

ANALYSIS & PDE

Volume 10

No. 2

2017

COSMIN BURTEA

**OPTIMAL WELL-POSEDNESS FOR
THE INHOMOGENEOUS INCOMPRESSIBLE NAVIER-STOKES
SYSTEM
WITH GENERAL VISCOSITY**

OPTIMAL WELL-POSEDNESS FOR THE INHOMOGENEOUS INCOMPRESSIBLE NAVIER–STOKES SYSTEM WITH GENERAL VISCOSITY

COSMIN BURTEA

In this paper we obtain new well-posedness results concerning a linear inhomogeneous Stokes-like system. These results are used to establish local well-posedness in the critical spaces for initial density ρ_0 and velocity u_0 such that $\rho_0 - \rho \in \dot{B}_{p,1}^{3/p}(\mathbb{R}^3)$, $u_0 \in \dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3)$, $p \in (\frac{6}{5}, 4)$ for the inhomogeneous incompressible Navier–Stokes system with variable viscosity. To the best of our knowledge, regarding the 3-dimensional case, this is the first result in a truly critical framework for which one does not assume any smallness condition on the density.

1. Introduction

In this paper we deal with the well-posedness of the inhomogeneous, incompressible Navier–Stokes system

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \operatorname{div}(\mu(\rho)D(u)) + \nabla P = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases} \quad (1-1)$$

In the above, $\rho > 0$ stands for the density of the fluid, $u \in \mathbb{R}^n$ is the fluid's velocity field, while P is the pressure. The viscosity coefficient μ is assumed to be a smooth, strictly positive function of the density, while

$$D(u) = \nabla u + Du$$

is the deformation tensor. This system is used to study fluids obtained as a mixture of two (or more) incompressible fluids that have different densities: fluids containing a melted substance, polluted air/water etc.

There is a very rich literature devoted to the study of the well-posedness of (1-1). Briefly, the question of existence of weak solutions with finite energy was first considered by Kažihov [1974] (see also [Antontsev et al. 1990]) in the case of constant viscosity. The case with a general viscosity law was treated in [Lions 1996]. Weak solutions for more regular data were considered in [Desjardins 1997]. Recently, weak solutions were investigated by Huang, Paicu and Zhang in [Huang et al. 2013c].

This work was partially supported by a grant of the Romanian National Authority for Scientific Research and Innovation, CNCS - UEFISCDI, project number PN-II-RU-TE-2014-4-0320.

MSC2010: 35Q30, 76D05.

Keywords: inhomogeneous Navier–Stokes system, critical regularity, Lagrangian coordinates.

The unique solvability of (1-1) was first addressed in the seminal work of Ladyzhenskaya and Solonnikov [1975]. More precisely, considering $u_0 \in W^{2-2/p,p}(\Omega)$, with $p > 2$, a divergence-free vector field that vanishes on $\partial\Omega$ and $\rho_0 \in C^1(\Omega)$ bounded away from zero, they construct a global strong solution in the 2-dimensional case and a local solution in the 3-dimensional case. Moreover, if u_0 is small in $W^{2-2/p,p}(\Omega)$ then global well-posedness holds true.

The question of weak-strong uniqueness was addressed in [Choe and Kim 2003] for the case of sufficiently smooth data with vanishing viscosity.

Over the last thirteen years, efforts were made to obtain well-posedness results in the so-called critical spaces, i.e., the spaces which have the same invariance with respect to time and space dilation as the system itself, namely

$$\begin{cases} (\rho_0(x), u_0(x)) \rightarrow (\rho_0(lx), lu_0(lx)), \\ (\rho(t, x), u(t, x)) \rightarrow (\rho(l^2t, lx), lu(l^2t, lx), l^2P(l^2t, lx)). \end{cases}$$

For more details and explanations for this classical approach we refer to [Danchin 2003] or [Danchin and Mucha 2015]. In the Besov space context, which includes in particular the more classical Sobolev spaces, these are

$$\rho_0 - \bar{\rho} \in \dot{B}_{p_1,r_1}^{n/p_1} \quad \text{and} \quad u_0 \in \dot{B}_{p_2,r_2}^{n/p_2-1}, \tag{1-2}$$

where $\bar{\rho}$ is some constant density state and n is the space dimension. Working with densities close (in some appropriate norm) to a constant has led to a rich literature. In [Danchin 2003] local and global existence results are obtained for the case of constant viscosity and by taking the initial data

$$\rho_0 - \bar{\rho} \in L^\infty \cap \dot{B}_{2,\infty}^{n/2} \quad \text{and} \quad u_0 \in \dot{B}_{2,1}^{n/2-1}$$

and under the assumption that $\|\rho_0 - \bar{\rho}\|_{L^\infty \cap \dot{B}_{2,\infty}^{n/2}}$ is sufficiently small. The case with variable viscosity and for initial data

$$\rho_0 - \bar{\rho} \in \dot{B}_{p,1}^{n/p} \quad \text{and} \quad u_0 \in \dot{B}_{p,1}^{n/p-1},$$

$p \in [1, 2n)$, is treated in [Abidi 2007]. However, uniqueness is guaranteed once $p \in [1, n)$. These results were further extended by H. Abidi and M. Paicu [2007] by noticing that $\rho_0 - \bar{\rho}$ can be taken in a larger Besov space. B. Haspot [2012] established results in the same spirit as those mentioned above (however, the results are obtained in the nonhomogeneous framework and thus do not fall into the critical framework) in the case where the velocity field is not Lipschitz. Using the Lagrangian formulation, R. Danchin and P. B. Mucha [2012], established local and global results for (1-1) with constant viscosity when $\rho_0 - \bar{\rho} \in \mathcal{M}(\dot{B}_{p,1}^{n/p-1})$, $u_0 \in \dot{B}_{p,1}^{n/p-1}$ and under the smallness condition

$$\|\rho_0 - \bar{\rho}\|_{\mathcal{M}(\dot{B}_{p,1}^{n/p-1})} \ll 1,$$

where $\mathcal{M}(\dot{B}_{p,1}^{n/p-1})$ stands for the multiplier space of $\dot{B}_{p,1}^{n/p-1}$. In particular, functions with small jumps enter this framework. Moreover, as a consequence of their approach, the range of Lebesgue exponents for which uniqueness of solutions holds is extended to $p \in [1, 2n)$. In [Paicu and Zhang 2012; Huang et al. 2013a; 2013b; 2013c] the authors improve the smallness assumptions used in order to obtain global

existence. To summarize, all the previous well-posedness results in critical spaces were established assuming the density is close in some sense to a constant state.

When the latter assumption is removed, one must impose more regularity on the data. For the case of constant viscosity, R. Danchin [2004] obtained local well-posedness and global well-posedness in dimension $n = 2$ for data drawn from the nonhomogeneous Sobolev spaces:

$$(\rho_0 - \bar{\rho}, u_0) \in H^{n/2+\alpha} \times H^{n/2-1+\beta}$$

with $\alpha, \beta > 0$. The same result for the case of the general viscosity law is established in [Abidi 2007]. For data with non-Lipschitz velocity results were established in [Haspot 2012]. Concerning rougher densities, considering $\rho_0 \in L^\infty(\mathbb{R}^d)$ bounded from below and $u_0 \in H^2(\mathbb{R}^d)$, Danchin and Mucha [2013] constructed a unique local solution. Again, supposing the density is close to some constant state, they proved global well-posedness. These results are generalized in [Paicu et al. 2013]. Taking the density as above, the authors construct a global unique solution provided that $u_0 \in H^s(\mathbb{R}^2)$ for any $s > 0$ in the 2-dimensional case and a local unique solution in the 3-dimensional case considering $u_0 \in H^1(\mathbb{R}^3)$. Moreover, assuming u_0 is suitably small, the solution constructed is global even in the 3-dimensional case.

In critical spaces of the Navier–Stokes system, i.e., (1-2) there are few well-posedness results. Very recently, in the 2-dimensional case and allowing variable viscosity, H. Xu, Y. Li and X. Zhai [2016] constructed a unique local solution to (1-1) provided that the initial data satisfy $\rho_0 - \bar{\rho} \in \dot{B}_{p,1}^{2/p}(\mathbb{R}^2)$ and $u_0 \in \dot{B}_{p,1}^{2/p-1}(\mathbb{R}^2)$. Moreover, if $\rho_0 - \bar{\rho} \in L^p \cap \dot{B}_{p,1}^{2/p}(\mathbb{R}^2)$ and the viscosity is supposed constant, their solution becomes global. In the 3-dimensional situation, to the best of our knowledge, the results that are closest to the critical regularity are those presented in [Abidi et al. 2012; 2013] (for a similar result in the periodic case one can consult [Poulon 2015]). More precisely, in three dimensions, assuming

$$\rho_0 - \bar{\rho} \in L^2 \cap \dot{B}_{2,1}^{3/2} \quad \text{and} \quad u_0 \in \dot{B}_{2,1}^{1/2}$$

and taking constant viscosity, H. Abidi, G. Gui and P. Zhang [Abidi et al. 2012] show the local well-posedness of system (1-1). Moreover, if the initial velocity is small then global well-posedness holds true. In [Abidi et al. 2013] they establish the same kind of result for initial data

$$\rho_0 - \bar{\rho} \in L^\lambda \cap \dot{B}_{\lambda,1}^{3/\lambda} \quad \text{and} \quad u_0 \in \dot{B}_{p,1}^{3/p-1},$$

where $\lambda \in [1, 2]$, $p \in [3, 4]$ are such that $\frac{1}{\lambda} + \frac{1}{p} > \frac{5}{6}$ and $\frac{1}{\lambda} - \frac{1}{p} \leq \frac{1}{3}$.

One of the goals of the present paper is to establish local well-posedness in the critical spaces

$$\rho_0 - \bar{\rho} \in \dot{B}_{p,1}^{3/p}(\mathbb{R}^3), \quad u_0 \in \dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3), \quad p \in \left(\frac{6}{5}, 4\right)$$

for system (1-1)

- with general smooth variable viscosity law,
- without any smallness assumption on the density,
- without any extra low frequencies assumption. In particular, we generalize the local existence and uniqueness result of [Abidi et al. 2012], thus achieving the critical regularity.

As in [Danchin and Mucha 2012], we will not work directly with system (1-1); instead we will use its Lagrangian formulation. By proceeding so, we are naturally led to consider the following Stokes problem with time-independent, nonconstant coefficients:

$$\begin{cases} \partial_t u - a \operatorname{div}(bD(u)) + a \nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u|_{t=0} = u_0. \end{cases} \tag{1-3}$$

We establish global well-posedness results for system (1-3). This can be viewed as a first step towards generalizing the results of Danchin and Mucha [2015, Chapter 4] for the case of general viscosity and without assuming the density is close to a constant state. Let us mention that the estimates we obtain for system (1-3) have a wider range of applications: in a forthcoming paper we will investigate the well-posedness issue of the Navier–Stokes–Korteweg system under optimal regularity assumptions.

To summarize all the above, our main result reads:

Theorem 1.1. *Consider $p \in (\frac{6}{5}, 4)$. Assume that there exist positive constants $(\bar{\rho}, \rho_\star, \rho^\star)$ such that $\rho_0 - \bar{\rho} \in \dot{B}_{p,1}^{3/p}(\mathbb{R}^3)$ and $0 < \rho_\star < \rho_0 < \rho^\star$. Furthermore, consider u_0 a divergence-free vector field with coefficients in $\dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3)$. Then, there exists a time $T > 0$ and a unique solution $(\rho, u, \nabla P)$ of system (1-1) with*

$$\rho - \bar{\rho} \in C_T(\dot{B}_{p,1}^{3/p}(\mathbb{R}^3)) \cap L_T^\infty(\dot{B}_{p,1}^{3/p}(\mathbb{R}^3)), \quad u \in C_T(\dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3)), \quad (\partial_t u, \nabla^2 u, \nabla P) \in L_T^1(\dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3)).$$

One salutary feature of the Lagrangian formulation is that the density becomes independent of time. More precisely, considering $(\rho, u, \nabla P)$ a solution of (1-1) and denoting by X the flow associated to the vector field u ,

$$X(t, y) = y + \int_0^t u(\tau, X(\tau, y)) \, d\tau.$$

We introduce the new Lagrangian variables

$$\bar{\rho}(t, y) = \rho(t, X(t, y)), \quad \bar{u}(t, y) = u(t, X(t, y)) \quad \text{and} \quad \bar{P}(t, y) = P(t, X(t, y)).$$

Then, using the chain rule and Proposition 3.23 we gather that $\bar{\rho}(t, \cdot) = \rho_0$ and

$$\begin{cases} \rho_0 \partial_t \bar{u} - \operatorname{div}(\mu(\rho_0) A_{\bar{u}} D_{A_{\bar{u}}}(\bar{u})) + A_{\bar{u}}^T \nabla \bar{P} = 0, \\ \operatorname{div}(A_{\bar{u}} \bar{u}) = 0, \\ \bar{u}|_{t=0} = u_0, \end{cases} \tag{1-4}$$

where $A_{\bar{u}}$ is the inverse of the differential of X , and

$$D_A(\bar{u}) = D\bar{u} A_{\bar{u}} + A_{\bar{u}}^T \nabla \bar{u}.$$

Note that we can give a meaning to (1-4) independently of the Eulerian formulation by stating

$$X(t, y) = y + \int_0^t \bar{u}(\tau, y) \, d\tau.$$

Theorem 1.1 will be a consequence of the following result:

Theorem 1.2. Consider $p \in (\frac{6}{5}, 4)$. Assume there exists positive $(\bar{\rho}, \rho_\star, \rho^\star)$ such that $\rho_0 - \bar{\rho} \in \dot{B}_{p,1}^{3/p}(\mathbb{R}^3)$ and $0 < \rho_\star < \rho_0 < \rho^\star$. Furthermore, consider u_0 a divergence-free vector field with coefficients in $\dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3)$. Then, there exists a time $T > 0$ and a unique solution $(\bar{u}, \nabla \bar{P})$ of system (1-4) with

$$\bar{u} \in C_T(\dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3)) \quad \text{and} \quad (\partial_t \bar{u}, \nabla^2 \bar{u}, \nabla \bar{P}) \in L_T^1(\dot{B}_{p,1}^{3/p-1}(\mathbb{R}^3)).$$

Moreover, there exists a positive constant $C = C(\rho_0)$ such that

$$\|u\|_{L_T^\infty(\dot{B}_{p,1}^{3/p-1})} + \|(\nabla^2 u, \nabla P)\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \leq \|u_0\|_{\dot{B}_{p,1}^{3/p-1}} \exp(C(T+1)).$$

The study of system (1-4) naturally leads to the Stokes-like system (1-3). In Section 2 we establish the global well-posedness of system (1-3). More precisely, we prove:

Theorem 1.3. Consider $n \in \{2, 3\}$ and $p \in (1, 4)$ if $n = 2$ or $p \in (\frac{6}{5}, 4)$ if $n = 3$. Assume there exist positive constants $(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b})$ such that $a - \bar{a} \in \dot{B}_{p,1}^{n/p}(\mathbb{R}^n)$, $b - \bar{b} \in \dot{B}_{p,1}^{n/p}(\mathbb{R}^n)$ and

$$0 < a_\star \leq a \leq a^\star, \quad 0 < b_\star \leq b \leq b^\star.$$

Furthermore, consider the vector fields u_0 and f with coefficients in $\dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n)$ and $L_{\text{loc}}^1(\dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n))$ respectively. Also, consider the vector field $R \in (S'(\mathbb{R}^n))^n$ with¹

$$\mathcal{Q}R \in C([0, \infty); \dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n)) \quad \text{and} \quad (\partial_t R, \nabla \operatorname{div} R) \in L_{\text{loc}}^1(\dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n))$$

such that

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, system (1-3) has a unique global solution $(u, \nabla P)$ with

$$u \in C([0, \infty), \dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n)) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_{\text{loc}}^1(\dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n)).$$

Moreover, there exists a constant $C = C(a, b)$ such that

$$\begin{aligned} \|u\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + \|(\partial_t u, \nabla^2 u, \nabla P)\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \\ \leq (\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})}) \exp(C(t+1)) \end{aligned} \quad (1-5)$$

for all $t \in [0, \infty)$.

The difficulty in establishing such a result comes from the fact that the pressure and velocity are “strongly” coupled as opposed to the case where ρ is close to a constant; see Remark 2.11 below. The key idea is to use the high-low frequency splitting technique first introduced in [Danchin 2007] combined with the particular structure of the divergence-free part of $a\nabla P$, i.e.,

$$\begin{aligned} \mathcal{P}(a\nabla P) &= \mathcal{P}((a - \bar{a})\nabla P) = \mathcal{P}((a - \bar{a})\nabla P) - (a - \bar{a})\mathcal{P}(\nabla P) \\ &:= [\mathcal{P}, a - \bar{a}]\nabla P, \end{aligned}$$

¹ \mathcal{P} is the Leray projector over divergence-free vector fields, $\mathcal{Q} = \operatorname{Id} - \mathcal{P}$.

which is, loosely speaking, more regular than ∇P . Let us mention that a similar principle holds for u , which is divergence free:² whenever we estimate some term of the form $\mathcal{Q}(bM(D)u)$, where b lies in an appropriate Besov space and $M(D)$ is some pseudodifferential operator, we may write it as

$$\mathcal{Q}(bM(D)u) = [\mathcal{Q}, b]M(D)u$$

and use the fact that the latter expression is more regular than $M(D)u$; see [Proposition 3.21](#).

