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A SECOND ORDER ESTIMATE FOR GENERAL COMPLEX HESSIAN EQUATIONS

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We consider the general complex Hessian equations with right-hand sides depending on gradients, which are motivated by the Fu–Yau equations arising from the study of Strominger systems. The second order estimate for the solution is crucial to solving the equation by the method of continuity. We obtain such an estimate for the χ-plurisubharmonic solutions.

1. Introduction

Let (X, ω) be a compact Kähler manifold of dimension $n \geq 2$. Let $u \in C^\infty(X)$ and consider a $(1, 1)$-form $\chi(z, u)$ possibly depending on u and satisfying the positivity condition $\chi \geq \varepsilon\omega$ for some $\varepsilon > 0$. We define

$$g = \chi(z, u) + i\partial\bar{\partial}u, \tag{1-1}$$

and u is called χ-plurisubharmonic if $g > 0$ as a $(1, 1)$ form. In this paper, we are concerned with the complex Hessian equation

$$(\chi(z, u) + i\partial\bar{\partial}u)^k \wedge \omega^{n-k} = \psi(z, Du, u)\,\omega^n \tag{1-2}$$

for $1 \leq k \leq n$, where $\psi(z, v, u) \in C^\infty(T^{1,0}(X)^* \times \mathbb{R})$ is a given strictly positive function.

The complex Hessian equation can be viewed as an intermediate equation between the Laplace equation and the complex Monge–Ampère equation. It encompasses the most natural invariants of the complex Hessian matrix of a real valued function, namely the elementary symmetric polynomials of its eigenvalues. When $k = 1$, the equation (1-2) is quasilinear, and the estimates follow from the classical theory of quasilinear PDEs. The real counterparts of (1-2) for $1 < k \leq n$, with ψ not depending on the gradient of u, have been studied extensively in the literature (see the survey paper [Wang 2009] and more recent related work [Guan 2014]), as these equations appear naturally and play very important roles in both classical and conformal geometry. When the right-hand side ψ depends on the gradient of the solution, even the real case has been a long-standing problem due to substantial difficulties in obtaining a priori C^2 estimates. This problem has recently been solved by Guan, Ren and Wang [Guan et al. 2015] for convex solutions of real Hessian equations.

In the complex case, equation (1-2) with $\psi = \psi(z, u)$ has been extensively studied in recent years, due to its appearance in many geometric problems, including the J-flow [Song and Weinkove 2008] and

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quaternionic geometry [Alesker and Verbitsky 2010]. The related Dirichlet problem for (1-2) on domains in $\mathbb{C}^n$ has been studied by Li [2004] and Błocki [2005]. The corresponding problem on compact Kähler or Hermitian manifolds has also been studied extensively; see, for example, [Dinew and Kołodziej 2014; Hou 2009; Kołodziej and Nguyen 2016; Lu and Nguyen 2015; Zhang 2010]. In particular, as a crucial step in the continuity method, C^2 estimates for complex Hessian type equations have been studied in various settings; see [Hou et al. 2010; Sun 2014; Székelyhidi 2015; Székelyhidi et al. 2015; Zhang 2015].

However, (1-2) with $\psi = \psi(z, Du, u)$ has been much less studied. An important case corresponding to $k = n = 2$, so that it is actually a Monge–Ampère equation in two dimensions, is central to the solution by Fu and Yau [2008; 2007] of a Strominger system on a toric fibration over a K3 surface. A natural generalization of this case to general dimension n was suggested by Fu and Yau [2008] and can be expressed as

$$((e^u + fe^{-u})\omega + ni\partial\bar{\partial}u)^2 \wedge \omega^{n-2} = \psi(z, Du, u)\,\omega^n, \tag{1-3}$$

where $\psi(z, v, u)$ is a function on $T^{1,0}(X)^* \times \mathbb{R}$ with a particular structure, and (X, ω) is a compact Kähler manifold. A priori estimates for this equation were obtained by the authors in [Phong et al. 2015].

In this paper, motivated by our previous work [Phong et al. 2015], we study a priori C^2 estimates for the equation (1-2) with general $\chi(z, u)$ and general right-hand side $\psi(z, Du, u)$. Building on the techniques developed in [Guan et al. 2015] (see also [Li et al. 2016] for real Hessian equations), we can prove the following theorem.

Theorem 1. *Let (X, ω) be a compact Kähler manifold of complex dimension n. Suppose $u \in C^4(X)$ is a solution of* (1-2) *with $g = \chi + i\partial\bar{\partial}u > 0$ and $\chi(z, u) \geq \varepsilon\omega$. Let $0 < \psi(z, v, u) \in C^\infty(T^{1,0}(X)^* \times \mathbb{R})$. Then we have the uniform second order derivative estimate*

$$|D\bar{D}u|_\omega \leq C, \tag{1-4}$$

where C is a positive constant depending only on $\varepsilon, n, k, \sup_X |u|, \sup_X |Du|$, and the C^2 norm of χ as a function of (u, z), the infimum of ψ, and the C^2 norm of ψ as a function of (z, Du, u), all restricted to the ranges in Du and u defined by the uniform upper bounds on $|u|$ and $|Du|$.

We remark that the above estimate is stated for χ-plurisubharmonic solutions, that is, $g = \chi + i\partial\bar{\partial}u > 0$. Actually, we only need to assume that g is in the Γ_{k+1} cone (see (3-11) below for the definition of the Garding cone Γ_k and also the discussion in Remark 2 at the end of the paper). However, a better condition would be $g \in \Gamma_k$, which is the natural cone for ellipticity. In fact, this is still an open problem even for real Hessian equations when $2 < k < n$. If $k = 2$, [Guan et al. 2015] removed the convexity assumption by investigating the structure of the operator. A simpler argument was given recently by Spruck and Xiao [2015]. However, the arguments are not applicable to the complex case due to the difference between the terms $|DDu|^2$ and $|D\bar{D}u|^2$ in the complex setting. When $k = 2$ in the complex setting, C^2 estimates for (1-3) were obtained in [Phong et al. 2015] without the plurisubharmonicity assumption, but the techniques rely on the specific right-hand side $\psi(z, Du, u)$ studied there.

We also note that if $k = n$, the condition $g = \chi + i\partial\bar{\partial}u > 0$ is the natural assumption for the ellipticity of equation (1-2). Thus, our result implies the a priori C^2 estimate for complex Monge–Ampère equations

with right-hand side depending on gradients:

$$(\chi(z,u)+i\partial\bar\partial u)^n=\psi(z,Du,u)\,\omega^n.$$

This generalizes the C^2 estimate for the equation studied by Fu and Yau [2008; 2007] mentioned above, which corresponds to $n=2$ and a specific form $\chi(z,u)$ as well as a specific right-hand side $\psi(z,Du,u)$. For dimension $n\geq 2$ and $k=n$, the estimate was obtained by Guan and Ma, in unpublished notes, using a different method where the structure of the Monge–Ampère operator plays an important role.

Compared to the estimates when $\psi=\psi(z,u)$, the dependence on the gradient of u in (1-2) creates substantial new difficulties. The main obstacle is the appearance of terms such as $|DDu|^2$ and $|D\bar Du|^2$ when one differentiates the equation twice. We adapt the techniques used in [Guan et al. 2015] and [Li et al. 2016] for real Hessian equations to overcome these difficulties. Furthermore, we also need to handle properly some subtle issues when dealing with the third-order terms due to complex conjugacy.

