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STRUCTURE OF MODULAR INVARIANT SUBALGEBRAS IN FREE ARAKI–WOODS FACTORS

RÉMI BOUTONNET AND CYRIL HOUDAYER

We show that any amenable von Neumann subalgebra of any free Araki–Woods factor that is globally invariant under the modular automorphism group of the free quasifree state is necessarily contained in the almost periodic free summand.

1. Introduction

Free Araki–Woods factors were introduced in [Shlyakhtenko 1997]. In the framework of Voiculescu's free probability theory, they can be regarded as the type III counterparts of free group factors using the free Gaussian functor [Voiculescu 1985; Voiculescu et al. 1992]. Following Shlyakhtenko, to any orthogonal representation $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$ on a real Hilbert space, one associates the *free Araki–Woods* von Neumann algebra $\Gamma(H_{\mathbb{R}}, U)''$. The von Neumann algebra $\Gamma(H_{\mathbb{R}}, U)''$ comes equipped with a unique *free quasifree state* φ_U which is always normal and faithful (see Section 2 for a detailed construction). We have $\Gamma(H_{\mathbb{R}}, U)'' \cong \mathrm{L}(\boldsymbol{F}_{\dim(H_{\mathbb{R}})})$ when $U = 1_{H_{\mathbb{R}}}$ and $\Gamma(H_{\mathbb{R}}, U)''$ is a full type III factor when $U \neq 1_{H_{\mathbb{R}}}$.

Let $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$ be any orthogonal representation. Using Zorn's lemma, we may decompose $H_{\mathbb{R}} = H_{\mathbb{R}}^{\mathrm{ap}} \oplus H_{\mathbb{R}}^{\mathrm{wm}}$ and $U = U^{\mathrm{wm}} \oplus U^{\mathrm{ap}}$, where $U^{\mathrm{ap}} : \mathbb{R} \curvearrowright H_{\mathbb{R}}^{\mathrm{ap}}$ is the *almost periodic*, and $U^{\mathrm{wm}} : \mathbb{R} \curvearrowright H_{\mathbb{R}}^{\mathrm{wm}}$ the *weakly mixing*, subrepresentation of $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$. Write $M = \Gamma(H_{\mathbb{R}}, U)''$, $N = \Gamma(H_{\mathbb{R}}^{\mathrm{ap}}, U^{\mathrm{ap}})''$ and $P = \Gamma(H_{\mathbb{R}}^{\mathrm{wm}}, U^{\mathrm{wm}})''$, so that we have the *free product* splitting

$$(M, \varphi_U) = (N, \varphi_{U^{\mathrm{ap}}}) * (P, \varphi_{U^{\mathrm{wm}}}).$$

Our main result provides a general structural decomposition for any von Neumann subalgebra $Q \subset M$ that is globally invariant under the modular automorphism group σ^{φ_U} and shows that when Q is also assumed to be amenable then Q sits inside N. It generalizes Theorem C of [Houdayer and Raum 2015] to *arbitrary* free Araki–Woods factors.

Main Theorem. *Keep the same notation as above. Let $Q \subset M$ be any unital von Neumann subalgebra that is globally invariant under the modular automorphism group σ^{φ_U}. Then there exists a unique central projection $z \in \mathcal{Z}(Q) \subset M^{\varphi_U} = N^{\varphi_{U^{\mathrm{ap}}}}$ such that*

- *Qz is amenable and $Qz \subset zNz$, and*
- *$Qz^{\perp}$ has no nonzero amenable direct summand and $(Q' \cap M^{\omega})z^{\perp} = (Q' \cap M)z^{\perp}$ is atomic for any nonprincipal ultrafilter $\omega \in \beta(\boldsymbol{N}) \setminus \boldsymbol{N}$.*

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In particular, for any unital amenable von Neumann subalgebra $Q \subset M$ that is globally invariant under the modular automorphism group σ^{φ_U}, we have $Q \subset N$.

Our main theorem should be compared to [Houdayer 2014b, Theorem D], which provides a similar result for crossed product II_1 factors arising from free Bogoljubov actions of amenable groups.

The core of our argument is Theorem 3.1 which generalizes [Houdayer and Raum 2015, Theorem 4.3] to arbitrary free Araki–Woods factors. Let us point out that Theorem 3.1 is reminiscent of Popa's asymptotic orthogonality property in free group factors [Popa 1983] which is based on the study of central sequences in the ultraproduct framework. Unlike other results on this theme [Houdayer 2014b; 2015; Houdayer and Ueda 2016], we do not assume here that the subalgebra $Q \subset M$ has a diffuse intersection with the free summand N of the free product splitting $(M, \varphi_U) = (N, \varphi_{U^{\mathrm{ap}}}) * (P, \varphi_{U^{\mathrm{wm}}})$, and so we cannot exploit commutation relations of Q-central sequences with elements in N. Instead, we use the facts that Q admits central sequences that are invariant under the modular automorphism group $\sigma^{\varphi_U^\omega}$ of the ultraproduct state φ_U^ω and that the modular automorphism group σ^{φ_U} is weakly mixing on P.

2. Preliminaries

For any von Neumann algebra M, we denote by $\mathcal{Z}(M)$ the center of M, by $\mathcal{U}(M)$ the group of unitaries in M, by $\mathrm{Ball}(M)$ the unit ball of M with respect to the uniform norm and by $(M, \mathrm{L}^2(M), J, \mathrm{L}^2(M)_+)$ the standard form of M. We say that an inclusion of von Neumann algebras $P \subset M$ is *with expectation* if there exists a faithful normal conditional expectation $\mathrm{E}_P : M \to P$. All the von Neumann algebras we consider in this paper are always assumed to be σ-finite.

Let M be any σ-finite von Neumann algebra with predual M_* and $\varphi \in M_*$ any faithful state. We write $\|x\|_\varphi = \varphi(x^*x)^{1/2}$ for all $x \in M$. Recall that on $\mathrm{Ball}(M)$, the topology given by $\|\cdot\|_\varphi$ coincides with the σ-strong topology. Denote by $\xi_\varphi \in \mathrm{L}^2(M)_+$ the unique representing vector of φ. The mapping $M \to \mathrm{L}^2(M) : x \mapsto x\xi_\varphi$ defines an embedding with dense image such that $\|x\|_\varphi = \|x\xi_\varphi\|_{\mathrm{L}^2(M)}$ for all $x \in M$. We denote by σ^φ the modular automorphism group of the state φ. The *centralizer* M^φ of the state φ is by definition the fixed point algebra of (M, σ^φ).