The proof of [Theorem 1.3](#) in the 3-dimensional case is more subtle. Loosely speaking, in order to close the estimates for system (1-3) one should work in a space on which the solution operator corresponding to the elliptic equation $\operatorname{div}(a\nabla P) = \operatorname{div} f$ is continuous. It is for this reason that we first prove a more restrictive result by demanding extra low-frequency information on the initial data. Then, using a perturbative version of Danchin and Mucha’s results [2015] we arrive at constructing a solution with the optimal regularity. Uniqueness is obtained by a duality method.

Once the estimates of [Theorem 1.3](#) are established, we proceed with the proof of [Theorem 1.2](#), which is the object of [Section 3](#). Finally, we show the equivalence between system (1-4) and system (1-1) thus achieving the proof of [Theorem 1.1](#). We end this paper with an [Appendix](#) where results of Littlewood–Paley theory used through the text are gathered.

We end this section with some observations regarding the global existence issue. As opposed to the case when ρ is supposed to be a small perturbation of a constant state, when considering the linearized system of the Lagrangian formulation, i.e., system (1-3), we obtain the estimates (1-5), which have a time-dependent right-hand side term. This in particular prevents us from adapting the arguments from [Danchin and Mucha 2012] to our situation and obtaining a global solution for system (1-4) and consequently for the system (1-1). In fact, even if we were able to construct such a solution for system (1-4), it is not clear how we could go back into the original formulation as passing from the Eulerian formulation to the Lagrangian one needs some smallness condition on the $\|\cdot\|_{L^1_t(\dot{B}^{n/p}_{p,1})}$ -norm of the velocity.

2. The Stokes system with nonconstant coefficients

Pressure estimates. Before handling system (1-3) we shall study the elliptic equation

$$\operatorname{div}(a\nabla P) = \operatorname{div} f. \tag{2-1}$$

For the reader’s convenience let us cite the following classical result, a proof of which can be found, for instance, in [Danchin 2010]:

Proposition 2.1. *Consider $a \in L^\infty(\mathbb{R}^n)$ and a constant a_\star such that*

$$a \geq a_\star > 0.$$

For all vector fields f with coefficients in $L^2(\mathbb{R}^n)$, there exists a tempered distribution P unique up to constant functions such that $\nabla P \in L^2(\mathbb{R}^n)$ and equation (2-1) is satisfied. In addition, we have

$$a_\star \|\nabla P\|_{L^2} \leq \|\mathcal{Q}f\|_{L^2}.$$

²and thus $\mathcal{Q}u = 0$.

Recently, regarding the 2-dimensional case, Xu et al. [2016], studied the elliptic equation (2-1) with the data $(a - \bar{a}, f)$ in Besov spaces. Using a different approach, we obtain estimates in both 2-dimensional and 3-dimensional situations. Let us also mention that our method allows to obtain a wider range of indices than the one of [Xu et al. 2016, Proposition 3.1(i)]. We choose to focus on the 3-dimensional case. We aim at establishing the following result:

Proposition 2.2. *Consider $p \in (\frac{6}{5}, 2)$ and $q \in [1, \infty)$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$. Assume there exist positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}_{q,1}^{3/q}(\mathbb{R}^3)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}_{p,2}^{3/p-3/2}(\mathbb{R}^3)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P \in \dot{B}_{p,2}^{3/p-3/2}(\mathbb{R}^3)$ and equation (2-1) is satisfied. Moreover, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim \left(\frac{1}{\bar{a}} + \left\| \frac{1}{a} - \frac{1}{\bar{a}} \right\|_{\dot{B}_{q,1}^{3/q}} \right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}} \right) \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{3/p-3/2}}. \quad (2-2)$$

Remark 2.3. Working in Besov spaces with third index $r = 2$ is enough in view of the applications that we have in mind. However, similar estimates do hold true when the third index is chosen in the interval $[1, 2]$.

Proof. Because $p < 2$, Proposition 3.7 ensures that $\dot{B}_{p,2}^{3/p-3/2} \hookrightarrow L^2 = \dot{B}_{2,2}^0$ and owing to Proposition 2.1, we get the existence of $P \in \mathcal{S}'(\mathbb{R}^3)$ with $\nabla P \in L^2$ and

$$a_\star \|\nabla P\|_{L^2} \leq \|\mathcal{Q}f\|_{L^2}. \quad (2-3)$$

Moreover, as \mathcal{Q} is a continuous operator on L^2 , we deduce from (2-1) that

$$\mathcal{Q}(a\nabla P) = \mathcal{Q}f. \quad (2-4)$$

Using the Bony decomposition (see Definition 3.14 and the remark that follows) and the fact that $\mathcal{P}(\nabla P) = 0$, we write

$$\mathcal{P}(a\nabla P) = \mathcal{P}(\dot{T}'_{\nabla P}(a - \bar{a})) + [\mathcal{P}, \dot{T}'_{a-\bar{a}}]\nabla P.$$

Using Proposition 3.16 along with Proposition 3.7 and relation (2-3), we get

$$\|\mathcal{P}(\dot{T}'_{\nabla P}(a - \bar{a}))\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim \|\nabla P\|_{L^2} \|a - \bar{a}\|_{\dot{B}_{p^\star,2}^{3/p-3/2}} \lesssim \frac{1}{a_\star} \|\mathcal{Q}f\|_{L^2} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}}, \quad (2-5)$$

where

$$\frac{1}{p} = \frac{1}{2} + \frac{1}{p^\star}.$$

Next, proceeding as in Proposition 3.20 we get

$$\|[\mathcal{P}, \dot{T}'_{a-\bar{a}}]\nabla P\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim \|\nabla a\|_{\dot{B}_{p^\star,2}^{3/p-5/2}} \|\nabla P\|_{L^2} \lesssim \frac{1}{a_\star} \|\mathcal{Q}f\|_{L^2} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}}. \quad (2-6)$$

Putting together relations (2-5) and (2-6) we get

$$\|\mathcal{P}(a\nabla P)\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim \frac{1}{a_\star} \|\mathcal{Q}f\|_{L^2} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}}.$$

Combining this with (2-4) and Proposition 3.7, we find

$$\|a\nabla P\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}}\right) \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{3/p-3/2}}.$$

Of course, writing

$$\nabla P = \frac{1}{a} a \nabla P,$$

using product rules one gets

$$\|\nabla P\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim \left(\frac{1}{\bar{a}} + \left\|\frac{1}{a} - \frac{1}{\bar{a}}\right\|_{\dot{B}_{q,1}^{3/q}}\right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}}\right) \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{3/p-3/2}} \tag{2-7}$$

This concludes the proof. □

Applying the same technique as above leads to the 2-dimensional estimate:

Proposition 2.4. *Consider $p \in (1, 2)$ and $q \in [1, \infty)$ such that $\frac{1}{p} - \frac{1}{q} \leq \frac{1}{2}$. Assume there exists positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}_{q,1}^{2/q}(\mathbb{R}^2)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}_{p,2}^{2/p-1}(\mathbb{R}^2)$. Then there exists a tempered distribution P unique up to constant functions such that $\nabla P \in \dot{B}_{p,2}^{2/p-1}(\mathbb{R}^2)$ and equation (2-1) is satisfied. Moreover, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}_{p,2}^{2/p-1}} \lesssim \left(\frac{1}{\bar{a}} + \left\|\frac{1}{a} - \frac{1}{\bar{a}}\right\|_{\dot{B}_{q,1}^{2/q}}\right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{2/q}}\right) \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{2/p-1}}. \tag{2-8}$$

Let us point out that the restriction $p > \frac{6}{5}$ comes from the fact that we need $\frac{3}{p} - \frac{5}{2} < 0$ in relation (2-6). In two dimensions, instead of $\frac{3}{p} - \frac{5}{2}$ we will have $\frac{2}{p} - 2$, which is negative provided $p > 1$.

The next result covers the range of integrability indices larger than 2:

Proposition 2.5. *Consider $p \in (2, 6)$ and $q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$. Assume there exist positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}_{q,1}^{3/q}(\mathbb{R}^3)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}_{p,2}^{3/p-3/2}(\mathbb{R}^3)$ and a tempered distribution P with $\nabla P \in \dot{B}_{p,2}^{3/p-3/2}(\mathbb{R}^3)$ such that equation (2-1) is satisfied. Then, the following estimate holds true:*

$$\|\nabla P\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim \left(\frac{1}{\bar{a}} + \left\|\frac{1}{a} - \frac{1}{\bar{a}}\right\|_{\dot{B}_{q,1}^{3/q}}\right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}}\right) \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{3/p-3/2}}. \tag{2-9}$$

Proof. Notice that p' , the conjugate Lebesgue exponent of p , satisfies $p' \in (\frac{6}{5}, 2)$ and $\frac{1}{p'} - \frac{1}{q} \leq \frac{1}{2}$. Thus, by Proposition 2.2, for any g belonging to the unit ball of $\mathcal{S} \cap \dot{B}_{p',2}^{3/p'-3/2}$ there exists a $P_g \in \mathcal{S}'(\mathbb{R}^3)$ with $\nabla P_g \in \mathcal{S} \cap \dot{B}_{p',2}^{3/p'-3/2}$ such that

$$\operatorname{div}(a\nabla P_g) = \operatorname{div} g$$

and

$$\|\nabla P_g\|_{\dot{B}_{p',2}^{3/p'-3/2}} \lesssim \left(\frac{1}{\bar{a}} + \left\|\frac{1}{a} - \frac{1}{\bar{a}}\right\|_{\dot{B}_{q,1}^{3/q}}\right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}}\right).$$

We write

$$\begin{aligned} \langle \nabla P, g \rangle &= -\langle P, \operatorname{div} g \rangle = -\langle P, \operatorname{div}(a\nabla P_g) \rangle \\ &= -\langle \operatorname{div} \mathcal{Q}f, P_g \rangle = \langle \mathcal{Q}f, \nabla P_g \rangle, \end{aligned}$$

and consequently

$$\begin{aligned} |\langle \nabla P, g \rangle| &\lesssim \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{3/p-3/2}} \|\nabla P g\|_{\dot{B}_{p',2}^{3/p'-3/2}} \\ &\lesssim \left(\frac{1}{\bar{a}} + \left\| \frac{1}{a} - \frac{1}{\bar{a}} \right\|_{\dot{B}_{q,1}^{3/q}} \right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{3/q}} \right) \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{3/p-3/2}}. \end{aligned}$$

Using [Proposition 3.8](#), we get that relation (2-9) holds true. \square

As in the previous situation, by applying the same technique we get a similar result in two dimensions:

Proposition 2.6. *Consider $p \in (2, \infty)$ and $q \in [1, \infty)$ such that $\frac{1}{p} + \frac{1}{q} \geq \frac{1}{2}$. Assume there exist positive constants $(\bar{a}, a_\star, a^\star)$ such that $a - \bar{a} \in \dot{B}_{q,1}^{2/q}(\mathbb{R}^2)$ and $0 < a_\star \leq a \leq a^\star$. Furthermore, consider $f \in \dot{B}_{p,2}^{2/p-1}(\mathbb{R}^2)$ and a tempered distribution P with $\nabla P \in \dot{B}_{p,2}^{2/p-1}(\mathbb{R}^2)$ such that equation (2-1) is satisfied. Then, following estimate holds true:*

$$\|\nabla P\|_{\dot{B}_{p,2}^{2/p-1}} \lesssim \left(\frac{1}{\bar{a}} + \left\| \frac{1}{a} - \frac{1}{\bar{a}} \right\|_{\dot{B}_{q,1}^{2/q}} \right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{q,1}^{2/q}} \right) \|\mathcal{Q}f\|_{\dot{B}_{p,2}^{2/p-1}}. \quad (2-10)$$

Some preliminary results. In this section we derive estimates for a Stokes-like problem with time-independent, nonconstant coefficients. Before proceeding to the actual proof, for the reader's convenience, let us cite the following results which were established by Danchin and Mucha [\[2009; 2015\]](#). These results correspond to the case where a and b are constants:

Proposition 2.7. *Consider $u_0 \in \dot{B}_{p,1}^{n/p-1}$ and $(f, \partial_t R, \nabla \operatorname{div} R) \in L_T^1(\dot{B}_{p,1}^{n/p-1})$ with $\mathcal{Q}R \in C_T(\dot{B}_{p,1}^{n/p-1})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, the system

$$\begin{cases} \partial_t u - \bar{a}\bar{b}\Delta u + \bar{a}\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u|_{t=0} = u_0 \end{cases}$$

has a unique solution $(u, \nabla P)$ with

$$u \in \mathcal{C}([0, T]; \dot{B}_{p,1}^{n/p-1}) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_T^1(\dot{B}_{p,1}^{n/p-1})$$

and the following estimate is valid:

$$\|u\|_{L_T^\infty(\dot{B}_{p,1}^{n/p-1})} + \|(\partial_t u, \bar{a}\bar{b}\nabla^2 u, \bar{a}\nabla P)\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})} \lesssim \|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \bar{a}\bar{b}\nabla \operatorname{div} R)\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})}.$$

As a consequence of the previous result, one can establish via a perturbation argument:

Proposition 2.8. *Consider $u_0 \in \dot{B}_{p,1}^{n/p-1}$ and $(f, \partial_t R, \nabla \operatorname{div} R) \in L_T^1(\dot{B}_{p,1}^{n/p-1})$ with $\mathcal{Q}R \in C_T(\dot{B}_{p,1}^{n/p-1})$ such that*

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, there exists an $\eta = \eta(\bar{a})$ small enough such that for all $c \in \dot{B}_{p,1}^{n/p}$ with

$$\|c\|_{\dot{B}_{p,1}^{n/p}} \leq \eta,$$

the system

$$\begin{cases} \partial_t u - \bar{a}\bar{b}\Delta u + (\bar{a} + c)\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u|_{t=0} = u_0 \end{cases}$$

has a unique solution $(u, \nabla P)$ with

$$u \in \mathcal{C}([0, T]; \dot{B}_{p,1}^{n/p-1}) \quad \text{and} \quad \partial_t u, \nabla^2 u, \nabla P \in L_T^1(\dot{B}_{p,1}^{n/p-1})$$

and the following estimate is valid:

$$\|u\|_{L_T^\infty(\dot{B}_{p,1}^{n/p-1})} + \|(\partial_t u, \bar{a}\bar{b}\nabla^2 u, \bar{a}\nabla P)\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})} \lesssim \|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \bar{a}\bar{b}\nabla \operatorname{div} R)\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})}.$$

In all that follows we denote by E_{loc} the space of $(u, \nabla P)$ such that

$$u \in \mathcal{C}([0, \infty); \dot{B}_{p,1}^{n/p-1}) \quad \text{and} \quad (\nabla^2 u, \nabla P) \in L_{\text{loc}}^1(\dot{B}_{p,1}^{n/p-1}) \times L_{\text{loc}}^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1}).$$

Additionally, we introduce the space E_T of $u \in \mathcal{C}_T(\dot{B}_{p,1}^{n/p-1})$ with $\nabla^2 u \in L_T^1(\dot{B}_{p,1}^{n/p-1})$ and $\nabla P \in L_T^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})$ such that

$$\|(u, \nabla P)\|_{E_T} = \|u\|_{L_T^\infty(\dot{B}_{p,1}^{n/p-1})} + \|\nabla^2 u\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})} + \|\nabla P\|_{L_T^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})} < \infty.$$

The first ingredient in proving [Theorem 1.3](#) is the following:

Proposition 2.9. Consider $n \in \{2, 3\}$ and $p \in (1, 4)$ if $n = 2$ or $p \in (\frac{6}{5}, 4)$ if $n = 3$. Assume there exist positive constants $(a_\star, b_\star, a^\star, b^\star, \bar{a}, \bar{b})$ such that $a - \bar{a} \in \dot{B}_{p,1}^{n/p}(\mathbb{R}^n)$, $b - \bar{b} \in \dot{B}_{p,1}^{n/p}(\mathbb{R}^n)$ and

$$0 < a_\star \leq a \leq a^\star, \quad 0 < b_\star \leq b \leq b^\star.$$

Furthermore, consider u_0, f vector fields with coefficients in $\dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n)$ and $L_{\text{loc}}^1(\dot{B}_{p,2}^{n/p-n/2}(\mathbb{R}^n) \cap \dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n))$ respectively and a vector field $R \in (S'(\mathbb{R}^n))^n$ with

$$(\partial_t R, \nabla \operatorname{div} R) \in L_{\text{loc}}^1(\dot{B}_{p,2}^{n/p-n/2}(\mathbb{R}^n) \cap \dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n)) \quad \text{and} \quad \mathcal{Q}R \in \mathcal{C}([0, \infty); \dot{B}_{p,1}^{n/p-1}(\mathbb{R}^n))$$

such that

$$\operatorname{div} u_0 = \operatorname{div} R(0, \cdot).$$

Then, there exists a constant C_{ab} depending on a and b such that any solution $(u, \nabla P) \in E_T$ of the Stokes system (1-3) will satisfy

$$\begin{aligned} \|u\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + \|\nabla^2 u\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} + \|\nabla P\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})} \\ \leq (\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(C_{ab}(t + 1)) \end{aligned} \quad (2-11)$$

for all $t \in (0, T]$.

Before proceeding with the proof, a few remarks are in order:

Remark 2.10. Proposition 2.9 is different from Theorem 1.3 when $n = 3$. Indeed, in the 3-dimensional case the theory is more subtle and thus, as a first step we construct a unique solution for the case of more regular initial data.

