$$

where $v_\pm$ denote the asymptotic states for the solution v. Note that the existence of $v_\pm$ is a consequence of the global space-time bounds for v, as discussed in [Colliander et al. 2008].

We first estimate the contribution of the higher-order terms appearing in e_n on the space-time slab $[-T, T] \times \mathbb{R}^3$. Defining

$$I_n = \{|t - t_n| \le \lambda_n^2 T_1\} \cap [-T, T] \quad \text{and} \quad I_n^c = \{|t - t_n| > \lambda_n^2 T_1\} \cap [-T, T],$$

we use Strichartz, Hölder, (4-14) and (4-16) to obtain

$$\begin{aligned}
&\Big\| \int_0^t e^{i(t-s)\Delta} \nabla \sum_{k=2}^4 N_k(\tilde{u}_n)(s)\,ds \Big\|_{L_t^\infty L_x^2 \cap L_t^{10} L_x^{30/13}} \\
&\qquad \lesssim \|\nabla N_2(\tilde{u}_n)\|_{L_t^{20/19} L_x^{30/16}} + \|\nabla N_3(\tilde{u}_n)\|_{L_t^{5/4} L_x^{30/19}} + \|\nabla N_4(\tilde{u}_n)\|_{L_t^{20/13} L_x^{30/22}} \\
&\qquad \lesssim \|\nabla \tilde{u}_n\|_{L_t^{10} L_x^{30/13}} \big\{ \|\tilde{u}_n\|_{L_t^{20/17} L_x^{10}(I_n \times \mathbb{R}^3)} + \|\tilde{u}_n\|_{L_t^{20/17} L_x^{10}(I_n^c \times \mathbb{R}^3)} \big\} \\
&\qquad\quad + \|\nabla \tilde{u}_n\|_{L_t^{10} L_x^{30/13}} \|\tilde{u}_n\|_{L_{t,x}^{10}} \big\{ \|\tilde{u}_n\|_{L_t^{5/3} L_x^{10}(I_n \times \mathbb{R}^3)} + \|\tilde{u}_n\|_{L_t^{5/3} L_x^{10}(I_n^c \times \mathbb{R}^3)} \big\} \\
&\qquad\quad + \|\nabla \tilde{u}_n\|_{L_t^{10} L_x^{30/13}} \|\tilde{u}_n\|_{L_{t,x}^{10}}^2 \big\{ \|\tilde{u}_n\|_{L_t^{20/7} L_x^{10}(I_n \times \mathbb{R}^3)} + \|\tilde{u}_n\|_{L_t^{20/7} L_x^{10}(I_n^c \times \mathbb{R}^3)} \big\} \\
&\qquad \lesssim_{\|\phi\|_{\dot{H}_x^1}} \sum_{k=2}^4 \big\{ (\lambda_n^2 T_1)^{\frac{5-k}{4}} + T^{\frac{5-k}{4}} \delta \big\}.
\end{aligned}$$

Taking δ sufficiently small depending on T and n sufficiently large, we see this contribution is acceptable.

Next we consider the contribution of the linear term appearing in e_n, again on the space-time slab $[-T,T] \times \mathbb{R}^3$. First, we observe that by Strichartz and (4-14), we have

$$\Big\| \int_0^t e^{i(t-s)\Delta} \overline{\tilde{u}_n(s)}\,ds \Big\|_{L_t^2 \dot{H}_x^{1,6}([-T,T]\times\mathbb{R}^3)} \lesssim \|\tilde{u}_n\|_{L_t^1 \dot{H}_x^1([-T,T]\times\mathbb{R}^3)} \lesssim_{\|\phi\|_{\dot{H}_x^1}} T. \tag{4-18}$$

To continue, using (4-17) we cover $\mathbb{R}$ by three disjoint intervals I_n^0 and $I_n^\pm$ such that

$$|I_n^0| \le 2\lambda_n^2 T_2 \quad \text{and} \quad \Big\| \tilde{u}_n - e^{-i\gamma t} e^{i(t-t_n)\Delta} \Big[\lambda_n^{-\frac12} v_\pm\Big(\frac{\cdot}{\lambda_n}\Big) \Big] \Big\|_{L_t^\infty \dot{H}_x^1(I_n^\pm \times \mathbb{R}^3)} < \delta. \tag{4-19}$$

By Strichartz, Hölder, (4-14), and (4-19), we have

$$\Big\| \int_0^t e^{i(t-s)\Delta} \chi_{I_n^0}(s) \overline{\tilde{u}_n(s)}\,ds \Big\|_{L_t^\infty \dot{H}_x^1([-T,T]\times\mathbb{R}^3)} \lesssim \|\tilde{u}_n\|_{L_t^1 \dot{H}_x^1(I_n^0 \times\mathbb{R}^3)} \lesssim_{\|\phi\|_{\dot{H}_x^1}} \lambda_n^2 T_2. \tag{4-20}$$

Using the triangle inequality, Strichartz, and (4-19),

$$\begin{aligned}
&\Big\| \int_0^t e^{i(t-s)\Delta} \chi_{I_n^\pm}(s) \overline{\tilde{u}_n(s)}\,ds \Big\|_{L_t^\infty \dot{H}_x^1([-T,T]\times\mathbb{R}^3)} \\
&\qquad \lesssim T\delta + \Big\| \int_0^t e^{i(t+t_n-2s)\Delta} \chi_{I_n^\pm}(s) e^{i\gamma s} \lambda_n^{-\frac12} \overline{v_\pm}\Big(\frac{\cdot}{\lambda_n}\Big)\,ds \Big\|_{L_t^\infty \dot{H}_x^1([-T,T]\times\mathbb{R}^3)}.
\end{aligned} \tag{4-21}$$

Now for any $-T \le a < b \le T$, an application of Plancherel gives

$$\begin{aligned}
\Big\| \int_a^b e^{is(\gamma-2\Delta)} \lambda_n^{-\frac12} \overline{v_\pm}\Big(\frac{\cdot}{\lambda_n}\Big)\,ds \Big\|_{\dot{H}_x^1} &= \Big\| \int_a^b e^{is(\gamma+2|\xi|^2)} |\xi| \lambda_n^{\frac52} \widehat{\overline{v_\pm}}(\xi\lambda_n)\,ds \Big\|_{L_\xi^2} \\
&\lesssim \big\| (\gamma+2|\xi|^2)^{-1} |\xi| \lambda_n^{\frac52} \widehat{\overline{v_\pm}}(\xi\lambda_n) \big\|_{L_\xi^2} \\
&\lesssim \Big\| \frac{\lambda_n^2}{2|\xi|^2 + \gamma\lambda_n^2} |\xi| \widehat{\overline{v_\pm}}(\xi) \Big\|_{L_\xi^2},
\end{aligned} \tag{4-22}$$

which converges to zero as $n \to \infty$ by the dominated convergence theorem. Collecting (4-20), (4-21), and (4-22), we obtain that

$$\left\| \int_0^t e^{i(t-s)\Delta} \overline{\tilde{u}_n(s)}\, ds \right\|_{L_t^\infty \dot H_x^1([-T,T]\times\mathbb{R}^3)} \lesssim_{\|\phi\|_{\dot H_x^1}} \lambda_n^2 T_2 + T\delta + o(1) \quad \text{as } n \to \infty.$$

Interpolating with (4-18) and taking δ sufficiently small depending on T and taking n sufficiently large, we see that the contribution of the linear term in e_n is also acceptable. This completes the proof of (4-10).

Finally, we turn to (4-11) and (4-12). For $\varepsilon > 0$, we approximate v by $\phi_\varepsilon, \psi_\varepsilon \in C_c^\infty(\mathbb{R}\times\mathbb{R}^3)$ such that

$$\|v - \phi_\varepsilon\|_{L_{t,x}^{10}(\mathbb{R}\times\mathbb{R}^3)} < \tfrac{1}{2}\varepsilon \quad \text{and} \quad \|\nabla v - \psi_\varepsilon\|_{L_{t,x}^{10/3}(\mathbb{R}\times\mathbb{R}^3)} < \tfrac{1}{2}\varepsilon$$

and take n sufficiently large so that

$$\|u_n - \tilde{u}_n\|_{L_{t,x}^{10} \cap L_t^{10/3} \dot H_x^{1,10/3}([-T,T]\times\mathbb{R}^3)} < \tfrac{1}{2}\varepsilon.$$

The two claims now follow easily from the triangle inequality. □

Finally we turn to the proof of Theorem 4.1.

Proof of Theorem 4.1. As mentioned above, by Lemma 4.2 and Theorem 3.3 we have that u and all of the u_n are global-in-time solutions to (1-6).

Fix $T > 0$. We will show that for any subsequence in n there exists a further subsequence so that along that subsequence, $u_n(t) \rightharpoonup u(t)$ weakly in $\dot H_x^1$ for all $t \in [-T, T]$. As the limit is independent of the original subsequence, this will prove the theorem.

Given a subsequence in n, we apply Proposition 4.3 to $u_n(0) - u_0$ and pass to a further subsequence to obtain the decomposition

$$u_n(0) - u_0 = \sum_{j=1}^J \phi_n^j + w_n^J \quad \text{with } \phi_n^j(x) := e^{-it_n^j\Delta}\left[(\lambda_n^j)^{-\frac{1}{2}} \phi^j\left(\frac{x - x_n^j}{\lambda_n^j}\right)\right],$$

which satisfies the conclusions of that proposition. By hypothesis, $u_n(0) - u_0 \rightharpoonup 0$ weakly in $\dot H_x^1$; using also (4-6) and (4-7), this implies that for all $j \geq 1$ we must have

$$w_n^J \rightharpoonup 0 \quad \text{weakly in } \dot H_x^1 \qquad \text{and} \qquad (\lambda_n^j)^{-1} + |t_n^j| + |x_n^j| \to \infty \quad \text{as } n \to \infty. \tag{4-23}$$

Indeed, one can first prove the divergence of the parameters by a contradiction argument. Briefly, if some $(\lambda_n^j)^{-1} + |t_n^j| + |x_n^j|$ were to remain bounded as $n \to \infty$ then one could use (4-6) and (4-7) to deduce that $\phi^j = 0$, a contradiction. Once the divergence of the parameters is established, the weak convergence of w_n^J to zero then follows.

Throughout the proof we write

$$\phi_n^0(x) := e^{-it_n^0\Delta}\left[(\lambda_n^0)^{-\frac{1}{2}} u_0\left(\frac{x - x_n^0}{\lambda_n^0}\right)\right] \quad \text{with parameters } \lambda_n^0 \equiv 1,\ t_n^0 \equiv 0,\ x_n^0 \equiv 0.$$

In view of (4-23), the decomposition

$$u_n(0) = \sum_{j=0}^{J} \phi_n^j + w_n^J$$

satisfies the conclusions of Proposition 4.3.

We next construct nonlinear profiles associated to each ϕ_n^j. If j conforms to the first scenario described in Proposition 4.3, then (4-3) and Lemma 4.2 guarantee that $\phi_n^j \in \mathcal{E}$ and moreover, $\|\nabla \operatorname{Re} \phi_n^j\|_{L_x^2} \le \delta_\gamma$ and $E(\phi_n^j) \le \frac{1}{4}\delta_\gamma$ if $\gamma \in \left(0, \frac{2}{3}\right)$. Thus by Theorem 3.3 there exists a unique solution u_n^j to (1-6) with data $u_n^j(0) = \phi_n^j$; in particular, $\|u_n^j\|_{\dot{S}^1([-T,T])} < \infty$. Note that u_n^0 is simply the solution u from the statement of Theorem 4.1.

If j conforms to the second scenario described in Proposition 4.3, we let u_n^j denote the solution to (1-6) with data $u_n^j(0) = \phi_n^j$ constructed in Proposition 4.5.

In either scenario, for all $\varepsilon > 0$ there exists $\phi_\varepsilon^j, \psi_\varepsilon^j \in C_c^\infty([-T,T] \times \mathbb{R}^3)$ such that

$$\limsup_{n\to\infty} \left\| u_n^j(t,x) - e^{-i\gamma t} (\lambda_n^j)^{-\frac{1}{2}} \phi_\varepsilon^j \left(\frac{t - t_n^j}{(\lambda_n^j)^2}, \frac{x - x_n^j}{\lambda_n^j} \right) \right\|_{L_{t,x}^{10}([-T,T]\times\mathbb{R}^3)} \le \varepsilon, \tag{4-24}$$

$$\limsup_{n\to\infty} \left\| \nabla u_n^j(t,x) - e^{-i\gamma t} (\lambda_n^j)^{-\frac{3}{2}} \psi_\varepsilon^j \left(\frac{t - t_n^j}{(\lambda_n^j)^2}, \frac{x - x_n^j}{\lambda_n^j} \right) \right\|_{L_{t,x}^{10/3}([-T,T]\times\mathbb{R}^3)} \le \varepsilon. \tag{4-25}$$

Note that the phase $e^{-i\gamma t}$ has no significance for j conforming to the first scenario described in Proposition 4.3; we simply incorporate it so as to treat both cases uniformly. For these j, both ϕ_ε^j and ψ_n^j are chosen to approximate $e^{i\gamma t} u_n^j$.

As a consequence of (4-24), (4-25), and the asymptotic orthogonality of parameters given by (4-7), for all $j \ne l$ we have

$$\|u_n^j u_n^l\|_{L_{t,x}^5} + \|u_n^j \nabla u_n^l\|_{L_t^5 L_x^{15/8}} + \|\nabla u_n^j \nabla u_n^l\|_{L_t^5 L_x^{15/13}} \to 0, \tag{4-26}$$

where all space-time norms are over $[-T,T] \times \mathbb{R}^3$.

We next claim that for $j \ge 1$ we have

$$u_n^j(t) \rightharpoonup 0 \quad \text{weakly in } \dot{H}_x^1(\mathbb{R}^3) \text{ as } n \to \infty \text{ for every } t \in [-T,T]. \tag{4-27}$$

Indeed, if j conforms to the first scenario, then (4-23) implies that $|x_n^j| \to \infty$ and hence (4-27) follows from the space-translation invariance of (1-6) together with uniqueness. If j conforms to the second scenario, then we have $\lambda_n^j \to 0$; however, as (1-6) is not scale invariant, the argument just described does not apply directly. For this case, we recall that according to the construction in Proposition 4.5, u_n^j are asymptotically close in $L_t^\infty \dot{H}_x^1$ (up to a phase factor) to rescaled solutions to the defocusing energy-critical NLS as $n \to \infty$. Using the scaling symmetry and uniqueness for (4-13), we see that these rescaled solutions converge weakly to 0 in $\dot{H}_x^1$ at each time; by construction, u_n^j inherit this property.