2. Preliminaries

Let σ_k be the k-th elementary symmetric function; that is, for $1\leq k\leq n$ and $\lambda=(\lambda_1,\dots,\lambda_n)\in\mathbb{R}^n$,

$$\sigma_k(\lambda)=\sum_{1<i_1<\cdots<i_k<n}\lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}.$$

Let $\lambda(a_{\bar ji})$ denote the eigenvalues of a Hermitian symmetric matrix $(a_{\bar ji})$ with respect to the background Kähler metric ω. We define $\sigma_k(a_{\bar ji})=\sigma_k(\lambda(a_{\bar ji}))$. This definition can be naturally extended to complex manifolds. Denoting by $A^{1,1}(X)$ the space of smooth real $(1,1)$-forms on a compact Kähler manifold (X,ω), we define, for any $g\in A^{1,1}(X)$,

$$\sigma_k(g)=\binom{n}{k}\frac{g^k\wedge\omega^{n-k}}{\omega^n}.$$

Using the above notation, we can rewrite (1-2) as follows:

$$\sigma_k(g)=\sigma_k(\chi_{\bar ji}+u_{\bar ji})=\psi(z,Du,u). \tag{2-1}$$

We use the notation

$$\sigma_k^{p\bar q}=\frac{\partial\sigma_k(g)}{\partial g_{\bar qp}},\quad \sigma_k^{p\bar q,r\bar s}=\frac{\partial^2\sigma_k(g)}{\partial g_{\bar qp}\bar\partial g_{\bar sr}}.$$

The symbol D indicates the covariant derivative with respect to the given metric ω. All norms and inner products are with respect to ω unless denoted otherwise. We denote by $\lambda_1,\lambda_2,\dots,\lambda_n$ the eigenvalues of $g_{\bar ji}=\chi_{\bar ji}+u_{\bar ji}$ with respect to ω, and use the ordering $\lambda_1\geq\lambda_2\geq\cdots\geq\lambda_n>0$. Our calculations are carried out at a point z on the manifold X, and we use coordinates such that at this point $\omega=i\sum\delta_{\ell k}\,dz^k\wedge d\bar z^\ell$ and $g_{\bar ji}$ is diagonal. We also use the notation

$$\mathcal{F}=\sum_p\sigma_k^{p\bar p}.$$

Differentiating (2-1) yields

$$\sigma_k^{p\bar q}D_{\bar j}g_{\bar qp}=D_{\bar j}\psi. \tag{2-2}$$

Differentiating the equation a second time gives

$$\sigma_k^{p\bar{q}} D_i D_{\bar{j}} g_{\bar{q}p} + \sigma_k^{p\bar{q},r\bar{s}} D_i g_{\bar{q}p} D_{\bar{j}} g_{\bar{s}r} = D_i D_{\bar{j}} \psi \geq -C(1+|DDu|^2+|D\bar{D}u|^2) + \sum_\ell \psi_{v_\ell} u_{\ell\bar{j}i} + \sum_\ell \psi_{\bar{v}_\ell} u_{\bar{\ell}\bar{j}i}. \quad (2\text{-}3)$$

We denote by C a uniform constant which depends only on (X,ω), n, k, $\|\chi\|_{C^2}$, $\inf\psi$, $\|u\|_{C^1}$ and $\|\psi\|_{C^2}$. We now compute the operator $\sigma_k^{p\bar{q}} D_p D_{\bar{q}}$ acting on $g_{\bar{j}i} = \chi_{\bar{j}i} + u_{\bar{j}i}$. Recalling that $\chi_{\bar{j}i}$ depends on u, we estimate

$$\sigma_k^{p\bar{q}} D_p D_{\bar{q}} g_{\bar{j}i} = \sigma_k^{p\bar{q}} D_p D_{\bar{q}} D_i D_{\bar{j}} u + \sigma_k^{p\bar{q}} D_p D_{\bar{q}} \chi_{\bar{j}i} \geq \sigma_k^{p\bar{q}} D_p D_{\bar{q}} D_i D_{\bar{j}} u - C(1+\lambda_1)\mathcal{F}. \quad (2\text{-}4)$$

Commuting derivatives

$$D_p D_{\bar{q}} D_i D_{\bar{j}} u = D_i D_{\bar{j}} D_p D_{\bar{q}} u - R_{\bar{q}i\bar{j}}{}^{\bar{a}} u_{\bar{a}p} + R_{\bar{q}p\bar{j}}{}^{\bar{a}} u_{\bar{a}i} = D_i D_{\bar{j}} g_{\bar{q}p} - D_i D_{\bar{j}} \chi_{\bar{q}p} - R_{\bar{q}i\bar{j}}{}^{\bar{a}} u_{\bar{a}p} + R_{\bar{q}p\bar{j}}{}^{\bar{a}} u_{\bar{a}i}. \quad (2\text{-}5)$$

Therefore, by (2-3),

$$\sigma_k^{p\bar{q}} D_p D_{\bar{q}} g_{\bar{j}i} \geq -\sigma_k^{p\bar{q},r\bar{s}} D_j g_{\bar{q}p} D_{\bar{j}} g_{\bar{s}r} + \sum \psi_{v_\ell} g_{\bar{j}i\ell} + \sum \psi_{\bar{v}_\ell} g_{\bar{j}i\bar{\ell}} - C(1+|DDu|^2+|D\bar{D}u|^2+(1+\lambda_1)\mathcal{F}). \quad (2\text{-}6)$$

We next compute the operator $\sigma_k^{p\bar{q}} D_p D_{\bar{q}}$ acting on $|Du|^2$, introducing the notation

$$|DDu|^2_{\sigma\omega} = \sigma_k^{p\bar{q}} \omega^{m\bar{\ell}} D_p D_m u D_{\bar{q}} D_{\bar{\ell}} u, \quad |D\bar{D}u|^2_{\sigma\omega} = \sigma_k^{p\bar{q}} \omega^{m\bar{\ell}} D_p D_{\bar{\ell}} u D_m D_{\bar{q}} u. \quad (2\text{-}7)$$

Then

$$\begin{aligned}\sigma_k^{p\bar{q}} |Du|^2_{\bar{q}p} &= \sigma_k^{p\bar{q}} (D_p D_{\bar{q}} D_m u D^m u + D_m u D_p D_{\bar{q}} D^m u) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega} \\ &= \sigma_k^{p\bar{q}} \big(D_m(g_{\bar{q}p} - \chi_{\bar{q}p}) D^m u + D_m u D^m (g_{\bar{q}p} - \chi_{\bar{q}p})\big) + \sigma_k^{p\bar{q}} R_{\bar{q}p}{}^{m\bar{\ell}} u_{\bar{\ell}} u_m \\ &\quad + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}. \end{aligned} \quad (2\text{-}8)$$

Using the differentiated equation, we obtain

$$\begin{aligned}\sigma_k^{p\bar{q}} |Du|^2_{\bar{q}p} &\geq 2\,\mathrm{Re}\langle Du, D\psi\rangle - C(1+\mathcal{F}) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega} \\ &\geq 2\,\mathrm{Re}\bigg(\sum_{p,m} (D_p D_m u D_{\bar{p}} u + D_p u D_{\bar{p}} D_m u)\psi_{v_m}\bigg) - C(1+\mathcal{F}) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}. \end{aligned}$$

To simplify the expression, we introduce the notation

$$\langle D|Du|^2, D_{\bar{v}}\psi\rangle = \sum_m (D_m D_p u D^p u\, \psi_{v_m} + D_p u D_m D^p u\, \psi_{v_m}). \quad (2\text{-}9)$$

We obtain

$$\sigma_k^{p\bar{q}} |Du|^2_{\bar{q}p} \geq 2\,\mathrm{Re}\langle D|Du|^2, D_{\bar{v}}\psi\rangle - C(1+\mathcal{F}) + |DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}. \quad (2\text{-}10)$$

We also compute

$$-\sigma_k^{p\bar{q}} u_{\bar{q}p} = \sigma_k^{p\bar{q}} (\chi_{\bar{q}p} - g_{\bar{q}p}) \geq \varepsilon\mathcal{F} - k\psi. \quad (2\text{-}11)$$

3. The C^2 estimate

In this section, we give the proof of the estimate stated in the theorem. When $k = 1$, (1-2) becomes

$$\Delta_\omega u + \mathrm{Tr}_\omega\, \chi(z, u) = n\psi(z, Du, u), \tag{3-1}$$

where Δ_ω and Tr_ω are the Laplacian and trace with respect to the background metric ω. It follows that $\Delta_\omega u$ is bounded, and the desired estimate follows in turn from the positivity of the metric g. Henceforth, we assume that $k \geq 2$. Motivated by the idea from [Guan et al. 2015] for real Hessian equations, we apply the maximum principle to the test function

$$G = \log P_m + mN|Du|^2 - mMu, \tag{3-2}$$

where $P_m = \sum_j \lambda_j^m$. Here, m, M and N are large positive constants to be determined later. We may assume that the maximum of G is achieved at some point $z \in X$. After rotating the coordinates, we may assume that the matrix $g_{\bar{j}i} = \chi_{\bar{j}i} + u_{\bar{j}i}$ is diagonal.