Remark 2.11. The difficulty when dealing with the Stokes system with nonconstant coefficients lies in the fact that the pressure and the velocity u are coupled. Indeed, in the constant coefficients case, in view of

$$\operatorname{div} u = \operatorname{div} R,$$

one can apply the divergence operator in the first equation of (1-3) in order to obtain the following elliptic equation verified by the pressure:

$$a\Delta P = \operatorname{div}(f - \partial_t R + 2ab\nabla \operatorname{div} R). \tag{2-12}$$

From (2-12) we can construct the pressure. Having built the pressure, the velocity satisfies a classical heat equation. In the nonconstant coefficient case, proceeding as above we find that

$$\operatorname{div}(a\nabla P) = \operatorname{div}(f - \partial_t R + a \operatorname{div}(bD(u))). \tag{2-13}$$

Therefore the strategy used in the previous case is not well-adapted. We will establish a priori estimates and use a continuity argument like in [Danchin 2014]. In order to be able to close the estimates on u , we have to bound $\|a\nabla P\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})}$ in terms of

$$\|u\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})}^\beta \|\nabla^2 u\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})}^{1-\beta}$$

for some $\beta \in (0, 1)$. Thus, the difficulty is to find estimates for the pressure which do not feature the time derivative of the velocity.

In view of Proposition 2.7, consider $(u_L, \nabla P_L)$, the unique solution of the system

$$\begin{cases} \partial_t u - \bar{a} \operatorname{div}(\bar{b}D(u)) + \bar{a}\nabla P = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u|_{t=0} = u_0, \end{cases} \tag{2-14}$$

with

$$u_L \in \mathcal{C}([0, \infty); \dot{B}_{p,1}^{n/p-1}) \quad \text{and} \quad (\partial_t u_L, \nabla^2 u_L, \nabla P_L) \in L_{\text{loc}}^1(\dot{B}_{p,1}^{n/p-1}).$$

Recall that for any $t \in [0, \infty)$ we have

$$\begin{aligned} \|u_L\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + \|(\partial_t u_L, \bar{a}\bar{b}\nabla^2 u_L, \bar{a}\nabla P_L)\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \\ \leq C (\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \bar{a}\bar{b}\nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})}). \end{aligned} \tag{2-15}$$

In what follows, we will use the notation

$$\tilde{u} = u - u_L, \quad \nabla \tilde{P} = \nabla P - \nabla P_L. \tag{2-16}$$

Obviously, we have

$$\operatorname{div} \tilde{u} = 0. \tag{2-17}$$

Thus, the system (1-3) is recast into

$$\begin{cases} \partial_t \tilde{u} - a \operatorname{div}(bD(\tilde{u})) + a \nabla \tilde{P} = \tilde{f}, \\ \operatorname{div} \tilde{u} = 0, \\ \tilde{u}|_{t=0} = 0, \end{cases} \quad (2-18)$$

where

$$\tilde{f} = a \operatorname{div}(bD(u_L)) - \bar{a} \operatorname{div}(\bar{b}D(u_L)) - (a - \bar{a}) \nabla P_L.$$

Using the last equality along with Proposition 3.17, we infer

$$\begin{aligned} \|\tilde{f}\|_{\dot{B}_{p,1}^{n/p-1}} &\leq \|a \operatorname{div}(bD(u_L)) - \bar{a} \operatorname{div}(\bar{b}D(u_L))\|_{\dot{B}_{p,1}^{n/p-1}} + \|(a - \bar{a}) \nabla P_L\|_{\dot{B}_{p,1}^{n/p-1}} \\ &\lesssim (\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{n/p}}) (\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{n/p}}) \|\nabla u_L\|_{\dot{B}_{p,1}^{n/p}} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{n/p}} \|\nabla P_L\|_{\dot{B}_{p,1}^{n/p-1}}. \end{aligned} \quad (2-19)$$

Let us estimate the pressure $a \nabla \tilde{P}$. First, we write

$$\|a \nabla \tilde{P}\|_{\dot{B}_{p,1}^{n/p-1}} \leq \|\mathcal{Q}(a \nabla \tilde{P})\|_{\dot{B}_{p,1}^{n/p-1}} + \|\mathcal{P}(a \nabla \tilde{P})\|_{\dot{B}_{p,1}^{n/p-1}}.$$

Applying the \mathcal{Q} operator in the first equation of (2-18) we get

$$\mathcal{Q}(a \nabla \tilde{P}) = \mathcal{Q}\tilde{f} + \mathcal{Q}(a \operatorname{div}(bD(\tilde{u}))).$$

Thus, we get

$$\|\mathcal{Q}(a \nabla \tilde{P})\|_{\dot{B}_{p,1}^{n/p-1}} \leq \|\mathcal{Q}\tilde{f}\|_{\dot{B}_{p,1}^{n/p-1}} + \|\mathcal{Q}(a \operatorname{div}(bD(\tilde{u})))\|_{\dot{B}_{p,1}^{n/p-1}}. \quad (2-20)$$

Let

$$\mathcal{Q}(a \operatorname{div}(bD(\tilde{u}))) = \mathcal{Q}(D(\tilde{u})\dot{S}_m(a \nabla b)) + \mathcal{Q}(\dot{S}_m(ab - \bar{a}\bar{b})\Delta \tilde{u}) \quad (2-21)$$

$$+ \mathcal{Q}(D(\tilde{u})(\operatorname{Id} - \dot{S}_m)(a \nabla b)) \quad (2-22)$$

$$+ \mathcal{Q}((\operatorname{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\Delta \tilde{u}), \quad (2-23)$$

where $m \in \mathbb{N}$ will be chosen later. According to Proposition 3.17 we have

$$\|\mathcal{Q}(D(\tilde{u})\dot{S}_m(a \nabla b))\|_{\dot{B}_{p,1}^{n/p-1}} \lesssim \|\dot{S}_m(a \nabla b)\|_{\dot{B}_{p,1}^{n/p-1/2}} \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{n/p-1/2}}. \quad (2-24)$$

Owing to the fact that \tilde{u} is divergence free we can write

$$\mathcal{Q}(\dot{S}_m(ab - \bar{a}\bar{b})\Delta \tilde{u}) = [\mathcal{Q}, \dot{S}_m(ab - \bar{a}\bar{b})]\Delta \tilde{u}, \quad (2-25)$$

such that applying Proposition 3.21 we get

$$\begin{aligned} \|\mathcal{Q}(\dot{S}_m(ab - \bar{a}\bar{b})\Delta \tilde{u})\|_{\dot{B}_{p,1}^{n/p-1}} &\lesssim \|(\dot{S}_m(a \nabla b), \dot{S}_m(b \nabla a))\|_{\dot{B}_{p,1}^{n/p-1/2}} \|\Delta \tilde{u}\|_{\dot{B}_{p,1}^{n/p-3/2}} \\ &\lesssim \|(\dot{S}_m(a \nabla b), \dot{S}_m(b \nabla a))\|_{\dot{B}_{p,1}^{n/p-1/2}} \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{n/p-1/2}}. \end{aligned} \quad (2-26)$$

The last two terms of (2-21)–(2-23) are estimated as follows:

$$\|\mathcal{Q}((\text{Id} - \dot{S}_m)(a\nabla b)D(\tilde{u})) + \mathcal{Q}((\text{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\Delta\tilde{u})\|_{\dot{B}_{p,1}^{n/p-1}} \quad (2-27)$$

$$\lesssim (\|(\text{Id} - \dot{S}_m)(a\nabla b)\|_{\dot{B}_{p,1}^{n/p-1}} + \|(\text{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\|_{\dot{B}_{p,1}^{n/p}})\|\nabla\tilde{u}\|_{\dot{B}_{p,1}^{n/p}}. \quad (2-28)$$

Thus, putting together relations (2-20)–(2-28) we get

$$\begin{aligned} \|\mathcal{Q}(a\nabla\tilde{P})\|_{\dot{B}_{p,1}^{n/p-1}} &\lesssim \|\mathcal{Q}\tilde{f}\|_{\dot{B}_{p,1}^{n/p-1}} + \|(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a))\|_{\dot{B}_{p,1}^{n/p-1/2}}\|\nabla\tilde{u}\|_{\dot{B}_{p,1}^{n/p-1/2}} \\ &\quad + \|\nabla\tilde{u}\|_{\dot{B}_{p,1}^{n/p}}(\|(\text{Id} - \dot{S}_m)(a\nabla b, b\nabla a)\|_{\dot{B}_{p,1}^{n/p-1}} + \|(\text{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\|_{\dot{B}_{p,1}^{n/p}}). \end{aligned} \quad (2-29)$$

Next, we turn our attention towards $\mathcal{P}(a\nabla\tilde{P})$. The 2-dimensional case and the 3-dimensional case have to be treated differently.

The 3-dimensional case. Noticing that

$$\mathcal{P}(a\nabla\tilde{P}) = \mathcal{P}((\text{Id} - \dot{S}_m)(a - \bar{a})\nabla\tilde{P}) + [\mathcal{P}, \dot{S}_m(a - \bar{a})]\nabla\tilde{P},$$

and using again Proposition 3.21 combined with Propositions 2.2 and 2.5 we get

$$\begin{aligned} \|\nabla\tilde{P}\|_{\dot{B}_{p,2}^{3/p-3/2}} + \|\mathcal{P}(a\nabla\tilde{P})\|_{\dot{B}_{p,1}^{3/p-1}} \\ \lesssim \|\nabla\tilde{P}\|_{\dot{B}_{p,2}^{3/p-3/2}} + \|\mathcal{P}((\text{Id} - \dot{S}_m)(a - \bar{a})\nabla\tilde{P})\|_{\dot{B}_{p,1}^{3/p-1}} + \|[\mathcal{P}, \dot{S}_m(a - \bar{a})]\nabla\tilde{P}\|_{\dot{B}_{p,1}^{3/p-1}} \end{aligned} \quad (2-30)$$

$$\lesssim \|(\text{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p}}\|\nabla\tilde{P}\|_{\dot{B}_{p,1}^{3/p-1}} + (1 + \|\dot{S}_m\nabla a\|_{\dot{B}_{p,2}^{3/p-1/2}})\|\nabla\tilde{P}\|_{\dot{B}_{p,2}^{3/p-3/2}} \quad (2-31)$$

$$\lesssim \|(\text{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p}}\left(\frac{1}{\bar{a}} + \left\|\frac{1}{a} - \frac{1}{\bar{a}}\right\|_{\dot{B}_{p,1}^{3/p}}\right)\|a\nabla\tilde{P}\|_{\dot{B}_{p,1}^{3/p-1}} \quad (2-32)$$

$$+ \tilde{C}(a)(1 + \|\dot{S}_m\nabla a\|_{\dot{B}_{p,2}^{3/p-1/2}})(\|\tilde{f}\|_{\dot{B}_{p,2}^{3/p-3/2}} + \|a\text{div}(bD(\tilde{u}))\|_{\dot{B}_{p,2}^{3/p-3/2}}), \quad (2-33)$$

where

$$\tilde{C}(a) = \left(\frac{1}{\bar{a}} + \left\|\frac{1}{a} - \frac{1}{\bar{a}}\right\|_{\dot{B}_{p,1}^{3/p}}\right)\left(1 + \frac{1}{a_\star}\|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}}\right).$$

We observe that

$$\|a\text{div}(bD(\tilde{u}))\|_{\dot{B}_{p,2}^{3/p-3/2}} \lesssim (\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}})(\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{3/p}})\|\nabla\tilde{u}\|_{\dot{B}_{p,1}^{3/p-1/2}}. \quad (2-34)$$

Putting together (2-30)–(2-33) along with (2-34) we get

$$\begin{aligned} \|\nabla\tilde{P}\|_{\dot{B}_{p,2}^{3/p-3/2}} + \|\mathcal{P}(a\nabla\tilde{P})\|_{\dot{B}_{p,1}^{3/p-1}} \\ \lesssim \|(\text{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p}}\left(\frac{1}{\bar{a}} + \left\|\frac{1}{a} - \frac{1}{\bar{a}}\right\|_{\dot{B}_{p,1}^{3/p}}\right)\|a\nabla\tilde{P}\|_{\dot{B}_{p,1}^{3/p-1}} + \tilde{C}(a)(1 + \|\dot{S}_m\nabla a\|_{\dot{B}_{p,2}^{3/p-1/2}}) \\ \times (\|\tilde{f}\|_{\dot{B}_{p,2}^{3/p-3/2}} + (\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}})(\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{3/p}})\|\nabla\tilde{u}\|_{\dot{B}_{p,1}^{3/p-1/2}}). \end{aligned} \quad (2-35)$$

Combining (2-29) with (2-35) yields

$$\begin{aligned} \|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{3/p-3/2}} + \|a\nabla \tilde{P}\|_{\dot{B}_{p,1}^{3/p-1}} &\lesssim T_m^1(a, b)\|a\nabla \tilde{P}\|_{\dot{B}_{p,1}^{3/p-1}} + T_m^2(a, b)\|\tilde{f}\|_{\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1}} \\ &\quad + T_m^3(a, b)\|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{3/p-1/2}} + T_m^4(a, b)\|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{3/p}}, \end{aligned}$$

where

$$\begin{aligned} T_m^1(a, b) &= \|(\text{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p}} \left(\frac{1}{\bar{a}} + \left\| \frac{1}{a} - \frac{1}{\bar{a}} \right\|_{\dot{B}_{p,1}^{3/p}} \right), \\ T_m^2(a, b) &= \tilde{C}(a)(1 + \|\dot{S}_m \nabla a\|_{\dot{B}_{p,2}^{3/p-1/2}}), \\ T_m^3(a, b) &= \|(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a))\|_{\dot{B}_{p,1}^{3/p-1/2}} \\ &\quad + \tilde{C}(a)(1 + \|\dot{S}_m \nabla a\|_{\dot{B}_{p,2}^{3/p-1/2}})(\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}})(\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{3/p}}), \\ T_m^4(a, b) &= \|(\text{Id} - \dot{S}_m)(a\nabla b, b\nabla a)\|_{\dot{B}_{p,1}^{3/p-1}} + \|(\text{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\|_{\dot{B}_{p,1}^{3/p}}. \end{aligned}$$

Observe that m could be chosen large enough such that $T_m^1(a, b)$ and $T_m^4(a, b)$ can be made arbitrarily small. Thus, there exists a constant C_{ab} depending on a and b such that

$$\|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1}} \leq C_{ab}(\|\tilde{f}\|_{\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1}} + \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{3/p-1/2}}) + \eta \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{3/p}}, \quad (2-36)$$

where η can be made arbitrarily small (of course, with the price of increasing the constant C_{ab}). Let us take a look at the $\dot{B}_{p,2}^{3/p-3/2}$ -norm of \tilde{f} ; we get

$$\begin{aligned} \|\tilde{f}\|_{\dot{B}_{p,2}^{3/p-3/2}} &\leq \|a \operatorname{div}(bD(u_L)) - \bar{a} \operatorname{div}(\bar{b}D(u_L))\|_{\dot{B}_{p,2}^{3/p-3/2}} + \|(a - \bar{a})\nabla P_L\|_{\dot{B}_{p,2}^{3/p-3/2}} \\ &\lesssim (\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}})(\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{3/p}})\|\nabla u_L\|_{\dot{B}_{p,1}^{3/p-1/2}} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}}\|\nabla P_L\|_{\dot{B}_{p,2}^{3/p-3/2}}. \end{aligned} \quad (2-37)$$

As $u_L \in C([0, \infty), \dot{B}_{p,1}^{3/p-1}) \cap L^1([0, \infty), \dot{B}_{p,1}^{3/p+1})$ and \mathcal{Q} is a continuous operator on homogeneous Besov spaces from

$$\operatorname{div}(u_L - R) = 0,$$

we deduce

$$\mathcal{P}(u_L - R) = u_L - R,$$

which implies

$$\mathcal{Q}u_L = \mathcal{Q}R.$$

By applying the operator \mathcal{Q} in the first equation of system (2-14) we get

$$\begin{aligned} \bar{a}\nabla P_L &= \mathcal{Q}f - \mathcal{Q}\partial_t u_L + \bar{a}\bar{b}\mathcal{Q}\Delta u_L + \bar{a}\bar{b}\nabla \operatorname{div} R \\ &= \mathcal{Q}f - \mathcal{Q}\partial_t R + 2\bar{a}\bar{b}\nabla \operatorname{div} R \end{aligned}$$

and thus

$$\|\nabla P_L\|_{\dot{B}_{p,2}^{3/p-3/2}} \leq \frac{1}{\bar{a}}\|\mathcal{Q}f\|_{\dot{B}_{p,2}^{3/p-3/2}} + \frac{1}{\bar{a}}\|\partial_t \mathcal{Q}R\|_{\dot{B}_{p,2}^{3/p-3/2}} + 2\bar{b}\|\nabla \operatorname{div} R\|_{\dot{B}_{p,2}^{3/p-3/2}}.$$

In view of (2-36), (2-19), (2-37) and interpolation we gather that there exists a constant C_{ab} such that

$$\begin{aligned} & \|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1}} \\ & \leq C_{ab} (\|\nabla u_L\|_{\dot{B}_{p,1}^{3/p-1/2}} + \|\nabla P_L\|_{\dot{B}_{p,2}^{3/p-3/2}} + \|(\nabla^2 u_L, \nabla P_L)\|_{\dot{B}_{p,1}^{3/p-1}} + \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{3/p-1/2}}) + \eta \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{3/p}} \end{aligned} \quad (2-38)$$

$$\leq C_{ab} \|\mathcal{Q}f, \partial_t \mathcal{Q}R, \nabla \operatorname{div} R\|_{\dot{B}_{p,2}^{3/p-3/2}} + C_{ab} \|u_L\|_{\dot{B}_{p,1}^{3/p-1}} \quad (2-39)$$

$$+ C_{ab} \|(\nabla^2 u_L, \nabla P_L)\|_{\dot{B}_{p,1}^{3/p-1}} + C_{ab} \|\tilde{u}\|_{\dot{B}_{p,1}^{3/p-1}} + 2\eta \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{3/p}}, \quad (2-40)$$

where, again, at the price of increasing C_{ab} , we can make η arbitrarily small.