To continue, we define

$$u_n^J(t) = \sum_{j=0}^{J} u_n^j(t) + V^{-1} e^{-itH} V w_n^J.$$

Note that $u_n^J(0) = u_n(0)$. In what follows we will prove that for n and J sufficiently large, u_n^J is an approximate solution to (1-6) with uniform space-time bounds on $[-T, T] \times \mathbb{R}^3$. An application of Proposition 3.5 will then yield that for any $\varepsilon > 0$ there exist n and J sufficiently large so that

$$\|u_n - u_n^J\|_{L_t^\infty \dot{H}_x^1([-T,T]\times\mathbb{R}^3)} \leq \varepsilon.$$

On the other hand, using (4-23) and (4-27) and recalling $u = u_n^0$, we see that for J fixed, $u_n^J(t) - u(t) \rightharpoonup 0$ weakly in $\dot{H}_x^1$ for all $t \in [-T, T]$. Thus by the triangle inequality, for any $\varphi \in C_c^\infty(\mathbb{R}^3)$, we have

$$\begin{aligned}\bigl|\langle u_n(t) - u(t), \varphi\rangle\bigr| &\leq \bigl|\langle u_n(t) - u_n^J(t), \varphi\rangle\bigr| + \bigl|\langle u_n^J(t) - u(t), \varphi\rangle\bigr| \\ &\leq \|u_n(t) - u_n^J(t)\|_{\dot{H}_x^1} \|\varphi\|_{\dot{H}_x^{-1}} + \bigl|\langle u_n^J(t) - u(t), \varphi\rangle\bigr| \\ &\lesssim_\varphi \varepsilon + o(1) \quad \text{as } n \to \infty,\end{aligned}$$

which proves the claim in Theorem 4.1.

Thus it remains to show that for n and J sufficiently large, u_n^J are approximate solutions to (1-6) with uniform space-time bounds on $[-T, T] \times \mathbb{R}^3$.

Our first step in this direction is the following lemma.

Lemma 4.6 (finite space-time bounds). *Given $T > 0$, we have*

$$\sup_J \limsup_{n\to\infty} \bigl[\|u_n^J\|_{L_{t,x}^{10}([-T,T]\times\mathbb{R}^3)} + \|\nabla u_n^J\|_{L_t^{10} L_x^{30/10}([-T,T]\times\mathbb{R}^3)}\bigr] \lesssim 1. \tag{4-28}$$

Moreover, for any $\eta > 0$ there exists $J' = J'(\eta)$ sufficiently large so that

$$\limsup_{n\to\infty} \biggl[\biggl\|\sum_{j=J'}^{J} u_n^j\biggr\|_{L_{t,x}^{10}([-T,T]\times\mathbb{R}^3)} + \biggl\|\sum_{j=J'}^{J} \nabla u_n^j\biggr\|_{L_t^{10} L_x^{30/10}([-T,T]\times\mathbb{R}^3)}\biggr] \leq \eta \tag{4-29}$$

uniformly in $J \geq J'$.

Proof. By the asymptotic decoupling of the $\dot{H}_x^1$-norm in (4-5), there exists $J_0 = J_0(T)$ such that for all $j \geq J_0$ we have $\|\phi^j\|_{\dot{H}_x^1} \leq \eta(T)$, where $\eta(T)$ is as in Corollary 3.6. In particular,

$$\|u_n^j\|_{\dot{S}^1([-T,T]} \lesssim_T \|\phi^j\|_{\dot{H}_x^1} \quad \text{for all } j \geq J_0. \tag{4-30}$$

On the space-time slab $[-T, T] \times \mathbb{R}^3$ we use Lemma 2.6 to estimate

$$\begin{aligned}\|u_n^J\|_{L_{t,x}^{10}}^2 &\lesssim \|V^{-1} e^{-itH} V w_n^J\|_{L_{t,x}^{10}}^2 + \biggl\|\biggl(\sum_{j=0}^{J} u_n^j\biggr)^2\biggr\|_{L_{t,x}^5} \\ &\lesssim_T \|w_n^J\|_{\dot{H}_x^1}^2 + \sum_{j=0}^{J} \|u_n^j\|_{L_{t,x}^{10}}^2 + \sum_{j\neq l} \|u_n^j u_n^l\|_{L_{t,x}^5}.\end{aligned}$$

This suffices to show that the first term on the left-hand side of (4-28) is finite. Indeed, we use (4-5) and (4-30) to bound the first two summands and (4-26) to bound the last (double) sum. An analogous argument yields that the second term on the left-hand side of (4-28) is also bounded.

To prove (4-29) one argues as above, taking $J' \geq J_0$ large enough that

$$\sum_{j \geq J'} \|\phi^j\|_{\dot{H}^1_x}^2 \lesssim \eta.$$

Note that this is possible because of (4-5). □

We next prove that the u_n^J are indeed approximate solutions to (1-6).

Lemma 4.7 (asymptotic solution to (1-6)). *We have*

$$\lim_{J \to J^*} \limsup_{n \to \infty} \big\| \nabla\big[(i\partial_t + \Delta - 2\gamma \operatorname{Re})u_n^J - N(u_n^J)\big]\big\|_{\dot{N}^0([-T,T])} = 0.$$

Proof. Throughout the proof of the lemma, all space-time norms will be over $[-T, T] \times \mathbb{R}^3$. Writing $\tilde{w}_n^J := V^{-1} e^{-itH} V w_n^J$, we have

$$\begin{aligned} e_n^J &:= (i\partial_t + \Delta - 2\gamma \operatorname{Re})u_n^J - N(u_n^J) \\ &= \sum_{j=0}^{J} N(u_n^j) - N\Big(\sum_{j=0}^{J} u_n^j\Big) + N\big(u_n^J - \tilde{w}_n^J\big) - N(u_n^J). \end{aligned}$$

Computations similar to those employed in the proof of Proposition 3.5 yield

$$\Big\| \nabla\Big[\sum_{j=0}^{J} N(u_n^j) - N\Big(\sum_{j=0}^{J} u_n^j\Big)\Big]\Big\|_{\dot{N}^0} \lesssim \sum_{k=2}^{5} \sum_{j \neq l} \sum_{m=0}^{J} T^{\frac{5-k}{4}} \|u_n^l \nabla u_n^j\|_{L_t^5 L_x^{15/8}} \|u_n^m\|_{L_{t,x}^{10}}^{k-2},$$

which converges to zero as $n \to \infty$ in view of (4-26) and (4-30).

Thus, it remains to show that

$$\lim_{J \to J^*} \limsup_{n \to \infty} \big\| \nabla\big[N(u_n^J - \tilde{w}_n^J) - N(u_n^J)\big]\big\|_{\dot{N}^0([-T,T])} = 0. \tag{4-31}$$

We argue as follows: First, we estimate

$$\begin{aligned} &\big\| \nabla\big[N(u_n^J - \tilde{w}_n^J) - N(u_n^J)\big]\big\|_{\dot{N}^0([-T,T])} \\ &\quad \lesssim \|\nabla u_n^J\|_{L_t^{10} L_x^{30/13}} \|\tilde{w}_n^J\|_{L_{t,x}^{10}} \sum_{k=2}^{5} T^{\frac{5-k}{4}} \big(\|u_n^J\|_{L_{t,x}^{10}}^{k-2} + \|\tilde{w}_n^J\|_{L_{t,x}^{10}}^{k-2}\big) \\ &\qquad + \|\nabla \tilde{w}_n^J\|_{L_t^{10} L_x^{30/13}} \sum_{k=2}^{5} T^{\frac{5-k}{4}} \|\tilde{w}_n^J\|_{L_{t,x}^{10}}^{k-1} + \|u_n^J \nabla \tilde{w}_n^J\|_{L_t^5 L_x^{15/8}} \sum_{k=2}^{5} T^{\frac{5-k}{4}} \|u_n^J\|_{L_{t,x}^{10}}^{k-2}. \end{aligned}$$

That the first two summands above go to zero as $n \to \infty$ and then $J \to \infty$ follows from (4-4) and Lemma 4.6. Thus, (4-31) will follow from Lemma 4.6 once we establish

$$\lim_{J \to J^*} \limsup_{n \to \infty} \|u_n^J \nabla \tilde{w}_n^J\|_{L_t^5 L_x^{15/8}([-T,T] \times \mathbb{R}^3)} = 0. \tag{4-32}$$

We will prove that the left-hand side of (4-32) is $\lesssim \eta$ for arbitrary $\eta > 0$. By the definition of u_n^J, the triangle inequality, and Hölder, we estimate

$$\|u_n^J \nabla \tilde{w}_n^J\|_{L_t^5 L_x^{15/8}} \lesssim \|\tilde{w}_n^J\|_{L_{t,x}^{10}} \|\nabla \tilde{w}_n^J\|_{L_t^{10} L_x^{30/13}} + \Big\|\sum_{j=J'}^{J} u_n^j\Big\|_{L_{t,x}^{10}} \|\nabla \tilde{w}_n^J\|_{L_t^{10} L_x^{30/13}} + \Big\|\sum_{j=0}^{J'-1} u_n^j \nabla \tilde{w}_n^J\Big\|_{L_t^5 L_x^{15/8}},$$

where $J' = J'(\eta)$ is as in the statement of Lemma 4.6. Using (4-4) and (4-29), we see that the contribution of the first two summands on the right-hand side of the formula above is acceptable.

It remains to prove that

$$\lim_{J\to J^*} \limsup_{n\to\infty} \|u_n^j \nabla \tilde{w}_n^J\|_{L_t^5 L_x^{15/8}([-T,T]\times\mathbb{R}^3)} = 0 \quad \text{for each } 0 \le j < J'. \tag{4-33}$$

Assume first that $0 \le j < J'$ conforms to the first scenario in Proposition 4.3. Fix $\varepsilon > 0$. Invoking (4-24) and using the triangle inequality, Hölder, interpolation, and Corollary 2.8, we estimate

$$\begin{aligned}
\|u_n^j \nabla \tilde{w}_n^J\|_{L_t^5 L_x^{15/8}} &\le \|u_n^j(t,x) - e^{-i\gamma t}\phi_\varepsilon^j(t, x - x_n^j)\|_{L_{t,x}^{10}} \|\nabla \tilde{w}_n^J\|_{L_t^{10} L_x^{30/13}} + \|\phi_\varepsilon^j \nabla \tilde{w}_n^J(x + x_n^j)\|_{L_t^5 L_x^{15/8}} \\
&\lesssim \varepsilon + \|\phi_\varepsilon^j\|_{L_t^\infty L_x^{12}} \|\nabla \tilde{w}_n^J(x+x_n^j)\|_{L_{t,x}^2(\operatorname{supp}\phi_\varepsilon^j)}^{\frac14} \|\nabla \tilde{w}_n^J\|_{L_t^{10} L_x^{30/13}}^{\frac34} \\
&\lesssim_{\phi_\varepsilon^j} \varepsilon + \|\tilde{w}_n^J\|_{L_{t,x}^{10}}^{\frac1{12}} \|w_n^J\|_{\dot H_x^1}^{\frac16} \|\nabla \tilde{w}_n^J\|_{L_t^{10} L_x^{30/13}}^{\frac34}.
\end{aligned}$$

By (4-4), we see that (4-33) follows in this case.

Now assume that $1 \le j < J'$ conforms to the second scenario in Proposition 4.3. We split $\tilde{w}_n^J$ into low and high frequencies and estimate them separately, starting with the low-frequency piece. Fix $\varepsilon > 0$. Arguing as before, using (4-24), Hölder, and Bernstein, we estimate

$$\begin{aligned}
\|u_n^j P_{\le(\lambda_n^j)^{-1}} \nabla \tilde{w}_n^J\|_{L_t^5 L_x^{15/8}} &\lesssim \varepsilon + \Big\|(\lambda_n^j)^{-\frac12} \phi_\varepsilon^j\Big(\frac{t - t_n^j}{(\lambda_n^j)^2}, \frac{x - x_n^j}{\lambda_n^j}\Big)\Big\|_{L_t^{10} L_x^{30/13}} \|P_{\le(\lambda_n^j)^{-1}} \nabla \tilde{w}_n^J\|_{L_{t,x}^{10}} \\
&\lesssim \varepsilon + \|\phi_\varepsilon^j\|_{L_t^{10} L_x^{30/13}} \|\tilde{w}_n^J\|_{L_{t,x}^{10}}.
\end{aligned}$$

In view of (4-4), this contribution is acceptable.

We now consider the high-frequency piece. Using (2-7) we can deduce

$$\|P_{\ge N} - P_{\ge N} e^{it(\gamma - \Delta)} V^{-1} e^{-itH} V\|_{\dot H_x^1 \to \dot H_x^1} \lesssim_T N^{-2}$$

uniformly for $N \ge 1$ and $t \in [-T, T]$. Thus

$$\begin{aligned}
\|u_n^j \nabla P_{\ge(\lambda_n^j)^{-1}} \tilde{w}_n^J\|_{L_t^5 L_x^{15/8}([-T,T]\times\mathbb{R}^3)} &\lesssim_T \|u_n^j P_{\ge(\lambda_n^j)^{-1}} \nabla e^{it\Delta} w_n^J\|_{L_t^5 L_x^{15/8}([-T,T]\times\mathbb{R}^3)} + (\lambda_n^j)^2 \|u_n^j\|_{L_{t,x}^{10}([-T,T]\times\mathbb{R}^3)} \|w_n^J\|_{\dot H_x^1} \\
&\lesssim_T \varepsilon + \|\phi_\varepsilon^j \nabla e^{it\Delta} f_n\|_{L_t^5 L_x^{15/8}(I_n\times\mathbb{R}^3)} + o(1) \quad \text{as } n \to \infty,
\end{aligned}$$

where

$$f_n(x) = P_{\ge 1} (\lambda_n^j)^{\frac12} w_n^J(\lambda_n^j x + x_n^j) \quad \text{and} \quad I_n = \{|t - t_n^j| \le (\lambda_n^j)^2 T\}.$$

To continue, we estimate in much the same manner as for j conforming to the first scenario:

$$\begin{aligned}\|\phi_\varepsilon^j \nabla e^{it\Delta} f_n\|_{L_t^5 L_x^{15/8}(I_n\times\mathbb{R}^3)} &\lesssim \|\phi_\varepsilon^j\|_{L_t^\infty L_x^{12}} \|\nabla e^{it\Delta} f_n\|^{\frac{1}{4}}_{L^2_{t,x}(\operatorname{supp}\phi_\varepsilon^j\cap I_n\times\mathbb{R}^3)} \|\nabla e^{it\Delta} f_n\|^{\frac{3}{4}}_{L_t^{10}L_x^{30/13}(I_n\times\mathbb{R}^3)}\\ &\lesssim_{\phi_\varepsilon^j} \|e^{it\Delta} f_n\|^{\frac{1}{12}}_{L^{10}_{t,x}(I_n\times\mathbb{R}^3)} \|f_n\|^{\frac{1}{6}}_{\dot H^1_x} \|\nabla e^{it\Delta} f_n\|^{\frac{3}{4}}_{L_t^{10}L_x^{30/13}(I_n\times\mathbb{R}^3)}\\ &\lesssim_{\phi_\varepsilon^j} \|e^{it\Delta} w_n^J\|^{\frac{1}{12}}_{L^{10}_{t,x}([-T,T]\times\mathbb{R}^3)} \|w_n^J\|^{\frac{1}{6}}_{\dot H^1_x} \|\nabla e^{it\Delta} w_n^J\|^{\frac{3}{4}}_{L_t^{10}L_x^{30/13}([-T,T]\times\mathbb{R}^3)},\end{aligned}$$

where we have again used Corollary 2.8. Recalling (4-4), we see that the contribution of the high-frequency piece is acceptable. This completes the proof of (4-33) and hence the proof of Lemma 4.7. □

The final step in checking the hypotheses of Proposition 3.5, which will finish the proof of Theorem 4.1, is to verify that

$$\limsup_{J\to J^*}\limsup_{n\to\infty} \|u_n^J\|_{L_t^\infty \dot H_x^1([-T,T]\times\mathbb{R}^3)} \lesssim_T 1.$$

In view of Lemma 2.6, we have

$$\|u_n^J\|_{L_t^\infty \dot H_x^1([-T,T]\times\mathbb{R}^3)} \lesssim_T \|u_n^J(0)\|_{\dot H^1_x} + \|\nabla e_n^J\|_{\dot N^0([-T,T])} + \|\nabla N(u_n^J)\|_{\dot N^0([-T,T])}.$$

The requisite bounds on the right-hand side now follow from Lemmas 4.6 and 4.7. □

5. Normal form transformation

In this section we discuss the normal form transformation that we use throughout the rest of the paper. The use of normal form transformations originates in work of Shatah [1985] and has since become a widely used technique in the setting of nonlinear dispersive equations. The transformation we use is similar to the one used by Gustafson et al. [2006; 2007; 2009] in the setting of the Gross–Pitaevskii equation.