Recall from [Houdayer 2014a, Section 2.1] that two subspaces $E, F \subset H$ of a Hilbert space are said to be *ε-orthogonal* for some $0 \leq \varepsilon \leq 1$ if $|\langle \xi, \eta\rangle| \leq \varepsilon \|\xi\| \|\eta\|$ for all $\xi \in E$ and all $\eta \in F$. We then simply write $E \perp_\varepsilon F$.

Ultraproduct von Neumann algebras. Let M be any σ-finite von Neumann algebra and $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ any nonprincipal ultrafilter. Define

$$\mathcal{I}_\omega(M) = \{(x_n)_n \in \ell^\infty(M) : x_n \to 0 \ *\text{-strongly as } n \to \omega\},$$
$$\mathcal{M}^\omega(M) = \{(x_n)_n \in \ell^\infty(M) : (x_n)_n \, \mathcal{I}_\omega(M) \subset \mathcal{I}_\omega(M) \text{ and } \mathcal{I}_\omega(M)\, (x_n)_n \subset \mathcal{I}_\omega(M)\}.$$

The *multiplier algebra* $\mathcal{M}^\omega(M)$ is a C^*-algebra and $\mathcal{I}_\omega(M) \subset \mathcal{M}^\omega(M)$ is a norm closed two-sided ideal. Following [Ocneanu 1985, §5.1], we define the *ultraproduct von Neumann algebra M^ω* by $M^\omega := \mathcal{M}^\omega(M)/\mathcal{I}_\omega(M)$, which is indeed known to be a von Neumann algebra. We denote the image of $(x_n)_n \in \mathcal{M}^\omega(M)$ by $(x_n)^\omega \in M^\omega$.

For every $x \in M$, the constant sequence $(x)_n$ lies in the multiplier algebra $\mathcal{M}^\omega(M)$. We then identify M with $(M + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and regard $M \subset M^\omega$ as a von Neumann subalgebra. The map

$$\mathrm{E}_\omega : M^\omega \to M, \quad (x_n)^\omega \mapsto \sigma\text{-weak} \lim_{n \to \omega} x_n$$

is a faithful normal conditional expectation. For every faithful state $\varphi \in M_*$, the formula $\varphi^\omega := \varphi \circ \mathrm{E}_\omega$ defines a faithful normal state on M^ω. Observe that $\varphi^\omega((x_n)^\omega) = \lim_{n\to\omega} \varphi(x_n)$ for all $(x_n)^\omega \in M^\omega$.

Let $Q \subset M$ be any von Neumann subalgebra with faithful normal conditional expectation $\mathrm{E}_Q : M \to Q$. Choose a faithful state $\varphi \in M_*$ in such a way that $\varphi = \varphi \circ \mathrm{E}_Q$. We have $\ell^\infty(Q) \subset \ell^\infty(M)$, $\mathcal{I}_\omega(Q) \subset \mathcal{I}_\omega(M)$ and $\mathcal{M}^\omega(Q) \subset \mathcal{M}^\omega(M)$. We then identify $Q^\omega = \mathcal{M}^\omega(Q)/\mathcal{I}_\omega(Q)$ with $(\mathcal{M}^\omega(Q) + \mathcal{I}_\omega(M))/\mathcal{I}_\omega(M)$ and may regard $Q^\omega \subset M^\omega$ as a von Neumann subalgebra. Observe that the norm $\|\cdot\|_{(\varphi|_Q)^\omega}$ on Q^ω is the restriction of the norm $\|\cdot\|_{\varphi^\omega}$ to Q^ω. Observe moreover that $(\mathrm{E}_Q(x_n))_n \in \mathcal{I}_\omega(Q)$ for all $(x_n)_n \in \mathcal{I}_\omega(M)$ and $(\mathrm{E}_Q(x_n))_n \in \mathcal{M}^\omega(Q)$ for all $(x_n)_n \in \mathcal{M}^\omega(M)$. Therefore, the mapping $\mathrm{E}_{Q^\omega} : M^\omega \to Q^\omega$ given by $(x_n)^\omega \mapsto (\mathrm{E}_Q(x_n))^\omega$ is a well-defined conditional expectation satisfying $\varphi^\omega \circ \mathrm{E}_{Q^\omega} = \varphi^\omega$. Hence, $\mathrm{E}_{Q^\omega} : M^\omega \to Q^\omega$ is a faithful normal conditional expectation. For more on ultraproduct von Neumann algebras, we refer the reader to [Ando and Haagerup 2014; Ocneanu 1985].

Free Araki–Woods factors. Let $H_\mathbb{R}$ be any real Hilbert space and $U : \mathbb{R} \curvearrowright H_\mathbb{R}$ any orthogonal representation. Denote by $H = H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = H_\mathbb{R} \oplus \mathrm{i} H_\mathbb{R}$ the complexified Hilbert space, by $I : H \to H : \xi + \mathrm{i}\eta \mapsto \xi - \mathrm{i}\eta$ the canonical anti-unitary involution on H and by A the infinitesimal generator of $U : \mathbb{R} \curvearrowright H$, that is, $U_t = A^{\mathrm{i}t}$ for all $t \in \mathbb{R}$. Moreover, we have $IAI = A^{-1}$. Observe that $j : H_\mathbb{R} \to H : \zeta \mapsto (2/(A^{-1}+1))^{1/2}\zeta$ defines an isometric embedding of $H_\mathbb{R}$ into H. Put $K_\mathbb{R} := j(H_\mathbb{R})$. It is easy to see that $K_\mathbb{R} \cap \mathrm{i}K_\mathbb{R} = \{0\}$ and that $K_\mathbb{R} + \mathrm{i}K_\mathbb{R}$ is dense in H. Write $T = IA^{-1/2}$. Then T is a conjugate-linear closed invertible operator on H satisfying $T = T^{-1}$ and $T^*T = A^{-1}$. Such an operator is called an *involution* on H. Moreover, we have $\mathrm{dom}(T) = \mathrm{dom}(A^{-1/2})$ and $K_\mathbb{R} = \{\xi \in \mathrm{dom}(T) : T\xi = \xi\}$. In what follows, we simply write

$$\overline{\xi + \mathrm{i}\eta} := T(\xi + \mathrm{i}\eta) = \xi - \mathrm{i}\eta, \quad \forall \xi, \eta \in K_\mathbb{R}.$$