Recall that if $F(A) = f(\lambda_1, \ldots, \lambda_n)$ is a symmetric function of the eigenvalues of a Hermitian matrix $A = (a_{\bar{j}i})$, then at a diagonal matrix A with distinct eigenvalues, we have

$$F^{i\bar{j}} = \delta_{ij} f_i, \tag{3-3}$$

$$F^{i\bar{j},r\bar{s}} w_{i\bar{j}k} w_{r\bar{s}\bar{k}} = \sum f_{ij} w_{i\bar{i}k} w_{j\bar{j}\bar{k}} + \sum_{p\neq q} \frac{f_p - f_q}{\lambda_p - \lambda_q} |w_{p\bar{q}k}|^2, \tag{3-4}$$

where

$$F^{i\bar{j}} = \frac{\partial F}{\partial a_{\bar{j}i}}, \qquad F^{i\bar{j},r\bar{s}} = \frac{\partial^2 F}{\partial a_{\bar{j}i}\partial a_{\bar{s}r}},$$

and $w_{i\bar{j}k}$ is an arbitrary tensor; see [Ball 1984]. Using these identities to differentiate G, we first obtain the critical equation

$$\frac{DP_m}{P_m} + mND|Du|^2 - mMDu = 0. \tag{3-5}$$

Differentiating G a second time and contracting with $\sigma_k^{p\bar{q}}$ yields

$$\begin{aligned} 0 \geq \frac{m}{P_m}\left(\sum_j \lambda_j^{m-1} \sigma_k^{p\bar{p}} D_p D_{\bar{p}} g_{\bar{j}j}\right) - \frac{|DP_m|_\sigma^2}{P_m^2} + mN\sigma_k^{p\bar{p}}|Du|_{\bar{p}p}^2 - mM\sigma_k^{p\bar{p}} u_{\bar{p}p} \\ + \frac{m}{P_m}\left((m-1)\sum_j \lambda_j^{m-2}\sigma_k^{p\bar{p}}|D_p g_{\bar{j}j}|^2 + \sigma_k^{p\bar{p}} \sum_{i\neq j} \frac{\lambda_i^{m-1} - \lambda_j^{m-1}}{\lambda_i - \lambda_j} |D_p g_{\bar{j}i}|^2\right). \end{aligned} \tag{3-6}$$

Here, we use the notation $|\eta|_\sigma^2 = \sigma_k^{p\bar{q}} \eta_p \eta_{\bar{q}}$. Substituting (2-6), (2-10) and (2-11), we obtain

$$
\begin{aligned}
0 \geq \frac{1}{P_m}\Big(-C\sum_j \lambda_j^{m-1}(1+|DDu|^2+|D\bar{D}u|^2+(1+\lambda_1)\mathcal{F})\Big)
&+\frac{1}{P_m}\Big(\sum_j \lambda_j^{m-1}(-\sigma_k^{p\bar{q},r\bar{s}} D_j g_{\bar{q}p} D_{\bar{j}} g_{\bar{s}r}+\sum_\ell \psi_{v_\ell} g_{\bar{j}j\ell}+\sum_\ell \psi_{\bar{v}_\ell} g_{\bar{j}j\bar{\ell}})\Big)\\
&+\frac{1}{P_m}\Big((m-1)\sum_j \lambda_j^{m-2}\sigma_k^{p\bar{p}}|D_p g_{\bar{j}j}|^2+\sigma_k^{p\bar{p}}\sum_{i\neq j}\frac{\lambda_i^{m-1}-\lambda_j^{m-1}}{\lambda_i-\lambda_j}|D_p g_{\bar{j}i}|^2\Big)\\
&-\frac{|DP_m|_\sigma^2}{mP_m^2}+N(|DDu|_{\sigma\omega}^2+|D\bar{D}u|_{\sigma\omega}^2)\\
&+N\langle D|Du|^2, D_{\bar{v}}\psi\rangle+N\langle D_{\bar{v}}\psi, D|Du|^2\rangle+(M\varepsilon-CN)\mathcal{F}-kM\psi. \qquad (3\text{-}7)
\end{aligned}
$$

From the critical equation (3-5), we have

$$
\frac{1}{P_m}\sum_{j,\ell}\lambda_j^{m-1} g_{\bar{j}j\ell}\psi_{v_\ell}=\frac{1}{m}\left\langle\frac{DP_m}{P_m}, D_{\bar{v}}\psi\right\rangle=-N\langle D|Du|^2, D_{\bar{v}}\psi\rangle+M\langle Du, D_{\bar{v}}\psi\rangle.
$$

It follows that

$$
\begin{aligned}
&\frac{1}{P_m}\sum_{j,\ell}(\psi_{v_\ell} g_{\bar{j}j\ell}+\psi_{\bar{v}_\ell} g_{\bar{j}j\bar{\ell}})+N\langle D|Du|^2, D_{\bar{v}}\psi\rangle+N\langle D_{\bar{v}}\psi, D|Du|^2\rangle\\
&\qquad = M(\langle Du, D_{\bar{v}}\psi\rangle+\langle D_{\bar{v}}\psi, Du\rangle)\geq -CM.
\end{aligned}
$$

Using (3-4), one can obtain the well-known identity

$$
-\sigma_k^{p\bar{q},r\bar{s}} D_j g_{\bar{q}p} D_{\bar{j}} g_{\bar{s}r}=-\sigma_k^{p\bar{p},q\bar{q}} D_j g_{\bar{p}p} D_{\bar{j}} g_{\bar{q}q}+\sigma_k^{p\bar{p},q\bar{q}}|D_j g_{\bar{p}q}|^2, \qquad (3\text{-}8)
$$

where

$$
\sigma_k^{p\bar{p},q\bar{q}}=\frac{\partial}{\partial\lambda_p}\frac{\partial}{\partial\lambda_q}\sigma_k(\lambda).
$$

We assume that $\lambda_1 \gg 1$, otherwise the C^2 estimate is complete. The main inequality (3-7) becomes

$$
\begin{aligned}
0 \geq \frac{-C}{\lambda_1}(1+|DDu|^2+|D\bar{D}u|^2)+\frac{1}{P_m}\Big(\sum_j \lambda_j^{m-1}\big(-\sigma_k^{p\bar{p},q\bar{q}} D_j g_{\bar{p}p} D_{\bar{j}} g_{\bar{q}q}+\sigma_k^{p\bar{p},q\bar{q}}|D_j g_{\bar{p}q}|^2\big)\Big)
&+\frac{1}{P_m}\Big((m-1)\sum_j \lambda_j^{m-2}\sigma_k^{p\bar{p}}|D_p g_{\bar{j}j}|^2+\sigma_k^{p\bar{p}}\sum_{i\neq j}\frac{\lambda_i^{m-1}-\lambda_j^{m-1}}{\lambda_i-\lambda_j}|D_p g_{\bar{j}i}|^2\Big)\\
&-\frac{|DP_m|_\sigma^2}{mP_m^2}+N(|DDu|_{\sigma\omega}^2+|D\bar{D}u|_{\sigma\omega}^2)+(M\varepsilon-CN-C)\mathcal{F}-CM. \qquad (3\text{-}9)
\end{aligned}
$$

The main objective is to show that the third-order terms on the right-hand side of (3-9) are nonnegative. To deal with this issue, we need a lemma from [Guan et al. 2015]; see also [Guan et al. 2012; Li et al. 2016].