Suppose u is a solution to (1-6). As mentioned in the Introduction, the quadratic terms in the nonlinearity are the most problematic when it comes to questions of long-time behavior; in particular, the worst terms are those containing $u_2 = \operatorname{Im} u$, since in the diagonal variables we have $u_2 = U^{-1}v_2$. We would like to find a normal form transformation that eliminates if not all, at least the worst quadratic terms.

To this end, we let $B_1[\cdot,\cdot]$ and $B_2[\cdot,\cdot]$ be arbitrary bilinear Fourier multiplier operators defined as in (2-1), with symmetric real-valued symbols $B_1(\xi_1,\xi_2)$ and $B_2(\xi_1,\xi_2)$. Then

$$\tilde u := u + B_1[u_1,u_1] + B_2[u_2,u_2]$$

satisfies the equation

$$\begin{aligned}(i\partial_t+\Delta)\tilde u - 2\gamma\tilde u_1 = {}&\\ &(3\gamma+4)u_1^2 - (2\gamma-\Delta)B_1[u_1,u_1] &&(5\text{-}1)\\ &+\gamma u_2^2 - (2\gamma-\Delta)B_2[u_2,u_2] &&(5\text{-}2)\\ &+2i\big(\gamma u_1u_2 + B_1[u_1,-\Delta u_2] - B_2[u_2,(2\gamma-\Delta)u_1]\big) &&(5\text{-}3)\\ &+\text{cubic and higher order terms.}\end{aligned}$$

While the symmetry of B_1 and B_2 makes it impossible to eliminate *all* of the quadratic terms, we see that if we choose

$$B_2(\xi_1,\xi_2)=\gamma(2\gamma+|\xi_1+\xi_2|^2)^{-1},\quad \text{i.e., } B_2[f,g]=\gamma\langle\nabla\rangle^{-2}(fg),$$

then (5-2) $=0$. This allows us to eliminate the worst quadratic term, namely, the one containing two copies of u_2. Moreover, choosing $B_1=B_2$ we get

$$\tilde{u}=u+\gamma\langle\nabla\rangle^{-2}|u|^2,$$

with

$$\text{(5-1)}=(2\gamma+4)u_1^2 \quad\text{and}\quad \text{(5-3)}=-4i\gamma\langle\nabla\rangle^{-2}\nabla\cdot[u_1\nabla u_2].$$

The derivative appearing in front of u_2 is a welcome addition in light of the problem at low frequencies.

Similarly one can compute the higher-order terms. In general, one finds that for $k\in\{3,4,5\}$ the terms of order k are given by

$$N_k(u)+2i\big\{B_1[u_1,\operatorname{Im}(N_{k-1}(u))]-B_2[u_2,\operatorname{Re}(N_{k-1}(u))]\big\},$$

where the N_k are as in (1-6). Notice that there are no sixth-order terms, since $B_1=B_2$ and $u_1\operatorname{Im}(N_5(u))=u_2\operatorname{Re}(N_5(u))$.

Finally, we employ the transformation $Vu=u_1+iUu_2$ to diagonalize the equation. Our normal form transformation is therefore given by

$$z:=M(u):=Vu+\gamma\langle\nabla\rangle^{-2}|u|^2, \tag{5-4}$$

and z satisfies the equation

$$(i\partial_t-H)z=N_z(u) \tag{5-5}$$

with

$$\begin{aligned}\operatorname{Re}[N_z(u)]&=U\operatorname{Re}[N(u)-\gamma|u|^2]\\&=U\big[(2\gamma+4)u_1^2+(\gamma+8)u_1^3+(\gamma+4)u_1u_2^2+(5u_1^4+6u_1^2u_2^2+u_2^4)+|u|^4u_1)\big],\\ \operatorname{Im}[N_z(u)]&=-\frac{\nabla}{\langle\nabla\rangle^2}\cdot\big[4\gamma u_1\nabla u_2+\nabla(\gamma|u|^2u_2+q^2u_2)\big]\\&=-\frac{\nabla}{\langle\nabla\rangle^2}\cdot\big[4\gamma u_1\nabla u_2\big]+U^2\big[(\gamma+4)u_1^2u_2+\gamma u_2^3+4u_1u_2|u|^2+|u|^4u_2)\big].\end{aligned}$$

We should briefly pause to point out the improvements present in (5-5) with respect to (1-6). Firstly, (5-5) does not contain a quadratic term involving two copies of u_2. Secondly, the remaining quadratic terms involving u_2 exhibit a derivative of this problematic term. Lastly, all the remaining terms appear with a derivative at low frequencies, which is helpful throughout.

We next discuss the invertibility of the transformation (5-4). Note that by using the definition of $\langle\nabla\rangle^{-2}$ we can rewrite the transformation as

$$M(u)=U^2u_1+\gamma\langle\nabla\rangle^{-2}q+iUu_2, \tag{5-6}$$

where $q=q(u)=2u_1+|u|^2$.

This normal form transformation is a homeomorphism from a neighborhood of zero in $\mathcal{E}$ onto a neighborhood of zero in H^1_x. To prove this, we make use of the neighborhoods

$$\mathcal{M}_{E_0,\varepsilon_0} := \{u \in \mathcal{E} : E(u) \le E_0^2,\ \|u\|_{L^6_x} \le \varepsilon_0\},$$
$$\mathcal{N}_{E_0',\varepsilon_0'} := \{f \in H^1_x : \|f\|_{H^1_x} \le CE_0',\ \|f\|_{L^6_x} \le C\varepsilon_0'\},$$

where C denotes an absolute constant depending on γ.

Proposition 5.1. *Fix $E_0 > 0$ and $\varepsilon_0 > 0$.*

(i) *If $\gamma \in \left(\frac{2}{3}, 1\right)$, then $M : \mathcal{M}_{E_0,\varepsilon_0} \to \mathcal{N}_{E_0,\varepsilon_0+\varepsilon_0^2}$ continuously.*

(ii) *If $\gamma = \frac{2}{3}$, then $M : \mathcal{M}_{E_0,\varepsilon_0} \to \mathcal{N}_{E_0+E_0^3,\varepsilon_0+\varepsilon_0^2}$ continuously.*

(iii) *If $\gamma \in \left(0, \frac{2}{3}\right)$, then $M : \mathcal{M}_{E_0,\varepsilon_0} \cap \{\|\nabla u_1\|_2^2 \le \delta_\gamma\} \to \mathcal{N}_{E_0,\varepsilon_0+\varepsilon_0^2}$ continuously, where δ_γ is as in Lemma 3.2.*

(iv) *Given $E_1 > 0$, there exists $\varepsilon_1 = \varepsilon_1(E_1)$ and a continuous mapping*

$$R : \mathcal{N}_{E_1,\varepsilon_1} \to \mathcal{E}$$

such that $M \circ R = \mathrm{Id}$ on $\mathcal{N}_{E_1,\varepsilon_1}$ and $\|R(f)\|_{\mathcal{E}} \lesssim E_1$ for $f \in \mathcal{N}_{E_1,\varepsilon_1}$.

(v) *Suppose $\gamma \ge \frac{2}{3}$. Given $E_2 > 0$, there exists $\varepsilon_2 = \varepsilon_2(E_2)$ so that M is a homeomorphism of $\mathcal{M}_{E_2,\varepsilon_2}$ onto a subset of $H^1_x(\mathbb{R}^3)$ and has inverse R. In particular, M is injective on $\mathcal{M}_{E_2,\varepsilon_2}$. If $\gamma < \frac{2}{3}$, then the analogous assertions hold on $\mathcal{M}_{E_2,\varepsilon_2} \cap \{\|\nabla u_1\|_2^2 \le \delta_\gamma\}$.*

Remark 5.2. We warn the reader that just because $M(u)$ is small in L^6_x, one cannot guarantee that $u = (R \circ M)(u)$. However, this would follow if u were sufficiently small in L^6_x. This subtlety contributes nontrivially to the complexity of the proof of Theorem 1.1.

Proof. The proofs of the first three claims parallel one another closely. We will only present the details when $\gamma \in \left(\frac{2}{3}, 1\right)$.

Let $u \in \mathcal{M}_{E_0,\varepsilon_0}$. Recall from Lemma 3.1 that

$$\|u\|_{\mathcal{E}}^2 \lesssim E(u) \lesssim E_0^2.$$

We first show that $M(u) \in \mathcal{N}_{E_0,\varepsilon_0+\varepsilon_0^2}$. Using the representation (5-6), we estimate

$$\|M(u)\|_{H^1_x} \lesssim \|U^2 u_1\|_{H^1_x} + \|\langle\nabla\rangle^{-2} q\|_{H^1_x} + \|U u_2\|_{H^1_x} \lesssim \|u\|_{\dot H^1_x} + \|q\|_{L^2_x} \lesssim E_0.$$

Using the representation (5-4) and Sobolev embedding, we estimate

$$\begin{aligned}\|M(u)\|_{L^6_x} &\lesssim \|u_1\|_{L^6_x} + \|U u_2\|_{L^6_x} + \big\|\langle\nabla\rangle^{-2}|u|^2\big\|_{L^6_x} \\ &\lesssim \|u\|_{L^6_x} + \big\||\nabla|^{1/2}\langle\nabla\rangle^{-2}|u|^2\big\|_{L^3_x} \\ &\lesssim \|u\|_{L^6_x} + \|u\|_{L^6_x}^2 \\ &\lesssim \varepsilon_0 + \varepsilon_0^2.\end{aligned}$$

Collecting these estimates, we conclude $M(u) \in \mathcal{N}_{E_0,\varepsilon_0+\varepsilon_0^2}$.

To prove the continuity of M, we note that for $u, v \in \mathcal{E}$ we may write

$$M(u) - M(v) = U^2(u_1 - v_1) + \gamma\langle\nabla\rangle^{-2}[q(u) - q(v)] + iU(u_2 - v_2).$$

Estimating as above we find

$$\|M(u) - M(v)\|_{H_x^1} \lesssim d_{\mathcal{E}}(u, v).$$

We turn now to the fourth claim in the statement of the proposition. Let $f \in \mathcal{N}_{E_1,\varepsilon_1}$. We aim to show that for $\varepsilon_1 = \varepsilon_1(E_1) > 0$ sufficiently small, we can find a unique $u \in \mathcal{E}$ such that $M(u) = f$, that is,

$$\begin{cases} u_2 = U^{-1} f_2, \\ u_1 = f_1 - \gamma\langle\nabla\rangle^{-2}[U^{-1} f_2]^2 - \gamma\langle\nabla\rangle^{-2} u_1^2. \end{cases}$$

To this end, we define

$$R_f(u_1) := f_1 - \gamma\langle\nabla\rangle^{-2}[U^{-1} f_2]^2 - \gamma\langle\nabla\rangle^{-2} u_1^2.$$

We will show that for $\varepsilon_1 = \varepsilon_1(E_1)$ sufficiently small, R_f is a contraction on

$$B := \big\{u_1 \in \dot H_x^1 : \|u_1\|_{\dot H_x^1} \le C E_1,\ \|u_1\|_{L_x^6} \le C(\varepsilon_1 + \varepsilon_1^{\frac12} E_1^{\frac32})\big\}$$

with respect to the metric $d(u_1, v_1) = \|u_1 - v_1\|_{\dot H_x^1}$, where C denotes an absolute constant depending on γ.

