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DECAY OF SOLUTIONS OF MAXWELL–KLEIN–GORDON EQUATIONS WITH ARBITRARY MAXWELL FIELD

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In the author's previous work, it has been shown that solutions of Maxwell–Klein–Gordon equations in $\mathbb{R}^{3+1}$ possess some form of global strong decay properties with data bounded in some weighted energy space. In this paper, we prove pointwise decay estimates for the solutions for the case when the initial data are merely small on the scalar field but can be arbitrarily large on the Maxwell field. This extends the previous result of Lindblad and Sterbenz, in which smallness was assumed both for the scalar field and the Maxwell field.

1. Introduction

In this paper, we study the pointwise decay of solutions to the Maxwell–Klein–Gordon equations on $\mathbb{R}^{3+1}$ with large Cauchy data. To define the equations, let $A = A_\mu\, dx^\mu$ be a 1-form. The covariant derivative associated to this 1-form is

$$D_\mu = \partial_\mu + \sqrt{-1}\, A_\mu,$$

which can be viewed as a $U(1)$ connection on the complex line bundle over $\mathbb{R}^{3+1}$ with the standard flat metric $m_{\mu\nu}$. Then the curvature 2-form F associated to this connection is given by

$$F_{\mu\nu} = -\sqrt{-1}\,[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu = (dA)_{\mu\nu}.$$

This is a closed 2-form, that is, F satisfies the Bianchi identity

$$\partial_\gamma F_{\mu\nu} + \partial_\mu F_{\nu\gamma} + \partial_\nu F_{\gamma\mu} = 0. \tag{1}$$

The Maxwell–Klein–Gordon equations (MKG) is a system for the connection field A and the complex scalar field ϕ:

$$\begin{cases} \partial^\nu F_{\mu\nu} = \Im(\phi \cdot \overline{D_\mu \phi}) = J_\mu, \\ D^\mu D_\mu \phi = \Box_A \phi = 0. \end{cases} \tag{MKG}$$

These are Euler–Lagrange equations of the functional

$$L[A,\phi] = \int_{\mathbb{R}^{3+1}} \left(\tfrac{1}{4} F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{2} D_\mu\phi \overline{D^\mu \phi}\right) dx\, dt.$$

A basic feature of this system is that it is gauge invariant under the gauge transformation

$$\phi \mapsto e^{i\chi}\phi, \qquad A \mapsto A - d\chi.$$

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More precisely, if (A, ϕ) solves (MKG), then $(A - d\chi, e^{i\chi}\phi)$ is also a solution for any potential function χ. Note that $U(1)$ is abelian. The Maxwell field F is invariant under the above gauge transformation, and (MKG) is said to be an *abelian gauge theory*. For the more general theory when $U(1)$ is replaced by a compact Lie group, the corresponding equations are referred to as *Yang–Mills–Higgs equations*.

In this paper, we consider the Cauchy problem to (MKG). The initial data set (E, H, ϕ_0, ϕ_1) consists of the initial electric field E and the magnetic field H, together with initial data (ϕ_0, ϕ_1) for the scalar field. In terms of the solution (F, ϕ), on the initial hypersurface, these are

$$F_{0i} = E_i, \quad {}^*F_{0i} = H_i, \quad \phi(0, x) = \phi_0, \quad D_t\phi(0, x) = \phi_1,$$

where ${}^*F$ is the Hodge dual of the 2-form F. In local coordinates (t, x),

$$(H_1, H_2, H_3) = (F_{23}, F_{31}, F_{12}).$$

The data set is said to be *admissible* if it satisfies the compatibility condition

$$\operatorname{div}(E) = \Im(\phi_0 \cdot \overline{\phi_1}) = J_0|_{t=0}, \quad \operatorname{div}(H) = 0, \tag{2}$$

where the divergence is taken on the initial hypersurface $\mathbb{R}^3$. For solutions of (MKG), the energy

$$E[F, \phi](t) := \int_{\mathbb{R}^3} |E|^2 + |H|^2 + |D\phi|^2\, dx$$

is conserved. Another important conserved quantity is the total charge

$$q_0 = \frac{1}{4\pi} \int_{\mathbb{R}^3} \Im(\phi \cdot \overline{D_t\phi})\, dx = \frac{1}{4\pi} \int_{\mathbb{R}^3} \operatorname{div}(E)\, dx, \tag{3}$$

which can be defined at any fixed time t. The existence of nonzero charge plays a crucial role in the asymptotic behavior of solutions of (MKG). It makes the analysis more complicated and subtle. This is obvious from the above definition as the electric field $E_i = F_{0i}$ has a tail $q_0 r^{-3} x_i$ at any fixed time t.

The Cauchy problem to (MKG) has been studied extensively. One of the most remarkable results is due to Eardley and Moncrief [1982a; 1982b], in which it was shown that there is always a global solution to the general Yang–Mills–Higgs equations for sufficiently smooth initial data. This was later improved to data merely bounded in the energy space for MKG by Klainerman and Machedon [1994] and for the nonabelian case of Yang–Mills equations in, e.g., [Klainerman and Machedon 1995; Oh 2015; Selberg and Tesfahun 2010]. Since then there has been extensive literature on generalizations and extensions of this classical result, aiming at improving the regularity of the initial data in order to construct a global solution; see [Krieger et al. 2015; Keel et al. 2011; Krieger and Lührmann 2015; Machedon and Sterbenz 2004; Oh and Tataru 2016; Rodnianski and Tao 2004] and references therein. A common feature of all these works is to construct a local solution with rough data. Then the global well-posedness follows by establishing a priori bounds for some appropriate norms of the solution. For example, a local solution was constructed in [Eardley and Moncrief 1982a], while in [Eardley and Moncrief 1982b], they showed that the L^∞ norm of the solution never blows up even though it may grow in time t. As a consequence, the solution can be extended to all time; however, the decay property of the solution is unknown. In view of this, although the solution of (MKG) exists globally with rough initial data, very little is known about the decay properties.

Asymptotic behavior and decay estimates are well understood for linear fields (see, e.g., [Christodoulou and Klainerman 1990]) and nonlinear fields with sufficiently small initial data (see, e.g., [Choquet-Bruhat and Christodoulou 1981b; Shu 1991]). These results rely on the conformal symmetry of the system, either by conformally compactifying the Minkowski space or by using the conformal Killing vector field $(t^2+r^2)\partial_t+2tr\partial_r$ as multiplier. Nevertheless the use of the conformal symmetry requires strong decay of the initial data, and thus in general does not allow the presence of nonzero charge except when the initial data are essentially compactly supported. For the case with nonzero charge, the first related work regarding the asymptotic properties was due to W. Shu [1992]. However, that work only considered the case when the solution is trivial outside a fixed forward light cone. Details for the general case were not carried out. A complete proof towards this program was later contributed by Lindblad and Sterbenz [2006]; also see the more recent work [Bieri et al. 2014].

The presence of nonzero charge has a long range effect on the asymptotic behavior of the solutions, at least in a neighborhood of the spatial infinity. This can be seen from the conservation law of the total charge as the electric field E decays at most r^{-2} as $r\to\infty$ at any fixed time. This weak decay rate makes the analysis more complicated even for small initial data. To deal with this difficulty, Lindblad–Sterbenz decomposed the Maxwell field into charged and chargeless components (see discussions in the end of this section) and made use of the fractional Morawetz estimates obtained by using the vector fields $u^p\partial_u+v^p\partial_v$ as multipliers. The latter work [Bieri et al. 2014] relied on the observation that the angular derivative of the Maxwell field has zero charge. The Maxwell field then can be estimated by using the Poincaré inequality.

The asymptotic behavior of solutions of MKG with general large data remains unknown until recently in [Yang 2015c] quantitative decay estimates were obtained for solutions with data bounded in some weighted energy space. Pointwise decay requires the energy estimates for the derivatives of the solution. However, commuting the equations with derivatives generates nonlinear terms. The aim of this paper is to identify a class of large data for MKG equations such that we can derive the pointwise decay of the solutions.

We define some necessary notations in order to state our main result. We use the standard polar local coordinate system (t,r,ω) of Minkowski space as well as the null coordinates $u=\frac{1}{2}(t-r)$, $v=\frac{1}{2}(t+r)$. Let $\overline{\nabla}$ denote the covariant derivative on $\mathbb{R}^3$ and Ω be the set of angular momentum vector fields $\Omega_{ij}=x_i\partial_j-x_j\partial_i$. Without loss of generality we only prove estimates in the future, i.e., $t\geq 0$. Next we introduce a null frame $\{L,\underline{L},e_1,e_2\}$, where

$$L=\partial_v=\partial_t+\partial_r,\quad \underline{L}=\partial_u=\partial_t-\partial_r$$

and $\{e_1,e_2\}$ is an orthonormal basis of the sphere with constant radius r. We use $\not{D}$ to denote the covariant derivative associated to the connection field A on the sphere with radius r. For any 2-form F, denote the null decomposition under the above null frame by

$$\alpha_i=F_{Le_i},\quad \underline{\alpha}_i=F_{\underline{L}e_i},\quad \rho=\tfrac{1}{2}F_{\underline{L}L},\quad \sigma=F_{e_1e_2},\quad i\in\{1,2\}. \tag{4}$$

We assume that the initial data set (E, H, ϕ_0, ϕ_1) is admissible. Let q_0 be the charge defined in (3) which is uniquely determined by the initial data of the scalar field (ϕ_0, ϕ_1). We assume that the data for the scalar field is small, but the data for the Maxwell field is arbitrary. However the data cannot be assigned freely. They satisfy the compatibility condition (2). To measure the size of the initial data for the scalar field and the Maxwell field, let $(E^{\mathrm{df}}, E^{\mathrm{cf}})$ be the Hodge decomposition of the electronic field E with E^{df} the divergence-free part and E^{cf} the curl-free part. Then the compatibility condition (2) on E is equivalent to

$$\operatorname{div} E^{\mathrm{cf}} = \Im(\phi_0 \cdot \overline{\phi_1}).$$

This implies that E^{cf} can be uniquely determined by (ϕ_0, ϕ_1) (with a suitable decay assumption on E). Therefore, for the initial data set (E, H, ϕ_0, ϕ_1) for (MKG) we can freely assign ϕ_0, ϕ_1 and E^{df}, H as long as $\operatorname{div} H = 0$, $\operatorname{div} E^{\mathrm{df}} = 0$. The total charge q_0 is a constant determined by (ϕ_0, ϕ_1).

We now define the norms of the initial data. For some positive constant $0 < \gamma_0 < 1$, we define the second-order weighted Sobolev norm respectively for the initial data of the Maxwell field (E, H) and the initial data of the scalar field (ϕ_0, ϕ):

$$\mathcal{M} := \sum_{l\le 2} \int_{\mathbb{R}^3} (1+r)^{1+\gamma_0}\big(|\Omega^l E^{\mathrm{df}}|^2 + |\Omega^l H|^2 + |\overline{\nabla}^l E^{\mathrm{df}}|^2 + |\overline{\nabla}^l H|^2\big)\, dx,$$

$$\mathcal{E} := \sum_{l\le 2} \int_{\mathbb{R}^3} (1+r)^{1+\gamma_0}\big(|\overline{\nabla}\Omega^l \phi_0|^2 + |\Omega^l \phi_1|^2 + |\overline{\nabla}^{l+1}\phi_0|^2 + |\overline{\nabla}^l \phi_1|^2 + |\phi_0|^2\big)\, dx.$$

We remark here that the definition for $\mathcal{E}$ is not gauge invariant. The gauge invariant norm depends on the connection field A, which up to a gauge transformation can be determined by the initial data of the Maxwell field (E^{df}, H). However, in our setting $\mathcal{M}$ is arbitrarily large while $\mathcal{E}$ is assumed to be small depending on $\mathcal{M}$. To measure the smallness of the scalar field, we choose the above gauge dependent norm for the scalar field. We will show later (see Lemma 58 in Section 5) that the gauge invariant norm is in fact equivalent to the above Sobolev norm up to a constant depending only on $\mathcal{M}$.

We now can state our main theorem:

Theorem 1. *Consider the Cauchy problem to (MKG) with admissible initial data set (E, H, ϕ_0, ϕ_1). There exists a positive constant ϵ_0, depending on $\mathcal{M}$ and γ_0, such that for all $\mathcal{E} < \epsilon_0$, the solution (F, ϕ) of (MKG) satisfies the following decay estimates*:

$$\begin{aligned}
|D_{\underline{L}}(r\phi)|^2(u, v, \omega) &\le C\mathcal{E}(1+|u|)^{-1-\gamma_0}, && |r\underline{\alpha}|^2(u, v, \omega) \le C(1+|u|)^{-1-\gamma_0};\\
r^p(|D_L(r\phi)|^2 + |\not{D}(r\phi)|^2)(u, v, \omega) &\le C\mathcal{E}(1+|u|)^{p-1-\gamma_0}, && 0 \le p \le 1+\gamma_0;\\
r^p(|r\alpha|^2 + |r\sigma|^2)(u, v, \omega) &\le C(1+|u|)^{p-1-\gamma_0}, && 0 \le p \le 1+\gamma_0;\\
r^{p+2}|\rho - q_0 r^{-2}\chi_{\{t+R\le r\}}|^2(u, v, \omega) &\le C(1+|u|)^{p-1-\gamma_0}, && 0 \le p < 1;\\
r^p|\phi|^2(u, v, \omega) &\le C\mathcal{E}(1+|u|)^{p-2-\gamma_0}, && 1 \le p \le 2;\\
|D\phi|^2(t, x) + |\phi|^2(t, x) &\le C\mathcal{E}(1+t)^{-1-\gamma_0}, && |F|^2(t, x) \le C(1+t)^{-1-\gamma_0},\ \forall |x| \le R;
\end{aligned}$$

for all $(u, v, \omega) \in \mathbb{R}^{3+1} \cap \{|x| \geq R\}$ *and for some constant* C *depending on* $\mathcal{M}$, γ_0, p. *Here* q_0 *is the total charge and* $\chi_{\{t+2\leq r\}}$ *is the characteristic function on the exterior region* $\{t+2 \leq r\}$.

We make several remarks.

Remark 2. The second-order derivatives of the initial data are the minimum regularity we need to derive the above pointwise decay of the solution. Similar decay estimates hold for the higher-order derivatives of the solution if higher-order weighted Sobolev norms of the initial data are known.

Remark 3. The restriction $0 < \gamma_0 < 1$ on γ_0 is merely for the sake of brevity. If $\gamma_0 \geq 1$, then the decay property of the solutions propagates in the exterior region ($t+2 \leq r$). In other words, we have the same decay estimates as in the theorem for $\tau \leq 0$. However in the interior region where $\tau > 0$, the maximal decay rate is τ_+^{-2} (corresponding to $\gamma_0 = 1$), that is, the decay rate in the interior region for $\gamma_0 \geq 1$ in general cannot be better than that of $\gamma_0 = 1$.

Compared to the previous result of Lindblad and Sterbenz [2006], we have made the following improvements: first of all, we obtain pointwise decay estimates for solutions of (MKG) for a class of large initial data. We only require smallness on the scalar field. In particular our initial data for (MKG) can be arbitrarily large. Combining the method in [Yang 2015a], we can even make the data on the scalar field large in the energy space. Secondly, we have lower regularity on the initial data. In [Lindblad and Sterbenz 2006], it was assumed that the derivative of the initial data decays one order better, that is, $\overline{\nabla}^k(E^{\mathrm{df}}, H)$, $D^k(D\phi_0, \phi_1)$ belong to the weighted Sobolev space with weights $(1+r)^{1+\gamma_0+2|k|}$, while in this paper we only assume that the angular derivatives of the data obey this improved decay (see the definition of $\mathcal{M}$, $\mathcal{E}$). For the other derivatives, the weight is merely $(1+r)^{1+\gamma_0}$. This makes the analysis more delicate. Moreover, as the solution decays weaker initially, our decay rate is weaker than that in [Lindblad and Sterbenz 2006] (only decay rate in u, the decay in r is the same). However if we assume the same decay of the initial data as in [Lindblad and Sterbenz 2006], then we are able to obtain the same decay for the solution.

We use a new approach developed in [Yang 2015c] to study the asymptotic behavior of solutions of (MKG). This new method was originally introduced by Dafermos and Rodnianski [2010] for the study of decay of linear waves on black hole spacetimes. This novel method starts by proving the energy flux decay of the solutions of linear equations through the forward light cone Σ_τ (see definitions in Section 2). The pointwise decay then follows by commuting the equation with ∂_t and the angular momentum Ω. In the abstract framework set by Dafermos and Rodnianski [2010], the energy flux decay relies on three kinds of basic ingredients and estimates: a uniform energy bound, an integrated local energy decay estimate and a hierarchy of r-weighted energy estimates in a neighborhood of the null infinity, which can be obtained by using the vector fields ∂_t, $f(r)\partial_r$, $r^p(\partial_t + \partial_r)$ as multipliers, respectively. Combining these three estimates, a pigeonhole argument then leads to the energy flux decay.

As the initial data for the scalar field is small, we can use the perturbation method to prove the pointwise decay of the solution. With a suitable bootstrap assumption on the nonlinearity $J[\phi] = \Im(\phi \cdot \overline{D\phi})$, we first can use the new method to prove energy decay estimates for the Maxwell field up to the second-order derivatives. Once we have these decay estimates for the Maxwell field, we then can show the energy decay as well as pointwise decay for the scalar field, which can then be used to improve the bootstrap assumption. The smallness of the scalar field is used here to close the bootstrap assumptions.

The existence of nonzero charge has a long-range effect on the asymptotic behavior of the solution in the exterior region $\{t+2 \le r\}$, which has been discussed in [Yang 2015c] when the charge is large. To deal with this difficulty, we define the chargeless 2-form

$$\bar{F} = F - \chi_{\{t+2\le r\}} q_0 r^{-2}\, dt \wedge dr.$$

We first carry out estimates for $\bar{F}$ on the exterior region $\{t+2 \le r\}$, which in particular controls the energy flux through $\{t+2 = r\}$ (the intersection of the interior region and the exterior region). We then can use the new method to obtain estimates for the Maxwell field F in the interior region. The Maxwell equation commutes with the Lie derivatives of F (see Lemma 4). It is not hard to obtain energy decay estimates for the derivatives of the Maxwell field under suitable bootstrap assumptions on the nonlinearity $J[\phi]$ by using the new approach.

The main difficulty lies in showing the energy decay estimates for the scalar field due to the fact that the covariant derivative D does not commute with the covariant wave operator $\Box_A$. The interaction terms of the Maxwell field and the scalar field arise from the commutator. To control those interaction terms, previous results [Bieri et al. 2014; Lindblad and Sterbenz 2006] rely on the smallness of the Maxwell field, and those terms could be absorbed. The key observation allowing the Maxwell field to be large in this paper is that the robust new method makes use of the decay in u (equivalent to τ up to a constant) and those terms could be controlled using Gronwall's inequality without any smallness assumption on the Maxwell field. Traditionally, Gronwall's inequality is used with respect to the foliation $t =$ constant. Therefore strong decay in t is necessary. As the new method foliates the spacetime by using the null hypersurfaces H_u, it enables us to make use of the weaker decay in u in order to apply Gronwall's inequality.

The paper is organized as follows. We define additional notations and derive the transport equations for the curvature components of the Maxwell field in Section 2. Since we only commute the equations with ∂_t or the angular momentum Ω, these transport equations will be used to recover the missing derivative in order to derive pointwise estimates for the Maxwell field. Section 3 is devoted to reviewing the energy estimates (an integrated local energy estimate and a hierarchy of r-weighted energy estimates) both for the scalar field verifying the linear covariant wave equation $\Box_A \phi = 0$ and the linear Maxwell field. The idea to prove these estimates is very similar to that in the author's other preprint [Yang 2015c], in which decay properties of solutions of MKG are discussed with data merely bounded in some weighted energy space. There the energy estimates are carried out for the full solution (A, ϕ) of the nonlinear MKG equations, and one of the difficulties is to deal with the arbitrarily large charge q_0. This paper aims at the pointwise decay of the solutions with some special initial data. In particular, energy decay estimates are also necessary for the derivatives of the solutions. We thus need energy estimates for the linearized equations. To make this paper self-contained, we give detailed proof for these energy estimates in Section 3. In Section 4, we use the new method to obtain decay estimates for the linear Maxwell field and the linear scalar field. More specifically, in Section 4.1, we derive energy flux decay estimates for the linear Maxwell fields under suitable assumptions on the inhomogeneous term $J_\mu = \nabla^\nu F_{\mu\nu}$. Then in Section 4.2, we obtain pointwise decay estimates by commuting the equation with vector fields in $\Gamma = \{\partial_t, \Omega\}$ merely twice. This lower regularity result relies on the elliptic estimates in the bounded region $\{r \le R\}$ and the transport equations for the curvature components when $r \ge R$. The most technical part of this paper lies in Section 4.3, in which energy decay estimates are obtained for the scalar field up to second-order derivatives. The main difficulty is that the covariant wave operator $\Box_A$ does not commute with the covariant derivative D. It heavily relies on the null structure of the commutators. Finally, in Section 5 we improve the bootstrap assumption and conclude our main theorem.

2. Preliminaries and notations

We define some additional notation used in the sequel. Recall the null frame $\{L, \underline{L}, e_1, e_2\}$ defined in the introduction. At any fixed point (t, x), we may choose e_1, e_2 such that

$$[L, e_i] = -\frac{1}{r} e_i, \quad [\underline{L}, e_i] = \frac{1}{r} e_i, \quad [e_1, e_2] = 0, \quad i \in \{1, 2\}.$$

This helps to compute those geometric quantities which are independent of the choice of the local coordinates. We then can compute the covariant derivatives for the null frame at any fixed point:

$$\begin{aligned} &\nabla_L L = 0, \quad \nabla_L \underline{L} = 0, \quad \nabla_L e_i = 0, \quad \nabla_{\underline{L}} \underline{L} = 0, \quad \nabla_{\underline{L}} e_i = 0, \\ &\nabla_{e_i} L = r^{-1} e_i, \quad \nabla_{e_i} \underline{L} = -r^{-1} e_i, \quad \nabla_{e_1} e_2 = \nabla_{e_2} e_1 = 0, \quad \nabla_{e_i} e_i = -r^{-1} \partial_r. \end{aligned} \tag{5}$$

Here ∇ is the covariant derivatives in Minkowski space and $\overline{\nabla}$ is the spatial component. We also use ∂ to abbreviate the partial derivatives $(\partial_t, \partial_1, \partial_2, \partial_3)$ in Minkowski space under the coordinates (t, x) and $\not{\nabla}$ to denote the covariant derivative on the sphere with radius r.

Now we define the foliation of the spacetime $\{t \ge 0\}$. Let H_u be the outgoing null hypersurface $\{t - r = 2u\}$ and $\underline{H}_v$ be the incoming null hypersurface $\{t + r = 2v\}$. Let $R > 1$ be a fixed constant. We now use this fixed constant R to define the foliation. For all $\tau \in \mathbb{R}$, denote

$$\tau^* = \frac{\tau - R}{2}.$$

In the exterior region where $t + R \le r$, we use the foliation

$$\Sigma_\tau := H_{\tau^*} \cap \{t \ge 0\}, \quad \tau \le 0,$$

while in the interior region where $t + R \ge r$, the foliation is defined as

$$\Sigma_\tau := \{t = \tau, |x| \le R\} \cup (H_{\tau^*} \cap \{|x| \ge R\}).$$

Unless we specify it, in the following the outgoing null hypersurface H_u stands for $H_u \cap \{t \ge 0\}$ in the exterior region and $H_u \cap \{|x| \ge R\}$ in the interior region. Note that the boundary of the region bounded by Σ_{τ_1} and Σ_{τ_2} is part of the future null infinity where the decay behavior of the solution is unknown. To make the energy estimates rigorous, we instead consider the finite truncated hypersurfaces

$$\Sigma_\tau^{v_0} := \Sigma_\tau \cap \{v \le v_0\}, \quad H_u^{v_0} := H_u \cap \{v \le v_0\}, \quad \underline{H}_v^{u_1, u_2} := \underline{H}_v \cap \{u_1 \le u \le u_2\}.$$

On the initial hypersurface $\{t=0\}$, we denote the annulus with radii $r_1 < r_2$ by

$$B_{r_1}^{r_2} = \{r_1 \le |x| \le r_2\}, \quad B_r = B_r^{\infty}.$$

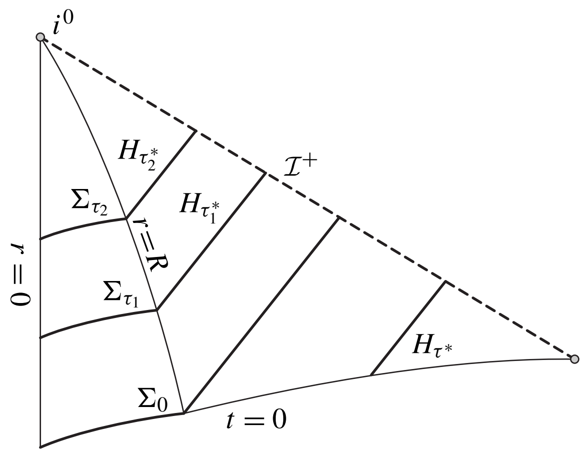
Next we define the domains. In the exterior region, for $\tau_2 \le \tau_1 \le 0$, define $\mathcal{D}_{\tau_1}^{\tau_2}$ to be the Cauchy development of the annulus $\{R-\tau_1 \le |x| \le R-\tau_2\}$, or more precisely

$$\mathcal{D}_{\tau_1}^{\tau_2} = \left\{(t,x) \mid \left||x| + \tau_1^* + \tau_2^*\right| + t \le \tau_1^* - \tau_2^*\right\}, \quad \tau_2 < \tau_1 \le 0,$$

while in the interior region, for any $\tau_2 > \tau_1 \ge 0$, we define $\mathcal{D}_{\tau_1}^{\tau_2}$ to be the region

$$\mathcal{D}_{\tau_1}^{\tau_2} = \{(t,x) \mid (t,x) \in \Sigma_\tau,\ 0 \le \tau_1 \le \tau \le \tau_2\}.$$

bounded by Σ_{τ_1} and Σ_{τ_2}. We may also use $\overline{\mathcal{D}}_{\tau_1}^{\tau_2} = \mathcal{D}_{\tau_1}^{\tau_2} \cap \{|x| \ge R\}$ to denote the region outside the cylinder $\{r \le R\}$. Finally, we write $\mathcal{D}_\tau$ for the region $\mathcal{D}_\tau^{+\infty}$ if $\tau \ge 0$ or the region $\mathcal{D}_\tau^{-\infty}$ when $\tau < 0$. The following Penrose diagram may be of help for the various pieces of notation described above.



We use $E[\phi](\Sigma)$ to denote the energy flux of the complex scalar field ϕ and $E[F](\Sigma)$ for the energy flux of the 2-form F through the hypersurface Σ in Minkowski space. The derivative on the scalar field is with respect to the covariant derivative D. For our interested hypersurfaces, we can compute

$$\begin{aligned}
E[\phi](\Sigma_\tau) &= \int_{\{t=\tau, r\le R\}} |D\phi|^2\,dx + \int_{H_{\tau^*}} (|D_L\phi|^2 + |\not{D}\phi|^2) r^2\,dv\,d\omega,\\
E[F](\Sigma_\tau) &= \int_{\{t=\tau, r\le R\}} \rho^2 + |\sigma|^2 + \frac{1}{2}(|\alpha|^2 + |\underline{\alpha}|^2)\,dx + \int_{H_{\tau^*}} (\rho^2 + |\sigma|^2 + |\alpha|^2) r^2\,dv\,d\omega,\\
E[\phi](\underline{H}_{\underline{u}}) &= \int_{\underline{H}_{\underline{u}}} (|D_{\underline{L}}\phi|^2 + |\not{D}\phi|^2) r^2\,dv\,d\omega, \quad E[F](\underline{H}_{\underline{u}}) = \int_{\underline{H}_{\underline{u}}} (\rho^2 + |\sigma|^2 + |\underline{\alpha}|^2) r^2\,dv\,d\omega.
\end{aligned}$$

Here $\rho, \sigma, \alpha, \underline{\alpha}$ are the null components of the 2-form F defined in line (4), and we recall that $\tau^* = \frac{1}{2}(\tau - R)$. Since we only consider estimates in the future when $t \ge 0$, the set $\{t=\tau, r\le R\}$ should be interpreted as the empty set when $\tau < 0$.

Next we define some useful weighted Sobolev norms either on domains or on surfaces. For any function f (scalar or vector valued or tensors) we denote the spacetime integral on $\mathcal{D}$ in Minkowski space

$$I_q^p[f](\mathcal{D}) := \int_{\mathcal{D}} u_+^q r_+^p |f|^2, \quad r_+ = 1+r, \quad u_+ = 1+|u|$$

for any real numbers p, q. Here $\mathcal{D}$ can be the domain or hypersurface in the Minkowski space. For example, when $\mathcal{D}$ is H_u, then

$$I_q^p[f](H_u) := \int_{H_u} r_+^p u_+^q |f|^2 r^2 \, dv \, d\omega.$$

To define the norms of the derivatives of the solution, we need vector fields used as commutators which, in this paper, are the Killing vector field ∂_t together with the angular momentum Ω with components $\Omega_{ij} = x_i \partial_j - x_j \partial_i$. We define the set

$$\Gamma = \{\partial_t, \Omega_{ij}\}.$$

For the scalar field, it is natural to take the covariant derivative $D_X = X^\mu D_\mu$ associated to the connection A for any vector field $X = X^\mu \partial_\mu$. This covariant derivative has already been defined for the purpose of defining the equations in the beginning of the introduction. For the Maxwell field F, which is a 2-form, we define the Lie derivative

$$(\mathcal{L}_Z F)_{\mu\nu} = Z(F_{\mu\nu}) - F(\mathcal{L}_Z \nabla_\mu, \nabla_\nu) - F(\nabla_\mu, \mathcal{L}_Z \nabla_\nu), \quad (\mathcal{L}_Z J)_\mu = Z(J_\mu) - J(\mathcal{L}_Z \nabla_\mu)$$

for any 2-form F and any 1-form J, respectively. Here $\mathcal{L}_Z X = [Z, X]$ for all vector fields Z, X.

If the vector field Z is Killing, that is, $\nabla^\mu Z^\nu + \nabla^\nu Z^\mu = 0$ for all μ, ν, then we can show that

$$\begin{aligned}
\nabla^\mu (\mathcal{L}_Z F)_{\mu\nu} &= Z(\nabla^\mu F_{\mu\nu}) + \nabla^\mu Z^\gamma \nabla_\gamma F_{\mu\nu} + \nabla_\mu Z^\gamma \nabla^\mu F_{\gamma\nu} + \nabla_\nu Z^\gamma \nabla^\mu F_{\mu\gamma} \\
&= Z(\nabla^\mu F_{\mu\nu}) + \nabla_\nu Z^\gamma \nabla^\mu F_{\mu\gamma} = (\mathcal{L}_Z \delta F)_\nu.
\end{aligned}$$

Here we denote $\delta F_\nu = \nabla^\mu F_{\mu\nu}$ as the divergence of the 2-form F. We use $\mathcal{L}_Z^k$ or D_Z^k to denote the k-th derivatives, that is,

$$\mathcal{L}_Z^k = \mathcal{L}_{Z^1} \mathcal{L}_{Z^2} \cdots \mathcal{L}_{Z^k}.$$

Similarly for D_Z^k. The vector fields Z^j are any vector fields in the set $\Gamma = \{\partial_t, \Omega_{ij}\}$.

Based on these calculations, we have the following commutator lemma.

Lemma 4. *For any Killing vector field Z, we have*

$$\begin{aligned}
[\Box_A, D_Z]\phi &= 2i Z^\nu F_{\mu\nu} D^\mu \phi + i \nabla^\mu (Z^\nu F_{\mu\nu})\phi, \\
\nabla^\mu (\mathcal{L}_Z G)_{\mu\nu} &= (\mathcal{L}_Z \delta G)_\nu
\end{aligned}$$

for any complex scalar field ϕ and any 2-form G.

For the energy estimates of the solutions of (MKG), the initial energies $\mathcal{M}$, $\mathcal{E}$ defined in the introduction cannot be used directly as $\mathcal{E}$ is not gauge invariant. Note that the vector fields used as commutators are $\Gamma = \{\partial_t, \Omega\}$. For any 2-form F satisfying the Bianchi identity and any scalar field ϕ, for the given connection field A, we define the weighted k-th order initial energies

$$
\begin{aligned}
E_0^k[F] &:= \sum_{l \le k} \int_{\mathbb{R}^3} r_+^{1+\gamma_0} |\mathcal{L}_Z^l F|^2(0, x)\, dx, \\
E_0^k[\phi] &:= \sum_{l \le k, j \le 3} \int_{\mathbb{R}^3} r_+^{1+\gamma_0} |D_Z^l D_j \phi|^2(0, x)\, dx.
\end{aligned} \tag{6}
$$

Here D_j denotes the spatial covariant derivative and $0 < \gamma_0 < 1$ is the constant in the main theorem. We remark here that F may not be the full Maxwell field of the solution of (MKG). In application, it can be the chargeless part of the full solution which also satisfies the Bianchi identity. However, the connection field A is associated to the full Maxwell field. In fact the full Maxwell field does not belong to this weighted Sobolev space due to the existence of nonzero charge.

We end this section by writing the Maxwell equation under the null frame $\{L, \underline{L}, e_1, e_2\}$. In other words, we derive the transport equations for the curvature components. Let $F_{\mu\nu}$ be the 2-form verifying the Bianchi identity. Let $J = \delta F$, that is, $J_\mu = \nabla^\nu F_{\mu\nu}$.

Lemma 5. *Under the null frame* $\{L, \underline{L}, e_1, e_2\}$, *the MKG equations are the following transport equations for the curvature components*:

$$\underline{L}(r^2\rho) - \operatorname{d\!\!/iv}(r^2\underline{\alpha}) = r^2 J_{\underline{L}}, \quad L(r^2\rho) + \operatorname{d\!\!/iv}(r^2\alpha) = r^2 J_L, \tag{7}$$

$$\nabla_L(r\underline{\alpha}_i) - r\nabla\!\!\!/_{e_i}\rho - r\nabla\!\!\!/_{e_j} F_{e_i e_j} = r J_{e_i}, \quad i = 1, 2, \tag{8}$$

$$\underline{L}(r^2\sigma) = r^2(e_2\underline{\alpha}_1 - e_1\underline{\alpha}_2), \quad L(r^2\sigma) = r^2(e_2\alpha_1 - e_1\alpha_2), \tag{9}$$

$$\nabla_{\underline{L}}(r\alpha_i) + r\nabla\!\!\!/_{e_i}\rho - r\nabla\!\!\!/_{e_j} F_{e_i e_j} = r J_{e_i}, \quad i = 1, 2. \tag{10}$$

Here $\operatorname{d\!\!/iv}$ *is the divergence operator on the sphere with radius* r.

Proof. From the Maxwell equation, $J_L = (\delta F)(L)$. Use the formula

$$(\nabla_X F)(Y, Z) = X F(Y, Z) - F(\nabla_X Y, Z) - F(Y, \nabla_X Z)$$

for all vector fields X, Y, Z. By using (5), we then can compute

$$
\begin{aligned}
-(\delta F)(\underline{L}) &= -\tfrac{1}{2}(\nabla_L F)(\underline{L}, \underline{L}) - \tfrac{1}{2}(\nabla_{\underline{L}} F)(L, \underline{L}) + (\nabla_{e_i} F)(e_i, \underline{L}) \\
&= \underline{L}\rho - e_i \underline{\alpha}_i - F(-2r^{-1}\partial_r, \underline{L}) - F(e_i, -r^{-1} e_i) \\
&= \underline{L}\rho - 2r^{-1}\rho - \operatorname{d\!\!/iv}(\underline{\alpha}).
\end{aligned}
$$

Multiply both sides by r^2. We then get the first equation of (7). The second equation follows similarly.

For (8) and (10), we need to use the Bianchi identity (1) which is equivalent to

$$(\nabla_X F)(Y, Z) + (\nabla_Y F)(Z, X) + (\nabla_Z F)(X, Y) = 0$$

for all vector fields X, Y, Z. Let's only prove (8). We can show that

$$\begin{aligned}
-(\delta F)(e_i) &= -\tfrac{1}{2}(\nabla_L F)(\underline{L}, e_i) - \tfrac{1}{2}(\nabla_{\underline{L}} F)(L, e_i) + (\nabla_{e_j} F)(e_j, e_i)\\
&= -\tfrac{1}{2}L\underline{\alpha}_i + \tfrac{1}{2}(\nabla_L F)(e_i, \underline{L}) + \tfrac{1}{2}(\nabla_{e_i} F)(\underline{L}, L) + e_j F_{e_j e_i} - F(-2r^{-1}\partial_r, e_i) - F(e_i, -r^{-1}\partial_r)\\
&= -L\underline{\alpha}_i + e_i\rho - \tfrac{1}{2}F(-r^{-1}e_i, L) - \tfrac{1}{2}F(\underline{L}, r^{-1}e_i) + e_j F_{e_j e_i} + r^{-1}F(\partial_r, e_i)\\
&= -L\underline{\alpha}_i + e_i\rho + e_j F_{e_j e_i} - r^{-1}\underline{\alpha}_i.
\end{aligned}$$

This leads to (8).

The first transport equation (9) for σ follows from the Bianchi identity:

$$\begin{aligned}
0 &= (\nabla_{\underline{L}} F)(e_1, e_2) + (\nabla_{e_1} F)(e_2, \underline{L}) + (\nabla_{e_2} F)(\underline{L}, e_1)\\
&= \underline{L}\sigma - e_1\underline{\alpha}_2 - F(e_2, -r^{-1}e_1) + e_2\underline{\alpha}_1 - F(-r^{-1}e_2, e_1)\\
&= \underline{L}\sigma - e_1\underline{\alpha}_2 + e_2\underline{\alpha}_1 - 2r^{-1}\sigma.
\end{aligned}$$

The dual one follows if we replace $\underline{L}$ with L. □

3. Energy method

In this section, we review the energy method for solutions of the covariant linear wave equations and Maxwell equations using the new method developed in [Yang 2015c]. This new method was originally introduced by Dafermos and Rodnianski [2010] for proving the decay of solutions of linear wave equations in Minkowski space. It has been successfully applied to MKG equations by the author in [Yang 2015c] to obtain the decay properties of the solutions for all data bounded in some weighted energy space. There the necessary new ingredients (see Propositions 7–10 in this section) were carried out for the full solution (ϕ, F). In this paper, the data for the scalar field are assumed to be small, and we also need to derive the decay estimates for the derivatives of the solutions in order to obtain the pointwise decay of the solutions. We thus need all the new ingredients both for the scalar field and the Maxwell field. The ideas to derive these new estimates are the same as those in [Yang 2015c]. For the readers' convenience, we repeat the proofs here.

