ANALYSIS & PDE

Volume 9 No. 7 2016

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ANALYSIS AND PDE
Vol. 9, No. 7, 2016
dx.doi.org/10.2140/apde.2016.9.1737

A DOUBLE WELL POTENTIAL SYSTEM

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A semilinear elliptic system of PDEs with a nonlinear term of double well potential type is studied in a cylindrical domain. The existence of solutions heteroclinic to the bottom of the wells as minima of the associated functional is established. Further applications are given, including the existence of multitransition solutions as local minima of the functional.

1. Introduction

In this paper, the system of partial differential equations

$$-\Delta u + V_u(x,u) = 0, \quad x \in \Omega, \tag{PDE}$$

where $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}^m$, will be studied. The set Ω is a cylindrical domain in $\mathbb{R}^n$ given by $\Omega = \mathbb{R} \times \mathcal{D}$, where $\mathcal{D}$ is a bounded open set in $\mathbb{R}^{n-1}$ with $\partial\mathcal{D} \in C^1$. On $\partial\Omega$, we require

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega = \mathbb{R} \times \partial\mathcal{D}, \tag{BC}$$

where ν is the outward-pointing unit normal to $\partial\mathcal{D}$. Later, Ω will be allowed to be a more general cylindrical domain which depends 1-periodically on x_1.

As to the function V, to begin assume:

(V_1) $V \in C^1(\overline{\Omega} \times \mathbb{R}^m, \mathbb{R})$ and $V(x_1+1, x_2, \dots, x_n, u) = V(x,u)$, i.e., V is 1-periodic in x_1.

(V_2) There are points $a^- \neq a^+$ such that $V(x, a^\pm) = 0$ for all $x \in \Omega$ and $V(x,u) > 0$ otherwise.

(V_3) There is a constant $\underline{V} > 0$ such that $\liminf_{|u|\to\infty} V(x,u) \geq \underline{V}$ uniformly in $x \in \Omega$.

(V_4) For $n \geq 2$, there exist constants $c_1, C_1 > 0$ such that

$$|V_u(x,u)| \leq c_1 + C_1|u|^p,$$

where $1 < p < (n+2)/(n-2)$ for $n \geq 3$ and there is no upper growth restriction on p if $n = 2$.

An example of V satisfying (V_1)–(V_4) is $V(x,u) = |u-a^-|^q\,|u-a^+|^q$ for $q \in (1, n/(n-2))$ and $a^+ \neq a^- \in \mathbb{R}^n$. By (V_2), V is a double well potential and we are interested in the existence of classical solutions of (PDE) that are heteroclinic in x_1 from a^- to a^+. If $n = 1$ and m is arbitrary, (PDE) reduces to a second-order Hamiltonian system of ordinary differential equations and conditions (V_1)–(V_3) suffice for such an existence result. For arbitrary n and m, conditions (V_1)–(V_4) enable us to show (PDE) possesses

MSC2010: primary 35J47; secondary 35J57, 58E30.
Keywords: elliptic system, double well potential, heteroclinic, minimization.

a weak solution. As is usual, we say that $U \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ is a weak solution of (PDE) and (BC) when for any $\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ having compact support in $\overline{\Omega}$,

$$\int_\Omega \big(\nabla U \cdot \nabla \varphi + V_u(x, U)\varphi\big)\, dx = 0. \tag{1.1}$$

The weak solution is a classical solution when $n = 1$. However, when $n > 1$, more regularity of V and $\partial\Omega$ is required to get a classical solution.

In Section 2, the functional

$$J(u) = \int_\Omega \big(\tfrac{1}{2}|\nabla u|^2 + V(x, u)\big)\, dx \equiv \int_\Omega L(u)\, dx, \tag{1.2}$$

whose formal Euler–Lagrange equation is (PDE), will be studied. Minimization arguments will be used to show that J has a critical point. In particular when $n = 1$, our first existence result for (PDE) is:

Theorem. *If V satisfies (V_1)–(V_3), then (PDE) possesses a solution heteroclinic from a^- to a^+.*

For $n > 1$, existence of solutions requires more work. In Section 3, a regularity theorem will be stated as a consequence of which we have:

Theorem. *If (V_1)–(V_4) hold, $V \in C^2$, and $\partial\Omega \in C^3$, there is a classical solution U of (PDE) and (BC) such that $\lim_{x_1 \to \pm\infty} U(x_1, \dots, x_n) = a^\pm$ uniformly for $(x_2, \dots, x_n) \in \mathcal{D}$.*

In Section 2, we find the solution by a minimization argument in an appropriate class of functions, Γ, and a detailed proof of the regularity will be given in Section 6.

Four generalizations of our existence results will be given in Section 4. The first, Theorem 4.1, essentially replaces conditions (V_2)–(V_4) by the requirement that V possesses a convex basin containing $a^\pm$ — see hypothesis (V_5) — to get an $L^\infty(\Omega, \mathbb{R}^m)$ bound for the minimizer of Section 2 and this bound leads in turn to the existence of a solution of (PDE) and (BC), which is heteroclinic in x_1 from a^- to a^+. This result gives the existence of the heteroclinic solution of (PDE) and (BC) for the example of $V(x, u) = |u - a^-|^q\, |u - a^+|^q$ mentioned earlier, but now for any $q > 1$.

The second replaces Ω by a more general domain varying periodically in x_1. The third considers a PDE perturbation of the case of $n = 1$. Finally for the fourth, the case of multiwell potentials will be discussed briefly.

In Section 5, it will be shown that variational gluing arguments in the spirit of [Montecchiari and Rabinowitz 2016] together with the basic heteroclinic minimizers of (1.2) as well as their counterparts when the roles of a^- and a^+ are reversed can be used to construct infinitely many multitransition homoclinic and heteroclinic solutions of (PDE). These solutions are local minima of (1.2) that as a function of x_1 transit back and forth between the two global minima, $a^\pm$, of V. Obtaining these solutions requires a mild nondegeneracy condition — see Proposition 5.10(ii) — on the set of heteroclinic minimizers of (1.2). Stated very informally, we will show:

Theorem. *If (V_1)–(V_4) are satisfied and a mild nondegeneracy condition on the heteroclinics in x_1 from $a^\pm$ to $a^\mp$ holds, then for each $k \in \mathbb{N} \cup \{\infty\}$, $k \geq 2$, there exist infinitely many k-transition solutions of (PDE) and (BC).*

As has been noted above, our existence results rely on minimization arguments from the calculus of variations. These arguments are elementary, but often delicately exploit (V_1)–(V_3). The regularity arguments where (V_4) and further smoothness of V and $\partial\mathcal{D}$ play their roles are of necessity rather technical.

To conclude this section, some of the literature on (PDE) and (BC) will be discussed. The earliest work we know of is for the case of $n = 1$, where of course $\mathcal{D} = \varnothing$ and (BC) is vacuous. Thus (PDE) becomes a second-order Hamiltonian system. Using geometrical arguments, the existence of heteroclinic solutions for $V = V(u)$ was studied for a more general class of potentials by Bolotin [1978]. See also the survey article by Kozlov [1985]. Subsequently other work was done, also for the autonomous case where $V \in C^3$ has nondegenerate minima and $m = 2$, by Sternberg [1991]. Rabinowitz [1993] treated $V = V(t, u)$ where $V \in C^2$ is periodic in t. He used minimization arguments from [Rabinowitz 1989], where $V = V(u)$ and is periodic in the components of u. Alikakos and Fusco [2008] also treated the autonomous case for a C^2 potential under a milder condition than the nondegeneracy of the minima.

For $m = 1$ and $n > 1$, where (BC) plays a role, minimization arguments similar to the ones used in [Rabinowitz 1994] were used in [Rabinowitz 2002] and generalized in [Rabinowitz 2004] to obtain heteroclinics in x_1. The case of $m, n > 1$ for (PDE) has been studied extensively in several papers by Alikakos and his collaborators, especially Fusco, mainly in the autonomous setting when V possesses symmetries and one seeks solutions possessing these symmetries [Alikakos 2012; 2013; Alikakos and Fusco 2008; 2009; 2011; 2015; Alikakos and Smyrnelis 2012]. In fact it was their recent paper, [Alikakos and Fusco 2015], together with our work [Montecchiari and Rabinowitz 2016] on systems like (PDE) but with potentials $V(x, u)$ that are periodic in the components of u that led to this paper. Alikakos and Fusco [2015] studied (PDE) and (BC), with Ω periodic in x_1, essentially under the C^2 version of (V_1), and stronger forms of (V_2) and (V_5). See the survey paper [Alikakos 2013] for many more references to and related questions for (PDE). For some other related results on entire solutions of systems of Allen–Cahn-type, see [Alessio 2013; Alessio and Montecchiari 2014; Bronsard and Reitich 1993; Gui and Schatzman 2008; Schatzman 2002].

2. The existence of a minimizer of J

In this section, as a first step towards finding heteroclinic solutions of (PDE), a minimizer will be obtained for the functional J, defined in (1.2). The functional will be studied on the Hilbert space

$$E \equiv \left\{ u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m) \;\middle|\; \|u\|^2 \equiv \int_\Omega |\nabla u|^2\, dx + \int_{T_0} |u|^2\, dx < \infty \right\},$$

where for $i \in \mathbb{Z}$, we set $T_i = (i, i+1) \times \mathcal{D}$. As the class of admissible functions, take

$$\Gamma = \left\{ u \in E \;\middle|\; \|u - a^\pm\|_{L^2(T_i, \mathbb{R}^m)} \to 0,\ i \to \pm\infty \right\}.$$

Define

$$c = \inf_{u \in \Gamma} J(u). \tag{2.1}$$

It is readily seen that $\Gamma \neq \varnothing$ and $0 \le c < \infty$. Then we have:

Theorem 2.2. *Suppose $\Omega = \mathbb{R} \times \mathcal{D}$ with $\mathcal{D} \subset \mathbb{R}^{n-1}$ a bounded domain and $\partial\mathcal{D} \in C^1$. If V satisfies (V_1)–(V_3), then there exists a $U \in \Gamma$ such that $J(U) = c > 0$. Moreover, there is a constant $M > 0$ such that for any minimizer U of (2.1),*

$$\sup_{i \in \mathbb{Z}} \|U\|_{W^{1,2}(T_i, \mathbb{R}^m)} \le M.$$

Before proving Theorem 2.2, the following result is useful.

Proposition 2.3. *Let V satisfy (V_1)–(V_3), $\partial\mathcal{D} \in C^1$, and $v \in E$ with $J(v) < \infty$. Then there are $\varphi^\pm \in \{a^-, a^+\}$ such that $\|v - \varphi^\pm\|_{L^2(T_i, \mathbb{R}^m)} \to 0$ as $i \to \pm\infty$.*

Proof. Their proofs being the same, we will prove the result for φ^+. For $x \in \overline{\Omega}$, set $x = (x_1, \hat{x})$ with $x_1 \in \mathbb{R}$ and $\hat{x} \in \overline{\mathcal{D}}$. For $x \in T_0$ and $k \in \mathbb{Z}$, set $v_k(x) = v(x_1 + k, \hat{x})$ so $v_k \in W^{1,2}(T_0, \mathbb{R}^m)$. Then (V_2) and $J(v) < \infty$ imply

$$\lim_{k\to\infty} \|\nabla v_k\|_{L^2(T_0, \mathbb{R}^m)} = \lim_{k\to\infty} \int_{T_0} V(x, v_k)\, dx = 0. \tag{2.4}$$

Consequently $\{\|\nabla v_k\|_{L^2(T_0,\mathbb{R}^m)}\}$ is bounded independently of $k \in \mathbb{Z}$. By the Poincaré inequality and the fact that $\mathcal{D} \in C^1$, there is a constant b so that

$$\|v_k - [v_k]\|_{L^2(T_0,\mathbb{R}^m)} \le b \|\nabla v_k\|_{L^2(T_0,\mathbb{R}^m)}, \tag{2.5}$$

where $[v_k]$ denotes the mean value of v_k on T_0. We claim that $\{v_k\}$ is bounded in $L^2(T_0, \mathbb{R}^m)$. If not, (2.5) shows $\{[v_k]\}$ is unbounded in $\mathbb{R}$. For a set $S \subset \mathbb{R}^n$, let $|S|$ denote the measure of S. By (2.5) again, the sequence $\{v_k - [v_k]\}$ converges to 0 in measure. Therefore for any $\delta > 0$, the measure of the set in T_0 where $|v_k - [v_k]| \le \delta$ is at least $\frac{1}{2}|T_0|$ for large k. Thus by (V_3), for large $k > 0$,

$$\int_{T_0} V(x, v_k)\, dx \ge \tfrac{1}{4}|T_0|\underline{V}. \tag{2.6}$$

But (2.6) is contrary to (2.4), so $\{v_k\}$ is bounded in $W^{1,2}(T_0, \mathbb{R}^m)$. Hence there is a $v^* \in W^{1,2}(T_0, \mathbb{R}^m)$ such that along a subsequence, v_k converges to v^* weakly in $W^{1,2}(T_0, \mathbb{R}^m)$ and strongly in $L^2(T_0, \mathbb{R}^m)$. By (2.5), $v^* = [v^*]$; i.e., v^* is a constant vector. Again $v_k \to v^*$ in measure along the subsequence as $k \to \infty$, so for any small δ, we have $|v_k - v^*| \le \delta$ on a subset of T_0 of measure $\ge \frac{1}{2}|T_0|$. Therefore

$$\int_{T_0} V(x, v_k)\, dx \ge \tfrac{1}{2}|T_0| \min_{z \in B_\delta(v^*)} V(x, z), \tag{2.7}$$

where $B_\delta(v)$ denotes an open ball of center v and radius δ in $\mathbb{R}^m$. If $v^* \notin \{a^-, a^+\}$, and δ is small enough, the right-hand side of (2.7) is positive. But as $k \to \infty$, the left-hand side of (2.7) goes to 0. Therefore $v^* \in \{a^-, a^+\}$. For notational convenience, suppose $v^* = a^-$.

It remains to show that the entire sequence $\{v_k\}$, rather than a subsequence, converges to a^-, i.e.,

$$\lim_{k\to\infty} v_k = a^-. \tag{2.8}$$

Otherwise, there exist subsequences $\{i_p\}, \{k_q\} \subset \mathbb{N}$, with $i_p \to \infty$ as $p \to \infty$, $k_q \to \infty$ as $q \to \infty$, $i_p < k_p < i_{p+1}$ for all p and such that

$$\lim_{p\to\infty} v_{i_p} = a^-, \quad \lim_{q\to\infty} v_{k_q} = a^+.$$

Set $\varepsilon = \frac{1}{3}|a^+ - a^-|\sqrt{|T_0|}$. Therefore there is a $\underline{p}$ such that for $p \geq \underline{p}$,

$$\|v_{i_p} - a^-\|_{L^2(T_0,\mathbb{R}^m)} < \varepsilon, \quad \|v_{k_p} - a^+\|_{L^2(T_0,\mathbb{R}^m)} < \varepsilon.$$

We claim that for $\underline{p}$ possibly still larger and all $p \geq \underline{p}$, there is an $s_p \in \mathbb{N}$ such that $i_p < s_p < k_p$ and

$$\|v_{s_p} - a^-\|_{L^2(T_0,\mathbb{R}^m)} \geq \varepsilon, \quad \|v_{s_p} - a^+\|_{L^2(T_0,\mathbb{R}^m)} \geq \varepsilon.$$

If not, for every t between i_p and k_p,

$$\|v_t - a^-\|_{L^2(T_0,\mathbb{R}^m)} < \varepsilon \quad \text{or} \quad \|v_t - a^+\|_{L^2(T_0,\mathbb{R}^m)} < \varepsilon.$$

Replace i_p and k_p by the smallest adjacent pair $j, j+1 \in \mathbb{N} \cap [i_p, k_p]$ such that

$$\|v_j - a^-\|_{L^2(T_0,\mathbb{R}^m)} < \varepsilon, \quad \|v_{j+1} - a^+\|_{L^2(T_0,\mathbb{R}^m)} < \varepsilon. \tag{2.9}$$

Next observe that

$$\begin{aligned}|v_{j+1}(x) - v_j(x)| &= \left|\int_0^1 v_{x_1}(x_1 + j + s, \hat{x})\,ds\right| \leq \left|\int_0^2 v_{x_1}(j + s, \hat{x})\,ds\right| \\ &\leq \sqrt{2}\left(\int_0^2 v_{x_1}(j+s,\hat{x})^2\,ds\right)^{1/2}.\end{aligned}$$

Therefore

$$\|v_{j+1} - v_j\|_{L^2(T_0,\mathbb{R}^m)} \leq \sqrt{2}\|v_{x_1}\|_{L^2(T_0\cup T_1,\mathbb{R}^m)}. \tag{2.10}$$

By (2.4), for $\underline{p}$ still larger, we can assume the right-hand side of (2.10) is $\leq \varepsilon$. On the other hand, by (2.9),

$$\begin{aligned}\|v_{j+1} - v_j\|_{L^2(T_0,\mathbb{R}^m)} &> \|a^+ - a^-\|_{L^2(T_0,\mathbb{R}^m)} - 2\varepsilon \\ &= |a^+ - a^-|\sqrt{|T_0|} - 2\varepsilon.\end{aligned} \tag{2.11}$$