Lemma 1 [Guan et al. 2015]. *Suppose $1 \le \ell < k \le n$, and let $\alpha = 1/(k-\ell)$. Let $W = (w_{\bar{q}p})$ be a Hermitian tensor in the Γ_k cone. Then for any $\theta > 0$,*

$$-\sigma_k^{p\bar{p},q\bar{q}}(W)w_{\bar{p}pi}w_{\bar{q}q\bar{i}} + \Big(1-\alpha+\frac{\alpha}{\theta}\Big)\frac{|D_i\sigma_k(W)|^2}{\sigma_k(W)} \\ \ge \sigma_k(W)(\alpha+1-\alpha\theta)\left|\frac{D_i\sigma_\ell(W)}{\sigma_\ell(W)}\right|^2 - \frac{\sigma_k}{\sigma_\ell}(W)\sigma_\ell^{p\bar{p},q\bar{q}}(W)w_{\bar{p}pi}w_{\bar{q}q\bar{i}}. \quad (3\text{-}10)$$

Here the Γ_k cone is defined as

$$\Gamma_k = \{\lambda \in \mathbb{R}^n \mid \sigma_m(\lambda) > 0,\ m = 1, \dots, k\}. \quad (3\text{-}11)$$

We say a Hermitian matrix $W \in \Gamma_k$ if $\lambda(W) \in \Gamma_k$.

It follows from the above lemma that, by taking $\ell = 1$, we have

$$-\sigma_k^{p\bar{p},q\bar{q}}D_i g_{\bar{p}p}D_{\bar{i}}g_{\bar{q}q} + K|D_i\sigma_k|^2 \ge 0, \quad (3\text{-}12)$$

for $K > (1-\alpha+\alpha/\theta)(\inf\psi)^{-1}$ if $2 \le k \le n$.

We denote

$$A_i = \frac{\lambda_i^{m-1}}{P_m}\big(K|D_i\sigma_k|^2 - \sigma_k^{p\bar{p},q\bar{q}}D_i g_{\bar{p}p}D_{\bar{i}}g_{\bar{q}q}\big),$$

$$B_i = \frac{1}{P_m}\bigg(\sum_p \sigma_k^{p\bar{p},i\bar{i}}\lambda_p^{m-1}|D_i g_{\bar{p}p}|^2\bigg), \qquad C_i = \frac{(m-1)\sigma_k^{i\bar{i}}}{P_m}\bigg(\sum_p \lambda_p^{m-2}|D_i g_{\bar{p}p}|^2\bigg),$$

$$D_i = \frac{1}{P_m}\bigg(\sum_{p\ne i}\sigma_k^{p\bar{p}}\frac{\lambda_p^{m-1}-\lambda_i^{m-1}}{\lambda_p-\lambda_i}|D_i g_{\bar{p}p}|^2\bigg), \qquad E_i = \frac{m\sigma_k^{i\bar{i}}}{P_m^2}\bigg|\sum_p \lambda_p^{m-1}D_i g_{\bar{p}p}\bigg|^2.$$

Define $T_{j\bar{p}q} = D_j\chi_{\bar{p}q} - D_q\chi_{\bar{p}j}$. For any $0 < \tau < 1$, we can estimate

$$\frac{1}{P_m}\bigg(\sum_p \lambda_p^{m-1}\sigma_k^{j\bar{j},i\bar{i}}|D_p g_{\bar{j}i}|^2\bigg) \ge \frac{1}{P_m}\bigg(\sum_p \lambda_p^{m-1}\sigma_k^{p\bar{p},i\bar{i}}|D_i g_{\bar{p}p} + T_{p\bar{p}i}|^2\bigg) \\ \ge \frac{1}{P_m}\bigg(\sum_p \lambda_p^{m-1}\sigma_k^{p\bar{p},i\bar{i}}\big((1-\tau)|D_i g_{\bar{p}p}|^2 - C_\tau|T_{p\bar{p}i}|^2\big)\bigg) \\ = (1-\tau)\sum_i B_i - \frac{C_\tau}{P_m}\sum_p \lambda_p^{m-2}\big(\lambda_p\sigma_k^{p\bar{p},i\bar{i}}\big)|T_{p\bar{p}i}|^2.$$

Now we use $\sigma_l(\lambda|i)$ and $\sigma_l(\lambda|ij)$ to denote the l-th elementary functions of

$$(\lambda|i) = (\lambda_1, \dots, \widehat{\lambda_i}, \dots, \lambda_n) \in \mathbb{R}^{n-1} \quad \text{and} \quad (\lambda|ij) = (\lambda_1, \dots, \widehat{\lambda_i}, \dots, \widehat{\lambda_j}, \dots, \lambda_n) \in \mathbb{R}^{n-2},$$

respectively. The following simple identities are used frequently:

$$\sigma_k^{i\bar{i}} = \sigma_{k-1}(\lambda|i) \quad \text{and} \quad \sigma_k^{p\bar{p},i\bar{i}} = \sigma_{k-2}(\lambda|pi).$$

Using the identity $\sigma_l(\lambda) = \sigma_l(\lambda|p) + \lambda_p \sigma_{l-1}(\lambda|p)$ for any $1 \le p \le n$, we obtain

$$\begin{aligned}\frac{1}{P_m}\Big(\sum_p \lambda_p^{m-1}\sigma_k^{j\bar{j},i\bar{i}}|D_p g_{\bar{j}i}|^2\Big) &\ge (1-\tau)\sum_i B_i - \frac{C_\tau}{P_m}\sum_p \lambda_p^{m-2}(\sigma_k^{i\bar{i}} - \sigma_{k-1}(\lambda|pi))|T_{p\bar{p}i}|^2 \\ &\ge (1-\tau)\sum_i B_i - \frac{C_\tau}{\lambda_1^2}\mathcal{F} \ge (1-\tau)\sum_i B_i - \mathcal{F}. \end{aligned} \tag{3-13}$$

We use the notation C_τ for a constant depending on τ. To get the last inequality above, we assume that $\lambda_1^2 \ge C_\tau$; otherwise, we already have the desired estimate $\lambda_1 \le C$. Similarly, we may estimate

$$\begin{aligned}\frac{1}{P_m}\sigma_k^{j\bar{j}}\sum_{i\ne p}\frac{\lambda_i^{m-1}-\lambda_p^{m-1}}{\lambda_i-\lambda_p}|D_j g_{\bar{p}i}|^2 &\ge \frac{1}{P_m}\sigma_k^{p\bar{p}}\sum_{p\ne i}\frac{\lambda_i^{m-1}-\lambda_p^{m-1}}{\lambda_i-\lambda_p}|D_i g_{\bar{p}p} + T_{p\bar{p}i}|^2 \\ &\ge \frac{1}{P_m}\sigma_k^{p\bar{p}}\sum_{p\ne i}\frac{\lambda_i^{m-1}-\lambda_p^{m-1}}{\lambda_i-\lambda_p}\big((1-\tau)|D_i g_{\bar{p}p}|^2 - C_\tau|T_{p\bar{p}i}|^2\big) \\ &\ge \sum_i (1-\tau)D_i - \frac{C_\tau}{\lambda_1^2}\mathcal{F} \ge \sum_i (1-\tau)D_i - \mathcal{F}. \end{aligned} \tag{3-14}$$