The 2-dimensional case. In this case, using again Proposition 3.21 combined with Propositions 2.4 and 2.6 we get

$$\begin{aligned} & \|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{2/p-1}} + \|\mathcal{P}(a\nabla \tilde{P})\|_{\dot{B}_{p,1}^{2/p-1}} \\ & \lesssim \|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{2/p-1}} + \|(\operatorname{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{2/p}} \|\nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p-1}} + \|[\mathcal{P}, \dot{S}_m(a - \bar{a})]\nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p-1}} \\ & \lesssim \|(\operatorname{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{2/p}} \|\nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p-1}} + (1 + \|\nabla \dot{S}_m a\|_{\dot{B}_{p,2}^{2/p}}) \|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{2/p-1}} \\ & \lesssim \|(\operatorname{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{2/p}} \|\nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p-1}} \\ & \quad + \tilde{C}(a)(1 + \|\nabla \dot{S}_m a\|_{\dot{B}_{p,2}^{2/p}}) (\|\tilde{f}\|_{\dot{B}_{p,2}^{2/p-1}} + \|\mathcal{Q}(a \operatorname{div}(bD(\tilde{u})))\|_{\dot{B}_{p,2}^{2/p-1}}) \end{aligned}$$

where, as before

$$\tilde{C}(a) = \left(\frac{1}{\bar{a}} + \left\| \frac{1}{a} - \frac{1}{\bar{a}} \right\|_{\dot{B}_{p,1}^{2/p}} \right) \left(1 + \frac{1}{a_\star} \|a - \bar{a}\|_{\dot{B}_{p,1}^{2/p}} \right).$$

As we have already estimated $\|\mathcal{Q}(a \operatorname{div}(bD(\tilde{u})))\|_{\dot{B}_{p,2}^{2/p-1}}$ in (2-29), we gather

$$\begin{aligned} & \|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{2/p-1}} + \|a\nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p}} \lesssim T_m^1(a, b) \|a\nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p-1}} + T_m^2(a, b) \|\tilde{f}\|_{\dot{B}_{p,1}^{2/p-1}} \\ & \quad + T_{m,M}^3(a, b) \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{2/p-1/2}} + T_{m,M}^4(a, b) \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{2/p}}, \end{aligned} \quad (2-41)$$

where

$$T_m^1(a, b) = \|(\operatorname{Id} - \dot{S}_m)(a - \bar{a})\|_{\dot{B}_{p,1}^{2/p}} \left(\frac{1}{\bar{a}} + \left\| \frac{1}{a} - \frac{1}{\bar{a}} \right\|_{\dot{B}_{p,1}^{2/p}} \right),$$

$$T_m^2(a, b) = \tilde{C}(a)(1 + \|\nabla \dot{S}_m a\|_{\dot{B}_{p,2}^{2/p}}),$$

$$T_{m,M}^3(a, b) = \|(\dot{S}_m(a\nabla b), \dot{S}_m(b\nabla a))\|_{\dot{B}_{p,1}^{2/p}} + \tilde{C}(a)(1 + \|\nabla \dot{S}_m a\|_{\dot{B}_{p,2}^{2/p}}) \|(\dot{S}_M(a\nabla b), \dot{S}_M(b\nabla a))\|_{\dot{B}_{p,1}^{2/p}},$$

$$\begin{aligned} T_{m,M}^4(a, b) & = \|(\operatorname{Id} - \dot{S}_m)(a\nabla b)\|_{\dot{B}_{p,1}^{2/p-1}} + \|(\operatorname{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\|_{\dot{B}_{p,1}^{2/p}} \\ & \quad + \tilde{C}(a)(1 + \|\nabla \dot{S}_m a\|_{\dot{B}_{p,2}^{2/p}}) (\|(\operatorname{Id} - \dot{S}_M)(a\nabla b)\|_{\dot{B}_{p,1}^{2/p-1}} + \|(\operatorname{Id} - \dot{S}_M)(ab - \bar{a}\bar{b})\|_{\dot{B}_{p,1}^{2/p-1}}). \end{aligned}$$

First, we fix an $\eta > 0$. Let us fix an $m \in \mathbb{N}$ such that $T_m^1(a, b) \|a\nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p-1}}$ can be “absorbed” by the left-hand side of (2-41) and such that

$$\|(\operatorname{Id} - \dot{S}_m)(a\nabla b)\|_{\dot{B}_{p,1}^{2/p-1}} + \|(\operatorname{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\|_{\dot{B}_{p,1}^{2/p}} \leq \frac{1}{2}\eta.$$

Next, we see that by choosing M large enough we have

$$T_{m,M}^4(a, b) \leq \eta.$$

Thus, using interpolation we can write

$$\|\nabla \tilde{P}\|_{\dot{B}_{p,2}^{2/p-1}} + \|a \nabla \tilde{P}\|_{\dot{B}_{p,1}^{2/p-1}} \leq C_{ab} (\|(\nabla^2 u_L, \nabla P_L)\|_{\dot{B}_{2,1}^{2/p-1}} + \|\tilde{u}\|_{\dot{B}_{p,1}^{2/p-1}}) + 2\eta \|\nabla^2 \tilde{u}\|_{\dot{B}_{p,1}^{2/p-1}}. \quad (2-42)$$

End of the proof of Proposition 2.9. Obviously, combining the two estimates (2-38)–(2-40) and (2-42) we can continue in a unified manner the rest of the proof of Proposition 2.9. First, choose $m \in \mathbb{N}$ large enough such that

$$\bar{a}\bar{b} + \dot{S}_m(ab - \bar{a}\bar{b}) \geq \frac{1}{2} a_\star b_\star.$$

We apply $\dot{\Delta}_j$ to (2-18) and we write

$$\begin{aligned} & \partial_t \tilde{u}_j - \operatorname{div}((\bar{a}\bar{b} + \dot{S}_m(ab - \bar{a}\bar{b})) \nabla \tilde{u}_j) \\ &= \tilde{f}_j - \dot{\Delta}_j(a \nabla \tilde{P}) + \dot{\Delta}_j \operatorname{div}((\operatorname{Id} - \dot{S}_m)(ab - \bar{a}\bar{b}) \nabla \tilde{u}) + \operatorname{div}[\dot{\Delta}_j, \dot{S}_m(ab - \bar{a}\bar{b})] \nabla \tilde{u} \\ & \quad + \dot{\Delta}_j(D\tilde{u} \dot{S}_m(b \nabla a)) + \dot{\Delta}_j(D\tilde{u}(\operatorname{Id} - \dot{S}_m)(b \nabla a)) + \dot{\Delta}_j(\nabla \tilde{u} \dot{S}_m(a \nabla b)) + \dot{\Delta}_j(\nabla \tilde{u}(\operatorname{Id} - \dot{S}_m)(a \nabla b)). \end{aligned}$$

Multiplying the last relation by $|\tilde{u}_j|^{p-1} \operatorname{sgn} \tilde{u}_j$, integrating and using Lemma 8 from Appendix B of [Danchin 2010], we get

$$\begin{aligned} \|\tilde{u}_j\|_{L^p} + a_\star b_\star 2^{2j} C \int_0^t \|\tilde{u}_j\|_{L^p} &\lesssim \int_0^t \|\tilde{f}_j\|_{L^p} + \int_0^t \|\dot{\Delta}_j(a \nabla \tilde{P})\|_{L^p} + \int_0^t \|\operatorname{div}[\dot{\Delta}_j, \dot{S}_m(ab - \bar{a}\bar{b})] \nabla \tilde{u}\|_{L^p} \\ & \quad + \int_0^t \|\dot{\Delta}_j \operatorname{div}((\operatorname{Id} - \dot{S}_m)(ab - \bar{a}\bar{b}) \nabla \tilde{u})\|_{L^p} \\ & \quad + \int_0^t \|\dot{\Delta}_j(D\tilde{u} \dot{S}_m(b \nabla a))\|_{L^p} + \int_0^t \|\dot{\Delta}_j(D\tilde{u}(\operatorname{Id} - \dot{S}_m)(b \nabla a))\|_{L^p} \\ & \quad + \int_0^t \|\dot{\Delta}_j(\nabla \tilde{u} \dot{S}_m(a \nabla b))\|_{L^p} + \int_0^t \|\dot{\Delta}_j(\nabla \tilde{u}(\operatorname{Id} - \dot{S}_m)(a \nabla b))\|_{L^p}. \end{aligned}$$

Multiplying the last relation by $2^{j(n/p-1)}$, performing an $\ell^1(\mathbb{Z})$ -summation and using Proposition 3.19 to deal with $\|\operatorname{div}[\dot{\Delta}_j, \dot{S}_m(ab - \bar{a}\bar{b})] \nabla \tilde{u}\|_{\dot{B}_{p,1}^{n/p-1}}$ along with (2-38)–(2-40) and (2-41) to deal with the pressure, we get

$$\begin{aligned} & \|\tilde{u}\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + a_\star b_\star C \|\nabla^2 \tilde{u}\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \\ & \lesssim \|\tilde{f}\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} + C \int_0^t \|a \nabla \tilde{P}\|_{\dot{B}_{p,1}^{n/p-1}} + \int_0^t \|(\dot{S}_m(b \nabla a), \dot{S}_m(a \nabla b))\|_{\dot{B}_{p,1}^{n/p}} \|\nabla \tilde{u}\|_{\dot{B}_{p,1}^{n/p-1}} \\ & \quad + T_m(a, b) \|\nabla^2 \tilde{u}\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \\ & \leq C_{ab}(1+t) (\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) + C_{ab} \int_0^t \|\tilde{u}\|_{\dot{B}_{p,1}^{n/p-1}} \\ & \quad + (T_m(a, b) + \eta) \|\nabla^2 \tilde{u}\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \quad (2-43) \end{aligned}$$

where

$$T_m(a, b) = \|(\operatorname{Id} - \dot{S}_m)(b \nabla a)\|_{\dot{B}_{p,1}^{n/p-1}} + \|(\operatorname{Id} - \dot{S}_m)(a \nabla b)\|_{\dot{B}_{p,1}^{n/p-1}} + \|(\operatorname{Id} - \dot{S}_m)(ab - \bar{a}\bar{b})\|_{\dot{B}_{p,1}^{n/p-1}}. \quad (2-44)$$

Assuming m is large enough and η is small enough, we can “absorb” $(T_m(a, b) + \eta)\|\nabla^2 u\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})}$ into the left-hand side of (2-43). Thus, we end up with

$$\begin{aligned} & \|\tilde{u}\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + a_\star b_\star \frac{1}{2} C \|\nabla^2 \tilde{u}\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \\ & \leq C_{ab}(1+t)(\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) + C_{ab} \int_0^t \|\tilde{u}\|_{\dot{B}_{p,1}^{n/p-1}} \end{aligned}$$

such that using Grönwall’s lemma, (2-15) and the classical inequality

$$1 + t^\alpha \leq C_\alpha \exp(t)$$

yields

$$\begin{aligned} & \|\tilde{u}\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + a_\star b_\star \frac{1}{2} C \|\nabla^2 \tilde{u}\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \\ & \leq C_{ab}(\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(C_{ab}t). \end{aligned} \quad (2-45)$$

Using the fact that $u = u_L + \tilde{u}$ along with (2-15) and (2-45) gives us

$$\begin{aligned} & \|u\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + a_\star b_\star \frac{1}{2} C \|\nabla^2 u\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} \\ & \leq C_{ab}(\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(C_{ab}t). \end{aligned} \quad (2-46)$$

Next, using (2-38)–(2-40) and (2-42) combined with (2-15), we infer

$$\begin{aligned} & \|\nabla P\|_{L_t^1((\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1}))} \\ & \leq C_a \|a \nabla P_L\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})} + C_a \|a \nabla \tilde{P}\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})} \end{aligned} \quad (2-47)$$

$$\leq C_{ab}(\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(C_{ab}t). \quad (2-48)$$

Combining (2-48) with (2-46) we finally get

$$\begin{aligned} & \|u\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + \|\nabla^2 u\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} + \|\nabla P\|_{L_t^1((\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1}))} \\ & \leq (\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(C_{ab}(t+1)). \end{aligned} \quad (2-49)$$

Obviously, by obtaining the last estimate we conclude the proof of [Proposition 2.9](#).

Next, let us deal with the existence part of the Stokes problem with the coefficients having regularity as in [Proposition 2.9](#). More precisely, we have:

Proposition 2.12. *Consider (a, b, u_0, f, R) as in the statement of [Proposition 2.9](#). Then, there exists a unique solution $(u, \nabla P) \in E_{\text{loc}}$ of the Stokes system (1-3). Furthermore, there exists a constant C_{ab} depending on a and b such that*

$$\begin{aligned} & \|u\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + \|\nabla^2 u\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} + \|\nabla P\|_{L_t^1((\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1}))} \\ & \leq (\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(C_{ab}(t+1)) \end{aligned} \quad (2-50)$$

for all $t > 0$.

Proof. The uniqueness property is a direct consequence of the estimates of Proposition 2.9. The proof of existence relies on Proposition 2.9 combined with a continuity argument as used in [Danchin 2014]; see also [Krylov 2008]. Let us introduce

$$(a_\theta, b_\theta) = (1 - \theta)(\bar{a}, \bar{b}) + \theta(a, b)$$

and consider the Stokes system

$$\begin{cases} \partial_t u - a_\theta(\operatorname{div}(b_\theta D(u)) - \nabla P) = f, \\ \operatorname{div} u = \operatorname{div} R, \\ u|_{t=0} = u_0. \end{cases} \tag{S_\theta}$$

First of all, a more detailed analysis of the estimates established in Proposition 2.9 enables us to conclude that the constant $C_{a_\theta b_\theta}$ appearing in (2-49) is uniformly bounded with respect to $\theta \in [0, 1]$ by a constant $c = c_{ab}$. Indeed, repeating the estimation process carried out in Proposition 2.9 with (a_θ, b_θ) instead of (a, b) amounts to replacing $(a - \bar{a})$ and $(b - \bar{b})$ with $\theta(a - \bar{a})$ and $\theta(b - \bar{b})$. Taking into account Proposition 3.12 and the remark that follows we get that there exists

$$c := \sup_{\theta \in [0,1]} C_{a_\theta b_\theta} < \infty.$$

Let us take $T > 0$ and consider \mathcal{E}_T , the set of those $\theta \in [0, 1]$ such that for any (u_0, f, R) as in the statement of Proposition 2.9 problem (S_θ) admits a unique solution $(u, \nabla P) \in E_T$ which satisfies

$$\begin{aligned} \|u\|_{L_t^\infty(\dot{B}_{p,1}^{n/p-1})} + \|\nabla^2 u\|_{L_t^1(\dot{B}_{p,1}^{n/p-1})} + \|\nabla P\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})} \\ \leq (\|u_0\|_{\dot{B}_{p,1}^{n/p-1}} + \|(f, \partial_t R, \nabla \operatorname{div} R)\|_{L_t^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(c(t+1)) \end{aligned} \tag{2-51}$$

for all $t \in [0, T]$. According to Proposition 2.7, $0 \in \mathcal{E}_T$.

Suppose $\theta \in \mathcal{E}_T$. First we denote by $(u_\theta, \nabla P_\theta) \in E_T$ the unique solution of (S_θ) . We consider the space

$$E_{T,\operatorname{div}} = \{(\tilde{w}, \nabla \tilde{Q}) \in E_T : \operatorname{div} \tilde{w} = 0\}$$

and let $S_{\theta\theta'}$ be the operator which associates to $(\tilde{w}, \nabla \tilde{Q}) \in E_{T,\operatorname{div}}$, the unique solution $(\tilde{u}, \nabla \tilde{P})$ of

$$\begin{cases} \partial_t \tilde{u} - a_\theta(\operatorname{div}(b_\theta D(\tilde{u})) - \nabla \tilde{P}) = g_{\theta\theta'}(u_\theta, \nabla P_\theta) + g_{\theta\theta'}(\tilde{w}, \nabla \tilde{Q}), \\ \operatorname{div} \tilde{u} = 0, \\ \tilde{u}|_{t=0} = 0, \end{cases} \tag{2-52}$$

where

$$g_{\theta\theta'}(u, \nabla P) = (a_\theta - a_{\theta'})\nabla P + a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u)). \tag{2-53}$$

Obviously, $S_{\theta\theta'}$ maps $E_{T,\operatorname{div}}$ into $E_{T,\operatorname{div}}$. We claim that there exists a positive quantity $\varepsilon = \varepsilon(T) > 0$ such that if $|\theta - \theta'| \leq \varepsilon(T)$ then $S_{\theta\theta'}$ has a fixed point $(\tilde{u}^*, \nabla \tilde{P}^*)$ in a suitable ball centered at the origin of the space $E_{T,\operatorname{div}}$. Obviously,

$$(\tilde{u}^* + u_\theta, \nabla \tilde{P}^* + \nabla P_\theta)$$

will solve $(S_{\theta'})$ in E_T .