We introduce the *full Fock space* of H:

$$\mathcal{F}(H) = \mathbb{C}\Omega \oplus \bigoplus_{n=1}^{\infty} H^{\otimes n}.$$

The unit vector Ω is known as the *vacuum vector*. For all $\xi \in H$, we define the *left creation* operator $\ell(\xi) : \mathcal{F}(H) \to \mathcal{F}(H)$ by

$$\begin{cases} \ell(\xi)\Omega = \xi, \\ \ell(\xi)(\xi_1 \otimes \cdots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \cdots \otimes \xi_n. \end{cases}$$

We have $\|\ell(\xi)\|_\infty = \|\xi\|$, and $\ell(\xi)$ is an isometry if $\|\xi\| = 1$. For all $\xi \in K_\mathbb{R}$, put $W(\xi) := \ell(\xi) + \ell(\xi)^*$. The crucial result of Voiculescu [Voiculescu et al. 1992, Lemma 2.6.3] is that the distribution of the self-adjoint operator $W(\xi)$ with respect to the vector state $\varphi_U = \langle \cdot\, \Omega, \Omega \rangle$ is the semicircular law of Wigner supported on the interval $[-\|\xi\|, \|\xi\|]$.

Definition 2.1 [Shlyakhtenko 1997]. Let $H_{\mathbb{R}}$ be any real Hilbert space and $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$ any orthogonal representation. The *free Araki–Woods* von Neumann algebra associated with $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$ is defined by

$$\Gamma(H_{\mathbb{R}}, U)'' := \{W(\xi) : \xi \in K_{\mathbb{R}}\}''.$$

We denote by $\Gamma(H_{\mathbb{R}}, U)$ the unital C*-algebra generated by 1 and by all the elements $W(\xi)$ for $\xi \in K_{\mathbb{R}}$.

The vector state $\varphi_U = \langle \cdot \Omega, \Omega\rangle$ is called the *free quasifree state* and is faithful on $\Gamma(H_{\mathbb{R}}, U)''$. Let $\xi, \eta \in K_{\mathbb{R}}$ and write $\zeta = \xi + \mathrm{i}\eta$. Put

$$W(\zeta) := W(\xi) + \mathrm{i}W(\eta) = \ell(\zeta) + \ell(\bar{\zeta})^*.$$

Note that the modular automorphism group σ^{φ_U} of the free quasifree state φ_U is given by $\sigma_t^{\varphi_U} = \mathrm{Ad}(\mathcal{F}(U_t))$, where $\mathcal{F}(U_t) = 1_{\mathbb{C}\Omega} \oplus \bigoplus_{n\geq 1} U_t^{\otimes n}$. In particular, it satisfies

$$\sigma_t^{\varphi_U}(W(\zeta)) = W(U_t\zeta), \quad \forall \zeta \in K_{\mathbb{R}} + \mathrm{i}K_{\mathbb{R}}, \forall t \in \mathbb{R}.$$

It is easy to see that for all $n \geq 1$ and all $\zeta_1, \dots, \zeta_n \in K_{\mathbb{R}} + \mathrm{i}K_{\mathbb{R}}$, $\zeta_1 \otimes \cdots \otimes \zeta_n \in \Gamma(H_{\mathbb{R}}, U)''\Omega$. When $\zeta_1, \dots, \zeta_n$ are all nonzero, we denote by $W(\zeta_1 \otimes \cdots \otimes \zeta_n) \in \Gamma(H_{\mathbb{R}}, U)''$ the unique element such that

$$\zeta_1 \otimes \cdots \otimes \zeta_n = W(\zeta_1 \otimes \cdots \otimes \zeta_n)\Omega.$$

Such an element is called a *reduced word*. By [Houdayer and Raum 2015, Proposition 2.1(i)] (see also [Houdayer 2014a, Proposition 2.4]), the reduced word $W(\zeta_1 \otimes \cdots \otimes \zeta_n)$ satisfies the *Wick formula* given by

$$W(\zeta_1 \otimes \cdots \otimes \zeta_n) = \sum_{k=0}^{n} \ell(\zeta_1) \cdots \ell(\zeta_k)\ell(\bar{\zeta}_{k+1})^* \cdots \ell(\bar{\zeta}_n)^*.$$

Note that since inner products are assumed to be linear in the first variable, for all $\xi, \eta \in H$ we have $\ell(\xi)^*\ell(\eta) = \overline{\langle \xi, \eta\rangle} 1 = \langle \eta, \xi\rangle 1$. In particular, the Wick formula from [Houdayer and Raum 2015, Proposition 2.1(ii)] is

$$\begin{aligned} &W(\xi_1 \otimes \cdots \otimes \xi_r)W(\eta_1 \otimes \cdots \otimes \eta_s) \\ &\quad = W(\xi_1 \otimes \cdots \otimes \xi_r \otimes \eta_1 \otimes \cdots \otimes \eta_s) + \overline{\langle \bar{\xi}_r, \eta_1\rangle}\, W(\xi_1 \otimes \cdots \otimes \xi_{r-1})W(\eta_2 \otimes \cdots \otimes \eta_s) \end{aligned}$$

for all $\xi_1, \dots, \xi_r, \eta_1, \dots, \eta_s \in K_{\mathbb{R}} + \mathrm{i}K_{\mathbb{R}}$. We repeatedly use this fact in the next section. We refer to [Houdayer and Raum 2015, Section 2] for further details.