We first show that $R_f : B \to B$. We have

$$\|R_f(u_1)\|_{L_x^6} \lesssim \|f_1\|_{L_x^6} + \big\|\langle\nabla\rangle^{-2}[U^{-1} f_2]^2\big\|_{L_x^6} + \|\langle\nabla\rangle^{-2} u_1^2\|_{L_x^6}. \tag{5-7}$$

The first term in (5-7) is controlled by ε_1 by assumption. For the second term in (5-7), we use Sobolev embedding, Bernstein, and interpolation to estimate

$$\begin{aligned}\|\langle\nabla\rangle^{-2}[U^{-1} f_2]^2\|_{L_x^6} &\lesssim \Big\|\frac{\nabla}{\langle\nabla\rangle^2}[P_{\text{lo}} U^{-1} f_2]^2\Big\|_{L_x^2} + \Big\|\frac{\nabla}{\langle\nabla\rangle^2}\big[(P_{\text{hi}} U^{-1} f_2) \emptyset (U^{-1} f_2)\big]\Big\|_{L_x^2} \\ &\lesssim \|\nabla P_{\text{lo}} U^{-1} f_2\|_{L_x^3} \|U^{-1} f_2\|_{L_x^6} + \|P_{\text{hi}} U^{-1} f_2\|_{L_x^3} \|U^{-1} f_2\|_{L_x^6} \\ &\lesssim \|f\|_{L_x^6}^{\frac12} \|f\|_{H_x^1}^{\frac32}.\end{aligned}$$

For the third term in (5-7), we have

$$\|\langle\nabla\rangle^{-2} u_1^2\|_{L_x^6} \lesssim \big\||\nabla|^{\frac12}\langle\nabla\rangle^{-2} u_1^2\big\|_{L_x^3} \lesssim \|u_1\|_{L_x^6}^2.$$

Thus, for $u_1 \in B$ and $\varepsilon_1 = \varepsilon_1(E_1)$ sufficiently small we obtain

$$\|R_f(u_1)\|_{L_x^6} \le C(\varepsilon_1 + \varepsilon_1^{\frac12} E_1^{\frac32}).$$

To continue, we estimate

$$\|R_f(u_1)\|_{\dot H_x^1} \lesssim \|f_1\|_{\dot H_x^1} + \big\|\langle\nabla\rangle^{-2}\big[U^{-1} f_2\big]^2\big\|_{\dot H_x^1} + \|\langle\nabla\rangle^{-2} u_1^2\|_{\dot H_x^1}. \tag{5-8}$$

The first term in (5-8) is controlled by E_1 by assumption. For the second term in (5-8), we argue as above to find

$$\|\langle\nabla\rangle^{-2}[U^{-1}f_2]^2\|_{\dot H^1_x} \lesssim \left\|\frac{\nabla}{\langle\nabla\rangle^2}[P_{\mathrm{lo}}U^{-1}f_2]^2\right\|_{L^2_x} + \left\|\frac{\nabla}{\langle\nabla\rangle^2}\big[(P_{\mathrm{hi}}U^{-1}f_2)\O(U^{-1}f_2)\big]\right\|_{L^2_x} \lesssim \|f\|_{L^6_x}^{\frac12}\|f\|_{H^1_x}^{\frac32}.$$

For the third term in (5-8) we estimate

$$\|\langle\nabla\rangle^{-2}u_1^2\|_{\dot H^1_x} \lesssim \||\nabla|^{\frac32}\langle\nabla\rangle^{-2}u_1^2\|_{L^{3/2}_x} \lesssim \|u_1\|_{L^6_x}\|\nabla u_1\|_{L^2_x}.$$

Thus for $u_1 \in B$ and $\varepsilon_1 = \varepsilon_1(E_1)$ sufficiently small we have

$$\|R_f(u_1)\|_{\dot H^1_x} \le CE_1.$$

Collecting these estimates, we conclude that $R_f : B \to B$.

Next we show that R_f is a contraction with respect to the $\dot H^1_x$-norm. We first use Sobolev embedding, Bernstein, and interpolation to estimate

$$\begin{aligned}
&\left\|\frac{1}{\langle\nabla\rangle^2}\big[(u_1+v_1)(u_1-v_1)\big]\right\|_{\dot H^1_x} \\
&\qquad\lesssim \left\|\frac{|\nabla|^{\frac32}}{\langle\nabla\rangle^2}\big[(u_1+v_1)P_{\mathrm{hi}}(u_1-v_1)\big]\right\|_{L^{3/2}_x} + \left\|\frac{1}{\langle\nabla\rangle^2}\big[P_{\mathrm{lo}}(u_1+v_1)\,P_{\mathrm{lo}}(u_1-v_1)\big]\right\|_{\dot H^1_x} \\
&\qquad\qquad + \left\|\frac{1}{\langle\nabla\rangle^2}\big[P_{\mathrm{hi}}(u_1+v_1)\,P_{\mathrm{lo}}(u_1-v_1)\big]\right\|_{\dot H^1_x} \\
&\qquad\lesssim \|u_1+v_1\|_{L^6_x}\|P_{\mathrm{hi}}(u_1-v_1)\|_{L^2_x} + \|\nabla P_{\mathrm{lo}}(u_1+v_1)\|_{L^3_x}\|u_1-v_1\|_{L^6_x} \\
&\qquad\qquad + \|u_1+v_1\|_{L^6_x}\|\nabla P_{\mathrm{lo}}(u_1-v_1)\|_{L^3_x} + \|P_{\mathrm{hi}}(u_1+v_1)\|_{L^3_x}\|u_1-v_1\|_{L^6_x} \\
&\qquad\lesssim \big(\|u_1\|_{L^6_x}^{\frac12}\|u_1\|_{\dot H^1_x}^{\frac12} + \|v_1\|_{L^6_x}^{\frac12}\|v_1\|_{\dot H^1_x}^{\frac12}\big)\|u_1-v_1\|_{\dot H^1_x}.
\end{aligned} \tag{5-9}$$

In particular, for $\varepsilon_1 = \varepsilon_1(E_1)$ sufficiently small we deduce that

$$\|R_f(u_1) - R_f(v_1)\|_{\dot H^1_x} \le \tfrac12\|u_1-v_1\|_{\dot H^1_x}.$$

Therefore, by the contraction mapping theorem there exists a unique $u_1 \in B$ such that $R_f(u_1) = u_1$. We define $R(f) := u_1 + iU^{-1}f_2$. By construction, we have $M(R(f)) = f$.

It remains to see that $u := R(f) \in \mathcal{E}$ with $\|u\|_{\mathcal{E}} \lesssim E_1$. As $u_1 \in B$, we have

$$\|u\|_{\dot H^1_x} \lesssim \|u_1\|_{\dot H^1_x} + \|U^{-1}f_2\|_{\dot H^1_x} \lesssim E_1 + \|f_2\|_{H^1_x} \lesssim E_1.$$

Moreover, by Hölder,

$$\begin{aligned}
\|q(u)\|_{L^2_x} = \|2f_1 + U^2|u|^2\|_{L^2_x} &\lesssim \|f\|_{L^2_x} + \|U(|u|^2)\|_{L^2_x} \\
&\lesssim E_1 + \|\nabla\O[(P_{\mathrm{lo}}u)^2]\|_{L^2_x} + \|\O(uP_{\mathrm{hi}}u)\|_{L^2_x} \\
&\lesssim E_1 + \|\nabla P_{\mathrm{lo}}u\|_{L^2_x}\|P_{\mathrm{lo}}u\|_{L^\infty_x} + \|u\|_{L^6_x}\|P_{\mathrm{hi}}u\|_{L^3_x} \\
&\lesssim E_1 + \|u\|_{\dot H^1_x}\big[\|P_{\mathrm{lo}}u\|_{L^\infty_x} + \|P_{\mathrm{hi}}u\|_{L^3_x}\big].
\end{aligned}$$

Using Bernstein, Hölder, and interpolation, we estimate

$$\|P_{\mathrm{lo}}u\|_{L_x^\infty} \lesssim \|u_1\|_{L_x^6} + \|P_{\mathrm{lo}}U^{-1}f_2\|_{L_x^{10}} \lesssim \|u_1\|_{L_x^6} + \|f_2\|_{L_x^{30/13}} \lesssim \|u_1\|_{L_x^6} + \|f_2\|_{L_x^6}^{\frac{1}{5}}\|f_2\|_{L_x^2}^{\frac{4}{5}}$$

and

$$\|P_{\mathrm{hi}}u\|_{L_x^3} \lesssim \|P_{\mathrm{hi}}u_1\|_{L_x^3} + \|P_{\mathrm{hi}}f_2\|_{L_x^3} \lesssim \|u_1\|_{L_x^6}^{\frac{1}{2}}\|u_1\|_{\dot H_x^1}^{\frac{1}{2}} + \|f_2\|_{L_x^6}^{\frac{1}{2}}\|f_2\|_{\dot H_x^1}^{\frac{1}{2}}.$$

Taking $\varepsilon_1 = \varepsilon_1(E_1)$ sufficiently small, this proves $\|q(u)\|_{L_x^2} \lesssim E_1$.

To complete the proof of the proposition, it remains to address part (v). From (5-4) and (5-9), we see that M is injective on $\mathcal{M}_{E_2,\varepsilon_2}$ provided ε_2 is sufficiently small depending on E_2. By shrinking ε_2, if necessary, we can further ensure that $M(\mathcal{M}_{E_2,\varepsilon_2})$ is contained in a region where R is defined (this relies on all the other parts of the proposition). It then follows that M is a homeomorphism on $M(\mathcal{M}_{E_2,\varepsilon_2})$ with inverse R. □

The last result of this section relates the energy and the inverse of the normal form transformation; this will be useful in the proof of Theorem 1.1.

Lemma 5.3. *Let $\{z_n\}_{n\geq 1} \subset H_x^1$ be uniformly bounded and assume that $z_n \to 0$ in L_x^6. Then*

$$E(R(z_n)) = \tfrac{1}{2}\|z_n\|_{H_x^1}^2 + o(1) \quad \text{as } n \to \infty.$$

Proof. By Proposition 5.1 (and its proof), we have that $R(z_n)$ exists for n large and

$$\limsup_n \|R(z_n)\|_{\mathcal{E}} \lesssim 1 \quad \text{and} \quad \lim_{n\to\infty} \|\operatorname{Re} R(z_n)\|_{L_x^6} = 0. \tag{5-10}$$

We first claim that

$$R(z_n) = V^{-1}z_n + o(1) \quad \text{in } \dot H_x^1 \text{ as } n \to \infty. \tag{5-11}$$

Indeed, from the construction of R via the fixed point argument in Proposition 5.1, this amounts to proving that

$$\|\langle\nabla\rangle^{-2}[U^{-1}\operatorname{Im} z_n]^2\|_{\dot H_x^1} + \|\langle\nabla\rangle^{-2}[\operatorname{Re} R(z_n)]^2\|_{\dot H_x^1} = o(1) \quad \text{as } n \to \infty.$$

To see this, we use the decomposition

$$[U^{-1}\operatorname{Im} z_n]^2 = [P_{\mathrm{lo}}U^{-1}\operatorname{Im} z_n]^2 + \emptyset\big[(U^{-1}\operatorname{Im} z_n)P_{\mathrm{hi}}U^{-1}\operatorname{Im} z_n\big] \tag{5-12}$$

together with Bernstein, Hölder, (5-10), and the hypotheses of the lemma to estimate

$$\begin{aligned}
\big\|\langle\nabla\rangle^{-2}[U^{-1}\operatorname{Im} z_n]^2\big\|_{\dot H_x^1} &\lesssim \big\||\nabla|[P_{\mathrm{lo}}U^{-1}\operatorname{Im} z_n]^2\big\|_{L_x^2} + \big\|(U^{-1}\operatorname{Im} z_n)P_{\mathrm{hi}}U^{-1}\operatorname{Im} z_n\big\|_{L_x^2}\\
&\lesssim \|\operatorname{Im} z_n\|_{L_x^3}\|U^{-1}\operatorname{Im} z_n\|_{L_x^6}\\
&\lesssim \|z_n\|_{L_x^6}^{\frac{1}{2}}\|z_n\|_{L_x^2}^{\frac{1}{2}}\|z_n\|_{H_x^1} = o(1) \quad \text{as } n \to \infty,
\end{aligned}$$

$$\begin{aligned}
\big\|\langle\nabla\rangle^{-2}[\operatorname{Re} R(z_n)]^2\big\|_{\dot H_x^1} &\lesssim \big\||\nabla|^{\frac{3}{2}}\langle\nabla\rangle^{-2}[\operatorname{Re} R(z_n)]^2\big\|_{L_x^{3/2}}\\
&\lesssim \|\nabla \operatorname{Re} R(z_n)\|_{L_x^2}\|\operatorname{Re} R(z_n)\|_{L_x^6} = o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

This completes the proof of (5-11).

We now turn our attention to the terms in the formula for $E(R(z_n))$ containing $q(R(z_n))$. Using the representation (5-6), we observe that

$$M(u) = \tfrac{1}{2}q(u) - \tfrac{1}{2}U^2(|u|^2) + iU\,\mathrm{Im}\,u \quad \text{and so} \quad q(R(z_n)) = 2\,\mathrm{Re}\,z_n + U^2(|R(z_n)|^2).$$

We next claim that

$$q(R(z_n)) = 2\,\mathrm{Re}\,z_n + o(1) \quad \text{in } L_x^2 \text{ as } n \to \infty. \tag{5-13}$$

To prove this, we note that $\mathrm{Im}\,R(z_n) = U^{-1}\,\mathrm{Im}\,z_n$ and use the decomposition (5-12), as well as the analogous decomposition for $\mathrm{Re}\,R(z_n)$. Arguing as for (5-11), we estimate

$$\begin{aligned}
\|U^2[U^{-1}\,\mathrm{Im}\,z_n]^2\|_{L_x^2} &\lesssim \big\||\nabla|[P_{\mathrm{lo}}U^{-1}\,\mathrm{Im}\,z_n]^2\big\|_{L_x^2} + \big\|(U^{-1}\,\mathrm{Im}\,z_n)P_{\mathrm{hi}}U^{-1}\,\mathrm{Im}\,z_n\big\|_{L_x^2} = o(1),\\
\|U^2[\mathrm{Re}\,R(z_n)]^2\|_{L_x^2} &\lesssim \big\||\nabla|[P_{\mathrm{lo}}\,\mathrm{Re}\,R(z_n)]^2\big\|_{L_x^{3/2}} + \big\|(\mathrm{Re}\,R(z_n))P_{\mathrm{hi}}\,\mathrm{Re}\,R(z_n)\big\|_{L_x^2}\\
&\lesssim \|\nabla\,\mathrm{Re}\,R(z_n)\|_{L_x^2}\|\,\mathrm{Re}\,R(z_n)\|_{L_x^6} + \|\,\mathrm{Re}\,R(z_n)\|_{L_x^6}\|P_{\mathrm{hi}}\,\mathrm{Re}\,R(z_n)\|_{L_x^3}\\
&\lesssim \|\nabla R(z_n)\|_{L_x^2}\|\,\mathrm{Re}\,R(z_n)\|_{L_x^6} = o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

This completes the proof of (5-13).

Finally, we note that

$$q(R(z_n)) = o(1) \quad \text{in } L_x^3 \text{ as } n \to \infty. \tag{5-14}$$

Indeed, arguing as above we find

$$\begin{aligned}
\|\,\mathrm{Re}\,z_n\|_{L_x^3} &\lesssim \|z_n\|_{L_x^2}^{\frac{1}{2}}\|z_n\|_{L_x^6}^{\frac{1}{2}} = o(1) \quad \text{as } n \to \infty,\\
\|U^2[\mathrm{Re}\,R(z_n)]^2\|_{L_x^3} &\lesssim \|\,\mathrm{Re}\,R(z_n)\|_{L_x^6}^2 = o(1) \quad \text{as } n \to \infty,\\
\|U^2[\mathrm{Im}\,R(z_n)]^2\|_{L_x^3} &\lesssim \big\||\nabla|[P_{\mathrm{lo}}U^{-1}\,\mathrm{Im}\,z_n]^2\big\|_{L_x^3} + \big\|(U^{-1}\,\mathrm{Im}\,z_n)P_{\mathrm{hi}}U^{-1}\,\mathrm{Im}\,z_n\big\|_{L_x^3}\\
&\lesssim \|\,\mathrm{Im}\,z_n\|_{L_x^6}\big\||\nabla|^{-1}\,\mathrm{Im}\,z_n\big\|_{L_x^6} + \|U^{-1}\,\mathrm{Im}\,z_n\|_{L_x^6}\|\,\mathrm{Im}\,z_n\|_{L_x^6}\\
&\lesssim \|z_n\|_{L_x^6}\|z_n\|_{H_x^1} = o(1) \quad \text{as } n \to \infty.
\end{aligned}$$

Putting together (5-11), (5-13), and (5-14) completes the proof of the lemma. □

6. Proof of the main result

In this section we prove the main result, Theorem 1.1. As discussed in the Introduction, the proof is based off of a strategy of Nakanishi; see especially Theorem 1.3 and the sketch of proof thereafter. For the convenience of the reader, we restate the main theorem here.