Since $3\varepsilon = |a^+ - a^-|\sqrt{|T_0|}$, (2.11) is not possible and therefore there exists a sequence $\{s_p\}$ as claimed. But then

$$J(v) \geq \sum_{\underline{p}}^{\infty} \int_{T_{s_p}} L(v)\,dx = \infty$$

and we have a contradiction, establishing Proposition 2.3. □

To prove Theorem 2.2, let $\{u_k\}$ be a minimizing sequence for (2.1). Thus there is a constant M_1 such that for all $k \in \mathbb{N}$,

$$J(u_k) \leq M_1. \tag{2.12}$$

Let $\rho \in \big(0, \frac{1}{4}|a^+ - a^-|\sqrt{|T_0|}\big)$. Noting that Γ and J are invariant under a unit phase shift in the x_1-direction, it can be assumed that

$$\|u_k - a^-\|_{L^2(T_i,\mathbb{R}^m)} \le \rho \quad \text{for all } i \le 0 \qquad \text{and} \qquad \|u_k - a^-\|_{L^2(T_1,\mathbb{R}^m)} > \rho \tag{2.13}$$

for all $k \in \mathbb{N}$. Now a few observations about any $u \in \Gamma$ are required. Set

$$\Gamma_1 \equiv \big\{u \in \Gamma \;\big|\; \min\{\|u - a^-\|_{L^2(T_0,\mathbb{R}^m)}, \|u - a^+\|_{L^2(T_0,\mathbb{R}^m)}\} \ge \rho\big\},$$
$$\Gamma_2 \equiv \big\{u \in \Gamma \;\big|\; \max\{\|u - a^-\|_{L^2(T_0,\mathbb{R}^m)}, \|u - a^+\|_{L^2(T_1,\mathbb{R}^m)}\} \le \rho\big\},$$
$$\Gamma_3 \equiv \big\{u \in \Gamma \;\big|\; \max\{\|u - a^+\|_{L^2(T_0,\mathbb{R}^m)}, \|u - a^-\|_{L^2(T_1,\mathbb{R}^m)}\} \le \rho\big\}.$$

Proposition 2.14. (1) *There is a constant* $\kappa_1 > 0$ *such that*

$$d_1 \equiv \inf_{u \in \Gamma_1} \int_{T_0} L(u)\, dx \ge \kappa_1.$$

(2) *There is a constant* $\kappa > 0$ *such that*

$$d \equiv \inf_{u \in \Gamma_2 \cup \Gamma_3} \int_{T_0 \cup T_1} L(u)\, dx \ge \kappa.$$

Proof. If $\kappa_1 = 0$, there is a sequence $\{v_k\}$ in Γ_1 such that

$$\int_{T_0} L(v_k)\, dx \to 0 \quad \text{as } k \to \infty. \tag{2.15}$$

Arguing as in the proof of Proposition 2.3, we again conclude (2.4)–(2.5) hold and either (i) both $\{v_k\}$ is bounded in $L^2(T_0, \mathbb{R}^m)$ and $\{[v_k]\}$ is bounded in $\mathbb{R}^m$ or (ii) both sequences are unbounded. If (i) occurs, as in the proof of Proposition 2.3, $\{v_k\}$ converges along a subsequence in $L^2(T_0, \mathbb{R}^m)$ to a constant function $v^* = [v^*]$ and for any small δ, for large k, we have $|v_k - v^*| \le \delta$ on a subset of T_0 of measure $\ge \frac{1}{2}|T_0|$. Thus (2.7) again holds. Noting that

$$|v - a^\pm||T_0|^{1/2} = \|v - a^\pm\|_{L^2(T_0,\mathbb{R}^m)} \ge \rho$$

for δ small compared to ρ, (V_2) and (V_3) show the right-hand side of (2.7) is positive independently of v. This contradicts (2.4) and this case is proved.

Next suppose that (ii) occurs. Then the argument centered around (2.6) again applies and this case is impossible. Thus (1) of the proposition is proved.

For the proof of (2), we use a similar argument. Assume to the contrary that $\kappa = 0$. Then there is a sequence $\{v_k\}$ in $\Gamma_2 \cup \Gamma_3$ such that

$$\int_{T_0 \cup T_1} L(v_k)\, dx \to 0 \quad \text{as } k \to \infty. \tag{2.16}$$

Taking a subsequence if necessary, it can be assumed that $\{v_k\} \subset \Gamma_2$ or $\{v_k\} \subset \Gamma_3$. Suppose $\{v_k\} \subset \Gamma_2$. Arguing as in the proof of (1), by (2.16),

$$\lim_{k\to\infty} \|\nabla v_k\|_{L^2(T_0 \cup T_1,\mathbb{R}^m)} = \lim_{k\to\infty} \int_{T_0 \cup T_1} V(x, v_k)\, dx = 0. \tag{2.17}$$

Again by the Poincaré inequality, there is a constant b_1 so that

$$\|v_k - [v_k]_1\|_{L^2(T_0 \cup T_1, \mathbb{R}^m)} \le b_1 \|\nabla v_k\|_{L^2(T_0 \cup T_1, \mathbb{R}^m)} \to 0 \quad \text{as } k \to \infty, \tag{2.18}$$

where $[v_k]_1$ denotes the mean value of v_k on $T_0 \cup T_1$. It follows as in case (ii) of (1) that $\{[v_k]_1\}$ is bounded. Taking a subsequence again if necessary, it can be assumed that $\lim_{k\to\infty}[v_k]_1 = a \in \mathbb{R}^m$. Then we see that

$$\begin{aligned} 2\rho &\ge \lim_{k\to\infty} \|v_k - a^-\|_{L^2(T_0,\mathbb{R}^m)} + \lim_{k\to\infty} \|v_k - a^+\|_{L^2(T_1,\mathbb{R}^m)} \\ &= \|a^- - a\|_{L^2(T_0,\mathbb{R}^m)} + \|a^+ - a\|_{L^2(T_1,\mathbb{R}^m)} \\ &= |a^- - a|\sqrt{|T_0|} + |a^+ - a|\sqrt{|T_1|} \\ &\ge |a^- - a^+|\sqrt{|T_0|}, \end{aligned}$$

which contradicts that $\rho < \frac{1}{4}|a^+ - a^-|\sqrt{|T_0|}$. In the remaining case where $\{v_k\} \subset \Gamma_3$, a contradiction follows by the same argument. This proves (2). □

Remark 2.19. Observe that for any $u \in \Gamma$ satisfying (2.12) and any $i \in \mathbb{N}$, either

$$\min\left\{\|u - a^-\|_{L^2(T_i,\mathbb{R}^m)}, \|u - a^+\|_{L^2(T_i,\mathbb{R}^m)}\right\} > \rho \tag{2.20}$$

or

$$\min\left\{\|u - a^-\|_{L^2(T_i,\mathbb{R}^m)}, \|u - a^+\|_{L^2(T_i,\mathbb{R}^m)}\right\} \le \rho. \tag{2.21}$$

Let $l(u)$ be the number of values of i for which (2.20) holds. By (2.12) and Proposition 2.14(1),

$$l(u)\kappa_1 \le M_1. \tag{2.22}$$

Thus (2.22) shows $l(u)$ is bounded from above independently of u; i.e., (2.20) holds for at most M_1/κ_1 values of i. Next let $l^*(u)$ denote the number of values of i for which

$$\max\left\{\|u - a^-\|_{L^2(T_i,\mathbb{R}^m)}, \|u - a^+\|_{L^2(T_{i+1},\mathbb{R}^m)}\right\} \le \rho$$

or

$$\max\left\{\|u - a^+\|_{L^2(T_i,\mathbb{R}^m)}, \|u - a^-\|_{L^2(T_{i+1},\mathbb{R}^m)}\right\} \le \rho.$$

Hence $l^*(u)$ represents the number of transitions of u from being "near" $a^\pm$ on T_i to being "near" $a^\mp$ on T_{i+1}. By Proposition 2.14(2),

$$l^*(u)\kappa \le M_1. \tag{2.23}$$

This means that the number of pairs of consecutive intervals on which u shifts from being near one of a^- or a^+ to the other is uniformly bounded for $u \in \Gamma$ satisfying (2.12).

Bounds for the functions u_k in the minimizing sequence are provided by the next result.

Proposition 2.24. *If V satisfies (V_1)–(V_3), then there is a constant M such that $\|u_k\|_{W^{1,2}(T_i,\mathbb{R}^m)} \le M$ for all $k \in \mathbb{N}$ and $i \in \mathbb{Z}$.*

Proof. We argue as in an analogous situation in [Montecchiari and Rabinowitz 2016]. It can be assumed that u_k satisfies the normalization (2.13). By (2.12),

$$J(u_k) = \sum_{i\in\mathbb{Z}} \int_{T_i} L(u_k)\,dx \le M_1. \tag{2.25}$$

Therefore (2.25) and (2.13) immediately yield the desired bound for some value of M, say M_2, for $i \le 0$. For any $i \in \mathbb{Z}$ for which $\|u_k - a^{\pm}\|_{L^2(T_i,\mathbb{R}^m)} \le \rho$, we get the $\|u_k\|_{W^{1,2}(T_i)}$ bound exactly as was done for $i \le 0$ and obtain the same upper bound, M_2. By Remark 2.19, there are at most l values of i that remain. They lie in

$$A_k = \big\{i \in \mathbb{N} \;\big|\; \|u_k - a^-\|_{L^2(T_i,\mathbb{R}^m)} \ge \rho,\ \|u_k - a^+\|_{L^2(T_i,\mathbb{R}^m)} \ge \rho\big\}.$$

Note that $A_k \subset \mathbb{N}$. Let $i \notin A_k$ and $i+1 \in A_k$. Let $\hat{x} = (x_2, \dots, x_n)$. For $(s,\hat{x}) \in T_i$ and $(\sigma,\hat{x}) \in T_{i+1}$,

$$u_k(\sigma,\hat{x}) = u_k(s,\hat{x}) + \int_s^\sigma \frac{\partial u_k(t,\hat{x})}{\partial t}\,dt,$$

so

$$|u_k(\sigma,\hat{x})|^2 \le 2|u_k(s,\hat{x})|^2 + 4\int_i^{i+2} |\nabla u_k(t,\hat{x})|^2\,dt. \tag{2.26}$$

Integrating (2.26) over $s, \sigma, \hat{x}$ gives

$$\|u_k\|^2_{L^2(T_{i+1},\mathbb{R}^m)} \le 2\|u_k\|^2_{L^2(T_i,\mathbb{R}^m)} + 4\|\nabla u_k\|^2_{L^2(T_i\cup T_{i+1},\mathbb{R}^m)}. \tag{2.27}$$

Therefore by (2.25) and the above remarks,

$$\|u_k\|^2_{W^{1,2}(T_{i+1},\mathbb{R}^m)} \le 2M_2^2 + 8M_1 \equiv M_3. \tag{2.28}$$

Then, if $i+2 \in A_k$, the argument of (2.27)–(2.28) can be repeated. Since the number of elements of A_k is bounded by $l \in \mathbb{N}$, the process stops in at most l steps, giving the desired bound with $M = M(l)$. □

Completion of the proof of Theorem 2.2. It is convenient to introduce some notions. A set $I \subset \mathbb{Z}$ will be called connected if for any $i, j \in I$ with $i \le j$, any integer between i and j is also an element in I. For two connected sets $I_1, I_2 \in \mathbb{Z}$ with $I_1 \cap I_2 = \varnothing$, we write $I_1 < I_2$ if $i_1 < i_2$ for any $i_1 \in I_1$ and $i_2 \in I_2$. For a connected set $I \subset \mathbb{Z}$, the length $|I|$ of I is defined by $|I| = \sup\{|i-j| \mid i, j \in I\}$. Now consider the minimizing sequence $\{u_k\}$ normalized by (2.13). By Remark 2.19, for each k, there are finitely many disjoint connected sets $I_1^k < \cdots < I_{l(k)}^k$ in $\mathbb{Z}$ satisfying

$$\big\{i \in \mathbb{Z} \;\big|\; \|u_k - a^-\|_{L^2(T_i,\mathbb{R}^m)} \le \rho\big\} = \bigcup_{j=1}^{l(k)} I_j^k.$$

The normalization (2.13) shows that for any integer $i \le 0$, we have $i \in I_1^k$ and $|I_j^k| < \infty$ for $j = 2, \dots, l(k)$. Remark 2.19 also implies that the sequence $\{l(k)\}$ is bounded. Taking a subsequence of $k \in \mathbb{N}$ if necessary, it can be assumed that $l(k)$ is a positive integer l independent of $k \in \mathbb{N}$. Define

$$p_0 \equiv \max\big\{i \in \{1, \dots, l\} \;\big|\; \limsup_{k\to\infty} |I_i^k| = \infty\big\}.$$

It is well-defined since $|I_1^k| = \infty$. Note that if $p_0 < l$,

$$\limsup_{k\to\infty} \sum_{j=p_0+1}^{l} |I_j^k| < \infty. \tag{2.29}$$

Now define $p(k)$ to be the largest $i \in I_{p_0}^k$. Set $v_k(x) = u_k(x_1 + p(k), x_2, \dots, x_n)$ for $k \in \mathbb{N}$, so $\{v_k\}$ is a new minimizing sequence. By Proposition 2.24, the set of norms $\{\|v_k\|_{W^{1,2}(T_i,\mathbb{R}^m)} \mid i \in \mathbb{Z},\ k \in \mathbb{N}\}$ is bounded. Since $\partial\Omega \in C^1$, taking a subsequence if necessary, we see that for some $U \in E$ and any $i \in \mathbb{Z}$, v_k converges weakly to U in $W^{1,2}(T_i)$, strongly to U in $L^2(T_i, \mathbb{R}^m)$ and pointwise a.e. to U on T_i as $k \to \infty$. Therefore $V(x, v_k) \to V(x, U)$ pointwise a.e. The weak lower semicontinuity of the $|\nabla u|^2$ term in J on bounded sets implies that for any $p < q \in \mathbb{Z}$,

$$\sum_{i=p}^{q} \left(\int_{T_i} |\nabla U|^2\, dx \right) \le \liminf_{k\to\infty} \sum_{i=p}^{q} \left(\int_{T_i} |\nabla v_k|^2\, dx \right).$$

By Fatou's lemma,

$$\sum_{i=p}^{q} \left(\int_{T_i} V(x, U)\, dx \right) \le \liminf_{k\to\infty} \sum_{i=p}^{q} \left(\int_{T_i} V(x, v_k)\, dx \right).$$

Combining these inequalities yields

$$\sum_{i=p}^{q} \left(\int_{T_i} L(U)\, dx \right) \le \liminf_{k\to\infty} \sum_{i=p}^{q} \left(\int_{T_i} L(v_k)\, dx \right) \le \liminf_{k\to\infty} J(v_k) \le c.$$

Letting $p \to -\infty$ and $q \to \infty$ gives

$$J(U) \le c. \tag{2.30}$$

Since $\lim_{k\to\infty} |I_{p_0}^k| = \infty$, we see that

$$\|U - a^-\|_{L^2(T_i,\mathbb{R}^m)} \le \rho \quad \text{for } i \le 0. \tag{2.31}$$

By (2.30), $\|\nabla U\|_{L^2(T_i,\mathbb{R}^m)} \to 0$ as $i \to \infty$. By the Poincaré inequality, there is a constant b, independent of $i \in \mathbb{Z}$, so that

$$\|U - [U]^i\|_{L^2(T_i,\mathbb{R}^m)} \le b\|\nabla U\|_{L^2(T_i,\mathbb{R}^m)} \to 0 \quad \text{as } i \to \infty, \tag{2.32}$$

where $[U]^i$ is the mean value of U on T_i. Since $\int_\Omega V(x, U)\, dx < \infty$, as in the proof of Proposition 2.14, it follows that $\lim_{i\to\infty} [U]^i = a^-$ or a^+. Thus,

$$\lim_{i\to\infty} \|U - a^-\|_{L^2(T_i,\mathbb{R}^m)} = 0 \quad \text{or} \quad \lim_{i\to\infty} \|U - a^+\|_{L^2(T_i,\mathbb{R}^m)} = 0.$$

If $\lim_{i\to\infty} \|U - a^-\|_{L^2(T_i,\mathbb{R}^m)} = 0$, this contradicts (2.29) since $\lim_{k\to\infty} \|U - v_k\|_{L^2(T_i,\mathbb{R}^m)} = 0$ for each $i \in \mathbb{Z}$. Consequently,

$$\lim_{i\to\infty} \|U - a^+\|_{L^2(T_i,\mathbb{R}^m)} = 0 \tag{2.33}$$

and $U \in \Gamma$. This with (2.30) shows U is a minimizer of J in (2.1). It is clear that $J(U) = c > 0$ and Theorem 2.2 is proved. □

If $n = 1$, then $\mathcal{D} = \varnothing$ and $\Omega = \mathbb{R}$ in the problem (PDE). Thus, in this case, (PDE) reduces to a second-order Hamiltonian system of ordinary differential equations. Moreover, we get a much stronger conclusion than Theorem 2.2:

Theorem 2.34. *Assume $n = 1$. If V satisfies (V_1)–(V_3) with $\mathcal{D} = \varnothing$ and $\Omega = \mathbb{R}$, then any minimizer U of (2.1) is a classical solution of (PDE).*

Proof. Since $n = 1$, the above $W^{1,2}_{\mathrm{loc}}$ bounds imply U is continuous. Its asymptotic behavior then shows $U \in L^\infty(\mathbb{R}, \mathbb{R}^m)$. Consider $\varphi \in W^{1,2}_{\mathrm{loc}}(\mathbb{R})$ having compact support in $\mathbb{R}$ and $t \in \mathbb{R}$. Then for $0 < |t|$ small, $U + t\varphi \in \Gamma$. Consequently, $J(U + t\varphi) \geq J(U)$ or

$$\int_{\operatorname{supp}\varphi} L(U + t\varphi) - L(U)\, dx \geq 0 \tag{2.35}$$

for all such t and φ. Hence

$$\int_\Omega \nabla U \cdot \nabla\varphi + V_u(x, U)\cdot\varphi\, dx = 0 \tag{2.36}$$

for all such φ, so U is a weak solution of (PDE). But for $n = 1$, the weak form of (PDE) implies U is a classical solution of (PDE). □

Remark 2.37. If the minimizer U of Theorem 2.2 lies in $L^\infty(\Omega, \mathbb{R}^m)$, the argument just given in (2.35)–(2.36) shows U is a weak solution of (PDE) even for $n > 1$.