3. Asymptotic orthogonality property in free Araki–Woods factors

Let $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$ be any orthogonal representation. By Zorn's lemma, we may decompose $H_{\mathbb{R}} = H_{\mathbb{R}}^{\mathrm{ap}} \oplus H_{\mathbb{R}}^{\mathrm{wm}}$ and $U = U^{\mathrm{wm}} \oplus U^{\mathrm{ap}}$, where $U^{\mathrm{ap}} : \mathbb{R} \curvearrowright H_{\mathbb{R}}^{\mathrm{ap}}$ is the *almost periodic*, and $U^{\mathrm{wm}} : \mathbb{R} \curvearrowright H_{\mathbb{R}}^{\mathrm{wm}}$ the *weakly mixing*, subrepresentation of $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}$. Write $M = \Gamma(H_{\mathbb{R}}, U)''$, $N = \Gamma(H_{\mathbb{R}}^{\mathrm{ap}}, U^{\mathrm{ap}})''$ and $P = \Gamma(H_{\mathbb{R}}^{\mathrm{wm}}, U_t^{\mathrm{wm}})''$, so that

$$(M, \varphi_U) = (N, \varphi_{U^{\mathrm{ap}}}) * (P, \varphi_{U^{\mathrm{wm}}}).$$

For notational convenience, we simply write $\varphi := \varphi_U$.

The main result of this section, Theorem 3.1 below, strengthens and generalizes [Houdayer and Raum 2015, Theorem 4.3].

Theorem 3.1. *Keep the same notation as above. Let $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$ be any nonprincipal ultrafilter. For all $a \in M \ominus N$, all $b \in M$ and all $x, y \in (M^\omega)^{\varphi^\omega} \cap (M^\omega \ominus M)$, we have*

$$\varphi^\omega(b^* y^* a x) = 0.$$

Proof. Denote, as usual, by $H := H_\mathbb{R} \otimes_\mathbb{R} \mathbb{C}$ the complexified Hilbert space and by $U : \mathbb{R} \curvearrowright H$ the corresponding unitary representation. Put $H^{\mathrm{ap}} := H_\mathbb{R}^{\mathrm{ap}} \otimes_\mathbb{R} \mathbb{C}$ and $H^{\mathrm{wm}} := H_\mathbb{R}^{\mathrm{wm}} \otimes_\mathbb{R} \mathbb{C}$. Put $K_\mathbb{R} := j(H_\mathbb{R})$, $K_\mathbb{R}^{\mathrm{ap}} = j(H_\mathbb{R}^{\mathrm{ap}})$ and $K_\mathbb{R}^{\mathrm{wm}} := j(H_\mathbb{R}^{\mathrm{wm}})$, where j is the isometric embedding $\xi \in H_\mathbb{R} \mapsto (2/(1+A^{-1}))^{1/2}\xi \in H$. Denote by $\mathcal{H} = \mathcal{F}(H)$ the full Fock space of H. For every $t \in \mathbb{R}$, put $\kappa_t = 1_{\mathbb{C}\Omega} \oplus \bigoplus_{n \geq 1} U_t^{\otimes n} \in \mathcal{U}(\mathcal{H})$. For every $t \in \mathbb{R}$ and every $x \in M$, we have $\sigma_t^\varphi(x)\Omega = \kappa_t(x\Omega)$. We implicitly identify the full Fock space $\mathcal{F}(H)$ with the standard Hilbert space $\mathrm{L}^2(M)$ and the vacuum vector $\Omega \in \mathcal{H}$ with the canonical representing vector $\xi_\varphi \in \mathrm{L}^2(M)_+$.

Put $K_{\mathrm{an}} := \bigcup_{\lambda > 1} \mathbf{1}_{[\lambda^{-1}, \lambda]}(A)(K_\mathbb{R} + \mathrm{i}K_\mathbb{R})$. Observe that $K_{\mathrm{an}} \subset K_\mathbb{R} + \mathrm{i}K_\mathbb{R}$ is a dense subspace of elements $\eta \in K_\mathbb{R} + \mathrm{i}K_\mathbb{R}$ for which the map $\mathbb{R} \to K_\mathbb{R} + \mathrm{i}K_\mathbb{R} : t \mapsto U_t\eta$ extends to a $(K_\mathbb{R} + \mathrm{i}K_\mathbb{R})$-valued entire analytic function, and that $\overline{K_{\mathrm{an}}} = K_{\mathrm{an}}$. For all $\eta \in K_{\mathrm{an}}$, the element $W(\eta)$ is analytic with respect to the modular automorphism group σ^φ and we have $\sigma_z^\varphi(W(\eta)) = W(A^{\mathrm{i}z}\eta)$ for all $z \in \mathbb{C}$.

Denote by $\mathcal{W}$ the set of reduced words of the form $W(\xi_1 \otimes \cdots \otimes \xi_n)$ for which $n \geq 1$ and $\xi_1, \dots, \xi_n \in K_{\mathrm{an}}$. By linearity/density, in order to prove Theorem 3.1, we may assume without loss of generality that a and b are reduced words in $\mathcal{W}$. Since moreover $a \in M \ominus N$, we can assume that at least one of its letters ξ_i lies in $K_\mathbb{R}^{\mathrm{wm}} + \mathrm{i}K_\mathbb{R}^{\mathrm{wm}}$. More precisely, we can write

$$a = a' \, W(\xi_1 \otimes \cdots \otimes \xi_p) \, a'',$$
$$b = b' \, W(\eta_1 \otimes \cdots \otimes \eta_q) \, b''$$

with $p \geq 1$, $q \geq 0$ and for reduced words a', a'', b', b'' in N with letters in $K_{\mathrm{an}} \cap (K_\mathbb{R}^{\mathrm{ap}} + \mathrm{i}K_\mathbb{R}^{\mathrm{ap}})$, and for $\xi_2, \dots, \xi_{p-1}, \eta_2, \dots, \eta_{q-1} \in K_{\mathrm{an}}$ and $\xi_1, \xi_p, \eta_1, \eta_q \in K_{\mathrm{an}} \cap (K_\mathbb{R}^{\mathrm{wm}} + \mathrm{i}K_\mathbb{R}^{\mathrm{wm}})$. By convention, when $q = 0$, $W(\eta_1 \otimes \cdots \otimes \eta_q)$ is the trivial word 1, so that $b = b'b''$.