With the introduced notation in place, the main inequality becomes

$$\begin{aligned}0 \ge \frac{-C(K)}{\lambda_1}\big(1+|DDu|^2+|D\bar{D}u|^2\big) - \tau\frac{|DP_m|_\sigma^2}{mP_m^2} \\ + \sum_i \big(A_i + (1-\tau)B_i + C_i + (1-\tau)D_i - (1-\tau)E_i\big) \\ + N\big(|DDu|_{\sigma\omega}^2 + |D\bar{D}u|_{\sigma\omega}^2\big) + (M\varepsilon - CN - C)\mathcal{F} - CM. \end{aligned} \tag{3-15}$$

Using the critical equation (3-5), we have

$$\begin{aligned}\tau\frac{|DP_m|_\sigma^2}{mP_m^2} &= \tau m\big|ND|Du|^2 - MDu\big|_\sigma^2 \le 2\tau m\big(N^2\big|D|Du|^2\big|_\sigma^2 + M^2|Du|_\sigma^2\big) \\ &\le C\tau m N^2\big(|DDu|_{\sigma\omega}^2 + |D\bar{D}u|_{\sigma\omega}^2\big) + C\tau m M^2\mathcal{F}. \end{aligned} \tag{3-16}$$

We thus have

$$\begin{aligned}0 \ge \frac{-C(K)}{\lambda_1}\big(1+|DDu|^2+|D\bar{D}u|^2\big) + (N - C\tau m N^2)\big(|DDu|_{\sigma\omega}^2 + |D\bar{D}u|_{\sigma\omega}^2\big) \\ + \sum_i \big(A_i + (1-\tau)B_i + C_i + (1-\tau)D_i - (1-\tau)E_i\big) \\ + (M\varepsilon - C\tau m M^2 - CN - C)\mathcal{F} - CM. \end{aligned} \tag{3-17}$$

3.1. ***Estimating the third-order terms.*** In this subsection, we will adapt the argument in [Li et al. 2016] to estimate the third-order terms.

Lemma 2. *For sufficiently large m, the following estimates hold*:

$$P_m^2(B_1 + C_1 + D_1 - E_1) \geq P_m \lambda_1^{m-2} \sum_{p \neq 1} \sigma_k^{p\bar{p}} |D_1 g_{\bar{p}p}|^2 - \lambda_1^m \sigma_k^{1\bar{1}} \lambda_1^{m-2} |D_1 g_{\bar{1}1}|^2, \tag{3-18}$$

and for any fixed $i \neq 1$,

$$P_m^2(B_i + C_i + D_i - E_i) \geq 0. \tag{3-19}$$

Proof. Fix $i \in \{1, 2, \ldots, n\}$. First, we compute

$$\begin{aligned} P_m(B_i + D_i) &= \sum_{p \neq i} \sigma_k^{p\bar{p},i\bar{i}} \lambda_p^{m-1} |D_i g_{\bar{p}p}|^2 + \sum_{p \neq i} \sigma_k^{p\bar{p}} \frac{\lambda_p^{m-1} - \lambda_i^{m-1}}{\lambda_p - \lambda_i} |D_i g_{\bar{p}p}|^2 \\ &= \sum_{p \neq i} \lambda_p^{m-2} \big((\lambda_p \sigma_k^{p\bar{p},i\bar{i}} + \sigma_k^{p\bar{p}}) |D_i g_{\bar{p}p}|^2 \big) + \bigg(\sum_{p \neq i} \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} {\lambda_p}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2 \bigg). \end{aligned}$$

Note that

$$\lambda_p \sigma_k^{p\bar{p},i\bar{i}} + \sigma_k^{p\bar{p}} \geq \sigma_k^{i\bar{i}}.$$

To see this, we write

$$\begin{aligned} \lambda_p \sigma_k^{p\bar{p},i\bar{i}} + \sigma_k^{p\bar{p}} &= \lambda_p \sigma_{k-2}(\lambda|pi) + \sigma_{k-1}(\lambda|p) \\ &= \sigma_{k-1}(\lambda|i) - \sigma_{k-1}(\lambda|ip) + \sigma_{k-1}(\lambda|p) \\ &= \sigma_{k-1}(\lambda|i) + \lambda_i \sigma_{k-2}(\lambda|ip) \geq \sigma_{k-1}(\lambda|i) = \sigma_k^{i\bar{i}}, \end{aligned}$$

where we used the standard identity $\sigma_l(\lambda) = \sigma_l(\lambda|p) + \lambda_p \sigma_{l-1}(\lambda|p)$ twice, to get the second and third equalities. Therefore

$$P_m(B_i + D_i) \geq \sigma_k^{i\bar{i}} \bigg(\sum_{p \neq i} \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 \bigg) + \bigg(\sum_{p \neq i} \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} {\lambda_p}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2 \bigg). \tag{3-20}$$

It follows that

$$\begin{aligned} P_m(B_i + C_i + D_i) \geq m \sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_p^{m-2} |D_i g_{\bar{p}p}|^2 + (m-1) \sigma_k^{i\bar{i}} \lambda_i^{m-2} |D_i g_{\bar{i}i}|^2 \\ + \sum_{p \neq i} \sigma_k^{p\bar{p}} \sum_{q=0}^{m-3} {\lambda_p}^q \lambda_i^{m-2-q} |D_i g_{\bar{p}p}|^2. \end{aligned} \tag{3-21}$$

Expanding out the definition of E_i,

$$P_m^2 E_i = m \sigma_k^{i\bar{i}} \sum_{p \neq i} \lambda_p^{2m-2} |D_i g_{\bar{p}p}|^2 + m \sigma_k^{i\bar{i}} \lambda_i^{2m-2} |D_i g_{\bar{i}i}|^2 + m \sigma_k^{i\bar{i}} \sum_p \sum_{q \neq p} \lambda_p^{m-1} \lambda_q^{m-1} D_i g_{\bar{p}p} D_{\bar{i}} g_{\bar{q}q}. \tag{3-22}$$

Therefore,

$$\begin{aligned}P_m^2(B_i+C_i+D_i-E_i)&\\ \geq\Big(m\sigma_k^{i\bar{i}}\sum_{p\neq i}(P_m-\lambda_p^m)\lambda_p^{m-2}|D_ig_{\bar{p}p}|^2-m\sigma_k^{i\bar{i}}\sum_{p\neq i}\sum_{q\neq p,i}\lambda_p^{m-1}\lambda_q^{m-1}D_ig_{\bar{p}p}D_{\bar{i}}g_{\bar{q}q}\Big)&\\ +P_m\sum_{p\neq i}\sigma_k^{p\bar{p}}\sum_{q=0}^{m-3}\lambda_p{}^q\lambda_i^{m-2-q}|D_ig_{\bar{p}p}|^2-2m\sigma_k^{i\bar{i}}\operatorname{Re}\sum_{q\neq i}\lambda_i^{m-1}\lambda_q^{m-1}D_ig_{\bar{i}i}D_{\bar{i}}g_{\bar{q}q}&\\ +((m-1)P_m-m\lambda_i^m)\sigma_k^{i\bar{i}}\lambda_i^{m-2}|D_ig_{\bar{i}i}|^2.\quad(3\text{-}23)\end{aligned}$$

We now estimate the expression on the second line above. First,

$$m\sigma_k^{i\bar{i}}\sum_{p\neq i}(P_m-\lambda_p^m)\lambda_p^{m-2}|D_ig_{\bar{p}p}|^2=m\sigma_k^{i\bar{i}}\sum_{p\neq i}\sum_{q\neq p,i}\lambda_q^m\lambda_p^{m-2}|D_ig_{\bar{p}p}|^2+m\sigma_k^{i\bar{i}}\sum_{p\neq i}\lambda_i^m\lambda_p^{m-2}|D_ig_{\bar{p}p}|^2.$$