3.1. ***Energy identity for the scalar field.*** Denote by $d\mathrm{vol}$ the volume form in the Minkowski space. In the local coordinate system (t, x), we have $d\mathrm{vol} = dx\,dt$. Here we have chosen t to be the time orientation. For any complex scalar field ϕ, we define the associated energy momentum tensor

$$T[\phi]_{\mu\nu} = \Re(\overline{D_\mu\phi}D_\nu\phi) - \tfrac{1}{2}m_{\mu\nu}\overline{D^\gamma\phi}D_\gamma\phi.$$

Here $m_{\mu\nu}$ is the flat metric of Minkowski spacetime and the covariant derivative D is defined with respect to the given connection field A. For any vector field X, we have the following identity

$$\nabla^\mu(T[\phi]_{\mu\nu}X^\nu) = \Re(\Box_A\phi X^\nu\overline{D_\nu\phi}) + X^\nu F_{\nu\gamma}J^\gamma[\phi] + T[\phi]^{\mu\nu}\pi^X_{\mu\nu},$$

where $\pi^X_{\mu\nu} = \tfrac{1}{2}\mathcal{L}_X m_{\mu\nu}$ is the deformation tensor of the vector field X in Minkowski space, $\Box_A$ is the covariant wave operator associated to the connection A, $F = dA$ is the exterior derivative of the 1-form

A which gives us a 2-form and $J^\gamma[\phi] = \Im(\phi \cdot \overline{D^\gamma \phi})$. For any function χ, we have

$$\tfrac{1}{2}\nabla^\mu(\chi \nabla_\mu |\phi|^2 - \nabla_\mu \chi\, |\phi|^2) = \chi\, \overline{D_\mu \phi} D^\mu \phi - \tfrac{1}{2}\Box \chi \cdot |\phi|^2 + \chi \Re(\Box_A \phi \cdot \bar{\phi}).$$

We now define the vector field $\tilde{J}^X[\phi]$ with components

$$\tilde{J}^X_\mu[\phi] = T[\phi]_{\mu\nu} X^\nu - \tfrac{1}{2}\nabla_\mu \chi \cdot |\phi|^2 + \tfrac{1}{2}\chi \nabla_\mu |\phi|^2 + Y_\mu \tag{11}$$

for some vector field Y which may also depend on the complex scalar field ϕ. We then have the equality

$$\nabla^\mu \tilde{J}^X_\mu[\phi] = \Re(\Box_A \phi(\overline{D_X \phi} + \chi \bar{\phi})) + \operatorname{div}(Y) + X^\nu F_{\nu\mu} J^\mu[\phi] + T[\phi]^{\mu\nu} \pi^X_{\mu\nu} + \chi\, \overline{D_\mu \phi} D^\mu \phi - \tfrac{1}{2}\Box \chi \cdot |\phi|^2.$$

Here the operator $\Box$ is the wave operator in Minkowski space. Now for any region $\mathcal{D}$ in $\mathbb{R}^{3+1}$, using Stokes' formula, we derive the energy identity

$$\begin{aligned}\iint_{\mathcal{D}} \Re(\Box_A \phi(\overline{D_X \phi} + \chi \bar{\phi})) + \operatorname{div}(Y) + X^\nu F_{\nu\gamma} J^\gamma[\phi] + T[\phi]^{\mu\nu} \pi^X_{\mu\nu} + \chi\, \overline{D_\mu \phi} D^\mu \phi - \tfrac{1}{2}\Box \chi \cdot |\phi|^2 \, d\mathrm{vol} \\ = \iint_{\mathcal{D}} \nabla^\mu \tilde{J}^X_\mu[\phi]\, d\mathrm{vol} = \int_{\partial \mathcal{D}} i_{\tilde{J}^X[\phi]}\, d\mathrm{vol},\end{aligned} \tag{12}$$

where $\partial\mathcal{D}$ denotes the boundary of the domain $\mathcal{D}$ and $i_Z\, d$vol denotes the contraction of the volume form $d$vol with the vector field Z which gives the surface measure of the boundary. For example, for any basis $\{e_1, e_2, \ldots, e_n\}$, we have

$$i_{e_1}(de_1 \wedge de_2 \wedge \cdots \wedge de_k) = de_2 \wedge de_3 \wedge \cdots \wedge de_k.$$

Throughout this paper, the domain $\mathcal{D}$ will be regular regions bounded by the t-constant slices, the outgoing null hypersurfaces H_u, the incoming null hypersurfaces $\underline{H}_v$ or the surface with constant r. We now compute $i_{\tilde{J}^X[\phi]}\, d$vol on each of these hypersurfaces.

On $t =$ constant slice, the surface measure is a function times dx. Recall the volume form

$$d\mathrm{vol} = dx\, dt = -dt\, dx.$$

Here note that dx is a 3-form. We thus can show that

$$i_{\tilde{J}^X[\phi]}\, d\mathrm{vol} = -(\tilde{J}^X[\phi])^0\, dx = -\big(\Re(\overline{D^t \phi} D_X \phi) - \tfrac{1}{2} X^0\, \overline{D^\gamma \phi} D_\gamma \phi - \tfrac{1}{2}\partial^t \chi\, |\phi|^2 + \tfrac{1}{2}\chi\, \partial^t |\phi|^2 + Y^0\big)\, dx. \tag{13}$$

On the surface with constant r, the surface measure is $r^2\, dt\, d\omega$. Therefore we have

$$i_{\tilde{J}^X[\phi]}\, d\mathrm{vol} = \big(\Re(\overline{D^r \phi} D_X \phi) - \tfrac{1}{2} X^r\, \overline{D^\gamma \phi} D_\gamma \phi - \tfrac{1}{2}\partial^r \chi\, |\phi|^2 + \tfrac{1}{2}\chi\, \partial^r |\phi|^2 + Y^r) r^2\, dt\, d\omega. \tag{14}$$

On the outgoing null hypersurface H_u, we can write the volume form

$$d\mathrm{vol} = dx\, dt = r^2\, dr\, dt\, d\omega = 2r^2\, dv\, du\, d\omega = -2\, du\, dv\, d\omega.$$

Here $d\omega$ is the standard surface measure on the unit sphere. Notice that $\underline{L} = \partial_u$. We can compute

$$i_{\tilde{J}^X[\phi]}\, d\mathrm{vol} = -2\big(\Re(\overline{D^{\underline{L}} \phi} D_X \phi) - \tfrac{1}{2} X^{\underline{L}}\, \overline{D^\gamma \phi} D_\gamma \phi - \tfrac{1}{2}\nabla^{\underline{L}} \chi\, |\phi|^2 + \tfrac{1}{2}\chi\, \nabla^{\underline{L}} |\phi|^2 + Y^{\underline{L}}\big) r^2\, dv\, d\omega. \tag{15}$$

Similarly, on the v-constant incoming null hypersurfaces $\underline{H}_{\underline{u}}$, we have

$$i_{\tilde{J}^X[\phi]}\, d\mathrm{vol} = 2\big(\Re(\overline{D^L\phi} D_X\phi) - \tfrac{1}{2}X^L \overline{D^\gamma\phi} D_\gamma\phi - \tfrac{1}{2}\nabla^L\chi|\phi|^2 + \tfrac{1}{2}\chi\nabla^L|\phi|^2 + Y^L\big) r^2\, du\, d\omega. \tag{16}$$

We remark here that the above formula hold for any vector fields X, Y and any function χ.

3.2. ***Energy identities for the Maxwell field.*** Let F be any 2-form satisfying the Bianchi identity (1). The associated energy momentum tensor is

$$T[F]_{\mu\nu} = F_{\mu\gamma}F_\nu^{\ \gamma} - \tfrac{1}{4}m_{\mu\nu}F_{\gamma\beta}F^{\gamma\beta}.$$

For any vector field X, we have the divergence formula

$$\nabla^\mu T[F]_{\mu\nu}X^\nu = \nabla^\mu F_{\mu\gamma}F_\nu^{\ \gamma}X^\nu + T[F]^{\mu\nu}\pi^X_{\mu\nu},$$

where as defined previously, $\pi^X_{\mu\nu} = \frac{1}{2}\mathcal{L}_X m_{\mu\nu}$ is the deformation tensor of the vector field X in Minkowski space. Define the vector field $J^X[F]$ by

$$J^X[F]_\mu = T[F]_{\mu\nu}X^\nu.$$

Then for any domain $\mathcal{D}$ in $\mathbb{R}^{3+1}$, we have the following energy identity for the Maxwell field F:

$$\iint_{\mathcal{D}} \nabla^\mu F_{\mu\gamma}F_\nu^{\ \gamma}X^\nu + T[F]^{\mu\nu}\pi^X_{\mu\nu}\, d\mathrm{vol} = \iint_{\mathcal{D}} \nabla^\mu J^X_\mu[F]\, d\mathrm{vol} = \int_{\partial\mathcal{D}} i_{J^X[F]}\, d\mathrm{vol}. \tag{17}$$

For the terms on the boundary, similar to (13)–(16), we can compute

$$\begin{aligned}
\int_{\{t=\text{const.}\}} i_{J^X[F]}\, d\mathrm{vol} &= -\int_{\{t=\text{const.}\}} \Big(F^{0\mu}F_{\nu\mu}X^\nu - \frac{1}{4}X^0 F_{\mu\nu}F^{\mu\nu}\Big)\, dx;\\
\int_{\{r=\text{const.}\}} i_{J^X[F]}\, d\mathrm{vol} &= \int_{\{r=\text{const.}\}} \Big(F^{r\mu}F_{\nu\mu}X^\nu - \frac{1}{4}X^r F_{\mu\nu}F^{\mu\nu}\Big) r^2\, dt\, d\omega;\\
\int_{H_u} i_{J^X[F]}\, d\mathrm{vol} &= -2\int_{H_u} \Big(F^{\underline{L}\mu}F_{\nu\mu}X^\nu - \frac{1}{4}X^{\underline{L}} F_{\mu\nu}F^{\mu\nu}\Big) r^2\, dv\, d\omega;\\
\int_{\underline{H}_v} i_{J^X[F]}\, d\mathrm{vol} &= 2\int_{\underline{H}_v} \Big(F^{L\mu}F_{\nu\mu}X^\nu - \frac{1}{4}X^{L} F_{\mu\nu}F^{\mu\nu}\Big) r^2\, du\, d\omega.
\end{aligned} \tag{18}$$

3.3. ***The integrated local energy estimates using the multiplier $f(r)\partial_r$.*** For the full solution (ϕ, F) of the Maxwell–Klein–Gordon equations, including the case with large charge, the integrated local energy estimates together with the r-weighted energy estimates in the next subsection have been studied in the author's work [Yang 2015c]. To obtain estimates for higher-order derivatives of the solutions, we need to commute the equations with derivatives, and hence nonlinear terms arise. Furthermore, in our setting, the data for the Maxwell field are large while the data for the complex scalar field are small. We thus need to obtain estimates separately for the Maxwell field and the scalar field.

We first consider the integrated local energy estimates for the scalar field. In the energy identity (12) for the scalar field, we choose the vector fields X, Y as follows:

$$X = f(r)\partial_r, \quad Y = 0$$

for some function $f(r)$. We then can compute

$$T[\phi]^{\mu\nu}\pi^X_{\mu\nu} + \chi\,\overline{D_\mu\phi}D^\mu\phi - \tfrac{1}{2}\Box\chi|\phi|^2$$
$$= \big(r^{-1}f - \chi + \tfrac{1}{2}f'\big)|D_t\phi|^2 + \big(\chi + \tfrac{1}{2}f' - r^{-1}f\big)|D_r\phi|^2 + \big(\chi - \tfrac{1}{2}f'\big)|\not{D}\phi|^2 - \tfrac{1}{2}\Box\chi|\phi|^2.$$

The idea is to choose the functions f, χ so that the coefficients are positive. Let ϵ be a small positive constant, depending only on γ_0 (e.g., $\epsilon = 10^{-3}\gamma_0$). Construct the functions f and χ so that

$$f(r) = 2\epsilon^{-1} - \frac{2\epsilon^{-1}}{(1+r)^\epsilon}, \quad \chi = r^{-1}f.$$

We can compute

$$\chi - r^{-1}f + \frac{1}{2}f' = r^{-1}f + \frac{1}{2}f' - \chi = \frac{1}{(1+r)^{1+\epsilon}}, \quad -\frac{1}{2}\Box\chi = \frac{1+\epsilon}{r(1+r)^{2+\epsilon}}.$$

When $r > 1$, we have the following improved estimate for $\chi - \frac{1}{2}f'$:

$$\chi - \frac{1}{2}f' \geq \frac{2\epsilon^{-1}}{r} - \frac{1+2\epsilon^{-1}}{r(1+r)^\epsilon} \geq \frac{1}{r}, \quad r \geq 1. \tag{19}$$

This improved estimate will be used to show the improved integrated local energy estimate for the covariant angular derivative of the scalar field ϕ.

From the above calculation, we see that for this particular choice of vector field X and the function χ, the last three terms in the first line of (12) have positive signs. We treat the first two terms as nonlinear terms. To get an integrated local energy estimate for the scalar field ϕ, it suffices to control the boundary terms arising from the Stokes' formula (12). This requires a version of Hardy's inequality. Before stating the lemma, we make a convention that the notation $A \lesssim_K B$ means that there exists a constant C, depending only on the constants R, γ_0, ϵ and the set K such that $A \leq CB$. For the particular case when K is empty, we omit the index K.

Lemma 6. *Assume $0 \leq \gamma < 1$ and the complex scalar field ϕ vanishes at null infinity, that is,*

$$\lim_{v\to\infty}\phi(v, u, \omega) = 0$$

for all u, ω. Then we have

$$\int_{H_u} r^\gamma|\phi|^2\,dv\,d\omega \lesssim \int_\omega r^{1+\gamma}|\phi|^2(u, v(u), \omega)\,d\omega + \int_{H_u} r^\gamma|D_L(r\phi)|^2\,dv\,d\omega \tag{20}$$

for all $u \in \mathbb{R}$. Here $v(u) = -u$ when $u \leq -\frac{1}{2}R$ that is in the exterior region and $v(u) = 2R + u$ when $u > -\frac{1}{2}R$ that is in the interior region. In particular, we have

$$\int_{H_u} |\phi|^2\,dv\,d\omega \lesssim E[\phi](H_u), \quad \int_{\Sigma_\tau} |\phi|^2\,dv'\,d\omega \lesssim E[\phi](\Sigma_\tau). \tag{21}$$

Here $v' = v$ when $r \geq R$ and $v' = r$ otherwise.

Proof. It suffices to notice that the connection D is compatible with the inner product $\langle\, ,\rangle$ on the complex plane. Then the proof when $\gamma = 0$ goes the same as the case when the connection field A is trivial; see, e.g., Lemma 2 of [Yang 2013] or Proposition 11.2 of [Dafermos and Rodnianski 2009]. Another quick way to reduce the proof of the lemma to the case with trivial connection field A is to choose a particular gauge such that the scalar field ϕ is real. We can do this is due to the fact that all the norms in this paper are gauge invariant. For general γ, based on the above argument, the proof goes similar to the proof of the standard Hardy's inequality. Let $\psi = r\phi$. Note that $\gamma < 1$. We can show that

$$\begin{aligned}\int_{v_0}^{\infty}\int_{\omega} r^{\gamma-2}|\psi|^2\,dv\,d\omega &= \frac{1}{\gamma-1}\int_{v_0}^{\infty}\int_{\omega}|\psi|^2\,d\omega\,dr^{\gamma-1}\\ &= \frac{1}{\gamma-1}r^{\gamma-1}\int_{\omega}|\psi|^2\,d\omega\Big|_{v_0}^{\infty} + \frac{2}{1-\gamma}\int_{v_0}^{\infty}\int_{\omega} r^{\gamma-1}D_L\psi\cdot\psi\,dv\,d\omega\\ &\le \frac{1}{1-\gamma}\int_{\omega} r^{1+\gamma}|\phi|^2(u,v_0,\omega)\,d\omega + \frac{1}{2}\int_{v_0}^{\infty}\int_{\omega} r^{\gamma-2}|\psi|^2\,dv\,d\omega\\ &\qquad + \frac{8}{(1-\gamma)^2}\int_{v_0}^{\infty}\int_{\omega} r^{\gamma}|D_L\psi|^2\,dv\,d\omega.\end{aligned}$$

The estimate (20) then follows by absorbing the second term and taking $v_0 = v(u)$. □

We then can derive the following integrated local energy estimate for the scalar field ϕ.

Proposition 7. *Assume the complex scalar field ϕ vanishes at null infinity and the spatial infinity initially. Then in the interior region $\{r \le R+t\}$, we have the energy estimates*

$$\begin{aligned}I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + E[\phi](\Sigma_{\tau_2}) + E[\phi](\underline{H}_v^{\tau_1^*,\tau_2^*}) + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} \frac{|\not{D}\phi|^2}{1+r}\,dx\,dt&\\ \lesssim E[\phi](\Sigma_{\tau_1}) + I_0^{1+\epsilon}[\Box_A\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{L\nu}J^{\nu}[\phi]| + |F_{\underline{L}\nu}J^{\nu}[\phi]|\,dx\,dt&\end{aligned}\tag{22}$$

for all $0 \le \tau_1 < \tau_2$ and $v \ge \frac{1}{2}(\tau_2 + R)$, where we let $\tilde{D}\phi = (D\phi, r_+^{-1}\phi)$ and $F = dA$. Similarly, in the exterior region $\{r > t + R\}$, we have

$$\begin{aligned}I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + E[\phi]\big(H_{\tau_1^*}^{-\tau_2^*}\big) + E[\phi]\big(\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}\big)&\\ \lesssim E[\phi](B_{R-\tau_1}) + I_0^{1+\epsilon}[\Box_A\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{L\nu}J^{\nu}[\phi]| + |F_{\underline{L}\nu}J^{\nu}[\phi]|\,dx\,dt&\end{aligned}\tag{23}$$

for all $\tau_2 \le \tau_1 \le 0$. Here see the notations in Section 2 and $J^{\mu}[\phi] = \Im(\phi\cdot\overline{D^{\mu}\phi})$, $\tau^* = \frac{1}{2}(\tau - R)$.

Proof. For all $v_0 \ge \frac{1}{2}(\tau_2 + R)$, take the region $\mathcal{D}$ to be $\mathcal{D}_{\tau_1}^{\tau_2} \cap \{v \le v_0\}$, which is bounded by the surfaces

$$\Sigma_{\tau_1}, \quad \Sigma_{\tau_2}, \quad \underline{H}_{v_0}^{\tau_1^*,\tau_2^*}$$

and the functions f, χ as above and the vector field $Y=0$ in the energy identity (12). The boundary terms can be controlled by the energy flux according to Hardy's inequality of Lemma 6. For more details regarding this bound, we refer to, e.g., Proposition 1 in [Yang 2013]. Therefore the above calculations lead to the following integrated local energy estimate:

$$\iint_{\mathcal{D}_{\tau_1}^{\tau_2}\cap\{v\le v_0\}} \frac{|D\phi|^2}{(1+r)^{1+\epsilon}}+\frac{|\not{D}\phi|^2}{1+r}+\frac{|\phi|^2}{r(1+r)^{2+\epsilon}}\,dx\,dt$$
$$\lesssim E[\phi](\Sigma_{\tau_1}^{v_0})+E[\phi](\Sigma_{\tau_2}^{v_0})+E[\phi]\big(\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}\big)+\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |\Box_A\phi(\overline{D_X\phi}+\chi\bar{\phi})|+|F_{r\nu}J^\nu[\phi]|\,dx\,dt.$$

Next, we take the vector fields $X=\partial_t$, $Y=0$ and the function $\chi=0$ in the energy identity (12) for the scalar field. Consider the region $\mathcal{D}_{\tau_1}^{\tau_2}\cap\{v\le v_0\}$. We retrieve the classical energy estimate

$$E[\phi](\Sigma_{\tau_2}^{v_0})+E[\phi]\big(\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}\big)=E[\phi](\Sigma_{\tau_1}^{v_0})-2\iint_{\mathcal{D}_{\tau_1}^{\tau_2}\cap\{v\le v_0\}}\Re(\Box_A\phi\overline{D_t\phi})+F_{0\nu}J^\nu[\phi]\,dx\,dt.$$

Combined with the previous integrated local energy estimate and letting $v_0\to\infty$, we derive that

$$I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}_{\tau_1}^{\tau_2})\lesssim E[\phi](\Sigma_{\tau_1})+\iint_{\mathcal{D}_{\tau_1}^{\tau_2}}\big|\Box_A\phi\overline{\tilde{D}\phi}\big|+|F_{L\nu}J^\nu[\phi]|+|F_{\underline{L}\nu}J^\nu[\phi]|\,dx\,dt.$$

We apply the Cauchy–Schwarz inequality to the integral of $\Box_A\phi\overline{\tilde{D}\phi}$:

$$2\big|\Box_A\phi\overline{\tilde{D}\phi}\big|\le\epsilon_1 r_+^{-1-\epsilon}|\tilde{D}\phi|^2+\epsilon_1^{-1}r_+^{1+\epsilon}|\Box_A\phi|^2,\quad \forall\epsilon_1>0.$$

Choose ϵ_1 to be sufficiently small depending only on ϵ, γ_0, R so that the integral of the first term can be absorbed. We thus can derive the integrated local energy estimate for the scalar field. Then in the above classical energy estimate, we can use the Cauchy–Schwarz inequality again to bound $\Re(\Box_A\phi\overline{D_t\phi})$ which gives control of the energy flux $E[\phi](H_{\tau_2^*})$. This energy estimate together with the previous integrated local energy estimate imply the energy estimate (22) of the proposition in the interior region. The improved estimate for the angular covariant derivative is due to the improve estimate (19).

The proof for the estimate (23) in the exterior region is similar. The only point we need to emphasize is that we use the fact that the ϕ goes to zero as $r\to\infty$ on the initial hypersurface. We thus can use the Hardy's inequality to control the integral of $|\phi|^2/(1+r)^2$. This is also the reason that we have $E[\phi](B_{R-\tau_1})$ instead of $E[\phi](B_{R-\tau_1}^{R-\tau_2})$ on the right-hand side of (23). $\square$

In our setting, F is the Maxwell field, which is no longer small. In particular this means that the integral of $|F_{L\nu}J^\nu[\phi]|$ on the right-hand side of (22), (23) could not be absorbed. The key to controlling those terms is to use the r-weighted energy estimates in the next section.

Let F be any 2-form satisfying the Bianchi identity (1). Let $J=\delta F$ or $J_\mu=\nabla^\nu F_{\nu\mu}$ be the divergence of F. This notation J can be viewed as the inhomogeneous term of the linear Maxwell equation. In (MKG), this J is identical to $J[\phi]$, which is quadratic in the scalar field ϕ. Under the null frame $\{L,\underline{L},e_1,e_2\}$, write $\not{J}=(J_{e_1},J_{e_2})$. We derive an analogue of Proposition 7 for the Maxwell field F.

Proposition 8. *In the interior region $\{r \le t+R\}$, we have the integrated local energy estimates*

$$I_0^{-1-\epsilon}[F](\mathcal{D}_{\tau_1}^{\tau_2}) + \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} \frac{\rho^2+|\sigma|^2}{1+r}\,dx\,d\tau + E[F]\big(\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}\big) + E[F](\Sigma_{\tau_2})$$
$$\lesssim E[F](\Sigma_{\tau_1}) + I_0^{1+\epsilon}[|J_L|+|\not{J}|](\mathcal{D}_{\tau_1}^{\tau_2}) + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |J_{\underline{L}}||\rho|\,dx\,dt \tag{24}$$

for all $0 \le \tau_1 < \tau_2$ and $v_0 \ge \frac{1}{2}(\tau_2+R)$. Similarly, in the exterior region $\{r \le R+t\}$, for all $\tau_2 < \tau_1 \le 0$ we have

$$I_0^{-1-\epsilon}[F](\mathcal{D}_{\tau_1}^{\tau_2}) + E[F]\big(\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}\big) + E[F]\big(H_{\tau_1^*}^{-\tau_2^*}\big)$$
$$\lesssim E[F](B_{R-\tau_1}^{R-\tau_2}) + I_0^{1+\epsilon}[|J_L|+|\not{J}|](\mathcal{D}_{\tau_1}^{\tau_2}) + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |J_{\underline{L}}||\rho|\,dx\,dt. \tag{25}$$

Proof. The idea to prove this proposition is the same as that of the previous proposition for the scalar field. However, the calculations are slightly different for the Maxwell field F. In the energy identity (17) for the Maxwell field, we take the vector field

$$X = f(r)\partial_r = 2\epsilon^{-1}(1-r_+^{-\epsilon})\partial_r.$$

Set $\omega_i = r^{-1}x_i$. We then can compute

$$\begin{aligned} T[F]^{\mu\nu}\pi_{\mu\nu}^X &= T[F]^{ij}(f'\omega_i\omega_j + r^{-1}f\delta_{ij} - r^{-1}f\omega_i\omega_j) \\ &= \tfrac{1}{4}(2r^{-1}f - f')F_{\mu\nu}F^{\mu\nu} + (f' - r^{-1}f)F_{r\nu}F^{r\nu} - r^{-1}fF_{0\nu}F^{0\nu}, \end{aligned}$$

where the Greek indices μ, ν run from 0 to 3 and the Latin indices i, j run from 1 to 3. Using the null decomposition of the 2-form under the null frame $\{L, \underline{L}, e_1, e_2\}$ defined in line (4), we can show that

$$\begin{aligned} F_{\mu\nu}F^{\mu\nu} &= -2\rho^2 - 2\alpha\cdot\underline{\alpha} + 2|\sigma|^2, \\ F_{0\nu}F^{0\nu} &= -\tfrac{1}{4}(4\rho^2 + 2\alpha\cdot\underline{\alpha} + |\alpha|^2 + |\underline{\alpha}|^2), \\ F_{r\nu}F^{r\nu} &= -\tfrac{1}{4}(4\rho^2 + 2\alpha\cdot\underline{\alpha} - |\alpha|^2 - |\underline{\alpha}|^2). \end{aligned}$$

Therefore we have

$$T[F]^{\mu\nu}\pi_{\mu\nu}^X = \big(r^{-1}f - \tfrac{1}{2}f'\big)(\rho^2+|\sigma|^2) + \tfrac{1}{4}f'(|\alpha|^2+|\underline{\alpha}|^2). \tag{26}$$

The calculations before line (19) imply that the coefficients $r^{-1}f - \frac{1}{2}f'$ and f' have positive signs. To obtain the similar integrated local energy estimates for the Maxwell field F, we need to control the boundary terms arising from the Stokes' formula (17). Using the formula (18), we can compute that

$$\begin{aligned} 2|i_{J^X[F]}\,d\mathrm{vol}| &= f\big||\alpha|^2 - |\underline{\alpha}|^2\big|\,dx \le |F|^2\,dx = 2f\,i_{J^{\partial_t}[F]}\,d\mathrm{vol}, \\ 2|i_{J^X[F]}\,d\mathrm{vol}| &= f\big|-\rho^2 + |\alpha|^2 - |\sigma|^2\big|r^2\,dv\,d\omega \le f(\rho^2+|\alpha|^2+|\sigma|^2)r^2\,dv\,d\omega = 2f\,i_{J^{\partial_t}[F]}\,d\mathrm{vol}, \\ 2|i_{J^X[F]}\,d\mathrm{vol}| &= f\big|-\rho^2 + |\underline{\alpha}|^2 - |\sigma|^2\big|r^2\,du\,d\omega \le f(\rho^2+|\underline{\alpha}|^2+|\sigma|^2)r^2\,du\,d\omega = 2f\,i_{J^{\partial_t}[F]}\,d\mathrm{vol}, \end{aligned}$$

on the $t =$ constant slice, the outgoing null hypersurface and the incoming null hypersurface, respectively, for all positive functions f. This in particular implies that the boundary terms corresponding to the vector field $f\partial_r$ can be bounded by the energy flux for all positive bounded functions f. Therefore, for the particular choice of vector field X, the energy identity (17) on the domain $\mathcal{D}_{\tau_1}^{\tau_2} \cap \{v \le v_0\}$ for all $0 \le \tau_1 < \tau_2$ and $v_0 \ge \frac{1}{2}(\tau_2 + R)$ leads to

$$\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau^{v_0}} \frac{|F|^2}{(1+r)^{1+\epsilon}} + \frac{\rho^2+|\sigma|^2}{1+r}\,dx\,d\tau \lesssim E[F](\Sigma_{\tau_1}^{v_0}) + E[F](\Sigma_{\tau_2}^{v_0}) + E[F]\big(\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}\big) + \int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau} |J^\gamma||F_{L\gamma} - F_{\underline{L}\gamma})|\,dx\,d\tau.$$

Here notice that we have the improved estimate (19) for the coefficient of $\rho^2 + |\sigma|^2$. If we take the vector field $X = \partial_t$ on the same domain, we then can derive the classical energy identity

$$\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau^{v_0}} J^\gamma(F_{L\gamma} + F_{\underline{L}\gamma})\,dx\,d\tau = E[F](\Sigma_{\tau_1}^{v_0}) - E[F]\big(\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}\big) - E[F](\Sigma_{\tau_2}^{v_0}).$$

Let $v_0 \to \infty$ and apply Cauchy–Schwarz to the inhomogeneous term $J^\mu(|F_{L\mu}| + |F_{\underline{L}\mu}|)$ for $\mu = \underline{L}, e_1, e_2$:

$$|J^{\underline{L}}||F_{L\underline{L}}| + |J^{e_i}|(|F_{Le_i}| + |F_{\underline{L}e_i}|) \lesssim \epsilon_1^{-1}(|J_L| + |\not{J}|)r_+^{1+\epsilon} + \epsilon_1|F|^2 r_+^{-1-\epsilon}, \quad \epsilon_1 > 0.$$

The integral of the second term could be absorbed for sufficiently small ϵ_1. For the component when $\mu = L$, we estimate

$$|J^L||F_{L\underline{L}}| \lesssim |J_{\underline{L}}||\rho|.$$

Then the above energy identity together with the integrated local energy estimates imply the integrated local energy estimate (24) in the interior region. The energy estimate (25) in the exterior region follows in a similar way. □

3.4. ***The r-weighted energy estimates using the multiplier r^pL.*** In this section, we establish the robust r-weighted energy estimates both for the scalar field and the Maxwell field. This estimate for solutions of linear wave equation in Minkowski space was first introduced by Dafermos and Rodnianski [2010]. We study the r-weighted energy estimate either in the exterior region $\{r \ge R + t\}$ for the domain $\mathcal{D}_{\tau_1}^{\tau_2}$ for $\tau_2 \le \tau_1 \le 0$ or in the interior region for domain $\bar{\mathcal{D}}_{\tau_1}^{\tau_2}$ for $0 \le \tau_1 < \tau_2$ which is bounded by the outgoing null hypersurfaces $H_{\tau_1^*}$, $H_{\tau_2^*}$ and the cylinder $\{r = R\}$.

Through out this paper, we denote $\psi = r\phi$ as the r-weighted scalar field. We have the following r-weighted energy estimates for the complex scalar field.

Proposition 9. *Assume that the complex scalar field ϕ vanishes at null infinity. Then in the interior region, for all $0 \le \tau_1 < \tau_2$ and $v_0 \ge \frac{1}{2}(\tau_2 + R)$, we have the r-weighted energy estimate*

$$\int_{H_{\tau_2^*}} r^p|D_L\psi|^2\,dv\,d\omega + \int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} r^{p-1}(p|D_L\psi|^2 + (2-p)|\not{D}\psi|^2)\,dv\,d\omega\,d\tau + \int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}} r^p|\not{D}\psi|^2\,du\,d\omega$$
$$\lesssim \int_{H_{\tau_1^*}} r^p|D_L\psi|^2\,dv\,d\omega + I_{\min\{1+\epsilon,p\}}^{\max\{1+\epsilon,p\}}[\Box_A\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + E[\phi](\Sigma_{\tau_1}) + I_0^{1+\epsilon}[\Box_A\phi](\mathcal{D}_{\tau_1}^{\tau_2})$$
$$+ \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{\underline{L}\mu}J^\mu[\phi]| + |F_{L\mu}J^\mu[\phi]|\,dx\,dt + \iint_{\bar{\mathcal{D}}_{\tau_1}^{\tau_2}} r^p|F_{L\mu}J^\mu[\phi]|\,dx\,dt \qquad (27)$$

for all $0 \le p \le 2$. *Similarly, in the exterior region, for all* $\tau_2 < \tau_1 \le 0$, *we have*

$$\int_{H_{\tau_1^*}^{-\tau_2^*}} r^p |D_L\psi|^2\, dv\, d\omega + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{p-1}(p|D_L\psi|^2 + (2-p)|\not{D}\psi|^2)\, dv\, d\omega\, du + \int_{\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}} r^p |\not{D}\psi|^2\, du\, d\omega$$
$$\lesssim \int_{B_{R-\tau_1}^{R-\tau_2}} r^p(|D_L\psi|^2 + |\not{D}\psi|^2)\, dr\, d\omega + I_{\min\{1+\epsilon,p\}}^{\max\{p,1+\epsilon\}}[\Box_A\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^p |F_{L\mu} J^\mu[\phi]|\, dx\, dt \tag{28}$$

for all $0 \le p \le 2$. *Here* $\psi = r\phi$.

Proof. Apply the energy identity (12) to the region $\overline{\mathcal{D}}_{\tau_1}^{\tau_2} \cap \{v \le v_0\}$, which is bounded by $H_{\tau_1^*}$, $H_{\tau_2^*}$, $\{r = R\}$ and $\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}$ with the vector fields X, Y and the function χ as follows:

$$X = r^p L, \quad Y = \tfrac{1}{2} p r^{p-2} |\phi|^2 L, \quad \chi = r^{p-1}.$$

Define $\psi = r\phi$ to be the weighted scalar field. We have the equalities

$$\begin{aligned} r^2 |D_L\phi|^2 &= |D_L\psi|^2 - L(r|\phi|^2), \\ r^2 |\not{D}\phi|^2 &= |\not{D}\psi|^2, \\ r^2 |D_{\underline{L}}\phi|^2 &= |D_{\underline{L}}\psi|^2 + \underline{L}(r|\phi|^2). \end{aligned}$$

We then can compute

$$\begin{aligned} &\mathrm{div}(Y) + T[\phi]^{\mu\nu}\pi_{\mu\nu}^X + \chi\, \overline{D^\mu\phi} D_\mu\phi - \tfrac{1}{2}\Box\chi\, |\phi|^2 \\ &\quad = \tfrac{1}{2} p r^{-2} L(r^p |\phi|^2) + \tfrac{1}{2} r^{p-1}\big(p|D_L\phi|^2 + (2-p)|\not{D}\phi|^2\big) - \tfrac{1}{2} p(p-1) r^{p-3} |\phi|^2 \\ &\quad = \tfrac{1}{2} r^{p-3}\big(p|D_L\psi|^2 + (2-p)|\not{D}\psi|^2\big). \end{aligned}$$

We next compute the boundary terms using the formula (18). We have

$$\begin{aligned} \int_{H_{\tau^*}^{v_0}} i_{\tilde{J}^X[\phi]}\, d\mathrm{vol} &= \int_{H_{\tau^*}^{v_0}} r^p |D_L\psi|^2 - \tfrac{1}{2} L(r^{p+1}\phi)\, dv\, d\omega, \\ \int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}} i_{\tilde{J}^X[\phi]}\, d\mathrm{vol} &= -\int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}} r^p |\not{D}\psi|^2 + \tfrac{1}{2}\underline{L}(r^{p+1}|\phi|^2)\, du\, d\omega, \\ \int_{\{r=R\}\cap\{\tau_1 \le t \le \tau_2\}} i_{\tilde{J}^X[\phi]}\, d\mathrm{vol} &= \int_{\tau_1}^{\tau_2}\int_\omega \tfrac{1}{2} r^p(|D_L\psi|^2 - |\not{D}\psi|^2) - \tfrac{1}{2}\partial_t(r^{p+1}|\phi|^2)\, d\omega\, dt. \end{aligned}$$

Now notice that there is a cancellation for the boundary terms:

$$-\int_{H_{\tau_1^*}^{v_0}} L(r^{p+1}|\phi|^2)\, dv\, d\omega - \int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}} \underline{L}(r^{p+1}|\phi|^2)\, du\, d\omega$$
$$+ \int_{H_{\tau_2^*}^{v_0}} L(r^{p+1}|\phi|^2)\, dv\, d\omega + \int_{\tau_1}^{\tau_2}\int_\omega \partial_t(r^{p+1}|\phi|^2)\, d\omega\, dt = 0.$$

Therefore in the interior region for the domain $\overline{\mathcal{D}}_{\tau_1}^{\tau_2} \cap \{v \le v_0\}$, the above calculations lead to the r-weighted energy identity

$$\int_{H_{\tau_2^*}^{v_0}} r^p |D_L\psi|^2 \, dv \, d\omega + \int_{\tau_1}^{\tau_2} \int_{H_{\tau^*}^{v_0}} r^{p-1}(p|D_L\psi|^2 + (2-p)|\not{D}\psi|^2) \, dv \, d\omega \, d\tau + \int_{\underline{H}_{v_0}^{\tau_1^*, \tau_2^*}} r^p |\not{D}\psi|^2 \, du \, d\omega$$
$$= \int_{H_{\tau_1^*}^{v_0}} r^p |D_L\psi|^2 \, dv \, d\omega - \frac{1}{2} \int_{\tau_1}^{\tau_2} \int_{\omega} r^p (|D_L\psi|^2 - |\not{D}\psi|^2) \, d\omega \, dt$$
$$- \int_{\tau_1}^{\tau_2} \int_{H_{\tau^*}^{v_0}} r^{p-1} \Re(\Box_A\phi \overline{D_L\psi}) + r^p F_{L\mu} J^\mu[\phi] \, dx \, dt. \quad (29)$$

Similarly, in the exterior region $\{r \le R + t\}$ for the domain $\mathcal{D}_{\tau_1}^{\tau_2}$ for all $\tau_2 < \tau_1 \le 0$, we have

$$\int_{H_{\tau_1^*}^{-\tau_2^*}} r^p |D_L\psi|^2 \, dv \, d\omega + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{p-1}(p|D_L\psi|^2 + (2-p)|\not{D}\psi|^2) \, dv \, d\omega \, du + \int_{\underline{H}_{-\tau_2^*}^{\tau_2^*, \tau_1^*}} r^p |\not{D}\psi|^2 \, du \, d\omega$$
$$= \frac{1}{2} \int_{B_{R-\tau_1}^{R-\tau_2}} r^p (|D_L\psi|^2 + |\not{D}\psi|^2) \, dr \, d\omega - \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{p-1} \Re(\Box_A\phi \overline{D_L\psi}) + r^p F_{L\mu} J^\mu[\phi] \, dx \, dt. \quad (30)$$

For the inhomogeneous term, when $p \ge 1 + \epsilon$, we apply the Cauchy–Schwarz inequality directly:

$$2r^{p+1} |\Box_A\phi \cdot \overline{D_L\psi}| \lesssim r^p u_+^{-1-\epsilon} |D_L\psi|^2 + r^{p+2} u_+^{1+\epsilon} |\Box_A\phi|^2.$$

The integral of the first term in the above inequality can be controlled using Gronwall's inequality both in (29) and (30). In particular this shows that estimate (28) follows from (30).