Denote by $L \subset K_\mathbb{R}^{\mathrm{wm}} + \mathrm{i}K_\mathbb{R}^{\mathrm{wm}}$ the finite dimensional subspace generated by $\xi_1, \xi_p, \eta_1, \eta_q$ and such that $\overline{L} = L$. If $q = 0$, then L is simply the subspace generated by $\xi_1, \xi_p, \bar{\xi}_1, \bar{\xi}_p$. Denote by

- $\mathcal{X}(1, r) \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $r \geq 0$, $n \geq r + 1$, $e_1, \dots, e_r \in K_\mathbb{R}^{\mathrm{ap}} + \mathrm{i}K_\mathbb{R}^{\mathrm{ap}}$ and $e_{r+1} \in L$;
- $\mathcal{X}(2, r) \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $r \geq 0$, $n \geq r + 1$, $e_{n-r} \in L$ and $e_{n-r+1}, \dots, e_n \in K_\mathbb{R}^{\mathrm{ap}} + \mathrm{i}K_\mathbb{R}^{\mathrm{ap}}$;
- $\mathcal{Y} \subset \mathcal{H}$ the closed linear subspace generated by all the reduced words of the form $e_1 \otimes \cdots \otimes e_n$ with $n \geq 1$ and $e_1, e_n \in L^\perp$.

When $r = 0$, we simply write $\mathcal{X}_1 := \mathcal{X}(1, 0)$ and $\mathcal{X}_2 := \mathcal{X}(2, 0)$. Observe that we have the orthogonal decomposition

$$\mathcal{H} = \mathbb{C}\Omega \oplus \overline{(\mathcal{X}_1 + \mathcal{X}_2)} \oplus \mathcal{Y}.$$

Claim 3.2. *Let $\varepsilon \geq 0$ and $t \in \mathbb{R}$ such that $U_t(L) \perp_{\varepsilon/\dim L} L$. Then for all $i \in \{1, 2\}$ and all $r \geq 0$, we have*

$$\kappa_t(\mathcal{X}(i, r)) \perp_\varepsilon \mathcal{X}(i, r).$$

Proof of Claim 3.2. Choose an orthonormal basis $(\zeta_1, \dots, \zeta_{\dim L})$ of L. We first prove the claim for $i = 1$. We identify $\mathcal{X}(1, r)$ with $L \otimes ((H^{\mathrm{ap}})^{\otimes r} \otimes \mathcal{H})$ using the unitary defined by

$$\mathcal{V}(1, r) : H \otimes (H^{\otimes r} \otimes \mathcal{H}) \to \mathcal{H} : \zeta \otimes \mu \otimes \nu \mapsto \mu \otimes \zeta \otimes \nu.$$

Observe that $\kappa_t \mathcal{V}(1, r) = \mathcal{V}(1, r)(U_t \otimes (U_t)^{\otimes r} \otimes \kappa_t)$ for every $t \in \mathbb{R}$. Let $\Xi_1, \Xi_2 \in \mathcal{X}(1, r)$ be such that $\Xi_1 = \sum_{i=1}^{\dim L} \zeta_i \otimes \Theta_i^1$ and $\Xi_2 = \sum_{j=1}^{\dim L} \zeta_j \otimes \Theta_j^2$ with $\Theta_i^1, \Theta_j^2 \in (H^{\mathrm{ap}})^{\otimes r} \otimes \mathcal{H}$. We have

$$\kappa_t(\Xi_1) = \sum_{i=1}^{\dim L} U_t(\zeta_i) \otimes \kappa_t(\Theta_i^1),$$

and hence

$$|\langle \kappa_t(\Xi_1), \Xi_2 \rangle| \leq \sum_{i,j=1}^{\dim L} |\langle U_t(\zeta_i), \zeta_j \rangle| \|\Theta_i^1\| \|\Theta_j^2\|.$$

Since $|\langle U_t(\zeta_i), \zeta_j \rangle| \leq \varepsilon/\dim L$, we obtain $|\langle \kappa_t(\Xi_1), \Xi_2 \rangle| \leq \varepsilon \|\Xi_1\| \|\Xi_2\|$ by the Cauchy–Schwarz inequality. The proof of the claim for $i = 2$ is entirely analogous. □

Given a closed subspace $\mathcal{K} \subset \mathcal{H}$, we denote by $P_{\mathcal{K}} : \mathcal{H} \to \mathcal{K}$ the orthogonal projection onto $\mathcal{K}$.

Claim 3.3. *Take $z = (z_n)^\omega \in (M^\omega)^{\varphi^\omega}$ and let $w_1, w_2 \in N$ be any elements of the following forms:*

- *Either $w_1 = 1$ or $w_1 = W(\zeta_1 \otimes \cdots \otimes \zeta_r)$ with $r \geq 1$ and $\zeta_1, \dots, \zeta_r \in K_{\mathrm{an}} \cap (K_{\mathbb{R}}^{\mathrm{ap}} + \mathrm{i} K_{\mathbb{R}}^{\mathrm{ap}})$.*
- *Either $w_2 = 1$ or $w_2 = W(\mu_1 \otimes \cdots \otimes \mu_s)$ with $s \geq 1$ and $\mu_1, \dots, \mu_s \in K_{\mathrm{an}} \cap (K_{\mathbb{R}}^{\mathrm{ap}} + \mathrm{i} K_{\mathbb{R}}^{\mathrm{ap}})$.*

Then for all $i \in \{1, 2\}$, we have $\lim_{n \to \omega} \|P_{\mathcal{X}_i}(w_1 z_n w_2 \Omega)\| = 0$.