Remark 2.38. This existence result for $n = 1$ under (V_1)–(V_3) seems to be new. It generalizes earlier such results, [Bolotin 1978; Kozlov 1985; Sternberg 1991; Rabinowitz 1989; 1993; 2012; Alikakos and Fusco 2015], which get the existence results under slightly stronger hypotheses on V in terms of smoothness and nondegenerate behavior of V at the equilibrium solutions $a^\pm$.

To conclude this section, as a corollary of Theorem 2.34, an explicit L^∞ bound for any minimizer U will be given. The bound will be useful in Section 4. First some notational preliminaries are needed. Since $U = U(x_1)$, writing t for x_1, by (2.1),

$$J(U) = c = \int_{\mathbb{R}} \left(\tfrac{1}{2}|U_t|^2 + V(t, U)\right) dt, \tag{2.39}$$

so

$$\int_{\mathbb{R}} |U_t|^2\, dt \leq 2c. \tag{2.40}$$

With $\rho \leq \frac{1}{2}|a^+ - a^-|$, let

$$T(\rho) \equiv \left\{t \in \mathbb{R} \;\middle|\; \min\{|U(t) - a^-|, |U(t) - a^+|\} \geq \rho\right\}.$$

By (V_1)–(V_3),

$$\beta(\rho) \equiv \inf\left\{V(t, u) \;\middle|\; t \in \mathbb{R},\ \min\{|u - a^-|, |u - a^+|\} \geq \rho\right\} > 0.$$

Therefore by (2.39),

$$|T(\rho)|\beta(\rho) \leq \int_{\mathbb{R}} V(t, U)\, dt \leq c. \tag{2.41}$$

Corollary 2.42. *If U is a minimizer of (2.1) as in Theorem 2.34, then*

$$\|U\|_{L^\infty(\mathbb{R},\mathbb{R}^m)} \le \rho + \max\{|a^-|,|a^+|\} + \left(\frac{2}{\beta(\rho)}\right)^{\frac{1}{2}} c \equiv K. \tag{2.43}$$

Proof. If $\|U\|_{L^\infty(\mathbb{R},\mathbb{R}^m)} \le \max\{|a^-|,|a^+\}$, the estimate holds. Thus we may assume that $\|U\|_{L^\infty(\mathbb{R},\mathbb{R}^m)} > \max\{|a^-|,|a^+|\}$. Then, the maximum of $|U|$ is achieved at some $z \in \mathbb{R}$. If $z \notin T(\rho)$, it follows that

$$|U(z)| \le \rho + \max\{|a^+|,|a^-|\}.$$

If $z \in T(\rho)$, we take ξ to be the closest boundary point of $T(\rho)$ to z. Then, we see from (2.40)–(2.41) that

$$\begin{aligned}\|U\|_{L^\infty(\mathbb{R},\mathbb{R}^m)} = |U(z)| &\le |U(\xi)| + \left|\int_\xi^z U_t(s)\,ds\right| \\ &\le |U(\xi)| + \left(|z-\xi|\int_\xi^z |U_t(s)|^2\,ds\right)^{\frac{1}{2}} \\ &\le |U(\xi)| + |T(\rho)|^{\frac{1}{2}}(2c)^{\frac{1}{2}} \le |U(\xi)| + \left(\frac{2}{\beta(\rho)}\right)^{\frac{1}{2}} c.\end{aligned}$$

Since $|U(\xi)| \le \rho + \max\{|a^-|,|a^+|\}$, (2.43) now follows. ☐

Remark 2.44. Suppose that V in Theorem 2.34 is modified for $|u| > K$ so that the resulting function, V^*, still satisfies (V_1)–(V_3) (for $n = 1$) and

$$\inf\{V^*(t,u) \mid t \in \mathbb{R},\ \min\{|u-a^-|,|u-a^+|\} \ge \rho\} \ge \beta(\rho).$$

Then the corresponding functional J^* has a minimizer $U^* \in \Gamma$ and since $V^*(t,u) = V(t,u)$ for $|u| \le K$, minimizing sequences $\{u_k\}$ for J^* can be assumed to satisfy $J^*(u_k) \le J(U)$. Consequently

$$J^*(U^*) \le J(U) \tag{2.45}$$

and (2.45) and the derivation of (2.43) show any minimizer U^* of the modified problem is also bounded in L^∞ by K. Thus such a modification produces no new minimizers.

3. The regularity of the weak solution

The regularity of any weak solution U of (PDE) that minimizes J on Γ will be discussed in this section. The special case of $n = 1$ has already been shown in Theorem 2.34. Therefore it will be assumed that $n \ge 2$ in what follows. Using standard terminology, a solution u of (PDE) and (BC) is called a strong solution if $u \in W^{2,2}_{\mathrm{loc}}(\bar{\Omega})$. Our main result is:

Theorem 3.1. *Suppose V satisfies (V_1)–(V_4).*

(1) *If $\partial\Omega = \mathbb{R} \times \partial\mathcal{D} \in C^2$, then any minimizer U of (2.1) is a weak solution of (PDE) and (BC). Moreover, any weak solution $U \in E$ of (PDE) and (BC) is a strong solution of (PDE) and (BC), and $U \in L^\infty(\Omega)$.*

(2) *If* $V_u \in C^1(\overline{\Omega} \times \mathbb{R}^m)$ *and* $\partial\Omega \in C^3$, *then* $U \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ *for any* $\alpha \in (0,1)$ *and* U *is a classical solution of* (PDE) *and* (BC) *with* $\lim_{x_1 \to \pm\infty} U(x_1, \hat{x}) = a^{\pm}$ *uniformly for* $\hat{x} \in \mathcal{D}$.

Regularity results for weak solutions of a single second-order quasilinear elliptic partial differential equation satisfying Dirichlet boundary conditions can be found in the literature; see, e.g., Chapters 8–9 of [Gilbarg and Trudinger 1983]. However, we do not know of a reference for such a result for the system (PDE) with (BC). Therefore for completeness we will provide a proof of Theorem 3.1 but postpone it until Section 6.

4. Some generalizations

In this section, Theorem 2.2 will be generalized in various ways. First we will show that the growth condition, (V_4), can be bypassed when a geometrical condition that leads to an L^∞ bound for minimizers of (2.1) is satisfied. Next the case of a more general domain Ω that is periodic in the x_1-direction will be treated. Then a perturbation result will be given. Lastly, the case when the potential V has multiple minima will be discussed briefly.

To begin the first result, for any set $A \in \mathbb{R}^m$ and $a > 0$, let $A^a \equiv \{y \in \mathbb{R}^m \mid \mathrm{dist}(y, A) < a\}$.

Theorem 4.1. *Suppose that* V *satisfies* (V_1) *and* (V_5), *where*:

(V_5) *There is a convex bounded open set* $O \subset \mathbb{R}^m$ *with* $\partial O \in C^2$ *such that*

(1) *there are two different points* a^- *and* a^+ *in* O *such that* $V(x, a^{\pm}) = 0$ *for all* $x \in \Omega$ *and* $V(x,u) > 0$ *for any* $u \in \overline{O} \setminus \{a^-, a^+\}$, $x \in \Omega$;

(2) *there is a constant* $\delta > 0$ *such that for the outward unit normal vector* $\mu = \mu(u)$ *to* ∂O,

$$V(x,u) \le V(x, u + t\mu(u)) \quad \text{when } x \in \Omega,\ u \in \partial O,\ t \in [0,\delta].$$

Then there is a weak solution, $U \in W^{1,2}_{\mathrm{loc}}(\Omega, \overline{O}) \cap \Gamma$ *of* (PDE) *and* (BC). *If further,* $V_u \in C^1(\overline{\Omega} \times \mathbb{R}^m)$ *and* $\partial\Omega \in C^3$, *the solution* U *is a classical solution of* (PDE) *and* (BC) *with* $U \in C^{2,\alpha}(\overline{\Omega})$ *for any* $\alpha \in (0,1)$ *with* $\lim_{x_1 \to \pm\infty} U(x_1, \hat{x}) = a^{\pm}$ *uniformly for* $\hat{x} \in \mathcal{D}$.

As a first step towards proving Theorem 4.1, a projection map $P : O^\delta \to \overline{O} = O \cup \partial O$ will be defined. Taking a smaller $\delta > 0$ if necessary shows that for each $u \in (\partial O)^\delta$, there exists a unique $s(u) \in \partial O$ with $|u - s(u)| = \min_{w \in \partial O} |u - w|$. This implies $s \in C^1((\partial O)^\delta, \partial O)$. Define a projection map $P : O^\delta \to \overline{O}$ by $P(u) = u$ for $u \in O$ and $P(u) = s(u) \in \partial O$ for $u \in O^\delta \setminus O$. Note that if $u \in O^\delta \setminus O$, then

$$u - s(u) = |u - s(u)|\,\mu(s(u)). \tag{4.2}$$

Making δ smaller if necessary, the implicit function theorem shows $P : O^\delta \setminus O \to \partial O$ is C^1.

Next to prove Theorem 4.1, a property of the function $s(u)$ is needed.

Lemma 4.3. *If* $u \in C^1(\Omega, \mathbb{R}^m)$ *and* $u(x) \in O^\delta$ *for some* $x \in \Omega$, *then, for each* $i = 1, \ldots, n$,

$$\left| \frac{\partial u(x)}{\partial x_i} \right| \ge \left| \frac{\partial (s \circ u)(x)}{\partial x_i} \right|.$$

Proof. It is a well-known result that the function s is a contraction; that is, $|s(z_1)-s(z_2)| \leq |z_1-z_2|$ for any $z_1, z_2 \in O^\delta$. Thus, for $y \in \Omega$ close to x,

$$\frac{|u(y)-u(x)|}{|y-x|} \geq \frac{|s(u(y))-s(u(x))|}{|y-x|}.$$

For $y = x + he_i$, $h \in \mathbb{R}$, letting $|h| \to 0$, we get the inequality. □

Proposition 4.4. *For any* $u \in C^1(\Omega, O^\delta) \cap W^{1,2}_{\mathrm{loc}}(\Omega, O^\delta)$, *it follows that*

$$P \circ u \in W^{1,2}_{\mathrm{loc}}(\Omega, \bar{O}) \quad \text{and} \quad J(P(u)) \leq J(u).$$

Proof. For each $z \in (\partial O)^\delta$, there exists a unique $s(z) \in \partial O$ with $|z - s(z)| = \min_{w \in \partial O} |z-w|$. For each $z \in \partial O$, we have $\mu(z)$ is the outward unit normal vector to ∂O at $z \in \partial O$. For each $z \in (\partial O)^\delta$, we define

$$\lambda(z) = \begin{cases} |z-s(z)| & \text{for } z \in (\partial O)^\delta \setminus \bar{O}, \\ -|z-s(z)| & \text{for } z \in (\partial O)^\delta \cap O, \end{cases}$$

and

$$\lambda_-(z) \equiv \min\{\lambda(z), 0\}.$$

Observe that

$$s \in C^1((\partial O)^\delta, \partial O), \quad \lambda \in C^1((\partial O)^\delta, \mathbb{R}), \quad \mu \in C^1(\partial O, \mathbb{R}^m),$$

and

$$z = s(z) + \lambda(z)\mu(s(z)).$$

For $z = u(x) \in (\partial O)^\delta$, we see that

$$P(u(x)) = s(u(x)) + \lambda_-(u(x))\mu(s(u(x))).$$

Define

$$f_\varepsilon(\lambda) \equiv \begin{cases} 0 & \text{for } \lambda \geq 0, \\ -(\lambda^2+\varepsilon^2)^{\frac{1}{2}} + \varepsilon & \text{for } \lambda < 0. \end{cases}$$

Approximating $P(u)(x)$ by $s(u(x)) + f_\varepsilon(\lambda(u(x)))\mu(s(u(x)))$ and letting $\varepsilon \to 0$ shows that $P(u) \in W^{1,2}_{\mathrm{loc}}(\Omega, \bar{O})$, and for $u(x) \in O^\delta \setminus O$, we have $\nabla P(u)(x) = \nabla s \circ u(x)$, while for $u(x) \in O$, we have $\nabla P(u)(x) = \nabla u(x)$. Now Lemma 4.3 implies that $|\nabla P(u)| \leq |\nabla u|$. Thus

$$\int_\Omega |\nabla P(u(x))|^2 \, dx \leq \int_\Omega |\nabla u(x)|^2 \, dx. \tag{4.5}$$

Moreover, hypothesis (V_5) implies that

$$\int_\Omega V(x, P(u(x)) \, dx \leq \int_\Omega V(x, u(x)) \, dx. \tag{4.6}$$

Then (4.5) and (4.6) show $J(P(u)) \leq J(u)$. □

Proof of Theorem 4.1. As a class of admissible functions, take

$$\Gamma(O^\delta) = \big\{u \in E \;\big|\; u(x) \in O^\delta \text{ for } x \in \Omega,\ \|u - a^\pm\|_{L^2(T_i, \mathbb{R}^m)} \to 0,\ i \to \pm\infty\big\}.$$

Define

$$c(O^\delta) = \inf_{u\in\Gamma(O^\delta)} J(u). \tag{4.7}$$

Since O is convex, it is readily seen that $\Gamma(O^\delta) \neq \varnothing$ and $0 \le c(O^\delta) < \infty$. Let $\{u_k\} \subset \Gamma(O^\delta)$ be a minimizing sequence for (4.7). By the density of $C^1(\Omega, \mathbb{R}^m) \cap \Gamma(O^\delta)$ in $\Gamma(O^\delta)$, we may assume that $\{u_k\} \subset C^1(\Omega, \mathbb{R}^m) \cap \Gamma(O^\delta)$. Since P is a contraction on O^δ and is the identity map on O, for any $z \in O^\delta$ and $w \in O$, we have $|P(z) - w| \le |z - w|$. Thus

$$\|P(u) - a^\pm\|_{L^2(T_i,\mathbb{R}^m)} \le \|u - a^\pm\|_{L^2(T_i,\mathbb{R}^m)} \to 0, \quad i \to \pm\infty.$$

Hence Proposition 4.4 implies that $\{P(u_k)\}$ is also a minimizing sequence for (4.7) which is contained in $W^{1,2}_{\mathrm{loc}}(\Omega, \overline{O}) \cap \Gamma(O^\delta)$. The proof of Theorem 2.2 shows that there exists a $p(k) \in \mathbb{Z}$ such that a subsequence of $\{P(u_k(\cdot + (p(k), 0, \dots, 0)))\}$ converges weakly in $W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$, strongly in $L^2_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and pointwise a.e. to a minimizer $U \in W^{1,2}_{\mathrm{loc}}(\Omega, \overline{O}) \cap \Gamma(O^\delta)$ of (2.1). Since $U(x) \in \overline{O}$ for any $x \in \Omega$ and O is bounded, by Remark 2.37, U is a weak solution of (PDE) and (BC).

Following the argument in the Completion of the Proof of Theorem 3.1, we get that if $V_u \in C^1$ and $\partial\Omega \in C^3$, then $U \in C^{2,\alpha}(\Omega, O) \cap \Gamma$ and U is a classical (PDE) and (BC) with $\lim_{x_1\to\pm\infty} U(x_1, \hat{x}) = a^\pm$ uniformly for $\hat{x} \in \mathcal{D}$. □

For our second result, as earlier, let e_i be a unit vector in the positive x_i-direction, $1 \le i \le n$. Assume:

(Ω_1) $\Omega \subset \mathbb{R} \times \mathcal{D}$ for some bounded set $\mathcal{D} \subset \mathbb{R}^{n-1}$, $\partial\Omega$ is a C^3 manifold, and for all $x \in \Omega$, we have $x \pm e_1 \in \Omega$.