Next, we can estimate

$$\begin{aligned}-m\sigma_k^{i\bar{i}}\sum_{p\neq i}\sum_{q\neq p,i}\lambda_p^{m-1}\lambda_q^{m-1}D_ig_{\bar{p}p}D_{\bar{i}}g_{\bar{q}q}&\geq-m\sigma_k^{i\bar{i}}\sum_{p\neq i}\sum_{q\neq p,i}\frac{1}{2}\big(\lambda_p^{m-2}\lambda_q^m|D_ig_{\bar{p}p}|^2+\lambda_p^m\lambda_q^{m-2}|D_ig_{\bar{q}q}|^2\big)\\ &=-m\sigma_k^{i\bar{i}}\sum_{p\neq i}\sum_{q\neq p,i}\lambda_p^{m-2}\lambda_q^m|D_ig_{\bar{p}p}|^2.\quad(3\text{-}24)\end{aligned}$$

We arrive at

$$\begin{aligned}P_m^2(B_i+C_i+D_i-E_i)&\\ \geq m\sigma_k^{i\bar{i}}\sum_{p\neq i}\lambda_i^m\lambda_p^{m-2}|D_ig_{\bar{p}p}|^2+P_m\sum_{p\neq i}\sigma_k^{p\bar{p}}\sum_{q=0}^{m-3}\lambda_p{}^q\lambda_i^{m-2-q}|D_ig_{\bar{p}p}|^2&\\ -2m\sigma_k^{i\bar{i}}\operatorname{Re}\Big(\lambda_i^{m-1}D_ig_{\bar{i}i}\sum_{q\neq i}\lambda_q^{m-1}D_{\bar{i}}g_{\bar{q}q}\Big)+((m-1)P_m-m\lambda_i^m)\sigma_k^{i\bar{i}}\lambda_i^{m-2}|D_ig_{\bar{i}i}|^2.\quad(3\text{-}25)\end{aligned}$$

The next step is to extract good terms from the second summation on the first line. We fix $p\neq i$.

Case 1: $\lambda_i\geq\lambda_p$. Then $\sigma_k^{p\bar{p}}\geq\sigma_k^{i\bar{i}}$. Hence

$$P_m\sigma_k^{p\bar{p}}\sum_{q=1}^{m-3}\lambda_p{}^q\lambda_i^{m-2-q}\geq\lambda_i^m\sigma_k^{i\bar{i}}\sum_{q=1}^{m-3}\lambda_p{}^q\lambda_p^{m-2-q}=(m-3)\sigma_k^{i\bar{i}}\lambda_i^m\lambda_p^{m-2}.\quad(3\text{-}26)$$

Case 2: $\lambda_i\leq\lambda_p$. Then $\lambda_p\sigma_k^{p\bar{p}}=\lambda_i\sigma_k^{i\bar{i}}+(\sigma_k(\lambda|i)-\sigma_k(\lambda|p))\geq\lambda_i\sigma_k^{i\bar{i}}$, and we obtain

$$P_m\sigma_k^{p\bar{p}}\sum_{q=1}^{m-3}\lambda_p{}^q\lambda_i^{m-2-q}\geq\lambda_p^m\sigma_k^{i\bar{i}}\sum_{q=1}^{m-3}\lambda_p{}^{q-1}\lambda_i^{m-1-q}\geq(m-3)\sigma_k^{i\bar{i}}\lambda_i^m\lambda_p^{m-2}.\quad(3\text{-}27)$$

Combining both cases, we have

$$P_m\sigma_k^{p\bar{p}}\sum_{q=0}^{m-3}\lambda_p{}^q\lambda_i^{m-2-q}|D_i g_{\bar{p}p}|^2 = P_m\sigma_k^{p\bar{p}}\sum_{q=1}^{m-3}\lambda_p{}^q\lambda_i^{m-2-q}|D_i g_{\bar{p}p}|^2 + P_m\sigma_k^{p\bar{p}}\lambda_i^{m-2}|D_i g_{\bar{p}p}|^2$$
$$\geq (m-3)\sigma_k^{i\bar{i}}\lambda_i^m\lambda_p^{m-2}|D_i g_{\bar{p}p}|^2 + P_m\sigma_k^{p\bar{p}}\lambda_i^{m-2}|D_i g_{\bar{p}p}|^2.$$

Substituting this estimate into inequality (3-25), we obtain

$$\begin{aligned}P_m^2(B_i + C_i + D_i - E_i)&\\ \geq (2m-3)\sigma_k^{i\bar{i}}\sum_{p\neq i}\lambda_i^m\lambda_p^{m-2}|D_i g_{\bar{p}p}|^2 - 2m\sigma_k^{i\bar{i}}\,\mathrm{Re}\Big(\lambda_i^{m-1}D_i g_{\bar{i}i}\sum_{p\neq i}\lambda_p^{m-1}D_{\bar{i}}g_{\bar{p}p}\Big)&\\ + P_m\lambda_i^{m-2}\sum_{p\neq i}\sigma_k^{p\bar{p}}|D_i g_{\bar{p}p}|^2 + ((m-1)P_m - m\lambda_i^m)\sigma_k^{i\bar{i}}\lambda_i^{m-2}|D_i g_{\bar{i}i}|^2.&\quad (3\text{-}28)\end{aligned}$$

Choose $m \gg 1$ such that

$$m^2 \leq (2m-3)(m-2). \tag{3-29}$$

We can therefore estimate

$$\begin{aligned}2m\sigma_k^{i\bar{i}}\,\mathrm{Re}\Big(\lambda_i^{m-1}D_i g_{\bar{i}i}\sum_{p\neq i}\lambda_p^{m-1}D_{\bar{i}}g_{\bar{p}p}\Big)&\\ \leq 2\sigma_k^{i\bar{i}}\sum_{p\neq i}\big((2m-3)^{1/2}\lambda_i^{m/2}\lambda_p^{(m-2)/2}|D_i g_{\bar{p}p}|\big)\big((m-2)^{1/2}\lambda_i^{(m-2)/2}\lambda_p^{m/2}|D_{\bar{i}}g_{\bar{i}i}|\big)&\\ \leq (2m-3)\sigma_k^{i\bar{i}}\sum_{p\neq i}\lambda_i^m\lambda_p^{m-2}|D_i g_{\bar{p}p}|^2 + (m-2)\sigma_k^{i\bar{i}}\sum_{p\neq i}\lambda_i^{m-2}\lambda_p^m|D_{\bar{i}}g_{\bar{i}i}|^2.&\quad (3\text{-}30)\end{aligned}$$

We finally arrive at

$$\begin{aligned}P_m^2(B_i + C_i + D_i - E_i) \geq P_m\lambda_i^{m-2}\sum_{p\neq i}\sigma_k^{p\bar{p}}|D_i g_{\bar{p}p}|^2 + ((m-1)P_m - m\lambda_i^m)\sigma_k^{i\bar{i}}\lambda_i^{m-2}|D_i g_{\bar{i}i}|^2&\\ - (m-2)\sigma_k^{i\bar{i}}\sum_{p\neq i}\lambda_i^{m-2}\lambda_p^m|D_{\bar{i}}g_{\bar{i}i}|^2.&\quad (3\text{-}31)\end{aligned}$$

If we let $i = 1$, we obtain inequality (3-18). For any fixed $i \neq 1$, this inequality yields

$$\begin{aligned}P_m^2(B_i + C_i + D_i - E_i) &\geq P_m\lambda_i^{m-2}\sum_{p\neq i}\sigma_k^{p\bar{p}}|D_i g_{\bar{p}p}|^2 + ((m-1)\lambda_1^m - \lambda_i^m)\sigma_k^{i\bar{i}}\lambda_i^{m-2}|D_i g_{\bar{i}i}|^2\\ &\quad + (m-1)\sum_{p\neq 1,i}\lambda_p^m\sigma_k^{i\bar{i}}\lambda_i^{m-2}|D_i g_{\bar{i}i}|^2 - (m-2)\sigma_k^{i\bar{i}}\sum_{p\neq i}\lambda_i^{m-2}\lambda_p^m|D_{\bar{i}}g_{\bar{i}i}|^2\\ &\geq P_m\lambda_i^{m-2}\sum_{p\neq i}\sigma_k^{p\bar{p}}|D_i g_{\bar{p}p}|^2 \geq 0.\end{aligned}$$

This completes the proof of Lemma 2. □

We observed in (3-12) that $A_i \geq 0$. Lemma 2 implies that for any $i \neq 1$,

$$A_i + B_i + C_i + D_i - E_i \geq 0.$$

Thus we have shown that for $i \neq 1$, the third-order terms in the main inequality (3-17) are indeed nonnegative. The only remaining case is when $i = 1$. By adapting once again the techniques from [Guan et al. 2015], we obtain the following lemma.