When $p < 1 + \epsilon$, we note that

$$2p - 1 - \epsilon < p \cdot \frac{p}{1+\epsilon} + (p-1)\Big(1 - \frac{p}{1+\epsilon}\Big).$$

Then we can estimate the inhomogeneous term as follows:

$$\begin{aligned} 2r^{p+1} |\Box_A\phi \cdot \overline{D_L\psi}| &\le \epsilon_1 r^{2p-1-\epsilon} u_+^{-p} |D_L\psi|^2 + \epsilon_1^{-1} r^{1+\epsilon+2} u_+^p |\Box_A\phi|^2 \\ &\le \epsilon_1 (r^p u_+^{-1-\epsilon})^{p/(1+\epsilon)} (r^{p-1})^{1-p/(1+\epsilon)} |D_L\psi|^2 + \epsilon_1^{-1} r^{1+\epsilon+2} u_+^p |\Box_A\phi|^2 \\ &\le \epsilon_1 r^p u_+^{-1-\epsilon} |D_L\psi|^2 + \epsilon_1 r^{p-1} |D_L\psi|^2 + \epsilon_1^{-1} r^{1+\epsilon+2} u_+^p |\Box_A\phi|^2 \end{aligned}$$

for all $\epsilon_1 > 0$. The integral of the first term can be controlled using Gronwall's inequality. The integral of the second term can be absorbed for sufficiently small ϵ_1. Then estimate (28) follows.

For the r-weighted energy estimate (27) in the interior region, we need to control the boundary term on $\{r = R\}$. It suffices to estimate it for $p = 0$ in (29) by making use of the energy estimate (22). From Hardy's inequality in Lemma 6, we note that

$$\int_{H_\tau} |D_L\psi|^2 \, d\omega \, dv \lesssim E[\phi](\Sigma_\tau).$$

By using the integrated local energy estimate (22), we therefore can show that

$$
\begin{aligned}
&\left|\int_{\tau_1}^{\tau_2}\int_{\omega} r^p(|D_L\psi|^2-|\not{D}\psi|^2)\,d\omega\,dt\right| \\
&\qquad\lesssim R^p\int_{H_{\tau_2^*}^{v_0}}|D_L\psi|^2\,dv\,d\omega+\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}^{v_0}} r^{-1}|\not{D}\psi|^2\,dv\,d\omega\,d\tau+\int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}}|\not{D}\psi|^2\,du\,d\omega \\
&\qquad\qquad+\int_{H_{\tau_1^*}^{v_0}}|D_L\psi|^2\,dv\,d\omega+\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}^{v_0}} r^{-1}|\Re(\Box_A\phi\overline{D_L\psi})|+|F_{L\mu}J^\mu[\phi]|\,dx\,dt \\
&\qquad\lesssim E[\phi](\Sigma_{\tau_1})+I_0^{1+\epsilon}[\Box_A\phi](\mathcal{D}_{\tau_1}^{\tau_2})+\iint_{\mathcal{D}_{\tau_1}^{\tau_2}}|F_{\underline{L}\mu}J^\mu[\phi]|+|F_{L\mu}J^\mu[\phi]|\,dx\,dt.
\end{aligned}
$$

The inhomogeneous term can be bounded using the Cauchy–Schwarz inequality together with the integrated local energy estimates. Once we have the bound for the boundary terms on $\{r=R\}$, the r-weighted energy estimate (27) follows from the identity (29) and Gronwall's inequality. □

Next we establish the r-weighted energy estimate for the Maxwell field.

Proposition 10. *Let F be any 2-form satisfying the Bianchi identity (1). Then in the interior region, for all $0\le\tau_1<\tau_2$ and $v_0\ge\frac{1}{2}(\tau_2+R)$, we have the r-weighted energy estimate*

$$
\begin{aligned}
&\int_{H_{\tau_2^*}} r^{p+2}|\alpha|^2\,dv\,d\omega \\
&\quad+\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} r^{p+1}(p|\alpha|^2+(2-p)(\rho^2+|\sigma|^2))\,dv\,d\omega\,d\tau+\int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}} r^{p+2}(\rho^2+|\sigma|^2)\,du\,d\omega \\
&\qquad\lesssim\int_{H_{\tau_1^*}} r^{p+2}|\alpha|^2\,dv\,d\omega+I_{\min\{1+\epsilon,p\}}^{\max\{p,1+\epsilon\}}[\not{J}](\mathcal{D}_{\tau_1}^{\tau_2})+(2-p)^{-1}I_0^{p+1}[J_L](\mathcal{D}_{\tau_1}^{\tau_2}) \\
&\qquad\qquad+E[F](\Sigma_{\tau_1})+I_0^{1+\epsilon}[|J_L|+|\not{J}|](\mathcal{D}_{\tau_1}^{\tau_2})+\iint_{\mathcal{D}_{\tau_1}^{\tau_2}}|J_{\underline{L}}||\rho|\,dx\,dt \qquad (31)
\end{aligned}
$$

for all $0\le p\le 2$. Similarly in the exterior region, for all $\tau_2<\tau_1\le 0$ and $0\le p\le 2$, we have

$$
\begin{aligned}
&\int_{H_{\tau_1^*}^{-\tau_2^*}} r^p|\alpha|^2r^2\,dv\,d\omega \\
&\quad+\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{p+1}(p|\alpha|^2+(2-p)(\rho^2+|\sigma|^2))\,dv\,d\omega\,du+\int_{\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}} r^p(\rho^2+|\sigma|^2)r^2\,du\,d\omega \\
&\qquad\lesssim\int_{B_{R-\tau_1}^{R-\tau_2}} r^p|F|^2\,dx+I_{\min\{p,1+\epsilon\}}^{\max\{p,1+\epsilon\}}[\not{J}](\mathcal{D}_{\tau_1}^{\tau_2})+(2-p)^{-1}I_0^{p+1}[J_L](\mathcal{D}_{\tau_1}^{\tau_2}). \qquad (32)
\end{aligned}
$$

Proof. Take the vector field

$$X=r^pL=f\partial_t+f\partial_r$$

in the energy identity (17) for the Maxwell field. Using the computations before (26), we have

$$
\begin{aligned}
T[F]^{\mu\nu}\pi^{X}_{\mu\nu} &= T[F]^{\mu\nu}\pi^{f\partial_r}_{\mu\nu} + T[F]^{\mu\nu}\pi^{f\partial_t} \\
&= \left(r^{-1}f - \tfrac{1}{2}f'\right)(\rho^2+|\sigma|^2) + \tfrac{1}{4}f'(|\alpha|^2+|\underline{\alpha}|^2) + \tfrac{1}{4}f'(|\alpha|^2-|\underline{\alpha}|^2) \\
&= \tfrac{1}{2}r^{p-1}((2-p)(\rho^2+|\sigma|^2)+p|\alpha|^2).
\end{aligned}
$$

For the boundary terms corresponding to the vector field $X = r^pL$, we have

$$
\begin{aligned}
&i_{J^X[F]}\,d\mathrm{vol} = \tfrac{1}{2}r^p(|\alpha|^2+\rho^2+|\sigma|^2)\,dx, && i_{J^X[F]}\,d\mathrm{vol} = \tfrac{1}{2}r^p(|\alpha|^2-\rho^2-|\sigma|^2)r^2\,dt\,d\omega, \\
&i_{J^X[F]}\,d\mathrm{vol} = r^p|\alpha|^2r^2\,dv\,d\omega, && i_{J^X[F]}\,d\mathrm{vol} = -r^p(\rho^2+|\sigma|^2)r^2\,du\,d\omega
\end{aligned}
$$

on $\{t=\tau\}$, $\{r=R\}$, H_u and $\underline{H}_v$, respectively. Therefore, for all $0\le\tau_1<\tau_2$ and $v_0\ge\frac{1}{2}(\tau_2+R)$, if we take the region $\mathcal{D}$ bounded by $H_{\tau_1^*}$, $H_{\tau_2^*}$, $\{r=R\}$, $\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}$, we get the r-weighted energy identity

$$
\begin{aligned}
&\int_{H_{\tau_2^*}^{v_0}} r^p|\alpha|^2r^2\,dv\,d\omega \\
&\qquad + \int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}^{v_0}} r^{p-1}(p|\alpha|^2+(2-p)(\rho^2+|\sigma|^2))r^2\,dv\,d\omega\,d\tau + \int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}} r^p(\rho^2+|\sigma|^2)r^2\,du\,d\omega \\
&\qquad = \int_{H_{\tau_1^*}^{v_0}} r^p|\alpha|^2r^2\,dv\,d\omega - \frac{1}{2}\int_{\tau_1}^{\tau_2}\int_{\omega} r^p(|\alpha|^2-\rho^2-|\sigma|^2)r^2\,d\omega\,dt - \int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}^{v_0}} r^pJ_\nu F_L^\nu\,dx\,dt. \qquad (33)
\end{aligned}
$$

Similarly, in the exterior region $\{r\ge R+t\}$, consider the region $\mathcal{D}_{\tau_1}^{\tau_2}$ for $\tau_2<\tau_1\le 0$. We have the following identity:

$$
\begin{aligned}
&\int_{H_{\tau_1^*}^{-\tau_2^*}} r^p|\alpha|^2r^2\,dv\,d\omega \\
&\qquad + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{p-1}(p|\alpha|^2+(2-p)(\rho^2+|\sigma|^2))\,dv\,d\omega\,du + \int_{\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}} r^p(\rho^2+|\sigma|^2)r^2\,du\,d\omega \\
&\qquad\qquad = \frac{1}{2}\int_{B_{R-\tau_1}^{R-\tau_2}} r^p(|\alpha|^2+\rho^2+|\sigma|^2)\,dx - \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^pJ_\nu F_L^\nu\,dx\,dt. \qquad (34)
\end{aligned}
$$

To obtain (32), we first note that under the null frame $\{L,\underline{L},e_1,e_2\}$,

$$
F_L^L = -\tfrac{1}{2}F_{L\underline{L}} = \rho, \qquad F_L^{\underline{L}} = 0, \qquad F_L^{e_j} = \alpha_j, \qquad j=1,2.
$$

We can use the same method to treat the term $r^p|J_{e_j}F_L^{e_j}|$ as that for $\Box_A\phi\cdot\overline{D_L\psi}$ in Proposition 9 (simply replace $\Box_A\phi$ with J_{e_j} and $\overline{D_L}\psi$ with $r\alpha_j$). For the term involving ρ, we estimate

$$
r^{p+2}|J_L\cdot\rho| \le \frac{1}{2}(2-p)r^{p+1}|\rho|^2 + \frac{2}{2-p}r^{p+3}|J_L|^2.
$$

The integral of the first term could be absorbed. Then the r-weighted energy estimate (32) follows from the above r-weighted energy identity (34).

We can treat the inhomogeneous term the same way for the r-weighted energy estimate in the interior region from the r-weighted energy identity (33). Like the case for the scalar field, the boundary term on $\{r = R\}$ can be bounded by taking $p = 0$ in (33) and then by making use of the integrated local energy estimate (24):

$$\left|\int_{\tau_1}^{\tau_2}\int_{\omega} r^p(|\alpha|^2-\rho^2-|\sigma|^2)r^2\,d\omega\,dt\right|$$
$$\lesssim \int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}^{v_0}}(\rho^2+|\sigma|^2)r\,dv\,d\omega\,d\tau+\int_{\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}}(\rho^2+|\sigma|^2)r^2\,du\,d\omega$$
$$+\int_{H_{\tau_2^*}^{v_0}}|\alpha|^2r^2\,dv\,d\omega+\int_{H_{\tau_1^*}^{v_0}}|\alpha|^2r^2\,dv\,d\omega+\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}^{v_0}}|J_\nu F_L^\nu|\,dx\,dt$$
$$\lesssim E[F](\Sigma_{\tau_1})+I_0^{1+\epsilon}[|J_L|+|\not{J}|](\mathcal{D}_{\tau_1}^{\tau_2})+\iint_{\mathcal{D}_{\tau_1}^{\tau_2}}|J_{\underline{L}}||\rho|\,dx\,dt.$$

This combined with Gronwall's inequality implies the r-weighted energy estimate for the Maxwell field in the interior region. □

4. Decay estimates for the linear solutions

In this section we derive energy flux decay for both the linear Maxwell field and the linear complex scalar field under appropriate assumptions. We use a bootstrap argument to construct global solutions of the nonlinear (MKG). The first step is to study the decay properties of the linear solutions. Recall that $F = dA$ with A the connection used to define the covariant derivative D. Our strategy is that we make assumptions on $J_\mu = \nabla^\nu F_{\mu\nu}$ to obtain estimates for the linear solution F. We then use these estimates to derive estimates for the solutions of the linear covariant wave equation $\Box_A\phi = 0$. As in (MKG) the nonlinearity $J[\phi]$ is quadratic in ϕ, so by making use of the smallness of the scalar field we then can improve the bootstrap assumption on J. The difficulties are that the Maxwell field F is no longer small and that there exists nonzero charge.

Assume that the Maxwell field $F = dA$ has charge q_0 and splits into the charge part and chargeless part

$$F = \chi_{\{r>t+R\}}q_0 r^{-2}\,dt\wedge dr+\overline{F}.$$

Let $J = \delta F$ be the divergence of F and $\not{J} = (J_{e_1}, J_{e_2})$ be the angular component. Let

$$\begin{aligned} m_k &= \sum_{l\le k} I_{1+\epsilon}^{1+\gamma_0}[\mathcal{L}_Z^l\not{J}](\{r\ge R\})+I_0^{2+\gamma_0}[\mathcal{L}_Z^l J_L](\{r\ge R\})+I_{1+\gamma_0}^{1+\epsilon}[|\mathcal{L}_Z^l\not{J}|+|\mathcal{L}_Z^l J_L|](\{t\ge 0\})\\ &\quad + I_{1+\gamma_0+2\epsilon}^{1-\epsilon}[\mathcal{L}_Z^l J_{\underline{L}}](\{t\ge 0\})+I_{1+\gamma_0}^{0}[\overline{\nabla}\mathcal{L}_Z^{l-1}J](\{r\le 2R\})+|q_0|\sup_{\tau\le 0}\tau_+^{1+\gamma_0}\iint_{\mathcal{D}_\tau^{-\infty}}|J_{\underline{L}}|r^{-2}\,dx\,dt,\\ M_k &= m_k+E_0^k[\overline{F}]+1+|q_0|, \end{aligned} \tag{35}$$

where we recall from (6) in Section 2 that $E_0^k[\bar{F}]$ denotes the weighted Sobolev norm of the Maxwell field $\bar{F}$ with weights $r_+^{1+\gamma_0}$ on the initial hypersurface $t=0$. The integral of $|J_{\underline{L}}|r^{-2}$ is used to control the interaction of the nonzero charge with the nonlinearity J in the exterior region.

To derive the energy decay for the Maxwell field, we assume that M_k is finite. This can be fulfilled as follows: the charge q_0 is a constant depending on the initial data of the scalar field. $E_0^k[\bar{F}]$ denotes the size of the initial data for the chargeless part of the Maxwell field. Recall that the nonlinearity J is quadratic in the scalar field ϕ. By using the bootstrap assumption, it is small.

4.1. ***Energy decay for the Maxwell field.*** We derive energy flux decay for the Maxwell field F under the assumption that M_k is finite.

Proposition 11. *In the interior region for all* $0\le\tau_1<\tau_2$ *and* $v_0\ge\frac{1}{2}(\tau_2+R)$, *we have the following energy flux decay for the Maxwell field*:

$$I_0^{-1-\epsilon}[F](\mathcal{D}_{\tau_1}^{\tau_2})+\int_{\tau_1}^{\tau_2}\int_{\Sigma_\tau}\frac{\rho^2+|\sigma|^2}{1+r}\,dx\,d\tau+E[F]\big(\underline{H}_{v_0}^{\tau_1^*,\tau_2^*}\big)+E[F](\Sigma_{\tau_1})\lesssim(\tau_1)_+^{-1-\gamma_0}M_0. \tag{36}$$

In the exterior region $\{r\le R+t\}$ *for all* $\tau_2<\tau_1\le 0$ *and* $0\le p\le 1+\gamma_0$, *we have*

$$I_0^{-1-\epsilon}[\bar{F}](\mathcal{D}_{\tau_1}^{\tau_2})+E[\bar{F}]\big(\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}\big)+E[\bar{F}](H_{\tau_1^*})+(\tau_1)_+^{-p}\int_{H_{\tau_1^*}}r^{p+2}|\alpha|^2\,dv\,d\omega\lesssim(\tau_1)_+^{-1-\gamma_0}M_0. \tag{37}$$

Here and throughout the paper, $\tau_+=1+|\tau|$ *for all real numbers* τ.

Proof. Let's first consider the estimates in the exterior region. By the definition of M_0, we derive that

$$\int_{B_{R-\tau_1}^{R-\tau_2}}r^p|\bar{F}|^2\,dx+I_{1+\epsilon}^{p}[\not{J}](\mathcal{D}_{\tau_1}^{\tau_2})+I_0^{p+1}[J_L](\mathcal{D}_{\tau_1}^{\tau_2})\lesssim(\tau_1)_+^{p-1-\gamma_0}M_0,\quad 0\le p\le 1+\gamma_0.$$

Here note that in the exterior region, $r\ge\frac{1}{2}u_+$. Then the r-weighted energy estimate (32) implies that

$$\int_{H_{\tau_1^*}^{-\tau_2^*}}r^{p+2}|\alpha|^2\,dv\,d\omega+\iint_{\mathcal{D}_{\tau_1}^{\tau_2}}r^{p+1}(|\alpha|^2+\bar{\rho}^2+|\sigma|^2)\,dv\,d\omega\,du\lesssim(\tau_1)_+^{p-1-\gamma_0}M_0.$$

This estimate can be used to bound the integral of $|J_{\underline{L}}||\rho|$ on the right-hand side of (25). Recall that $\rho=q_0r^{-2}+\bar{\rho}$ when $r\ge R+t$. We then can show that

$$\iint_{\mathcal{D}_{\tau_1}^{\tau_2}}|J_{\underline{L}}||\rho|\,dx\,dt\lesssim\iint_{\mathcal{D}_{\tau_1}^{\tau_2}}(|q_0||J_{\underline{L}}|r^{-2}+|\bar{\rho}|^2r^{\epsilon-1}u_+^{-\epsilon}+|J_{\underline{L}}|^2r^{1-\epsilon}u_+^{\epsilon})\,dx\,dt\lesssim M_0(\tau_1)_+^{-1-\gamma_0}.$$

The decay estimate (37) then follows from the energy estimate (25) as

$$E[\bar{F}](B_{R-\tau_1}^{R-\tau_2})+I_0^{1+\epsilon}[|\not{J}|+|J_L|](\mathcal{D}_{\tau_1}^{\tau_2})\lesssim(\tau_1)_+^{-1-\gamma_0}M_0.$$

For the decay estimates in the interior region, we use the pigeonhole argument in [Dafermos and Rodnianski 2010]. First, by interpolation, we derive from the definition of M_0 that

$$I_{\min\{p,1+\epsilon\}}^{\max\{p,1+\epsilon\}}[\not{J}](\mathcal{D}_{\tau_1}^{\tau_2})+I_0^{p+1}[J_L](\mathcal{D}_{\tau_1}^{\tau_2})\lesssim(\tau_1)_+^{p-1-\gamma_0}M_0$$

for all $\epsilon \le p \le 1+\gamma_0$. To bound $|J_{\underline{L}}||\rho|$, we use the Cauchy–Schwarz inequality:

$$\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |J_{\underline{L}}||\rho|\,dx\,dt \lesssim \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} (\epsilon_1|\rho|^2 r_+^{\epsilon-1} + \epsilon_1^{-1} r_+^{1-\epsilon}|J_{\underline{L}}|^2)\,dx\,dt, \quad \forall \epsilon_1 > 0.$$

Here note that in the interior region, $\rho = \bar{\rho}$. For $\epsilon \le p \le 1+\gamma_0$ and sufficiently small ϵ_1 the first term could be absorbed from the r-weighted energy estimates (31) and the second term is bounded above by $M_0(\tau_1)_+^{-1-\gamma_0}$ by the definition of M_0.

To apply the pigeonhole argument, we need to control the weighted energy flux through the initial hypersurface Σ_0 of the interior region. Note that $H_{-R/2} = H_{0^*}$. The bound for the weighted energy flux through H_{0^*} follows from the decay estimate (37) in the exterior region:

$$E[F](H_{0^*}) + \int_{H_{0^*}} r^{3+\gamma_0}|\alpha|^2\,dv\,d\omega \lesssim M_0.$$

Here we note that on the boundary H_{0^*} the charge part has bounded energy. Hence take $p = 1+\gamma_0$, $\tau_1 = 0$ in the r-weighted energy estimate (31). We derive that

$$\int_{H_{\tau_2^*}} r^{3+\gamma_0}|\alpha|^2\,dv\,d\omega + \int_0^{\tau_2}\int_{H_{\tau^*}} r^{\gamma_0+2}(|\alpha|^2+|\sigma|^2+\rho^2)\,dv\,d\omega \lesssim M_0, \quad \forall \tau_2 \ge 0.$$

We conclude that there exists a dyadic sequence $\{\tau_n\}$, $n \ge 3$ such that

$$\int_{H_{\tau_n^*}} r^{\gamma_0+2}|\alpha|^2\,dv\,d\omega \lesssim (\tau_n)_+^{-1} M_0, \quad \lambda^{-1}\tau_n \le \tau_{n+1} \le \lambda\tau_n$$

for some constant λ depending only on γ_0, ϵ, R. Interpolation implies that

$$\int_{H_{\tau_n^*}} r^{1+2}|\alpha|^2\,dv\,d\omega \lesssim (\tau_n)_+^{-\gamma_0} M_0.$$

To bound $|J_{\underline{L}}||\rho|$ on the right-hand side of the energy estimate (24), we interpolate $|\rho|$ between the integrated local energy estimate and the above r-weighted energy estimate:

$$\begin{aligned}\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |J_{\underline{L}}||\rho|\,dx\,dt &\lesssim \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} \big(\epsilon_1|\rho|^2(r_+^{-\epsilon-1} + r_+^{\gamma_0}\tau_+^{-1-\gamma_0}) + \epsilon_1^{-1}\tau_+^{2\epsilon} r_+^{1-\epsilon}|J_{\underline{L}}|^2\big)\,dx\,dt \\ &\lesssim \epsilon_1 I_0^{-1-\epsilon}[F](\mathcal{D}_{\tau_1}^{\tau_2}) + \epsilon_1^{-1} M_0(\tau_1)_+^{-1-\gamma_0}, \quad \forall 1 > \epsilon_1 > 0.\end{aligned}$$

Here we have used the bound

$$r_+^{\epsilon-1}\tau_+^{-2\epsilon} \le r_+^{-1-\epsilon} + \tau_+^{-1-\gamma_0} r_+^{\gamma_0}.$$

Take ϵ_1 to be sufficiently small. From the energy estimate (24), we then obtain

$$I_0^{-1-\epsilon}[F](\mathcal{D}_{\tau_1}^{\tau_2}) + E[F](\Sigma_{\tau_2}) \lesssim E[F](\Sigma_{\tau_1}) + (\tau_1)_+^{-1-\gamma_0} M_0$$

for all $0 \le \tau_1 < \tau_2$ and $0 < \epsilon_1 < 1$. In particular, we have

$$\int_{\tau_n}^{\tau_2}\int_{\{r\le R\}\cap\{t=\tau\}} |F|^2\,dx\,d\tau \lesssim E[F](\Sigma_{\tau_1}) + (\tau_1)_+^{-1-\gamma_0} M_0.$$

Then combine this integrated local energy estimate with the r-weighted energy estimate (31) with $p=1$. For all $\tau_n \le \tau_2$, we derive that

$$\begin{aligned}\int_{\tau_n}^{\tau_2} E[F](\Sigma_\tau)\,d\tau &\lesssim \int_{\tau_n}^{\tau_2}\int_{\{r\le R\}\cap\{t=\tau\}} |F|^2\,dx\,d\tau + \int_{\tau_n}^{\tau_2}\int_{H_{\tau^*}} (|\alpha|^2+|\sigma|^2+\rho^2)r^2\,dv\,d\omega \\ &\lesssim \int_{H_{\tau_n^*}} r^{1+2}|\alpha|^2\,dv\,d\omega + E[F](\Sigma_{\tau_n}) + (\tau_n)_+^{-\gamma_0}M_0 \\ &\lesssim E[F](\Sigma_{\tau_n}) + (\tau_n)_+^{-\gamma_0}M_0.\end{aligned}$$

On the other hand, for all $\tau < \tau_2$, we have

$$E[F](\Sigma_{\tau_2}) \le E[F](\tau) + (\tau)_+^{-1-\gamma_0}M_0 \lesssim E[F](\Sigma_0) + M_0 \lesssim M_0.$$

Then from the previous estimate, we can show that

$$(\tau_2-\tau_n)E[F](\Sigma_{\tau_2}) \lesssim E[F](\Sigma_{\tau_n}) + (\tau_n)_+^{-\gamma_0}M_0 \lesssim M_0.$$

The above estimate holds for all $\tau_2 \ge \tau_n$. In particular, we obtain the coarse bound

$$E[F](\Sigma_\tau) \lesssim \tau_+^{-1}M_0, \quad \forall \tau \ge 0.$$

Based on this coarse bound, we can take $\tau_2 = \tau_{n+1}$ in the previous estimate. We then can show that

$$(\tau_{n+1}-\tau_n)E[F](\Sigma_{\tau_{n+1}}) \lesssim (\tau_n)^{-\gamma_0}M_0.$$

As $\{\tau_n\}$ is dyadic, we conclude that

$$E[F](\Sigma_{\tau_n}) \lesssim (\tau_n)^{-1-\gamma_0}M_0, \quad \forall n \ge 3.$$

Then using the energy estimate, we can show that for $\tau \in [\tau_n, \tau_{n+1}]$ we have

$$E[F](\Sigma_\tau) \lesssim E[F](\tau_n) + (\tau_n)_+^{-1-\gamma_0}M_0 \lesssim (\tau_n)_+^{-1-\gamma_0}M_0 \lesssim \tau_+^{-1-\gamma_0}M_0.$$

Having this energy flux decay, the integrated local energy decay (36) follows from the integrated local energy estimate (24). □

Since the Lie derivative $\mathcal{L}_Z$ commutes with the linear Maxwell equation from the commutator by Lemma 4, as a corollary of the above energy decay proposition, we also have the energy decay estimates for the higher-order derivatives of the Maxwell field.

Corollary 12. *We have the following energy flux decay for the k-th derivative of the Maxwell field*:

$$E[\mathcal{L}_Z^k \bar{F}](\Sigma_\tau) \lesssim (\tau)_+^{-1-\gamma_0}M_k, \quad \forall \tau \in \mathbb{R}. \tag{38}$$

This decay estimate then leads to the integrated local energy and r-weighted energy estimates for the Maxwell field.

Remark 13. By using the finite speed of propagation, the estimates in the above proposition and corollary in the exterior region depend only on the data and J in the exterior region $\{t + R \le r\}$ instead of the whole spacetime. Therefore the quantity M_k can be replaced by the corresponding one defined in the exterior region. However, the estimates in the interior region rely on the data in the whole space.

4.2. ***Pointwise bounds for the Maxwell field.*** The energy decay estimates derived in the previous section are sufficient to obtain pointwise bounds for the Maxwell field F after commuting the equation with vector fields in $\Gamma = \{\partial_t, \Omega\}$ sufficiently many times; e.g., in [Yang 2015b], four derivatives were used to show the pointwise bound for the solution. The aim of this section is to derive the pointwise bound for the Maxwell field F merely assuming M_2 is finite, that is, we commute the equation with Γ only twice. The difficulty is that we are not able to use Klainerman–Sobolev embedding to derive the decay of the solution directly as in [Lindblad and Sterbenz 2006]. Our idea is that in the inner region $\{r \le R\}$ we rely on elliptic estimates. In the outer region $\{r \ge R\}$, we analyze the solutions under the null coordinates (u, v, ω). The angular momentum Ω can be viewed as the derivative on ω. The pointwise bound then follows by using a trace theorem on the null hypersurfaces and a Sobolev embedding on the sphere. Since we do not commute the equation with L nor $\underline{L}$, those necessary energy estimates heavily rely on the null equations given in Lemma 5.

Let's first consider the pointwise bound for the Maxwell field in the inner region $\{r \le R\}$. To derive the pointwise bound, we use the vector fields ∂_t and the angular momentum Ω as commutators. Note that the angular momentum vanishes at $r = 0$. In particular we are not able to get the robust estimates for the solution in the bounded region $\{r \le R\}$ merely from the angular momentum. We thus rely on the Killing vector field ∂_t and elliptic estimates. The following proposition gives the estimates for the Maxwell field F on the bounded region $\{r \le R\}$.

Proposition 14. *For all* $0 \le \tau$ *and* $0 \le \tau_1 < \tau_2$, *we have*

$$\int_{\tau_1}^{\tau_2} \sup_{|x| \le R} |F|^2(\tau, x)\, d\tau \lesssim \int_{\tau_1}^{\tau_2} \int_{r \le R} |\nabla^2 F|^2\, dx\, dt \lesssim M_2 (\tau_1)_+^{-1-\gamma_0}, \tag{39}$$

$$|F|^2(\tau, x) \lesssim M_2 \tau_+^{-1-\gamma_0}, \quad \forall |x| \le R. \tag{40}$$

Remark 15. Estimate (40) gives the pointwise bound for F in the inner region $\{r \le R\}$ but it is weaker than the integral version (39) in the sense of decay rate. It is this integrated decay estimate that allows us to control the nonlinearities in the inner region. In other words, it is not necessary to show the improved decay of the solution in the inner region by using our approach; see, e.g., [Luk 2010]. However this does not mean that our method is not able to obtain the improved decay in the inner region. The improved decay can be derived by commuting the equation with the vector field L. For details about this, we refer to [Schlue 2013].

Proof of Proposition 14. We use elliptic estimates to prove this proposition. At fixed time t, let E and H be the electric and magnetic parts of the Maxwell field F. Let B_r be the ball with radius r, that is, $B_r = \{t \mid |x| \le r\}$. The Maxwell equation can be written as

$$\mathrm{div}(E) = J_0, \quad \partial_t H + \mathrm{curl}(E) = 0,$$
$$\mathrm{div}(H) = 0, \quad \partial_t E - \mathrm{curl}(H) = \bar{J},$$

where $\bar{J} = (J_1, J_2, J_3)$ is the spatial part of J. Therefore, using elliptic theory we derive that

$$\sum_{k\le 1} \|\partial_t^k F\|^2_{H_x^1(B_{3R/2})} \le \sum_{k\le 1} \|\partial_t^k H\|^2_{H_x^1(B_{3R/2})} + \|\partial_t^k E\|^2_{H_x^1(B_{3R/2})} \lesssim \sum_{k\le 1} \|\partial_t^k J\|^2_{L_x^2(B_{2R})} + \|\partial_t^{k+1} F\|^2_{L_x^2(B_{2R})}.$$

Make use of the above estimates with $k = 1$. Differentiate the linear Maxwell equation with the spatial covariant derivative $\overline{\nabla}$. Using elliptic estimates again, we then obtain

$$\|\nabla F\|^2_{H_x^1(B_R)} \lesssim \|\nabla J\|^2_{L_x^2(B_{2R})} + \|\partial_t^2 F\|^2_{L_x^2(B_{2R})}.$$

Here we omitted the lower-order terms. Integrate the above inequality from time τ_1 to τ_2. We derive

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\int_{r\le R} |\nabla^2 F|^2\,dx\,dt &\lesssim \int_{\tau_1}^{\tau_2}\int_{r\le 2R} |\partial_t^2 F|^2 + |\nabla J|^2\,dx\,dt\\ &\lesssim I_0^{-1-\epsilon}[\partial_t^2 F](\mathcal{D}_{\tau_1^+}^{\tau_2}) + I_0^{-1-\epsilon}[\partial_t J](\mathcal{D}_{\tau_1^+}^{\tau_2}) + I_0^0[\overline{\nabla} J](\mathcal{D}_{\tau_1}^{\tau_2} \cap \{r \le 2R\})\\ &\lesssim M_2(\tau_1)_+^{-1-\gamma_0}.\end{aligned}$$

Here $\tau_1^+ = \max\{\tau_1 - R, 0\}$. The estimate (39) then follows using Sobolev embedding.

For the pointwise bound (40), first we note that

$$\int_{r\le 2R} |\overline{\nabla} J|^2\,dx \lesssim \sum_{k\le 1}\int_\tau^{\tau+1} |\overline{\nabla}\mathcal{L}_Z^k J|^2\,dx\,dt \lesssim M_2\tau_+^{-1-\gamma_0}.$$

Consider the energy estimate on the region $\mathcal{D}_1$ bounded by Σ_{τ^+}, $\tau^+ = \max\{\tau - R, 0\}$ and $t = \tau$, $\tau \ge 0$. From the energy estimate (24), we conclude that

$$\int_{r\le 2R} |\mathcal{L}_Z^2 F|^2\,dx = E[\mathcal{L}_Z^2 F](r \le 2R) \lesssim E[\mathcal{L}_Z^2 F](\Sigma_{\tau^+}) + I_0^{1+\epsilon}[\mathcal{L}_Z^2 J](\mathcal{D}_1) \lesssim M_2\tau_+^{-1-\gamma_0}.$$

Thus the pointwise bound (40) holds. □

To show the decay of the solution via the energy flux through the null hypersurface, we rely on the following trace theorem.

Lemma 16. *Let $f(r, \omega)$ be a smooth function defined on $[a, b] \times \mathbb{S}^2$. Then*

$$\left(\int_\omega |f|^4(r_0, \omega)\,d\omega\right)^{1/2} \le C\int_a^b\int_\omega |f|^2 + |\partial_r f|^2 + |\partial_\omega f|^2\,d\omega\,dr, \quad \forall r_0 \in [a, b] \tag{41}$$

for some constant C independent of r_0.

Proof. The condition implies that $f \in H^1_{r,\omega}$. By using the trace theorem, we have

$$\|f(r_0, \cdot)\|_{H_\omega^{1/2}} \le C\|f\|_{H^1_{r,\omega}}, \quad \forall r_0 \in [a, b].$$

The lemma then follows using Sobolev embedding on the sphere. □

Using this lemma, we are now able to show the pointwise bound for the Maxwell field when $\{r \geq R\}$.

Proposition 17. *Let $\bar{\mathcal{D}}_{\tau_1} = \mathcal{D}_{\tau_1} \cap \{r \geq R\}$. Then we have*

$$\|r\mathcal{L}_Z^k \underline{\alpha}\|^2_{L_u^2 L_v^\infty L_\omega^2(\bar{\mathcal{D}}_{\tau_1})} \lesssim M_2(\tau_1)_+^{-1-\gamma_0+2\epsilon}, \quad k = 0, 1, \tag{42}$$

$$|r\underline{\alpha}|^2(\tau, v, \omega) \lesssim M_2\tau_+^{-1-\gamma_0}, \tag{43}$$

$$r^p(|r\alpha|^2 + |r\sigma|^2)(\tau, v, \omega) \lesssim M_2\tau_+^{p-1-\gamma_0}, \quad 0 \leq p \leq 1+\gamma_0, \tag{44}$$

$$r^p|r\bar{\rho}|^2(\tau, v, \omega) \lesssim M_2\tau_+^{p-1-\gamma_0}, \quad 0 \leq p \leq 1-\epsilon, \tag{45}$$

$$\|r\mathcal{L}_Z^k \sigma\|^2_{L_v^2 L_u^\infty L_\omega^2(\bar{\mathcal{D}}_\tau)} \lesssim M_2\tau_+^{-1-\gamma_0+\epsilon}, \quad k \leq 1. \tag{46}$$

Here recall that Z is a vector field in the set $\Gamma = \{\partial_t, \Omega_{ij}\}$.

Remark 18. In terms of decay rate, the integral version (42) is stronger than the pointwise bound (43). We are not able to improve the u decay of the Maxwell field due to the weak decay rate of the initial data. However the integral version improves one order of decay in u (or τ as $u = \frac{1}{2}(\tau - R)$). This is the key point that allows us to construct the global solution with the weak decay rate of the initial data.

Proof of Proposition 17. For the integral estimate (42), we rely on the transport equation (8) for $\underline{\alpha}$. For the case in the exterior region, one can choose the initial hypersurface $\{t = 0\}$. In the interior region, for all $0 \leq \tau_1 < \tau_2$, we can choose the incoming null hypersurface $\underline{H}^{\tau_1^*, \tau_2^*}_{(\tau_2+R)/2}$. Let's only consider the case in the interior region. From (8) for $\underline{\alpha}$ under the null frame, for $k = 0$ or 1, we can show that

$$\begin{aligned}
\|r\mathcal{L}_Z^k \underline{\alpha}\|^2_{L_u^2 L_v^\infty L_\omega^2(\bar{\mathcal{D}}_{\tau_1}^{\tau_2})} &\lesssim E[\mathcal{L}_Z^k \underline{\alpha}]\big(\underline{H}^{\tau_1^*, \tau_2^*}_{(\tau_2+R)/2}\big) + I_0^{-1-\epsilon}[\mathcal{L}_Z^k \underline{\alpha}](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}) + \|r^{(1+\epsilon)/2} L\mathcal{L}_Z^k (r\underline{\alpha})\|^2_{L_u^2 L_v^2 L_\omega^2(\bar{\mathcal{D}}_{\tau_1}^{\tau_2})} \\
&\lesssim M_2(\tau_1)_+^{-1-\gamma_0} + \|r^{(1+\epsilon)/2}(|\mathcal{L}_Z^{k+1}\rho| + |\mathcal{L}_Z^{k+1}\sigma|)\|^2_{L_u^2 L_v^2 L_\omega^2(\bar{\mathcal{D}}_{\tau_1}^{\tau_2})} + I_0^{1+\epsilon}[\mathcal{L}_Z^k \not{J}](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}) \\
&\lesssim M_2(\tau_1)_+^{-1-\gamma_0+2\epsilon}.
\end{aligned}$$

Here we use interpolation to bound ρ and σ. Indeed, the integrated local energy estimate implies that

$$\iint_{\bar{\mathcal{D}}_{\tau_1}^{\tau_2}} r^{-\epsilon+1}(|\mathcal{L}_Z^{k+1}\rho|^2 + |\mathcal{L}_Z^{k+1}\sigma|^2)\, du\, dv\, d\omega \lesssim M_2(\tau_1)_+^{-1-\gamma_0}.$$

On the other hand, the r-weighted energy estimate shows that

$$\iint_{\bar{\mathcal{D}}_{\tau_1}^{\tau_2}} r^{2+\gamma_0}(|\mathcal{L}_Z^{k+1}\rho|^2 + |\mathcal{L}_Z^{k+1}\sigma|^2)\, du\, dv\, d\omega \lesssim M_2.$$

Interpolation then implies the estimate for ρ and σ. Thus estimate (42) holds.