First, we note that, as a consequence of [Proposition 2.9](#), we have

$$\begin{aligned} & \|(\tilde{u}, \nabla \tilde{P})\|_{E_T} \\ & \leq (\|g_{\theta\theta'}(u_\theta, \nabla P_\theta)\|_{L_T^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})} + \|g_{\theta\theta'}(\tilde{w}, \nabla \tilde{Q})\|_{L_T^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}) \exp(c(T+1)). \end{aligned} \quad (2-54)$$

Observe that

$$\|(a_\theta - a_{\theta'})\nabla P\|_{L_T^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})} \leq |\theta - \theta'| \|a - \bar{a}\|_{\dot{B}_{p,1}^{n/p}} \|\nabla P\|_{L_T^1(\dot{B}_{p,2}^{n/p-n/2} \cap \dot{B}_{p,1}^{n/p-1})}. \quad (2-55)$$

Next, we write

$$a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u)) = (a_{\theta'} - a_\theta) \operatorname{div}(b_{\theta'} D(u)) + a_\theta \operatorname{div}((b_{\theta'} - b_\theta) D(u)).$$

The first term of the last identity is estimated as follows:

$$\|(a_{\theta'} - a_\theta) \operatorname{div}(b_{\theta'} D(u))\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})} \leq |\theta - \theta'| \|a - \bar{a}\|_{\dot{B}_{p,1}^{n/p}} (\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{n/p}}) \|D(u)\|_{L_T^1(\dot{B}_{p,1}^{n/p})}.$$

Regarding the second term, we have

$$\|a_\theta \operatorname{div}((b_{\theta'} - b_\theta) D(u))\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})} \leq |\theta - \theta'| \|b - \bar{b}\|_{\dot{B}_{p,1}^{n/p}} (\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{n/p}}) \|D(u)\|_{L_T^1(\dot{B}_{p,1}^{n/p})}$$

and thus

$$\begin{aligned} & \|a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u))\|_{L_T^1(\dot{B}_{p,1}^{n/p-1})} \\ & \leq |\theta - \theta'| (\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{n/p}}) (\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{n/p}}) \|Du\|_{L_T^1(\dot{B}_{p,1}^{n/p})}. \end{aligned} \quad (2-56)$$

The only thing left is to treat the $L_T^1(\dot{B}_{p,2}^{3/p-3/2})$ -norm of $a_{\theta'} \operatorname{div}(b_{\theta'} D(u)) - a_\theta \operatorname{div}(b_\theta D(u))$ in the case where $n = 3$. Using the fact that $\nabla u \in L_T^{4/3}(\dot{B}_{p,1}^{3/p-1/2})$, we can write

$$\begin{aligned} & \|(a_{\theta'} - a_\theta) \operatorname{div}(b_{\theta'} D(u))\|_{L_T^1(\dot{B}_{p,2}^{3/p-3/2})} \\ & \leq |\theta - \theta'| \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}} \|\operatorname{div}(b_{\theta'} D(u))\|_{L_T^1(\dot{B}_{p,1}^{3/p-3/2})} \\ & \leq |\theta - \theta'| \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}} (\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{3/p}}) \|Du\|_{L_T^1(\dot{B}_{p,1}^{3/p-1/2})} \end{aligned} \quad (2-57)$$

$$\leq |\theta - \theta'| \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}} (\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{3/p}}) T^{1/4} \|u\|_{L_T^\infty(\dot{B}_{p,1}^{3/p-1})}^{1/4} \|u\|_{L_T^1(\dot{B}_{p,1}^{3/p+1})}^{3/4} \quad (2-58)$$

$$\leq |\theta - \theta'| C(T, a, b) (\|u\|_{L_T^\infty(\dot{B}_{p,1}^{3/p-1})} + \|\nabla^2 u\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}) \quad (2-59)$$

and, proceeding in a similar manner, we can estimate $\|a_\theta \operatorname{div}((b_{\theta'} - b_\theta) D(u))\|_{L_T^1(\dot{B}_{p,2}^{3/p-3/2})}$.

Combining (2-55), (2-56) along with (2-59) we get

$$\begin{aligned} & \|g_{\theta\theta'}(u, \nabla P)\|_{L_T^1(\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1})} \\ & \leq |\theta - \theta'| C(T, a, b) (\|u\|_{L_T^\infty(\dot{B}_{p,1}^{3/p-1})} + \|\nabla^2 u\|_{L_T^1(\dot{B}_{p,1}^{3/p+1})} + \|\nabla P\|_{L_T^1(\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1})}). \end{aligned} \quad (2-60)$$

Substituting this into (2-54), we get

$$\|(\tilde{u}, \nabla \tilde{P})\|_{E_T} \leq |\theta - \theta'| C(T, a, b) (\|(u_\theta, \nabla P_\theta)\|_{E_T} + \|(\tilde{w}, \nabla \tilde{Q})\|_{E_T}),$$

and by linearity

$$\|(\tilde{u}^1 - \tilde{u}^2, \nabla \tilde{P}^1 - \nabla \tilde{P}^2)\|_{E_T} \leq |\theta - \theta'| C(T, a, b) \|(\tilde{w}^1 - \tilde{w}^2, \nabla \tilde{Q}^1 - \nabla \tilde{Q}^2)\|_{E_T},$$

where for $k = 1, 2$,

$$(\tilde{u}^i, \nabla \tilde{P}^i) = S_{\theta\theta'}((\tilde{w}^i, \nabla \tilde{Q}^i)).$$

Thus one can choose $\varepsilon(T)$ small enough such that $|\theta - \theta'| \leq \varepsilon(T)$ gives us a fixed point of the solution operator $S_{\theta\theta'}$ in $B_{E_T, \text{div}}(0, 2\|(u_\theta, \nabla P_\theta)\|_{E_T})$.

Thus, for all $T > 0$, we have $E_T = [0, 1]$ and owing to the uniqueness property and to [Proposition 2.9](#), we can construct a unique solution $(u, \nabla P) \in E_{\text{loc}}$ to (1-3) such that for all $t > 0$ the estimate (2-11) is valid. \square

The proof of Theorem 1.3 in the case $n = 3$. As discussed earlier, in dimension $n = 3$, [Proposition 2.9](#) is weaker than [Theorem 1.3](#), as one requires additional low-frequency information on the data

$$(f, \partial_t R, \nabla \text{div } R) \in L_t^1(\dot{B}_{p,2}^{3/p-3/2}).$$

Thus, we have to bring an extra argument in order to conclude the validity of [Theorem 1.3](#). This is the object of interest of this section.

Existence. We begin by taking $m \in \mathbb{N}$ large enough and owing to [Proposition 2.8](#) we can consider $(u^1, \nabla P^1)$, the unique solution with $u^1 \in \mathcal{C}(\mathbb{R}^+; \dot{B}_{p,1}^{3/p-1})$ and $(\partial_t u^1, \nabla^2 u^1, \nabla P^1) \in L_{\text{loc}}^1(\dot{B}_{p,1}^{3/p-1})$ of the system

$$\begin{cases} \partial_t u - \bar{a}\bar{b} \text{div } D(u) + (\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P = f, \\ \text{div } u = \text{div } R, \\ u|_{t=0} = u_0, \end{cases}$$

which also satisfies

$$\begin{aligned} \|u^1\|_{L_T^\infty(\dot{B}_{p,1}^{3/p-1})} + \|(\partial_t u^1, \bar{a}\bar{b}\nabla^2 u^1, \bar{a}\nabla P^1)\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\ \leq C(\|u_0\|_{\dot{B}_{p,1}^{3/p-1}} + \|(f, \partial_t R, \bar{a}\bar{b}\nabla \text{div } R)\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}) \end{aligned}$$

for all $T > 0$. Let us consider

$$G(u^1, \nabla P^1) = a \text{div}(bD(u^1)) - \bar{a} \text{div}(\bar{b}D(u^1)) - ((\text{Id} - \dot{S}_{-m})(a - \bar{a}))\nabla P^1.$$

We claim $G(u^1, \nabla P^1) \in L_{\text{loc}}^1(\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1})$. Indeed

$$a \text{div}(bD(u^1)) - \bar{a} \text{div}(\bar{b}D(u^1)) = (a - \bar{a}) \text{div}(bD(u^1)) + \bar{a} \text{div}((b - \bar{b})D(u^1))$$

and proceeding as in (2-56) and (2-58) we get

$$\begin{aligned} \|a \text{div}(bD(u^1)) - \bar{a} \text{div}(\bar{b}D(u^1))\|_{L_t^1(\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1})} \\ \leq C_{ab}(1 + t^{1/4})(\|u^1\|_{L_t^\infty(\dot{B}_{p,1}^{3/p-1})} + \|u^1\|_{L_t^1(\dot{B}_{p,1}^{3/p+1})}) \\ \leq \exp(C_{ab}(t + 1))(\|u_0\|_{\dot{B}_{p,1}^{3/p-1}} + \|(f, \partial_t R, \nabla \text{div } R)\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}). \end{aligned} \tag{2-61}$$

Next, we obviously have

$$\|((\text{Id} - \dot{S}_{-m})(a - \bar{a}))\nabla P^1\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})} \leq C \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}} \|\nabla P^1\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})}. \quad (2-62)$$

Using the fact that the product maps $\dot{B}_{p,1}^{3/p-1/2} \times \dot{B}_{p,1}^{3/p-1} \rightarrow \dot{B}_{p,2}^{3/p-3/2}$, we get

$$\|((\text{Id} - \dot{S}_{-m})(a - \bar{a}))\nabla P^1\|_{L_t^1(\dot{B}_{p,2}^{3/p-3/2})} \leq C \|(\text{Id} - \dot{S}_{-m})(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p-1/2}} \|\nabla P^1\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})}. \quad (2-63)$$

Of course

$$\begin{aligned} \|(\text{Id} - \dot{S}_{-m})(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p-1/2}} &\leq C \sum_{j \geq -m} 2^{j(3/p-1/2)} \|\dot{\Delta}_j(a - \bar{a})\|_{L^2} \leq C 2^{m/2} \sum_{j \geq -m} 2^{3/pj} \|\dot{\Delta}_j(a - \bar{a})\|_{L^2} \\ &\leq C 2^{m/2} \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}} \end{aligned}$$

so that the first term on the right-hand side of (2-63) is finite. We thus gather from (2-61), (2-62) and (2-63) that $G(u^1, \nabla P^1) \in L_{\text{loc}}^1(\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1})$ and that for all $t > 0$ there exists a constant C_{ab} such that

$$\|G(u^1, \nabla P^1)\|_{L_t^1(\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1})} \leq (\|u_0\|_{\dot{B}_{p,1}^{3/p-1}} + \|(f, \partial_t R, \nabla \text{div} R)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})}) \exp(C_{ab}(t+1)).$$

According to [Proposition 2.12](#), there exists a unique solution $(u^2, \nabla P^2) \in E_{\text{loc}}$ of the system

$$\begin{cases} \partial_t u - a \text{div}(bD(u)) + a\nabla P = G(u^1, \nabla P^1), \\ \text{div} u = 0, \\ u|_{t=0} = 0, \end{cases}$$

which satisfies the estimate

$$\begin{aligned} \|u^2\|_{L_t^\infty(\dot{B}_{p,1}^{3/p-1})} + \|(\nabla^2 u^2, \nabla P^2)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})} \\ \leq \|G(u^1, \nabla P^1)\|_{L_t^1(\dot{B}_{p,2}^{3/p-3/2} \cap \dot{B}_{p,1}^{3/p-1})} \exp(C_{ab}(t+1)) \\ \leq (\|u_0\|_{\dot{B}_{p,1}^{3/p-1}} + \|(f, \partial_t R, \nabla \text{div} R)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})}) \exp(C_{ab}(t+1)). \end{aligned}$$

We observe that

$$(u, \nabla P) := (u^1 + u^2, \nabla P^1 + \nabla P^2)$$

is a solution of (1-3) which satisfies

$$\begin{aligned} \|u\|_{L_t^\infty(\dot{B}_{p,1}^{3/p-1})} + \|(\nabla^2 u, \nabla P)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})} \\ \leq (\|u_0\|_{\dot{B}_{p,1}^{3/p-1}} + \|(f, \partial_t R, \nabla \text{div} R)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})}) \exp(C_{ab}(t+1)). \quad (2-64) \end{aligned}$$

Of course, using again the first equation of (1-3) we get

$$\|\partial_t u\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})} \leq C_{ab} \|(f, \nabla^2 u, \nabla P)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})}$$

and thus, we get the estimate

$$\begin{aligned} \|u\|_{L_t^\infty(\dot{B}_{p,1}^{3/p-1})} + \|(\partial_t u, \nabla^2 u, \nabla P)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})} \\ \leq (\|u_0\|_{\dot{B}_{p,1}^{3/p-1}} + \|(f, \partial_t R, \nabla \text{div} R)\|_{L_t^1(\dot{B}_{p,1}^{3/p-1})}) \exp(C_{ab}(t+1)). \quad (2-65) \end{aligned}$$

Uniqueness. Next, let us prove the uniqueness property. Let us suppose there exists a $T > 0$ and a pair $(u, \nabla P)$ that solves

$$\begin{cases} \partial_t u - a \operatorname{div}(bD(u)) + a \nabla P = 0, \\ \operatorname{div} u = 0, \\ u|_{t=0} = 0, \end{cases} \tag{2-66}$$

with

$$u \in C_T(\dot{B}_{p,1}^{3/p-1}) \text{ and } (\partial_t u, \nabla^2 u, \nabla P) \in L_T^1(\dot{B}_{p,1}^{3/p-1}).$$

Observe that we cannot directly conclude to the uniqueness property by appealing to [Proposition 2.12](#) because the pressure does not belong (a priori) to $L_T^1(\dot{B}_{p,2}^{3/p-3/2})$. Recovering this low-frequency information is done in the following lines. Suppose $3 < p < 4$. Applying the operator \mathcal{Q} in the first equation of (2-66) we can write

$$\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P) = \mathcal{Q}(a \operatorname{div}(bD(u))) - \mathcal{Q}((\operatorname{Id} - \dot{S}_{-m})(a - \bar{a})\nabla P),$$

where $m \in \mathbb{N}$ will be fixed later. We observe that

$$\begin{aligned} & \|\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P)\|_{L_T^1(\dot{B}_{p,1}^{3/p-3/2})} \\ & \lesssim \|\mathcal{Q}(a \operatorname{div}(bD(u)))\|_{L_T^1(\dot{B}_{p,1}^{3/p-3/2})} + \|\mathcal{Q}((\operatorname{Id} - \dot{S}_{-m})(a - \bar{a})\nabla P)\|_{L_T^1(\dot{B}_{p,1}^{3/p-3/2})} \\ & \lesssim T^{1/4}(\bar{a} + \|a - \bar{a}\|_{\dot{B}_{p,1}^{3/p}})(\bar{b} + \|b - \bar{b}\|_{\dot{B}_{p,1}^{3/p}})\|\nabla u\|_{L_T^{4/3}(\dot{B}_{p,1}^{3/p-1/2})} \\ & \quad + \|(\operatorname{Id} - \dot{S}_{-m})(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p-1/2}}\|\nabla P\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}. \end{aligned}$$

Consequently, we get

$$\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P) \in L_T^1(\dot{B}_{p,1}^{3/p-3/2}). \tag{2-67}$$

Let us observe that the condition $p \in (3, 4)$ ensures that $\dot{B}_{p,1}^{3/p}$ is contained in the multiplier space of $\dot{B}_{p',2}^{-3/p+1} = \dot{B}_{p',2}^{3/p'-2}$. More precisely, we get:

Proposition 2.13. *Consider $p \in (3, 4)$ and $(u, v) \in \dot{B}_{p,1}^{3/p} \times \dot{B}_{p',2}^{-3/p+1}$. Then $uv \in \dot{B}_{p',2}^{-3/p+1}$ and*

$$\|uv\|_{\dot{B}_{p',2}^{-3/p+1}} \lesssim \|u\|_{\dot{B}_{p,1}^{3/p}} \|v\|_{\dot{B}_{p',2}^{-3/p+1}}.$$

Proof. Indeed, considering $(u, v) \in \dot{B}_{p,1}^{3/p} \times \dot{B}_{p',2}^{-3/p+1}$ and using the Bony decomposition we get

$$\|\dot{T}_u v\|_{\dot{B}_{p',2}^{-3/p+1}} \lesssim \|u\|_{L^\infty} \|v\|_{\dot{B}_{p',2}^{-3/p+1}}.$$

Next, considering

$$\frac{1}{p'} = \frac{1}{2} + \frac{1}{p^*},$$

we see

$$\begin{aligned} 2^{j(-3/p+1)} \|\dot{\Delta}_j \dot{T}'_v u\|_{L^{p'}} & \lesssim \sum_{\ell \geq j-3} 2^{(-3/p+1)(j-\ell)} 2^{(-3/p+1)\ell} \|S_{\ell+1} v\|_{L^2} \|\dot{\Delta}_\ell u\|_{L^{p^*}} \\ & = \sum_{\ell \geq j-3} 2^{(-3/p+1)(j-\ell)} 2^{-1/2\ell} \|S_{\ell+1} v\|_{L^2} 2^{3/p^* \ell} \|\dot{\Delta}_\ell u\|_{L^{p^*}} \end{aligned}$$

such that, with the help of [Proposition 3.10](#), we get

$$\|\dot{T}'_v u\|_{\dot{B}_{p',2}^{-3/p+1}} \lesssim \|v\|_{\dot{H}^{-1/2}} \|u\|_{\dot{B}_{p^*,1}^{3/p^*}} \lesssim \|v\|_{\dot{B}_{p',2}^{-3/p+1}} \|u\|_{\dot{B}_{p,1}^{3/p}}. \quad \square$$

Proposition 2.14. *Consider $p \in (3, 4)$. Furthermore, consider a constant $\bar{c} > 0$ and $c \in \dot{B}_{p,1}^{3/p}$. Then there exists a universal constant $\eta > 0$ such that if*

$$\|c\|_{\dot{B}_{p,1}^{3/p}} \leq \eta,$$

then for any $\psi \in \dot{B}_{p',2}^{3/p'-3/2} \cap \dot{B}_{p',2}^{3/p'-2}$ there exists a unique solution $\nabla P \in \dot{B}_{p',2}^{3/p'-3/2} \cap \dot{B}_{p',2}^{3/p'-2}$ of the elliptic equation

$$\operatorname{div}((\bar{c} + c)\nabla P) = \operatorname{div} \psi.$$

Moreover, the following estimate holds true:

$$\|\nabla P\|_{\dot{B}_{p',2}^{3/p'-\sigma}} \lesssim \|\mathcal{Q}\psi\|_{\dot{B}_{p',2}^{3/p'-\sigma}},$$

where $\sigma \in \{\frac{3}{2}, 2\}$.