Theorem 6.1. *Suppose* $\gamma \in \big[\frac{2}{3}, 1\big)$. *For any* $u_+ \in H^1_{\mathrm{real}} + i\dot{H}^1_{\mathrm{real}}$, *there exists a global solution* $u \in C(\mathbb{R}; \mathcal{E})$ *to* (1-6) *such that*

$$\lim_{t\to\infty} \|u(t) - u_{\mathrm{lin}}(t)\|_{\dot{H}_x^1} = 0, \tag{6-1}$$

where $u_{\mathrm{lin}}(t) := V^{-1}e^{-itH}Vu_+$. *Moreover,*

$$\lim_{t\to\infty} d_{\mathcal{E}}\big(u(t), u_{\mathrm{lin}}(t) - \gamma\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big) = 0. \tag{6-2}$$

For $\gamma \in \big(0, \frac{2}{3}\big)$, *these conclusions hold if* $\|u_+\|_{H^1_{\mathrm{real}}+i\dot H^1_{\mathrm{real}}}$ *is sufficiently small.*

Proof of Theorem 6.1. Let $u_+ \in H^1_{\mathrm{real}} + i\dot H^1_{\mathrm{real}}$. We define $z_+ = Vu_+ \in H^1_x$ and we let $E_0 := \|z_+\|_{H^1_x}$.

We first claim that

$$\lim_{t\to\infty} \|e^{-itH}z_+\|_{L^6_x} = 0. \tag{6-3}$$

Indeed, given $\eta > 0$ we may find $\varphi \in \mathcal{S}(\mathbb{R}^3)$ such that $\|z_+ - \varphi\|_{\dot H^1_x} < \eta$. Using the dispersive estimate (2-3) and Sobolev embedding, we find

$$\|e^{-itH}z_+\|_{L^6_x} \lesssim \|e^{-itH}\varphi\|_{L^6_x} + \|z_+ - \varphi\|_{\dot H^1_x} \lesssim |t|^{-1}\|\varphi\|_{L^{6/5}_x} + \eta,$$

which yields (6-3).

Next, we choose ε_0 sufficiently small depending on E_0 as in Proposition 5.1. By (6-3), there exists $T_0 > 0$ such that $e^{-itH}z_+ \in \mathcal{N}_{E_0,\varepsilon_0}$ for $t \geq T_0$; thus for any $T \geq T_0$, we may define $R(e^{-iTH}z_+) \in \mathcal{E}$ so that $M(R(e^{-iTH}z_+)) = e^{-iTH}z_+$, with $\|R(e^{-iTH}z_+)\|_{\mathcal{E}} \lesssim E_0$.

By Theorem 3.3, there exists a global solution $u^T \in C(\mathbb{R};\mathcal{E})$ to (1-6) with $u^T(T) = R(e^{-iTH}z_+)$. Note that when $\gamma \in \big(0, \frac{2}{3}\big)$, we require E_0 to be sufficiently small to guarantee that

$$\|\nabla\,\mathrm{Re}(u^T(0))\|^2_{L^2_x} \leq \delta_\gamma \quad\text{and}\quad E(u^T(0)) \leq \tfrac{1}{4}\delta_\gamma \tag{6-4}$$

uniformly in T, where δ_γ is as in Theorem 3.3. We define

$$q^T := q(u^T) = 2u_1^T + |u^T|^2 \quad\text{and}\quad z^T := M(u^T).$$

Note that (u^T, z^T) solves (5-4)–(5-5) with $z^T(T) = e^{-iTH}z_+$. Furthermore, we have

$$\|z^T(t)\|_{H^1_x} + \|u^T(t)\|_{\dot H^1_x} + \|q^T(t)\|_{L^2_x\cap L^3_x} + \|u_1^T(t)\|_{L^3_x\cap L^6_x} \lesssim_{E_0} 1 \tag{6-5}$$

uniformly in t and T.

As a consequence of (6-5), there exists a sequence $T_n \to \infty$ and a function $u_0 \in \dot H^1_x$ such that $u^{T_n}(0) \rightharpoonup u_0$ weakly in $\dot H^1_x$. As (6-5) and (6-4) imply that $\{u^{T_n}(0)\}$ satisfy the hypotheses of Theorem 4.1, we may apply this theorem to deduce that

$$u^{T_n}(t) \rightharpoonup u^\infty(t) \quad\text{weakly in } \dot H^1_x \text{ for all } t \in \mathbb{R}, \tag{6-6}$$

where $u^\infty \in C(\mathbb{R};\mathcal{E})$ denotes the solution to (1-6) with initial data $u^\infty(0) = u_0 \in \mathcal{E}$.

We define $z^\infty := M(u^\infty)$ and note that (u^∞, z^∞) solves (5-4)–(5-5). We will prove that u^∞ is a solution to (1-6) that satisfies the conclusions of Theorem 1.1. A first step in this direction is the following weak convergence result.

Proposition 6.2. *We have*

$$e^{itH}z^\infty(t) \rightharpoonup z_+ \quad \textit{weakly in } H^1_x \textit{ as } t \to \infty.$$

Assuming Proposition 6.2 for now, we proceed with the proof of Theorem 1.1. We begin by upgrading the weak convergence from Proposition 6.2 to strong convergence, namely,

$$\lim_{t\to\infty} \|z^\infty(t) - e^{-itH} z_+\|_{H^1_x} = 0. \tag{6-7}$$

Using Lemma 4.2 combined with (6-6) and Lemma 5.3 combined with (6-3), we can first write

$$E(u^\infty) \le \liminf_{n\to\infty} E(u^{T_n}) = \liminf_{n\to\infty} E(R(e^{-iT_n H} z_+)) = \tfrac{1}{2}\|z_+\|^2_{H^1_x}. \tag{6-8}$$

At this moment, it is tempting to attempt a Radon–Riesz-style argument. Recall that the Radon–Riesz theorem says that if $x_n \rightharpoonup x$ weakly in some Banach space X and $\limsup F(x_n) \le F(x)$ for some uniformly convex function $F : X \to \mathbb{R}$, then $x_n \to x$ in norm. (This is most often quoted in the case of a uniformly convex Banach space with F being the norm.)

The ideas just sketched were adapted beautifully to the Gross–Pitaevskii setting treated in [Gustafson et al. 2009]. As discussed in the Introduction, those authors exploit

$$E_{GP}(u) = \tfrac{1}{2}\|M(u)\|^2_{H^1_x} + \tfrac{1}{4}\|U|u|^2\|^2_{L^2_x} \ge \tfrac{1}{2}\|M(u)\|^2_{H^1_x},$$

which holds under no additional hypotheses. As also discussed there (see (1-23), in particular) the energy functional for the cubic-quintic problem admits no such global inequality. Correspondingly, we need to keep track of the structure of $z^\infty(t_n)$ as $t_n \to \infty$ and then demonstrate the requisite coercivity is available in this particular limiting regime. To achieve this goal we will use the following lemma. Note that the result on $\tilde{E}$ plays a key role in controlling the kinetic energy of the real part when $\gamma < \frac{2}{3}$.

Lemma 6.3. *Let $\{u_n\}_{n\ge1} \subset \mathcal{E}$ be uniformly bounded. Assume that we may write $u_n = \xi_n + r_n$, where ξ_n satisfies*

$$\sup_n \|\xi_n\|_{H^1_{\text{real}} + i\dot{H}^1_{\text{real}}} \lesssim 1 \quad \textit{and} \quad \lim_{n\to\infty} \|\xi_n\|_{L^3_x \cap L^6_x} = 0.$$

Then

$$E(u_n) = E(r_n) + \tfrac{1}{2}\|V\xi_n\|^2_{H^1_x} + \operatorname{Re}\langle M(r_n), V\xi_n\rangle_{H^1_x} + o(1) \quad \textit{as } n \to \infty. \tag{6-9}$$

Furthermore, if $\tilde{E}$ denotes the reduced energy defined via

$$\tilde{E}(f) := \int \tfrac{1}{4}|\nabla f|^2 + \tfrac{1}{8}\gamma |q(f)|^2\,dx = \tfrac{1}{2}E(f) - \tfrac{1}{12}\int q(f)^3\,dx,$$

then

$$\tilde{E}(u_n) = \tilde{E}(r_n) + \tfrac{1}{4}\|V\xi_n\|^2_{H^1_x} + \tfrac{1}{2}\operatorname{Re}\langle M(r_n), V\xi_n\rangle_{H^1_x} + o(1) \quad \textit{as } n \to \infty. \tag{6-10}$$

Proof. We will only prove (6-9). Claim (6-10) can be read off from the proof we give below.

To begin we observe that

$$q(u_n) = q(r_n) + 2\operatorname{Re}\xi_n + |\xi_n|^2 + 2\operatorname{Re}(\bar{\xi}_n r_n).$$

By hypothesis, $r_n = u_n - \xi_n$ is uniformly bounded in L^6_x. Using this and our assumptions on ξ_n, we see

$$\begin{aligned} q(u_n) &= q(r_n) + 2\operatorname{Re}\xi_n + o(1) && \text{in } L^2_x \text{ as } n\to\infty,\\ q(u_n) &= q(r_n) + o(1) && \text{in } L^3_x \text{ as } n\to\infty. \end{aligned}$$

Moreover, as u_n is bounded in $\mathcal{E}$ and $\operatorname{Re}\xi_n$ is bounded in L^2_x, we deduce that $q(r_n)$ is uniformly bounded in both L^2_x and L^3_x.

Therefore, we obtain

$$\begin{aligned} E(u_n) &= E(r_n) + \int \tfrac{1}{2}|\nabla\xi_n|^2 + \operatorname{Re}(\nabla\bar{\xi}_n\nabla r_n) + \gamma q(r_n)\operatorname{Re}\xi_n + \gamma(\operatorname{Re}\xi_n)^2\,dx + o(1) \\ &= E(r_n) + \tfrac{1}{2}\|V\xi_n\|^2_{H^1_x} + \operatorname{Re}\big\langle(2\gamma-\Delta)M(r_n), V\xi_n\big\rangle_{L^2_x} + o(1) \quad \text{as } n\to\infty. \qquad \square \end{aligned}$$

We return now to the proof of (6-7). Let us begin by showing that

$$E(u^\infty) \geq \tfrac{1}{2}\|z_+\|^2_{H^1_x}, \tag{6-11}$$

which combined with (6-8) fully identifies $E(u^\infty)$. While natural, this is not (in and of itself) essential to the argument; it does, however, force us to control the contributions of parts of the energy with the unhelpful sign. It will be this control that will ultimately allows us to complete the proof of (6-7).

Let $t_n\to\infty$ be an arbitrary sequence. We apply Lemma 6.3 with

$$u_n := u^\infty(t_n) \quad \text{and} \quad \xi_n := (\operatorname{Id}\oplus P_{\geq N_n})V^{-1}e^{-it_nH}z_+, \tag{6-12}$$

where $N_n\in 2^{\mathbb{Z}}$ converges to zero sufficiently slowly to guarantee that

$$\|\xi_n\|_{L^3_x\cap L^6_x}\to 0 \quad \text{as } n\to\infty. \tag{6-13}$$

Note that this is possible because of (6-3). In view of (6-9), we obtain

$$\begin{aligned} E(u^\infty) = E(r_n) + \tfrac{1}{2}\big\|(\operatorname{Id}\oplus P_{\geq N_n})e^{-it_nH}z_+\big\|^2_{H^1_x} \\ + \operatorname{Re}\big\langle M(r_n), (\operatorname{Id}\oplus P_{\geq N_n})e^{-it_nH}z_+\big\rangle_{H^1_x} + o(1) \quad \text{as } n\to\infty. \end{aligned} \tag{6-14}$$

By Proposition 6.2, $e^{it_nH}M(u^\infty(t_n)) = e^{it_nH}z^\infty(t_n) \rightharpoonup z_+$ weakly in H^1_x. On the other hand, by (6-13), we have

$$\begin{aligned} M(u^\infty(t_n)) &= e^{-it_nH}z_+ + M(r_n) - P_{\leq N_n}\operatorname{Im}e^{-it_nH}z_+ + \gamma\langle\nabla\rangle^{-2}[|\xi_n|^2 + 2\operatorname{Re}(\bar{\xi}_n r_n)] \\ &= e^{-it_nH}z_+ + M(r_n) + o(1) \quad \text{in } H^1_x \text{ as } n\to\infty. \end{aligned} \tag{6-15}$$

Thus, we may deduce that

$$e^{it_nH}M(r_n)\rightharpoonup 0 \quad \text{weakly in } H^1_x \text{ as } n\to\infty.$$

Combining this with the dominated convergence theorem (which allows us to replace $P_{\geq N_n}$ by Id), (6-14) becomes

$$E(u^\infty) = E(r_n) + \tfrac{1}{2}\|z_+\|^2_{H^1_x} + o(1) \quad \text{as } n\to\infty. \tag{6-16}$$

Arguing similarly and using (6-10) in place of (6-9), we obtain

$$\tilde{E}(u^\infty) = \tilde{E}(r_n) + \tfrac{1}{4}\|z_+\|^2_{H^1_x} + o(1) \quad \text{as } n\to\infty. \tag{6-17}$$

Note that (6-11) follows immediately from (6-16), provided that $E(r_n) \geq 0$. By Lemma 3.1, this is immediate if $\gamma \in \left[\frac{2}{3}, 1\right)$. In view of Lemma 3.2, if $\gamma \in \left(0, \frac{2}{3}\right)$ we simply have to verify that $\|\nabla \operatorname{Re} r_n\|_{L_x^2}^2 \leq \delta_\gamma$. This, however, follows from (6-8) and (6-17), provided E_0 is chosen sufficiently small depending on γ.