For the pointwise bound (43) for $\underline{\alpha}$, we rely on the energy flux on the incoming null hypersurface together with Lemma 16. Consider the point (τ, v, ω). In the exterior region when $\tau < 0$, let $\underline{H}_\tau = \underline{H}_v^{\tau^*, -v}$ be the incoming null hypersurface extending to the initial hypersurface $\{t = 0\}$. In the interior region when $\tau \geq 0$, we instead let $\underline{H}_\tau$ be $\underline{H}_v^{\tau, 2v-R}$, which is the incoming null hypersurface truncated by $\{r = R\}$.

From the energy estimates (24) and (25), we conclude that

$$\int_{\underline{H}_\tau} |r\mathcal{L}_Z^k \underline{\alpha}|^2 \, du \, d\omega \lesssim E[\mathcal{L}_Z^k F](\underline{H}_\tau) \lesssim M_2 \tau_+^{-1-\gamma_0}, \quad \forall k \le 2.$$

As Z may be ∂_t or the angular momentum Ω, to apply Lemma 16, we need the energy flux of the tangential derivative $\underline{L}(\underline{\alpha})$. We make use of the structure equation (8), which implies that

$$\begin{aligned}
\int_{\underline{H}_\tau} |\underline{L}\mathcal{L}_Z^k(r\underline{\alpha})|^2 \, du \, d\omega &\lesssim \int_{\underline{H}_\tau} (|L\mathcal{L}_Z^k(r\underline{\alpha})|^2 + |\mathcal{L}_{\partial_t}\mathcal{L}_Z^k(r\underline{\alpha})|^2) \, du \, d\omega \\
&\lesssim \int_{\underline{H}_\tau} (|\mathcal{L}_Z^{k+1}\rho|^2 + |\mathcal{L}_Z^{k+1}\sigma|^2 + |\mathcal{L}_Z^k(r\not{J})|^2 + |\mathcal{L}_Z^{k+1}(r\underline{\alpha})|^2) \, du \, d\omega \\
&\lesssim E[\mathcal{L}_Z^{k+1} F](\underline{H}_\tau) + I_0^0[\mathcal{L}_Z^{k+1}\not{J}](\mathcal{D}_\tau) \\
&\lesssim M_2 \tau_+^{-1-\gamma_0}, \quad k \le 1.
\end{aligned}$$

Here note that $\Omega = (re_1, re_2)$. Then by Lemma 16, for all v and fixed τ,

$$\left(\int_\omega |r\mathcal{L}_Z \underline{\alpha}|^4 (\tau, v, \omega) \, d\omega \right)^{1/2} \lesssim M_2 \tau_+^{-1-\gamma_0}, \quad k \le 1.$$

Estimate (43) then follows using Sobolev embedding on the sphere.

For the pointwise bound (44), (45) for α, σ, $\bar{\rho}$, the proof for α is slightly different from that of σ and $\bar{\rho}$. However, the idea is the same. Let's consider α first. Consider H_{τ^*}, $\tau \in \mathbb{R}$. The r-weighted energy estimates (31), (32) imply that

$$\int_{H_{\tau^*}} r^p |r\mathcal{L}_Z^k \alpha|^2 \, dv \, d\omega \lesssim M_2 \tau_+^{p-1-\gamma_0}, \quad \forall 0 \le p \le 1+\gamma_0, \quad k \le 2.$$

To apply Lemma 16, we need the energy flux of the tangential derivative $L(r\alpha)$. Similar to the case of $\underline{\alpha}$, we make use of (10) and the ∂_t derivative:

$$\begin{aligned}
\int_{H_{\tau^*}} r^p |L(r\mathcal{L}_Z^k \alpha)|^2 \, dv \, d\omega &\lesssim \int_{H_{\tau^*}} r^p (|\underline{L}(r\mathcal{L}_Z^k \alpha)|^2 + |r\partial_t \mathcal{L}_Z^k \alpha|^2) \, dv \, d\omega \\
&\lesssim \int_{H_{\tau^*}} r^p (|\mathcal{L}_Z^{k+1}\rho|^2 + |\mathcal{L}_Z^{k+1}\sigma|^2 + |\mathcal{L}_Z^k(r\not{J})|^2 + |\mathcal{L}_Z^{k+1}(r\alpha)|^2) \, dv \, d\omega \\
&\lesssim M_2 \tau_+^{p-1-\gamma_0} + \int_{H_{\tau^*}} r^2 (|\mathcal{L}_Z^{k+1}\rho|^2 + |\mathcal{L}_Z^{k+1}\sigma|^2) + r^p |\mathcal{L}_Z^k(r\not{J})|^2 \, dv \, d\omega \\
&\lesssim M_2 \tau_+^{p-1-\gamma_0} + E[\mathcal{L}_Z^k F](H_{\tau^*}) + I_0^p[\mathcal{L}_Z^{k+1}\not{J}](\mathcal{D}_\tau) \\
&\lesssim M_2 \tau_+^{p-1-\gamma_0}
\end{aligned}$$

for $k \le 1$. The estimate for α then follows from Lemma 16 together with Sobolev embedding on the unit sphere.

For $\bar{\rho}$, σ, we make use of the r-weighted energy estimates (31), (32) through the incoming null hypersurface $\underline{H}_\tau$ defined as above. First, we have

$$\int_{\underline{H}_\tau} r^{p-2}(|\mathcal{L}_Z^k(r^2\bar{\rho})|^2 + |\mathcal{L}_Z^k(r^2\sigma)|^2)\,du\,d\omega \lesssim M_2\tau_+^{p-1-\gamma_0}, \quad k \le 2.$$

To derive the tangential derivative $\underline{L}(r^2\bar{\rho})$, $\underline{L}(r^2\sigma)$, we use the equations (7) and (9). We can show that

$$\begin{aligned}\int_{\underline{H}_\tau} r^{p-2}(|\underline{L}(r^2\mathcal{L}_Z^k\bar{\rho})|^2\,du\,d\omega &\lesssim \int_{\underline{H}_\tau} r^{p-2}(|r\mathcal{L}_Z^{k+1}\underline{\alpha}|^2 + |r^2\mathcal{L}_Z^k J_{\underline{L}}|^2)\,du\,d\omega\\ &\lesssim E[\mathcal{L}_Z^{k+1}F](\underline{H}_\tau) + I_0^p[\mathcal{L}_Z^{k+1}J_{\underline{L}}](\mathcal{D}_\tau)\\ &\lesssim M_2\tau_+^{p-1-\gamma_0}, \quad k = 0, 1\end{aligned}$$

for all $0 \le p \le 1-\epsilon$. We cannot extend p to the full range of $[0, 1+\gamma_0]$ due the weak assumption on $J_{\underline{L}}$. The equation (9) for σ does not involve $J_{\underline{L}}$. We hence have the full range $0 \le p \le 1+\gamma_0$ for σ. Lemma 16 and Sobolev embedding on the sphere then lead to the pointwise bound for $\bar{\rho}$ and σ. We thus have shown estimates (44), (45).

Finally, for the integrated decay estimates (46), we proceed by integrating along the incoming null hypersurface. In the interior region case we integrate from $\{r = R\}$, while in the exterior region we integrate from the initial hypersurface $\{t = 0\}$. Let's only prove (46) for the interior region case. In particular, take $\overline{\mathcal{D}}_\tau$ to be $\overline{\mathcal{D}}_{\tau_1}^{\tau_2}$ for $0 \le \tau_1 < \tau_2$. First, using the decay estimate (39) for F when $r \le R$, we can show that on the boundary $\{r = R\}$,

$$\int_{\tau_1}^{\tau_2}\int_\omega |\mathcal{L}_Z^k F|^2(\tau, R, \omega)\,d\omega\,d\tau \lesssim \int_{\tau_1}^{\tau_2}\int_{r\le R} |\nabla\mathcal{L}_Z^k F|^2\,dx\,d\tau \lesssim M_2(\tau_1)_+^{-1-\gamma_0}.$$

Then from the transport equations (7) and (9), we can show that

$$\begin{aligned}\|r\mathcal{L}_Z^k\sigma\|^2_{L_v^2L_u^\infty L_\omega^2(\overline{\mathcal{D}}_{\tau_1}^{\tau_2})} &\lesssim \int_{\tau_1}^{\tau_2} |\mathcal{L}_Z^k F|^2(\tau, R, \omega)\,d\omega\,d\tau + \iint_{\overline{\mathcal{D}}_{\tau_1}^{\tau_2}} (r|\mathcal{L}_Z^k\sigma|^2 + |\mathcal{L}_Z^k\sigma\cdot\underline{L}(r^2\mathcal{L}_Z^k\sigma)|)\,du\,dv\,d\omega\\ &\lesssim M_2(\tau_1)_+^{-1-\gamma_0} + \iint_{\overline{\mathcal{D}}_{\tau_1}^{\tau_2}} (r^{1+\epsilon}|\mathcal{L}_Z^k\sigma|^2 + r^{1-\epsilon}|\mathcal{L}_Z^{k+1}\underline{\alpha}|^2)\,du\,dv\,d\omega\\ &\lesssim M_2(\tau_1)_+^{-1-\gamma_0} + M_2(\tau_1)_+^{-1-\gamma_0+\epsilon} \lesssim M_2(\tau_1)_+^{-1-\gamma_0+\epsilon}.\end{aligned}$$

Here we have used the r-weighted energy estimates for σ with $p = \epsilon$ and the integrated local energy estimates to bound $\underline{\alpha}$. This proves (46). □

4.3. ***Energy decay for the scalar field.*** In this section, we study the energy decay for the complex scalar field ϕ satisfying the linear covariant wave equation. When the connection field A is trivial, the energy decay has been well studied using the new approach; see, e.g., [Yang 2013]. For a general connection field A, presumably not small, new difficulty arises as there are interaction terms between the curvature dA and the scalar field. In the previous subsection, we derived the energy flux decay for the Maxwell field $F = dA$ with appropriate bound on $J = \delta F$. The purpose of this section is to derive energy flux decay for the complex scalar field.

In addition to the assumption that M_k is finite, for the general complex scalar field ϕ, we assume the inhomogeneous term $\Box_A\phi$ and the initial data are bounded in the norm

$$\mathcal{E}_k[\phi] = E_0^k[\phi] + \sum_{l\le k} + I_{1+\epsilon}^{1+\gamma_0}[D_Z^l \Box_A \phi](\{t\ge 0\}) + I_{1+\gamma_0}^{1+\epsilon}[D_Z^l \Box_A \phi](\{t\ge 0\}). \tag{47}$$

Here in this section we will estimate the general complex scalar field ϕ in terms of the initial data and the inhomogeneous term $\Box_A\phi$. For solutions of (MKG), the complex scalar field ϕ verifies the linear covariant wave equation $\Box_A\phi = 0$. In particular, if (ϕ, A) solves (MKG), then $\mathcal{E}_k[\phi] = E_0^k[\phi]$, which denotes the weighted Sobolev norm of the initial data for the complex scalar field.

As the estimates in the interior region require information on the boundary Σ_0, which contains the boundary H_{0^*} of the exterior region, we need first to obtain the energy decay estimates in the exterior region. The main difficulty in the presence of a nontrivial connection field is to control the interaction term $(dA)_{\mu\nu}J^\nu[\phi]$ under mild assumptions on the curvature dA. In the integrated local energy estimate (23) for the scalar field, it is not possible to control or absorb those terms as there is no smallness assumption on dA. The idea is to make use of the null structure of $J^\nu[\phi]$ together with the r-weighted energy estimate (28). More precisely, we first control those terms in the r-weighted energy estimate via Gronwall's inequality. Then we estimate those terms in the integrated local energy estimates. Once we have control of those interaction terms, the decay of the energy flux follows from the standard argument of the new approach, similar to that of the energy decay for the Maxwell field in the previous section.

We first prove a lemma used to control the scalar field ϕ by using the r-weighted energy.

Lemma 19. *Assume ϕ vanishes at null infinity. In the exterior region on H_u, we have*

$$\int_\omega |r\phi|^2(u, v, \omega)\, d\omega \lesssim \int_\omega |r\phi|^2(u, -u, \omega)\, d\omega + \beta^{-1}u_+^{-\beta}\int_{-u}^{v}\int_\omega r^{1+\beta}|D_L(r\phi)|^2\, dv\, d\omega, \quad \forall \beta > 0. \tag{48}$$

In the interior region on Σ_τ, for $1\le p\le 2$, we have

$$\int_\omega r^p|\phi|^2\, d\omega \lesssim (E[\phi](\Sigma_\tau))^{\delta_p}(I_0^{1+\gamma_0}[r^{-1}D_L(r\phi)](H_{\tau^*}))^{1-\delta_p}, \qquad \delta_p = \frac{2+\gamma_0-p}{1+\gamma_0}. \tag{49}$$

Moreover on Σ_τ, $\tau\in\mathbb{R}$, we have

$$r\int_\omega |\phi|^2\, d\omega \lesssim \epsilon_1^{-1}\int_\Sigma |\phi|^2\, d\tilde{v}\, d\omega + \epsilon_1 E[\phi](\Sigma_\tau) \tag{50}$$

for all $0<\epsilon_1\le 1$. Here $(\tilde{v}, \omega) = (v, \omega)$ when $r\ge R$ or (r, ω) when $r < R$.

Proof. Estimate (48) follows from the inequality

$$|r\phi|(u, v, \omega) \le |r\phi|(u, -u, \omega) + \int_{-u}^{v} |D_L(r\phi)|\, dv$$

followed by the Cauchy–Schwarz inequality.

In the interior region, the problem is that we cannot integrate from the initial hypersurface nor the boundary H_{0^*} nor the null infinity as the behavior of $r\phi$ at null infinity is unknown (generically not zero). However, the scalar field ϕ vanishes at null infinity. We thus can bound $r|\phi|^2$ by the energy flux through Σ_τ. More precisely, on Σ we can show that

$$\begin{aligned} r\int_\omega |\phi|^2\,d\omega &\lesssim \int_{\Sigma_\tau} |\phi|^2\,d\tilde{v}\,d\omega + \int_{\Sigma_\tau} r|D_{\tilde{v}}\phi|\,|\phi|\,d\tilde{v}\,d\omega \\ &\lesssim \epsilon_1 \int_{\Sigma_\tau} |D_{\tilde{v}}\phi|^2 r^2\,d\tilde{v}\,d\omega + (\epsilon_1^{-1}+1)\int_{\Sigma_\tau} |\phi|^2\,d\tilde{v}\,d\omega \\ &\lesssim \epsilon_1 E[\phi](\Sigma_\tau) + \epsilon_1^{-1}\int_{\Sigma_\tau} |\phi|^2\,d\tilde{v}\,d\omega. \end{aligned}$$

This gives estimate (50). In particular, for $\epsilon_1 = 1$, from Hardy's inequality (21) we conclude that estimate (49) holds for $p = 1$. To prove it for all $1 \le p \le 2$, it suffices to show the estimate with $p = 2$. Consider the sphere with radius $r = \frac{1}{2}(\tau^* + v)$ on $H_{\tau^*} \subset \Sigma_\tau$. Choose the sphere with radius $r_1 = \frac{1}{2}(\tau^* + v_1)$ such that

$$r_1^{1+\gamma_0} = E[\phi](\Sigma_\tau)^{-1}\int_{H_{\tau^*}} r^{1+\gamma_0}|D_L(r\phi)|^2\,dv\,d\omega.$$

If $r \le r_1$, then (49) with $p = 2$ follows from (49) with $p = 1$. Otherwise, we have $r_1 < r$. Then

$$\begin{aligned} \int_\omega |r\phi|^2(\tau^*, v, \omega)\,d\omega &\lesssim \int_\omega |r\phi|^2(\tau^*, v_1, \omega) + r_1^{-\gamma_0}\int_{H_{\tau^*}} r^{1+\gamma_0}|D_L(r\phi)|^2\,dv\,d\omega \\ &\lesssim r_1 E[\phi](\Sigma_\tau) + r_1^{-\gamma_0} I_0^{1+\gamma_0}[r^{-1}D_L\psi](H_{\tau^*}) \\ &\lesssim (E[\phi](\Sigma_\tau))^{\gamma_0/(1+\gamma_0)}(I_0^{1+\gamma_0}[r^{-1}D_L\psi](H_{\tau^*}))^{1/(1+\gamma_0)}. \end{aligned}$$

Here we recall the notation I defined in Section 2. □

The following lemma is very simple but it turns out to be very useful.

Lemma 20. *Suppose $f(\tau)$ is smooth. Then for any $\beta \neq 0$, we have the identity*

$$\int_{\tau_1}^{\tau_2} s^\beta f(s)\,ds = \beta\int_{\tau_1}^{\tau_2} \tau^{\beta-1}\int_\tau^{\tau_2} f(s)\,ds\,d\tau + \tau_1^\beta\int_{\tau_1}^{\tau_2} f(s)\,ds.$$

4.3.1. *Energy decay in the exterior region.* In the exterior region, as $r \ge \frac{1}{3}u_+$, it suffices to consider the r-weighted energy estimate for the largest $p = 1 + \gamma_0$. First we can show the following proposition.

Proposition 21. *In the exterior region, for all $\tau_2 < \tau_1 \le 0$, we have*

$$\begin{aligned} \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{1+\gamma_0}|F_{L\mu}J^\mu[\phi]|\,dx\,dt \lesssim M_2 E_0^0[\phi] + M_2\int_u u_+^{-1-\epsilon}\int_v r^{1+\gamma_0}|D_L\psi|^2\,dv\,d\omega\,du \\ + |q_0|\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{\gamma_0}(|D_L(r\phi)|^2 + |\not{D}(r\phi)|^2)\,dv\,du\,d\omega. \end{aligned} \tag{51}$$

Proof. As $F = dA$ has different decay properties for different components, we estimate the integral according to the index μ. Denote $\psi = r\phi$. Note that $r^2 J[\phi] = J[r\phi]$. For $\mu = \underline{L}$, we have

$$|F_{L\underline{L}} J^{\underline{L}}[\phi]| \lesssim r^{-2}|q_0|\,|D_L\psi||\psi| + |\bar{\rho}||D_L\psi||\psi|. \tag{52}$$

The first term on the right-hand side will be absorbed with the smallness assumption on the charge q_0 (as the data for the scalar field is small). Indeed, using Lemma 6 we can show that

$$\begin{aligned} 2\iint r^{\gamma_0-1}|D_L\psi||\psi|\,du\,dv\,d\omega &\le \iint r^{\gamma_0}|D_L\psi|^2\,dv\,du\,d\omega + \iint r^{\gamma_0}|\phi|^2\,dv\,du\,d\omega \\ &\lesssim \iint r^{\gamma_0}|D_L\psi|^2\,dv\,du\,d\omega + \int_u\int_\omega (r^{1+\gamma_0}|\phi|^2)(u,-u,\omega)\,d\omega\,du \\ &\lesssim \iint r^{\gamma_0}|D_L\psi|^2\,dv\,du\,d\omega + E_0^0[\phi]. \end{aligned}$$

For the second term on the right-hand side of (52), the idea is that we use the Cauchy–Schwarz inequality and make use of the r-weighted energy estimate. First, we can estimate that

$$2r^{1+\gamma_0}|\bar{\rho}||D_L\psi||\psi| \le r^{1+\gamma_0}|D_L\psi|^2 u_+^{-1-\epsilon} + u_+^{1+\epsilon} r^2|\bar{\rho}|^2 r^{1+\gamma_0}|\phi|^2.$$

The first term will be controlled through Gronwall's inequality. For the second term, we can first use Sobolev embedding on the unit sphere to bound $\bar{\rho}$ and then apply Lemma 19:

$$\begin{aligned} &\iint u_+^{1+\epsilon} r^2|\bar{\rho}|^2 r^{1+\gamma_0}|\phi|^2\,du\,dv\,d\omega \\ &\qquad\lesssim \int_u u_+^{1+\epsilon}\int_v \sum_{j\le 2} r^2\int_\omega |\mathcal{L}_\Omega^j\bar{\rho}|^2\,d\omega\cdot\int_\omega r^{1+\gamma_0}|\phi|^2\,d\omega\,dv\,du \\ &\qquad\lesssim \int_u u_+^{1+\epsilon-1} E[\mathcal{L}_Z^2\bar{F}](H_u)\bigg(u_+^{\gamma_0}\int_\omega |r\phi|^2(u,-u,\omega)\,d\omega + \int_v\int_\omega r^{1+\gamma_0}|D_L\psi|^2\,dv\,d\omega\bigg)\,du \\ &\qquad\lesssim M_2\int_{|x|\ge R} r_+^{1+\gamma_0-\epsilon-2}|\phi|^2(0,x)\,dx + M_2\int_u u_+^{-1-\epsilon}\int_v r^{1+\gamma_0}|D_L\psi|^2\,dv\,d\omega\,du. \end{aligned}$$

The first term is bounded by the weighted Sobolev norm of the initial data. The second term can be controlled by using Gronwall's inequality. Thus estimate (51) holds for the case $\mu = L$.

For $\mu = e_1$ or e_2, first we can bound

$$r^{1+\gamma_0}|F_{Le_j}||J^{e_j}[\psi]| \le \epsilon_1 r^{\gamma_0}|\nabla\!\!\!/\psi|^2 + \epsilon_1^{-1} r^{3+\gamma_0}|\alpha|^2 r|\phi|^2, \quad \forall\epsilon_1 > 0.$$

We choose sufficiently small ϵ_1 so that the integral of the first term can be absorbed. For the second term, we first use Sobolev embedding on the unit sphere to bound α and then Lemma 19 to control ϕ:

$$
\begin{aligned}
&\iint r^{3+\epsilon}|\alpha|^2 r^{1+\gamma_0-\epsilon}|\phi|^2\,du\,dv\,d\omega\\
&\lesssim \int_u u_+^{\gamma_0-\epsilon-1}\int_v r^{3+\epsilon}\sum_{j\le 2}\int_\omega |\mathcal{L}_\Omega^j\alpha|^2\,d\omega\cdot\left(\int_\omega |r\phi|^2(u,-u,\omega)\,d\omega+u_+^{-\gamma_0}\int_v\int_\omega r^{1+\gamma_0}|D_L\psi|^2\,dv\,d\omega\right)du\\
&\lesssim M_2\int_{|x|\ge R} r_+^{1+\gamma_0-\epsilon-2}|\phi|^2(0,x)\,dx+M_2\int_u u_+^{-1-\epsilon}\int_v r^{1+\gamma_0}|D_L\psi|^2\,dv\,d\omega\,du\\
&\lesssim M_2E_0^0[\phi]+M_2\int_u u_+^{-1-\epsilon}\int_v r^{1+\gamma_0}|D_L\psi|^2\,dv\,d\omega\,du.
\end{aligned}
$$

As the data for the scalar field is small, the charge is also small. In particular, we can choose $\epsilon_1=|q_0|$ (if $q_0=0$, let ϵ_1 be small depending only on ϵ, γ_0 and R). Therefore estimate (51) holds for the case when $\mu=e_1$ or e_2. This completes the proof. □

As a corollary, we show the r-weighted energy flux decay of the scalar field in the exterior region.

Corollary 22. *Assume that the charge q_0 is sufficiently small, depending only on ϵ, R, γ_0. Then in the exterior region, we have the energy flux decay*

$$
\begin{aligned}
&\int_{H_{\tau_1^*}} r^p|D_L\psi|^2\,dv\,d\omega+\iint_{\mathcal{D}_{\tau_1}} r^{p-1}(p|D_L\psi|^2+|\not{D}\psi|^2)\,dv\,d\omega\,du+\int_{\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}} r^p|\not{D}\psi|^2\,du\,d\omega\\
&\qquad\lesssim_{M_2}\mathcal{E}_0[\phi](\tau_1)_+^{p-1-\gamma_0},\quad \forall 0\le p\le 1+\gamma_0,\quad \forall\tau_2\le\tau_1\le 0,\quad \psi=r\phi. \qquad (53)
\end{aligned}
$$

Proof. It suffices to prove the corollary for $p=1+\gamma_0$. For sufficiently small q_0 depending only on ϵ, γ_0 and R, from the r-weighted energy estimate (28) and the estimate (51) for the error term, the integral of $r^{\gamma_0}(|D_L(r\phi)|^2+|\not{D}(r\phi)|^2)$ can be absorbed. Then estimate (53) follows from Gronwall's inequality. □

Next we make use of the r-weighted energy decay to show the energy flux decay and the integrated energy decay for the scalar field in the exterior region. From the integrated energy estimate (23), it suffices to bound the interaction term of the gauge field and the scalar field.

Proposition 23. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $\tau_2<\tau_1\le 0$, we have*

$$
\begin{aligned}
&\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{L\nu}J^\nu[\phi]|+|F_{\underline{L}\nu}J^\nu[\phi]|\,dx\,dt\\
&\qquad\lesssim \epsilon_1 I_0^{-1-\epsilon}[D\phi](\mathcal{D}_{\tau_1}^{\tau_2})+C_{M_2,\epsilon_1}\left(\mathcal{E}_0[\phi](\tau_1)_+^{-1-\gamma_0}+(\tau_1)_+^{\epsilon}\int_{-\tau_1^*}^{-\tau_2^*} v^{-1-\epsilon}E[\phi]\big(\underline{H}_v^{-v,\tau_1^*}\big)\,dv\right) \qquad (54)
\end{aligned}
$$

for all $\epsilon_1>0$ and some constant C_{M_2,ϵ_1} depending on M_2 and ϵ_1.

Proof. The integral of $(dA)_{L\nu}J^\nu[\phi]$ has been controlled in the previous Proposition 21 as Corollary 22 implies that the right-hand side of (51) can be bounded by a constant depending on M_2, ϵ, γ_0 and R.

Since in the exterior region $r \geq \frac{1}{3}u_+$, we easily obtain the desired bound:

$$\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{L\nu} J^\nu[\phi]|\, dx\, dt \lesssim_{M_2} (\tau_1)_+^{-1-\gamma_0} \mathcal{E}_0[\phi].$$

It remains to estimate the integral of $F_{\underline{L}\nu} J^\nu[\phi]$. The r-weighted energy decay gives control for the "good" derivative of the scalar field. The problem is that we do not have any control for the "bad" derivative $D_{\underline{L}}\phi$. In addition, since the charge is nonzero, we are not able to absorb the charge part $q_0 r^{-2} J_{\underline{L}}[\phi]$ in the integrated local energy estimate (23) as there is a small ϵ loss of decay in $I_0^{-1-\epsilon}[\tilde{D}\phi]$ on the left-hand side. The idea to treat this term is to make use of the energy flux on the incoming null hypersurface $\underline{H}_{-u_2}^{u_2,u_1}$ and then apply Gronwall's inequality. Let's first consider the easier terms in the integral of $F_{\underline{L}\nu} J^\nu[\phi]$. For $\nu = e_1$ or e_2, we have

$$|F_{\underline{L}\nu} J^\nu[\phi]| \lesssim |\underline{\alpha}|\, |\not{D}\phi|\, |\phi|.$$

Note that from estimate (48) of Lemma 19 and Corollary 22, we obtain

$$\int_\omega |r\phi|^2(u, v, \omega)\, d\omega \lesssim_{M_2} u_+ \int_\omega (-2u)|\phi(0, -2u, \omega)|^2\, d\omega + \mathcal{E}_0[\phi] u_+^{-\gamma_0}.$$

Here we parametrize ϕ in (t, r, ω) coordinates. We then use Sobolev embedding on the initial hypersurface $\{t = 0\}$ to derive the decay of ϕ:

$$\int_\omega |r\phi|^2(u, v, \omega)\, d\omega \lesssim_{M_2} \mathcal{E}_0[\phi] u_+^{-\gamma_0}. \tag{55}$$

From the r-weighted energy estimate (53), we have an estimate for the weighted angular derivative of the scalar field on the incoming null hypersurface:

$$\int_{\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}} r^{1+\gamma_0} |\not{D}(r\phi)|^2\, du\, d\omega \lesssim_{M_2} \mathcal{E}_0[\phi].$$

In the exterior region, note that $r \geq \frac{1}{2}v$. Therefore we can show that

$$\begin{aligned}
&\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{\underline{L}e_j}|\, |J^{e_j}[\phi]|\, dx\, dt \\
&\quad\lesssim \int_{-\tau_1^*}^{-\tau_2^*} \int_{-v}^{\tau_1^*} \int_\omega r^2 |\underline{\alpha}|\, |\not{D}\phi|\, |\phi|\, d\omega\, du\, dv \\
&\quad\lesssim \int_{-\tau_1^*}^{-\tau_2^*} \int_{-v}^{\tau_1^*} r^{-\frac{1}{2}(3+\gamma_0)} \Big(r^2 \sum_{j\leq 2} \int_\omega |\mathcal{L}_\Omega^j \underline{\alpha}|^2\, d\omega \Big)^{\frac{1}{2}} \Big(r^{3+\gamma_0} \int_\omega |\not{D}\phi|^2\, d\omega \cdot \int_\omega |r\phi|^2\, d\omega \Big)^{\frac{1}{2}}\, du\, dv \\
&\quad\lesssim_{M_2} \mathcal{E}_0[\phi]^{\frac{1}{2}} (\tau_1)_+^{-\frac{1}{2}\gamma_0} \int_{-\tau_1^*}^{-\tau_2^*} v^{-\frac{1}{2}(3+\gamma_0)} \big(E[\mathcal{L}_Z^2 dA]\big(\underline{H}_v^{-v,\tau_1^*}\big)\big)^{\frac{1}{2}} \mathcal{E}_0[\phi]^{\frac{1}{2}}\, dv \\
&\quad\lesssim_{M_2} \mathcal{E}_0[\phi] (\tau_1)_+^{-\frac{1}{2}\gamma_0 - \frac{1}{2}(1+\gamma_0) - \frac{1}{2}} \lesssim_{M_2} \mathcal{E}_0[\phi] (\tau_1)_+^{-1-\gamma_0}.
\end{aligned}$$

When $\nu = L$, first we have

$$|F_{\underline{L}L}||J^L[\phi]| \lesssim |q_0| r^{-2} |D_{\underline{L}}\phi||\phi| + |\bar{\rho}||D_{\underline{L}}\phi||\phi|.$$

The second term is easy to bound. We may use the Cauchy–Schwarz inequality. Indeed,

$$2|\bar{\rho}||D_{\underline{L}}\phi||\phi| \le \epsilon_1 |D_{\underline{L}}\phi|^2 r^{-1-\epsilon} + \epsilon_1^{-1}|\bar{\rho}|^2|\phi|^2 r^{1+\epsilon}, \quad \forall \epsilon_1 > 0.$$

For sufficiently small ϵ_1, the integral of the first term on the right-hand side can be absorbed from the integrated energy estimate (23). For the second term, we make use of estimate (55) to show that

$$\begin{aligned}
\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |\bar{\rho}|^2 r^{3+\epsilon}|\phi|^2\, dv\, du\, d\omega &\lesssim \int_{\tau_1^*}^{\tau_2^*}\int_{-u}^{-\tau_2^*} \sum_{j\le 2}\int_\omega r^2|\mathcal{L}_\Omega^j\bar{\rho}|^2\, d\omega \cdot r^{1+\epsilon}\int_\omega |\phi|^2\, d\omega\, dv\, du \\
&\lesssim_{M_2} \int_{\tau_1^*}^{\tau_2^*} E^2[\bar{F}]\big(H_u^{-\tau_2^*}\big) u_+^{-1+\epsilon-\gamma_0}\mathcal{E}_0[\phi]\, du \\
&\lesssim_{M_2} \mathcal{E}_0[\phi]\int_{u_1^*}^{u_2^*} u_+^{-2-\gamma_0}\, du \lesssim_{M_2} \mathcal{E}_0[\phi](\tau_1)_+^{-1-\gamma_0}.
\end{aligned}$$

Finally, we need to bound the charge part, namely the integral of $|q_0| r^{-2}|D_{\underline{L}}\phi||\phi|$. As we have explained previously, this term cannot be absorbed even though the charge q_0 is small due to the loss of decay in the integrated local energy $I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}_{\tau_1}^{\tau_2})$ in (23). The idea is to make use of the energy flux in the incoming null hypersurface $\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}$ and then apply Gronwall's inequality. From estimate (55) and noting that $r \ge \frac{1}{2}v$ in the exterior region, we can show that

$$\begin{aligned}
\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} r^{-2}|D_{\underline{L}}\phi||\phi|\, dx\, dt &\lesssim \int_{-\tau_1^*}^{-\tau_2^*}\int_{-v}^{\tau_1}\int_\omega |D_{\underline{L}}\phi||\phi|\, d\omega\, du\, dv \\
&\lesssim \int_{-\tau_1^*}^{-\tau_2^*}\int_{-v}^{\tau_1} r^{-2}\Big(r^2\int_\omega |D_{\underline{L}}\phi|^2\, d\omega \cdot \int_\omega |r\phi|^2 d\omega\Big)^{\frac{1}{2}} du\, dv \\
&\lesssim_{M_2} \mathcal{E}_0[\phi]^{\frac{1}{2}}\int_{-\tau_1^*}^{-\tau_2^*} v^{-\frac{1}{2}(3+\gamma_0-\epsilon)}\int_{-v}^{\tau_1^*} r^{-\frac{1}{2}(1-\gamma_0+\epsilon)}\Big(r^2\int_\omega |D_{\underline{L}}\phi|^2\, d\omega\Big)^{\frac{1}{2}} u_+^{-\frac{1}{2}\gamma_0}\, du\, dv \\
&\lesssim_{M_2} \mathcal{E}_0[\phi]^{\frac{1}{2}}\int_{-\tau_1^*}^{-\tau_2^*} v^{-\frac{1}{2}(3+\gamma_0-\epsilon)}\big(E[\phi]\big(\underline{H}_v^{-v,\tau_1^*}\big)\big)^{\frac{1}{2}}(\tau_1)_+^{-\frac{1}{2}\epsilon}\, dv \\
&\lesssim_{M_2} \mathcal{E}_0[\phi](\tau_1)_+^{-1-\gamma_0} + (\tau_1)_+^{\epsilon}\int_{-\tau_1^*}^{-\tau_2^*} v^{-1-\epsilon} E[\phi]\big(\underline{H}_v^{-v,\tau_1^*}\big)\, dv.
\end{aligned}$$

Combining all the previous estimates, we then have shown (54). □

As a corollary we then can show the energy flux decay as well as the integrated local energy decay of the scalar field in the exterior region.

Corollary 24. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $\tau_2 < \tau_1 \le 0$, we have*

$$I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + E[\phi]\big(H_{\tau_1^*}^{-\tau_2^*}\big) + E[\phi]\big(\underline{H}_{-\tau_2^*}^{\tau_2^*,\tau_1^*}\big) \lesssim_{M_2} (\tau_1)_+^{-1-\gamma_0}\mathcal{E}_0[\phi]. \tag{56}$$

Proof. First choose ϵ_1 in the estimate (54) to be sufficiently small, depending only on ϵ, γ_0 and R, so that after combining estimate (54) and the integrated energy estimate (23), the term $\epsilon_1 I_0^{-1-\epsilon}[D\phi](\mathcal{D}_{\tau_1}^{\tau_2})$ on the right-hand side of (54) can be absorbed by $I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}_{\tau_1}^{\tau_2})$ on the left-hand side of (23). Then notice that we have the uniform bound

$$(\tau_1)_+^{\epsilon}\int_{-\tau_1^*}^{-\tau_2^*} v^{-1-\epsilon}\,dv \lesssim 1, \qquad \forall \tau_2 < \tau_1 \le 0.$$

Using Gronwall's inequality (fix $\tau_1 \le 0$ and take $\tau_2 \le \tau_1$ as variable), we then obtain (56). □

4.3.2. *Energy decay in the interior region.* Once we have the energy flux and the r-weighted energy decay estimates for the scalar field in the exterior region, we in particular have the energy flux bound for the scalar field on the boundary $H_{-R/2}$. This is necessary to consider the energy flux decay in the interior region. Compared to the case in the exterior region, the charge is not a problem as the charge only effects the decay property of the Maxwell field in the exterior region. However, new difficulties arise in the interior region case. First of all there is no lower bound for r/τ_+. That means we may need estimates for general p for the r-weighted energy estimates instead of simply the largest p. Secondly, as we have explained before, we are not able to absorb the interaction term between the gauge field A and the scalar field due to the fact that dA is no longer small in our setting. Thus we need to rely on the r-weighted energy estimates and make use of the null structure of $J[\phi]$. In the exterior region, the idea is first to derive the r-weighted energy decay and then to obtain the integrated local energy and energy flux decay. In the interior region, we see from the r-weighted energy estimates (27) that the term $|F_{\underline{L}\mu}J^{\mu}[\phi]|$ also appears on the right-hand side. This suggests that we have to consider the r-weighted energy estimate and the integrated local energy estimates simultaneously.

We first estimate the interaction terms of dA and $J[\phi]$ in the r-weighted energy estimate (27).

Proposition 25. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then in the interior region, for all $0 \le \tau_1 < \tau_2$ and $1 \le p \le 1+\gamma_0$, we have*

$$\begin{aligned}\iint_{\overline{\mathcal{D}}_{\tau_1}^{\tau_2}} r^p|F_{L\mu}J^{\mu}[\phi]|^2\,dx\,dt \lesssim \epsilon_1 \iint_{\overline{\mathcal{D}}_{\tau_1}^{\tau_2}} r^{p-1}|\not{D}(r\phi)|^2\,dv\,d\omega\,d\tau + I_{-1-\epsilon}^{p}[r^{-1}D_L(r\phi)](\overline{\mathcal{D}}_{\tau_1}^{\tau_2})\\ + M_2\epsilon_1^{-1}\Big(\delta_p\int_{\tau_1}^{\tau_2} E[\phi](\Sigma_\tau)\tau_+^{\delta_p^{-1}-1-\epsilon}\,d\tau + (1-\delta_p)I_{-1-\epsilon}^{1+\gamma_0}[r^{-1}D_L(r\phi)](\overline{\mathcal{D}}_{\tau_1}^{\tau_2})\Big)\end{aligned} \tag{57}$$

for all $\epsilon_1 > 0$. Here $\delta_p = (2+\gamma_0-p)/(1+\gamma_0)$ is given in Lemma 19 in line (49).