(Ω_2) Ω is a connected set.

Define the functional J as earlier with this new choice of Ω and for $i, j \in \mathbb{Z}$ with $i < j$, set $T_i = \{x \in \Omega \mid i < x_1 < i+1\}$ and $T_i^j = \{x \in \Omega \mid i < x_1 < j\}$.

Then we have:

Theorem 4.8. *Suppose that V satisfies (V_1)–(V_3) and Ω satisfies (Ω_1), (Ω_2). Let*

$$\Gamma_1 = \big\{u \in E \;\big|\; \|u - a^\pm\|_{L^2(T_i,\mathbb{R}^m)} \to 0,\ i \to \pm\infty\big\}.$$

Then there is a $U \in \Gamma_1$ such that

$$J(U) = \inf_{u\in\Gamma_1} J(u). \tag{4.9}$$

Proof. The proof of Theorem 2.2 uses Proposition 2.14 and Remark 2.19 to show that a minimizing sequence $\{u_k\}$ satisfying the normalization (2.13) and the bounds given by Proposition 2.24 has a subsequence which converges to a minimizer U of the functional J on Γ. Since Proposition 2.14 and Remark 2.19 can be proved in the same manner for a domain Ω satisfying (Ω_1) and (Ω_2), the proof carries over to the present setting provided that the bounds of Proposition 2.24 are also valid here; i.e., if $\{u_k\}$ is a minimizing sequence for (4.9), there is a constant $M > 0$ such that

$$\|u_k\|_{W^{1,2}(T_i,\mathbb{R}^m)} \le M \tag{4.10}$$

for all $k \in \mathbb{N}$ and $i \in \mathbb{Z}$. We will show that this is the case. The proof uses the following result.

Lemma 4.11. *Assume that* (Ω_1) *and* (Ω_2) *hold. Then for any fixed* $k \geq 3$*, there exists a constant* $C > 0$*, independent of* $i \in \mathbb{Z}$*, such that for any* $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ *and* $j \in \{i+1, \dots, i+k-2\}$*,*

$$\|u\|_{L^2(T_j,\mathbb{R}^m)} \leq C\big(\|\nabla u\|_{L^2(T_i^{i+k},\mathbb{R}^m)} + \|u\|_{W^{1,2}(T_i,\mathbb{R}^m)} + \|u\|_{L^2(T_{i+k-1},\mathbb{R}^m)}\big).$$

Proof. By a translation in $\mathbb{Z}e_1$, it suffices to show that there exists a constant $C > 0$ such that for any $u \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and $j \in \{1, \dots, k-2\}$,

$$\|u\|_{L^2(T_j,\mathbb{R}^m)} \leq C\big(\|\nabla u\|_{L^2(T_0^{k},\mathbb{R}^m)} + \|u\|_{L^2(T_0,\mathbb{R}^m)} + \|u\|_{L^2(T_{k-1},\mathbb{R}^m)}\big).$$

To the contrary, suppose that the inequality above does not hold. Then there is a sequence $\{w_l\} \subset W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ and $j \in \{1, \dots, k-2\}$ such that

$$\|w_l\|_{L^2(T_j,\mathbb{R}^m)} = 1 \tag{4.12}$$

and

$$\|\nabla w_l\|_{L^2(T_0^k,\mathbb{R}^m)} + \|w_l\|_{L^2(T_0,\mathbb{R}^m)} + \|u\|_{L^2(T_{k-1},\mathbb{R}^m)} \to 0 \quad \text{as } l \to \infty. \tag{4.13}$$

Let Ω_0^j be a connected component of T_j and Ω^j a connected component of T_0^k containing Ω_0^j. If $\Omega^j \cap (T_0 \cup T_{k-1}) = \varnothing$, then Ω^j is an isolated connected component of Ω. But $k \geq 3$, so this contradicts the connectedness of Ω. Thus $\Omega^j \cap (T_0 \cup T_{k-1}) \neq \varnothing$. Assume that $\Omega^j \cap T_0 \neq \varnothing$. Then by the Poincaré inequality, there exists $c > 0$, independent of l, such that

$$\|w_l - [w_l]_j\|_{L^2(\Omega^j,\mathbb{R}^m)} \leq c\|\nabla w_l\|_{L^2(\Omega^j,\mathbb{R}^m)}, \tag{4.14}$$

where $[w_l]_j = (1/|\Omega_j|)\int_{\Omega_j} w_l\,dx$. Since $\lim_{l\to\infty}\|w_l\|_{L^2(T_0,\mathbb{R}^m)} = 0$ and

$$\begin{aligned}\|w_l - [w_l]_j\|_{L^2(\Omega^j\cap T_0,\mathbb{R}^m)} &\leq \|w_l - [w_l]_j\|_{L^2(\Omega^j,\mathbb{R}^m)} \leq c\|\nabla w_l\|_{L^2(\Omega^j,\mathbb{R}^m)}\\ &\leq c\|\nabla w_l\|_{L^2(T_0^k,\mathbb{R}^m)},\end{aligned}$$

(4.13) implies that $\lim_{l\to\infty}[w_l]_j = 0$. Then (4.14) shows that

$$\|w_l\|_{L^2(\Omega_0^j,\mathbb{R}^m)} \leq \|w_l\|_{L^2(\Omega^j,\mathbb{R}^m)} \to 0 \quad \text{as } l \to \infty.$$

If $\Omega^j \cap T_k \neq \varnothing$, we obtain the same conclusion. Thus, for each connected component Ω_0^j, we have $\lim_{l\to\infty}\|w_l\|_{L^2(\Omega_0^j,\mathbb{R}^m)} = 0$. This implies $\lim_{l\to\infty}\|w_l\|_{L^2(T_j,\mathbb{R}^m)} = 0$, contradicting (4.12) and completing the proof. □

Now, we argue as in the proof of Proposition 2.14. Since Proposition 2.14 and Remark 2.19 hold for a domain Ω satisfying (Ω_1) and (Ω_2), there exists $L \in \mathbb{N}$, independent of k, such that the number of elements of

$$A_k = \big\{i \in \mathbb{N} \;\big|\; \|u_k - a^-\|_{L^2(T_i,\mathbb{R}^m)} \geq \rho,\ \|u_k - a^+\|_{L^2(T_i,\mathbb{R}^m)} \geq \rho\big\}$$

is bounded by L for each $k \in \mathbb{N}$. Note that if $i \notin A_k$,

$$\|u_k\|_{L^2(T_i,\mathbb{R}^m)} \leq \rho + \max\{|a^{-1}|, |a^+|\}|T_0|^{\frac{1}{2}}.$$

Then, applying Lemma 4.11, we get the boundedness (4.10). For the completion of the proof of Theorem 4.8, we follow exactly the same argument as in the Completion of the Proof of Theorem 2.2. Then, we get a minimizer $U \in \Gamma_1$ of J. □

As a consequence of Theorem 4.8 and Theorem 3.1, we have:

Corollary 4.15. *If in addition to the hypotheses of Theorem 4.8, (V_4) is satisfied, $V_u \in C^1(\overline{\Omega} \times \mathbb{R}^m)$ and $\partial\Omega \in C^3$, then $U \in C^{2,\alpha}(\overline{\Omega}, \mathbb{R}^m)$ for any $\alpha \in (0,1)$ and U is a classical solution of (PDE) and (BC) with $\lim_{x_1 \to \pm\infty} U(x_1, \hat{x}) = a^{\pm}$ uniformly for $\hat{x} \in \mathcal{D}$. If V satisfies (V_5), then (PDE) and (BC) possess a solution $U \in C^{2,\alpha}(\Omega, \mathcal{O}) \cap \Gamma_1$.*

Theorem 3.1 and Corollary 4.15 require condition (V_4), which allows us to get an L^∞ bound for the solution. When $n = 1$, condition (V_4) is not required; conditions (V_1)–(V_3) suffice. Next an example will be given showing that a PDE perturbation of that case without any further conditions other than $V \in C^2$ gives classical solutions of (PDE) and (BC). Thus consider (PDE) and (BC) for $n = 1$. To better distinguish between the cases of $n = 1$ and the general case, set

$$\Gamma_0 = \left\{u \in W^{1,2}_{\text{loc}}(\mathbb{R}, \mathbb{R}^m) \;\middle|\; \|u - a_\pm\|_{L^2([i,i+1],\mathbb{R}^m)} \to 0,\ \pm i \to \infty\right\}$$

and

$$J_0(u) = \int_{\mathbb{R}} \left(\tfrac{1}{2}|u_{x_1}|^2 + V(x_1, u)\right) dx_1.$$

Then in Section 2, it was shown that

$$c_0 = \inf_{u \in \Gamma_0} J_0(u)$$

has a minimizer, $U_0 = U_0(x_1)$, which is a classical solution of (PDE). With the same choice of V, take any bounded domain $\mathcal{D} \subset \mathbb{R}^{n-1}$ with $\Omega = \mathbb{R} \times \mathcal{D}$ and Γ as in Section 2, J as in (1.2) and c as in (2.1). Note that $U_0 \in \Gamma$ so $J(U_0) = |\mathcal{D}| J_0(U_0) \geq c$.

Proposition 4.16. *$J(U_0) = c$ and any minimizer $U \in C^{2,\alpha}(\overline{\Omega}) \cap L^\infty(\Omega)$ of (2.1) depends only on x_1.*

Proof. Let $\{u_k\}$ be a minimizing sequence for (2.1). Write $x = (x_1, \hat{x})$ for $x \in \mathbb{R}^n$ and fix $k \in \mathbb{N}$. Then by Fubini's theorem, there exists a set $A_k \subset \mathcal{D}$ with $|A_k| = |\mathcal{D}|$ such that for any $\hat{x} \in A_k$,

$$\int_{\Omega} \left(\tfrac{1}{2}|\nabla u_k(x_1, \hat{x})|^2 + V(x_1, u_k(x_1, \hat{x}))\right) dx_1 < \infty. \tag{4.17}$$

Therefore by Proposition 2.3, there exist $e_k^{\pm}(\hat{x}) \in \{a^-, a^+\}$ such that

$$\lim_{i \to \pm\infty} \|u_k(\cdot, \hat{x}) - e_k^{\pm}(\hat{x})\|_{L^2([i,i+1],\mathbb{R}^m)} = 0. \tag{4.18}$$

We claim that $e_k^{\pm}(\hat{x}) = a^{\pm}$ for all $\hat{x} \in A_k$. Indeed for each $i \in \mathbb{Z}$, set

$$f_i^{\pm}(\hat{x}) = \int_{[i,i+1]} |u_k(x_1, \hat{x}) - a^{\pm}|^2 \, dx_1.$$

Then each function $f_i^{\pm}$ is measurable on $\mathcal{D}$ and by Fubini's theorem again,

$$\lim_{i\to\pm\infty}\int_{\mathcal{D}} f_i^{\pm}(\hat{x})\,d\hat{x} = \lim_{i\to\pm\infty}\int_{\mathcal{D}}\int_{[i,i+1]} |u_k(x_1,\hat{x})-a^{\pm}|^2\,dx_1\,d\hat{x}$$
$$= \lim_{i\to\pm\infty}\|u_k-a^{\pm}\|_{L^2(T_i)} = 0$$

since $u_k \in \Gamma$. But $f_i^{\pm}(\hat{x}) \geq 0$, so $f_i^{\pm} \to 0$ in $L^1(\mathcal{D})$ as $i \to \pm\infty$. Hence there exist subsequences $i_j^{\pm} \to \pm\infty$ such that

$$f_{i_j^{\pm}}^{\pm}(\hat{x}) \to 0 \quad \text{for a.e. } x \in \mathcal{D}. \tag{4.19}$$

Comparing (4.19) to (4.18) shows the existence of a set $B_k \subset A_k$ with $|B_k| = |A_k| = |\mathcal{D}|$ and $e_k^{\pm}(\hat{x}) = a^{\pm}$ for all $\hat{x} \in B_k$. Defining $B = \bigcap_k B_k$, we have $|B| = |\mathcal{D}|$, and for any $\hat{x} \in B$ and $k \in \mathbb{N}$, we have

$$u_k(\,\cdot\,,\hat{x}) \in W^{1,2}_{\mathrm{loc}}(\mathbb{R},\mathbb{R}^m) \quad \text{and} \quad \lim_{i\to\pm\infty}\|u_k(\,\cdot\,,\hat{x})-a^{\pm}\|_{L^2([i,i+1],\mathbb{R}^m)} \to 0.$$

This implies that for each $\hat{x} \in B$, we have $u_k(x_1,\hat{x}) \in \Gamma_0$. Therefore, for each $\hat{x} \in B$,

$$J_0(u_k(\,\cdot\,,\hat{x})) \geq J_0(U_0). \tag{4.20}$$

Integrating (4.20) over $\mathcal{D}$ then shows $J(u_k) \geq J(U_0)$, which implies $J(U_0) = c$, yielding the first part of the proposition.

For the second part, suppose that c is attained by $U \in C^{2,\alpha}(\overline{\Omega}) \cap L^{\infty}(\Omega)$. As in (4.20), for a.e. $\hat{x} \in \mathcal{D}$,

$$J_0(U(\,\cdot\,,\hat{x})) \geq J_0(U_0).$$

Since $J(U) = c$, this implies that for a.e. $\hat{x} \in \mathcal{D}$,

$$J_0(U(\,\cdot\,,\hat{x})) = J_0(U_0).$$

Then, for a.e. $\hat{x} \in \mathcal{D}$,

$$\frac{\partial^2 U(x_1,\hat{x})}{\partial x_1^2} - V_u(x_1, U(x_1,\hat{x})) = 0.$$

This implies that

$$\Delta_{\hat{x}} U \equiv \Delta U - \frac{\partial^2 U(x_1,\hat{x})}{\partial x_1^2} = 0 \quad \text{for any } x_1 \in \mathbb{R};$$

i.e., $U(x_1,\hat{x})$ as a function of $\hat{x}$ is harmonic. Thus using the boundary condition (BC) shows that $U(x_1,\hat{x})$ does not depend on $\hat{x} \in \mathcal{D}$. This completes the proof. □

Now the perturbation result can be formulated. Suppose:

(V_0) For some $\varepsilon_0 > 0$, there exists a function $W \in C^1((-\varepsilon,\varepsilon)\times\overline{\Omega}\times\mathbb{R}^m)$ such that for each $\varepsilon \in (-\varepsilon_0,\varepsilon_0)$, $W(\varepsilon,\cdot)$ satisfies (V_1)–(V_3) and $W(0,x,u) = V(x_1,u)$.