Combining (6-8) with (6-11) and (6-16), we deduce that

$$E(u^\infty) = \tfrac{1}{2}\|z_+\|_{H_x^1}^2 \qquad \text{and} \qquad E(r_n) \to 0 \quad \text{as } n \to \infty.$$

By the argument in the preceding paragraph, this implies

$$\|r_n\|_{\mathcal{E}} \to 0 \quad \text{as } n \to \infty. \tag{6-18}$$

Therefore, using the representation (5-6) for M, we see that

$$\begin{aligned}\|M(r_n)\|_{H_x^1} &\lesssim \|U \operatorname{Im} r_n\|_{H_x^1} + \|U^2 \operatorname{Re} r_n\|_{H_x^1} + \|\langle\nabla\rangle^{-2} q(r_n)\|_{H_x^1} \\ &\lesssim \|r_n\|_{\dot{H}_x^1} + \|q(r_n)\|_{L_x^2} \to 0 \quad \text{as } n \to \infty.\end{aligned}$$

Combining this with (6-15), we get

$$\|z^\infty(t_n) - e^{-it_n H} z_+\|_{H_x^1} \to 0 \quad \text{as } n \to \infty.$$

As the sequence $t_n \to \infty$ was arbitrary, this completes the proof of (6-7).

We next prove that (6-7) implies the conclusions of Theorem 6.1. We first show that (6-7) implies (6-1). Let $t_n \to \infty$ be an arbitrary sequence and define u_n and ξ_n as in (6-12). Using (6-13) and (6-18), we deduce that $u^\infty(t_n) \to 0$ in L_x^6. Furthermore, by (6-7) and (6-3), we have that $z^\infty(t_n) \to 0$ in L_x^6. Using Proposition 5.1(v), we find that $u^\infty(t_n) = R(z^\infty(t_n))$ for n sufficiently large. Arguing as in Lemma 5.3 and using (5-11), we may write $u^\infty(t_n) = V^{-1} z^\infty(t_n) + o(1)$ in $\dot{H}_x^1$, which together with (6-7) yields (6-1).

We now turn to (6-2). We begin with the following strengthening of (6-3):

$$\lim_{t\to\infty} \|U^{-1} e^{-itH} z_+\|_{L_x^6} = 0. \tag{6-19}$$

Given $0 < N < 1$, we have

$$\begin{aligned}\|U^{-1} P_{\leq N} e^{-itH} z_+\|_{L_x^6} &\lesssim \|\nabla U^{-1} P_{\leq N} e^{-itH} z_+\|_{L_x^2} \lesssim \|P_{\leq N} z_+\|_{L_x^2}, \\ \|U^{-1} P_{>N} e^{-itH} z_+\|_{L_x^6} &\lesssim N^{-1} \|e^{-itH} z_+\|_{L_x^6}.\end{aligned}$$

In view of (6-3), choosing N sufficiently small and then sending $t \to \infty$ yields (6-19).

Using (6-19), we now show that the modification $\gamma\langle\nabla\rangle^{-2}|u_{\text{lin}}|^2$ appearing in (6-2) is negligible in the $\dot{H}_x^1$-norm. Indeed, we have the stronger statement

$$\begin{aligned}\big\|\langle\nabla\rangle^{-1}|u_{\text{lin}}(t)|^2\big\|_{\dot{H}_x^1} &\lesssim \big\||\nabla|^{\frac{1}{2}} u_{\text{lin}}(t)\big\|_{L_x^3} \|u_{\text{lin}}(t)\|_{L_x^6} \\ &\lesssim \|\nabla u_{\text{lin}}(t)\|_{L_x^2} \|U^{-1} e^{-itH} z_+\|_{L_x^6} \\ &\lesssim \|z_+\|_{H_x^1} \|U^{-1} e^{-itH} z_+\|_{L_x^6} \to 0 \quad \text{as } t \to \infty.\end{aligned} \tag{6-20}$$

It remains to show

$$\lim_{t\to\infty}\big\|q\big(u^\infty(t)\big)-q\big(u_{\mathrm{lin}}(t)-\gamma\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big)\big\|_{L_x^2}=0. \tag{6-21}$$

As demonstrated above, $u^\infty(t)=R(z^\infty(t))$ for t sufficiently large and $z^\infty(t)\to 0$ in L_x^6 as $t\to\infty$. Thus, arguing as for (5-13) and using (6-7), we deduce that

$$q(u^\infty(t))=2\,\mathrm{Re}\,u_{\mathrm{lin}}(t)+o(1)\quad \text{in } L_x^2 \text{ as } t\to\infty.$$

On the other hand, a straightforward computation yields

$$\begin{aligned}q\big(&u_{\mathrm{lin}}(t)-\gamma\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big)\\&=2\,\mathrm{Re}\,u_{\mathrm{lin}}(t)+U^2|u_{\mathrm{lin}}(t)|^2+\big[\gamma\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big]^2-2\gamma\big[\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big]\,\mathrm{Re}\,u_{\mathrm{lin}}(t).\end{aligned}$$

Thus, to prove (6-21) it suffices to show that the last three terms on the right-hand side above are $o(1)$ in L_x^2 as $t\to\infty$. Indeed, we may estimate

$$\begin{aligned}\|U^2|u_{\mathrm{lin}}(t)|^2\|_{L_x^2}&\lesssim\|\langle\nabla\rangle^{-1}|u_{\mathrm{lin}}(t)|^2\|_{\dot H_x^1},\\ \big\|\big[\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big]^2\big\|_{L_x^2}&\lesssim\big\||\nabla|^{\frac14}\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big\|_{L_x^3}^2\lesssim\|U^{-1}e^{-itH}z_+\|_{L_x^6}^4,\\ \big\|\big[\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big]\,\mathrm{Re}\,u_{\mathrm{lin}}(t)\big\|_{L_x^2}&\lesssim\|U^{-1}e^{-itH}z_+\|_{L_x^6}^3,\end{aligned}$$

and so by (6-19) and (6-20), we have

$$q\big(u_{\mathrm{lin}}(t)-\gamma\langle\nabla\rangle^{-2}|u_{\mathrm{lin}}(t)|^2\big)=2\,\mathrm{Re}\,u_{\mathrm{lin}}(t)+o(1)\quad\text{in } L_x^2 \text{ as } t\to\infty.$$

This completes the proof of (6-21) and hence that of Theorem 1.1. □

It remains to prove Proposition 6.2.

Proof of Proposition 6.2. We first claim that

$$z^{T_n}(t)\rightharpoonup z^\infty(t)\quad\text{weakly in } \dot H_x^1 \text{ for all } t\in\mathbb{R}. \tag{6-22}$$

This relies in an essential way on Theorem 4.1 via (6-6). Henceforth, we let $t\in\mathbb{R}$ be fixed. Using (6-6) and Rellich–Kondrashov and passing to a subsequence, we have $u^{T_n}(t)\to u^\infty(t)$ strongly in $L_x^2(K)$ for any compact $K\subset\mathbb{R}^3$. Now fix $\varphi\in C_c^\infty(\mathbb{R}^3)$. Then $\langle\nabla\rangle^{-2}\varphi\in\mathcal{S}(\mathbb{R}^3)$; in particular, for any $\varepsilon>0$ there exists $\tilde\varphi_\varepsilon\in C_c^\infty(\mathbb{R}^3)$ such that

$$\|\langle\nabla\rangle^{-2}\varphi-\tilde\varphi_\varepsilon\|_{L_x^{3/2}}\le\varepsilon.$$

Using this, Hölder, (6-5), and (6-6), we obtain

$$\begin{aligned}\langle z^{T_n}(t),\varphi\rangle&=\langle u^{T_n}(t),V\varphi\rangle+\gamma\langle|u^{T_n}(t)|^2,\tilde\varphi_\varepsilon\rangle+\gamma\big\langle|u^{T_n}(t)|^2,\langle\nabla\rangle^{-2}\varphi-\tilde\varphi_\varepsilon\big\rangle\\&=\langle u^{T_n}(t),V\varphi\rangle+\gamma\langle|u^{T_n}(t)|^2,\tilde\varphi_\varepsilon\rangle+O(\varepsilon)\\&\to\langle u^\infty(t),V\varphi\rangle+\gamma\langle|u^\infty(t)|^2,\tilde\varphi_\varepsilon\rangle+O(\varepsilon)\\&=\langle z^\infty(t),\varphi\rangle+O(\varepsilon).\end{aligned}$$

As $\varepsilon>0$ was arbitrary, this proves (6-22).

To continue, we write

$$\begin{aligned}&e^{itH}z^{\infty}(t)-z_+\\&=\big[e^{itH}z^{\infty}(t)-e^{iT_0H}z^{\infty}(T_0)\big]+\big[e^{iT_0H}z^{\infty}(T_0)-e^{iT_0H}z^{T_n}(T_0)\big]+\big[e^{iT_0H}z^{T_n}(T_0)-e^{iT_nH}z^{T_n}(T_n)\big].\end{aligned}$$

As the above is bounded in H_x^1, it suffices to prove weak convergence when testing against a dense set of functions in H_x^{-1}. In this role, we take $\varphi \in \mathcal{S}(\mathbb{R}^3)$ with $\hat{\varphi} \in C_c^{\infty}(\mathbb{R}^3 \setminus \{0\})$. To continue, we choose $N_0 \in 2^{\mathbb{Z}}$ so that $\operatorname{supp}\hat{\varphi} \subset \{|\xi| \geq 100N_0\}$ and fix $\varepsilon > 0$. By (6-22), there exists n sufficiently large (depending on T_0) so that

$$\big|\langle e^{iT_0H}z^{\infty}(T_0) - e^{iT_0H}z^{T_n}(T_0), \varphi\rangle\big| \leq \varepsilon. \tag{6-23}$$

To handle the remaining two differences, we will prove the inequality

$$\big|\langle e^{it_2H}z(t_2) - e^{it_1H}z(t_1), \varphi\rangle\big| \lesssim_{\varphi} |t_1|^{-\frac{1}{4}} \tag{6-24}$$

uniformly for $t_2 > t_1$, where z denotes any of the functions z^{T_n}. In view of (6-22), we see that (6-24) also holds (with the same implicit constant) for $z = z^{\infty}$. Thus, taking T_0 large enough depending on ε and then n large enough so that $T_n > T_0$ and (6-23) holds, we get

$$\sup_{t>T_0} |\langle e^{itH}z^{\infty}(t) - z_+, \varphi\rangle| \lesssim_{\varphi} \varepsilon.$$

As $\varepsilon > 0$ was arbitrary, this proves Proposition 6.2.

It remains to verify (6-24). By Duhamel's formula, we have

$$\big|\langle e^{it_2H}z(t_2) - e^{it_1H}z(t_1), \varphi\rangle\big| \leq \int_{t_1}^{t_2} |\langle N_z(u(s)), e^{-isH}\varphi\rangle|\, ds. \tag{6-25}$$

To continue, we decompose the nonlinearity as

$$N_z(u) = N_z^1(u) - \gamma N_z^2(u),$$

where

$$\begin{aligned}N_z^1(u) &= U\big[\tfrac{1}{2}\gamma+2q^2+q^2u_1-\gamma u_1^3-\tfrac{1}{2}\gamma|u|^4\big]-2\gamma i\langle\nabla\rangle^{-2}\nabla\cdot[q\nabla u_2-u_1^2\nabla u_2]+iU^2[\gamma u_1^2u_2+q^2u_2],\\ N_z^2(u) &= U(u_1u_2^2)-\tfrac{1}{3}iU^2(u_2^3).\end{aligned}$$

We first estimate the contribution of $N_z^1(u)$ to (6-25). By Hölder and the dispersive estimate (2-3), we can estimate

$$\begin{aligned}\int_{t_1}^{t_2} \big|\langle N_z^1(u(s)), e^{-isH}\varphi\rangle\big|\, ds &\lesssim \int_{t_1}^{t_2} \|N_z^1(u(s))\|_{L_x^{12/11}}\|e^{-isH}\varphi\|_{L_x^{12}}\, ds\\ &\lesssim_{\varphi} \int_{t_1}^{t_2} |s|^{-\frac{5}{4}}\|N_z^1(u(s))\|_{L_x^{12/11}}\, ds.\end{aligned}$$

Most of the terms appearing in $\operatorname{Re}(N_z^1)$ can be handled using Hölder and (6-5):

$$\big\|U\big[\tfrac{1}{2}\gamma + 2q^2 + q^2u_1 - \gamma u_1^3\big]\big\|_{L_x^{12/11}} \lesssim \|q\|_{L_x^{24/11}}^2 + \|q\|_{L_x^{8/3}}^2\|u_1\|_{L_x^6} + \|u_1\|_{L_x^{36/11}}^3 \lesssim 1.$$

To estimate the remaining term in $\mathrm{Re}(N_z^1)$ we also use the fractional chain rule and Sobolev embedding:

$$\|U(|u|^4)\|_{L_x^{12/11}} \lesssim \big\||\nabla|^{\frac{3}{4}}(|u|^4)\big\|_{L_x^{12/11}} \lesssim \big\||\nabla|^{\frac{3}{4}}u\big\|_{L_x^{12/5}}\|u\|_{L_x^6}^3 \lesssim \|\nabla u\|_{L_x^2}^4 \lesssim 1.$$

To estimate the terms in $\mathrm{Im}(N_z^1)$, we use Hölder and (6-5):

$$\begin{aligned}\big\|\langle\nabla\rangle^{-2}\nabla\cdot[q\nabla u_2 - u_1^2\nabla u_2]\big\|_{L_x^{12/11}} &\lesssim \|\nabla u_2\|_{L_x^2}\|q\|_{L_x^{12/5}} + \|\nabla u_2\|_{L_x^2}\|u_1\|_{L_x^{24/5}}^2 \lesssim 1,\\ \big\|U^2[\gamma u_1^2 u_2 + q^2 u_2]\big\|_{L_x^{12/11}} &\lesssim \|\nabla(u_1^2 u_2)\|_{L_x^{12/11}} + \|q^2 u_2\|_{L_x^{12/11}}\\ &\lesssim \|u_1\|_{L_x^4}\|u\|_{L_x^6}\|\nabla u\|_{L_x^2} + \|q\|_{L_x^{8/3}}^2\|u_2\|_{L_x^6} \lesssim 1.\end{aligned}$$

Putting everything together, we find

$$\int_{t_1}^{t_2} \big|\langle N_z^1(u(s)), e^{-isH}\varphi\rangle\big|\,ds \lesssim_\varphi |t_1|^{-\frac{1}{4}}.$$

We turn now to estimating the contribution of $N_z^2(u)$ to (6-25). To complete the proof of (6-24) and so that of the proposition, we must show that

$$\int_{t_1}^{t_2} \big|\langle N_z^2(u(s)), e^{-isH}\varphi\rangle\big|\,ds \lesssim_\varphi |t_1|^{-\frac{1}{4}}. \tag{6-26}$$

Recalling that $\mathrm{supp}\,\hat{\varphi} \subset \{|\xi| \geq 100N_0\}$, we see that

$$\langle P_{\leq 20N_0} N_z^2(u(s)), e^{-isH}\varphi\rangle \equiv 0.$$

Writing $u_2 = P_{\leq N_0}u_2 + P_{>N_0}u_2$, we may decompose the remaining part of $N_z^2(u)$ as

$$\begin{aligned}P_{>20N_0}N_z^2(u) = P_{>20N_0}U\emptyset\big(u_1u_2P_{>N_0}u_2\big) + P_{>20N_0}U^2\emptyset\big(u_2[P_{>N_0}u_2]^2\big)\\ + P_{>20N_0}U\big\{[P_{>8N_0}u_1][P_{\leq N_0}u_2]^2 - iU\big([P_{>8N_0}u_2][P_{\leq N_0}u_2]^2\big)\big\}.\end{aligned}$$

Writing $u_1 = \bar{v} + iUu_2$ (with $v = Vu$) and $a := [P_{\leq N_0}u_2]^2$, we arrive at the decomposition

$$P_{>20N_0}N_z^2(u) =$$

$$P_{>20N_0}U\emptyset\big(u_1u_2P_{>N_0}u_2\big) + P_{>20N_0}U^2\emptyset\big(u_2[P_{>N_0}u_2]^2\big) \tag{6-27}$$

$$+ iP_{>20N_0}U\big\{aU(P_{>8N_0}u_2) - U\big(aP_{>8N_0}u_2\big)\big\} \tag{6-28}$$

$$+ P_{>20N_0}U\big\{aP_{>8N_0}\bar{v}\big\}. \tag{6-29}$$

As we will see, the terms in (6-27) and (6-28) are small. However, there is no reason to believe that (6-29) is small pointwise in time; instead, we will show that this term is nonresonant.