Proof. Denote $\psi = r\phi$ and $F = dA$. First we have

$$2r^p|F_{L\mu}J^{\mu}[\phi]|r^2 \le r^p|D_L\psi|^2\tau_+^{-1-\epsilon} + r^p|\rho|^2|\psi|^2\tau_+^{1+\epsilon} + \epsilon_1 r^{p-1}|\not{D}\psi|^2 + \epsilon_1^{-1}r^{p+3}|\alpha|^2|\phi|^2$$

for all $\epsilon_1 > 0$. The first term can be absorbed using Gronwall's inequality. The third term will be absorbed for sufficiently small ϵ_1 depending only on ϵ, γ_0 and R. For the second term, we use the energy flux of ρ

on H_{τ^*} to bound ρ, and estimate (49) of Lemma 19 to bound ϕ. For the last term, we use the r-weighted energy estimate to bound α. Then similarly to the proof of Proposition 21 we can show that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} \tau_+^{1+\epsilon} r^p |\rho|^2|\psi|^2 + r^{p+3}|\alpha|^2|\phi|^2\, dv\, d\omega\, d\tau \\
&\qquad\lesssim \int_{\tau_1}^{\tau_2}\int_{2R+\tau^*}^{\infty} \sum_{j\le 2}\int_\omega \tau_+^{1+\epsilon} r^2|\mathcal{L}_\Omega^j\rho|^2 + r^3|\mathcal{L}_\Omega^j\alpha|^2\, d\omega\cdot\int_\omega r^p|\phi|^2\, d\omega\, dv\, du \\
&\qquad\lesssim M_2\int_{\tau_1}^{\tau_2} \tau_+^{-\epsilon}(E[\phi](\Sigma_\tau))^\delta (I_0^{1+\gamma_0}[r^{-1}D_L\psi](H_{\tau^*}))^{1-\delta}\, d\tau \\
&\qquad\lesssim M_2\Big(\delta\int_{\tau_1}^{\tau_2} E[\phi](\Sigma_\tau)\tau_+^{\delta^{-1}-1-\epsilon}\, d\tau + (1-\delta) I_{-1-\epsilon}^{1+\gamma_0}[r^{-1}D_L\psi](\bar{\mathcal{D}}_{\tau_1}^{\tau_2})\Big).
\end{aligned}$$

The proposition then follows. □

Next we estimate the interaction terms in the energy estimate (22). We show the following:

Proposition 26. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then we have*

$$\begin{aligned}
&\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{L\nu}J^\nu[\phi]| + |F_{\underline{L}\nu}J^\nu[\phi]|\, dx\, dt \\
&\qquad\lesssim \epsilon_1 I_0^{-1-\epsilon}[D\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + \epsilon_1^{-1}\int_{\tau_1}^{\tau_2} g(\tau)E[\phi](\Sigma_\tau)\, d\tau + I_{-2-\gamma_0}^{1+\gamma_0}[r^{-1}D_L(r\phi)](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}) \quad (58)
\end{aligned}$$

for all $0<\epsilon_1<1$, where

$$g(\tau) := \sum_{j\le 2} I_{1+2\epsilon}^{-1-\epsilon}[\mathcal{L}_\Omega^j F](\Sigma_\tau) + \sum_{j\le 2}\int_{H_{\tau^*}} r^{2+\epsilon}(|\mathcal{L}_\Omega^j\alpha|^2 + |\mathcal{L}_\Omega^j\rho|^2)\, dv\, d\omega + \sup_{|x|\le R}|F|^2(\tau,x).$$

Proof. For the integral on $\{r\ge R\}$, we use Sobolev embedding on the unit sphere to bound the curvature, and the proof is quite similar to that of the previous proposition. On the finite region $\{r\le R\}$, we make use of the $L_t^2 L_x^\infty$ norm of the curvature given in Proposition 14. For the case when $r\ge R$, first we have

$$\begin{aligned}
|F_{L\nu}J^\nu[\phi]| + |F_{\underline{L}\nu}J^\nu[\phi]| &\lesssim (|\rho|+|\alpha|)|D\phi||\phi| + |\underline{\alpha}||\not{D}\phi||\phi| \\
&\lesssim \epsilon_1 r_+^{-1-\epsilon}|D\phi|^2 + \epsilon_1^{-1}(|\rho|^2+|\alpha|^2)r_+^{1+\epsilon}|\phi|^2 + |\underline{\alpha}||\not{D}\phi||\phi|.
\end{aligned}$$

The first term can be absorbed in the energy estimate (22) for sufficiently small ϵ_1. For the second term, we can use estimate (49) to bound ϕ by the energy flux through H_{τ^*} and the r-weighted energy to control the curvature terms. The last term is the most difficult one to control. The reason is that we do not have powerful estimates for $\underline{\alpha}$. The estimates we have are the integrated local energy estimate and the energy flux decay through the incoming null hypersurface. Unlike the case in the exterior region, where we can make use of the energy flux through the incoming null hypersurface for $\underline{\alpha}$, that method fails in the interior region. The main reason is that the energy flux $E[F]\big(\underline{H}_v^{\tau_1^*,\tau_2^*}\big)$ decays in τ_1 instead of v. A possible way to solve this issue is to assume a pointwise bound for $\underline{\alpha}$. However the problem is that the pointwise decay for $\underline{\alpha}$ is too weak (due to the assumption on the initial data, as explained in the introduction) to be useful. We thus can only rely on the integrated local energy estimate for $\underline{\alpha}$. As there is an r^ϵ decay loss in the integrated local energy estimate for $\underline{\alpha}$, we are not able to bound ϕ simply by using the energy flux through $H_{\Sigma_{\tau^*}}$. Instead, we need to make use of the r-weighted energy estimate. This means that we cannot obtain a uniform energy bound from the energy estimate (22). We need to combine it with the r-weighted energy estimate.

For the integral of $|\underline{\alpha}||\not{D}\phi||\phi|$, from estimate (49) with $p=1+\epsilon$, we can show that

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} |\underline{\alpha}||\not{D}\phi||\phi| r^2\,d\omega\,dv\,d\tau \\
&\quad\lesssim \int_{\tau_1}^{\tau_2}\int_{2R+\tau^*}^{\infty}\Big(\sum_{j\le 2}\int_\omega r^{1-\epsilon}|\mathcal{L}_\Omega^j\underline{\alpha}|^2\,d\omega\Big)^{\frac12}\Big(\int_\omega r^2|\not{D}\phi|^2\,d\omega\cdot\int_\omega r^{1+\epsilon}|\phi|^2\,d\omega\Big)^{\frac12}dv\,d\tau \\
&\quad\lesssim \sum_{j\le 2}\int_{\tau_1}^{\tau_2}(I_0^{-1-\epsilon}[\mathcal{L}_\Omega^j\underline{\alpha}](\Sigma_\tau)E[\phi](\Sigma_\tau))^{\frac12}(E[\phi](\Sigma_\tau))^{\frac12\delta}(I_0^{1+\gamma_0}[r^{-1}D_L(r\phi)](H_{\tau^*}))^{\frac12-\frac12\delta}\,d\tau \\
&\quad\lesssim \sum_{j\le 2}\int_{\tau_1}^{\tau_2} I_{1+2\epsilon}^{-1-\epsilon}[\mathcal{L}_\Omega^j\underline{\alpha}](\Sigma_\tau)E[\phi](\Sigma_\tau)\,d\tau + \int_{\tau_1}^{\tau_2}\tau_+^{-1-\epsilon}E[\phi](\Sigma_\tau)\,d\tau + I_{-2-\gamma_0}^{1+\gamma_0}[r^{-1}D_L(r\phi)](\overline{\mathcal{D}}_{\tau_1}^{\tau_2}).
\end{aligned}$$

Here $\delta=(1+\gamma_0-\epsilon)/(1+\gamma_0)$, and in the last step we have used Jensen's inequality as well as the relation

$$\frac12+\epsilon-\frac12\delta(1+\epsilon)-(2+\gamma_0)\Big(\frac12-\frac12\delta\Big)=\frac12\frac{\epsilon}{1+\gamma_0}>0.$$

In the above estimate the first two terms will be estimated using Gronwall's inequality. We keep the last term involving the r-weighted energy estimates. For the integral of $(|\rho|^2+|\alpha|^2)r_+^{1+\epsilon}|\phi|^2$, we use estimate (49) to bound ϕ. We have

$$\begin{aligned}
&\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}}(|\rho|^2+|\alpha|^2)r_+^{1+\epsilon}|\phi|^2 r^2\,d\omega\,dv\,d\tau \\
&\qquad\lesssim \int_{\tau_1}^{\tau_2}\int_{2R+\tau^*}^{\infty}\sum_{j\le 2}\int_\omega r^{2+\epsilon}(|\mathcal{L}_\Omega^j\alpha|^2+|\mathcal{L}_\Omega^j\rho|^2)\,d\omega\cdot\int_\omega r|\phi|^2\,d\omega\,dv\,d\tau \\
&\qquad\lesssim \sum_{j\le 2}\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} r^{2+\epsilon}(|\mathcal{L}_\Omega^j\alpha|^2+|\mathcal{L}_\Omega^j\rho|^2)\,dv\,d\omega\cdot E[\phi](\Sigma_\tau)\,d\tau.
\end{aligned}$$

This term will be controlled in the energy estimate (22) using Gronwall's inequality.

For the integral on the region $\{r\le R\}$, we can show that

$$\begin{aligned}
\int_{\tau_1}^{\tau_2}\int_{r\le R}|F_{Lv}J^v[\phi]|+|F_{\underline{L}v}J^v[\phi]|\,dx\,d\tau &\lesssim \epsilon_1\int_{\tau_1}^{\tau_2}\int_{r\le R}|D\phi|^2\,dx\,d\tau+\epsilon_1^{-1}\int_{\tau_1}^{\tau_2}\int_{r\le R}|F|^2|\phi|^2\,dx\,d\tau \\
&\lesssim \epsilon_1\int_{\tau_1}^{\tau_2}\int_{r\le R}\frac{|D\phi|^2}{r_+^{1+\epsilon}}\,dx\,d\tau+\epsilon_1^{-1}\int_{\tau_1}^{\tau_2}\sup_{|x|\le R}|F|^2\cdot E[\phi](\Sigma_\tau)\,d\tau
\end{aligned}$$

for all $\epsilon_1>0$. The first term will be absorbed for small ϵ_1. The second term can be controlled using Gronwall's inequality. Combining all these estimates above, we thus have shown estimate (58). □

As a corollary, the energy estimate (22) leads to the following:

Corollary 27. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then in the interior region, we have the estimate*

$$I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}_{\tau_1}^{\tau_2}) + E[\phi](\Sigma_{\tau_2}) + \iint_{\mathcal{D}_{\tau_1}^{\tau_2}} |F_{L\nu} J^\nu[\phi]| + |F_{\underline{L}\nu} J^\nu[\phi]|\, dx\, dt$$
$$\lesssim_{M_2} E[\phi](\Sigma_{\tau_1}) + (\tau_1)_+^{-1-\gamma_0}\mathcal{E}_0[\phi] + I_{-2-\gamma_0}^{1+\gamma_0}[r^{-1}D_L(r\phi)](\overline{\mathcal{D}}_{\tau_1}^{\tau_2}). \quad (59)$$

Proof. First choose ϵ_1 sufficiently small in the estimate (58) so that combining the energy estimate (22) with (58), the integrated local energy term $I_0^{-1-\epsilon}[D\phi](\mathcal{D}_{\tau_1}^{\tau_2})$ could be absorbed. By our notation, the smallness of ϵ_1 depends only on ϵ, γ_0 and R. Then for the second term on the right-hand side of (58), to apply Gronwall's inequality, we show that $g(\tau)$ (defined after line (58)) is integrable. From the integrated local energy estimates (36) and the r-weighted energy estimates (31) for the Maxwell field, we conclude from the previous section that

$$I_0^{-1-\epsilon}[\mathcal{L}_Z^k F](\mathcal{D}_{\tau_1}^{\tau_2}) \lesssim M_k(\tau_1)_+^{-1-\gamma_0},$$
$$\int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} r^{2+\epsilon}(|\mathcal{L}_Z^k\alpha|^2 + |\mathcal{L}_Z^k\rho|^2)\, dv\, d\omega\, d\tau \lesssim M_k(\tau_1)_+^{-\gamma_0+\epsilon}.$$

Therefore, using Lemma 20 and Proposition 14, we can show that

$$\begin{aligned}
\int_{\tau_1}^{\tau_2} g(\tau)\, d\tau &\lesssim M_2(\tau_1)_+^{-\gamma_0+\epsilon} + \sum_{j\le 2} I_{1+2\epsilon}^{-1-\epsilon}[\mathcal{L}_\Omega^j F](\mathcal{D}_{\tau_1}^{\tau_2}) \\
&\lesssim M_2(\tau_1)_+^{-\gamma_0+\epsilon} + \sum_{j\le 2}\int_{\tau_1}^{\tau_2} \tau_+^{2\epsilon} I_0^{-1-\epsilon}[\mathcal{L}_\Omega^j F](\mathcal{D}_{\tau}^{\tau_2})\, d\tau + (\tau_1)_+^{1+2\epsilon} I_0^{-1-\epsilon}[\mathcal{L}_\Omega^j F](\mathcal{D}_{\tau_1}^{\tau_2}) \\
&\lesssim M_2(\tau_1)_+^{-\gamma_0+\epsilon} + M_2\int_{\tau_1}^{\tau_2} \tau_+^{-1-\gamma_0+2\epsilon}\, d\tau + M_2(\tau_1)_+^{-\gamma_0+2\epsilon} \\
&\lesssim M_2(\tau_1)_+^{-\gamma_0+2\epsilon}.
\end{aligned}$$

By using this uniform bound, the second term on the right-hand side of (58) can be absorbed using Gronwall's inequality. The corollary then follows. □

We now can use Proposition 25 and the above corollary to obtain the necessary r-weighted energy estimates. To derive energy decay estimates, we at least need the r-weighted energy estimates with $p = 1$ and $p = 1 + \gamma_0$ (some p bigger than one, the decay rate depending on this largest p). In any case, we first choose ϵ_1 in estimate (57) sufficiently small, so that combining it with the r-weighted energy estimate (27), the first term on the right-hand side of (57) can be absorbed (note that $\gamma_0 < 1$). The second term on the right-hand side of (57) can be controlled using Gronwall's inequality. Let's first combine the r-weighted energy estimate (27) for $p = 1$ with the integrated local energy estimate (59) to derive the bound for the integral of the energy flux.

Proposition 28. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then in the interior region, for all $0 \le \tau_1 < \tau_2$, we have*

$$\int_{\tau_1}^{\tau_2} E[\phi](\Sigma_\tau)\,d\tau \lesssim_{M_2} \int_{H_{\tau_1^*}} r|D_L\psi|^2\,dv\,d\omega + E[\phi](\Sigma_{\tau_1}) + (\tau_1)_+^{-\gamma_0}\mathcal{E}_0[\phi] + I^{1+\gamma_0}_{-2-\gamma_0}[r^{-1}D_L(r\phi)](\bar{\mathcal{D}}^{\tau_2}_{\tau_1}). \quad (60)$$

Proof. In (t, r, ω) coordinates, using Sobolev embedding, we have

$$\int_\omega |\phi|^2(\tau, R, \omega)\,d\omega \lesssim \int_{r\le R} |\phi|^2 + |D\phi|^2\,dx.$$

Then we can show that

$$\begin{aligned}\int_{\tau_1}^{\tau_2} E[\phi](\Sigma_\tau)\,d\tau &\lesssim \int_{\tau_1}^{\tau_2}\int_{r\le R} |D\phi|^2\,dx\,d\tau + \int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} |D_L(r\phi)|^2 + |\not{D}(r\phi)|^2\,dv\,d\omega\,d\tau \\ &\qquad + \int_{\tau_1}^{\tau_2}\int_\omega |\phi|^2(\tau, R, \omega)\,d\omega \\ &\lesssim I_0^{-1-\epsilon}[\tilde{D}\phi](\mathcal{D}^{\tau_2}_{\tau_1}) + \int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} |D_L(r\phi)|^2 + |\not{D}(r\phi)|^2\,dv\,d\omega\,d\tau.\end{aligned}$$

Therefore, take $p = 1$ in the r-weighted energy estimate (27). From the above argument, we obtain the following bound for the integral of the energy flux:

$$\begin{aligned}\int_{\tau_1}^{\tau_2} E[\phi](\Sigma_\tau)d\tau &\lesssim \int_{H_{\tau_1^*}} r|D_L\psi|^2\,dv\,d\omega + M_2\int_{\tau_1}^{\tau_2} E[\phi](\Sigma_\tau)\tau_+^{-\epsilon}\,d\tau \\ &\quad + C_{M_2}\big(E[\phi](\Sigma_{\tau_1}) + (\tau_1)_+^{-\gamma_0}\mathcal{E}_0[\phi] + I^{1+\gamma_0}_{-2-\gamma_0}[r^{-1}D_L(r\phi)](\bar{\mathcal{D}}^{\tau_2}_{\tau_1})\big)\end{aligned}$$

for some constant C_{M_2} depending on M_2. For the second term, we further can bound

$$\tau_+^{-\epsilon} = (\epsilon_1^{-1/\epsilon}\tau_+^{-1-\epsilon})^{\epsilon/(1+\epsilon)}\cdot(\epsilon_1)^{1/(1+\epsilon)} \le \frac{\epsilon}{1+\epsilon}\epsilon_1^{-1/\epsilon}\tau_+^{-1-\epsilon} + \frac{\epsilon_1}{1+\epsilon}, \quad \forall \epsilon_1 > 0.$$

Choose ϵ_1 sufficiently small, so that the second term can be absorbed. Then the first term can be bounded using Corollary 27. Therefore, the previous estimate amounts to estimate (60). □

We have the following corollary.

Corollary 29. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then we have*

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \tau_+^{\gamma_0-\epsilon} E[\phi](\Sigma_\tau)\,d\tau &\lesssim_{M_2} \int_{H_{\tau_1^*}} r^{1+\gamma_0}|D_L\psi|^2\,dv\,d\omega + (\tau_1)_+^{1+\gamma_0-\epsilon}E[\phi](\Sigma_{\tau_1}) \\ &\quad + \mathcal{E}_0[\phi] + I^{1+\gamma_0}_{-2-\epsilon}[r^{-1}D_L(r\phi)](\bar{\mathcal{D}}^{\tau_2}_{\tau_1}). \quad (61)\end{aligned}$$

Proof. Using estimate (49) of Lemma 19, we have the bound

$$\int_{H_{\tau^*}} |D_L(r\phi)|^2\,dv\,d\omega \le \int_{H_{\tau^*}} |D_L\phi|^2 r^2\,dv\,d\omega + \lim_{r\to\infty}\int_\omega r|\phi|^2\,d\omega \lesssim E[\phi](\Sigma_\tau).$$

For all $\epsilon_1 > 0$, we have the inequality

$$\tau_+^{\gamma_0-1-\epsilon} r = (\epsilon_1^{-\gamma_0} r^{1+\gamma_0} \tau_+^{-1-\epsilon})^{1/(1+\gamma_0)} (\epsilon_1 \tau_+^{\gamma_0-\epsilon})^{\gamma_0/(1+\gamma_0)} \leq \frac{\epsilon_1^{-\gamma_0} r^{1+\gamma_0} \tau_+^{-1-\epsilon}}{1+\gamma_0} + \frac{\gamma_0 \epsilon_1 \tau_+^{\gamma_0-\epsilon}}{1+\gamma_0}.$$

In particular, the above inequality holds for $r = 1$. Moreover, we also have

$$(\tau_1)_+^{\gamma_0-\epsilon} r = (r^{1+\gamma_0})^{1/(1+\gamma_0)} \big((\tau_1)_+^{1+\gamma_0-\epsilon(1+\gamma_0)/\gamma_0}\big)^{\gamma_0/(1+\gamma_0)} \leq r^{1+\gamma_0} + (\tau_1)_+^{1+\gamma_0-\epsilon}.$$

Denote $\psi = r\phi$. From estimate (60), we can show that

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \tau_+^{\gamma_0-1-\epsilon} \bigg(\int_{H_{\tau^*}} r |D_L \psi|^2 \, dv \, d\omega + E[\phi](\Sigma_\tau) + \tau_+^{-\gamma_0} \mathcal{E}_0[\phi] + I_{-2-\gamma_0}^{1+\gamma_0} [r^{-1} D_L(r\phi)](\bar{\mathcal{D}}_\tau^{\tau_2}) \bigg) d\tau \\ \lesssim \epsilon_1^{-\gamma_0} \int_{\tau_1}^{\tau_2} \tau_+^{-1-\epsilon} \int_{H_{\tau^*}} r^{1+\gamma_0} |D_L \psi|^2 \, dv \, d\omega \, d\tau + \epsilon_1 \int_{\tau_1}^{\tau_2} \tau_+^{\gamma_0-\epsilon} E[\phi](\Sigma_\tau) \, d\tau \\ + \epsilon_1^{-\gamma_0} \int_{\tau_1}^{\tau_2} \tau_+^{-1-\epsilon} E[\phi](\Sigma_\tau) \, d\tau + \mathcal{E}_0[\phi] + I_{-2-\epsilon}^{1+\gamma_0} [r^{-1} D_L(r\phi)](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}).\end{aligned}$$

On the right-hand side of the above estimate, the first term can be grouped with the last term. The second term will be absorbed for small ϵ_1. The third term can be bounded using estimate (59). Therefore, using Lemma 20 and Proposition 28, we can show that

$$\begin{aligned}\int_{\tau_1}^{\tau_2} \tau_+^{\gamma_0-\epsilon} E[\phi](\Sigma_\tau) \, d\tau \lesssim_{M_2} \epsilon_1 \int_{\tau_1}^{\tau_2} \tau_+^{\gamma_0-\epsilon} E[\phi](\Sigma_\tau) \, d\tau + \epsilon_1^{-\gamma_0} \int_{H_{\tau_1^*}} r^{1+\gamma_0} |D_L \psi|^2 \, dv \, d\omega + \epsilon_1^{-\gamma_0} \mathcal{E}_0[\phi] \\ + \epsilon_1^{-\gamma_0} (\tau_1)_+^{1+\gamma_0-\epsilon} E[\phi](\Sigma_{\tau_1}) + \epsilon_1^{-\gamma_0} I_{-2-\epsilon}^{1+\gamma_0} [r^{-1} D_L(r\phi)](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}).\end{aligned}$$

Let ϵ_1 be sufficiently small, depending on M_2, ϵ, γ_0 and R. We obtain estimate (61). □

Estimate (61) can now be used to derive the r-weighted energy estimate with $p = 1 + \gamma_0$.

Proposition 30. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then we have*

$$\begin{aligned}\int_{H_{\tau_2^*}} r^{1+\gamma_0} |D_L \psi|^2 \, dv \, d\omega + \int_{\tau_1}^{\tau_2} \int_{H_{\tau^*}} r^{\gamma_0} (|D_L \psi|^2 + |\not{D} \psi|^2) \, dv \, d\omega \, d\tau \\ \lesssim_{M_2} \int_{H_{\tau_1^*}} r^{1+\gamma_0} |D_L \psi|^2 \, dv \, d\omega + \mathcal{E}_0[\phi] + (\tau_1)_+^{1+\gamma_0-\epsilon} E[\phi](\Sigma_{\tau_1}),\end{aligned} \tag{62}$$

where $\psi = r\phi$.

Proof. By taking ϵ_1 in estimate (57) to be sufficiently small and combining it with the r-weighted energy estimate (27) for $p = 1 + \gamma_0$, from Corollary 27 we obtain

$$\begin{aligned}\int_{H_{\tau_2^*}} r^{1+\gamma_0} |D_L \psi|^2 \, dv \, d\omega + \int_{\tau_1}^{\tau_2} \int_{H_{\tau^*}} r^{\gamma_0} (|D_L \psi|^2 + |\not{D} \psi|^2) \, dv \, d\omega \, d\tau \\ \lesssim \int_{H_{\tau_1^*}} r^{1+\gamma_0} |D_L \psi|^2 \, dv \, d\omega + \mathcal{E}_0[\phi] + M_2 \bigg(\int_{\tau_1}^{\tau_2} E[\phi](\Sigma_\tau) \tau_+^{\gamma_0-\epsilon} \, d\tau + I_{-1-\epsilon}^{1+\gamma_0} [r^{-1} D_L \psi](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}) \bigg) \\ + C_{M_2} \big(E[\Phi](\Sigma_{\tau_1}) + (\tau_1)_+^{-1-\gamma_0} \mathcal{E}_0[\phi] + I_{-2-\gamma_0}^{1+\gamma_0} [r^{-1} D_L \psi](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}) \big)\end{aligned}$$

for some constant C_{M_2} depending on M_2. Estimate (62) then follows from estimate (61) together with Gronwall's inequality. □

Take $\tau_1 = 0$ in (62). From the energy estimate (56) and the r-weighted energy estimate (53) in the exterior region, we conclude that the right-hand side of (62) is bounded. Since $\tau_2 > \tau_1$ is arbitrary there, we in particular have the r-weighted energy estimate for the scalar field in the interior region.

Corollary 31. *Let $\psi = r\phi$. Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $0 \le \tau_1 < \tau_2$, we have*

$$\int_{H_{\tau_2^*}} r^{1+\gamma_0}|D_L\psi|^2\, dv\, d\omega + \int_{\tau_1}^{\tau_2}\int_{H_{\tau^*}} r^{\gamma_0}(|D_L\psi|^2 + |\not{D}\psi|^2)\, dv\, d\omega\, d\tau \lesssim_{M_2} \mathcal{E}_0[\phi]. \tag{63}$$

Proof. From the r-weighted energy estimate (53) in the exterior region with $p = 1+\gamma_0$, $\tau_1 = 0$, we derive

$$\int_{H_{0^*}} r^{1+\gamma_0}|D_L\psi|^2\, dv\, d\omega = \int_{H_{-R/2}} r^{1+\gamma_0}|D_L\psi|^2 \lesssim_{M_2} \mathcal{E}_0[\phi].$$

The energy estimate (56) in the exterior region implies that

$$E[\phi](\Sigma_0) = E[\phi](\{t=0, r \le R\}) + E[\phi](H_{-R/2}) \lesssim \mathcal{E}_0[\phi].$$

Then estimate (63) follows from (62) by taking $\tau_1 = 0$. □

This uniform bound for the r-weighted energy estimate in the interior region is crucial for the energy flux decay. It in particular implies that the terms involving the r-weighted energy flux on the right-hand side of the energy estimate (59) and the integral of the energy flux estimate (60) have the right decay in order to show the energy flux decay.

Proposition 32. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then in the interior region, we have the energy flux decay*

$$E[\phi](\Sigma_\tau) \lesssim_{M_2} \mathcal{E}_0[\phi]\tau_+^{-1-\gamma_0}, \quad \forall \tau \ge 0. \tag{64}$$

Proof. Estimate (63) implies that

$$I^{1+\gamma_0}_{-2-\gamma_0}[r^{-1}D_L\psi](\bar{\mathcal{D}}_{\tau_1}^{\tau_2}) \lesssim_{M_2} (\tau_1)_+^{-1-\gamma_0}\mathcal{E}_0[\phi], \quad \forall 0 \le \tau_1 < \tau_2.$$

Then using a pigeonhole argument like in the proof of Proposition 11 for the energy flux decay of the Maxwell field in the interior, the energy decay estimate (64) for the scalar field follows from the energy estimate (59), the integral of the energy flux estimate (60) and the r-weighted energy estimate (63). For a detailed proof for this, we refer to Proposition 2 of [Yang 2015b]. □

4.3.3. *Energy decay estimates for the first-order derivative of the scalar field.* In this section, we derive the energy flux decay estimates for the derivative of the scalar field. The difficulty is that the covariant wave operator $\Box_A$ does not commute with D_Z. Commutators are quadratic in the Maxwell field and the scalar field. In our setting, the Maxwell field is large. In particular, those terms cannot be absorbed. The idea is to exploit the null structure of the commutators and to use Gronwall's inequality adapted to our foliation Σ_τ.

In the following, we always use ψ to denote the weighted scalar field $r\phi$, that is, $\psi = r\phi$. The first-order derivative of ϕ is abbreviated ϕ_1, and the second-order derivative ϕ_2. More precisely, we denote $\phi_1 = D_Z\phi$, $\phi_2 = D_Z^2\phi$ with Z any vector field in the set $\Gamma = \{\partial_t, \Omega_{ij} = x_i\partial_j - x_j\partial_i\}$. We use the same notation for the weighted scalar field ψ, e.g., $\psi_1 = rD_Z\phi$. For any function f, under the null coordinates (u, v, ω), we define

$$\|f\|^2_{L^2_v L^\infty_u L^2_\omega(\mathcal{D})} := \int_v \sup_u \int_\omega |f|^2 \, d\omega \, dv,$$

where (u, v, ω) are the null coordinates on the region $\mathcal{D}$. Similarly, we have the notation $\|f\|_{L^2_u L^\infty_v L^2_\omega(\mathcal{D})}$. We can also define $L^p_u L^q_v L^r_\omega$ norms for general p, q, r.

To apply Corollary 24 for the exterior region and Proposition 32 for the interior region, it suffices to control the commutator terms. However, we are not able to bound the commutator terms directly by using the zero's order energy estimates. One has to make use of the energy flux of the first-order derivative of the solution and then apply Gronwall's inequality. However, for the energy estimate for the first-order derivative of the solution, the key is to understand the commutator $[\Box_A, D_Z]$ with $Z = \partial_t$ or the angular momentum. The cases of ∂_t and the angular momentum are quite different. The main reason is that the angular momentum contains weights in r while ∂_t does not. For the case when $Z = \partial_t$, it is easy to bound $[\Box_A, D_{\partial_t}]\phi$. The only place we need to be careful is the charge part. For the case of $Z = \Omega$, the problem is that the commutator $[\Box_A, D_\Omega]$ produces a term of the form $Z^\nu F_{\mu\nu} D^\mu\phi$ which cannot be written as a linear combination of $D_Z\phi$. The estimate for the commutator terms heavily rely on the null structure. We first show the following lemma for the commutator terms.

Lemma 33. *When $|x| \geq R$, we have*

$$|[\Box_A, D_Z]\phi| \lesssim |\alpha||D_{\underline{L}}\psi| + (|\underline{\alpha}| + r^{-1}|\rho|)|D_L\psi| + |F||\not{D}\phi| + (|J| + r|\not{J}| + |\sigma| + r^{-1}|\rho|)|\phi|. \tag{65}$$

When $r \leq R$, we have

$$|[\Box_A, D_Z]\phi| \lesssim |F||\bar{D}\phi| + |J||\phi|. \tag{66}$$

Here $F = dA$ and $J = \delta F$.

Remark 34. In this paper, all the quantities involving Z should be interpreted as the sum of the quantity for all possible vector fields Z in Γ unless otherwise specified.

Proof. Let $\psi = r\phi$. First, from Lemma 4 we can write

$$[\Box_A, D_Z]\phi = 2ir^{-1}Z^\nu F_{\mu\nu}D^\mu\psi + i\nabla^\mu F_{\mu\nu}Z^\nu\phi + i\phi(-2Z^\nu F_{\mu\nu}r^{-1}\nabla^\mu r + \nabla^\mu Z^\nu F_{\mu\nu}). \tag{67}$$

We need to exploit the null structure of the above commutator terms. The first term is the main one. Since we will rely on the r-weighted energy estimates, it suggests writing the main term in terms of the weighted solution $r\phi$. The second term is easy, as $\nabla^\mu F_{\mu\nu}$ is a nonlinear term of ϕ by the Maxwell equation. Let's first estimate the third term. When $Z = \Omega$, note that $r^{-1}\Omega$ is a linear combination of e_1 and e_2. We then can show that

$$|r^{-1}Z^\nu F_{\mu\nu}D^\mu(r\phi)| \lesssim |\alpha||D_{\underline{L}}(r\phi)| + |\underline{\alpha}||D_L(r\phi)|.$$

This is the null structure we need: the "bad" component $\underline{\alpha}$ of the curvature does not interact with the "bad" component $D_{\underline{L}}(r\phi)$ of the scalar field. Similarly, when $Z = \partial_t$, the "bad" term $r^{-1}\underline{\alpha} D_{\underline{L}}(r\phi)$ does not appear. More precisely, we have

$$|r^{-1} Z^\nu F_{\mu\nu} D^\mu(r\phi)| \lesssim r^{-1}(|\alpha| + |\underline{\alpha}|)|\not{D}(r\phi)| + r^{-1}|\rho||D_r(r\phi)|.$$

For the second term on the right-hand side of (67), we note that $\nabla^\mu F_{\mu\nu}$ is a nonlinear term of ϕ. We have

$$|\nabla^\mu F_{\mu\nu} Z^\nu \phi| \lesssim (|J| + r|\not{J}|)|\phi|.$$

For the third term on the right-hand side of (67), we show that

$$|i\phi(-2Z^\nu F_{\mu\nu} r^{-1}\nabla^\mu r + \nabla^\mu Z^\nu F_{\mu\nu})| \lesssim (|\sigma| + r^{-1}|\rho|)|\phi|.$$

The case when $Z = \partial_t$ is trivial. To check the above inequality for the case when $Z = \Omega$, it suffices to prove it for the component $\Omega_{jk} = x_j\partial_k - x_k\partial_j$. Then we can show that

$$\begin{aligned} -2\Omega^\nu F_{\mu\nu} r^{-1}\nabla^\mu r + \nabla^\mu \Omega^\nu F_{\mu\nu} &= 2F_{jk} - 2F(\partial_r, \Omega_{ij}) \\ &= 2F(\omega_j\partial_r + \partial_j - \omega_j\partial_r, \omega_k\partial_r + \partial_k - \omega_k\partial_r) - 2F(\partial_r, \Omega_{jk}) \\ &= 2F(\partial_j - \omega_j\partial_r, \partial_k - \omega_k\partial_r). \end{aligned}$$

Here recall that $\omega_j = r^{-1}x_j$. Since $\partial_j - \omega_j\partial_r$ is orthogonal to L and $\underline{L}$ for all $j = 1, 2, 3$, we conclude that $\partial_j - \omega_j\partial_r$ is a linear combination of e_1 and e_2. The desired estimate then follows, as the norm of the vector fields $\partial_j - \omega_j\partial_r$ is less than 1. □

We begin a series of propositions in order to estimate the weighted spacetime norm of the commutators. The estimates in the bounded region $\{r \le R\}$ are easy to obtain as the weights are finite. We now concentrate on the region $\{r \ge R\}$. Let $\bar{\mathcal{D}}_\tau = \mathcal{D}_\tau \cap \{|x| \ge R\}$ and recall that $D_\tau = D_\tau^{+\infty}$ when $\tau \ge 0$, or $D_\tau = D_\tau^{-\infty}$ otherwise. We first consider $|\alpha||D_{\underline{L}}(r\phi)|$.

Proposition 35. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $\epsilon_1 > 0$, we have*

$$\|D_{\underline{L}}(r\phi)\|_{L_u^2 L_v^\infty L_\omega^2(\bar{\mathcal{D}}_\tau)} \lesssim_{M_2} \mathcal{E}_0[\phi]\epsilon_1^{-1}\tau_+^{-1-\gamma_0} + \epsilon_1 I_0^{1+\epsilon}[r^{-1} D_L D_{\underline{L}}(r\phi)](\bar{\mathcal{D}}_\tau). \tag{68}$$

Proof. The idea is to bound $\sup|D_{\underline{L}}(r\phi)|$ by the L^2 norm of $D_L D_{\underline{L}}(r\phi)$. In the exterior region when $\bar{\mathcal{D}}_\tau = \mathcal{D}_\tau^{-\infty}$, we can integrate from the initial hypersurface $\{t = 0\}$. In the interior region, choose the incoming null hypersurface $\underline{H}_{(\tau_2+R)/2}^{\tau_1^*, \tau_2^*}$ as the starting surface. Denote $\psi = r\phi$. We show estimate (68) for the interior region case, that is, when $0 \le \tau_1 < \tau_2$. On the outgoing null hypersurface H_{τ^*}, for all $0 \le \tau_1 \le \tau \le \tau_2$, we have

$$\begin{aligned} &\sup_{v \ge (\tau+R)/2} \int_\omega |D_{\underline{L}}(r\phi)|^2(\tau^*, v, \omega)\, d\omega \\ &\qquad \lesssim \int_\omega |D_{\underline{L}}(r\phi)|^2\Big(\tau^*, \frac{\tau_2 + R}{2}, \omega\Big)\, d\omega + \int_{H_{\tau^*}} |D_L D_{\underline{L}}(r\phi)| \cdot |D_{\underline{L}}(r\phi)|\, dv\, d\omega. \end{aligned}$$

Integrate the above estimate from τ_1 to τ_2 and apply the Cauchy–Schwarz inequality to the last term. From the integrated local energy estimate (59) and the energy decay estimate (64), we then derive

$$\int_{\tau_1}^{\tau_2} \sup_{v\geq(R+\tau)/2} \int_\omega |D_{\underline{L}}(r\phi)|^2\, d\omega\, d\tau \lesssim_{M_2} \mathcal{E}_0[\phi]\epsilon_1^{-1}(\tau_1)_+^{-1-\gamma_0} + \epsilon_1 I_0^{1+\epsilon}[r^{-1}D_L D_{\underline{L}}\psi](\bar{\mathcal{D}}_{\tau_1}^{\tau_2})$$

for all $\epsilon_1 > 0$. The case in the exterior region follows in a similar way. □

We also need the analogous estimate for $D_L(r\phi)$.

Proposition 36. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $\epsilon_1 > 0$ and $0 \leq p \leq 1+\gamma_0$, we have*

$$\|r^{p/2}D_L(r\phi)\|^2_{L_v^2 L_u^\infty L_\omega^2(\bar{\mathcal{D}}_\tau)} \lesssim_{M_2} \epsilon_1^{-1}\mathcal{E}_0[\phi](\tau)_+^{-1-\gamma_0} + \epsilon_1 I_{p_2}^{p_1}[r^{-1}D_{\underline{L}}D_L(r\phi)](\bar{\mathcal{D}}_\tau). \tag{69}$$

Here $p_1 = \max\{1+\epsilon, p\}$ and $p_2 = \min\{1+\frac{1}{2}\epsilon, p\}$.