Proof. The proof is standard. Under some smallness condition on $c \in \dot{B}_{p,1}^{3/p}$, the operator

$$\nabla R \rightarrow \nabla P = \frac{1}{\bar{c}} \mathcal{Q}(\psi - c\nabla R)$$

has a fixed point in a suitable chosen ball of the space $\dot{B}_{p',2}^{3/p'-3/2} \cap \dot{B}_{p',2}^{3/p'-2}$. □

Choose $m \in \mathbb{N}$ such that $\|\dot{S}_{-m}(a - \bar{a})\|_{\dot{B}_{p,1}^{3/p}}$ is small enough that we can apply [Proposition 2.14](#) with \bar{a} and $\dot{S}_{-m}(a - \bar{a})$ instead of \bar{c} and c , and we consider ψ a vector field with coefficients in \mathcal{S} . As the Schwartz class is included in $\dot{B}_{p',2}^{3/p'-3/2} \cap \dot{B}_{p',2}^{3/p'-2}$, let us consider $\nabla P_\psi \in \dot{B}_{p',2}^{3/p'-3/2} \cap \dot{B}_{p',2}^{3/p'-2}$, the solution of the equation

$$\operatorname{div}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P_\psi) = \operatorname{div} \psi,$$

the existence of which is granted by [Proposition 2.14](#). Then, using [Propositions 3.8](#) and [3.9](#), we can write³

$$\begin{aligned} & \langle \nabla P, \psi \rangle_{\mathcal{S}' \times \mathcal{S}} \\ &= \sum_j \langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j \psi \rangle = \sum_j -\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div} \psi \rangle \end{aligned} \quad (2-68)$$

$$= \sum_j -\langle \dot{\Delta}_j P, \tilde{\Delta}_j \operatorname{div}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P_\psi) \rangle = \sum_j \langle \dot{\Delta}_j \nabla P, \tilde{\Delta}_j((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P_\psi) \rangle \quad (2-69)$$

$$= \sum_j \langle \dot{\Delta}_j(\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P, \tilde{\Delta}_j \nabla P_\psi \rangle = \sum_j \langle \dot{\Delta}_j \mathcal{Q}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P), \tilde{\Delta}_j \nabla P_\psi \rangle \quad (2-70)$$

$$\lesssim \|\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P)\|_{\dot{B}_{p',2}^{3/p-3/2}} \|\nabla P_\psi\|_{\dot{B}_{p',1}^{3/p'-3/2}} \quad (2-71)$$

$$\lesssim \|\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P)\|_{\dot{B}_{p',2}^{3/p-3/2}} \|\psi\|_{\dot{B}_{p',1}^{3/p'-3/2}}. \quad (2-72)$$

³We define $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$.

Taking the supremum over all $\psi \in \mathcal{S}$ with $\|\psi\|_{\dot{B}_{p',2}^{3/p'-3/2}} \leq 1$, by (2-67) and Proposition 3.8, it follows that $\nabla P \in L_T^1(\dot{B}_{p,2}^{3/p-3/2})$ and that

$$\|\nabla P\|_{L_T^1(\dot{B}_{p,2}^{3/p-3/2})} \lesssim \|\mathcal{Q}((\bar{a} + \dot{S}_{-m}(a - \bar{a}))\nabla P)\|_{L_T^1(\dot{B}_{p,2}^{3/p-3/2})}.$$

According to the uniqueness property of Proposition 2.12 we conclude that $(u, \nabla P) = (0, 0)$.

Observe that in the case $p \in (\frac{6}{5}, 3]$, owing to the fact that $\dot{B}_{p,1}^{3/p-1} \hookrightarrow \dot{B}_{q,1}^{3/q-1}$ for any $q \in (3, 4)$ and $u \in C_T(\dot{B}_{p,1}^{3/p-1})$ along with $(\partial_t u, \nabla^2 u, \nabla P) \in L_T^1(\dot{B}_{p,1}^{3/p-1})$, we get $u \in C_T(\dot{B}_{q,1}^{3/q-1})$ along with $(\partial_t u, \nabla^2 u, \nabla P) \in L_T^1(\dot{B}_{q,1}^{3/q-1})$. Thus, by the uniqueness property for the case $q \in (3, 4)$, we conclude that $(u, \nabla P)$ is identically null for $p \in (\frac{6}{5}, 3]$.

3. Proof of Theorem 1.2

In the rest of the paper we aim to prove Theorem 1.2. Thus, from now on we will work in a 3-dimensional framework.

The linear theory. Let us fix some notation. The space \widetilde{F}_T consists of $(\tilde{w}, \nabla \tilde{Q})$ with $\tilde{w} \in C_T(\dot{B}_{p,1}^{3/p-1})$ and $(\partial_t \tilde{w}, \nabla^2 \tilde{w}, \nabla \tilde{Q}) \in L_T^1(\dot{B}_{p,1}^{3/p-1})$ with the norm

$$\|(\tilde{w}, \nabla \tilde{Q})\|_{\widetilde{F}_T} = \|\tilde{w}\|_{L_T^\infty(\dot{B}_{p,1}^{3/p-1})} + \|(\partial_t \tilde{w}, \nabla^2 \tilde{w}, \nabla \tilde{Q})\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}. \tag{3-1}$$

For any time-dependent vector field \bar{v} we define

$$X_{\bar{v}}(t, x) = x + \int_0^t \bar{v}(\tau, x) d\tau,$$

and $A_{\bar{v}} = (DX_{\bar{v}})^{-1}$. Also, let us denote by $\text{adj}(DX_{\bar{v}})$ the adjugate matrix (i.e., the transpose of the cofactor matrix) of $DX_{\bar{v}}$ and $J_{\bar{v}} = \det(DX_{\bar{v}})$.

Before attacking the well-posedness of (1-4), we first have to solve the linear system

$$\begin{cases} \rho_0 \partial_t \bar{u} - \text{div}(\mu(\rho_0) A_{\bar{v}} D A_{\bar{v}}(\bar{u})) + A_{\bar{v}}^T \nabla \bar{P} = 0, \\ \text{div}(\text{adj}(DX_{\bar{v}})\bar{u}) = 0, \\ \bar{u}|_{t=0} = u_0, \end{cases} \tag{3-2}$$

where $\bar{v} \in C_T(\dot{B}_{p,1}^{3/p-1})$ with $\nabla \bar{v} \in L_T^1(\dot{B}_{p,1}^{3/p}) \cap L_T^2(\dot{B}_{p,1}^{3/p-1})$ is such that

$$\|\nabla \bar{v}\|_{L_T^2(\dot{B}_{p,1}^{3/p-1})} + \|\nabla \bar{v}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \leq 2\alpha \tag{3-3}$$

for a suitably small α . Obviously, this will be achieved using the estimates of the Stokes system established in the previous section; see Theorem 1.3. Let us write (3-2) in the form

$$\begin{cases} \partial_t \bar{u} - \frac{1}{\rho_0} \text{div}(\mu(\rho_0) D(\bar{u})) + \frac{1}{\rho_0} \nabla \bar{P} = \frac{1}{\rho_0} F_{\bar{v}}(\bar{u}, \nabla \bar{P}), \\ \text{div} \bar{u} = \text{div}((\text{Id} - \text{adj}(DX_{\bar{v}}))\bar{u}), \\ \bar{u}|_{t=0} = u_0, \end{cases}$$

with

$$F_{\bar{v}}(\bar{w}, \nabla \bar{Q}) := \operatorname{div}(\mu(\rho_0) A_{\bar{v}} D_{A_{\bar{v}}}(\bar{w}) - \mu(\rho_0) D(\bar{w})) + (\operatorname{Id} - A_{\bar{v}}^T) \nabla \bar{Q}.$$

Consider $(u_L, \nabla P_L)$ with $u_L \in \mathcal{C}(\mathbb{R}^+, \dot{B}_{p,1}^{3/p-1})$ and $(\partial_t u_L, \nabla^2 u_L, \nabla P_L) \in L_{\text{loc}}^1(\dot{B}_{p,1}^{3/p-1})$, the unique solution of

$$\begin{cases} \partial_t u_L - \frac{1}{\rho_0} \operatorname{div}(\mu(\rho_0) D(u_L)) + \frac{1}{\rho_0} \nabla P_L = 0, \\ \operatorname{div} u_L = 0, \\ u_L|_{t=0} = u_0, \end{cases} \quad (3-4)$$

for which we know that

$$\|(u_L, \nabla P_L)\|_{E_T} \leq \|u_0\|_{\dot{B}_{p,1}^{3/p-1}} \exp(C_{\rho_0}(T+1)).$$

Moreover, T can be chosen small enough such that

$$\|\nabla u_L\|_{L_T^2(\dot{B}_{p,1}^{3/p-1})} + \|(\partial_t u_L, \nabla^2 u_L, \nabla P_L)\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \leq \alpha. \quad (3-5)$$

Following the idea in [Danchin and Mucha 2012], and owing to Theorem 1.3, we consider the operator

$$\Phi(\tilde{w}, \nabla \tilde{Q}) = (\tilde{u}, \nabla \tilde{P}), \quad (3-6)$$

which associates to $(\tilde{w}, \nabla \tilde{Q}) \in \tilde{F}_T$ the unique solution $(\tilde{u}, \nabla \tilde{P}) \in \tilde{F}_T$ of

$$\begin{cases} \partial_t \tilde{u} - \frac{1}{\rho_0} \operatorname{div}(\mu(\rho_0) D(\tilde{u})) + \frac{1}{\rho_0} \nabla \tilde{P} = \frac{1}{\rho_0} F_{\bar{v}}(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}), \\ \operatorname{div} \tilde{u} = \operatorname{div}((\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}))(u_L + \tilde{w})), \\ \tilde{u}|_{t=0} = 0. \end{cases}$$

We will show in the following that for any $R > 0$ there exists a sufficiently small $T > 0$ such that there exists a fixed point for Φ in the ball of radius R centered at the origin of \tilde{F}_T . More precisely, according to Theorem 1.3 we get

$$\begin{aligned} \|\Phi(\tilde{w}, \nabla \tilde{Q})\|_{\tilde{F}_T} &\leq \left\| \frac{1}{\rho_0} F_{\bar{v}}(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}) \right\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\ &\quad + \|\partial_t (\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}))(u_L + \tilde{w})\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\ &\quad + \|\nabla \operatorname{div}((\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}))(u_L + \tilde{w}))\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}. \end{aligned} \quad (3-7)$$

We begin by treating the first term:

$$\begin{aligned} &\left\| \frac{1}{\rho_0} F_{\bar{v}}(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}) \right\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\ &\quad \lesssim \left(\frac{1}{\bar{\rho}} + \left\| \frac{1}{\rho_0} - \frac{1}{\bar{\rho}} \right\|_{\dot{B}_{p,1}^{3/p}} \right) \|F_{\bar{v}}(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q})\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}. \end{aligned} \quad (3-8)$$

We write

$$\begin{aligned} T_1 &= \operatorname{div}(\mu(\rho_0)A_{\bar{v}}D_{A_{\bar{v}}}(u_L + \tilde{w})) - \operatorname{div}(\mu(\rho_0)D(u_L + \tilde{w})) \\ &= \operatorname{div}(\mu(\rho_0)(A_{\bar{v}} - \operatorname{Id})D_{A_{\bar{v}}}(u_L + \tilde{w})) + \operatorname{div}(\mu(\rho_0)D_{A_{\bar{v}} - \operatorname{Id}}(u_L + \tilde{w})) \\ &= \operatorname{div}(\mu(\rho_0)(A_{\bar{v}} - \operatorname{Id})D_{A_{\bar{v}} - \operatorname{Id}}(u_L + \tilde{w})) + \operatorname{div}(\mu(\rho_0)(A_{\bar{v}} - \operatorname{Id})D(u_L + \tilde{w})) + \operatorname{div}(\mu(\rho_0)D_{A_{\bar{v}} - \operatorname{Id}}(u_L + \tilde{w})). \end{aligned}$$

Thus, using (3-22) of Proposition 3.27 along with product laws in Besov spaces (see Proposition 3.17) we get the following bound for T_1 :

$$\begin{aligned} \|T_1\|_{L^1_T(\dot{B}^{3/p-1})} &\lesssim C_{\rho_0} \|A_{\bar{v}} - \operatorname{Id}\|_{L^\infty_T(\dot{B}^{3/p})} (1 + \|A_{\bar{v}} - \operatorname{Id}\|_{L^\infty_T(\dot{B}^{3/p})}) (\|\nabla u_L\|_{L^1_T(\dot{B}^{3/p})} + \|\nabla \tilde{w}\|_{L^1_T(\dot{B}^{3/p})}) \\ &\lesssim C_{\rho_0} \|\nabla \bar{v}\|_{L^1_T(\dot{B}^{3/p})} (1 + \|\nabla \bar{v}\|_{L^1_T(\dot{B}^{3/p})}) (\|\nabla u_L\|_{L^1_T(\dot{B}^{3/p})} + \|\nabla \tilde{w}\|_{L^1_T(\dot{B}^{3/p})}) \\ &\lesssim C_{\rho_0} \alpha (\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}). \end{aligned} \tag{3-9}$$

The second term is estimated as

$$\begin{aligned} \|(\operatorname{Id} - A_{\bar{v}}^T)(\nabla P_L + \nabla \tilde{Q})\|_{L^1_T(\dot{B}^{3/p-1})} &\lesssim \|\nabla \bar{v}\|_{L^1_T(\dot{B}^{3/p})} (\|\nabla P_L\|_{L^1_T(\dot{B}^{3/p-1})} + \|\nabla \tilde{Q}\|_{L^1_T(\dot{B}^{3/p-1})}) \\ &\lesssim \alpha (\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}) \end{aligned} \tag{3-10}$$

so that combining (3-8), (3-9) and (3-10) we get

$$\left\| \frac{1}{\rho_0} F_{\bar{v}}(u_L + \tilde{w}, \nabla P_L + \nabla \tilde{Q}) \right\|_{L^1_T(\dot{B}^{3/p-1})} \lesssim C_{\rho_0} \alpha (\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}). \tag{3-11}$$

In order to treat the second term of (3-7) we use the estimates (3-23) and (3-24) of Proposition 3.27 along with Hölder’s inequality in order to obtain

$$\begin{aligned} &\|\partial_t(\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}))(u_L + \tilde{w})\|_{L^1_T(\dot{B}^{3/p-1})} \\ &\lesssim \|(u_L + \tilde{w})\partial_t \operatorname{adj}(DX_{\bar{v}})\|_{L^1_T(\dot{B}^{3/p-1})} + \|(\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}))(\partial_t u_L + \partial_t \tilde{w})\|_{L^1_T(\dot{B}^{3/p-1})} \end{aligned} \tag{3-12}$$

$$\begin{aligned} &\lesssim \|\partial_t \operatorname{adj}(DX_{\bar{v}})\|_{L^2_T(\dot{B}^{3/p-1})} \|u_L + \tilde{w}\|_{L^2_T(\dot{B}^{3/p})} + \|\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}})\|_{L^\infty_T(\dot{B}^{3/p})} \|\partial_t u_L + \partial_t \tilde{w}\|_{L^1_T(\dot{B}^{3/p-1})} \\ &\lesssim \|\nabla \bar{v}\|_{L^2_T(\dot{B}^{3/p-1})} (\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}) + \alpha (\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}) \\ &\lesssim \alpha (\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}). \end{aligned} \tag{3-13}$$

Treating the last term of (3-7) is done with the aid of Corollary 3.24:

$$\begin{aligned} \operatorname{div}((\operatorname{Id} - \operatorname{adj}(DX_{\bar{v}}))(u_L + \tilde{w})) &= (Du_L + D\tilde{w}) : (\operatorname{Id} - J_{\bar{v}}A_{\bar{v}}) \\ &= J_{\bar{v}}(Du_L + D\tilde{w}) : (\operatorname{Id} - A_{\bar{v}}) + (1 - J_{\bar{v}})(\operatorname{div} u_L + \operatorname{div} \tilde{w}). \end{aligned}$$

Thus, using the estimates (3-22) and (3-26) of Proposition 3.27, we may write

$$\begin{aligned}
& \|\nabla \operatorname{div}((\operatorname{Id} - \operatorname{adj}(DX_{\tilde{v}}))(u_L + \tilde{w}))\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\
& \lesssim \|J_{\tilde{v}}(Du_L + D\tilde{w}) : (\operatorname{Id} - A_{\tilde{v}})\|_{L_T^1(\dot{B}_{p,1}^{3/p})} + \|(1 - J_{\tilde{v}})(\operatorname{div} u_L + \operatorname{div} \tilde{w})\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \\
& \lesssim (1 + \|J_{\tilde{v}} - 1\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})}) \|\operatorname{Id} - A_{\tilde{v}}\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})} \|Du_L + D\tilde{w}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \\
& + \|(J_{\tilde{v}} - 1)\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})} \|\operatorname{div} u_L + \operatorname{div} \tilde{w}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \\
& \lesssim \alpha(1 + \alpha)(\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}).
\end{aligned} \tag{3-14}$$

Combining the estimates (3-11), (3-13) and (3-14) we get

$$\|\Phi(\tilde{w}, \nabla \tilde{Q})\|_{\tilde{F}_T} \lesssim \alpha(\alpha + \|(\tilde{w}, \nabla \tilde{Q})\|_{F_T}). \tag{3-15}$$

Thus, for a suitably small α the operator Φ maps the ball of radius R centered at the origin of \tilde{F}_T into itself. Due to the linearity of Φ , one can repeat the above arguments in order to show that Φ is a contraction for small values of α . This concludes the existence of a fixed point of Φ , say $(\tilde{u}^*, \nabla \tilde{P}^*) \in \tilde{F}_T$. Of course,

$$(\bar{u}, \nabla \bar{P}) = (\tilde{u}^*, \nabla \tilde{P}^*) + (u_L, \nabla P_L)$$

is a solution of (3-2).