For $|\varepsilon| < \varepsilon_0$, consider the family of equations

$$-\Delta u + W_u(\varepsilon, x, u) = 0, \quad x \in \Omega, \tag{4.21}$$

with boundary conditions

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{4.22}$$

Then we have:

Theorem 4.23. *Suppose (V_0) is satisfied and $\partial\Omega \in C^3$. Then there is an $\varepsilon_1 \in (0, \varepsilon_0)$ such that the problem (4.21)–(4.22) has a classical solution U_ε for each $|\varepsilon| \le \varepsilon_1$.*

Proof. Let u_0 be any minimizer of J_0 on Γ_0. Then (2.43) provides an upper bound K for $\|u_0\|_{L^\infty(\mathbb{R},\mathbb{R}^m)}$ and any such u_0. To obtain the solutions U_ε, the family of functions $W(\varepsilon, \cdot)$ will be truncated. Let $W_K \in C((-\varepsilon, \varepsilon) \times \overline{\Omega} \times \mathbb{R}^m)$ satisfy (V_0) with

(1) $W_K(0, x, u)$ independent of $\hat{x}$,

(2) $W_K(\varepsilon, x, u) = W(\varepsilon, x, u)$ for $|u| \le 2K$,

(3) $|(W_K)_u(\varepsilon, x, u)| \le K_1$ for some constant K_1,

(4) $\liminf_{|u|\to\infty} W_K(\varepsilon, x, u) \ge \underline{V} > 0$ uniformly for $x \in \Omega$ and $|\varepsilon| \le \varepsilon_0$,

(5) $\inf\{W_K(0, x, u) \mid t \in \mathbb{R}, \min\{|u - a^-|, |u - a^+|\} \ge \rho\} \ge \beta(\rho)$,

where $\beta(\rho) = \inf\{V(x_1, u) \mid x_1 \in \mathbb{R}, \min\{|u - a^-|, |u - a^+|\} \ge \rho\}$. It is straightforward to construct such a family of functions. By (V_0) and Theorem 2.2, the functional

$$J_{\varepsilon,K}(u) \equiv \int_\Omega \left(\tfrac{1}{2}|\nabla u|^2 + W_K(\varepsilon, x, u)\right) dx \equiv \int_\Omega L_{\varepsilon,K}(u)\, dx,$$

corresponding to (4.21) with W replaced by W_K has a minimizer $U_{\varepsilon,K} \in \Gamma$ for each $|\varepsilon| \le \varepsilon_0$. By (3) of the properties of W_K and Theorem 3.1(1), there is a constant $\overline{M}_1$ that is independent of ε but depends on K such that

$$\|U_{\varepsilon,K}\|_{L^\infty(\Omega,\mathbb{R}^m)} \le \overline{M}_1.$$

Thus by Theorem 3.1(2), each component of $U_{\varepsilon,K}$ can be viewed as a $C^{2,\alpha}(\Omega, \mathbb{R})$ solution of a linear elliptic equation of the form

$$-\Delta v = f, \quad x \in \Omega,$$

with $\partial v/\partial \nu = 0$, $x \in \partial\Omega$ and $f \in L^\infty(\Omega, \mathbb{R})$. Applying the $W^{2,q}_{\text{loc}}$ estimates for such equations gives a constant $\overline{M}_2 > 0$ that is independent of ε but depends on K such that

$$\|U_{\varepsilon,K}\|_{W^{2,q}(\Omega,\mathbb{R})} \le \overline{M}_2.$$

Taking $q > n$ and applying the Schauder estimates for each component yields a constant $\overline{M} > 0$ that is independent of ε but depends on K such that

$$\|U_{\varepsilon,K}\|_{C^{2,\alpha}(\Omega,\mathbb{R}^m)} \le \overline{M}. \tag{4.24}$$

Thus, $U_{\varepsilon,K}$ is a classical solution of (4.21)–(4.22). It can be assumed that the functions $U_{\varepsilon,K}$ are normalized as in (2.13). We claim there is an $\varepsilon_1 \in (0,\varepsilon_0]$ such that for $|\varepsilon| \le \varepsilon_1$, we have $\|U_{\varepsilon,K}\|_{L^\infty(\Omega,\mathbb{R}^m)} < 2K$. If so, $W_K(\varepsilon,x,U_{\varepsilon,K}(x)) = W(\varepsilon,x,U_{\varepsilon,K}(x))$ and $U_{\varepsilon,K}$ is the desired solution of (4.21)–(4.22) for $|\varepsilon| \le \varepsilon_1$. To show that such an ε_1 exists, suppose that there exists $\varepsilon_l \in (-\varepsilon_0,\varepsilon_0)$ with $\lim_{l\to\infty}\varepsilon_l = 0$ such that

$$\limsup_{l\to\infty} \|U_{\varepsilon_l,K}\|_{L^\infty(\Omega,\mathbb{R}^m)} \ge 2K. \tag{4.25}$$

By (4.24), it can be assumed that $U_{\varepsilon_l,K}$ converges in $C^2_{\mathrm{loc}}(\overline{\Omega},\mathbb{R}^m)$ to a solution U^* of (4.21)–(4.22) for $\varepsilon = 0$. Due to equations (4.21)–(4.22) again, the convergence is in $C^{2,\alpha}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ so by (4.24),

$$\|U^*\|_{C^{2,\alpha}(\Omega,\mathbb{R}^m)} \le \overline{M}. \tag{4.26}$$

Suppose for the moment that U^* minimizes $J_{0,K}$ on Γ. Then by Remark 2.44, $\|U^*\|_{L^\infty(\Omega,\mathbb{R}^m)} \le K$ and (4.26) is in contradiction to (4.25). Hence ε_1 exists and the theorem is proved.

It remains to verify that U^* minimizes $J_{0,K}$ on Γ. As a first step, let $w \in C^1(\mathbb{R},\mathbb{R}^m)$ with $w(t) = a^-$ for $t \le -1$ and $w(t) = a^+$ for $t \ge 1$. We define $\tilde{w}(x_1,\hat{x}) = w(x_1)$. Then there is a constant M_1 independent of ε but depending on K such that

$$J_{\varepsilon,K}(\hat{w}) \le M_1. \tag{4.27}$$

Thus $J_{\varepsilon,K}(U_{\varepsilon,K}) \le M_1$ for $|\varepsilon| \le \varepsilon_0$. Now for any $R > 0$, due to the C^1_{loc} convergence of $U_{\varepsilon,K}$,

$$\int_{[-R,R]\times\mathcal{D}} L_{0,K}(U^*)\,dx = \lim_{\varepsilon\to 0}\int_{[-R,R]\times\mathcal{D}} L_{\varepsilon,K}(U_{\varepsilon,K})\,dx \le M_1.$$

Thus letting $R \to \infty$ shows

$$J_{0,K}(U^*) \le M_1. \tag{4.28}$$

By (4.28), as $|i| \to \infty$,

$$\int_{T_i} L_{0,K}(U^*)\,dx \to 0. \tag{4.29}$$

Due to the bounds (4.26) and the Poincaré inequality,

$$\|U^* - a^\pm\|_{W^{1,2}(T_i,\mathbb{R}^m)} \to 0, \quad |i| \to \infty. \tag{4.30}$$

Employing the bounds, (4.26) again with (4.30) and an interpolation inequality shows

$$\|U^* - a^\pm\|_{C^1(T_i,\mathbb{R}^m)} \to 0, \quad i \to \pm\infty. \tag{4.31}$$

The estimate (4.31) also holds for any u_0 minimizing $J_{0,K}$ on Γ. Let $\sigma > 0$. By (4.31), there is a $q = q(\sigma) \in \mathbb{N}$ such that for $u = U^*$ or $u = u_0$,

$$\|u - a^\pm\|_{C^1(T_i,\mathbb{R}^m)} \le \sigma, \quad \pm i \ge q. \tag{4.32}$$

By (4.28) again, by taking q larger if need be, it can be assumed that

$$\int_{\{|x_1|\ge q+1\}\times\mathcal{D}} L_{0,K}(U^*)\,dx \le \sigma \quad\text{and}\quad \int_{\{|x_1|\ge q+1\}\times\mathcal{D}} L_{0,K}(u_0)\,dx \le \sigma. \tag{4.33}$$

Next observe that U^*, being a limit of minimizers, possesses a minimality property. Indeed since $U_{\varepsilon,K}$ minimizes $J_{\varepsilon,K}$ over Γ, for any $\varphi \in W^{1,2}(\Omega, \mathbb{R}^m)$ having compact support,

$$\int_\Omega \big(L_{\varepsilon,K}(U_{\varepsilon,K}+\varphi) - L_{\varepsilon,K}(U_{\varepsilon,K})\big)\, dx = \int_{\operatorname{supp}\varphi} \big(L_{\varepsilon,K}(U_{\varepsilon,K}+\varphi) - L_{\varepsilon,K}(U_{\varepsilon,K})\big)\, dx \geq 0. \tag{4.34}$$

Thus taking $\varepsilon \to 0$ in (4.34) yields

$$\int_{\operatorname{supp}\varphi} \big(L_{0,K}(U^*+\varphi) - L_{0,K}(U^*)\big)\, dx \geq 0. \tag{4.35}$$

Taking $q = q(\sigma)$, choose $\varphi = f_q$, where

$$f_q(x) = \begin{cases} u_0 - U^* & \text{for } |x_1| \leq q, \\ (x_1 - q - 1)(U^* - u_0) & \text{for } q \leq x_1 \leq q+1, \\ (-q-1-x_1)(U^*-u_0) & \text{for } -q-1 \leq x_1 \leq -q, \\ 0 & \text{for } |x_1| \geq q+1. \end{cases}$$

With this choice of φ, (4.35) becomes

$$\int_{[-q,q]\times\mathcal{D}} L_{0,K}(u_0)\, dx + \int_{T_{-q-1}\cup T_q} L_{0,K}(U^*+f_q)\, dx \geq \int_{[-q-1,q+1]\times\mathcal{D}} L_{0,K}(U^*)\, dx. \tag{4.36}$$

The choice of f_q and (4.32) show

$$\int_{T_{-q-1}\cup T_q} L_{0,K}(U^*+f_q)\, dx \leq \gamma(\sigma),$$

where $\gamma(\sigma) \to 0$ as $\sigma \to 0$. Recall that by Remark 2.44 and Proposition 4.16,

$$c_{0,K} \equiv \inf_{u\in\Gamma} J_{0,K}(u) = \inf_{u\in\Gamma} J(u) \equiv c.$$

Consequently, letting $\sigma \to 0$, $q \to \infty$ and (4.36) implies

$$c = J(u_0) \geq J_{0,K}(U^*) \geq c_{0,K}$$

and Theorem 4.23 is proved. □

Remark 4.37. One can also allow for perturbations of the domain in the setting of Theorem 4.23. For example, with a condition like:

(Ω_0) For some $\varepsilon_0 > 0$ and each $|\varepsilon| \leq \varepsilon_0$, there is a domain $\Omega_\varepsilon \subset \mathbb{R}^n$, where Ω_ε satisfies (Ω_1)–(Ω_2), the map $\varepsilon \to \Omega_\varepsilon$ is continuous, and $\Omega_0 = \mathbb{R}\times\mathcal{D}$.

To conclude this section, we will briefly mention the case of (V_2) replaced by:

(V_2') There are points $a^i \in \mathbb{R}^m$ such that $V(x, a^i) = 0$, $1 \leq i \leq s$, for all $x \in \Omega$, and $V(x,u) > 0$ otherwise,

i.e., V is a multiwell potential. Existence and multiplicity results for such multiwell potentials and even infinite well potentials have been studied, e.g., in [Montecchiari and Rabinowitz 2016]. Using the methods of this paper, such treatments can readily be extended to the current setting. For example, suppose that V is an s-well potential and set

$$A = \{a^1, \dots, a^s\}.$$

Then it is straightforward to show:

Theorem 4.38. *Suppose that V satisfies (V_1), (V_2'), (V_3), (V_4), $V_u \in C^1$, $\Omega = \mathbb{R} \times \mathcal{D}$ with $\mathcal{D} \subset \mathbb{R}^{n-1}$ a bounded open set and $\partial\mathcal{D}$ a C^3 manifold. Then:*

(1) *For any $a^i \in A$, there exists an $a^j \in A$ with $i \neq j$ and corresponding classical solution $U_{i,j}$ of (PDE) and (BC) such that U_{ij} is heteroclinic in x_1 from a^i to a^j and $U_{i,j}$ minimizes J over the set*

$$\Big\{u \in E \,\Big|\, \lim_{k\to-\infty} \|u - a^i\|_{L^2(T_k,\mathbb{R}^m)} = \lim_{k\to\infty} \|u - a^j\|_{L^2(T_k,\mathbb{R}^m)} = 0 \text{ for some } j \neq i\Big\}.$$

(2) *For any $a^i, a^j \in A$, with $i \neq j$, there exists a (minimal) heteroclinic chain of solutions U_{i,p_1}, $U_{p_1,p_2}, \dots, U_{p_t,j}$ of (PDE) and (BC), where $U_{k,l}$ are as in (1) and the integers $i, p_1, \dots, p_t, j$ are distinct. Moreover, if*

$$c_{i,j} = \inf_{u\in\Gamma_{i,j}} J(u),$$

where

$$\Gamma_{i,j} = \Big\{u \in E \,\Big|\, \|u - a^i\|_{L^2(T_k,\mathbb{R}^m)} \to 0,\ k \to -\infty;\ \|u - a^j\|_{L^2(T_k,\mathbb{R}^m)} \to 0,\ k \to \infty\Big\},$$

then

$$c_{i,j} = J(U_{i,p_1}) + \dots + J(U_{p_t,j}).$$

5. Multitransition solutions

In this section, it will be shown how the approach of [Montecchiari and Rabinowitz 2016] can be mirrored to construct multitransition homoclinic and heteroclinic solutions of (PDE). More precisely, we seek solutions of (PDE) that as a function of x_1 make multiple transitions between small neighborhoods of a^- and a^+. In order to find such solutions, we need a mild nondegeneracy condition on the set of minimizing heteroclinics given by Theorem 2.2. To make this precise, we replace Γ by $\Gamma(a^-, a^+)$ and c by $c(a^-, a^+)$. Thus interchanging the roles of a^- and a^+ gives us $\Gamma(a^+, a^-)$ and $c(a^+, a^-)$. For $\xi \in \{a^+, a^-\}$, and $\eta \in \{a^+, a^-\} \setminus \{\xi\}$, set

$$\mathcal{M}(\xi, \eta) \equiv \{u \in \Gamma(\xi, \eta) \mid J(u) = c(\xi, \eta)\}.$$

Define

$$\mathcal{S}(\xi, \eta) \equiv \{u|_{T_0} \mid u \in \mathcal{M}(\xi, \eta)\}$$

and put the $W^{1,2}(T_0, \mathbb{R}^m)$ topology on this set. Then we have:

Proposition 5.1. *Suppose V satisfies (V_1)–(V_4), $V_u \in C^1(\bar{\Omega} \times \mathbb{R}^m)$ and $\partial\Omega \in C^3$. Then*

(1) $\bar{\mathcal{S}}(\xi, \eta) = \mathcal{S}(\xi, \eta) \cup \{\xi\} \cup \{\eta\}$,

(2) *$\bar{\mathcal{S}}(\xi, \eta)$ is compact.*

Proof. Due to the asymptotic behavior of the members u of $\mathcal{M}(\xi,\eta)$, we know $u(x_1+j,x_2,\dots,x_n)$ converges in $L^2(T_0,\mathbb{R}^m)$ to η as $j\to\infty$ and to ξ as $j\to-\infty$. Then, by the L^∞ uniform boundedness of minimizers $U\in S(\xi,\eta)$ in Proposition 6.2 and elliptic estimates, we see that $\{\xi\}\cup\{\eta\}\in\bar{\mathcal{S}}(\xi,\eta)$.

Let $\{w_j\}$ be a sequence in $\mathcal{S}(\xi,\eta)$. Then the proof of (1)–(2) is complete if a subsequence of $\{w_j\}$ converges to a member of $\mathcal{S}(\xi,\eta)\cup\{\xi\}\cup\{\eta\}$. If a subsequence of w_j converges to ξ or η, we are done. Thus suppose this is not the case. For any j, we have $w_j=W_j|_{T_0}$, where $W_j\in\mathcal{M}(\xi,\eta)$, so $J(W_j)=c(\xi,\eta)$. By Proposition 6.2 and elliptic estimates, there exists $K>0$ such that $\|W_j\|_{C^{2,\alpha}(\Omega,\mathbb{R}^m)}\le K$. Then, a subsequence of W_j converges in $C^2_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ to a function $W\in E\cap C^2(\Omega,\mathbb{R}^m)$ and W is a classical solution of (PDE). In particular $w_j\to w=W|_{T_0}\ne\xi,\eta$. Since for each $p<q\in\mathbb{Z}$,

$$\sum_{i=p}^{q}\int_{T_i}L(W)\,dx=\lim_{j\to\infty}\sum_{i=p}^{q}\int_{T_i}L(W_j)\,dx\le J(W_j)=c(\xi,\eta),$$

letting $q,-p\to\infty$ shows

$$\sum_{i\in\mathbb{Z}}\int_{T_i}L(W)\,dx=J(W)\le c(\xi,\eta). \tag{5.2}$$

Equation (5.2) and (V_2) imply there are points $\xi^\pm\in\{\xi,\eta\}$ such that $\|W-\xi^\pm\|_{L^2(T_i,\mathbb{R}^m)}\to 0$ as $i\to\pm\infty$, respectively. We must show $\xi^-=\xi$ and $\xi^+=\eta$. Arguing indirectly, suppose that $\xi^-\ne\xi$, so $\xi^-=\eta$. Let $\varepsilon>0$. Then there is a negative integer $i_0=i_0(\varepsilon)\in\mathbb{Z}$ such that $\|W-\eta\|_{W^{1,2}(T_i,\mathbb{R}^m)}\le\varepsilon$ for all $i\le i_0+2$. For large $k=k(i_0)$ and $i\in\{i_0-1,\dots,i_0+2\}$, we have $\|W_k-\eta\|_{W^{1,2}(T_i,\mathbb{R}^m)}\le 2\varepsilon$. Define $f_k\in\Gamma(\xi,\eta)$ by

$$f_k=\begin{cases}W_k & \text{for } x_1\le i_0-1,\\ (x_1-i_0+1)\eta+(i_0-x_1)W_k & \text{for } i_0-1\le x_1\le i_0,\\ \eta & \text{for } i_0\le x_1\le i_0+1,\\ (x_1-i_0-1)W_k+(2+i_0-x_1)\eta & \text{for } i_0+1\le x_1\le i_0+2,\\ W_k & \text{for } i_0+2\le x_1.\end{cases} \tag{5.3}$$

Note that

$$|J(W_k)-J(f_k)|=\left|\int_{\bigcup_{i=i_0-1}^{i_0+2}T_i}(L(W_k)-L(f_k))\,dx\right|\le\kappa(\varepsilon), \tag{5.4}$$

where $\kappa(\varepsilon)\to0$ as $\varepsilon\to0$. Therefore by (5.4),

$$c(\xi,\eta)=J(W_k)\ge J(f_k)-\kappa(\varepsilon). \tag{5.5}$$

Further define functions $g_k\in\Gamma(\xi,\eta)$ and h_k via

$$g_k=\begin{cases}f_k & \text{for } x_1\le i_0,\\ \eta & \text{for } x_1\ge i_0,\end{cases} \tag{5.6}$$

$$h_k=\begin{cases}\eta & \text{for } x_1\le i_0+1,\\ f_k & \text{for } x_1\ge i_0+1,\end{cases} \tag{5.7}$$

so by construction,

$$J(f_k)=J(g_k)+J(h_k). \tag{5.8}$$

By (5.5) and (5.8),

$$c(\xi,\eta) \geq J(g_k) + J(h_k) - \kappa(\varepsilon) \geq c(\xi,\eta) + J(h_k) - \kappa(\varepsilon).$$

Thus, we get

$$\kappa(\varepsilon) \geq J(h_k) \geq \sum_{i=i_0+1}^{\infty} \int_{T_i} L(f_k)\,dx \geq \sum_{i=0}^{\infty} \int_{T_i} L(f_k)\,dx, \tag{5.9}$$

where the last inequality follows since i_0 is negative. But on T_0, we have $f_k = w_k \to w$ in $W^{1,2}(T_0, \mathbb{R}^m)$ as $k \to \infty$ and $w \neq a_\pm$. Therefore $\int_{T_0} L(f_k)\,dx \geq \omega > 0$ for all large k. Since the left-hand side of (5.9) goes to 0 as $\varepsilon \to 0$, we have a contradiction. Thus $\xi^- = \xi$. Similarly, $\xi^+ = \eta$ and the proposition is proved. □

Next define $\mathcal{C}_\xi(\xi,\eta)$ to be the connected component of $\bar{\mathcal{S}}(\xi,\eta)$ to which ξ belongs and define $\mathcal{C}_\eta(\xi,\eta)$ similarly. Then the following alternative holds.