Proof. Similar to the proof of the previous proposition, we choose the starting surface for $D_L(r\phi)$ to be $H_{\tau_1^*}$ in the interior region and the initial hypersurface $\{t=0\}$ in the exterior region. We only prove the proposition for the exterior region case. Denote $\psi = r\phi$. On $\underline{H}_v^{-v,\tau^*}$, $v \geq -\tau^*$, we can show that

$$\begin{aligned} r^p\int_\omega |D_L\psi|^2\, d\omega \lesssim &\int_\omega (r^p|D_L\psi|^2)(-v,v,\omega)\, d\omega \\ &+ \int_{\underline{H}_v^{-v,\tau^*}} (r^{p-1}|D_L\psi|^2 + r^p|D_L\psi||D_{\underline{L}}D_L(r\phi)|)\, du\, d\omega. \end{aligned}$$

The integral of the first term can be bounded by the assumption on the data. We control the second term by using the r-weighted energy estimate. We bound the last term as follows:

$$r^p|D_L\psi||D_{\underline{L}}D_L\psi| \lesssim \epsilon_1 r^{p_1}u_+^{p_2}|D_{\underline{L}}D_L\psi|^2 + \epsilon_1^{-1}r^{2p-p_1}u_+^{-p_2}|D_L\psi|^2, \quad \forall \epsilon_1 > 0.$$

When $2p \geq p_1$, we can use the r-weighted energy estimate (53) to bound the weighted integral of $|D_L\psi|$. Otherwise one can use interpolation and the integrated local energy decay estimate (56). For any case, from the energy decay estimates (53), (56), (63) and (64) for ϕ, one can always show that

$$\iint_{\mathcal{D}_\tau} r^{2p-p_1}u_+^{-p_2}|D_L\psi|^2\, du\, dv\, d\omega \lesssim_{M_2} \mathcal{E}_0[\phi]\tau_+^{p-1-\gamma_0}.$$

Another way to understand the above estimate is to use interpolation. It suffices to show the above estimate with $p=0$ and $p = 1+\gamma_0$. The former case follows by using the integrated local energy estimates for ϕ, while the later situation relies on the r-weighted energy estimate. Estimate (69) for the exterior region case then follows. The interior region case holds in a similar way. □

As we only commute the equation with ∂_t or the angular momentum Ω, to estimate the weighted spacetime integral of $D_L D_{\underline{L}}(r\phi)$ in terms of $D_Z\phi$, we use the equation of ϕ under the null frame.

Lemma 37. *Under the null frame, we can write the covariant wave operator $\Box_A$ as*

$$r\Box_A\phi = rD^\mu D_\mu\phi = -D_L D_{\underline{L}}(r\phi) + \not{D}^2(r\phi) - i\rho\cdot r\phi = -D_{\underline{L}}D_L(r\phi) + \not{D}^2(r\phi) + i\rho\cdot r\phi \tag{70}$$

for any complex scalar field ϕ. Here $\not{D}^2 = \not{D}^{e_1}\not{D}_{e_1} + \not{D}^{e_2}\not{D}_{e_2}$ and $\rho = \frac{1}{2}(dA)_{\underline{L}L}$.

Proof. The lemma follows by direct computation. □

This lemma leads to the following estimate for $D_L D_{\underline{L}}(r\phi)$ and $D_{\underline{L}} D_L(r\phi)$.

Proposition 38. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $1+\epsilon\le p\le 1+\gamma_0$, we have*

$$I^{p}_{2+\gamma_0-p-2\epsilon}[r^{-1}(|D_L D_{\underline{L}}\psi|+|D_L D_{\underline{L}}\psi|)](\bar{\mathcal{D}}_\tau)\lesssim_{M_2}\mathcal{E}_0[\phi]+I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[D\phi_1](\bar{\mathcal{D}}_\tau)+I^{\gamma_0}_{0}[\not{D}\psi_1](\bar{\mathcal{D}}_\tau). \quad (71)$$

Here $\phi_1=D_Z\phi$, $\psi_1=D_Z(r\phi)$ and $\psi=r\phi$.

Proof. Let's only consider the estimate for $D_L D_{\underline{L}}(r\phi)$ in the interior region. The proof easily implies the estimates for $D_{\underline{L}} D_L(r\phi)$. The case in the exterior region is easier since in that region $r\ge\frac{1}{3}u_+$. It hence suffices to show the estimate for $p=1+\gamma_0$, which is similar to the proof for the interior region case. Take $\bar{\mathcal{D}}_\tau$ to be $\bar{\mathcal{D}}^{\tau_2}_{\tau_1}$ for $0\le\tau_1=\tau<\tau_2$. From the equation (70) for ϕ under the null frame, we derive

$$r^p|D_L D_{\underline{L}}(r\phi)|^2\lesssim r^p|\Box_A\phi|^2r^2+r^p|r\phi\rho|^2+r^p|r^{-1}\not{D}D_\Omega\psi|^2.$$

Here we note that $|\not{D}^2\psi|^2\lesssim|r^{-1}\not{D}D_\Omega\psi|$. The integral of the first term on the right-hand side can be bounded by $\mathcal{E}_0[\phi]$. For the second term, we control ϕ by using Lemma 19. The last term is favorable as it is a form of $\not{D}D_Z\psi$. We absorb those terms with the help of the small constant ϵ_1 from Propositions 35 and 36. According to our notation in this section, let $\psi_1=D_\Omega\psi$. For all $1+\epsilon\le p\le 1+\gamma_0$, we have

$$\tau_+^{2+\gamma_0-p-2\epsilon}r^{p-2}\lesssim r^{\gamma_0}+\tau_+^{1+\gamma_0-\epsilon}r^{-1-\epsilon},\quad r\ge R.$$

Since the energy flux for ϕ decays from Proposition 32, using Lemma 19 we conclude that

$$\int_\omega r^p|\phi|^2\,d\omega\lesssim_{M_2}\mathcal{E}_0[\phi](\tau_1)_+^{p-2-\gamma_0}.$$

Therefore, for all $1+\epsilon\le p\le 1+\gamma_0$ we can show that

$$\begin{aligned}\iint_{\bar{\mathcal{D}}^{\tau_2}_{\tau_1}}&\tau_+^{2+\gamma_0-p-2\epsilon}r^p|D_L D_{\underline{L}}(r\phi)|^2\,dv\,du\,d\omega\\ &\lesssim I^{p}_{2+\gamma_0-p-2\epsilon}[\Box_A\phi](\bar{\mathcal{D}}^{\tau_2}_{\tau_1})+I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[D\phi_1](\bar{\mathcal{D}}^{\tau_2}_{\tau_1})+I^{\gamma_0}_{0}[\not{D}\psi_1](\bar{\mathcal{D}}^{\tau_2}_{\tau_1})\\ &\qquad+\int_{\tau_1}^{\tau_2}\tau_+^{2+\gamma_0-p-2\epsilon}\int_{\frac{1}{2}(\tau+R)}^{\infty}\int_\omega r^p|\phi|^2\,d\omega\cdot\sum_{j\le 2}\int_\omega r^2|\mathcal{L}^j_\Omega\bar{\rho}|^2\,d\omega\,dv\,du\\ &\lesssim_{M_2}\mathcal{E}_0[\phi]+I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[D\phi_1](\bar{\mathcal{D}}^{\tau_2}_{\tau_1})+I^{\gamma_0}_{0}[\not{D}\psi_1](\bar{\mathcal{D}}^{\tau_2}_{\tau_1})\end{aligned}$$

This finishes the proof. □

Next we estimate the weighted spacetime norm of $|\alpha||D_{\underline{L}}(r\phi)|$.

Proposition 39. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Denote $\psi=r\phi$. For all $1+\epsilon\le p\le 1+\gamma_0$, $\epsilon_1>0$, we have*

$$\iint_{\bar{\mathcal{D}}_\tau}u_+^{2+\gamma_0+\epsilon-p}r^p|\alpha|^2|D_{\underline{L}}(r\phi)|^2\,dx\,dt\lesssim_{M_2}\mathcal{E}_0[\phi]\epsilon_1^{-1}\tau_+^{-\gamma_0+\epsilon}+\epsilon_1 I^{1+\epsilon}_{1+\epsilon}[r^{-1}D_L D_{\underline{L}}(r\phi)](\bar{\mathcal{D}}_\tau). \quad (72)$$

Proof. Make use of Proposition 35. For all $1+\epsilon \le p \le 1+\gamma_0$, we can show that

$$\begin{aligned}\iint_{\overline{\mathcal{D}}_\tau} r^p |\alpha|^2 |D_{\underline{L}}(r\phi)|^2\, dx\, dt &\lesssim \|D_{\underline{L}}\psi\|^2_{L_u^2 L_v^\infty L_\omega^2(\overline{\mathcal{D}}_\tau)} \cdot \big\|r^{(p/2)+1}\alpha\big\|^2_{L_u^\infty L_v^2 L_\omega^\infty(\overline{\mathcal{D}}_\tau)} \\ &\lesssim \|D_{\underline{L}}\psi\|^2_{L_u^2 L_v^\infty L_\omega^2(\overline{\mathcal{D}}_\tau)} \cdot \sum_{j\le 2} \big\|r^{(p/2)+1}\mathcal{L}_\Omega^j \alpha\big\|^2_{L_u^\infty L_v^2 L_\omega^2(\overline{\mathcal{D}}_\tau)} \\ &\lesssim_{M_2} \mathcal{E}_0[\phi]\epsilon_1^{-1}\tau_+^{p-2-2\gamma_0} + \epsilon_1 \tau_+^{p-1-\gamma_0} I_0^{1+\epsilon}[r^{-1}D_L D_{\underline{L}}\psi](\overline{\mathcal{D}}_\tau)\end{aligned}$$

for all $\epsilon_1 > 0$. As the above estimate holds for all $\tau \in \mathbb{R}$, from Lemma 20, we conclude that

$$\iint_{\overline{\mathcal{D}}_\tau} u_+^{2+\gamma_0+\epsilon-p} r^p |\alpha|^2 |D_{\underline{L}}(r\phi)|^2\, dx\, dt \lesssim_{M_2} \mathcal{E}_0[\phi]\epsilon_1^{-1}\tau_+^{-\gamma_0+\epsilon} + \epsilon_1 I_{1+\epsilon}^{1+\epsilon}[r^{-1}D_L D_{\underline{L}}\psi](\overline{\mathcal{D}}_\tau).$$

This finishes the proof for estimate (72). □

Next we estimate the weighted spacetime integral of $(|\underline{\alpha}| + r^{-1}|\rho|)|D_L(r\phi)|$. One possible way to bound this term, in particular $\underline{\alpha}$, is to make use of the energy flux through the incoming null hypersurface. It turns out that we lose a little bit of decay in u and we are not able to close the bootstrap argument later. An alternative way is to use $\sup_v \int_\omega |\underline{\alpha}|^2\, d\omega$, which has to exploit the equation for F. For $\tau \in \mathbb{R}$, denote

$$h(\tau) = \sum_{k\le 1} \|\mathcal{L}_\Omega^k (r\underline{\alpha})\|^2_{L_v^\infty L_\omega^2(H_{\tau^*})} + \sum_{k\le 2} \int_{\Sigma_\tau} \frac{|\mathcal{L}_Z^k \overline{F}|^2}{r_+^{1+\epsilon}} r^2\, d\tilde{v}\, d\omega. \tag{73}$$

Here $(\tilde{v}, \omega)$ are coordinates of Σ_τ, that is, $(\tilde{v}, \omega) = (r, \omega)$ when $r \le R$ and $(\tilde{v}, \omega) = (v, \omega)$ otherwise. We cannot show that $h(\tau)$ decays in τ. However, we can show the following:

Corollary 40. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then the function $h(\tau)$ is integrable in τ:*

$$\int_{\tau_1}^{\tau_2} \tau_+^{1+\epsilon} h(\tau)\, d\tau \lesssim M_2 \tau_+^{-\gamma_0+3\epsilon}, \qquad \int_{\tilde{\tau}\le\tau} \tilde{\tau}_+^{1+\epsilon} h(\tilde{\tau})\, d\tilde{\tau} \lesssim M_2 \tau_+^{-\gamma_0+3\epsilon} \tag{74}$$

for all $0 \le \tau_1 < \tau_2$ and $\tau \le 0$.

Proof. Using Lemma 20, the corollary follows from estimate (42) and the integrated local energy estimates (24) and (25) for the Maxwell field F. □

We now can estimate the weighted spacetime integral of $|\underline{\alpha}|\,|D_L\psi|$.

Proposition 41. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $1+\epsilon \le p \le 1+\gamma_0$, $\epsilon_1 > 0$, we have*

$$\begin{aligned}&\iint_{\overline{\mathcal{D}}_\tau} u_+^{2+\gamma_0+\epsilon-p} |\underline{\alpha}|^2 |D_L(r\phi)|^2 r^p\, dx\, dt \\ &\qquad \lesssim_{M_2} \epsilon_1 \iint_{\overline{\mathcal{D}}_\tau} \tilde{\tau}_+^{2+\gamma_0+\epsilon-p} h(\tilde{\tau}) r^p |D_L(r D_Z\phi)|^2\, dv\, d\omega\, d\tilde{\tau} + \epsilon_1^{-1}\mathcal{E}_0[\phi]\tau_+^{-\gamma_0+3\epsilon}.\end{aligned} \tag{75}$$

Proof. Let $\psi = r\phi$ and $\psi_1 = D_Z(r\phi)$. We first use Sobolev embedding on the unit sphere to bound

$$\big\| |r\underline{\alpha}| |D_L\psi| \big\|^2_{L^2_\omega} \lesssim \big(\|r\underline{\alpha}\|^2_{L^2_\omega} + \|r\mathcal{L}_\Omega\underline{\alpha}\|^2_{L^2_\omega}\big) \cdot \big(\epsilon_1^{-1}\|D_L\psi\|^2_{L^2_\omega} + \epsilon_1\|D_\Omega D_L\psi\|^2_{L^2_\omega}\big), \quad \epsilon_1 > 0.$$

The proof for this estimate for all connections A follows from the case when A is trivial, as the norm is gauge invariant. We in particular can choose a gauge so that the function is real, then make use of estimate (42) of Proposition 17. We therefore can show that

$$\begin{aligned}
&\big\| r^{p/2} u_+^{(2+\gamma_0+\epsilon-p)/2} |r\underline{\alpha}| |D_L\psi| \big\|^2_{L^2_u L^2_v L^2_\omega(\overline{\mathcal{D}}_\tau)} \\
&\quad\lesssim \Big\| r^{p/2} \sum_{k\le 1} \|\mathcal{L}^k_\Omega(r\underline{\alpha})\|_{L^2_\omega} \cdot u_+^{(2+\gamma_0+\epsilon-p)/2} \big(\epsilon_1^{1/2}\|D_\Omega D_L\psi\|_{L^2_\omega} + \epsilon_1^{-1/2}\|D_L\psi\|_{L^2_\omega}\big)\Big\|^2_{L^2_u L^2_v} \\
&\quad\lesssim \big\| \tilde{\tau}_+^{2+\gamma_0+\epsilon-p} h(\tilde{\tau}) \big(\epsilon_1 \|r^{p/2} D_L D_\Omega\psi\|^2_{L^2_v L^2_\omega} + \epsilon_1\|r^{p/2} r\alpha\psi\|^2_{L^2_v L^2_\omega} + \epsilon_1^{-1}\|r^{p/2}D_L\psi\|^2_{L^2_v L^2_\omega}\big)\big\|_{L^1_u} \\
&\quad\lesssim \epsilon_1 \int_{\tilde{\tau}} \tilde{\tau}_+^{2+\gamma_0+\epsilon-p} h(\tilde{\tau}) \int_{H_{\tilde{\tau}^*}} r^p |D_L\psi_1|^2 \, dv\, d\omega\, d\tilde{\tau} + \epsilon_1^{-1}\|\tilde{u}_+^{1+\epsilon} h(\tilde{u})\|_{L^1_u} \|u_+^{1+\gamma_0-p/2} r^{p/2} D_L\psi\|^2_{L^\infty_u L^2_v L^2_\omega} \\
&\qquad\qquad + \epsilon_1 \|\tilde{u}_+^{1+\epsilon-\gamma_0} h(\tilde{u})\|_{L^1_u} \|u_+^{1+\gamma_0-p/2} r^{p/2} r\alpha\|^2_{L^\infty_u L^2_v L^\infty_\omega} \|u_+^{\gamma_0/2}\psi\|^2_{L^\infty_u L^\infty_v L^2_\omega} \\
&\quad\lesssim_{M_2} \epsilon_1 \iint_{\overline{\mathcal{D}}_\tau} \tilde{\tau}_+^{2+\gamma_0+\epsilon-p} h(\tilde{\tau}) r^p |D_L\psi_1|^2 \, dv\, d\omega\, du + \epsilon_1^{-1}\mathcal{E}_0[\phi]\tau_+^{-\gamma_0+3\epsilon}.
\end{aligned}$$

Here we have used the r-weighted energy estimates (27) and (28) and estimate (49) to bound ϕ. □

For $|r^{-1}\rho||D_L\psi|$, we have extra decay in r, which allows us to use Proposition 36.

Proposition 42. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $\epsilon_1 > 0$, we have*

$$\iint_{\overline{\mathcal{D}}_\tau} u_+^{1+\gamma_0} |r^{-1}\rho|^2 |D_L(r\phi)|^2 r^{1+\gamma_0}\, dx\, dt \lesssim_{M_2} \epsilon_1 I_\epsilon^{1+\epsilon}[r^{-1}D_{\underline{L}}D_L(r\phi)](\mathcal{D}_{\tau_1}^{\tau_2}) + \epsilon_1^{-1}\mathcal{E}_0[\phi]. \tag{76}$$

Proof. The idea is that we bound ρ by using the energy flux through the incoming null hypersurface and $D_L(r\phi)$ by using Proposition 36. In the exterior region, we need to specially consider the effect of the nonzero charge. Other than that, the proof is the same for the interior region case. We thus take $\overline{\mathcal{D}}_\tau$ to be $\mathcal{D}_\tau$ with $\tau \le 0$. Let $\psi = r\phi$. For all $1+\epsilon \le p \le 1+\gamma_0$, we can show that

$$\begin{aligned}
\iint_{\mathcal{D}_\tau} |r^{-1}\rho|^2 |D_L\psi|^2 r^p\, dx\, dt &\lesssim \iint_{\mathcal{D}_\tau} |\bar{\rho}|^2 |D_L\psi|^2 r^p\, du\, dv\, d\omega + \iint_{\mathcal{D}_\tau} |q_0|^2 |D_L\psi|^2 r^{p-4}\, du\, dv\, d\omega \\
&\lesssim_{M_2} \|D_L\psi\|^2_{L^2_v L^\infty_u L^2_\omega(\mathcal{D}_\tau)} \|r\bar{\rho}\|^2_{L^\infty_v L^2_u L^\infty_\omega(\mathcal{D}_\tau)} + \mathcal{E}_0[\phi]\tau_+^{-1-2\gamma_0} \\
&\lesssim_{M_2} \epsilon_1 \tau_+^{-1-\gamma_0} I_0^{1+\epsilon}[r^{-1}D_{\underline{L}}D_L\psi](\mathcal{D}_\tau) + \epsilon_1^{-1}\mathcal{E}_0[\phi]\tau_+^{-1-2\gamma_0}.
\end{aligned}$$

The above estimate also holds for the interior region case when $\overline{\mathcal{D}}_\tau = \mathcal{D}_{\tau_1}^{\tau_2}$ for all $0 \le \tau = \tau_1 < \tau_2$. From Lemma 20, we then can show, taking the interior region for example, that

$$\begin{aligned}
&\iint_{\mathcal{D}_{\tau_1}^{\tau_2}} \tau_+^{1+\gamma_0} |r^{-1}\rho|^2 |D_L\psi|^2 r^p \, dx\, dt \\
&\qquad\qquad \lesssim_{M_2} \epsilon_1 I_0^{1+\epsilon}[r^{-1} D_{\underline{L}} D_L \psi](\mathcal{D}_{\tau_1}^{\tau_2}) + \epsilon_1 \int_{\tau_1}^{\tau_2} \tau_+^{-1} I_0^{1+\epsilon}[r^{-1} D_{\underline{L}} D_L\psi](\mathcal{D}_\tau^{\tau_2})\, d\tau + \epsilon_1^{-1}\mathcal{E}_0[\phi] \\
&\qquad\qquad \lesssim_{M_2} \epsilon_1 I_\epsilon^{1+\epsilon}[r^{-1} D_{\underline{L}} D_L\psi](\mathcal{D}_{\tau_1}^{\tau_2}) + \epsilon_1^{-1}\mathcal{E}_0[\phi].
\end{aligned}$$

Here we note that $\ln \tau_+ \lesssim \tau_+^\epsilon$. □

Next we estimate $r^{-1}|F||\not{D}(r\phi)|$.

Proposition 43. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $\epsilon_1 > 0$, we have*

$$\iint_{\overline{\mathcal{D}}_\tau} u_+^{1+\gamma_0} |r^{-1}F|^2 |\not{D}(r\phi)|^2 r^{1+\gamma_0}\, dx\, dt \lesssim_{M_2} \epsilon_1^{-1}\mathcal{E}_0[\phi] + \epsilon_1 \int_{\tilde{\tau}} \tilde{\tau}_+^{1+\gamma_0} h(\tilde{\tau}) E[D_Z\phi](H_{\tilde{\tau}^*})\, d\tilde{\tau}. \tag{77}$$

Proof. The idea is to use the energy flux through the outgoing null hypersurface to bound $\not{D}(r\phi) = D_\Omega\phi$ and the integrated local energy estimate to control F. We only show the estimate in the exterior region. Take $\overline{\mathcal{D}_\tau}$ to be $\mathcal{D}_\tau$ for any $\tau \le 0$. In the exterior region we have the relation $r \ge \frac{1}{3}u_+$. Therefore, from estimate (50) and the definition (73) of $h(\tau)$, we can show that

$$\begin{aligned}
&\iint_{\mathcal{D}_\tau} u_+^{1+\gamma_0} |F|^2 |\not{D}(r\phi)|^2 r^{1+\gamma_0}\, du\, dv\, d\omega \\
&\quad \lesssim \int_u u_+^{1+\gamma_0} \int_v \sum_{k\le 2} \left(r^{1-\epsilon} \int_\omega \left(|\mathcal{L}_\Omega^k \overline{F}|^2 + |q_0 r^{-2}|^2\right) d\omega \right) \cdot \int_\omega r |D_\Omega\phi|^2\, d\omega\, dv\, du \\
&\quad \lesssim |q_0|^2 \iint_{\mathcal{D}_\tau} \frac{|\not{D}\phi|^2}{r^{1+\epsilon}}\, dx\, dt + \int_{\tilde{\tau}} (\tilde{\tau})_+^{1+\gamma_0} h(\tilde{\tau}) \left(\epsilon_1^{-1} \int_{H_{\tilde{\tau}^*}} |\not{D}\phi|^2 r^2\, dv\, d\omega + \epsilon_1 E[D_Z\phi](H_{\tilde{\tau}^*}) \right) d\tilde{\tau} \\
&\quad \lesssim_{M_2} \mathcal{E}_0[\phi]\tau_+^{-1-\gamma_0} + \int_{\tilde{\tau}} (\tilde{\tau})_+^{1+\gamma_0} h(\tilde{\tau}) \epsilon_1^{-1} E[\phi](H_{\tilde{\tau}^*})\, d\tilde{\tau} + \epsilon_1 \int_u h(2u+R) E[D_Z\phi](H_u)\, du \\
&\quad \lesssim_{M_2} \epsilon_1^{-1}\mathcal{E}_0[\phi]\tau_+^{-1-\gamma_0+2\epsilon} + \epsilon_1 \int_{\tilde{\tau}} \tilde{\tau}_+^{1+\gamma_0} h(\tilde{\tau}) E[D_Z\phi](H_{\tilde{\tau}^*})\, d\tilde{\tau}.
\end{aligned}$$

Here we assumed that $\gamma_0 < 1$ and ϵ is sufficiently small. For the case $\gamma_0 = 1$, the above estimate also holds but in a different form where we have to rely on the r-weighted energy estimate. For the sake of simplicity, we do not discuss this in detail when $\gamma_0 \ge 1$. □

Finally, we estimate the weighted spacetime norm of $(|J| + |r\not{J}| + |\sigma| + |r^{-1}\rho|)|\phi|$. We show:

Proposition 44. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $1+\epsilon \le p \le 1+\gamma_0$, we have*

$$\iint_{\overline{\mathcal{D}}_\tau} (|J|^2 + |r\not{J}|^2 + |\sigma|^2 + |r^{-1}\rho|^2)|\phi|^2 r^p u_+^{2+\gamma_0+\epsilon-p}\, dx\, dt \lesssim_{M_2} \mathcal{E}_0[\phi]\tau_+^{-\gamma_0+\epsilon}. \tag{78}$$

Proof. Let's first consider $(|J|^2+|\sigma|^2+|r^{-1}\rho|^2)|\phi|^2$. The idea is that we bound ϕ by the energy flux. Note that the nonzero charge only affects the estimates in the exterior region where $r \geq \frac{1}{3}u_+$. From the embedding (49) and the energy decay estimates (56) and (64), we can show that

$$\begin{aligned}
&\iint_{\bar{\mathcal{D}}_\tau} (|J|^2+|\sigma|^2+|r^{-1}\rho|^2)|\phi|^2 r^{p+2}u_+^{2+\gamma_0+\epsilon-p}\,du\,dv\,d\omega \\
&\quad\lesssim \int_u u_+^{2+\gamma_0+\epsilon-p}\int_v \sum_{k\leq 2} r^{p+1}\int_\omega \big(|\mathcal{L}_\Omega^k J|^2+|\mathcal{L}_\Omega^k\sigma|^2+|r^{-1}\mathcal{L}_\Omega^k\bar{\rho}|^2+|q_0 r^{-3}|^2\big)\,d\omega\cdot\int_\omega r|\phi|^2\,d\omega\,dv\,du \\
&\quad\lesssim_{M_2} \mathcal{E}_0[\phi]\int_u u_+^{1+\epsilon-p}\int_v \sum_{k\leq 2} r^{p+1}\int_\omega \big(|\mathcal{L}_\Omega^k J|^2+|\mathcal{L}_\Omega^k\sigma|^2+|r^{-1}\mathcal{L}_\Omega^k\bar{\rho}|^2+|q_0 r^{-3}|^2\big)\,d\omega\,dv\,du \\
&\quad\lesssim_{M_2} \tau_+^{-\gamma_0+\epsilon}.
\end{aligned}$$

Here we used the r-weighted energy estimates (31) and (32) to bound the curvature components and the definition for M_2 to control J. For $|r\underline{\alpha}|^2|\phi|^2$, the only difference is that we need to put more r weights on ϕ. By using the embedding inequality (49) and the energy decay estimates (56) and (64), we conclude that

$$\int_\omega r^{1+p-\gamma_0}|\phi|^2\,d\omega \lesssim_{M_2} \tau_+^{p-1-2\gamma_0}.$$

Therefore, we have

$$\begin{aligned}
&\iint_{\bar{\mathcal{D}}_\tau} u_+^{2+\gamma_0+\epsilon-p}|r\underline{\alpha}|^2|\phi|^2 r^{p+2}\,du\,dv\,d\omega \\
&\qquad\lesssim \int_u u_+^{2+\gamma_0+\epsilon-p}\int_v \sum_{k\leq 2} r^{3+\gamma_0}\int_\omega |\mathcal{L}_\Omega^k\underline{\alpha}|^2\,d\omega\cdot\int_\omega r^{1+p-\gamma_0}|\phi|^2\,d\omega\,du\,dv \\
&\qquad\lesssim_{M_2} \mathcal{E}_0[\phi]\sum_{k\leq 2}\iint_{\bar{\mathcal{D}}_\tau} u_+^{1-\gamma_0+\epsilon} r^{3+\gamma_0}|\mathcal{L}_\Omega^k\underline{\alpha}|^2\,d\omega\,dv\,du \\
&\qquad\lesssim_{M_2} \mathcal{E}_0[\phi]\tau_+^{-\gamma_0}.
\end{aligned}$$

Estimate (78) follows. □

Now it remains to consider the spacetime norm on the bounded region $\{r \leq R\}$.

Proposition 45. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then on the bounded region $\{r \leq R\}$, for all $0 \leq \tau_1 < \tau_2$ we have*

$$\int_{\tau_1}^{\tau_2} \tau_+^{1+\gamma_0}\int_{r\leq R} |[\Box_A, D_Z]\phi|^2\,dx\,dt \lesssim_{M_2} \mathcal{E}_0[\phi](\tau_1)_+^{-1-\gamma_0}. \tag{79}$$

Proof. First we conclude from the energy estimate (64) that the energy flux of the scalar field decays:

$$E[\phi](\Sigma_\tau) \lesssim_{M_2} \tau_+^{-1-\gamma_0}, \quad \forall \tau \geq 0.$$

From the commutator estimate (66), we have

$$|[\Box_A, D_Z]\phi| \lesssim |F||\tilde{D}\phi| + |J||\phi|.$$

For the first term, we make use of estimate (39):

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\tau_+^{1+\gamma_0}\int_{r\le R}|F|^2|\tilde{D}\phi|\,dx\,dt &\lesssim \int_{\tau_1}^{\tau_2}\sup_{|x|\le R}|F|^2(\tau,x)E[\phi](\Sigma_\tau)\tau_+^{1+\gamma_0}\,d\tau\\ &\lesssim_{M_2}\mathcal{E}_0[\phi]\int_{\tau_1}^{\tau_2}\sup_{|x|\le R}|F|^2(\tau,x)\,d\tau\\ &\lesssim_{M_2}\mathcal{E}_0[\phi](\tau_1)_+^{-1-\gamma_0}.\end{aligned}$$

For $|J||\phi|$, we use Sobolev embedding on the ball B_R with radius R at fixed time τ:

$$\begin{aligned}\int_{\tau_1}^{\tau_2}\tau_+^{1+\gamma_0}\int_{r\le R}|J|^2|\phi|^2\,dx\,dt &\lesssim \int_{\tau_1}^{\tau_2}\tau_+^{1+\gamma_0}\|J\|^2_{H_x^1(B_R)}\cdot\|\phi\|^2_{H_x^1(B_R)}\,d\tau\\ &\lesssim_{M_2}\int_{\tau_1}^{\tau_2}\mathcal{E}_0[\phi]\int_{r\le R}|\overline{\nabla}J|^2+|J|^2\,dx\,d\tau\\ &\lesssim_{M_2}\mathcal{E}_0[\phi](\tau_1)_+^{-1-\gamma_0}.\end{aligned}$$

Thus, estimate (79) holds. □

Now, from Lemma 33, combine estimates (72) and (75)-(79). We can bound the first-order commutator.

Corollary 46. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all positive constants $\epsilon_1 < 1$, we have*

$$\begin{aligned}I_{1+\epsilon}^{1+\gamma_0}&[[\Box_A, D_Z]\phi](\{t\ge 0\}) + I_{1+\epsilon}^{1+\gamma_0}[[\Box_A, D_Z]\phi](\{t\ge 0\})\\ &\lesssim_{M_2}\epsilon_1 I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[D\phi_1](\{t\ge 0\}) + \epsilon_1 I_0^{\gamma_0}[\not{D}\psi_1](\{t\ge 0\}\cap\{r\ge R\}) + \mathcal{E}_0[\phi]\epsilon_1^{-1}\\ &\quad+\epsilon_1\int_{\mathbb{R}}\tau_+^{1+\gamma_0}h(\tau)E[D_Z\phi](\Sigma_\tau)\,d\tau + \epsilon_1\int_{\mathbb{R}}\int_{H_{\tau^*}}\tau_+^{2+\gamma_0+\epsilon-p}h(\tau)r^p|D_L\psi_1|^2\,dv\,d\omega\,d\tau. \quad (80)\end{aligned}$$

Here $\phi_1 = D_Z\phi$, $\psi_1 = D_Z(r\phi)$ and $Z\in\Gamma\{\partial_t,\Omega_{ij}\}$.

Proof. From Lemma 33, estimate (80) is a consequence of estimates (72), (75), (77), (76), (78) and (79). The term $I_{1+\epsilon}^{1+\epsilon}[r^{-1}D_{\underline{L}}D_L\psi](D)$ can further be controlled by using Proposition 38 with $p = 1+\epsilon$. □

Now we are able to derive the energy decay estimates for the first-order derivative of the scalar field. Based on the result for the decay estimates for ϕ in the previous subsection, it suffices to bound $\mathcal{E}_0[D_Z\phi]$.

Proposition 47. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then we have the bound*

$$\mathcal{E}_0[D_Z\phi]\lesssim_{M_2}\mathcal{E}_1[\phi]. \quad (81)$$

Proof. First, by definition,

$$\mathcal{E}_0[D_Z\phi]\lesssim\mathcal{E}_1[\phi] + I_{1+\epsilon}^{1+\gamma_0}[[\Box_A, D_Z]\phi](\{t\ge 0\}) + I^{1+\epsilon}_{1+\gamma_0}[[\Box_A, D_Z]\phi](\{t\ge 0\}).$$

Then from the previous estimate (80), the above inequality leads to

$$\mathcal{E}_0[D_Z\phi] \lesssim_{M_2} \epsilon_1 I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[DD_Z\phi](\{t\ge 0\}) + \epsilon_1 I^{\gamma_0}_0[\not{D}\psi_1](\{t\ge 0\}\cap\{r\ge R\}) + \mathcal{E}_1[\phi]\epsilon_1^{-1}$$
$$+\epsilon_1 \int_{\mathbb{R}} \tau_+^{1+\gamma_0} h(\tau) E[D_Z\phi](\Sigma_\tau)\,d\tau + \epsilon_1 \int_{\mathbb{R}}\int_{H_{\tau^*}} \tau_+^{2+\gamma_0+\epsilon-p} h(\tau) r^p |D_L\psi_1|^2\,dv\,d\omega\,d\tau$$

for all $0<\epsilon_1<1$. Here $\phi_1 = D_Z\phi$, $\psi_1 = D_Z(r\phi)$ and the implicit constant is independent of ϵ_1.

Now from the integrated local energy estimates (56) and (59) combined with Lemma 20, we can show that

$$I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[DD_Z\phi](\{t\ge 0\}) \lesssim_{M_2} \mathcal{E}_0[D_Z\phi].$$

By the energy decay estimates (23) and (64), we have the energy decay for $D_Z\phi$:

$$E[D_Z\phi](\Sigma_\tau) \lesssim_{M_2} \mathcal{E}_0[D_Z\phi]\tau_+^{-1-\gamma_0}, \quad \forall \tau\in\mathbb{R}.$$

Moreover, the r-weighted energy estimates (53) and (63) imply that

$$\tau_+^{1+\gamma_0-p}\int_{H_{\tau^*}} r^p |D_L(rD_Z\phi)|^2\,dv\,d\omega + \int_{\mathbb{R}}\int_{H_{\tau^*}} r^{\gamma_0}|\not{D}(rD_Z\phi)|^2\,dv\,d\omega\,d\tau \lesssim_{M_2} \mathcal{E}_0[D_Z\phi].$$

Recall the definition for $h(\tau)$ in line (73). By Corollary 40, we then can demonstrate that

$$\int_{\mathbb{R}} \tau_+^{1+\gamma_0} h(\tau) E[D_Z\phi](\Sigma_\tau)\,d\tau \lesssim_{M_2} \mathcal{E}_0[D_Z\phi]\int_{\mathbb{R}} h(\tau)\,d\tau \lesssim_{M_2} \mathcal{E}_0[D_Z\phi],$$
$$\int_{\mathbb{R}}\int_{H_{\tau^*}} \tau_+^{2+\gamma_0+\epsilon-p} h(\tau) r^p |D_L\psi_1|^2\,dv\,d\omega\,d\tau \lesssim_{M_2} \mathcal{E}_0[D_Z\phi]\int_{\mathbb{R}} \tau_+^{1+\epsilon} h(\tau)\,d\tau \lesssim_{M_2} \mathcal{E}_0[D_Z\phi].$$

We therefore derive that

$$\mathcal{E}_0[D_Z\phi] \lesssim_{M_2} \epsilon_1\mathcal{E}_0[D_Z\phi] + \epsilon_1^{-1}\mathcal{E}_1[\phi], \quad \forall 0<\epsilon_1<1.$$

Take ϵ_1 to be sufficiently small, depending only on M_2, γ_0, R and ϵ. We then obtain estimate (81). □

The above argument implies all the desired energy decay estimates for the first-order derivative of the scalar field in terms of $\mathcal{E}_1[\phi]$. Moreover, estimate (80) can be improved as follows:

Corollary 48. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all positive constants $\epsilon_1 < 1$, we have*

$$I^{1+\gamma_0}_{1+\epsilon}[[\Box_A, D_Z]\phi](\{t\ge 0\}) + I^{1+\gamma_0}_{1+\epsilon}[[\Box_A, D_Z]\phi](\{t\ge 0\}) \lesssim_{M_2} \epsilon_1\mathcal{E}_1[\phi] + \mathcal{E}_0[\phi]\epsilon_1^{-1}. \tag{82}$$

4.3.4. *Energy decay estimates for the second-order derivatives of the scalar field.* In this subsection, we establish the energy decay estimates for the second-order derivative of the scalar field. Note that the definition of M_2 records the size and regularity of the connection field A, which is independent of the scalar field. In particular, Proposition 47 and Corollary 48 apply to $\phi_1 = D_Z\phi$:

$$\mathcal{E}_0[D_Z\phi_1] \lesssim_{M_2} \mathcal{E}_1[\phi_1],$$
$$I^{1+\gamma_0}_{1+\epsilon}[[\Box_A, D_Z]\phi_1](\{t\ge 0\}) + I^{1+\gamma_0}_{1+\epsilon}[[\Box_A, D_Z]\phi_1](\{t\ge 0\}) \lesssim_{M_2} \epsilon_1\mathcal{E}_1[\phi_1] + \mathcal{E}_1[\phi]\epsilon_1^{-1}$$

for all $0<\epsilon_1<1$. Here $\mathcal{E}_0[\phi_1]\lesssim_{M_2}\mathcal{E}_1[\phi]$ by Proposition 47. To derive the energy decay estimates for the second-order derivative of the solution, it suffices to bound $\mathcal{E}_1[\phi_1]$. As $\phi_1=D_Z\phi$, by definition

$$\begin{aligned}
\mathcal{E}_1[\phi_1]&=\mathcal{E}_0[\phi_1]+E_0^1[\phi_1]+I_{1+\epsilon}^{1+\gamma_0}[D_Z\Box_A\phi_1](\{t\geq 0\})+I_{1+\gamma_0}^{1+\epsilon}[D_Z\Box_A\phi_1](\{t\geq 0\})\\
&\lesssim \mathcal{E}_2[\phi]+I_{1+\epsilon}^{1+\gamma_0}[D_Z[\Box_A,D_Z]\phi](\{t\geq 0\})+I_{1+\gamma_0}^{1+\epsilon}[D_Z[\Box_A,D_Z]\phi](\{t\geq 0\})\\
&\lesssim \mathcal{E}_2[\phi]+I_{1+\epsilon}^{1+\gamma_0}[[D_Z,[\Box_A,D_Z]]\phi](\{t\geq 0\})+I_{1+\gamma_0}^{1+\epsilon}[[D_Z,[\Box_A,D_Z]]\phi](\{t\geq 0\})\\
&\qquad +I_{1+\epsilon}^{1+\gamma_0}[[\Box_A,D_Z]D_Z\phi](\{t\geq 0\})+I_{1+\gamma_0}^{1+\epsilon}[[\Box_A,D_Z]D_Z\phi](\{t\geq 0\})\\
&\lesssim_{M_2} \mathcal{E}_2[\phi]+I_{1+\epsilon}^{1+\gamma_0}[[D_Z,[\Box_A,D_Z]]\phi](\{t\geq 0\})+I_{1+\gamma_0}^{1+\epsilon}[[D_Z,[\Box_A,D_Z]]\phi](\{t\geq 0\})\\
&\qquad +\epsilon_1\mathcal{E}_1[\phi_1]+\mathcal{E}_1[\phi]\epsilon_1^{-1}
\end{aligned}$$

for $0<\epsilon_1<1$. Let ϵ_1 be sufficiently small. We then conclude that

$$\mathcal{E}_1[\phi_1]\lesssim_{M_2}\mathcal{E}_2[\phi]+I_{1+\epsilon}^{1+\gamma_0}[[D_Z,[\Box_A,D_Z]]\phi](\{t\geq 0\})+I_{1+\gamma_0}^{1+\epsilon}[[D_Z,[\Box_A,D_Z]]\phi](\{t\geq 0\}).$$

Therefore, bounding $\mathcal{E}_1[\phi_1]$ is reduced to controlling the second-order commutator $[D_Z,[\Box_A,D_Z]]\phi$.