We first consider (6-27). Using Hölder, Bernstein, and (6-5), we estimate

$$\begin{aligned}\|(6\text{-}27)\|_{L_x^{12/11}} &\lesssim \|u_1\|_{L_x^4}\|u_2\|_{L_x^6}\|P_{>N_0}u_2\|_{L_x^2} + \|u_2\|_{L_x^6}\|P_{>N_0}u_2\|_{L_x^{8/3}}^2\\ &\lesssim_{N_0} \|u_1\|_{L_x^4}\|\nabla u_2\|_{L_x^2}^2 + \|\nabla u_2\|_{L_x^2}^3 \lesssim_{N_0} 1.\end{aligned}$$

Thus, the contribution of this term to (6-26) is acceptable.

We now turn to (6-28), which includes the commutator $[a, U]$. We regard this term as a bilinear operator $T(a, P_{>8N_0}u_2)$ with symbol given by

$$\Big[1-\phi\Big(\frac{\xi}{20N_0}\Big)\Big]U(\xi)[U(\xi_2)-U(\xi)]\phi\Big(\frac{\xi_1}{4N_0}\Big) = \Big\{-2\gamma\frac{U(\xi)(\xi_1+2\xi_2)}{\langle\xi\rangle\langle\xi_2\rangle(|\xi_2|\langle\xi\rangle+|\xi|\langle\xi_2\rangle)}\Big[1-\phi\Big(\frac{\xi}{20N_0}\Big)\Big]\phi\Big(\frac{\xi_1}{4N_0}\Big)\Big\}\cdot\xi_1,$$

where ϕ denotes the standard Littlewood–Paley multiplier. Observing that the multiplier inside the braces is amenable to Lemma 2.3, we may estimate

$$\|(6\text{-}28)\|_{L_x^{12/11}} \lesssim \|\nabla a\|_{L_x^{3/2}}\|P_{>8N_0}u_2\|_{L_x^4} \lesssim_{N_0} \|\nabla P_{\le N_0}u_2\|_{L_x^2}\|P_{\le N_0}u_2\|_{L_x^6}\|\nabla u_2\|_{L_x^2}.$$

In view of (6-5), the contribution of this term to (6-26) is acceptable.

Finally, we consider (6-29). Using (1-11), we find

$$i\partial_t\Big\langle aP_{>8N_0}\bar v, e^{-itH}\frac{U}{2H}\varphi\Big\rangle = \langle U(aP_{>8N_0}\bar v), e^{-itH}\varphi\rangle + \Big\langle aP_{>8N_0}\overline{N_v(u)}, e^{-itH}\frac{U}{2H}\varphi\Big\rangle + \Big\langle [a,H]P_{>8N_0}\bar v, e^{-itH}\frac{U}{2H}\varphi\Big\rangle + i\Big\langle \dot a P_{>8N_0}\bar v, e^{-itH}\frac{U}{2H}\varphi\Big\rangle.$$

By the fundamental theorem of calculus, we may thus estimate the contribution of (6-29) to (6-26) as

$$\begin{aligned}\int_{t_1}^{t_2} \big|\langle U(aP_{>8N_0}\bar v), e^{-isH}\varphi\rangle\big|\,ds &\lesssim \sup_{t\ge t_1}\big|\langle aP_{>8N_0}\bar v, e^{-itH}\langle\nabla\rangle^{-2}\varphi\rangle\big| + \int_{t_1}^{t_2}\big|\langle aP_{>8N_0}\overline{N_v(u)}, e^{-isH}\langle\nabla\rangle^{-2}\varphi\rangle\big|\,ds \\ &\quad + \int_{t_1}^{t_2}\big|\langle\langle\nabla\rangle^{-2}([a,H]P_{>8N_0}\bar v), e^{-isH}\varphi\rangle\big|\,ds + \int_{t_1}^{t_2}\big|\langle\dot a P_{>8N_0}\bar v, e^{-isH}\langle\nabla\rangle^{-2}\varphi\rangle\big|\,ds. \end{aligned}\tag{6-30}$$

To estimate the terms on the right-hand side of (6-30), we note that in view of (6-5),

$$\|\nabla a(t)\|_{L_x^{3/2}\cap L_x^\infty} + \|a(t)\|_{L_x^3\cap L_x^\infty} + \|\nabla v(t)\|_{L_x^2} + \|v(t)\|_{L_x^3\cap L_x^6} \lesssim_{N_0} 1 \tag{6-31}$$

uniformly for $t\in\mathbb{R}$. Using this, Hölder, and (2-3), we estimate the first term on the right-hand side of (6-30) as

$$\sup_{t\ge t_1}\big|\langle aP_{>8N_0}\bar v, e^{-itH}\langle\nabla\rangle^{-2}\varphi\rangle\big| \lesssim \|a\|_{L_t^\infty L_x^3}\|v\|_{L_t^\infty L_x^3}\sup_{t\ge t_1}\|e^{-itH}\langle\nabla\rangle^{-2}\varphi\|_{L_x^3} \lesssim_{\varphi,N_0} |t_1|^{-\frac12}.$$

Thus, the contribution of this term to (6-26) is acceptable.

Next we consider the second term on the right-hand side of (6-30). By Hölder and (2-3),

$$\int_{t_1}^{t_2}\big|\langle aP_{>8N_0}\overline{N_v(u)}, e^{-isH}\langle\nabla\rangle^{-2}\varphi\rangle\big|\,ds \lesssim_\varphi |t_1|^{-\frac12}\|aP_{>8N_0}N_v(u)\|_{L_t^\infty L_x^1}.$$

Note that

$$N_v(u) = \sum_{k=2}^5 U\text{Ø}(u^k) + \text{Ø}(u^k).$$

To estimate the contribution of the quadratic terms in $N_v(u)$, we use Bernstein, (6-5), and (6-31):

$$\begin{aligned}\big\| a P_{>8N_0}\big[U\O(u^2)+\O(u^2)\big]\big\|_{L_t^\infty L_x^1} &\lesssim_{N_0} \|a\|_{L_t^\infty L_x^3}\|\nabla \O(u^2)\|_{L_t^\infty L_x^{3/2}} \\ &\lesssim_{N_0} \|a\|_{L_t^\infty L_x^3}\|\nabla u\|_{L_t^\infty L_x^2}\|u\|_{L_t^\infty L_x^6} \lesssim_{N_0} 1.\end{aligned}$$

Similarly, we can estimate the cubic terms in $N_v(u)$ via

$$\begin{aligned}\big\| a P_{>8N_0}\big[U\O(u^3)+\O(u^3)\big]\big\|_{L_t^\infty L_x^1} &\lesssim_{N_0} \|a\|_{L_t^\infty L_x^6}\|\nabla \O(u^3)\|_{L_t^\infty L_x^{6/5}} \\ &\lesssim_{N_0} \|a\|_{L_t^\infty L_x^6}\|\nabla u\|_{L_t^\infty L_x^2}\|u\|^2_{L_t^\infty L_x^6} \lesssim_{N_0} 1.\end{aligned}$$

We estimate the quartic and quintic terms in $N_v(u)$ using Hölder, (6-5), and (6-31):

$$\begin{aligned}\big\| a P_{>8N_0}\big[U\O(u^4)+\O(u^4)\big]\big\|_{L_t^\infty L_x^1} &\lesssim \|a\|_{L_t^\infty L_x^3}\|u\|^4_{L_t^\infty L_x^6} \lesssim 1, \\ \big\| a P_{>8N_0}\big[U\O(u^5)+\O(u^5)\big]\big\|_{L_t^\infty L_x^1} &\lesssim \|a\|_{L_t^\infty L_x^6}\|u\|^5_{L_t^\infty L_x^6} \lesssim 1.\end{aligned}$$

Putting everything together, we see that the contribution of the second term on the right-hand side of (6-30) to (6-26) is acceptable.

We now turn to the third term on the right-hand side of (6-30). By Hölder and (2-3),

$$\int_{t_1}^{t_2} \big|\big\langle \langle\nabla\rangle^{-2}\big([a,H]P_{>8N_0}\bar v\big), e^{-isH}\varphi\big\rangle\big|\,ds \lesssim_\varphi |t_1|^{-\frac14}\big\|\langle\nabla\rangle^{-2}\big([a,H]P_{>8N_0}\bar v\big)\big\|_{L_t^\infty L_x^{12/11}}.$$

We regard the term on the right-hand side above as a bilinear operator $T(a,v)$ with symbol given by

$$\frac{H(\xi_2)-H(\xi)}{\langle\xi\rangle^2}\phi\Big(\frac{\xi_1}{4N_0}\Big)\Big[1-\phi\Big(\frac{\xi_2}{8N_0}\Big)\Big] = m(\xi_1,\xi_2)\cdot\xi_1,$$

where

$$m(\xi_1,\xi_2) = -\frac{(2\gamma+|\xi|^2+|\xi_2|^2)(\xi_1+2\xi_2)}{\langle\xi\rangle^2(|\xi_2|\langle\xi_2\rangle+|\xi|\langle\xi\rangle)}\phi\Big(\frac{\xi_1}{4N_0}\Big)\Big[1-\phi\Big(\frac{\xi_2}{8N_0}\Big)\Big]$$

is a bounded bilinear multiplier in view of Lemma 2.3. Using also (6-31), we get

$$\big\|\langle\nabla\rangle^{-2}\big([a,H]P_{>8N_0}\bar v\big)\big\|_{L_t^\infty L_x^{12/11}} \lesssim \|\nabla a\|_{L_t^\infty L_x^{3/2}}\|v\|_{L_t^\infty L_x^4} \lesssim_{N_0} 1.$$

Thus, the contribution of the third term on the right-hand side of (6-30) to (6-26) is acceptable.

We now turn to the fourth and last term on the right-hand side of (6-30). By Hölder, (2-3), and Bernstein,

$$\begin{aligned}\int_{t_1}^{t_2} \big|\big\langle \dot a P_{>8N_0}\bar v, e^{-isH}\langle\nabla\rangle^{-2}\varphi\big\rangle\big|\,ds &\lesssim_\varphi |t_1|^{-\frac12}\|\dot a\|_{L_t^\infty L_x^2}\|P_{>8N_0}v\|_{L_t^\infty L_x^2} \\ &\lesssim_{\varphi,N_0} |t_1|^{-\frac12}\|P_{\le N_0}\dot u_2\|_{L_t^\infty L_x^3}\|u_2\|_{L_t^\infty L_x^6}\|\nabla v\|_{L_t^\infty L_x^2}.\end{aligned}$$

In view of (6-5) and (6-31), we need only bound $P_{\leq N_0}\dot{u}_2$ in $L_t^\infty L_x^3$. To this end, we use (1-6), Bernstein, and (6-5):

$$\|P_{\leq N_0}\dot{u}_2\|_{L_t^\infty L_x^3} \lesssim \|P_{\leq N_0}(2\gamma-\Delta)u_1\|_{L_t^\infty L_x^3} + \sum_{k=2}^{5}\|P_{\leq N_0}\mathcal{O}(u^k)\|_{L_t^\infty L_x^3}$$

$$\lesssim_{N_0} \|u_1\|_{L_t^\infty L_x^3} + \sum_{k=2}^{5}\|u\|_{L_t^\infty L_x^6}^k \lesssim_{N_0} 1.$$

Thus, the contribution of the fourth term on the right-hand side of (6-30) to (6-26) is acceptable. This completes the justification of (6-26) and so the proof of Proposition 6.2. □

7. Proof of Theorem 1.4

In this section we prove Theorem 1.4 and Corollary 1.7. We recall the norm

$$\|u\|_{X_T} := \sup_{t\geq T} t^{\frac{1}{2}}\|u(t)\|_{H_x^{1,3}(\mathbb{R}^3)}.$$

The proof of Theorem 1.4 will be effected by running a contraction mapping argument simultaneously for u and $z = M(u)$. The necessity of exploiting the normal form transformation can be seen when one endeavors to estimate the quadratic terms appearing in the nonlinearity.

Proof of Theorem 1.4.. We define maps

$$\begin{cases} [\Phi_1(u,z)](t) = V^{-1}z(t) - \gamma\langle\nabla\rangle^{-2}|u(t)|^2, \\ [\Phi_2(u)](t) = e^{-itH}Vu_+ + i\displaystyle\int_t^\infty e^{-i(t-s)H}N_z(u(s))\,ds, \end{cases}$$

where N_z is as in (5-5).

We will show that the map $(u,z)\mapsto \Phi(u,z) := (\Phi_1(u,z),\Phi_2(u))$ is a contraction on a suitable complete metric space, and so deduce that Φ has a unique fixed point (u,z) in this space, which then necessarily solves (5-4)–(5-5).