Proposition 5.10. *One of the following items holds*:

(i) $\mathcal{C}_\xi(\xi,\eta) = \mathcal{C}_\eta(\xi,\eta)$;

(ii) $\mathcal{C}_\xi(\xi,\eta) = \{\xi\}$ *and* $\mathcal{C}_\eta(\xi,\eta) = \{\eta\}$.

If (ii) *holds*, *there exist nonempty disjoint compact sets* $K_\xi(\xi,\eta),\ K_\eta(\xi,\eta) \subset \bar{\mathcal{S}}(\xi,\eta)$ *such that*

(a) $\xi \in K_\xi(\xi,\eta),\quad \eta \in K_\eta(\xi,\eta)$,

(b) $\bar{\mathcal{S}}(\xi,\eta) = K_\xi(\xi,\eta) \cup K_\eta(\xi,\eta)$,

(c) $\operatorname{dist}(K_\xi(\xi,\eta), K_\eta(\xi,\eta)) \equiv 5r(\xi,\eta) > 0$.

Proof. The proofs of these statements are exactly the same as their counterparts in Proposition 2.43 of [Montecchiari and Rabinowitz 2016]. □

Remark 5.11. Note that Proposition 5.10(i) occurs if V is independent of x_1.

To continue, we assume that the nondegeneracy condition, alternative (ii) of Proposition 5.10, holds for both $\mathcal{C}_\xi(\xi,\eta)$ and $\mathcal{C}_\xi(\eta,\xi)$. Since the arguments are very close to those of [Montecchiari and Rabinowitz 2016], we will give the proof for the simplest case of two transition solutions and merely set up the variational problem that finds the multitransition solutions as local minima of J, referring to [Montecchiari and Rabinowitz 2016] for further results and details.

Recalling the definition of ρ given after (2.12), by Proposition 5.10,

$$\bar{r} = \min\bigl(\rho, r(a_-,a_+), r(a_+,a_-)\bigr) > 0.$$

Define the set

$$\Lambda(\xi,\eta) = \bigl\{u \in \Gamma(\xi,\eta) \bigm| \|u - K_\xi(\xi,\eta)\|_{W^{1,2}(T_0,\mathbb{R}^m)} = \bar{r} \text{ or } \|u - K_\eta(\xi,\eta)\|_{W^{1,2}(T_0,\mathbb{R}^m)} = \bar{r}\bigr\}$$

and

$$d(\xi,\eta) = \inf_{u \in \Lambda(\xi,\eta)} J(u). \tag{5.12}$$

Arguing as in the proof of Proposition 2.47 of [Montecchiari and Rabinowitz 2016] shows

$$d(\xi,\eta) > c(\xi,\eta). \tag{5.13}$$

To set up the variational framework to find the simplest two transition solutions of (PDE) and (BC), following [Montecchiari and Rabinowitz 2016], let $\boldsymbol{m} = (m_1, \dots, m_4) \in \mathbb{Z}^4$ and $l \in \mathbb{N}$ be such that

$$m_1 + 2l < m_2 - 2l < m_2 + 2l < m_3 - 2l < m_3 + 2l < m_4 - 2l.$$

Finally define

$$\mathcal{A}_2 = \mathcal{A}_2(\boldsymbol{m}, l) = \{u \in E \mid u \text{ satisfies (5.14)}\},$$

where

$$u(\cdot + je_1)|_{T_0} \in \begin{cases} N_r(K_{a_-}(a_-, a_+)), & j < m_1 + l, \\ N_r(K_{a_+}(a_-, a_+)), & m_2 - l \le j < m_2 + l, \\ N_r(K_{a_+}(a_+, a_-)), & m_3 - l \le j < m_3 + l, \\ N_r(K_{a_-}(a_+, a_-)), & m_4 - l \le j. \end{cases} \tag{5.14}$$

Here $N_r(A) \equiv \{u \in W^{1,2}(T_0, \mathbb{R}^m) \mid \text{dist}_{W^{1,2}(T_0,\mathbb{R}^m)}(u, A) \le r\}$ for any $A \subset W^{1,2}(T_0\mathbb{R}^m)$.

We seek 2-transition solutions as minima of J on $\mathcal{A}_2$. Define

$$b_2 = b_2(\boldsymbol{m}, l) = \inf_{u \in \mathcal{A}_2} J(u). \tag{5.15}$$

Theorem 5.16. *Suppose (V_1)–(V_4) are satisfied and that Proposition 5.10(ii) holds for $\mathcal{C}_\xi(\xi,\eta)$ whenever $\xi \ne \eta \in \{a_-, a_+\}$. There exists an $m_0 \in \mathbb{N}$ such that if $l \ge m_0$ and $m_{i+1} - m_i - 6l \ge m_0$ for $i = 1, 2, 3$, then*

$$\mathcal{M}(b_2) \equiv \{u \in \mathcal{A}_2 \mid J(u) = b_2\} \ne \varnothing.$$

Moreover, any $U \in \mathcal{M}(b_2)$ is a classical solution of (PDE) satisfying (BC) and $\|U - a_-\|_{W^{1,2}(T_p,\mathbb{R}^m)} \to 0$ as $p \to \pm\infty$.

Proof. Let $\{u_k\} \subset \mathcal{A}_2$ be such that $J(u_k) \to b_2$. Arguments similar to the ones used to prove Propositions 2.14 and 2.24 show that $\{\|u_k\|_{W^{1,2}(T_i,\mathbb{R}^m)}\}_{i\in\mathbb{Z},k\in\mathbb{N}}$ is bounded. Then, along a subsequence (denoted again by $\{u_k\}$), $u_k \to U$ weakly in E. Since $\mathcal{A}_2$ is weakly closed, we have $U \in \mathcal{A}_2$ and J is weakly lower semicontinuous, so $J(U) = b_2$. Since $J(U) < +\infty$,

$$\text{dist}_{W^{1,2}(T_p,\mathbb{R}^m)}(U, \{a_-, a_+\}) \to 0 \quad \text{as } p \to \pm\infty,$$

and by the definition of $\mathcal{A}_2$, it follows that $\lim_{\pm\to\infty} \|U - a_-\|_{W^{1,2}(T_p,\mathbb{R}^m)} = 0$. To show that U is a classical solution of (PDE) satisfying (BC), the arguments of Section 3 can be applied here once we have verified that U is a weak solution of (PDE), i.e.,

$$\int_\Omega \nabla U \cdot \nabla\varphi + V_u(x, U)\cdot\varphi \, dx = 0 \quad \text{for any } \varphi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^m). \tag{5.17}$$

To verify (5.17), it suffices to show that if m_0 is large enough, U satisfies the inequalities defining $\mathcal{A}_2$ with strict inequalities. Towards this end, define $K_1 = K_{a_-}(a_-, a_+)$, $K_2 = K_{a_+}(a_-, a_+)$, $K_3 = K_{a_+}(a_+, a_-)$, $K_4 = K_{a_-}(a_+, a_-)$, and $a_1 = a_4 = a_-$, $a_2 = a_3 = a_+$. If U does not satisfy one of the inequalities defining $\mathcal{A}_2$ with strict inequality, then

(∗) there exist integers $j \in \{1, \dots, 4\}$ and

$$p_j \in \begin{cases} (-\infty, m_1 + l] \cap \mathbb{Z} & \text{if } j = 1, \\ [m_j - l, m_j + l) \cap \mathbb{Z} & \text{if } 1 < j < 4, \\ [m_4 - l, +\infty) \cap \mathbb{Z} & \text{if } j = 4, \end{cases}$$

for which

$$\bar{r} = \operatorname{dist}_{W^{1,2}(T_0,\mathbb{R}^m)}(U(\cdot + p_j e_1)|_{T_0}, K_j).$$

We show here below how (∗) is not possible if m_0 is large enough. The arguments are slightly different depending on whether $j = 1, 4$ (the "boundary" case) or $j = 2, 3$ (the "interior" case). We will show here how to get a contradiction only for the interior case, the other being very similar (and simpler).

Assume that for some $\bar{j} \in \{2, 3\}$ there exists a $p \in [m_{\bar{j}} - l, m_{\bar{j}} + l)$ such that

$$\bar{r} = \operatorname{dist}_{W^{1,2}(T_0,\mathbb{R}^m)}(U(\cdot + pe_1)|_{T_0}, K_{\bar{j}}). \tag{5.18}$$

Let $\varepsilon \in (0, \bar{r})$. First note that if m_0 is sufficiently large then

$$b_2 < c(a_-, a_+) + c(a_+, a_-) + 2 \tag{5.19}$$

independently of the choice on $\boldsymbol{m}$ — see the simple argument at the beginning of the proof of Proposition 3.29 in [Montecchiari and Rabinowitz 2016]. Moreover, taking m_0 larger if need be, observe that for any $j \in \{1, \dots, 4\}$ there exists $\iota_j \in [m_j - l + 2, m_j + l - 2) \cap \mathbb{Z}$ such that

$$\|U - a_j\|_{W^{1,2}(T_i,\mathbb{R}^m)} < \varepsilon \quad \text{for } i \in [\iota_j - 2, \iota_j + 2] \cap \mathbb{Z}. \tag{5.20}$$

Indeed, suppose for every $X_i \equiv \bigcup_{k=i-2}^{i+2} T_k \subset [m_j - l, m_j + l] \times \mathcal{D}$, there exists $T_j \subset X_i$ such that $\|U - a_j\|_{W^{1,2}(T_j,\mathbb{R}^m)} \geq \varepsilon$. Since $U \in \mathcal{A}_2$, it follows that $\operatorname{dist}_{W^{1,2}(T_j,\mathbb{R}^m)}(U, \{a_-, a_+\}) \geq \varepsilon$. Then, the argument in Proposition 2.14 shows $\int_{T_j} L(U)\, dx \geq \beta(\varepsilon) > 0$. Therefore

$$b_2 = J_1(U) \geq \tfrac{1}{5}(2l + 1)\beta(\varepsilon) \geq \tfrac{2}{5} m_0 \beta(\varepsilon),$$

which is in contradiction with (5.19) for large values of m_0.

By (5.20), there are integers $i_- \in (m_{\bar{j}-1} - l + 2, m_{\bar{j}-1} + l - 2)$ and $i_+ \in (m_{\bar{j}+1} - l + 2, m_{\bar{j}+1} + l - 2)$ and corresponding regions X_{i_-} and X_{i_+} such that if $T_l \subset X_{i_-}$ and $T_k \subset X_{i_+}$, then

$$\|U - a_{\bar{j}-1}\|_{W^{1,2}(T_l,\mathbb{R}^m)} < \varepsilon \quad \text{and} \quad \|U - a_{\bar{j}+1}\|_{W^{1,2}(T_k,\mathbb{R}^m)} < \varepsilon. \tag{5.21}$$

Define

$$f = \begin{cases} a_{\bar{j}-1} & \text{for } x_1 \leq i_-, \\ U & \text{for } i_- + 1 \leq x_1 \leq i_+ - 1, \\ a_{\bar{j}+1} & \text{for } i_+ \leq x_1, \end{cases} \tag{5.22}$$

with interpolations as in (5.3) in the other regions.

By construction, $f \in \Gamma(a_{\bar{j}-1}, a_{\bar{j}+1})$ and since $f = U$ on T_p, by (5.18) we have $f \in \Lambda(a_{\bar{j}-1}, a_{\bar{j}+1})$. Then, by (5.13) and (5.22),

$$\begin{aligned} d(a_{\bar{j}-1}, a_{\bar{j}+1}) &\le J(f) \\ &\le \int_{\bigcup_{i_-+1}^{i_+-2} T_i} L(U)\,dx + \int_{T_{i_-}} L(f)\,dx + \int_{T_{i_+-1}} L(f)\,dx \\ &\le \int_{\bigcup_{i_-+1}^{i_+-2} T_i} L(U)\,dx + 2\kappa(\varepsilon). \end{aligned} \tag{5.23}$$

If m_0 is large enough, there exists $u \in \mathcal{M}(a_{\bar{j}-1}, a_{\bar{j}+1})$ such that

$$\|u - a_{\bar{j}-1}\|_{W^{1,2}(T_q,\mathbb{R}^m)} \le \varepsilon \quad \text{for any } q \le m_{\bar{j}-1} + l,$$
$$\|u - a_{\bar{j}+1}\|_{W^{1,2}(T_q,\mathbb{R}^m)} \le \varepsilon \quad \text{for any } q \ge m_{\bar{j}+1} - l.$$

Define

$$\Phi = \begin{cases} U & \text{for } x_1 \le i_- - 2, \\ a_{\bar{j}-1} & \text{for } i_- - 1 \le x_1 \le i_-, \\ u & \text{for } i_- + 1 \le x_1 \le i_+ - 1, \\ a_{\bar{j}+1} & \text{for } i_+ \le x_1 \le i_+ + 1, \\ U & \text{for } x_1 \ge i_+ + 2, \end{cases} \tag{5.24}$$

making the usual interpolations in the remaining regions. Observe that $\Phi \in \mathcal{A}_2$. Consequently, with the aid of (5.23), we obtain

$$\begin{aligned} 0 \le J(\Phi) - J(U) &= \int_{\bigcup_{i=i_--2}^{i_++1} T_i} L(\Phi) - L(U)\,dx \\ &\le \int_{\bigcup_{i=i_-+1}^{i_+-1} T_i} L(u)\,dx + 2\kappa(\varepsilon) - \int_{\bigcup_{i=i_--2}^{i_++1} T_i} L(U)\,dx \\ &\le c(a_{\bar{j}-1}, a_{\bar{j}+1}) - d(a_{\bar{j}-1}, a_{\bar{j}+1}) + 4\kappa(\varepsilon), \end{aligned}$$

a contradiction to (5.13) if $4\kappa(\varepsilon) < d(a_{\bar{j}-1}, a_{\bar{j}+1}) - c(a_{\bar{j}-1}, a_{\bar{j}+1})$. An analogous argument leads to a contradiction in the boundary case. Thus $(*)$ cannot occur and the theorem is proved. □

Remark 5.25. Varying the values of $\boldsymbol{m}$, Theorem 5.16 provides the existence of infinitely many 2-transition solutions of (PDE) homoclinic to a_-. Reversing the roles of a_- and a_+, an analogous result is obtained giving infinitely many solutions homoclinic to a_+.

As in [Montecchiari and Rabinowitz 2016], Theorem 5.16 can be generalized also to the case of k-transition and infinite transition solutions. We state here the case of k-transition solutions referring to [Montecchiari and Rabinowitz 2016] for more details.