First, we have the following analogue of Lemma 33.

Lemma 49. *For all $X,Y\in\Gamma$, when $r\geq R$, we have*

$$|[D_X,[\Box_A,D_Y]]\phi|\lesssim|[\Box_{\mathcal{L}_ZA},D_Z]\phi|+\big(|F|^2+|r\alpha||r\underline{\alpha}|+|r\sigma|^2+|r\rho|(|\underline{\alpha}|+|\alpha|)\big)|\phi|. \tag{83}$$

When $r\leq R$, we have

$$|[D_X,[\Box_A,D_Y]]\phi|\lesssim|[\Box_{\mathcal{L}_ZA},D_Z]\phi|+|[\Box_A,D_Z]\phi|+|F|^2|\phi|. \tag{84}$$

Here we note that $\mathcal{L}_ZF=\mathcal{L}_Z\,dA=d\mathcal{L}_ZA$.

Proof. First, from Lemma 4, we can write

$$[\Box_A,D_X]\phi=2iX^\nu F_{\mu\nu}D^\mu\phi+i\nabla^\mu(F_{\mu\nu}X^\nu)\phi.$$

We need to compute the double commutator $[D_Y,[\Box_A,D_X]]\phi$ for $X,Y\in\Gamma$. We can compute that

$$\begin{aligned}
[D_Y,[\Box_A,D_X]]\phi&=\mathcal{L}_Y(2iX^\nu F_{\mu\nu}D^\mu+i\nabla^\mu(F_{\mu\nu}X^\nu))\phi\\
&=2i(\mathcal{L}_YF)(D\phi,X)+2iF([D_Y,D\phi],X)+2iF(D\phi,[Y,X])+iY(\nabla^\mu(F_{\mu\nu}X^\nu))\phi.
\end{aligned}$$

Here

$$\begin{aligned}
[D_Y,D\phi]&=D^\mu\phi[Y,\nabla_\mu]+[D_Y,D^\mu]\phi\partial_\mu\\
&=iF_{Y\nabla_\mu}\phi\nabla^\mu-(\nabla^\mu Y^\nu+\nabla^\nu Y^\mu)D_\mu\phi\partial_\nu.
\end{aligned}$$

As $X,Y\in\Gamma$ for $\Gamma=\{\partial_t,\Omega_{ij}\}$, we conclude that X,Y are Killing:

$$\nabla^\mu X^\nu+\nabla^\nu X^\mu=0,\qquad \nabla^\mu Y^\nu+\nabla^\nu Y^\mu=0.$$

This implies that the following term can be simplified:

$$Y(\nabla^\mu(F_{\mu\nu}X^\nu)) = [Y, \nabla^\mu]F(\nabla_\mu, X) + \nabla^\mu(\mathcal{L}_Y F)(\nabla_\mu, X) + \nabla^\mu F(\mathcal{L}_Y \nabla_\mu, X) + \nabla^\mu F(\nabla_\mu, \mathcal{L}_Y X)$$

$$= \nabla^\mu(\mathcal{L}_Y F)(\nabla_\mu, X) + \nabla^\mu F(\nabla_\mu, [Y, X]).$$

Therefore, we can write the double commutator as

$$[D_Y, [\Box_A, D_X]]\phi = 2i(\mathcal{L}_Y F)(D\phi, X) + i\nabla^\mu(\mathcal{L}_Y F)(\nabla_\mu, X) + 2iF(D\phi, [Y, X]) + i\nabla^\mu F(\nabla_\mu, [Y, X])\phi - 2F_X^\mu F_{Y\mu}\phi.$$

Note that $[X, Y] \in \mathrm{span}\{\Gamma\}$ for $X, Y \in \Gamma = \{\partial_t, \Omega_{ij}\}$. We thus can write

$$2iF(D\phi, [Y, X]) + i\partial^\mu F(\partial_\mu, [Y, X])\phi = [\Box_A, D_{[Y,X]}]\phi,$$

which can be bounded using Lemma 33. The term

$$2i(\mathcal{L}_Y F)(D\phi, X) + i\nabla^\mu(\mathcal{L}_Y F)(\nabla_\mu, X)$$

has the same form with $[\Box_A, D_X]\phi$ if we replace F with $\mathcal{L}_Y F$. In particular, the bound follows from Lemma 33. Therefore, to show this lemma, it remains to control $F_X^\mu F_{Y\mu}\phi$ for $X, Y \in \Gamma$. This term has crucial null structure we need to exploit when $r \geq R$. The main difficulty is that the angular momentum Ω contains weights in r. If both $X, Y \in \Omega$, then

$$|F_X^\mu F_{Y\mu}| \lesssim |r\alpha||r\underline{\alpha}| + |r\sigma|^2.$$

If $X = Y = \partial_t$, then

$$|F_X^\mu F_{Y\mu}| \lesssim |F|^2.$$

If one and only one of X, Y is ∂_t, then the null structure is as follows:

$$|F_X^\mu F_{Y\mu}| \lesssim r|F_L^\mu F_{e_i\mu}| + r|F_{\underline{L}}^\mu F_{e_i\mu}|$$

$$\lesssim r(|\rho| + |\sigma|)(|\underline{\alpha}| + |\alpha|).$$

We see that the "bad" term $r|\underline{\alpha}|^2$ does not appear on the right-hand side. Hence

$$|F_X^\mu F_{Y\mu}| \lesssim |F|^2 + |r\alpha||r\underline{\alpha}| + |r\sigma|^2 + |r\rho|^2, \quad \forall X, Y \in \Gamma.$$

Therefore, estimate (83) holds. On the bounded region $\{r \leq R\}$, null structure is not necessary and estimate (84) follows trivially. $\square$

The above lemma shows that the double commutator $[D_Z, [\Box_A, D_Z]]\phi$ consists of the quadratic part $[\Box_{\mathcal{L}_Z^k A}, D_Z]\phi$, which can be bounded similarly to $[\Box_A, D_Z]\phi$ as we can put one more derivative D_Z on the scalar field ϕ when we do Sobolev embedding. It thus suffices to control those cubic terms in (83).

Proposition 50. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then we have*

$$\sum_{k\le 1} I^{1+\epsilon}_{1+\gamma_0}[[\Box_{\mathcal{L}^k_Z A}, D_Z]\phi](\{t\ge 0\}) + I^{1+\gamma_0}_{1+\epsilon}[[\Box_{\mathcal{L}^k_Z A}, D_Z]\phi](\{t\ge 0\})$$

$$\lesssim_{M_2} \epsilon_1 I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[D\phi_2](\{t\ge 0\}) + \epsilon_1 I^{\gamma_0}_0[\not{D}\psi_2](\{t\ge 0\}\cap\{r\ge R\}) + \mathcal{E}_1[\phi]\epsilon_1^{-1}$$

$$+\epsilon_1\int_{\mathbb{R}} \tau_+^{1+\gamma_0} h(\tau) E[\phi_2](\Sigma_\tau)\,d\tau + \epsilon_1 \int_{\mathbb{R}}\int_{H_{\tau^*}} \tau_+^{2+\gamma_0+\epsilon-p} h(\tau) r^p |D_L\psi_2|^2\,dv\,d\omega\,d\tau \tag{85}$$

for all positive constants ϵ_1. Here $\phi_2 = D_Z^2\phi$, $\psi_2 = D_Z^2(r\phi)$. The function $h(\tau)$ is defined in (73).

Proof. From Corollary 46 and the decay estimates for the first-order derivative of the scalar field, it suffices to consider estimate (85) with $k=1$. The difference between estimate (80) and estimate (85) is that F is replaced with $\mathcal{L}_Z F$ in (85). However, we are allowed to put one more derivative on the scalar field ($\phi_1 = D_Z\phi$ is replaced with $D_Z^2\phi$). Note that for the proof of estimate (80), the higher-order derivative comes in when we use Sobolev embedding on the sphere to bound $\|F\cdot D\phi\|_{L^2_\omega}$:

$$\|F\cdot D\phi\|_{L^2_\omega} \lesssim \sum_{k\le 2}\|\mathcal{L}^k_Z F\|_{L^2_\omega}\cdot\|D\phi\|_{L^2_\omega} \quad\text{or}\quad \sum_{k\le 1}\|\mathcal{L}^k_Z F\|_{L^2_\omega}\cdot\|DD^k_Z\phi\|_{L^2_\omega}.$$

For estimate (85), the corresponding term $\mathcal{L}_Z F\cdot D\phi$ can be bounded as follows:

$$\|\mathcal{L}_Z F\cdot D\phi\|_{L^2_\omega} \lesssim \sum_{k\le 1}\|\mathcal{L}^k_Z\mathcal{L}_Z F\|_{L^2_\omega}\cdot\|DD^k_Z\phi\|_{L^2_\omega} \quad\text{or}\quad \|\mathcal{L}_Z F\|_{L^2_\omega}\cdot\sum_{k\le 2}\|DD^k_Z\phi\|_{L^2_\omega}.$$

This is how we can transfer one derivative on F to the scalar field ϕ. In particular, estimate (85) holds. ☐

From Lemma 49, to bound the double commutator, it suffices to control the cubic terms in (83) and (84). We rely on the pointwise bound for the Maxwell field summarized in Propositions 14 and 17.

Proposition 51. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $1+\epsilon\le p\le 1+\gamma_0$, we have*

$$I^p_{2+\gamma_0+\epsilon-p}[(|F|^2+|r\alpha||r\underline{\alpha}|+|r\sigma|^2+|r\rho|(|\underline{\alpha}|+|\alpha|))|\phi|](\{t\ge 0,\ r\ge R\})$$

$$+ I^p_{2+\gamma_0+\epsilon-p}[|F|^2|\phi|](\{t\ge 0,\ r\le R\}) \lesssim_{M_2} \mathcal{E}_1[\phi]. \tag{86}$$

Proof. On the bounded region $\{r\le R\}$, the weights r^p have an upper bound. The Maxwell field F can be bounded by using the pointwise estimate (40). We then can estimate the scalar field by using the integrated local energy estimates. Indeed, for all $0\le\tau_1<\tau_2$, we can show that

$$\int_{\tau_1}^{\tau_2}\int_{r\le R}\tau_+^{1+\gamma_0}|F|^4|\phi|^2\,dx\,d\tau \lesssim \int_{\tau_1}^{\tau_2}\tau_+^{1+\gamma_0}\sup|F|^4|\phi|^2\,dx\,d\tau$$

$$\lesssim_{M_2} (\tau_1)_+^{-2-2\gamma_0}\mathcal{E}_1[\phi].$$

For the cubic terms on the region $\{r \geq R\}$, let's first consider $|r\alpha||r\underline{\alpha}||\phi|$. We use the r-weighted energy estimates (31) and (32) for the Maxwell field to control α, and the integrated decay estimate (42) of Proposition 17 to bound $\underline{\alpha}$. The reason that we cannot use the pointwise bound (43) is the weak decay rate there. The scalar field ϕ can be bounded by using Lemma 19. Indeed, for $1+\epsilon \leq p \leq 1+\gamma_0$, we can show that

$$
\begin{aligned}
&I^p_{2+\gamma_0+\epsilon-p}[|r\alpha||r\underline{\alpha}||\phi|](\{t \geq 0\} \cap \{r \geq R\}) \\
&\qquad \lesssim \iint u_+^{2+\gamma_0+\epsilon-p} r^{p+2} |r\alpha|^2 |r\underline{\alpha}|^2 |\phi|^2 \, du \, dv \, d\omega \\
&\qquad \lesssim \sum_{k\leq 1} \int_u \int_v u_+^{2+\gamma_0+\epsilon-p} r^{1+\gamma_0} \int_\omega |r\mathcal{L}_Z^k \alpha|^2 \, d\omega \cdot \int_\omega |r\mathcal{L}_Z^k \underline{\alpha}|^2 \, d\omega \cdot \int_\omega r^{p+1-\gamma_0} |\mathcal{L}_Z^k \phi|^2 \, d\omega \, dv \, du \\
&\qquad \lesssim_{M_2} \mathcal{E}_1[\phi] \sum_{k\leq 1} \int_u \int_v u_+^{1+\epsilon-\gamma_0} r^{1+\gamma_0} \int_\omega |r\mathcal{L}_Z^k \alpha|^2 \, d\omega \cdot \int_\omega |r\mathcal{L}_Z^k \underline{\alpha}|^2 \, d\omega \, dv \, du \\
&\qquad \lesssim_{M_2} \mathcal{E}_1[\phi] \sum_{k\leq 1} \int_u u_+^{1+\epsilon-\gamma_0} \int_v r^{1+\gamma_0} \int_\omega |r\mathcal{L}_Z^k \alpha|^2 \, d\omega \, dv \cdot \sup_v \int_\omega |r\mathcal{L}_Z^k \underline{\alpha}|^2 \, d\omega \, du \\
&\qquad \lesssim_{M_2} \mathcal{E}_1[\phi] \int_{\mathbb{R}} \tau_+^{1+\epsilon-\gamma_0} h(\tau) \, d\tau \\
&\qquad \lesssim_{M_2} \mathcal{E}_1[\phi].
\end{aligned}
$$

Here recall the definition of $h(\tau)$ in (73), and the last step follows from Corollary 40.

For $|F|^2|\phi|$, we use the pointwise estimates (43) and (44) of Proposition 17 to bound the Maxwell field F. The scalar field ϕ can be bounded using Lemma 19 as above. In the exterior region where the Maxwell field contains the charge part $q_0 r^{-2} \, dt \wedge dr$, we have the relation $r_+ \geq \frac{1}{2} u_+$. We can show that

$$
\begin{aligned}
&I^p_{2+\gamma_0+\epsilon-p}[|F|^2 \cdot \phi](\{t \geq 0\} \cap \{r \geq R\}) \\
&\qquad \lesssim \iint u_+^{2+\gamma_0+\epsilon-p} r^{p+2} |\bar{F}|^4 |\phi|^2 \, du \, dv \, d\omega + |q_0|^2 \iint_{t+R\leq r} u_+^{2+\gamma_0+\epsilon-p} r^{p+2-8} |\phi|^2 \, du \, dv \, d\omega \\
&\qquad \lesssim_{M_2} \int_u \int_v u_+^{2+\gamma_0+\epsilon-p-2-2\gamma_0} r^{p+2-4-1} \int_\omega r|\phi|^2 \, d\omega \, dv \, du + \mathcal{E}_0[\phi] \\
&\qquad \lesssim_{M_2} \mathcal{E}_1[\phi] \int_u \int_v u_+^{\epsilon-p-1-2\gamma_0} r^{p-3} \, dv \, du \\
&\qquad \lesssim_{M_2} \mathcal{E}_1[\phi].
\end{aligned}
$$

For $|r\sigma|^2|\phi|$, for the same reason as in the case of $|r\alpha||r\underline{\alpha}||\phi|$, we are not allowed to use the pointwise bound (44) to control σ due to the strong r weights here. Instead, we use the r-weighted energy estimate for σ on the incoming null hypersurface together with the integrated decay estimate (46). We can show that

$$
\begin{aligned}
\iint_{\overline{\mathcal{D}}_{\tau_1}} r^p |r\sigma|^4 |\phi|^2 \, dx\, dt &\lesssim \sum_{k\le 1} \int_u \int_v r^{1+\gamma_0} \int_\omega |r\mathcal{L}_Z^k \sigma|^2 \, d\omega \cdot \int_\omega |r\mathcal{L}_Z^k\sigma|^2 \, d\omega \cdot \int_\omega r^{p+1-\gamma_0} |\mathcal{L}_Z^k \phi|^2 \, d\omega \, dv\, du \\
&\lesssim_{M_2} \mathcal{E}_1[\phi](\tau_1)_+^{-1+p-2\gamma_0} \sum_{k\le 1} \int_v \int_u r^{1+\gamma_0} \int_\omega |r\mathcal{L}_Z^k\sigma|^2 \, d\omega\, du \cdot \sup_u \int_\omega |r\mathcal{L}_Z^k \sigma|^2 \, d\omega\, dv \\
&\lesssim_{M_2} \mathcal{E}_1[\phi](\tau_1)_+^{-1+p-2\gamma_0} \sum_{k\le 1} \| r\mathcal{L}_Z^k \sigma\|^2_{L_v^2 L_u^\infty L_\omega^2(\overline{\mathcal{D}}_{\tau_1})} \\
&\lesssim_{M_2} \mathcal{E}_1[\phi](\tau_1)_+^{-2+p+\epsilon-3\gamma_0}.
\end{aligned}
$$

This holds for all $\tau_1 \in \mathbb{R}$. Since

$$
2+3\gamma_0-\epsilon-p > 2+\gamma_0+\epsilon-p, \quad 0 \le p \le 1+\gamma_0,
$$

from Lemma 20, we obtain

$$
I^p_{2+\gamma_0+\epsilon-p}[|r\sigma|^2\phi](\{t\ge 0\}\cap\{r \ge R\}) \lesssim_{M_2} \mathcal{E}_1[\phi].
$$

Finally, for $|r\rho|(|\underline{\alpha}|+|\alpha|)|\phi|$, we need to take into consideration the charge effect in the exterior region. Except for this charge, the proof for the interior region case is the same. Let's merely estimate this cubic term in the exterior region. In particular, take $\overline{\mathcal{D}}_{\tau_1}$ to be $\mathcal{D}_{\tau_1}$ for some $\tau_1 < 0$. By using the r-weighted energy estimate for $\bar{\rho}$ and the pointwise bound (43) and (44) for F, for $0 \le p \le 1+\gamma_0$ we then can show that

$$
\begin{aligned}
&\iint_{\mathcal{D}_{\tau_1}} r^{p+2}|r\rho|^2(|\alpha|^2+|\underline{\alpha}|^2)|\phi|^2 \, du\, dv\, d\omega \\
&\quad\lesssim \iint_{\mathcal{D}_{\tau_1}} |q_0| r^p(|\alpha|^2+|\underline{\alpha}|^2)|\phi|^2 \, du\, dv\, d\omega + \iint_{\mathcal{D}_{\tau_1}} r^{p+2}|r\bar{\rho}|^2(|\alpha|^2+|\underline{\alpha}|^2)|\phi|^2 \, du\, dv\, d\omega \\
&\quad\lesssim_{M_2} \mathcal{E}_1[\phi](\tau_1)_+^{-1-2\gamma_0} + \sum_{k\le 1}\int_u\int_v r^{p-1}\int_\omega |r\mathcal{L}_Z^k\bar{\rho}|^2 \, d\omega \cdot \sup(|r\underline{\alpha}|^2+|r\alpha|^2)\cdot \int_\omega r|\mathcal{L}_Z^k\phi|^2 \, d\omega\, dv\, du \\
&\quad\lesssim_{M_2} \mathcal{E}_1[\phi](\tau_1)_+^{-1-2\gamma_0} + \mathcal{E}_1[\phi](\tau_1)_+^{-2-2\gamma_0} \sum_{k\le 1} \iint_{\mathcal{D}_{\tau_1}} r^{p-1}|r\mathcal{L}_Z^k\bar{\rho}|^2 \, du\, dv\, d\omega \\
&\quad\lesssim_{M_2} \mathcal{E}_1[\phi](\tau_1)_+^{-1-2\gamma_0}.
\end{aligned}
$$

Here the last term is bounded by using the r-weighted energy estimates for $\bar{\rho}$. As τ_1 is arbitrary, from Lemma 20, we derive that

$$
I^p_{2+\gamma_0+\epsilon-p}[r\rho\cdot(|\underline{\alpha}|+|\alpha|)\cdot\phi](\{t\ge 0\}\cap\{r\ge R\}) \lesssim_{M_2} \mathcal{E}_1[\phi], \quad 1+\epsilon \le p \le 1+\gamma_0.
$$

To summarize, we have shown (86). □

Propositions 50 and 51 together with Lemma 49 lead to the desired estimates for the double commutator $[D_X, [\Box_A, D_Y]]$ for $X, Y \in \Gamma$. Then by the argument at the beginning of this section, we have control of

$\mathcal{E}_0[D_X D_Y \phi]$. By using the same argument as Proposition 47, we then can bound $\mathcal{E}_0[D_X D_Y \phi]$ by $\mathcal{E}_2[\phi]$. This then implies the decay of the second-order derivative of the scalar field.

Proposition 52. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then for all $X, Y \in \Gamma$, we have the bound*

$$\mathcal{E}_0[D_X D_Y \phi] \lesssim_{M_2} \mathcal{E}_2[\phi]. \tag{87}$$

Proof. From the argument at the beginning of this section (before Lemma 49), we derive that

$$\mathcal{E}_0[D_X D_Y \phi] \lesssim_{M_2} \mathcal{E}_2[\phi] + I^{1+\gamma_0}_{1+\epsilon}[[D_X, [\Box_A, D_Y]]\phi](\{t \geq 0\}) + I^{1+\epsilon}_{1+\gamma_0}[[D_X, [\Box_A, D_Y]]\phi](\{t \geq 0\}).$$

Then by Lemma 49 and Proposition 50, for all $0 < \epsilon_1 < 1$ and $X, Y \in \Gamma$, we conclude that

$$\begin{aligned}\mathcal{E}_0[D_X D_Y \phi] \lesssim_{M_2} & \epsilon_1 I^{-1-\epsilon}_{1+\gamma_0-\epsilon}[D\phi_2](\{t \geq 0\}) + \epsilon_1 I^{\gamma_0}_0[\not{D}\psi_2](\{t \geq 0\} \cap \{r \geq R\}) \\ & + \mathcal{E}_1[\phi]\epsilon_1^{-1} + \epsilon_1 \int_{\mathbb{R}} \tau_+^{1+\gamma_0} g(\tau) E[\phi_2](\Sigma_\tau)\, d\tau + \epsilon_1 \int_{\mathbb{R}} \int_{H_{\tau^*}} \tau_+^{2+\gamma_0+\epsilon-p} g(\tau) r^p |D_L \psi_2|^2 \, dv\, d\omega\, d\tau,\end{aligned}$$

where $\phi_2 = D_X D_Y \phi$ and $\psi_2 = D_X D_Y (r\phi)$. The proposition then follows by the same argument as Proposition 47. ☐

4.4. *Pointwise bound for the scalar field.* Once we have the bound (87), from Proposition 32 and Corollary 24, we obtain the energy flux decay estimates as well as the r-weighted energy estimates for the second-order derivatives of the scalar field. In other words, simply assuming M_2 is finite (see the definition of M_2 in (35)) and the charge q_0 is small, we then can derive the energy decay estimates for the second-order derivatives of the scalar field. For the MKG equations, $J = \delta F = J[\phi]$ is quadratic in ϕ. To construct global solutions, we need to bound these nonlinear terms. In this section, we show the pointwise bound for the scalar field with the assumption that M_2 is finite.

We start with an analogue of Proposition 14 regarding the pointwise bound of the scalar field in the finite region $\{r \leq R\}$. Similarly to the pointwise bound of the Maxwell field, we use elliptic estimates. However as the connection field A is general, we are not able to apply the elliptic estimates for the flat case directly. We therefore establish an elliptic lemma for the operator $\Delta_A = \sum_{i=1}^3 D_i D_i$ first. Let B_{R_1} be the ball with radius R_1 in $\mathbb{R}^3$. Define

$$\|\phi\|_{H^k(B_{R_1})} = \sum_{1 \leq j_l \leq 3} \|D_{j_1} D_{j_2} \cdots D_{j_k} \phi\|_{L^2(B_{R_1})} + \|\phi\|_{H^{k-1}(B_{R_1})}, \quad k \geq 1.$$

Then we have the following lemma.

Lemma 53. *We have the elliptic estimates*

$$\|\phi\|_{H^2(B_{R_1})} \lesssim_{M_2, R_1, R_2} \|\Delta_A \phi\|_{L^2(B_{R_2})} + \big(1 + \|F\|_{L^\infty(B_{R_2})} + \|J\|_{H^1(B_{R_2})}\big) \|\phi\|_{H^1(B_{R_2})} \tag{88}$$

for all $R_1 < R_2$. Here the constant M_2 is defined in line (35) and $J = \delta(dA)$ or $J_j = \partial^i (dA)_{ij}$.

Proof. The proof is similar to the case when the connection field A is trivial. For the case when the scalar field ϕ is compactly supported in some ball B_{R_1}, using integration by parts we can show that

$$\begin{aligned}\int_{B_{R_1}} D_i D_j\phi\cdot\overline{D_i D_j\phi}\,dx &= -\int_{B_{R_1}} D_i D_i D_j\phi\cdot\overline{D_j\phi}\,dx\\ &= -\int_{B_{R_1}} D_j D_i D_i\phi\cdot\overline{D_j\phi}\,dx-\int_{B_{R_1}}[D_iD_i,\,D_j]\phi\cdot\overline{D_j\phi}\,dx\\ &= \int_{B_{R_1}}|\Delta_A\phi|^2\,dx-\int_{B_{R_1}}\sqrt{-1}\,(2F_{ij}D_i\phi+\partial_iF_{ij}\phi)\cdot\overline{D_j\phi}\,dx.\end{aligned}$$

Estimate (88) then follows.

For a general complex function ϕ, we can choose a real cut-off function χ which is supported on the ball B_{R_2} and equal to 1 on the smaller ball B_{R_1}. By direct computation, we can show that

$$\begin{aligned}\|\Delta_A(\chi\phi)\|_{L^2(B_{R_2})} &= \|\chi\Delta_A\phi+2\partial_i\chi\cdot D_i\phi+\Delta\chi\cdot\phi\|_{L^2(B_{R_2})}\\ &\lesssim \|\Delta_A\phi\|_{L^2(B_{R_2})}+\|\phi\|_{H^1(B_{R_2})}.\end{aligned}$$

The lemma then follows from the above argument for the compactly supported case. □

We assume $\Box_A\phi$ verifies the extra bound

$$\int_{\tau_1}^{\tau_2}\int_{r\le 2R}|D\Box_A\phi|^2+|D_Z D\Box_A\phi|^2\,dx\,d\tau\le C\mathcal{E}_2[\phi](\tau_1)_+^{-1-\gamma_0},\quad 0\le\tau_1<\tau_2 \tag{89}$$

for some constant C depending only on R. For solutions of (MKG), one has $\Box_A\phi=0$ and the above bound trivially holds. The above elliptic estimate adapted to the connection field A implies the following pointwise bound for the scalar field ϕ on the compact region $\{r\le R\}$.

Proposition 54. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds and the inhomogeneous term $\Box_A\phi$ verifies the bound (89). Then for all $0\le\tau$ and $0\le\tau_1<\tau_2$, we have*

$$\int_{\tau_1}^{\tau_2}\sup_{|x|\le R}(|D\phi|^2+|\phi|^2)(\tau,x)\,d\tau\lesssim\int_{\tau_1}^{\tau_2}\int_{r\le R}|D^2D\phi|^2+|\phi|^2\,dx\,dt\lesssim_{M_2}\mathcal{E}_2[\phi](\tau_1)_+^{-1-\gamma_0}, \tag{90}$$

$$|D\phi|^2(\tau,x)+|\phi|^2(\tau,x)\lesssim_{M_2}\mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0},\quad\forall|x|\le R. \tag{91}$$

Proof. At the fixed time $\tau\ge 0$, consider the elliptic equation for the scalar field $\phi_k=D_Z^k\phi$:

$$\Delta_A\phi_k=D_tD_t\phi_k+D_k\Box_A\phi+[\Box_A,\,D_Z^k]\phi.$$

Proposition 14 together with Proposition 17 indicate that the Maxwell field F is bounded. The definition of M_2 shows that

$$\|J\|_{H^1(B_{2R})}^2\lesssim\int_\tau^{\tau+1}|\overline{\nabla}J|^2+|\partial_t\overline{\nabla}J|^2+|J|^2+|\partial_tJ|^2\,dx\,dt\lesssim M_2.$$

Here B_{R_1} denotes the ball with radius R_1 at time τ. Then by the previous Lemma 53, we conclude that

$$\|\phi_k\|^2_{H^2(B_{3R/2})} \lesssim_{M_2} \|D_t D_t \phi_k\|^2_{L^2(B_{2R})} + \|D_Z^k \Box_A \phi\|^2_{L^2(B_{2R})} + \|[\Box_A, D_Z^k]\phi\|^2_{L^2(B_{2R})} + \|\phi_k\|^2_{H^1(B_{2R})}.$$

This gives the H^2 estimates for $D_t\phi$ and ϕ. To obtain estimates for $D_j\phi$, commute the equation with D_j:

$$\Delta_A D_j\phi = D_j D_t D_t \phi + D_j \Box_A \phi + [\Delta_A, D_j]\phi = D_j D_t D_t \phi + D_j \Box_A \phi + \sqrt{-1}(2F_{ij} D_i \phi + \partial_i F_{ij}\phi).$$

Then using Lemma 53 again, we obtain

$$\begin{aligned}\|D_j\phi\|^2_{H^2(B_R)} &\lesssim_{M_2} \|D_j\phi\|^2_{H^1(B_{3R/2})} + \|\Delta_A D_j\phi\|^2_{L^2(B_{3R/2})} \\ &\lesssim_{M_2} \|\phi\|^2_{H^2(B_{3R/2})} + \|D_j D_t^2\phi\|^2_{L^2(B_{3R/2})} + \|D_j\Box_A\phi\|^2_{L^2(B_{3R/2})}.\end{aligned}$$

Here we have used the facts $|F|^2 \lesssim M_2$ and $\|J\|^2_{H^1(B_{2R})} \lesssim M_2$. Then for the pointwise bound (91), we need to show the energy flux decay through B_{2R} at time τ. This can be fulfilled by considering the energy estimate obtained by using the vector field ∂_t as multiplier on the region bounded by $\{t = \tau\}$ and $\Sigma_{\tau-R}$ (recall that $\Sigma_\tau = H_{\tau^*}$ for negative $\tau < 0$). Corollary 27 together with Propositions 32 and 52 then imply that

$$E[D_Z^k\phi](B_{2R}) \lesssim E[D_Z^k\phi](\Sigma_{\tau-R}) + (\tau - R)_+^{-1-\gamma_0}\mathcal{E}_0[D_Z^k\phi] \lesssim_{M_2} \mathcal{E}_k[\phi]\tau_+^{-1-\gamma_0}, \quad k \le 2.$$

For the flux of the inhomogeneous term $D\Box_A\phi$ and the commutator term $[D_Z, \Box_A]\phi$, we can make use of the integrated local energy estimates. More precisely, combine the above H^2 estimates for $\phi_k = D_Z^k\phi$, $k = 0, 1$, and $D_j\phi$. We can show that

$$\begin{aligned}&\|D_j\phi\|^2_{H^2(B_R)} + \sum_{k\le 1}\|\phi_k\|^2_{H^2(B_R)} \\ &\qquad\lesssim_{M_2} \sum_{l\le 2} E[D_Z^l\phi](B_{2R}) + \|D\Box_A\phi\|^2_{L^2(B_{2R})} + \|[\Box_A, D_Z]\phi\|^2_{L^2(B_{2R})} \\ &\qquad\lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0} + \int_\tau^{\tau+1}\int_{r\le 2R} |D\Box_A\phi|^2 + |D_t D\Box_A\phi|^2\,dx\,d\tau + I_0^0[D_Z[\Box_A, D_Z]\phi](D_{\tau-R}^\tau) \\ &\qquad\lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0}.\end{aligned}$$

Here we have used the bound

$$I^{1+\epsilon}_{1+\gamma_0}[D_Z^k[\Box_A, D_Z]\phi](\{t \ge 0\}) \lesssim_{M_2} \mathcal{E}_{k+1}[\phi], \quad k = 0, 1,$$

which is a consequence of the proof in the previous section (see the argument in the beginning of Section 4.3.4). Then Sobolev embedding implies the pointwise bound (91) for ϕ.

For the integrated decay estimate (90), we integrate the H^2 norm of $D_j\phi$ from time τ_1 to τ_2:

$$\int_{\tau_1}^{\tau_2}\|D\phi\|^2_{H^2(B_R)}\,d\tau \lesssim_{M_2} \int_{\tau_1}^{\tau_2}\sum_{l\le 2}\|D_Z^l\phi\|^2_{L^2(B_{2R})}+\|D\Box_A\phi\|^2_{L^2(B_{2R})}+\|[\Box_A, D_Z]\phi\|^2_{L^2(B_{2R})}\,d\tau$$

$$\lesssim_{M_2}\sum_{l\le 2} I_0^{-1-\epsilon}[D_Z^l\phi](D_{\tau_1-R}^{\tau_2})+\int_{\tau_1}^{\tau_2}\int_{r\le 2R}|D\Box_A\phi|^2\,dx\,d\tau+I_0^0[[D_Z,\Box_A]\phi](D_{\tau_1-R}^{\tau_2})$$

$$\lesssim_{M_2}\mathcal{E}_2[\phi](\tau_1)_+^{-1-\gamma_0}.$$

Here we have used the integrated local energy estimates for the second-order derivative of the scalar field. Then Sobolev embedding implies the integrated decay estimate (90). □

Remark 55. For the Sobolev embedding adapted to the connection A, it suffices to establish the L^p embedding in terms of the H^1 norm. As the norm is gauge invariant, we can choose a particular gauge so that the function is real. For a real function f we have the trivial bound $\|D_A f\|_{L^2}\ge\|\partial f\|_{L^2}$. This explains the Sobolev embedding we have used in this paper adapted to the general connection field A.

Next we consider the pointwise bound for the scalar field outside the cylinder $\{r\le R\}$. The decay estimate for ϕ easily follows from Lemma 19, as we have energy decay estimates for second-order derivatives of ϕ. However, this does not apply to the derivative of ϕ due to the limited regularity (only two derivatives). Like with the Maxwell field in Proposition 17, we rely on Lemma 16.

Proposition 56. *Assume that the charge q_0 is sufficiently small, so that Corollary 22 holds. Then we have the pointwise bound*

$$\|D_{\underline{L}}(rD_Z^k\phi)\|_{L_u^2L_v^\infty L_\omega^2(\bar{\mathcal{D}}_\tau)}\lesssim_{M_2}\mathcal{E}_{k+1}[\phi](\tau_1)_+^{-1-\gamma_0+3\epsilon},\qquad k=0,1, \tag{92}$$

$$\|r^{p/2}D_L(rD_Z^k\phi)\|^2_{L_v^2L_u^\infty L_\omega^2(\bar{\mathcal{D}}_\tau)}\lesssim_{M_2}\mathcal{E}_{k+1}[\phi](\tau_1)_+^{p+4\epsilon-1-\gamma_0},\quad 0\le p\le 1+\gamma_0-4\epsilon,\ k=0,1, \tag{93}$$

$$r^p(|D_L(r\phi)|^2+|\not{D}(r\phi)|^2)(\tau,v,\omega)\lesssim_{M_2}\mathcal{E}_2[\phi]\tau_+^{p-1-\gamma_0},\qquad 0\le p\le 1+\gamma_0, \tag{94}$$

$$|D_{\underline{L}}(r\phi)|^2(\tau,v,\omega)\lesssim_{M_2}\mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0}, \tag{95}$$

$$r^p|\phi|^2(\tau,v,\omega)\lesssim_{M_2}\mathcal{E}_2[\phi]\tau_+^{p-2-\gamma_0},\qquad 1\le p\le 2. \tag{96}$$

Remark 57. If we have one more derivative (assume M_3), then we have a better estimate for $\not{D}(r\phi)$, as we can write it as $D_Z\phi$.

Proof. Estimate (92) follows from (68) and (71) together with the r-weighted energy and integrated local energy estimates for the scalar field $D_Z^k\phi$, $k\le 2$. Estimate (93) is a consequence of (69) and (71).