Lemma 3. *Let $1 < k \leq n$. Suppose there exists $0 < \delta \leq 1$ such that $\lambda_\mu \geq \delta\lambda_1$ for some $\mu \in \{1, 2, \ldots, k-1\}$. There exists a small $\delta' > 0$ such that if $\lambda_{\mu+1} \leq \delta'\lambda_1$, then*

$$A_1 + B_1 + C_1 + D_1 - E_1 \geq 0.$$

Proof. By Lemma 2, we have

$$P_m^2(A_1 + B_1 + C_1 + D_1 - E_1) \geq P_m^2 A_1 + P_m \lambda_1^{m-2} \sum_{p \neq 1} \sigma_k^{p\bar{p}} |D_1 g_{\bar{p}p}|^2 - \lambda_1^m \sigma_k^{1\bar{1}} \lambda_1^{m-2} |D_1 g_{\bar{1}1}|^2. \quad (3\text{-}32)$$

The key insight in [Guan et al. 2015], used also in [Li et al. 2016], is to extract a good term involving $|D_1 g_{\bar{1}1}|^2$ from A_1. By the inequality in Lemma 1 (with $\theta = 1/2$), we have for $\mu < k$

$$\begin{aligned} P_m^2 A_1 &\geq \frac{P_m \lambda_1^{m-1} \sigma_k}{\sigma_\mu^2} \left(\left(1 + \frac{\alpha}{2}\right) \left| \sum_p \sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p} \right|^2 - \sigma_\mu \sigma_\mu^{p\bar{p},q\bar{q}} D_1 g_{\bar{p}p} D_{\bar{1}} g_{\bar{q}q} \right) \\ &= \frac{P_m \lambda_1^{m-1} \sigma_k}{\sigma_\mu^2} \left(\sum_p \left(1 + \frac{\alpha}{2}\right) |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 + \sum_{p \neq q} \frac{\alpha}{2} \sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p} \sigma_\mu^{q\bar{q}} D_{\bar{1}} g_{\bar{q}q} \right. \\ &\qquad \left. + \sum_{p \neq q} (\sigma_\mu^{p\bar{p}} \sigma_\mu^{q\bar{q}} - \sigma_\mu \sigma_\mu^{p\bar{p},q\bar{q}}) D_1 g_{\bar{p}p} D_{\bar{1}} g_{\bar{q}q} \right) \\ &\geq \frac{P_m \lambda_1^{m-1} \sigma_k}{\sigma_\mu^2} \left(\sum_p |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 - \sum_{p \neq q} |F^{pq} D_1 g_{\bar{p}p} D_{\bar{1}} g_{\bar{q}q}| \right), \end{aligned} \quad (3\text{-}33)$$

where we defined $F^{pq} = \sigma_\mu^{p\bar{p}} \sigma_\mu^{q\bar{q}} - \sigma_\mu \sigma_\mu^{p\bar{p},q\bar{q}}$. Notice that if $\mu = 1$, then $F^{pq} = 1$. If $\mu \geq 2$, then the Newton–Maclaurin inequality implies

$$F^{pq} = \sigma_{\mu-1}^2(\lambda|pq) - \sigma_\mu(\lambda|pq)\sigma_{\mu-2}(\lambda|pq) \geq 0. \quad (3\text{-}34)$$

We split the sum involving F^{pq} in the following way:

$$\sum_{p \neq q} |F^{pq} D_1 g_{\bar{p}p} D_{\bar{1}} g_{\bar{q}q}| = \sum_{\substack{p \neq q \\ p,q \leq \mu}} F^{pq} |D_1 g_{\bar{p}p}| |D_{\bar{1}} g_{\bar{q}q}| + \sum_{(p,q) \in J} F^{pq} |D_1 g_{\bar{p}p}| |D_{\bar{1}} g_{\bar{q}q}|, \quad (3\text{-}35)$$

where J is the set of indices where at least one of $p \neq q$ is strictly greater than μ. The summation of terms in J can be estimated by

$$\begin{aligned} -\sum_{(p,q)\in J} F^{pq}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| &\geq -\sum_{(p,q)\in J} \sigma_\mu^{p\bar{p}}\sigma_\mu^{q\bar{q}}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| \\ &\geq -\epsilon \sum_{p\leq\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 - C\sum_{p>\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \end{aligned} \tag{3-36}$$

If $\mu = 1$, the first term on the right-hand side of (3-35) vanishes and this estimate applies to all terms on the right hand side of (3-35).

If $\mu \geq 2$, we have for $p, q \leq \mu$,

$$\sigma_{\mu-1}(\lambda|pq) \leq C\frac{\lambda_1 \cdots \lambda_{\mu+1}}{\lambda_p \lambda_q} \leq C\frac{\sigma_\mu^{p\bar{p}}\lambda_{\mu+1}}{\lambda_q}. \tag{3-37}$$

Using (3-34) and (3-37), for δ' small enough we can control

$$\begin{aligned} -\sum_{\substack{p\neq q\\ p,q\leq\mu}} F^{pq}|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| &\geq -\sum_{\substack{p\neq q\\ p,q\leq\mu}} \sigma_{\mu-1}^2(\lambda|pq)|D_1 g_{\bar{p}p}||D_{\bar{1}} g_{\bar{q}q}| \\ &\geq -C\lambda_{\mu+1}^2 \sum_{\substack{p\neq q\\ p,q\leq\mu}} \frac{\sigma_\mu^{p\bar{p}}}{\lambda_p}|D_1 g_{\bar{p}p}|\frac{\sigma_\mu^{q\bar{q}}}{\lambda_q}|D_{\bar{1}} g_{\bar{q}q}| \\ &\geq -C\sum_{p\leq\mu} \frac{\lambda_{\mu+1}^2}{\lambda_p^2}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 \\ &\geq -C\sum_{p\leq\mu} \frac{\delta'^2}{\delta^2}|\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 \geq -\epsilon\sum_{p\leq\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \end{aligned} \tag{3-38}$$

Combining all cases, we have

$$-\sum_{p\neq q} |F^{pq} D_1 g_{\bar{p}p} D_{\bar{1}} g_{\bar{q}q}| \geq -2\epsilon\sum_{p\leq\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 - C\sum_{p>\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \tag{3-39}$$

Using this inequality in (3-33) yields

$$\begin{aligned} P_m^2 A_1 &\geq \frac{P_m \lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\left((1-2\epsilon)\sum_{p\leq\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2 - C\sum_{p>\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2\right) \\ &\geq (1-2\epsilon)\frac{P_m \lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}|\sigma_\mu^{1\bar{1}} D_1 g_{\bar{1}1}|^2 - C\frac{P_m \lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\sum_{p>\mu} |\sigma_\mu^{p\bar{p}} D_1 g_{\bar{p}p}|^2. \end{aligned} \tag{3-40}$$