Proof of Theorem 1.2. Consider T small enough such that $(u_L, \nabla P_L)$, the solution of (3-4), satisfies

$$\|\nabla u_L\|_{L_T^2(\dot{B}_{p,1}^{3/p-1})} + \|\nabla u_L\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \leq \alpha,$$

and consider the closed set

$$\tilde{F}_T(\alpha) = \{(\tilde{v}, \nabla \tilde{Q}) \in F_T : \tilde{v}|_{t=0} = 0, \|(\tilde{v}, \nabla \tilde{Q})\|_{F_T} \leq R\alpha\}$$

with R sufficiently small such that

$$\|\nabla \tilde{v}\|_{L_T^2(\dot{B}_{p,1}^{3/p-1})} + \|\nabla \tilde{v}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \leq \alpha. \tag{3-16}$$

Let us consider the operator S which associates to $(\tilde{v}, \nabla \tilde{Q}) \in \tilde{F}_T(\alpha)$, the solution of

$$\begin{cases}
\partial_t \tilde{u} - \frac{1}{\rho_0} \operatorname{div}(\mu(\rho_0) D(\tilde{u})) + \frac{1}{\rho_0} \nabla \tilde{P} = \frac{1}{\rho_0} F_{(u_L + \tilde{v})}(u_L + \tilde{u}, \nabla P_L + \nabla \tilde{P}), \\
\operatorname{div}(\operatorname{adj}(DX_{u_L + \tilde{v}})(u_L + \tilde{u})) = 0, \\
\tilde{u}|_{t=0} = 0
\end{cases}$$

constructed in the previous section. We will show that for suitably small T and α , the operator S maps the closed set $\tilde{F}_T(\alpha)$ into itself and that S is a contraction. First of all, recalling that $(\tilde{u}, \nabla \tilde{P})$ is in fact the fixed point of the operator Φ defined in (3-6) and using the estimates established in the last section, we conclude that

$$\|(\tilde{u}, \nabla \tilde{P})\|_{\tilde{F}_T} = \|S(\tilde{v}, \nabla \tilde{Q})\|_{F_T} \leq R\alpha \tag{3-17}$$

for some small enough T .

Next, we will deal with the stability estimates. For $i = 1, 2$, let us consider $(\tilde{v}_i, \nabla \tilde{Q}_i) \in \tilde{F}_T(\alpha)$ and $(\tilde{u}_i, \nabla \tilde{P}_i) = S(\tilde{v}_i, \nabla \tilde{Q}_i) \in \tilde{F}_T(\alpha)$. Defining

$$\begin{aligned} (\delta \tilde{v}, \nabla \delta \tilde{Q}) &= (\tilde{v}_1 - \tilde{v}_2, \nabla \tilde{Q}_1 - \nabla \tilde{Q}_2), \\ (\delta \tilde{u}, \nabla \delta \tilde{P}) &= (\tilde{u}_1 - \tilde{u}_2, \nabla \tilde{P}_1 - \nabla \tilde{P}_2), \end{aligned}$$

we see

$$\begin{cases} \partial_t \delta \tilde{u} - \frac{1}{\rho_0} \operatorname{div}(\mu(\rho_0) D(\delta \tilde{u})) + \frac{1}{\rho_0} \nabla \delta \tilde{P} = \frac{1}{\rho_0} \tilde{F}, \\ \operatorname{div} \delta \tilde{u} = \operatorname{div} \tilde{G}, \\ \delta \tilde{u}|_{t=0} = 0, \end{cases}$$

where

$$\begin{aligned} \tilde{F} &= F_1(\delta \tilde{v}, u_L + \tilde{u}_1) + F_1(u_L + \tilde{v}_2, \delta \tilde{u}) + F_2(\delta \tilde{v}, \nabla P_L + \nabla \tilde{P}_1) + F_2(u_L + \tilde{v}_2, \nabla \delta \tilde{P}), \\ \tilde{G} &= -(\operatorname{adj}(DX_{(u_L + \tilde{v}_1)}) - \operatorname{Id})\delta \tilde{u} - (\operatorname{adj}(DX_{(u_L + \tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L + \tilde{v}_2)}))(u_L + \tilde{u}_2) := \tilde{G}_1 + \tilde{G}_2, \end{aligned}$$

and

$$\begin{aligned} F_1(\tilde{v}, \tilde{w}) &= \operatorname{div}(\mu(\rho_0) A_{\tilde{v}} D_{A_{\tilde{v}}}(\tilde{w}) - \mu(\rho_0) D(\tilde{w})), \\ F_2(\tilde{v}, \nabla \tilde{Q}) &= (\operatorname{Id} - A_{\tilde{v}}^T) \nabla \tilde{Q}. \end{aligned}$$

According to [Theorem 1.3](#) we get

$$\|(\delta \tilde{u}, \nabla \delta \tilde{P})\|_{\tilde{F}_T} \lesssim C_{\rho_0} (\|\tilde{F}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} + \|\nabla \operatorname{div} \tilde{G}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} + \|\partial_t \tilde{G}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}). \tag{3-18}$$

Proceeding as in relations [\(3-8\)](#) and [\(3-9\)](#) we get

$$\begin{aligned} \|\tilde{F}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} &\lesssim \|\nabla \delta \tilde{v}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \|\nabla u_L + \nabla \tilde{u}_1\|_{L_T^1(\dot{B}_{p,1}^{3/p})} + \|\nabla u_L + \nabla \tilde{v}_2\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \|\nabla \delta \tilde{u}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \\ &\quad + \|\nabla \delta \tilde{v}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \|\nabla P_L + \nabla \tilde{P}_1\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\ &\quad + \|\nabla u_L + \nabla \tilde{v}_2\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \|\nabla \delta \tilde{P}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\ &\lesssim \alpha \|(\nabla \delta \tilde{v}, \nabla \delta \tilde{Q})\|_{L_T^1(\dot{B}_{p,1}^{3/p})} + \alpha \|(\delta \tilde{u}, \nabla \delta \tilde{P})\|_{\tilde{F}_T}. \end{aligned} \tag{3-19}$$

Of course, we will use the smallness of α to absorb $\alpha \|(\nabla \delta \tilde{u}, \nabla \delta \tilde{P})\|_{L_T^1(\dot{B}_{p,1}^{3/p})}$ into the left-hand side of [\(3-18\)](#).

Next, we treat $\|\nabla \operatorname{div} \tilde{G}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})}$. Using [Proposition 3.23](#), we can write

$$\begin{aligned} -\operatorname{div} \tilde{G}_2 &= \operatorname{div}((\operatorname{adj}(DX_{(u_L + \tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L + \tilde{v}_2)}))(u_L + \tilde{u}_2)) \\ &= \operatorname{div}(\operatorname{adj}(DX_{(u_L + \tilde{v}_1)})(u_L + \tilde{u}_2)) - \operatorname{div}(\operatorname{adj}(DX_{(u_L + \tilde{v}_2)})(u_L + \tilde{u}_2)) \\ &= J_{u_L + \tilde{v}_1} D(u_L + \tilde{u}_2) : A_{(u_L + \tilde{v}_1)} - J_{(u_L + \tilde{v}_2)} D(u_L + \tilde{u}_2) : A_{(u_L + \tilde{v}_2)} \\ &= (J_{u_L + \tilde{v}_1} - J_{(u_L + \tilde{v}_2)}) D(u_L + \tilde{u}_2) : A_{(u_L + \tilde{v}_1)} + J_{(u_L + \tilde{v}_2)} D(u_L + \tilde{u}_2) : (A_{(u_L + \tilde{v}_1)} - A_{(u_L + \tilde{v}_2)}), \end{aligned}$$

and thus, using Propositions 3.27 and 3.28 we get

$$\begin{aligned}
& \|\nabla \operatorname{div} \widetilde{G}_2\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\
& \lesssim \|(J_{u_L+\tilde{v}_1} - J_{(u_L+\tilde{v}_2)})\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})} \|(Du_L, D\tilde{u}_2)\|_{L_T^1(\dot{B}_{p,1}^{3/p})} (1 + \|\operatorname{Id} - A_{(u_L+\tilde{v}_1)}\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})}) \\
& \quad + (1 + \|(J_{u_L+\tilde{v}_2} - 1)\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})}) \|(Du_L, D\tilde{u}_2)\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \|A_{(u_L+\tilde{v}_1)} - A_{(u_L+\tilde{v}_2)}\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})} \\
& \lesssim \alpha \|\nabla \delta \tilde{v}\|_{L_T^1(\dot{B}_{p,1}^{3/p})}.
\end{aligned} \tag{3-20}$$

Next, using again Proposition 3.23 we see

$$\begin{aligned}
-\operatorname{div} \widetilde{G}_1 &= \operatorname{div}((\operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \operatorname{Id})\delta\tilde{u}) = D\delta\tilde{u} : (J_{u_L+\tilde{v}_1} A_{(u_L+\tilde{v}_1)} - \operatorname{Id}) \\
&= J_{u_L+\tilde{v}_1} D\delta\tilde{u} : (A_{(u_L+\tilde{v}_1)} - \operatorname{Id}) + (J_{u_L+\tilde{v}_1} - 1) \operatorname{div} \delta\tilde{u}
\end{aligned}$$

and consequently

$$\begin{aligned}
& \|\nabla \operatorname{div} \widetilde{G}_1\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\
& \lesssim \|J_{u_L+\tilde{v}_1} D\delta\tilde{u} : (A_{(u_L+\tilde{v}_1)} - \operatorname{Id})\|_{L_T^1(\dot{B}_{p,1}^{3/p})} + \|(J_{u_L+\tilde{v}_1} - 1) \operatorname{div} \delta\tilde{u}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \\
& \lesssim (1 + \|J_{u_L+\tilde{v}_1} - 1\|_{L_T^1(\dot{B}_{p,1}^{3/p})}) \|D\delta\tilde{u} : (A_{(u_L+\tilde{v}_1)} - \operatorname{Id})\|_{L_T^1(\dot{B}_{p,1}^{3/p})} + \|J_{u_L+\tilde{v}_1} - 1\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \|\operatorname{div} \delta\tilde{u}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \\
& \lesssim \alpha \|(\delta\tilde{u}, \nabla \delta P)\|_{\widetilde{F}_T}.
\end{aligned} \tag{3-21}$$

Combining (3-20) with (3-21) yields

$$\|\nabla \operatorname{div} \widetilde{G}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \lesssim \alpha \|\nabla \delta \tilde{v}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} + \alpha \|(\delta\tilde{u}, \nabla \delta P)\|_{\widetilde{F}_T}. \tag{3-22}$$

Again, we will use the smallness of α to absorb $\alpha \|(\nabla \delta \tilde{u}, \nabla \delta \widetilde{P})\|_{L_T^1(\dot{B}_{p,1}^{3/p})}$ into the left-hand side of (3-22).

Finally, we write

$$\begin{aligned}
& \partial_t [(\operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L+\tilde{v}_2)}))(u_L + \tilde{u}_2)] \\
& = (\partial_t \operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \partial_t \operatorname{adj}(DX_{(u_L+\tilde{v}_2)}))(u_L + \tilde{u}_2) \\
& \quad + \operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L+\tilde{v}_2)}) (\partial_t u_L + \partial_t \tilde{u}_2).
\end{aligned}$$

Using Proposition 3.28 gives us

$$\begin{aligned}
& \|\partial_t [(\operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L+\tilde{v}_2)}))(u_L + \tilde{u}_2)]\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\
& \lesssim \|\partial_t \operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \partial_t \operatorname{adj}(DX_{(u_L+\tilde{v}_2)})\|_{L_T^2(\dot{B}_{p,1}^{3/p-1})} \|u_L\|_{L_T^2(\dot{B}_{p,1}^{3/p})} \\
& \quad + \|\partial_t \operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \partial_t \operatorname{adj}(DX_{(u_L+\tilde{v}_2)})\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \|\tilde{u}_2\|_{L_T^\infty(\dot{B}_{p,1}^{3/p-1})} \\
& \quad + \|\operatorname{adj}(DX_{(u_L+\tilde{v}_1)}) - \operatorname{adj}(DX_{(u_L+\tilde{v}_2)})\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})} \|\partial_t u_L + \partial_t \tilde{u}_2\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \\
& \lesssim \alpha \|\delta \tilde{v}\|_{L_T^2(\dot{B}_{p,1}^{3/p})}.
\end{aligned}$$

The conclusion is

$$\|\partial_t \widetilde{G}\|_{L_T^1(\dot{B}_{p,1}^{3/p-1})} \lesssim \alpha \|\delta \tilde{v}\|_{L_T^2(\dot{B}_{p,1}^{3/p})}. \tag{3-23}$$

Combining (3-19), (3-22) and (3-23) we get if α is chosen sufficiently small then

$$\|((\delta\tilde{u}, \nabla\delta\tilde{P}))\|_{\tilde{F}_T} \leq \frac{1}{2}\|((\delta\tilde{v}, \nabla\delta\tilde{Q}))\|_{F_T} \tag{3-24}$$

and the operator S is also a contraction over $\tilde{F}_T(\alpha)$. Thus, according to Banach’s theorem there exists a fixed point $(\bar{u}^*, \nabla\bar{P}^*)$ of S . Obviously,

$$(\bar{u}, \nabla\bar{P}) = (u_L, \nabla P_L) + (\bar{u}^*, \nabla\bar{P}^*)$$

is a solution of

$$\begin{cases} \rho_0\partial_t\bar{u} - \operatorname{div}(\mu(\rho_0)A_{\bar{u}}DA_{\bar{u}}(\bar{u})) + A_{\bar{u}}^T\nabla\bar{P} = 0, \\ \operatorname{div}(\operatorname{adj}(DX_{\bar{u}})\bar{u}) = 0, \\ \bar{u}|_{t=0} = u_0. \end{cases} \tag{3-25}$$

In view of Proposition 3.26 we also get $J_{\bar{u}} = 1$. Thus, the second equation of (3-25) becomes

$$\operatorname{div}(A_{\bar{u}}\bar{u}) = 0.$$

The only thing left to prove is the uniqueness property. Consider $(\bar{u}^1, \nabla\bar{P}^1), (\bar{u}^2, \nabla\bar{P}^2) \in F_T$, two solutions of (3-25) with the same initial data $u_0 \in \dot{B}_{p,1}^{3/p-1}$. With $(u_L, \nabla P_L)$ defined above, we let

$$(\tilde{u}^i, \nabla\tilde{P}^i) = (\bar{u}^i, \nabla\bar{P}^i) - (u_L, \nabla P_L) \quad \text{for } i = 1, 2$$

such that the system verified by $(\tilde{u}^i, \nabla\tilde{P}^i)$ is

$$\begin{cases} \partial_t\tilde{u}^i - \frac{1}{\rho_0}\operatorname{div}(\mu(\rho_0)D(\tilde{u}^i)) + \frac{1}{\rho_0}\nabla\tilde{P}^i = \frac{1}{\rho_0}F_{(u_L+\tilde{u}^i)}(u_L + \tilde{u}^i, \nabla P_L + \nabla\tilde{P}^i), \\ \operatorname{div}(A_{(u_L+\tilde{u}^i)}(u_L + \tilde{u}^i)) = 0, \\ \tilde{u}^i|_{t=0} = 0. \end{cases}$$

We are now in the position of performing exactly the same computations as above so that we obtain a time T' sufficiently small such that

$$(\bar{u}^1, \nabla\bar{P}^1) = (\bar{u}^2, \nabla\bar{P}^2) \quad \text{on } [0, T'].$$

It is classical that the above local uniqueness property extends to all of $[0, T]$. □

Proof of Theorem 1.1. Considering $(\rho_0, u_0) \in \dot{B}_{p,1}^{3/p} \times \dot{B}_{p,1}^{3/p-1}$ and applying Theorem 1.2, there exists a positive $T > 0$ such that we may construct a solution $(\bar{u}, \nabla\bar{P})$ to the system (1-4) in F_T . Then, working with a smaller T if needed and considering $X_{\bar{u}}$, the “flow” of \bar{u} defined by (3-17), by using Proposition 3.26 from the Appendix, one obtains that $X_{\bar{u}}$ is a measure preserving C^1 -diffeomorphism over \mathbb{R}^n for all $t \in [0, T]$. Thus we may introduce the Eulerian variable:

$$\rho(t, x) = \rho_0(X_{\bar{u}}^{-1}(t, x)), \quad u(t, x) = \bar{u}(t, X_{\bar{u}}^{-1}(t, x)) \quad \text{and} \quad P(t, x) = \bar{P}(t, X_{\bar{u}}^{-1}(t, x)).$$

Then, Proposition 3.23 ensures that $(\rho, u, \nabla P)$ is a solution of (1-1). As $DX_{\bar{u}} - \operatorname{Id}$ belongs to $\dot{B}_{p,1}^{3/p}$, using Proposition 3.22, we may conclude that $(\rho, u, \nabla P)$ has the announced regularity.