For the pointwise bound for the scalar field, let $\phi_k=D_Z^k\phi$, $\psi_k=D_Z^k(r\phi)$, $k\le 2$. First, the r-weighted energy estimates (53) and (63) imply that

$$\int_{H_{\tau^*}} r^p|D_L\psi_k|^2\,dv\,d\omega\lesssim_{M_2}\mathcal{E}_k[\phi]\tau_+^{p-1-\gamma_0},\quad k\le 2,\quad 0\le p\le 1+\gamma_0.$$

From the r-weighted energy estimate for F and Lemma 19, we can bound the commutator:

$$\begin{aligned}\int_{H_{\tau^*}} r^p |[D_Z^2, D_L]\psi|^2\, dv\, d\omega &\lesssim \int_{H_{\tau^*}} r^p (|F_{ZL} D_Z\psi|^2 + |\mathcal{L}_Z F_{ZL}\psi|^2)\, dv\, d\omega \\ &\lesssim \sum_{l\le 1} \int_{H_{\tau^*}} r^p (|\mathcal{L}_Z^l \rho \psi_{1-l}|^2 + |r\mathcal{L}_Z^l \alpha||\psi_{l-1}|)\, dv\, d\omega \\ &\lesssim_{M_2} \mathcal{E}_2[\phi]|q_0|\tau_+^{p-3-\gamma_0} + \mathcal{E}_2[\phi]\tau_+^{p-1-2\gamma_0} \\ &\lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{p-1-\gamma_0}.\end{aligned}$$

Here the charge part only appears when $\tau < 0$. The previous two estimates lead to

$$\int_{H_{\tau^*}} r^p |D_Z^k D_L D_Z^l \psi|^2\, dv\, d\omega \lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{p-1-\gamma_0}, \quad k+l\le 2, \quad 0\le p\le 1+\gamma_0.$$

To apply Lemma 16, we need the energy flux for $D_L D_L \psi$. From the null equation (70) for the scalar field, on the outgoing null hypersurface H_{τ^*}, for $k = 0, 1$, we can show that

$$\begin{aligned}\int_{H_{\tau^*}} r^p |D_{\underline{L}} D_L \psi_k|^2\, dv\, d\omega &\lesssim \int_{H_{\tau^*}} r^p (|\rho\cdot r\phi_k|^2 + |r^{-1}\not{D} D_\Omega \psi_k|^2 + |r\Box_A \phi_k|^2)\, dv\, d\omega \\ &\lesssim_{M_2} \mathcal{E}_{k+1}[\phi]\tau_+^{p-1-\gamma_0}.\end{aligned}$$

Here the first term $\rho\cdot r\psi_k$ has been bounded in the above commutator estimate for $[D_Z^2, D_L]\psi$. The second term $|r^{-1}\not{D} D_\Omega \psi_k|^2$ can be bounded by the energy flux of $\mathcal{L}_Z^2 F$ through H_{τ^*} as $p\le 2$. The bound for $\Box_A\phi_k$ follows from the argument in Section 4.3.4 where we have shown that $\mathcal{E}_1[\phi_k] \lesssim_{M_2} \mathcal{E}_2[\phi_{k-1}]$ for $k = 0, 1$. Now commute D_L with $\psi_k = D_Z^k\psi$. First, we can show that

$$\begin{aligned}|D_L[D_L, D_Z]\psi| &\lesssim |LF_{LZ}||\psi| + |F_{ZL}||D_L\psi| \\ &\lesssim (|\underline{L}(r\alpha)| + |r\mathcal{L}_Z\alpha| + |L\rho|)|\psi| + (|\rho| + |r\alpha|)|D_L\psi|.\end{aligned}$$

On the right-hand side, the second term is easy to bound as we can control the Maxwell field ρ, $r\alpha$ by the L^∞ norm shown in Proposition 17 and the scalar field ψ by the r-weighted energy estimates. For the first term, we have to use the null structure equations of Lemma 5 to control $L(r\alpha)$, $L\rho$. Indeed, we can show that

$$\begin{aligned}\int_{H_{\tau^*}} r^p |D_L[D_L, D_Z]\psi|^2\, dv\, d\omega &\lesssim_{M_2} \int_{H_{\tau^*}} r^p |D_Z^2 D_L\psi|^2 + r^p (|\underline{L}(r\alpha)|^2 + |r\mathcal{L}_Z\alpha|^2 + |L\rho|^2)\mathcal{E}_2[\phi]\, dv\, d\omega \\ &\lesssim_{M_2} \mathcal{E}_2[\phi]\Big(\tau_+^{p-1-\gamma_0} + \int_{H_{\tau^*}} r^p (|\mathcal{L}_\Omega(\rho,\sigma,\alpha)|^2 + |rJ|^2 + |\rho|^2)\, dv\, d\omega\Big) \\ &\lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{p-1-\gamma_0}.\end{aligned}$$

Here we can bound ρ, α, σ by the energy flux as $p \le 2$. For the inhomogeneous term J we can use one more derivative $\mathcal{L}_{\partial_t}$. In particular, we can show that

$$\sum_{k\le 1}\int_{H_{\tau^*}} r^p(|D_L D_Z^k D_L\psi|^2 + |D_\Omega D_Z^k D_L\psi|^2 + |D_Z^k D_L\psi|^2)\,dv\,d\omega$$

$$\lesssim \sum_{l\le 2}\int_{H_{\tau^*}} r^p(|D_Z^l D_L\psi|^2 + |D_L[D_Z, D_L]\psi|^2 + |D_{\underline{L}} D_L D_Z\psi|^2 + |D_{\partial_t} D_L D_Z\psi|^2)\,dv\,d\omega$$

$$\lesssim_{M_2} \mathcal{E}_{k+1}[\phi]\tau_+^{p-1-\gamma_0}.$$

Then using Lemma 16 and Sobolev embedding, we derive the pointwise estimate for $D_L\psi$ (see Remark 55 for the Sobolev embedding adapted to the connection A). This proves the first part of (94).

For $D_{\underline{L}}\psi$ and $\not{D}(r\phi)$, we make use of the energy flux through the incoming null hypersurface $\underline{H}_\tau$, which is defined as $\underline{H}_v^{-v,\tau^*}$ when $\tau < 0$ or $\underline{H}_v^{\tau^*,\,v}$ when $\tau \ge 0$. From the energy estimates (53), (56), (63) and (64), we obtain the energy flux decay

$$\int_{\underline{H}_\tau} |D_{\underline{L}} D_Z^k\psi|^2 + |\not{D} D_Z^k\psi|^2 + \tau_+^{-p} r^p|D_\Omega D_Z^k\phi|^2 + r^2|D_{\underline{L}} D_Z^k\phi|^2\,du\,d\omega \lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0}$$

for $k \le 2$ and $0 \le p \le 1+\gamma_0$. As $\not{D}(r\phi) = D_\Omega\phi$, the above estimates together with Lemma 16 indicate that

$$r^p|D_\Omega\phi|^2 \lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{p-1-\gamma_0}, \quad 0 \le p \le 1+\gamma_0.$$

Thus the second part of (94) holds.

For $D_{\underline{L}}\psi$, we need to pass the $D_{\underline{L}}$ derivative to ψ. We can compute the commutator:

$$|[D_Z^2, D_{\underline{L}}]\psi| \lesssim (|r\mathcal{L}_Z\underline{\alpha}| + |\mathcal{L}_Z\rho|)|\psi| + (|r\underline{\alpha}| + |\rho|)|D_Z\psi|.$$

We can bound ψ using Lemma 19 and $\rho, \underline{\alpha}$ using the energy flux through $\underline{H}_\tau$. Then the previous energy estimate implies that

$$\int_{\underline{H}_\tau} |D_Z^k D_{\underline{L}} D_Z^l\psi|^2 + |r^{-1}D_Z^{k+1}\psi|^2\,du\,d\omega \lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0}, \quad k+l \le 2. \tag{97}$$

To apply Lemma 16, we also need an estimate for $D_{\underline{L}} D_{\underline{L}}\psi$. We use the null equation (70) to show that

$$\int_{\underline{H}_\tau} |D_L D_{\underline{L}}\psi_k|^2\,du\,d\omega \lesssim_{M_2} \mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0}, \quad k \le 1.$$

The proof of this estimate is similar to that through the outgoing null hypersurface we have done above. To pass the $D_{\underline{L}}$ derivative to ψ, we commute $D_{\underline{L}}$ with $\psi_1 = D_Z\psi$:

$$|D_{\underline{L}}[D_{\underline{L}}, D_Z]\psi| \lesssim |D_{\underline{L}}\psi|(|r\underline{\alpha}| + |\rho|) + |\psi|(|\underline{L}\rho| + |L(r\underline{\alpha})| + |\partial_t(r\underline{\alpha})|).$$

Again we can bound $D_{\underline{L}}\psi$ using the energy flux and $r\underline{\alpha}$, ρ by the L^∞ norm. For the second term, ψ can be bounded using Lemma 19, and the curvature components $\underline{L}\rho$, $L(r\underline{\alpha})$ are controlled by using the null structure equations (7) and (8). More precisely, we can show that

$$\sum_{k\le 1}\int_{\underline{H}_\tau}|D_{\underline{L}}D_Z^kD_{\underline{L}}\psi|^2\,du\,d\omega\lesssim\int_{\underline{H}_\tau}|D_{\underline{L}}[D_Z,D_{\underline{L}}]\psi|^2+|D_LD_{\underline{L}}D_Z^k\psi|^2+|D_{\partial_t}D_{\underline{L}}D_Z^k\psi|^2\,du\,d\omega$$

$$\lesssim_{M_2}\mathcal{E}_2[\phi]\tau_+^{-1-\gamma_0}.$$

This estimate and (97) combined with Lemma 16 imply the pointwise bound (95) for $D_{\underline{L}}\psi$.

The pointwise bound (96) for ϕ follows from Lemma 19:

$$\int_\omega r^p|D_Z^k\phi|^2(\tau,v,\omega)\,d\omega\lesssim_{M_2}\mathcal{E}_k[\phi]\tau_+^{p-2-\gamma_0},\quad k\le 2,\quad 1\le p\le 2,$$

together with Sobolev embedding on the sphere. □

5. Bootstrap argument

We use a bootstrap argument to prove the main theorem. In the exterior region, we decompose the full Maxwell field F into the chargeless part and the charge part:

$$F=\bar{F}+q_0\chi_{\{r\ge t+R\}}r^{-2}\,dt\wedge dr.$$

We make the bootstrap assumption

$$m_2\le 2\mathcal{E}\tag{98}$$

on the nonlinearity $J_\mu=\nabla^\nu F_{\nu\mu}=\Im(\phi\cdot\overline{D_\mu\phi})=J_\mu[\phi]$. Here recall the definition of m_2 in (35). Since the nonlinearity J is quadratic in ϕ, m_2 has size $\mathcal{E}^2$. By assuming that $\mathcal{E}$ is sufficiently small, we then can improve the above bootstrap assumption and hence conclude our main theorem. The smallness of $\mathcal{E}$ depends on $\mathcal{M}$. Without loss of generality, we assume $\mathcal{E}\le 1$ and $\mathcal{M}>1$.

In the definition (35) for M_2, the main contribution is $E_0^2[\bar{F}]$ with $\bar{F}$ the chargeless part of the Maxwell field on the initial hypersurface $\{t=0\}$. As the scalar field ϕ solves the linear equation $\Box_A\phi=0$, we derive from the definition (47) for $\mathcal{E}_2[\phi]$ that $\mathcal{E}_2[\phi]=E_0^2[\phi]$. The definition for $E_0^k[\bar{F}]$ and $E_0^k[\phi]$ has been given in (6). To proceed, we need to bound $E_0^2[\bar{F}]$ and $\mathcal{E}_2[\phi]$ in terms of $\mathcal{M}$ and $\mathcal{E}$, which is shown in the following lemma.

Lemma 58. *Assume that the initial data set (E,H,ϕ_0,ϕ_1) satisfies the compatibility condition (2) and that the norms $\mathcal{M}$, $\mathcal{E}$ defined before Theorem 1 are finite. Then we can bound $E_0^2[\bar{F}]$ and $E_0^2[\phi]$ as follows*:

$$E_0^2[\bar{F}]\lesssim\mathcal{M},\quad E_0^2[\phi]\lesssim_{\mathcal{M}}\mathcal{E}.$$

Proof. To define the norm $E_0^k[\phi]$, we need to know the connection field A on the initial hypersurface $\{t=0\}$. As the norm $E_0^k[\phi]$ is gauge invariant, we may choose a particular gauge. Let $\bar{A}=(A_1,A_2,A_3)(0,x)$, $A_0=A_0(0,x)$. We want to choose a particular connection field $(A_0,\bar{A})$ on the initial hypersurface to define the gauge invariant norm $E_0^k[\phi]$.

It is convenient to choose the Coulomb gauge to make use of the divergence-free part E^{df} and the curl-free part E^{cf} of E. More precisely, on the initial hypersurface $\{t=0\}$, we choose $(A_0,\bar{A})$ so that

$\mathrm{div}(\bar{A}) = 0$. Then the compatibility condition (2) is equivalent to

$$\Delta A_0 = -\Im(\phi_0 \cdot \overline{\phi_1}) = -J_0(0), \quad \overline{\nabla} \times \bar{A} = H.$$

Define the weighed Sobolev space

$$W^p_{s,\delta} := \Big\{ f \;\Big|\; \sum_{|\beta| \le s} \big\| (1+|x|)^{\delta+|\beta|} |\partial^\beta f| \big\|_{L^p} < \infty \Big\}.$$

For the special case $p = 2$, let $H_{s,\delta} = W^2_{s,\delta}$. Denote $\tilde{\Gamma} = \{\Omega, \partial_j\}$, $\delta = \frac{1}{2}(1+\gamma_0)$. By the definition of $\mathcal{M}$,

$$\|\mathcal{L}^k_{\tilde{Z}} H\|_{H_{0,\delta}} \lesssim \mathcal{M}^{1/2}, \quad k \le 2, \quad \tilde{Z} \in \tilde{\Gamma}.$$

Then from Theorem 0 of [McOwen 1979] or Theorem 5.1 of [Choquet-Bruhat and Christodoulou 1981a], we conclude that

$$\|\mathcal{L}^k_{\tilde{Z}} \bar{A}\|_{H_{1,\delta-1}} \lesssim \mathcal{M}^{1/2}, \quad k \le 2, \quad \tilde{Z} \in \tilde{\Gamma}.$$

This is the desired estimate for the gauge field $\bar{A}$. With this connection field $\bar{A}$, we then can define the covariant derivative $\bar{D} = \overline{\nabla} + \sqrt{-1}\,\bar{A}$ in the spatial direction. Therefore,

$$\begin{aligned}\|D\phi(0,\cdot)\|_{H_{0,\delta}} &= \|\bar{D}\phi_0\|_{H_{0,\delta}} + \|\phi_1\|_{H_{0,\delta}} \lesssim \mathcal{E}^{1/2} + \|\bar{A}\|_{W^3_{0,\delta}} \|\phi_0\|_{W^6_{0,\delta}} \\ &\lesssim \mathcal{E}^{1/2}(1+\mathcal{M}^{1/2}) \lesssim \mathcal{E}^{1/2}\mathcal{M}^{1/2}.\end{aligned}$$

By the same argument, and commuting the equations with $D_{\tilde{Z}}$, we obtain the same estimates for $D_{\tilde{Z}}\phi$:

$$\|DD^k_{\tilde{Z}}\phi(0,\cdot)\|_{H_{0,\delta}} \lesssim \mathcal{E}^{1/2}\mathcal{M}^{1/2}, \quad k \le 2.$$

To define the covariant derivative D_0, we need estimates for A_0. The difficulty is the nonzero charge. Take a cut-off function $\chi(x) = \chi(|x|)$ such that $\chi = 1$ when $|x| \ge R$ and vanishes for $|x| \le \frac{1}{2}R$. Denote the chargeless part of A_0 and J_0 as follows:

$$\bar{A}_0 = A_0 + \chi q_0 r^{-1}, \quad \bar{J}_0(0) := J_0 - \Delta(\chi q_0 r^{-1}).$$

By the definition of the charge q_0, we then have

$$\Delta \bar{A}_0 = -\bar{J}_0(0), \quad \int_{\mathbb{R}^3} \bar{J}_0(0)\, dx = 0.$$

Recall that $J_0(0) = \Im(\phi_0 \cdot \overline{\phi_1})$. Using Sobolev embedding, we can bound

$$\|\bar{J}_0(0)\|_{W^{3/2}_{0,2\delta}} \lesssim |q_0| + \|\phi_1\|_{W^2_{0,\delta}} \|\phi_0\|_{W^6_{0,\delta}} \lesssim |q_0| + \|\phi_1\|_{W^2_{0,\delta}} \|\phi_0\|_{W^2_{1,\delta}} \lesssim \mathcal{E}.$$

Then from Theorem 0 of [McOwen 1979] again, we conclude that

$$\|\bar{A}_0\|_{W^{3/2}_{2,2\delta-2}} \lesssim \mathcal{E}.$$

Here the condition that $\bar{A}_0$ is chargeless guarantees $\bar{A}_0$ to belong to the above weighted Sobolev space. Then using the Gagliardo–Nirenberg interpolation inequality, we derive that

$$\|\overline{\nabla}\bar{A}_0\|_{H_{0,2\delta-1/2}} \lesssim \|\overline{\nabla}\bar{A}_0\|^{1/2}_{W^{3/2}_{0,2\delta-1}} \cdot \|\overline{\nabla}\overline{\nabla}\bar{A}_0\|^{1/2}_{W^{3/2}_{0,2\delta}} \lesssim \mathcal{E}.$$

By definition, one has $E = \partial_t \bar{A} - \overline{\nabla} A_0$. By our gauge choice, $\partial_t \bar{A}$ is divergence-free and $\overline{\nabla} A_0$ is curl-free. In particular, we derive that $E^{\mathrm{df}} = \partial_t \bar{A}$ and $E^{\mathrm{cf}} = -\overline{\nabla} A_0$. Take the chargeless part. We obtain that $\bar{E}^{\mathrm{cf}} = \overline{\nabla}\bar{A}_0$ when $|x| \geq R$. Therefore, we can bound the weighted Sobolev norm of the chargeless part of the Maxwell field $\bar{F}$ on the initial hypersurface as follows:

$$\begin{aligned}
\|\bar{F}\|_{H_{0,\delta}} &\leq \|F\chi_{\{|x|\leq R\}}\|_{H_{0,\delta}} + \|(\bar{E}, H)\chi_{\{|x|\geq R\}}\|_{H_{0,\delta}} \\
&\lesssim \|F\chi_{\{|x|\leq R\}}\|_{H_{0,\delta}} + \|(E^{\mathrm{df}}, H)\chi_{\{|x|\geq R\}}\|_{H_{0,\delta}} + \|\bar{E}^{\mathrm{cf}}\chi_{\{|x|\geq R\}}\|_{H_{0,\delta}} \\
&\lesssim \mathcal{M}^{1/2} + \|\overline{\nabla}\bar{A}_0\|_{H_{0,\delta}} \lesssim \mathcal{M}^{1/2}.
\end{aligned}$$

Similarly, we have the same estimates for $\mathcal{L}^k_{\tilde{Z}}\bar{F}$, $k \leq 2$, that is,

$$\|\mathcal{L}^k_{\tilde{Z}}\bar{F}\|_{H_{0,\delta}} \lesssim \mathcal{M}^{1/2}, \quad k \leq 2.$$

To derive estimates for $D^k_Z\phi$ and $\mathcal{L}^k_Z\bar{F}$ on the initial hypersurface, we use the equations

$$\partial_t E - \overline{\nabla} \times H = \Im(\phi \cdot \tilde{D}\phi), \quad \partial_t H + \overline{\nabla} \times E = 0, \quad D_t\phi_1 = \bar{D}\bar{D}\phi$$

to replace the time derivatives with the spatial derivatives. The inhomogeneous term $\Im(\phi \cdot \bar{D}\phi)$ or the commutators $[D_t, \bar{D}]$ could be controlled using Sobolev embedding together with Hölder's inequality. The lemma then follows. □

The above lemma then leads to the following corollary:

Corollary 59. *Let* (ϕ, A) *be the solution of* (MKG). *Under the bootstrap assumption* (98), *we have*

$$M_2 \lesssim \mathcal{M}, \quad \mathcal{E}_2[\phi] \lesssim_{\mathcal{M}} \mathcal{E}.$$

Proof. The corollary follows from the definition of M_2 and $\mathcal{E}_2[\phi]$ in (35) and (47) together Lemma 58. □

From now on, we allow the implicit constant in $\lesssim$ to also depend on $\mathcal{M}$, that is, $B \lesssim K$ means that $B \leq CK$ for some constant C depending on γ_0, R, ϵ and $\mathcal{M}$. The rest of this section is devoted to improving the bootstrap assumption.

To improve the bootstrap assumption, we need to estimate m_2 defined in (35). On the finite region $\{r \leq R\}$, the null structure of $J[\phi]$ is not necessary as the weights of r are bounded above. When $r \geq R$, the null structure of $J[\phi]$ plays a crucial role. Note that J_L and $\not{J} = (J_{e_1}, J_{e_2})$ are easy to control as they already contain "good" components $\not{D}\phi$ or $D_L(r\phi)$. The difficulty is to exploit the null structure of the component $J_{\underline{L}}$ which is not a standard null form as defined in [Klainerman 1984; 1986]. The null structure of the system is that $J_{\underline{L}}$ does not interact with the "bad" component $\underline{\alpha}$ of the Maxwell field.

For nonnegative integers k, write $\phi_k = D^k_Z\phi$, $\psi_k = D^k_Z(r\phi)$, $F_k = \mathcal{L}^k_Z F$ in this section. First we expand the second-order derivative of $J[\phi] = \Im(\phi \cdot \overline{D\phi})$.

Lemma 60. *Let X be L, $\underline{L}$, e_1, e_2. Then we have*

$$|\mathcal{L}_Z^2 J| + |\overline{\nabla}\mathcal{L}_Z J| \lesssim |D\phi_1||D\phi| + |\phi_1||D^2\phi| + |\phi||D^2\phi_1| + |\nabla F||\phi|^2 + |F||D\phi||\phi|, \quad |x| \le R;$$

$$r^2|\mathcal{L}_Z^2 J_X| \lesssim \sum_{k\le 2} |\psi_k||D_X\psi_{2-k}| + \sum_{l_1+l_2+l_3\le 1} |\mathcal{L}_Z^{l_1} F_{ZX}||\psi_{l_2}||\psi_{l_3}|, \qquad |x| > R.$$

Proof. By the definition of the Lie derivative $\mathcal{L}_Z$, we can compute

$$\begin{aligned}\mathcal{L}_Z J_X = Z(J_X) - J_{\mathcal{L}_Z X} &= \Im(D_Z\phi\cdot\overline{D_X\phi} + \phi\cdot\overline{D_Z D_X\phi} - \phi\cdot\overline{D_{[Z,X]\phi}})\\ &= \Im(\phi_1\cdot\overline{D_X\phi} + \phi\cdot\overline{D_X\phi_1} + \phi\cdot\overline{([D_Z, D_X] - D_{[Z,X]})\phi})\\ &= \Im(\phi_l\cdot\overline{D_X\phi_{1-l}}) - F_{ZX}|\phi|^2.\end{aligned}$$

Here we note that $[D_Z, D_X] - D_{[Z,X]} = \sqrt{-1}\,F_{ZX}$ for any vector fields Z, X, and we omitted the summation sign for $l = 0, 1$. Take one more derivative $\overline{\nabla}$ (recall that $\overline{\nabla}$ is the covariant derivative in the spatial direction). The estimate on the region $\{r \le R\}$ then follows.

Similarly, the second-order derivative expands as follows:

$$\begin{aligned}\mathcal{L}_Y\mathcal{L}_Z J_X &= Y\mathcal{L}_Z J_X - \mathcal{L}_Z J_{[Y,X]}\\ &= Y\Im(\phi_l\cdot\overline{D_X}\phi_{1-l}) - Y(F_{ZX}|\phi|^2) - \Im(\phi_l\cdot\overline{D_{[Y,X]}}\phi_{1-l}) + F_{Z[Y,X]}|\phi|^2\\ &= \Im(\phi_k\cdot\overline{D_X}\phi_{2-k}) - (\mathcal{L}_Y F_{ZX} + F_{[Y,Z]X})|\phi|^2 + \Im(\sqrt{-1}\phi_l\cdot\overline{F_{YX}\phi_{1-l}}) - F_{ZX}Y|\phi|^2\end{aligned}$$

for any vector fields $X, Y, Z \in \Gamma$. Here we have omitted the summation sign for $k = 0, 1, 2$ and $l = 0, 1$. Note that

$$\Im(\phi\cdot\overline{D_X\phi}) = r^{-2}\Im(r\phi\cdot\overline{D_X(r\phi)}), \quad [Y, Z] = 0 \text{ or } \in \Gamma.$$

The estimate on the region $\{r \ge R\}$ then follows. Thus the proof of the lemma is finished. □

Next we use the above bound for $J[\phi]$ to improve the bootstrap assumption.

Proposition 61. *Assume that the charge q_0 is sufficiently small, depending only on ϵ, R and γ_0, so that Corollary 22 holds. Then we have*

$$m_2 \le C\mathcal{E}^2 \tag{99}$$

for some constant C depending on $\mathcal{M}$, ϵ, R and γ_0.

Proof. Since $M_2 \lesssim \mathcal{M}$, all the estimates in the previous section hold. In particular, we have the energy flux and the r-weighted energy decay estimates for the scalar field and the chargeless part of the Maxwell field up to second-order derivatives. Moreover, the pointwise estimates in Propositions 14, 17, 54 and 56 hold.

Let's first consider the estimate of $|J_{\underline{L}}|r^{-2}$ in the exterior region. We have the simple bound that $|J_{\underline{L}}| \le |D_{\underline{L}}\phi||\phi|$. We can control $D_{\underline{L}}\phi$ by using the energy flux through the incoming null hypersurface

$\underline{H}_v$ and ϕ by the L^∞ norm. In particular, for any $\tau < 0$ we can show that

$$\iint_{\mathcal{D}_\tau^{-\infty}} |J_{\underline{L}}| r^{-2}\, dx\, dt \lesssim \int_{-\tau^*}^{\infty} \Big(\int_{\underline{H}_v} |D_{\underline{L}}\phi|^2 r^2\, du\, d\omega \Big)^{1/2} \cdot \Big(\int_{\underline{H}_v} |\phi|^2 r^{-2}\, du\, d\omega \Big)^{1/2} dv$$
$$\lesssim \mathcal{E} \int_{-\tau^*}^{\infty} \tau_+^{-(1+\gamma_0)/2} \Big(\int_{\underline{H}_v} r^{-4} \tau_+^{-\gamma_0}\, du\, d\omega \Big)^{1/2} dv$$
$$\lesssim \mathcal{E} \int_{-\tau^*}^{\infty} \tau_+^{-(1+2\gamma_0)/2} r^{-3/2}\, dv \lesssim \mathcal{E} \tau_+^{-1-\gamma_0}.$$

We remark here that we cannot use the integrated local energy to bound the above term due to the exact total decay rate of $|J_{\underline{L}}| r^{-2}$. As $|q_0| \lesssim \mathcal{E}$, we therefore obtain

$$|q_0| \sup_{\tau \le 0} \tau_+^{1+\gamma_0} \iint_{\mathcal{D}_\tau^{-\infty}} |J_{\underline{L}}| r^{-2}\, dx\, dt \lesssim \mathcal{E}^2, \quad \forall \tau \le 0.$$

Next we consider the estimates on the compact region $\{r \le 2R\}$. As $|\phi_1| = |D_Z\phi| \lesssim |D\phi|$ when $|x| \le R$, we can bound ϕ_1, ϕ, $D\phi$ and F by the L^∞ norm obtained in (40) and (91). Then $D^2\phi_1$ and ∇F can be controlled using the integral decay estimates (39) and (90) on $\{r \le R\}$. To derive estimates for $D^2\phi_k$ or ∇F on the region $\{R \le r \le 2R\}$, we use (MKG). From Lemma 37 and Lemma 5, we can show that

$$|D^2\phi_1| + \mathcal{E}|\nabla F| \lesssim |D\phi_2| + |D_L D_{\underline{L}}\psi_1| + |F||\phi_1| + \mathcal{E}\big(|\mathcal{L}_Z F| + |L(r^2\rho, r^2\sigma, r\underline{\alpha})| + |L(r\underline{\alpha})|\big)$$
$$\lesssim |D\phi_2| + |\Box_A\phi_1| + |F||\phi_1| + \mathcal{E}(|\mathcal{L}_Z F| + |J|).$$

Here we omitted the easier lower-order terms. On the region $\{R \le r \le 2R\}$, the set Γ only misses one derivative, which could be recovered from the equation. From Lemma 60, we can show that

$$I^0_{1+\gamma_0+2\epsilon}[|\mathcal{L}_Z^2 J| + |\overline{\nabla}\mathcal{L}_Z J|](\{r \le 2R\})$$
$$\lesssim \mathcal{E} \int_0^\infty \tau_+^{2\epsilon} \int_{r \le 2R} |D^2\phi_1|^2 + \mathcal{E}|\nabla F|^2 + |D\phi|^2\, dx\, d\tau$$
$$\lesssim \mathcal{E}^2 + \mathcal{E} \int_0^\infty \tau_+^{2\epsilon} \int_{R \le r \le 2R} |D\phi_2|^2 + |\Box_A\phi_1|^2 + |F|^2|\phi_1|^2 + \mathcal{E}(|\mathcal{L}_Z F|^2 + |J|^2)\, dx\, d\tau$$
$$\lesssim \mathcal{E}^2 + \mathcal{E} I^0_{2\epsilon}[\Box_A\phi_1](\{r \ge R\}) + \mathcal{E}^2 I^0_{2\epsilon}[J](\{r \ge R\}) \lesssim \mathcal{E}^2.$$

Here the implicit constant also depends on $\mathcal{M}$ and we only consider the highest-order terms. The second to last step follows as the integral from time τ_1 to τ_2 decays in τ_1. Hence the spacetime integral is bounded, using Lemma 20. The bound for $\Box_A\phi_1$ follows from Proposition 47 and the spacetime norm for J is controlled by the bootstrap assumption.

Next, we consider the case when $|x| \ge R$, where the null structure of J plays a crucial role. For $|\mathcal{L}_Z^2 J_{\underline{L}}|$, Lemma 60 implies that

$$r^2 |\mathcal{L}_Z^2 J_{\underline{L}}| \lesssim |\psi_k||D_{\underline{L}}\psi_{2-k}| + (|r\mathcal{L}_Z^{l_1}\underline{\alpha}| + |\mathcal{L}_Z^{l_1}\rho|)|\psi_{l_2}||\psi_{1-l_1-l_2}|.$$

Here the indices $k, 2-k, 1-l_1-l_2, l_1, l_2$ are nonnegative integers and we only consider the highest-order term as the lower-order terms are easier and could be bounded in a similar way. On the right-hand side of the above inequality, after using Sobolev embedding on the sphere, we can bound $|\psi|$ using Lemma 19 and $D_{\underline{L}}\psi$, $|\underline{\alpha}|$, ρ using the integrated local energy estimates. Indeed we can show that

$$\begin{aligned}
&I^{1-\epsilon}_{1+\gamma_0+2\epsilon}[\mathcal{L}_Z^2 J_{\underline{L}}](\{r\geq R\})\\
&= \int_\tau \int_{H_{\tau^*}} r_+^{-\epsilon-1}\tau_+^{1+\gamma_0+2\epsilon}|r^2\mathcal{L}_Z^2 J_{\underline{L}}|^2\,du\,d\omega\,d\tau\\
&\lesssim \int_\tau\int_v r_+^{-1-\epsilon}\tau_+^{1+\gamma_0+2\epsilon}\int_\omega |\psi_2|^2\,d\omega\cdot\int_\omega |D_{\underline{L}}\psi_2|^2\,d\omega+\int_\omega |r\mathcal{L}_Z^2\underline{\alpha}|^2+|\mathcal{L}_Z^2\bar{\rho}|^2\,d\omega\cdot\biggl(\int_\omega|\psi_2|^2\,d\omega\biggr)^2 dv\,d\tau\\
&\qquad+\int_{\tau\leq 0}\int_v r^{-1-\epsilon}|q_0|^2 r^{-4}\tau_+^{1+\gamma_0+2\epsilon}\biggl(\int_\omega|\psi_2|^2 d\omega\biggr)^{1/2} dv\,d\tau\\
&\lesssim \mathcal{E}\int_\tau \tau_+^{1+2\epsilon}\int_{H_{\tau^*}}\frac{|\tilde{D}\phi_2|^2}{r_+^{1+\epsilon}}\,dx\,d\tau+\mathcal{E}^2\int_\tau\tau_+^{1+2\epsilon}\int_{H_{\tau^*}}\frac{|\mathcal{L}_Z^2\bar{F}|^2}{r_+^{1+\epsilon}}\,dx\,d\tau+\mathcal{E}^2|q_0|^2\int_{\tau\leq 0}\int_v r_+^{-4+\epsilon+\gamma_0}\,dv\,d\tau\\
&\lesssim \mathcal{E}I^{-1-\epsilon}_{1+2\epsilon}[\tilde{D}\phi_2](\{t\geq 0\})+\mathcal{E}^2 I^{-1-\epsilon}_{1+2\epsilon}[\mathcal{L}_Z^2\bar{F}](\{t\geq 0\})+|q_0|^2\mathcal{E}^2\lesssim\mathcal{E}^2.
\end{aligned}$$

Here, after using Sobolev embedding on the sphere, we dropped the lower-order terms like ψ_1, ψ. In the above estimate, we have used the decay estimates $\int_\omega |\psi_k|^2\,d\omega\lesssim\mathcal{E}\tau_+^{-\gamma_0}$ by Lemma 19. The last step follows from the integrated local energy decay (see, e.g., estimate (64)) and Lemma 20. We also note that in the exterior region, $r_+\geq\frac{1}{2}\tau_+$.

For J_L, Lemma 60 indicates that

$$r^2|\mathcal{L}_Z^2 J_L|\lesssim|\psi_k||D_L\psi_{2-k}|+(|r\mathcal{L}_Z^{l_1}\alpha|+|\mathcal{L}_Z^{l_1}\rho|)|\psi_{l_2}||\psi_{1-l_1-l_2}|.$$

Similarly, after using Sobolev embedding, we control ψ_k by using Lemma 19. Then for $D_L\psi_k$, $|\mathcal{L}_Z^k\alpha|$ we can apply the r-weighted energy estimates. For ρ, we split it into the charge part $q_0 r^{-2}$ and the chargeless part which can be bounded by using the energy flux decay estimates. More precisely, for $\epsilon\leq p\leq 1+\gamma_0$, using the estimate $r^{-1}\int_\omega|\psi_k|^2\,d\omega\lesssim\mathcal{E}\tau_+^{-1-\gamma_0}$ we can show that

$$\begin{aligned}
&I^{1+p}_{1+\gamma_0+\epsilon-p}[\mathcal{L}_Z^2 J_L](\{r\geq R\})\\
&\qquad= \int_\tau\int_{H_{\tau^*}} r_+^{p-1}\tau_+^{1+\gamma_0+\epsilon-p}|r^2\mathcal{L}_Z^2 J_L|^2\,dv\,d\omega\,d\tau\\
&\qquad\lesssim \mathcal{E}\int_\tau\int_{H_{\tau^*}} r_+^p\tau_+^{\epsilon-p}|D_L\psi_2|^2\,d\omega\,dv\,d\tau+\mathcal{E}^2\int_\tau\int_{H_{\tau^*}} r_+^p\tau_+^{\epsilon-p-\gamma_0}(|r\mathcal{L}_Z^2\alpha|^2+|\mathcal{L}_Z^2\bar{\rho}|^2)\,d\omega\,dv\,d\tau\\
&\qquad\qquad+\mathcal{E}^2\int_{\tau\leq 0}\int_v r^{p-1}|q_0|^2 r^{-4}\tau_+^{1-\gamma_0+\epsilon-p}\,dv\,d\tau\\
&\qquad\lesssim \mathcal{E}^2\int_\tau \tau_+^{\epsilon-p-1-\gamma_0+p}+\tau_+^{\epsilon-p-\gamma_0-1-\gamma_0+p}+\tau_+^{\epsilon-p-1-\gamma_0}\,d\tau+\mathcal{E}^2|q_0|^2\lesssim\mathcal{E}^2.
\end{aligned}$$

Next, for $\not{J}$, Lemma 60 shows that

$$r^2|\mathcal{L}_Z^2 \not{J}| \lesssim |\psi_k||\not{D}\psi_{2-k}| + (|r\mathcal{L}_Z^{l_1}\sigma| + |\mathcal{L}_Z^{l_1}\alpha| + |\mathcal{L}_Z^{l_1}\underline{\alpha}|)|\psi_{l_2}||\psi_{1-l_1-l_2}|.$$

Like the previous estimates for $J_{\underline{L}}$, J_L, for all $\epsilon \le p \le \gamma_0$ we can show that

$$\begin{aligned}
I^{1+p}_{1+\gamma_0+\epsilon-p}&[\mathcal{L}_Z^2 \not{J}_L](\{r \ge R\}) \\
&= \int_\tau \int_{H_{\tau^*}} r_+^{p-1} \tau_+^{1+\gamma_0+\epsilon-p} |r^2 \mathcal{L}_Z^2 \not{J}|^2 \, dv \, d\omega \, d\tau \\
&\lesssim \mathcal{E} \int_\tau \int_{H_{\tau^*}} r_+^p \tau_+^{\epsilon-p} |\not{D}\psi_2|^2 \, d\omega \, dv \, d\tau + \mathcal{E}^2 \int_\tau \int_{H_{\tau^*}} r_+^p \tau_+^{\epsilon-p-\gamma_0} (|r\mathcal{L}_Z^{l_1}\sigma|^2 + |\mathcal{L}_Z^{l_1}(\underline{\alpha}, \alpha)|^2) \, d\omega \, dv \, d\tau \\
&\lesssim \mathcal{E} \int_\tau \int_{H_{\tau^*}} r_+^{\gamma_0} (|\not{D}\psi_2|^2 + \mathcal{E}|r\mathcal{L}_Z^{l_1}(\sigma, \alpha)|^2) \, d\omega \, dv \, d\tau + \mathcal{E}^2 \int_\tau \int_{H_{\tau^*}} r_+^{1-\epsilon} |\mathcal{L}_Z^{l_1}\underline{\alpha}|^2 \, d\omega \, dv \, d\tau \\
&\lesssim \mathcal{E}^2.
\end{aligned}$$

Here $l_1 \le 1$. The last term is bounded by using the integrated local energy estimates. This relies on the assumption that $\gamma_0 \le 1 - \epsilon < 1$. For $\gamma_0 \ge 1$, we then can use the improved integrated local energy estimate for the angular derivatives of ϕ or σ, or we can move the r weights to ϕ_k.

Combining the above estimates, we have (99). □

By choosing $\mathcal{E}$ sufficiently small depending only on $\mathcal{M}$, ϵ, R and γ_0, we then can improve the bootstrap assumption (98). To prove Theorem 1, we can choose $R = 2$. Then for sufficiently small $\mathcal{E}$, we can bound m_2 and M_2. The pointwise estimates in the main Theorem 1 follow from Propositions 14, 17, 54 and 56.

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