For $k \in \mathbb{N}$, let $\{a_1, \dots, a_{2k}\} \in \{a_-, a_+\}^{2k}$ be such that

$$a_1 \ne a_2 = a_3 \ne \cdots \ne a_{2k-2} = a_{2k-1} \ne a_{2k}.$$

Consider also the family of sets $\{K_1, \dots, K_{2k}\}$ defined as

$$K_{2j-1} = K_{a_{2j-1}}(a_{2j-1}, a_{2j}) \quad \text{and} \quad K_{2j} = K_{a_{2j}}(a_{2j-1}, a_{2j}), \qquad j = 1, \dots, k.$$

Given $l \in \mathbb{N}$ and $\boldsymbol{m} = (m_1, \dots, m_{2k}) \in \mathbb{Z}^{2k}$ with $m_j - m_{j-1} > 2l$ for $j = 2, \dots, 2k$, consider the set

$$\mathcal{A}(k, \boldsymbol{m}, l) = \{u \in E \mid u \text{ satisfies (5.26)}\},$$

where

$$u(\cdot + pe_1)|_{T_0} \in \begin{cases} N_{\bar{r}}(K_1), & p \in (-\infty, m_1 + l) \cap \mathbb{Z}, \\ N_{\bar{r}}(K_j), & p \in [m_j - l, m_j + l) \cap \mathbb{Z},\ 2 \le j \le 2k-1, \\ N_{\bar{r}}(K_{2k}), & p \in [m_{2k} - l, +\infty) \cap \mathbb{Z}, \end{cases} \tag{5.26}$$

and let

$$b_k = b(k, \boldsymbol{m}, l) = \inf_{u \in \mathcal{A}(k, \boldsymbol{m}, l)} J(u). \tag{5.27}$$

Theorem 5.28. *Under the hypotheses of Theorem 5.16, there is an $m_0 \in \mathbb{N}$ for which if $k \in \mathbb{N}$, $l \ge m_0$ and $m_{i+1} - m_i - 6l \ge m_0$ for $i = 1, \dots, 2k-1$, then*

$$\mathcal{M}(b_k) \equiv \{u \in \mathcal{A}(k, \boldsymbol{m}, l) \mid J(u) = b(k, \boldsymbol{m}, l)\} \ne \varnothing.$$

Moreover, any $U \in \mathcal{M}(b_k)$ is a classical solution of (PDE) satisfying (BC).

6. Proof of Theorem 3.1

In this section the proof of Theorem 3.1 will be carried out. It is similar to the proof of the corresponding scalar case. The proof consists of several steps. First note that since (V_1)–(V_3) are satisfied and $\partial\Omega = \mathbb{R} \times \partial\mathcal{D} \in C^1$, by Theorem 2.2, there exists a minimizer $U \in \Gamma$ of (2.1). For any $\varphi \in W^{1,2}_{\text{loc}}(\Omega)$ with compact support in $\bar{\Omega}$ and $t \in \mathbb{R}$, we see that $U + t\varphi \in \Gamma$. Since $V(x, \cdot) \in C^1(\mathbb{R}^m)$ for each $x \in \Omega$ and (V_4) holds, $\lim_{t\to 0}(J(U + t\varphi) - J(U))/t$ exists. Since $J(U) \le J(U + t\varphi)$ for any $t \in \mathbb{R}$, we see that

$$\lim_{t\to 0} \frac{J(U + t\varphi) - J(U)}{t} = \int_\Omega \nabla U \cdot \nabla\varphi + V_u(x, U)\varphi\, dx = 0.$$

This implies that U is a weak solution of (PDE) and (BC).

Now two rather technical steps are required and will be stated as separate propositions. The first provides an L^∞ bound for any weak solution U of (PDE) and (BC). When $m = 1$, such results are well known; see, e.g., [Gilbarg and Trudinger 1983]. In general, they are not true for systems, but we will show that due to the semilinear structure of (PDE) and (V_4), variants of arguments in [Gilbarg and Trudinger 1983], that in turn go back to work of Moser, can be modified to treat the current setting.

Note that for any $U \in E$, there is a constant $M_4 > 0$ depending on U such that

$$\int_\Omega |\nabla U|^2\, dx + \sup_{i\in\mathbb{Z}} \int_{T_i} |U|^2\, dx \le M_4. \tag{6.1}$$

Proposition 6.2. *Suppose V satisfies (V_1)–(V_4), and that $\partial\Omega = \mathbb{R} \times \partial\mathcal{D} \in C^1$. Then for any weak solution $U \in E$ of (PDE) and (BC), there exists a constant $M_5 > 0$ depending on U such that*

$$\|U\|_{L^\infty(\Omega, \mathbb{R}^m)} \le M_5.$$

If U is a minimizer of (2.1), M_4 and M_5 are independent of U.

Proof. First observe that by Proposition 2.24, M_4 can be chosen independently of U if U is a minimizer of (2.1). Let $\eta \in C^1(\mathbb{R}, [0,1])$ have compact support. Then η extends to a C^1-function on Ω by defining $\eta(x_1, \dots, x_n) = \eta(x_1)$. For each $\sigma > 0$ and $i = 1, \dots, m$, define a function U_i^σ by $U_i^\sigma(x) = U_i(x)$ if $|U_i(x)| < \sigma$, by $U_i^\sigma(x) = \sigma$ if $U_i(x) \geq \sigma$ and by $U_i^\sigma(x) = -\sigma$ if $U_i(x) \leq -\sigma$. If $U = (U_1, \dots, U_m)$, set $U^\sigma = (U_1^\sigma, \dots, U_m^\sigma)$. Let $\beta > 0$ and take $\varphi_j = \eta^2 U_j |U_j^\sigma|^{2\beta}$, $1 \leq j \leq m$. Then, taking $\varphi = \varphi_j e_j$, with e_j the j-th unit vector in $\mathbb{R}^m$, we see that $\varphi \in W^{1,2}_{\mathrm{loc}}(\Omega)$ and the support of φ is compact. Thus, (1.1) implies that for $1 \leq j \leq m$,

$$\int_\Omega \nabla U_j \cdot \nabla(\eta^2 U_j |U_j^\sigma|^{2\beta}) + V_{u_j}(x, U)\eta^2 U_j |U_j^\sigma|^{2\beta}\, dx = 0. \tag{6.3}$$

Note that

$$\nabla U_j \cdot \nabla(\eta^2 U_j |U_j^\sigma|^{2\beta}) = \eta^2 |U_j^\sigma|^{2\beta} |\nabla U_j|^2 + 2\beta\eta^2 U_j |U_j^\sigma|^{2\beta-1} \nabla U_j \cdot \nabla |U_j^\sigma| + 2\eta U_j |U_j^\sigma|^{2\beta} \nabla U_j \cdot \nabla\eta. \tag{6.4}$$

Observing that the middle term on the right in (6.4) satisfies

$$2\beta\eta^2 U_j |U_j^\sigma|^{2\beta-1} \nabla U_j \cdot \nabla |U_j^\sigma| \geq 0, \tag{6.5}$$

substituting (6.4)–(6.5) in (6.3) and using (V_4) shows for some constant $C_2 > 0$, independent of σ, j, β,

$$\int_\Omega \eta^2 |U_j^\sigma|^{2\beta} |\nabla U_j|^2\, dx \leq 2\int_\Omega \eta |U_j| |U_j^\sigma|^{2\beta} |\nabla U_j| |\nabla\eta|\, dx + C_2 \int_\Omega \eta^2 |U_j| |U_j^\sigma|^{2\beta} (1 + |U|^p)\, dx. \tag{6.6}$$

Simplifying the right-hand side of (6.6) gives

$$\int_\Omega \eta^2 |U_j^\sigma|^{2\beta} |\nabla U_j|^2\, dx \leq \frac{1}{2}\int_\Omega \eta^2 |U_j^\sigma|^{2\beta} |\nabla U_j|^2\, dx + 8\int_\Omega |U_j|^2 |U_j^\sigma|^{2\beta} |\nabla\eta|^2\, dx + C_2 \int_\Omega \eta^2 |U_j^\sigma|^{2\beta} (|U_j| + |U|^{p+1})\, dx.$$

Hence there is a constant $C_3 > 0$, independent of σ, j, β, such that

$$\int_\Omega \eta^2 |U_j^\sigma|^{2\beta} |\nabla U_j|^2\, dx \leq C_3 \int_{\Omega\cap\mathrm{supp}(\eta)} \left[(1 + |\nabla\eta|_{L^\infty})^2 (|U_j| + |U_j|^2) |U_j^\sigma|^{2\beta} + \eta^2 |U_j^\sigma|^{2\beta} |U|^{p+1}\right] dx. \tag{6.7}$$

Since

$$\left|\nabla(\eta U_j |U_j^\sigma|^\beta)\right|^2 \leq 2(\beta+1)^2 \eta^2 |U_j^\sigma|^{2\beta} |\nabla U_j|^2 + 2(U_j)^2 |U_j^\sigma|^{2\beta} |\nabla\eta|^2,$$

using this estimate in (6.7) shows there is a constant $C_4 > 0$, independent of σ, j, β, such that

$$\int_\Omega |\nabla(\eta U_j |U_j^\sigma|^\beta)|^2 + \eta^2 (U_j)^2 |U_j^\sigma|^{2\beta}\, dx \leq C_4(\beta+1)^2 \int_{\Omega\cap\mathrm{supp}(\eta)} \left[(1 + |\nabla\eta|_{L^\infty})^2 (|U_j| + |U_j|^2) |U_j^\sigma|^{2\beta} + \eta^2 |U_j^\sigma|^{2\beta} |U|^{p+1}\right] dx. \tag{6.8}$$

Due to the Sobolev inequality and (6.8), there exists a constant $C_5 > 0$, independent of σ, j, β, such that

$$\left(\int_\Omega (\eta|U_j^\sigma|^{\beta+1})^{\frac{2n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \le C_5(\beta+1)^2\int_{\Omega\cap\operatorname{supp}(\eta)}\left[(1+|\nabla\eta|_{L^\infty})^2|U_j|^2|U_j^\sigma|^{2\beta}+\eta^2|U_j^\sigma|^{2\beta}|U|^{p+1}\right]dx \equiv M(j,\sigma,\beta). \tag{6.9}$$

Suppose for the moment that $n \ge 3$. Define β_1 by

$$2(\beta_1+1)+p-1=\frac{2n}{n-2}.$$

The restriction on p in (V_4) implies $\beta_1 > 0$. By this choice of β_1, (6.1) and the Sobolev inequality, $M(j,\sigma,\beta_1)$ is bounded independently of σ. Consequently letting $\sigma \to \infty$ and choosing η so that $\eta(x_1) = 1$ for $|x_1 - i| \le l$, (6.9) shows there is a constant $K = K(\beta_1, l)$ such that for each $l \ge 1$,

$$\int_{\Omega\cap\{|x_1-i|\le l\}}(|U|^{\beta_1+1})^{\frac{2n}{n-2}}\,dx \le K(\beta_1,l) \tag{6.10}$$

independently of i. For $t \in \mathbb{N}$ with $t \ge 2$, define β_t via

$$2(\beta_t+1)+p-1=2(\beta_{t-1}+1)\frac{n}{n-2}$$

and repeat the above argument, obtaining

$$\int_{\Omega\cap\{|x_1-i|\le l\}}(|U|^{\beta_t+1})^{\frac{2n}{n-2}}\,dx \le K(\beta_{t-1},l+t-1) \tag{6.11}$$

independently of i. Since $\beta_{t+1}-\beta_t = n/(n-2)(\beta_t-\beta_{t-1})$ and $\beta_2-\beta_1 > 0$, it follows that $\beta_t \to \infty$ as $t \to \infty$. Thus for each fixed $q > 0$ and $l \ge 1$,

$$\int_{\Omega\cap\{|x_1-i|\le l\}}|U|^q\,dx$$

is bounded independently of i, the bound depending on q, n, M_4 and the constants in (V_4).

Now the Moser iteration argument will be used to get the L^∞ bound of the proposition. Returning to (6.9), our above observations show there is a constant $C_6 > 0$, independent of η and $\beta > 0$, such that

$$\left(\int_\Omega (\eta^2|U|^{(2\beta+2)})^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \le C_6(\beta+1)^2\int_{\Omega\cap\operatorname{supp}(\eta)}\left[(1+|\nabla\eta|_{L^\infty})^2(1+|U|)|U|^{2\beta+1}+\eta^2|U|^{2\beta+2+p-1}\right]dx. \tag{6.12}$$

Consider the last term in (6.12). Let $h > 0$. Note that

$$\int_{\Omega\cap\operatorname{supp}(\eta)}\eta^2|U|^{2\beta+2+p-1}\,dx = \int_{R_1}\eta^2|U|^{2\beta+2+p-1}\,dx + \int_{R_2}\eta^2|U|^{2\beta+2+p-1}\,dx \equiv I_1+I_2, \tag{6.13}$$

where $R_1 = \{x \in \Omega \cap \operatorname{supp}(\eta) \mid |U| \le h\}$ and $R_2 = \{x \in \Omega \cap \operatorname{supp}(\eta) \mid |U| > h\}$. Then

$$I_1 \le h^{p-1} \int_\Omega \eta^2 |U|^{2\beta+2}\,dx. \tag{6.14}$$

By Hölder's inequality,

$$I_2 \le \left(\int_{R_2} |U|^{n(p-1)/2}\,dx\right)^{\frac{2}{n}} \left(\int_\Omega (\eta^2|U|^{2\beta+2})^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \equiv I_3 I_4. \tag{6.15}$$

Noting that $\frac{1}{2}n(p-1) < 2n/(n-2)$ and setting $d = \frac{1}{4}(p-1)(n-2)$, another application of Hölder's inequality implies

$$I_3^{\frac{n}{2}} \le |R_2|^{1-d}\left(\int_{R_2} |U|^{2n/(n-2)}\,dx\right)^d. \tag{6.16}$$

Since

$$|R_2| = \Big|\{x \in \Omega \cap \operatorname{supp}(\eta) \,\Big|\, |U| > h\}\Big| \le h^{-2n/(n-2)} \int_{\Omega\cap\operatorname{supp}(\eta)} |U|^{2n/(n-2)}\,dx,$$

(6.16) can be rewritten as

$$I_3 \le h^{-\frac{4(1-d)}{n-2}} \left(\int_{\Omega\cap\operatorname{supp}(\eta)} |U|^{\frac{2n}{n-2}}\,dx\right)^{\frac{2}{n}}. \tag{6.17}$$

Combining (6.13)–(6.17), (6.12) becomes

$$\begin{aligned}
&\left(\int_\Omega (\eta^2|U|^{2\beta+2})^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}} \\
&\quad\le C_6(\beta+1)^2 \int_{\Omega\cap\operatorname{supp}(\eta)} (1+|\nabla\eta|_{L^\infty})^2 (|U|^{2\beta+1}+|U|^{2\beta+2})\,dx \\
&\qquad + C_6(\beta+1)^2\Bigg[h^{p-1}\int_\Omega \eta^2|U|^{2\beta+2}\,dx \\
&\qquad\qquad + h^{-\frac{4(1-d)}{n-2}}\left(\int_{\Omega\cap\operatorname{supp}(\eta)} |U|^{\frac{2n}{n-2}}\,dx\right)^{\frac{2}{n}} \left(\int_\Omega (\eta^2|U|^{2\beta+2})^{\frac{n}{n-2}}\,dx\right)^{\frac{n-2}{n}}\Bigg].
\end{aligned} \tag{6.18}$$

Using the freedom in the choice of h, we require that

$$C_6(\beta+1)^2 h^{-\frac{4(1-d)}{n-2}} \left(\int_{\Omega\cap\operatorname{supp}(\eta)} |U|^{\frac{2n}{n-2}}\,dx\right)^{\frac{2}{n}} = \frac{1}{2}$$

or equivalently

$$h = \left(2C_6(\beta+1)^2\left(\int_{\Omega\cap\operatorname{supp}(\eta)} |U|^{\frac{2n}{n-2}}\,dx\right)^{\frac{2}{n}}\right)^{\frac{n-2}{4(1-d)}}.$$

This makes the coefficient of the last integral term in (6.18) equal to $\frac{1}{2}$ so it can be absorbed on the left-hand side of the inequality (6.18). Thus (6.18) becomes

$$\begin{aligned}\Big(\int_\Omega (\eta^2|U|^{(2\beta+2)})^{\frac{n}{n-2}}\,dx\Big)^{\frac{n-2}{n}} &\le 2C_6(\beta+1)^2\int_{\Omega\cap\mathrm{supp}(\eta)}(1+\|\nabla\eta\|_{L^\infty})^2(|U|^{2\beta+1}+|U|^{2\beta+2})\,dx\\ &+(2C_6(\beta+1)^2)^{1+\frac{(p-1)(n-2)}{4(1-d)}}\Big(\int_{\Omega\cap\mathrm{supp}(\eta)}|U|^{\frac{2n}{n-2}}\,dx\Big)^{\frac{2(n-2)(p-1)}{4n(1-d)}}\int_\Omega \eta^2|U|^{2\beta+2}\,dx. \quad (6.19)\end{aligned}$$

Now for each $i\in\mathbb{Z}$ and $j\in\mathbb{N}$, choose $\eta_j\in C^\infty(\mathbb{R})$ such that $\eta_j(x_1)=1$ for $|x_1-i|\le 1+2^{-j-1}$, $\eta_j(x_1)=0$ for $|x_1-i|\ge 1+2^{-j}$ and $\|\nabla\eta_j\|_{L^\infty}\le 2^{j+2}$. By (6.1) and the Sobolev inequality, there exists a constant $M'>0$, independent of $i\in\mathbb{Z}$ and $j\in\mathbb{N}$, such that

$$\int_{\Omega\cap\mathrm{supp}(\eta_j)}|U|^{\frac{2n}{n-2}}\,dx\le M'.$$

Set $\theta=n/(n-2)$ and for each $j\in\mathbb{N}$, define γ_j by $\gamma_j=2\theta^j$. Then, taking $\eta=\eta_j$ and $2\beta+2=\gamma_j$ in (6.19), simple estimates show there is a constant $C_7\ge 1$, independent of $j\in\mathbb{N}$ and $i\in\mathbb{Z}$, so that

$$\begin{aligned}\Big(\int_{\{x\in\Omega\,|\,|x_1-i|\le 1+2^{-j-1}\}}|U|^{2\theta^{j+1}}\,dx\Big)^{\frac{n-2}{n}} &\le C_7\big((2\theta)^{2j}+\theta^{2j(1+\frac{(p-1)(n-2)}{4(1-d)})}\big)\int_{\{x\in\Omega\,|\,|x_1-i|\le 1+2^{-j}\}}(|U|^{2\theta^j-1}+|U|^{2\theta^j})\,dx\\ &\le C_7\big((2\theta)+\theta^{1+\frac{(p-1)(n-2)}{4(1-d)}}\big)^{2j}\int_{\{x\in\Omega\,|\,|x_1-i|\le 1+2^{-j}\}}(|U|^{2\theta^j-1}+|U|^{2\theta^j})\,dx. \quad (6.20)\end{aligned}$$