For $0<\eta<1$ and $T>1$ to be determined below, we define

$$B_1 = \big\{u : \|u\|_{L_t^\infty(H^1_{\text{real}}+i\dot{H}^1_{\text{real}})} \leq 4\|u_+\|_{H^1_{\text{real}}+i\dot{H}^1_{\text{real}}},\ \|u\|_{X_T}\leq 4\eta\big\},$$

and

$$B_2 = \big\{z : \|z\|_{L_t^\infty H_x^1} \leq 2\|u_+\|_{H^1_{\text{real}}+i\dot{H}^1_{\text{real}}},\ \|V^{-1}z\|_{X_T}\leq 2\eta\big\},$$

where here and in what follows all space-time norms are taken over $(T,\infty)\times\mathbb{R}^3$ unless stated otherwise. We define $B = B_1\times B_2$ and equip B with the metric

$$d((u,z),(\tilde{u},\tilde{z})) = \|u-\tilde{u}\|_{X_T} + 8\|V^{-1}(z-\tilde{z})\|_{X_T}.$$

We first show that $\Phi : B\to B$. By Sobolev embedding, for $(u,z)\in B$ and $t>T\geq 1$,

$$\big\|\langle\nabla\rangle^{-2}|u(t)|^2\big\|_{H_x^1} + \big\|\langle\nabla\rangle^{-2}|u(t)|^2\big\|_{H_x^{1,3}} \lesssim \big\||u(t)|^2\big\|_{L_x^{3/2}} \lesssim \|u(t)\|_{L_x^3}^2 \lesssim \eta^2 t^{-1}.$$

Thus choosing $T = T(\|u_+\|_{H^1_{\text{real}}+i\dot H^1_{\text{real}}})$ large enough, we have

$$\big\|[\Phi_1(u,z)](t)\big\|_{H^1_{\text{real}}+i\dot H^1_{\text{real}}} \le \|V^{-1}z(t)\|_{H^1_{\text{real}}+i\dot H^1_{\text{real}}} + \gamma\big\|\langle\nabla\rangle^{-2}|u(t)|^2\big\|_{H^1_x} \le 2\|z\|_{L^\infty_t H^1_x}.$$

Similarly,

$$\big\|[\Phi_1(u,z)](t)\big\|_{H^{1,3}_x} \le \|V^{-1}z(t)\|_{H^{1,3}_x} + \gamma\big\|\langle\nabla\rangle^{-2}|u(t)|^2\big\|_{H^{1,3}_x} \le 4\eta t^{-\frac12},$$

provided η is chosen small enough. Thus $\Phi_1 : B \to B_1$.

We next show that $\Phi_2 : B_1 \to B_2$. We first estimate $N_z(u)$, which satisfies

$$U^{-1}N_z(u) = Ø(u^2+u^3+u^4+u^5) + U^{-1}\frac{\nabla}{\langle\nabla\rangle^2}\cdot Ø(u\nabla u) + UØ(u^3+u^4+u^5). \tag{7-1}$$

We estimate the quadratic terms at fixed time $t > T \ge 1$, as

$$\Big\|\Big[Ø(u^2) + U^{-1}\frac{\nabla}{\langle\nabla\rangle^2}\cdot Ø(u\nabla u)\Big](t)\Big\|_{H^{1,3/2}_x} \lesssim t^{-1}\|u\|^2_{X_T} \lesssim t^{-1}\eta^2.$$

Similarly, for $k \in \{2,3,4\}$ we have

$$\begin{aligned}\big\|\big[Ø(u^{k+1}) + UØ(u^{k+1})\big](t)\big\|_{H^{1,3/2}_x} &\lesssim \|u(t)\|^k_{L^{3k}_x}\|u(t)\|_{H^{1,3}_x}\\ &\lesssim \||\nabla|^{1-\frac1k}u(t)\|^k_{L^3_x}\|u(t)\|_{H^{1,3}_x} \lesssim t^{-\frac{k+1}{2}}\|u\|^{k+1}_{X_T} \lesssim t^{-\frac{k+1}{2}}\eta^{k+1}.\end{aligned} \tag{7-2}$$

Combining the above, we deduce that

$$\|U^{-1}N_z(u(t))\|_{H^{1,3/2}_x} \lesssim \sum_{k=1}^4 t^{-\frac{k+1}{2}}\eta^{k+1} \quad \text{uniformly for } t > T \ge 1. \tag{7-3}$$

To continue, we use Strichartz and (7-3) to estimate

$$\begin{aligned}\|\Phi_2(u)\|_{L^\infty_t H^1_x} &\le \|Vu_+\|_{L^\infty_t H^1_x} + C\|N_z(u)\|_{L^{4/3}_t H^{1,3/2}_x}\\ &\le \|u_+\|_{H^1_{\text{real}}+i\dot H^1_{\text{real}}} + C\sum_{k=1}^4 T^{-\frac{2k-1}{4}}\eta^{k+1} \le 2\|u_+\|_{H^1_{\text{real}}+i\dot H^1_{\text{real}}},\end{aligned}$$

provided $T = T(\|u_+\|_{H^1_{\text{real}}+i\dot H^1_{\text{real}}})$ is chosen sufficiently large.

We turn to estimating $V^{-1}\Phi_2(u)$ in the X-norm for $u \in B_1$. By hypothesis, the dispersive estimate (2-3), and (7-3), for $t > T \ge 1$ we have

$$\begin{aligned}\|[V^{-1}\Phi_2(u)](t)\|_{H^{1,3}_x} &\le \|V^{-1}e^{-itH}Vu_+\|_{H^{1,3}_x} + \Big\|\int_t^\infty V^{-1}[ie^{-i(t-s)H}N_z(u(s))]\,ds\Big\|_{H^{1,3}_x}\\ &\le \eta t^{-\frac12} + \int_t^\infty \big\|e^{-i(t-s)H}U^{-1}N_z(u(s))]\big\|_{H^{1,3}_x}\,ds\\ &\le \eta t^{-\frac12} + C\int_t^\infty |t-s|^{-\frac12}\sum_{k=1}^4 s^{-\frac{k+1}{2}}\eta^{k+1}\,ds \le \eta t^{-\frac12} + C\sum_{k=1}^4 t^{-\frac{k}{2}}\eta^{k+1} \le 2\eta t^{-\frac12},\end{aligned}$$

provided η is chosen sufficiently small. This completes the proof that $\Phi : B \to B$.

We next claim that Φ is a contraction with respect to the metric defined above. First, for $(u,z),(\tilde{u},\tilde{z})\in B$, we estimate

$$\begin{aligned}\big\|\Phi_1(u,z)-\Phi_1(\tilde{u},\tilde{z})\big\|_{X_T} &\le \|V^{-1}(z-\tilde{z})\|_{X_T}+\gamma\big\|\langle\nabla\rangle^{-2}(|u|^2-|\tilde{u}|^2)\big\|_{X_T}\\ &\le \tfrac{1}{8}d((u,z),(\tilde{u},\tilde{z}))+C\sup_{t\ge T} t^{\frac{1}{2}}\big\|(u+\tilde{u})(t)(u-\tilde{u})(t)\big\|_{L_x^{3/2}}\\ &\le \tfrac{1}{8}d((u,z),(\tilde{u},\tilde{z}))+C\eta T^{-\frac{1}{2}}\|u-\tilde{u}\|_{X_T}\\ &\le \tfrac{1}{4}d((u,z),(\tilde{u},\tilde{z})),\end{aligned}$$

provided η is sufficiently small.

By (2-3), for $t>T\ge 1$ we estimate

$$\begin{aligned}\big\|V^{-1}[\Phi_2(u)-\Phi_2(\tilde{u})](t)\big\|_{H_x^{1,3}} &\le \bigg\|\int_t^\infty V^{-1}\big(ie^{-i(t-s)H}[N_z(u(s))-N_z(\tilde{u}(s))]\big)\,ds\bigg\|_{H_x^{1,3}}\\ &\le \int_t^\infty |t-s|^{-\frac{1}{2}}\big\|U^{-1}[N_z(u(s))-N_z(\tilde{u}(s))]\big\|_{H_x^{1,3/2}}\,ds.\end{aligned}$$

Writing w to indicate that either u or $\tilde{u}$ may appear, we have

$$U^{-1}[N_z(u)-N_z(\tilde{u})]=\sum_{k=1}^{4}\emptyset[w^k(u-\tilde{u})]+\frac{U^{-1}\nabla}{\langle\nabla\rangle^2}\cdot[(u-\tilde{u})\nabla w+w\nabla(u-\tilde{u})]+U\sum_{k=2}^{4}\emptyset[w^k(u-\tilde{u})].$$

We estimate the contribution of the quadratic terms via

$$\begin{aligned}&\int_t^\infty |t-s|^{-\frac{1}{2}}\bigg\|\emptyset[w(u-\tilde{u})](s)+\frac{U^{-1}\nabla}{\langle\nabla\rangle^2}\cdot[(u-\tilde{u})\nabla w+w\nabla(u-\tilde{u})](s)\bigg\|_{H_x^{1,3/2}}\,ds\\ &\qquad\lesssim \|w\|_{X_T}\|u-\tilde{u}\|_{X_T}\int_t^\infty |t-s|^{-\frac{1}{2}}s^{-1}\,ds\lesssim t^{-\frac{1}{2}}\eta\|u-\tilde{u}\|_{X_T}.\end{aligned}$$

Arguing as in (7-2), we obtain

$$\begin{aligned}&\int_t^\infty |t-s|^{-\frac{1}{2}}\bigg\|\sum_{k=2}^{4}\emptyset[w^k(u-\tilde{u})](s)+U\emptyset[w^k(u-\tilde{u})](s)\bigg\|_{H_x^{1,3/2}}\,ds\\ &\qquad\lesssim \|w\|_{X_T}^k\|u-\tilde{u}\|_{X_T}\int_t^\infty |t-s|^{-\frac{1}{2}}s^{-\frac{k+1}{2}}\,ds\lesssim t^{-\frac{k}{2}}\eta^k\|u-\tilde{u}\|_{X_T}.\end{aligned}$$

Thus for η sufficiently small we get

$$8\big\|V^{-1}[\Phi_2(u)-\Phi_2(\tilde{u})]\big\|_{X_T}\le \tfrac{1}{4}d((u,z),(\tilde{u},\tilde{z})).$$

This completes the proof that Φ is a contraction on B. Hence there exists a unique $(u,z)\in B$ such that $\Phi(u,z)=(u,z)$. In particular $z=M(u)$ and (u,z) solves (5-4)–(5-5) on $(T,\infty)\times\mathbb{R}^3$. We note that by construction we have $u_1\in H_x^1$ and $u\in L_x^3\cap L_x^6$. In particular, $q(u)=|u|^2+2u_1\in L_x^2$ and hence $u(t)\in\mathcal{E}$ for $t>T$.

For $\gamma \in \left[\frac{2}{3}, 1\right)$, Theorem 3.3 guarantees that the solution u can be extended (in a unique way) to be global in time. For $\gamma \in \left(0, \frac{2}{3}\right)$, global existence follows from [Killip et al. 2012, Theorem 1.3], while uniqueness in the energy space follows from Theorem 3.3 (see also Remark 3.4).

Next we show that (1-25) holds; indeed, we prove the stronger claim (1-26). We first note that Strichartz combined with (7-3) gives

$$\|z(t) - e^{-itH} V u_+\|_{H_x^1} \lesssim t^{-\frac{1}{4}},$$

which in turn implies

$$\left\| V^{-1} z(t) - V^{-1} e^{-itH} V u_+ \right\|_{H^1_{\mathrm{real}} + i \dot{H}^1_{\mathrm{real}}} \lesssim t^{-\frac{1}{4}}.$$

As $z = M(u)$, for $t > T$ we have

$$\left\| V^{-1} z(t) - u(t) \right\|_{H^1_{\mathrm{real}} + i \dot{H}^1_{\mathrm{real}}} \lesssim \left\| \langle\nabla\rangle^{-2} |u(t)|^2 \right\|_{H^1_x} \lesssim \left\| |u(t)|^2 \right\|_{L_x^{3/2}} \lesssim t^{-1} \|u\|_{X_T}^2 .$$

Therefore, by the triangle inequality we may conclude that

$$\left\| u(t) - V^{-1} e^{-itH} V u_+ \right\|_{H^1_{\mathrm{real}} + i \dot{H}^1_{\mathrm{real}}} \lesssim t^{-\frac{1}{4}}.$$

By the arguments presented so far, it is clear that u is the unique solution in B_1 that obeys (1-25). This is slightly weaker than is claimed in Theorem 1.4, which places no restrictions on the $H^1_{\mathrm{real}} + i \dot{H}^1_{\mathrm{real}}$-norm of alternate solutions $v(t)$, nor any restriction on the value of T for which $\|v\|_{X_T} \le 4\eta$; however, any solution $v(t)$ obeying (1-25) must have

$$\|v\|_{L_t^\infty([T,\infty); H^1_{\mathrm{real}} + i \dot{H}^1_{\mathrm{real}})} \le 4 \|u_+\|_{H^1_{\mathrm{real}} + i \dot{H}^1_{\mathrm{real}}}$$

for some T large enough. Thus the equality of $v(t)$ and $u(t)$ follows from the contraction mapping argument above with T large enough combined with uniqueness in the energy space. □

Finally, we prove Corollary 1.7.

Proof of Corollary 1.7. The proof consists of showing that smallness of the weighted norms implies the smallness condition (1-24). In view of (2-4), it suffices to show

$$\|e^{\pm itH} u_+\|_{H_x^{1,3}} \lesssim |t|^{-\frac{1}{2}} \eta \quad \text{and} \quad \|e^{\pm itH} U^{-1} \operatorname{Re} u_+\|_{H_x^{1,3}} \lesssim |t|^{-\frac{1}{2}} \eta.$$

By the dispersive estimate (2-3) and Hölder,

$$\|e^{\pm itH} u_+\|_{H_x^{1,3}} \lesssim |t|^{-\frac{1}{2}} \|\langle\nabla\rangle u_+\|_{L_x^{3/2}} \lesssim |t|^{-\frac{1}{2}} \left\| \langle x\rangle^{\frac{1}{2}+} \langle\nabla\rangle u_+ \right\|_{L_x^2}$$

and

$$\|e^{\pm itH} U^{-1} \operatorname{Re} u_+\|_{H_x^{1,3}} \lesssim |t|^{-\frac{1}{2}} \left\| U^{-\frac{5}{6}} \langle\nabla\rangle \operatorname{Re} u_+ \right\|_{L_x^{3/2}}.$$

Using Hölder and Sobolev embedding, we obtain

$$\begin{aligned} \|\nabla U^{-\frac{5}{6}} \operatorname{Re} u_+\|_{L_x^{3/2}} &\lesssim \|\langle\nabla\rangle u_+\|_{L_x^{3/2}} \lesssim \|\langle x\rangle^{\frac{1}{2}+} \langle\nabla\rangle u_+\|_{L_x^2}, \\ \|U^{-\frac{5}{6}} \operatorname{Re} u_+\|_{L_x^{3/2}} &\lesssim \left\| |\nabla|^{\frac{5}{6}} U^{-\frac{5}{6}} \operatorname{Re} u_+ \right\|_{L_x^{18/17}} \lesssim \left\| \langle x\rangle^{\frac{4}{3}+} \langle\nabla\rangle^{\frac{5}{6}} \operatorname{Re} u_+ \right\|_{L_x^2}. \end{aligned}$$ □

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