Proof of Claim 3.3. Observe that $w_1 z_n w_2 \Omega = w_1 J \sigma_{-\mathrm{i}/2}^\varphi(w_2^*) J z_n \Omega$. Firstly, we have

$$P_{\mathcal{X}(1,r)}(J \sigma_{-\mathrm{i}/2}^\varphi(w_2^*) J z_n \Omega) = J \sigma_{-\mathrm{i}/2}^\varphi(w_2^*) J P_{\mathcal{X}(1,r)}(z_n \Omega),$$
$$P_{\mathcal{X}(2,s)}(w_1 z_n \Omega) = w_1 P_{\mathcal{X}(2,s)}(z_n \Omega).$$

Secondly, for all $\Xi \in \mathcal{H}$, we have

$$P_{\mathcal{X}_1}(w_1 \Xi) = P_{\mathcal{X}_1}(w_1 P_{\mathcal{X}(1,r)}(\Xi)),$$
$$P_{\mathcal{X}_2}(J \sigma_{-\mathrm{i}/2}^\varphi(w_2^*) J \Xi) = P_{\mathcal{X}_2}(J \sigma_{-\mathrm{i}/2}^\varphi(w_2^*) J P_{\mathcal{X}(2,s)}(\Xi)).$$

This implies that

$$P_{\mathcal{X}_1}(w_1 z_n w_2 \Omega) = P_{\mathcal{X}_1}(w_1 J \sigma_{-\mathrm{i}/2}^\varphi(w_2^*) J P_{\mathcal{X}(1,r)}(z_n \Omega)),$$
$$P_{\mathcal{X}_2}(w_1 z_n w_2 \Omega) = P_{\mathcal{X}_2}(w_1 J \sigma_{-\mathrm{i}/2}^\varphi(w_2^*) J P_{\mathcal{X}(2,s)}(z_n \Omega)),$$

and we are left to show that $\lim_{n \to \omega} \|P_{\mathcal{X}(1,r)}(z_n \Omega)\| = \lim_{n \to \omega} \|P_{\mathcal{X}(2,s)}(z_n \Omega)\| = 0$.

Let $i \in \{1, 2\}$ and $k \in \{r, s\}$. Fix $N \geq 0$. Since the orthogonal representation $U : \mathbb{R} \curvearrowright H_{\mathbb{R}}^{\mathrm{wm}}$ is weakly mixing and $L \subset H^{\mathrm{wm}}$ is a finite dimensional subspace, we may choose inductively $t_1, \dots, t_N \in \mathbb{R}$ such that $U_{t_{j_1}}(L) \perp_{(N \dim(L))^{-1}} U_{t_{j_2}}(L)$ for all $1 \leq j_1 < j_2 \leq N$. By Claim 3.2, this implies that

$$\kappa_{t_{j_1}}(\mathcal{X}(i,k)) \perp_{1/N} \kappa_{t_{j_2}}(\mathcal{X}(i,k)), \quad \forall 1 \leq j_1 < j_2 \leq N.$$

For all $t \in \mathbb{R}$ and all $n \in \boldsymbol{N}$, we have

$$\begin{aligned} \|P_{\mathcal{X}(i,k)}(z_n\Omega)\|^2 &= \langle P_{\mathcal{X}(i,k)}(z_n\Omega), z_n\Omega\rangle \\ &= \langle \kappa_t(P_{\mathcal{X}(i,k)}(z_n\Omega)), \kappa_t(z_n\Omega)\rangle \qquad (\text{since } \kappa_t \in \mathcal{U}(\mathcal{H})) \\ &= \langle P_{\kappa_t(\mathcal{X}(i,k))}(\kappa_t(z_n\Omega)), \kappa_t(z_n\Omega)\rangle. \end{aligned}$$

By [Ando and Haagerup 2014, Theorem 4.1], for all $t \in \mathbb{R}$, we have $(z_n)^\omega = z = \sigma_t^{\varphi^\omega}(z) = (\sigma_t^\varphi(z_n))^\omega$. This implies that $\lim_{n\to\omega} \|\sigma_t^\varphi(z_n) - z_n\|_\varphi = 0$, and hence $\lim_{n\to\omega} \|\kappa_t(z_n\Omega) - z_n\Omega\| = 0$ for all $t \in \mathbb{R}$. In particular, since the sequence $(z_n\Omega)_n$ is bounded in $\mathcal{H}$, we deduce that for all $t \in \mathbb{R}$,

$$\lim_{n\to\omega} \|P_{\mathcal{X}(i,k)}(z_n\Omega)\|^2 = \lim_{n\to\omega} \langle P_{\kappa_t(\mathcal{X}(i,k))}(z_n\Omega), z_n\Omega\rangle.$$

Applying this equality to our well chosen reals $(t_j)_{1\leq j\leq N}$, taking a convex combination and applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned} \lim_{n\to\omega} \|P_{\mathcal{X}(i,k)}(z_n\Omega)\|^2 &= \lim_{n\to\omega} \frac{1}{N}\sum_{j=1}^N \langle P_{\kappa_{t_j}(\mathcal{X}(i,k))}(z_n\Omega), z_n\Omega\rangle \\ &= \lim_{n\to\omega} \frac{1}{N}\Big\langle \sum_{j=1}^N P_{\kappa_{t_j}(\mathcal{X}(i,k))}(z_n\Omega), z_n\Omega\Big\rangle \\ &\leq \lim_{n\to\omega} \frac{1}{N}\Big\| \sum_{j=1}^N P_{\kappa_{t_j}(\mathcal{X}(i,k))}(z_n\Omega)\Big\| \|z_n\|_\varphi. \end{aligned}$$

Then for all $n \in \boldsymbol{N}$, we have

$$\begin{aligned} \Big\|\sum_{j=1}^N P_{\kappa_{t_j}(\mathcal{X}(i,k))}(z_n\Omega)\Big\|^2 &= \sum_{j_1,j_2=1}^N \big\langle P_{\kappa_{t_{j_1}}(\mathcal{X}(i,k))}(z_n\Omega), P_{\kappa_{t_{j_2}}(\mathcal{X}(i,k))}(z_n\Omega)\big\rangle \\ &\leq \sum_{j=1}^N \|P_{\kappa_{t_j}(\mathcal{X}(i,k))}(z_n\Omega)\|^2 + \sum_{j_1\neq j_2}^N \frac{\|z_n\|_\varphi^2}{N} \\ &\leq N\|z_n\|_\varphi^2 + N^2\frac{\|z_n\|_\varphi^2}{N} \\ &= 2N\|z_n\|_\varphi^2. \end{aligned}$$

Altogether, we have obtained the inequality $\lim_{n\to\omega} \|P_{\mathcal{X}(i,k)}(z_n\Omega)\|^2 \leq \sqrt{2}\|z\|_{\varphi^\omega}^2/\sqrt{N}$. As N is arbitrarily large, this finishes the proof of Claim 3.3. The above argument is inspired by [Wen 2016, Lemma 10]. Alternatively, we could have used [Houdayer 2014a, Proposition 2.3]. ☐

Claim 3.4. *The subspaces* $W(\xi_1 \otimes \cdots \otimes \xi_p)\mathcal{Y}$ *and* $J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))J\mathcal{Y}$ *are orthogonal in* $\mathcal{H}$. *Here, in the case* $q = 0$, *the vector space* $J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))J\mathcal{Y}$ *is nothing but* $\mathcal{Y}$.