Thus setting

$$A_j\equiv\Big(\int_{\{x\in\Omega\,|\,|x_1-i|\le 1+2^{-j}\}}|U|^{2\theta^j}\,dx\Big)^{\frac{1}{2\theta^j}},$$

by the Hölder inequality,

$$\int_{\{x\in\Omega\,|\,|x_1-i|\le 1+2^{-j}\}}|U|^{2\theta^j-1}\,dx\le\Big|\{x\in\Omega\mid |x_1-i|\le 2\}\Big|^{\frac{1}{2\theta^j}}(A_j)^{2\theta^j-1}. \quad (6.21)$$

With the aid of (6.21), there is a constant $C_8\ge 1$, independent of $j\in\mathbb{N}$ and $i\in\mathbb{Z}$, such that (6.20) yields the simpler inequality

$$A_{j+1}\le (C_8)^{\frac{j}{2\theta^j}}\Big(1+\frac{1}{A_j}\Big)^{\frac{1}{2\theta^j}}A_j. \quad (6.22)$$

Since $\Big(1+\frac{1}{A_j}\Big)^{\frac{1}{2\theta^j}}\le 1+\frac{1}{2\theta^j}\frac{1}{A_j}$, (6.22) implies

$$A_{j+1}\le (C_8)^{\frac{j}{2\theta^j}}\Big(A_j+\frac{1}{2\theta^j}\Big). \quad (6.23)$$

This inequality can be further rewritten as

$$A_{j+1} \leq (C_8)^{\frac{j}{2\theta^j}} A_j + \frac{C_9}{2\theta^j}, \tag{6.24}$$

where the constant $C_9 \geq 1$ is independent of $j \in \mathbb{N}$ and $i \in \mathbb{Z}$. Since $D \equiv \sum_{j\in\mathbb{N}} j/(2\theta^j) < \infty$, by (6.24),

$$A_{j+1} \leq (C_8)^{\frac{j}{2\theta^j}} A_j + \frac{C_9}{2\theta^j} \leq (C_8)^{\frac{j}{2\theta^j}+\frac{j-1}{2\theta^{j-1}}} A_{j-1} + (C_8)^{\frac{j}{2\theta^j}} \frac{C_9}{2\theta^{j-1}} + \frac{C_9}{2\theta^j}$$
$$\leq \cdots \leq (C_8)^D \Big(A_1 + C_9 \sum_{j=1}^{\infty} \theta^{-j}\Big) < \infty. \tag{6.25}$$

Consequently there exists a constant $M_5 > 0$, independent of $i \in \mathbb{Z}$, such that for any weak solution U of (1.1),

$$\|U\|_{L^\infty(\{x\in\Omega \mid |x_1-i|\leq 1\})} = \lim_{j\to\infty} A_j \leq M_5$$

as claimed. Note that M_5 depends on n, M_4, and the constants in (V_4).

When $n = 2$, by the Sobolev inequality, for any $q > 2$, there exists a constant, C_{10}, depending on q but independent of i, such that

$$\|U\|_{L^q(\{x\in\Omega \mid |x_1-i|\leq 2\})} \leq C_{10} \|U\|_{W^{1,2}(\{x\in\Omega \mid |x_1-i|\leq 2\})}. \tag{6.26}$$

Therefore the case of $n \geq 3$ can be simplified and modified by, e.g., replacing our earlier φ_j by $\eta^2 U_j |U_j|^{2(\beta+1)}$. This leads to a simpler version of (6.8). Then employing (6.26) in going from (6.8) to (6.9) leads to the following variant of (6.12)with q replacing $n/(n-2)$:

$$\Big(\int_\Omega (\eta|U_j|^{2(\beta+1)})^q \, dx\Big)^{\frac{1}{q}} \leq C_{11}(\beta+1)^2 \int_{\Omega\cap \mathrm{supp}(\eta)} \big[(1+|\nabla\eta|_{L^\infty})^2 |U_j|^2 |U_j|^{2\beta} + \eta^2 |U_j|^{2\beta} |U|^{p+1}\big] \, dx$$

for any $q > 2$, where C_{11} depends on q. Then continuing as earlier completes the proof for this case. □

As the next step in the proof of Theorem 3.1, we have:

Proposition 6.27. *Suppose that V satisfies (V_1)–(V_4) and $\partial\Omega \in C^1$. If $U \in E$ is a weak solution of* (PDE) *and* (BC), *then:*

(1) *For any $\Omega' \subset\subset \Omega$, we have $U \in W^{2,2}(\Omega')$.*

(2) *If $V_u \in C^1(\Omega \times \mathbb{R}^m, \mathbb{R}^m)$, then $U \in C^{2,\alpha}_{\mathrm{loc}}(\Omega, \mathbb{R}^m)$ for any $\alpha \in (0,1)$ and satisfies* (PDE) *in Ω.*

(3) *If $\partial\Omega \in C^2$, then $U \in W^{2,2}_{\mathrm{loc}}(\overline{\Omega})$ and U is a strong solution of* (PDE) *and* (BC).

Proof. First since U is a weak solution of (PDE) and by Theorem 3.1, $V(x,U) \in L^2(\Omega', \mathbb{R}^m)$, (1) follows from Theorem 8.8 of [Gilbarg and Trudinger 1983]. Moreover, this additional differentiability shows U is a strong solution of (PDE). Next by Theorem 3.1 again, $V_u(\cdot, U) \in L^q(\Omega', \mathbb{R}^m)$ for any $q > 1$ so by Theorem 9.11 of [Gilbarg and Trudinger 1983], $U \in W^{2,q}(\Omega', \mathbb{R}^m)$. The Sobolev inequality then implies $U \in C^{1,\alpha}(\Omega', \mathbb{R}^m)$ for any $\alpha \in (0,1)$. Then, since $V_u(x,U) \in C^1(\Omega')$, invoking the linear Schauder theory then gives $U \in C^{2,\alpha}(\Omega', \mathbb{R}^m)$ and (2) holds. Lastly the proof of Theorem 4 in §6.3.2 of [Evans

1998] with the modification that $U \in W^{1,2}$ rather than $U \in W_0^{1,2}$ yields the first part of (3). For the second, taking any $\varphi \in W^{1,2}_{\text{loc}}(\Omega)$ with compact support in $\overline{\Omega}$, by (2.36) and integration by parts due to the fact $U \in W^{2,2}_{\text{loc}}(\overline{\Omega}) \hookrightarrow W^{1,2}_{\text{loc}}(\partial\Omega)$, we get

$$\int_\Omega (-\Delta U + V_u(x,U)) \cdot \varphi \, dx - \int_{\partial\Omega} \frac{\partial U}{\partial \nu} \cdot \varphi \, dS = 0.$$

Thus since U satisfies (PDE),

$$\int_{\partial\Omega} \frac{\partial U}{\partial \nu} \cdot \varphi \, dS = 0 \tag{6.28}$$

for all $\varphi \in W^{1,2}_{\text{loc}}(\Omega) \hookrightarrow L^2_{\text{loc}}(\partial\Omega)$ having compact support and (6.28) implies (BC). Thus, U is a strong solution of (PDE) and (BC). □

Completion of proof of Theorem 3.1. It remains to show the regularity of U in a neighborhood of $\partial\Omega$ when $\partial\Omega \in C^3$. Since in Section 4 we consider a more general domain than $\Omega = \mathbb{R} \times \mathcal{D}$, the special nature of Ω will be suppressed here so that our argument also adapts easily to the case treated elsewhere. Let $\mathbb{R}^n_+ \equiv \{x \in \mathbb{R}^n \mid x_n > 0\}$ and $z \in \partial\Omega = \mathbb{R} \times \partial\mathcal{D}$. Slightly modifying the proof of Theorem 8.12 of [Gilbarg and Trudinger 1983], there exists a C^3 diffeomorphism Ψ defined on $B_R(z)$ such that $\Psi(B_R(z) \cap \Omega) \subset \mathbb{R}^n_+$, $\Psi(B_R(z) \cap \partial\Omega) \subset \partial\mathbb{R}^n_+$. Choose $\sigma < R$ and set $B^+ = B_\sigma(z) \cap \Omega$, $D' = \Psi(B_\sigma(z))$, and $D^+ = \Psi(B^+)$. Then setting $\Phi = \Psi^{-1}$ and $w \equiv U \circ \Phi$, (PDE) in B^+ is transformed into the equation

$$-\sum_{1 \le i,j \le n} a_{ij}(y) \frac{\partial^2 w}{\partial y_i \partial y_j} + \sum_{j=1}^n b_i(y) \frac{\partial w}{\partial y_i} + V_u(\Phi(y), w) = 0 \quad \text{in } D', \tag{6.29}$$

where

$$a_{ij}(y) = \sum_{l=1}^n \frac{\partial \Psi_i}{\partial x_l}(\Phi(y)) \frac{\partial \Psi_j}{\partial x_l}(\Phi(y)), \quad 1 \le i, j \le n,$$
$$b_i(y) = (\Delta \Psi_i)(\Phi(y)), \qquad i = 1, \dots, n.$$

Moreover, since $U \in W^{1,2}_{\text{loc}}(\Omega, \mathbb{R}^m)$, we know $w \in W^{1,2}(D^+, \mathbb{R}^m)$.

Next we will show that an appropriate choice of Ψ, or equivalently of Φ, allows us to get the regularity of U near z and satisfy (BC). Translate and rotate variables for convenience so that z becomes 0 and ∂R^n_+ is the tangent space to $\partial\Omega$ at $z = 0$. Since $\partial\Omega$ is a C^3 manifold, for r small, there is a C^3 $\mathbb{R}$-valued map ϕ defined on $B_r(0) \cap \partial\mathbb{R}^n_+$ with $\phi(0) = 0 = |\nabla\phi(0)|$ and such that near 0, the boundary $\partial\Omega$ is given by

$$\{(y', \phi(y')) \mid y' \in B_r(0) \cap \partial\mathbb{R}^n_+\}.$$

Then, for $y = (y', y_n) = (y_1, \dots, y_n)$, extend ϕ to $\Phi = (\Phi_1, \dots, \Phi_n) : B_r(0) \to \mathbb{R}^n$ with $\Phi : B_r(0) \cap \overline{\mathbb{R}^n_+} \to \overline{\Omega}$ via

$$\Phi_j(y) = \begin{cases} y_j - y_n \dfrac{\partial \phi}{\partial y_j}(y') & \text{for } j = 1, \dots, n-1, \\ y_n + \phi(y') & \text{for } j = n. \end{cases}$$

This extension of ϕ makes Φ a C^2 function with $\Phi(0) = 0$ and $\Phi'(0) = I$, the identity matrix. Thus Φ is a diffeomorphism in $B_r(0)$ for r small and

$$\frac{\partial \Phi}{\partial y_n}(y', 0) = \left(-\frac{\partial \psi}{\partial y_1}(y'), \dots, -\frac{\partial \psi}{\partial y_{n-1}}(y'), 1 \right)$$

is the inward normal to $\partial\Omega$. Hence for small $r > 0$ and an open neighborhood N of 0 in $\mathbb{R}^n$, the map $\Phi : \overline{\mathbb{R}^n_+} \cap B_r(0) \to N \cap \bar{\Omega}$ is a diffeomorphism. Let $\Psi : N \cap \bar{\Omega} \to \overline{\mathbb{R}^n_+} \cap B_r(0)$ be the inverse of the map Φ. Note that $\nabla \Psi_n(\Phi(y', 0))$ is orthogonal to the surface $\{(y', \psi(y')) \mid y' \in B_r(0) \cap \partial\mathbb{R}^n_+\} \subset \partial\Omega$ since Ψ_n vanishes on the surface $\{(y', \psi(y')) \mid y' \in B_r(0) \cap \partial\mathbb{R}^n_+\}$. For $i \in \{1, \dots, n-1\}$, we know $\nabla \Psi_i(\Phi(y', 0))$ is orthogonal to the surface

$$\left\{ (y' - y_n \nabla \phi(y'), y_n + \phi(y')) \;\middle|\; y' \in B_r(0) \cap \partial\mathbb{R}^n_+, y_n \ge 0 \text{ and fixed } y_i \right\}$$

at $(y', \phi(y'))$. This implies that $\nabla \Psi_i(\Phi(y', 0))$ is in the tangent space of $\partial\Omega$ at $\Phi(y', 0)$. Thus for $i \in \{1, \dots, n-1\}$,

$$\nabla \Psi_n(\Phi(y', 0)) \cdot \nabla \Psi_i(\Phi(y', 0)) = 0.$$

Hence $a_{in} = a_{ni} = 0$ when $y_n = 0$ and $i = 1, \dots, n-1$. Now for $(y', y_n) \in B_r(0)$, we define

$$\begin{aligned} \bar{a}_{ij}(y', y_n) &= a_{ij}(y', |y_n|) && \text{if } i, j \le n-1, \\ \bar{a}_{in}(y', y_n) &= \frac{y_n}{|y_n|} a_{ij}(y', |y_n|) && \text{if } 1 \le i \le n-1, \\ \bar{a}_{nn}(y', y_n) &= a_{nn}(y', |y_n|). \end{aligned}$$

We also define $b_i(y', y_n) = b_i(y', |y_n|)$ for $i = 1, \dots, n-1$, and $b_n(y', y_n) = -b_n(y', y_n)$. For the solution w, we define a function $\bar{w}$ on $B_r(0)$ by $\bar{w}(y', y_n) = w(y', |y_n|)$. Then, we see that $\bar{w}$ is a strong solution of

$$-\sum_{1 \le i, j \le n} \overline{a_{ij}}(y) \frac{\partial^2 \bar{w}}{\partial y_i \partial y_j} + \sum_{j=1}^{n} \overline{b_i}(y) \frac{\partial \bar{w}}{\partial y_i} + V_u(\bar{\Phi}(y), \bar{w}) = 0, \qquad y \in B_r(0).$$

Since $\overline{a_{ij}}$ is continuous and $\overline{b_i}, V_u \in L^\infty$, Theorem 9.11 in [Gilbarg and Trudinger 1983] shows first that $\bar{w} \in W^{2,p}(B_{\frac{r}{2}}(0))$ for any $p > 1$, and then $\bar{w} \in C^{1,\alpha}(B_{\frac{r}{2}}(0))$. This implies that $U \in C^{2,\alpha}(\Omega) \cap C^{1,\alpha}(\bar{\Omega})$.

Now, returning to the original equation (PDE) and applying Theorem 6.31 of [Gilbarg and Trudinger 1983], we get the regularity $U \in C^{2,\alpha}(\bar{\Omega})$.

It remains to show

$$\lim_{x_1 \to \pm\infty} U(x_1, \hat{x}) = a^\pm \quad \text{uniformly for } \hat{x} \in \mathcal{D}. \tag{6.30}$$

From the proof of Theorem 2.2, we have

$$\lim_{i \to \pm\infty} \left(\|\nabla(U - a^\pm)\|_{L^2(T_i, \mathbb{R}^m)} + \|U - a^\pm\|_{L^2(T_i, \mathbb{R}^m)} \right) = 0.$$

By Theorem 3.1, Proposition 6.2, and (PDE), there is a constant, $M^* > 0$ such that $\|U\|_{C^{2,\alpha}(\Omega, \mathbb{R}^m)} \le M^*$. Therefore standard interpolation inequalities imply $\|U - a^\pm\|_{L^\infty(T_i)} \to 0$, $i \to \pm\infty$, which gives (6.30) and completes the proof of Theorem 3.1. □

Remark 6.31. The arguments we have given to establish the regularity of solutions of (PDE) and (BC), in particular Proposition 6.2 obtaining an L^∞ bound for the solution, Proposition 6.27 giving interior regularity, and the final arguments establishing regularity up to the boundary, work equally well for any divergence structure semilinear elliptic system of PDEs satisfying (V_4) provided the coefficients are sufficiently smooth.

Acknowledgments

Byeon was supported by Mid-career Researcher Program through the National Research Foundation of Korea funded by the Ministry of Science, ICT and Future Planning (NRF-2013R1A2A2A05006371).

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Received 23 Mar 2016. Revised 26 Jun 2016. Accepted 30 Jul 2016.

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Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

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ANALYSIS & PDE

Volume 9 No. 7 2016