We estimate

$$\begin{aligned}(1-2\epsilon)\frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}|\sigma_\mu^{1\bar 1}D_1 g_{\bar 1 1}|^2 &= (1-2\epsilon)\frac{P_m\lambda_1^{m-2}\sigma_k}{\lambda_1}\left(\frac{\lambda_1\sigma_\mu^{1\bar 1}}{\sigma_\mu}\right)^2|D_1 g_{\bar 1 1}|^2\\ &\geq (1-2\epsilon)P_m\lambda_1^{m-2}\frac{\sigma_k}{\lambda_1}\left(1-C\frac{\lambda_{\mu+1}}{\lambda_1}\right)^2|D_1 g_{\bar 1 1}|^2\\ &\geq (1-2\epsilon)(1-C\delta')^2 P_m\lambda_1^{m-2}\sigma_k^{1\bar 1}|D_1 g_{\bar 1 1}|^2\\ &\geq (1-2\epsilon)(1-C\delta')^2(1+\delta^m)\lambda_1^{2m-2}\sigma_k^{1\bar 1}|D_1 g_{\bar 1 1}|^2. \end{aligned} \tag{3-41}$$

For δ' and ϵ small enough, we obtain

$$P_m^2 A_1 \geq \lambda_1^m\sigma_k^{1\bar 1}\lambda_1^{m-2}|D_1 g_{\bar 1 1}|^2 - C\frac{P_m\lambda_1^{m-1}\sigma_k}{\sigma_\mu^2}\sum_{p>\mu}|\sigma_\mu^{p\bar p}D_1 g_{\bar p p}|^2. \tag{3-42}$$

We see that the $|D_1 g_{\bar 1 1}|^2$ term cancels from the inequality (3-32) and we are left with

$$P_m^2(A_1+B_1+C_1+D_1-E_1) \geq P_m\lambda_1^{m-2}\sum_{p>\mu}\left(\sigma_k^{p\bar p}-C\frac{\lambda_1\sigma_k(\sigma_\mu^{p\bar p})^2}{\sigma_\mu^2}\right)|D_1 g_{\bar p p}|^2. \tag{3-43}$$

For δ' small enough, the above expression is nonnegative. Indeed, for any $p>\mu$, we have

$$(\lambda_1\sigma_\mu^{p\bar p})^2 \leq \frac{1}{\delta^2}(\lambda_\mu\sigma_\mu^{p\bar p})^2 \leq C\frac{(\sigma_\mu)^2}{\delta^2}. \tag{3-44}$$

Therefore,

$$C\frac{\lambda_1\sigma_k(\sigma_\mu^{p\bar p})^2}{\sigma_\mu^2} \leq \frac{C}{\delta^2}\frac{\sigma_k}{\lambda_1}. \tag{3-45}$$

On the other hand, we notice that if $p>k$, then

$$\sigma_k^{p\bar p} \geq \lambda_1\cdots\lambda_{k-1} \geq c_n\frac{\sigma_k}{\lambda_k} \geq \frac{c_n}{\delta'}\frac{\sigma_k}{\lambda_1}.$$

If $\mu<p\leq k$, then

$$\sigma_k^{p\bar p} \geq \frac{\lambda_1\cdots\lambda_k}{\lambda_p} \geq c_n\frac{\sigma_k}{\lambda_p} \geq \frac{c_n}{\delta'}\frac{\sigma_k}{\lambda_1}.$$

It follows that for δ' small enough we have

$$\sigma_k^{p\bar p} \geq C\frac{\lambda_1\sigma_k(\sigma_\mu^{p\bar p})^2}{\sigma_\mu^2}. \tag{3-46}$$

This completes the proof of Lemma 3. □

3.2. ***Completing the proof.*** With Lemma 2 and Lemma 3 at our disposal, we claim that we may assume in inequality (3-17) that

$$A_i + B_i + C_i + D_i - E_i \geq 0, \quad \forall i = 1, \ldots, n. \tag{3-47}$$

Indeed, first set $\delta_1 = 1$. If $\lambda_2 \leq \delta_2 \lambda_1$ for $\delta_2 > 0$ small enough, then by Lemma 3 we see that (3-47) holds. Otherwise, $\lambda_2 \geq \delta_2 \lambda_1$. If $\lambda_3 \leq \delta_3 \lambda_1$ for $\delta_3 > 0$ small enough, then by Lemma 3 we see that (3-47) holds. Otherwise, $\lambda_3 \geq \delta_3 \lambda_1$. Proceeding iteratively, we may arrive at $\lambda_k \geq \delta_k \lambda_1$. But in this case, the C^2 estimate follows directly from the equation as

$$C \geq \sigma_k \geq \lambda_1 \cdots \lambda_k \geq (\delta_k)^{k-1} \lambda_1. \tag{3-48}$$

Therefore we may assume (3-47), and inequality (3-17) becomes

$$\begin{aligned} 0 \geq \frac{-C(K)}{\lambda_1}\big(1 + |DDu|^2 + |D\bar{D}u|^2\big) + (N - C\tau m N^2)(|DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega}) \\ + (M\varepsilon - C\tau m M^2 - CN - C)\mathcal{F} - CM. \end{aligned} \tag{3-49}$$

Since $\sigma_k^{i\bar{i}} \geq \sigma_k^{1\bar{1}} \geq \dfrac{k}{n}\dfrac{\sigma_k}{\lambda_1} \geq \dfrac{1}{C\lambda_1}$ for fixed i, we can estimate

$$|DDu|^2_{\sigma\omega} + |D\bar{D}u|^2_{\sigma\omega} \geq \frac{1}{C\lambda_1}(|DDu|^2 + |D\bar{D}u|^2) \geq \frac{1}{C\lambda_1}|DDu|^2 + \frac{\lambda_1}{C}. \tag{3-50}$$

This leads to

$$\begin{aligned} 0 \geq \Big(\frac{N}{C} - C\tau m N^2 - C(K)\Big)\lambda_1 + \frac{1}{\lambda_1}\Big(\frac{N}{C} - C\tau m N^2 - C(K)\Big)(1 + |DDu|^2) \\ + (M\varepsilon - C\tau m M^2 - CN - C)\mathcal{F} - CM. \end{aligned}$$

By choosing τ small, for example $\tau = 1/(NM)$, we have

$$\begin{aligned} 0 \geq \Big(\frac{N}{C} - \frac{Cm}{M}N - C(K)\Big)\lambda_1 + \frac{1}{\lambda_1}\Big(\frac{N}{C} - \frac{Cm}{M}N - C(K)\Big)(1 + |DDu|^2) \\ + \Big(M\varepsilon - \frac{Cm}{N}M - CN - C\Big)\mathcal{F} - CM. \end{aligned}$$

Taking N and M large enough, we can ensure that the coefficients of the first three terms are positive. For example, if we let $M = N^2$ for N large, then

$$\begin{aligned} \frac{N}{C} - \frac{Cm}{M}N - C(K) = \frac{N}{C} - \frac{Cm}{N} - C(K) > 0, \\ M\varepsilon - \frac{Cm}{N}M - CN - C = N^2\varepsilon - CmN - CN - C > 0. \end{aligned}$$

Thus, an upper bound of λ_1 follows. □

Remark 2. In the above estimate, we assume that $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Gamma_n$. Indeed, our estimate still works with $\lambda \in \Gamma_{k+1}$. It was observed in [Li et al. 2016, Lemma 7] that if $\lambda \in \Gamma_{k+1}$, then $\lambda_1 \geq \cdots \geq \lambda_n > -K_0$ for some positive constant K_0. Thus, we can replace λ by $\tilde{\lambda} = \lambda + K_0 I$ in our test function G in (3-2).

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