Proof of Claim 3.4. Let $m, n \geq 1$ and $e_1, \dots, e_m, f_1, \dots, f_n \in H$ with $e_1, e_m, f_1, f_n \in L^{\perp}$, so that the vectors $e_1 \otimes \cdots \otimes e_m$ and $f_1 \otimes \cdots \otimes f_n$ belong to $\mathcal{Y}$. Since $\bar{\xi}_p \perp e_1$, $\bar{f}_n \perp \eta_1$ and $\xi_1 \perp f_1$, we have

$$\begin{aligned}
&\left\langle W(\xi_1 \otimes \cdots \otimes \xi_p)(e_1 \otimes \cdots \otimes e_m), J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))J(f_1 \otimes \cdots \otimes f_n)\right\rangle \\
&\qquad = \left\langle W(\xi_1 \otimes \cdots \otimes \xi_p)W(e_1 \otimes \cdots \otimes e_m)\Omega, J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))JW(f_1 \otimes \cdots \otimes f_n)\Omega\right\rangle \\
&\qquad = \left\langle W(\xi_1 \otimes \cdots \otimes \xi_p)W(e_1 \otimes \cdots \otimes e_m)\Omega, W(f_1 \otimes \cdots \otimes f_n)W(\eta_1 \otimes \cdots \otimes \eta_q)\Omega\right\rangle \\
&\qquad = \left\langle W(\xi_1 \otimes \cdots \otimes \xi_p \otimes e_1 \otimes \cdots \otimes e_m)\Omega, W(f_1 \otimes \cdots \otimes f_n \otimes \eta_1 \otimes \cdots \otimes \eta_q)\Omega\right\rangle \\
&\qquad = \left\langle \xi_1 \otimes \cdots \otimes \xi_p \otimes e_1 \otimes \cdots \otimes e_m, f_1 \otimes \cdots \otimes f_n \otimes \eta_1 \otimes \cdots \otimes \eta_q\right\rangle \\
&\qquad = 0.
\end{aligned}$$

Note that in the case $q = 0$, the above calculation still makes sense. Indeed, we have

$$\left\langle W(\xi_1 \otimes \cdots \otimes \xi_p)(e_1 \otimes \cdots \otimes e_m), (f_1 \otimes \cdots \otimes f_n)\right\rangle = \left\langle \xi_1 \otimes \cdots \otimes \xi_p \otimes e_1 \otimes \cdots \otimes e_m, f_1 \otimes \cdots \otimes f_n\right\rangle = 0.$$

Since the linear span of all such reduced words $e_1 \otimes \cdots \otimes e_m$ generate $\mathcal{Y}$ (and likewise the span of the words $f_1 \otimes \cdots \otimes f_n$), we obtain that the subspaces $W(\xi_1 \otimes \cdots \otimes \xi_p)\mathcal{Y}$ and $J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))J\mathcal{Y}$ are orthogonal in $\mathcal{H}$. □

Let $x, y \in (M^{\omega})^{\varphi^{\omega}} \cap (M^{\omega} \ominus M)$. We have

$$\begin{aligned}
\varphi^{\omega}(b^* y^* a x) &= \langle a x \xi_{\varphi^{\omega}}, y b \xi_{\varphi^{\omega}}\rangle \\
&= \lim_{n \to \omega} \langle a x_n \xi_{\varphi}, y_n b \xi_{\varphi}\rangle \\
&= \lim_{n \to \omega} \left\langle a' W(\xi_1 \otimes \cdots \otimes \xi_p) a'' x_n \Omega, y_n b' W(\eta_1 \otimes \cdots \otimes \eta_q) b'' \Omega\right\rangle \\
&= \lim_{n \to \omega} \left\langle W(\xi_1 \otimes \cdots \otimes \xi_p) a'' x_n \sigma^{\varphi}_{-\mathrm{i}}((b'')^*)\Omega, J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))J(a')^* y_n b' \Omega\right\rangle.
\end{aligned}$$

Put $z_n = a'' x_n \sigma^{\varphi}_{-\mathrm{i}}((b'')^*)$ and $z'_n = (a')^* y_n b'$. By Claim 3.3, we have that

$$\lim_{n \to \omega} \|P_{\mathcal{X}_i}(z_n \Omega)\| = \lim_{n \to \omega} \|P_{\mathcal{X}_i}(z'_n \Omega)\| = 0, \quad \forall i \in \{1, 2\}.$$

Since moreover $\mathrm{E}_{\omega}(x) = \mathrm{E}_{\omega}(y) = 0$, we see that $\lim_{n \to \omega} \|P_{\mathbb{C}\Omega}(z_n \Omega)\| = \lim_{n \to \omega} \|P_{\mathbb{C}\Omega}(z'_n \Omega)\| = 0$. Since $\mathcal{H} = \mathbb{C}\Omega \oplus \overline{(\mathcal{X}_1 + \mathcal{X}_2)} \oplus \mathcal{Y}$, we obtain

$$\lim_{n \to \omega} \|z_n \Omega - P_{\mathcal{Y}}(z_n \Omega)\| = 0 \quad \text{and} \quad \lim_{n \to \omega} \|z'_n \Omega - P_{\mathcal{Y}}(z'_n \Omega)\| = 0.$$

By Claim 3.4, we finally obtain

$$\begin{aligned}
\varphi^{\omega}(b^* y^* a x) &= \lim_{n \to \omega} \left\langle W(\xi_1 \otimes \cdots \otimes \xi_p) z_n \Omega, J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))J z'_n \Omega\right\rangle \\
&= \lim_{n \to \omega} \left\langle W(\xi_1 \otimes \cdots \otimes \xi_p) P_{\mathcal{Y}}(z_n \Omega), J\sigma^{\varphi}_{-\mathrm{i}/2}(W(\bar{\eta}_q \otimes \cdots \otimes \bar{\eta}_1))J P_{\mathcal{Y}}(z'_n \Omega)\right\rangle = 0.
\end{aligned}$$

This finishes the proof of Theorem 3.1. □

4. Proof of the Main Theorem

We start by proving the following intermediate result.

Theorem 4.1. *Let $(M, \varphi) = (\Gamma(H_{\mathbb{R}}, U)'', \varphi_U)$ be any free Araki–Woods factor endowed with its free quasifree state. Keep the same notation as in the introduction. Let $q \in M^{\varphi} = N^{\varphi_U \text{ap}}$ be any nonzero projection. Write $\varphi_q = \varphi(q \cdot q)/\varphi(q)$. Then for any amenable von Neumann subalgebra $Q \subset qMq$ that is globally invariant under the modular automorphism group σ^{φ_q}, we have $Q \subset qNq$.*

Proof. We may assume that Q has separable predual. Indeed, let $x \in Q$ be any element and denote by $Q_0 \subset Q$ the von Neumann subalgebra generated by $x \in Q$ that is globally invariant under the modular automorphism group σ^{φ_q}. Then Q_0 is amenable and has separable predual. Therefore, we may assume without loss of generality that $Q_0 = Q$, that is, Q has separable predual.

Special case. We first prove the result when $Q \subset qMq$ is globally invariant under σ^{φ_q} and is an irreducible subfactor, meaning that $Q' \cap qMq = \mathbb{C}q$.

Let $a \in Q$ be any element. Since Q is amenable and has separable predual, $Q' \cap (qMq)^{\omega}$ is diffuse and so is $Q' \cap ((qMq)^{\omega})^{\varphi_q^{\omega}}$ by [Houdayer and Raum 2015, Theorem 2.3]. In particular, there exists a unitary $u \in \mathcal{U}\big(Q' \cap ((qMq)^{\omega})^{\varphi_q^{\omega}}\big)$ such that $\varphi_q^{\omega}(u) = 0$. Note that $\mathrm{E}_{\omega}(u) \in Q' \cap qMq = \mathbb{C}q$, and hence $\mathrm{E}_{\omega}(u) = \varphi_q^{\omega}(u) = 0$, so that $u \in (M^{\omega})^{\varphi^{\omega}} \cap (M^{\omega} \ominus M)$. Theorem 3.1 yields $\varphi^{\omega}(a^*u^*(a - \mathrm{E}_N(a))u) = 0$. Since moreover $au = ua$ and $u \in \mathcal{U}\big((qMq)^{\varphi_q^{\omega}}\big)$, we have

$$\begin{aligned}\|a\|_{\varphi}^2 = \|au\|_{\varphi^{\omega}}^2 &= \varphi^{\omega}(u^*a^*au) = \varphi^{\omega}(a^*u^*au)\\ &= \varphi^{\omega}(a^*u^*\,\mathrm{E}_N(a)u) = \varphi^{\omega}(ua^*u^*\,\mathrm{E}_N(a)) = \varphi(a^*\,\mathrm{E}_N(a)) = \|\mathrm{E}_N(a)\|_{\varphi}^2.\end{aligned}$$

This shows that $a = \mathrm{E}_N(a) \in N$.

General case. We next prove the result when $Q \subset qMq$ is any amenable subalgebra globally invariant under σ^{φ_q}.

Denote by $z \in \mathcal{Z}(Q) \subset N^{\varphi}$ the unique central projection such that Qz is atomic and $Q(1-z)$ is diffuse. Since Qz is atomic and globally invariant under the modular automorphism group σ^{φ_z}, we have that $\varphi_z|_{Qz}$ is almost periodic and hence $Qz \subset N$. It remains to prove that $Q(1-z) \subset N$. Cutting down by $1-z$ if necessary, we may assume that Q itself is diffuse.

Since $Q \subset qMq$ is diffuse and with expectation and since M is solid (see [Houdayer and Raum 2015, Theorem A] and [Houdayer and Isono 2016, Theorem 7.1], which does not require separability of the predual), the relative commutant $Q' \cap qMq$ is amenable. Up to replacing Q by $Q \vee Q' \cap qMq$, which is still amenable and globally invariant under the modular automorphism group σ^{φ_q}, we may assume that $Q' \cap qMq = \mathcal{Z}(Q)$. Denote by $(z_n)_n$ a sequence of central projections in $\mathcal{Z}(Q)$ such that $\sum_n z_n = q$, $(Qz_0)' \cap z_0Mz_0 = \mathcal{Z}(Q)z_0$ is diffuse and $(Qz_n)' \cap z_nMz_n = \mathbb{C}z_n$ for every $n \geq 1$.

- By the special case above, we know that $Qz_n \subset N$ for all $n \geq 1$.
- Since $\mathcal{Z}(Q)z_0 \oplus (1-z_0)N(1-z_0)$ is diffuse and with expectation in N, its relative commutant inside M is contained in N by [Houdayer and Ueda 2016, Proposition 2.7(1)]. In particular, $Qz_0 \subset N$.

Therefore, we have $Q \subset N$. □

Proof of the main theorem. Put $\varphi := \varphi_U$. Denote by $z \in \mathcal{Z}(Q) \subset M^\varphi = N^\varphi$ the unique central projection such that Qz is amenable and $Qz^\perp$ has no nonzero amenable direct summand. By Theorem 4.1, we have $Qz \subset zNz$. Fix any nonprincipal ultrafilter $\omega \in \beta(\mathbf{N}) \setminus \mathbf{N}$. Then $(Q' \cap M^\omega)z^\perp = (Q' \cap M)z^\perp$ is atomic, by [Houdayer and Raum 2015, Theorem A] (see also [Houdayer and Isono 2016, Theorem 7.1]). □

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RÉMI BOUTONNET: remi.boutonnet@math.u-bordeaux1.fr
Institut de Mathématiques de Bordeaux, CNRS, Université Bordeaux I, 33405 Talence, France

CYRIL HOUDAYER: cyril.houdayer@math.u-psud.fr
Laboratoire de Mathématiques d'Orsay, Université Paris-Sud, CNRS, 91405 Orsay, France

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