The uniqueness property comes from the fact that considering two solutions $(\rho^i, u^i, \nabla P^i)$ of (1-1), $i = 1, 2$, and considering Y_{u^i} , the flow of u^i , we find that $(u^i(t, Y_{u^i}(t, y)), \nabla P^i(t, Y_{u^i}(t, y)))$ are solutions of the system (1-4) with the same data. Thus, they are equal according to the uniqueness property

announced in [Theorem 1.2](#). Thus, on some nontrivial interval $[0, T'] \subset [0, T]$ (chosen such that condition (3-19) holds), the solutions $(\rho^i, u^i, \nabla P^i)$ are equal. This local uniqueness property obviously entails uniqueness on all of $[0, T]$. □

Appendix

We present here a few results of Fourier analysis used through the text. The full proofs along with other complementary results can be found in [\[Bahouri et al. 2011, Chapter 2\]](#).

Let us introduce the dyadic partition of the space:

Proposition A.1. *Let \mathcal{C} be the annulus $\{\xi \in \mathbb{R}^n : \frac{3}{4} \leq |\xi| \leq \frac{8}{3}\}$. There exists a radial function $\varphi \in \mathcal{D}(\mathcal{C})$ valued in the interval $[0, 1]$ such that*

$$\text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1, \tag{A-1}$$

$$2 \leq |j - j'| \implies \text{Supp}(\varphi(2^{-j} \cdot)) \cap \text{Supp}(\varphi(2^{-j'} \cdot)) = \emptyset. \tag{A-2}$$

Also, the following inequality holds:

$$\text{for all } \xi \in \mathbb{R}^n \setminus \{0\}, \quad \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j}\xi) \leq 1. \tag{A-3}$$

From now on we fix functions χ and φ satisfying the assertions of the above proposition and denote by \tilde{h} and h their Fourier inverses.

The homogeneous dyadic blocks $\dot{\Delta}_j$ and the homogeneous low-frequency cut-off operators \dot{S}_j are

$$\begin{aligned} \dot{\Delta}_j u &= \varphi(2^{-j} D)u = 2^{jn} \int_{\mathbb{R}^n} h(2^j y)u(x - y) dy, \\ \dot{S}_j u &= \chi(2^{-j} D)u = 2^{jn} \int_{\mathbb{R}^n} \tilde{h}(2^j y)u(x - y) dy \end{aligned}$$

for all $j \in \mathbb{Z}$.

Definition A.2. We denote by S'_h the space of tempered distributions such that

$$\lim_{j \rightarrow -\infty} \|\dot{S}_j u\|_{L^\infty} = 0.$$

Let us now define the homogeneous Besov spaces:

Definition A.3. Let s be a real number and $(p, r) \in [1, \infty]$. The homogeneous Besov space $\dot{B}^s_{p,r}$ is the subset of tempered distributions $u \in S'_h$ such that

$$\|u\|_{\dot{B}^s_{p,r}} := \left\| (2^{js} \|\dot{\Delta}_j u\|_{L^2})_{j \in \mathbb{Z}} \right\|_{\ell^r(\mathbb{Z})} < \infty.$$

The next propositions gather some basic properties of Besov spaces.

Proposition A.4. *Let us consider $s \in \mathbb{R}$ and $p, r \in [1, \infty]$ such that*

$$s < \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad r = 1. \tag{A-4}$$

Then $(\dot{B}^s_{p,r}, \|\cdot\|_{\dot{B}^s_{p,r}})$ is a Banach space.

Proposition 3.5. *A tempered distribution $u \in S'_h$ belongs to $\dot{B}^s_{p,r}(\mathbb{R}^n)$ if and only if there exists a sequence $(c_j)_j$ such that $(2^{js}c_j)_j \in \ell^r(\mathbb{Z})$ with norm 1 and a constant $C = C(u) > 0$ such that for any $j \in \mathbb{Z}$ we have*

$$\|\dot{\Delta}_j u\|_{L^p} \leq Cc_j.$$

Proposition 3.6. *Consider s_1 and s_2 two real numbers such that $s_1 < s_2$ and $\theta \in (0, 1)$. Then, there exists a constant $C > 0$ such that for all $r \in [1, \infty]$ we have*

$$\begin{aligned} \|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,r}} &\leq \|u\|_{\dot{B}^{s_1}_{p,r}}^\theta \|u\|_{\dot{B}^{s_2}_{p,r}}^{1-\theta}, \\ \|u\|_{\dot{B}^{\theta s_1+(1-\theta)s_2}_{p,1}} &\leq \frac{C}{s_2-s_1} \left(\frac{1}{\theta} + \frac{1}{1-\theta} \right) \|u\|_{\dot{B}^{s_1}_{p,\infty}}^\theta \|u\|_{\dot{B}^{s_2}_{p,\infty}}^{1-\theta}. \end{aligned}$$

Proposition 3.7. (1) *Let $1 \leq p_1 \leq p_2 \leq \infty$ and $1 \leq r_1 \leq r_2 \leq \infty$. Then, for any real number s , the space $\dot{B}^s_{p_1,r_1}$ is continuously embedded in $\dot{B}^{s-n(1/p_1-1/p_2)}_{p_2,r_2}$.*

(2) *Let $1 \leq p < \infty$. Then, $\dot{B}^{n/p}_{p,1}$ is continuously embedded in $(C_0(\mathbb{R}^n), \|\cdot\|_{L^\infty})$, the space of continuous functions vanishing at infinity.*

Proposition 3.8. *For all $1 \leq p, r \leq \infty$ and $s \in \mathbb{R}$,*

$$\begin{cases} \dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'} \rightarrow \mathbb{R}, \\ (u, v) \rightarrow \sum_j \langle \dot{\Delta}_j u, \tilde{\Delta}_j v \rangle, \end{cases} \tag{3-5}$$

where $\tilde{\Delta}_j := \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}$, defines a continuous bilinear functional on $\dot{B}^s_{p,r} \times \dot{B}^{-s}_{p',r'}$. Denote by $Q^{-s}_{p',r'}$ the set of functions $\varphi \in \mathcal{S} \cap \dot{B}^{-s}_{p',r'}$ such that $\|\varphi\|_{\dot{B}^{-s}_{p',r'}} \leq 1$. If $u \in S'_h$, then we have

$$\|u\|_{\dot{B}^s_{p,r}} \lesssim \sup_{\varphi \in Q^{-s}_{p',r'}} \langle u, \varphi \rangle_{S' \times \mathcal{S}}.$$

Proposition 3.9. *Consider $1 < p, r < \infty$ and $s \in \mathbb{R}$. Furthermore, let $u \in \dot{B}^s_{p,r}$, $v \in \dot{B}^{-s}_{p',r'}$ and $\rho \in L^\infty \cap \mathcal{M}(\dot{B}^s_{p,r}) \cap \mathcal{M}(\dot{B}^{-s}_{p',r'})$. Then, we have*

$$(\rho u, v) = \sum_j \sum_j \langle \dot{\Delta}_j u, \tilde{\Delta}_j(\rho v) \rangle = (u, \rho v). \tag{3-6}$$

The proof of Proposition 3.9 follows from a density argument. Relation (3-6) clearly holds for functions from the Schwartz class: then we may write

$$\int_{\mathbb{R}^n} \rho uv = (\rho u, v) = (u, \rho v).$$

The conditions $1 < p, r < \infty$ and $s \in \mathbb{R}$ ensure that u and v may be approximated by Schwartz functions.

An important feature of Besov spaces with negative index of regularity is the following:

Proposition 3.10. *Let $s < 0$ and $1 \leq p, r \leq \infty$. Let u be a distribution in S'_h . Then, u belongs to $\dot{B}^s_{p,r}$ if and only if*

$$(2^{js} \|\dot{S}_j u\|_{L^p})_{j \in \mathbb{Z}} \in \ell^r(\mathbb{Z}).$$

Moreover, there exists a constant C depending only on the dimension n such that

$$C^{-|s|+1} \|u\|_{\dot{B}_{p,r}^s} \leq \|(2^{js} \|\dot{S}_j u\|_{L^p})_{j \in \mathbb{Z}}\|_{\ell^r(\mathbb{Z})} \leq C \left(1 + \frac{1}{|s|}\right) \|u\|_{\dot{B}_{p,r}^s}.$$

The next proposition tells us how certain multipliers act on Besov spaces.

Proposition 3.11. *Consider A a smooth function on $\mathbb{R}^n \setminus \{0\}$ which is homogeneous of degree m . Then, for any $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ such that*

$$s - m < \frac{n}{p} \quad \text{or} \quad s - m = \frac{n}{p} \quad \text{and} \quad r = 1,$$

the operator⁴ $A(D)$ maps $\dot{B}_{p,r}^s$ continuously into $\dot{B}_{p,r}^{s-m}$.

The next proposition describes how smooth functions act on homogeneous Besov spaces.

Proposition 3.12. *Let f be a smooth function on \mathbb{R} which vanishes at 0. Consider $(s, p, r) \in \mathbb{R} \times [1, \infty]^2$ such that*

$$0 < s < \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad r = 1.$$

Then for any real-valued function $u \in \dot{B}_{p,r}^s \cap L^\infty$, the function $f \circ u$ is in $\dot{B}_{p,r}^s \cap L^\infty$ and we have

$$\|f \circ u\|_{\dot{B}_{p,r}^s} \leq C(f', \|u\|_{L^\infty}) \|u\|_{\dot{B}_{p,r}^s}.$$

Remark 3.13. The constant $C(f', \|u\|_{L^\infty})$ appearing above can be taken to be

$$\sup_{i \in \mathbb{1}, [s]+1} \|f^{(i)}\|_{L^\infty([-M\|u\|_{L^\infty}, -M\|u\|_{L^\infty})},$$

where M is a constant depending only on the dimension n .

Commutator and product estimates. Next, we want to see how the product acts in Besov spaces. The Bony decomposition, introduced in [Bony 1981], offers a mathematical framework to obtain estimates of the product of two distributions, when the latter is defined.

Definition 3.14. Given two tempered distributions $u, v \in S'_h$, the homogeneous paraproduct of v by u is defined as

$$\dot{T}_u v = \sum_{j \in \mathbb{Z}} \dot{S}_{j-1} u \dot{\Delta}_j v. \quad (3-7)$$

The homogeneous remainder of u and v is defined by

$$\dot{R}(u, v) = \sum_{j \in \mathbb{Z}} \dot{\Delta}_j u \dot{\Delta}'_j v, \quad (3-8)$$

where

$$\dot{\Delta}'_j = \dot{\Delta}_{j-1} + \dot{\Delta}_j + \dot{\Delta}_{j+1}.$$

⁴ $A(D)w = \mathcal{F}^{-1}(A\mathcal{F}w)$.

Remark 3.15. Notice that at a formal level, one has the following decomposition of the product of two (sufficiently well-behaved) distributions:

$$uv = \dot{T}_u v + \dot{T}_v u + \dot{R}(u, v) = \dot{T}_u v + \dot{T}'_v u.$$

The next result describes how the paraproduct and remainder behave.

Proposition 3.16. (1) Assume $(s, p, p_1, p_2, r) \in \mathbb{R} \times [1, \infty]^4$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad s < \frac{n}{p} \quad \text{or} \quad s = \frac{n}{p} \quad \text{and} \quad r = 1.$$

Then, the paraproduct maps $L^{p_1} \times \dot{B}_{p_2, r}^s$ into $\dot{B}_{p, r}^s$ and the following estimate holds:

$$\|\dot{T}_f g\|_{\dot{B}_{p, r}^s} \lesssim \|f\|_{L^{p_1}} \|g\|_{\dot{B}_{p_2, r}^s}.$$

(2) Assume $(s, p, p_1, p_2, r, r_1, r_2) \in \mathbb{R} \times [1, \infty]^6$ and $v > 0$ such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \quad \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - v \quad \text{or} \quad s = \frac{n}{p} - v \quad \text{and} \quad r = 1.$$

Then, the paraproduct maps $\dot{B}_{p_1, r_1}^{-v} \times \dot{B}_{p_2, r_2}^{s+v}$ into $\dot{B}_{p, r}^s$ and the following estimate holds:

$$\|\dot{T}_f g\|_{\dot{B}_{p, r}^s} \lesssim \|f\|_{\dot{B}_{p_1, r_1}^{-v}} \|g\|_{\dot{B}_{p_2, r_2}^{s+v}}.$$

(3) Consider $(s_1, s_2, p, p_1, p_2, r, r_1, r_2) \in \mathbb{R}^2 \times [1, \infty]^6$ such that

$$0 < s_1 + s_2 < \frac{n}{p} \quad \text{or} \quad s_1 + s_2 = \frac{n}{p} \quad \text{and} \quad r = 1.$$

Then, the remainder maps $\dot{B}_{p_1, r_1}^{s_1} \times \dot{B}_{p_2, r_2}^{s_2}$ into $\dot{B}_{p, r}^{s_1+s_2}$ and

$$\|\dot{R}(f, g)\|_{\dot{B}_{p, r}^{s_1+s_2}} \leq \|f\|_{\dot{B}_{p_1, r_1}^{s_1}} \|g\|_{\dot{B}_{p_2, r_2}^{s_2}}.$$

As a consequence we obtain the following product rules in Besov space:

Proposition 3.17. Consider $p \in [1, \infty]$ and the real numbers $v_1 \geq 0$ and $v_2 \geq 0$ with

$$v_1 + v_2 < \frac{n}{p} + \min \left\{ \frac{n}{p}, \frac{n}{p'} \right\}.$$

Then, the following estimate holds:

$$\|fg\|_{\dot{B}_{p, 1}^{n/p-v_1-v_2}} \lesssim \|f\|_{\dot{B}_{p, 1}^{n/p-v_1}} \|g\|_{\dot{B}_{p, 1}^{n/p-v_2}}.$$

Proposition 3.18. Consider θ a C^1 function on \mathbb{R}^n such that $(1 + |\cdot|)\hat{\theta} \in L^1$. Let us also consider $p, q \in [1, \infty]$ such that

$$\frac{1}{r} := \frac{1}{p} + \frac{1}{q} \leq 1.$$

Then, there exists a constant C such that for any Lipschitz function a with gradient in L^p , any function $b \in L^q$ and any positive λ ,

$$\|[\theta(\lambda^{-1}D), a]b\|_{L^r} \leq C\lambda^{-1}\|\nabla a\|_{L^p}\|b\|_{L^q}.$$

In particular, when $\theta = \varphi$ and $\lambda = 2^j$ we get

$$\|[\dot{\Delta}_j, a]b\|_{L^r} \leq C2^{-j}\|\nabla a\|_{L^p}\|b\|_{L^q}.$$

Proposition 3.19. Assume s, ν and $p \in [1, \infty]$ are such that

$$0 \leq \nu \leq \frac{n}{p} \quad \text{and} \quad -1 - \min\left\{\frac{n}{p}, \frac{n}{p'}\right\} < s \leq \frac{n}{p} - \nu.$$

Then, there exists a constant C depending only on s, ν, p and n such that for all $l \in \overline{1, n}$ we have for some sequence $(c_j)_{j \in \mathbb{Z}}$ with $\|(c_j)_{j \in \mathbb{Z}}\|_{\ell^1(\mathbb{Z})} = 1$,

$$\|\partial_l[a, \dot{\Delta}_j]w\|_{L^p} \leq Cc_j2^{-js}\|\nabla a\|_{\dot{B}_{p,1}^{n/p-\nu}}\|w\|_{\dot{B}_{p,1}^{s+\nu}}$$

for all $j \in \mathbb{Z}$.

For a proof of the above results we refer the reader to the Appendix of [Danchin 2014, Lemmas A.5 and A.6].

Proposition 3.20. Consider a homogeneous function $A : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ of degree 0. Let us consider $s \in \mathbb{R}$, $0 < \nu \leq 1$ and $p, r, r_1, r_2 \in [1, \infty]$ such that

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$s < \frac{n}{p} - \nu \quad \text{or} \quad s = \frac{n}{p} - \nu \quad \text{and} \quad r_2 = 1. \quad (3-9)$$

Moreover, assume $w \in \dot{B}_{p,r_2}^{s+\nu}$ and $a \in L^\infty$ with $\nabla a \in \dot{B}_{\infty,r_1}^{-\nu}$. Then, the following estimate holds:

$$\|[A(D), \dot{T}_a]w\|_{\dot{B}_{p,r}^{s+1}} \lesssim \|\nabla a\|_{\dot{B}_{\infty,r_1}^{-\nu}}\|w\|_{\dot{B}_{p,r_2}^{s+\nu}}. \quad (3-10)$$

As this result is of great importance in the analysis of the pressure term, we present a sketched proof below (see also [Bahouri et al. 2011, Chapter 2, Lemma 2.99]).

Proof. The fact that $a \in L^\infty$, along with relation (3-9), guarantees that $A(D)w \in \dot{B}_{p,r}^{s+\nu}$ and that the paraproducts $\dot{T}_a w$ and $\dot{T}_a A(D)w$ are well-defined. We observe that there exists a function $\tilde{\varphi}$ supported in some annulus which equals 1 on the support of φ such that one may write (of course it is here that we use the homogeneity of A)

$$[A(D), \dot{T}_a]w = \sum_j [(A\tilde{\varphi})(2^{-j}D), \dot{S}_{j-1}a] \dot{\Delta}_j w.$$

But according to Proposition 3.18 we have

$$2^{j(s+1)}\|[A\tilde{\varphi})(2^{-j}D), \dot{S}_{j-1}a] \dot{\Delta}_j w\|_{L^p} \lesssim 2^{-j\nu}\|\nabla \dot{S}_{j-1}a\|_{L^\infty}2^{j(s+\nu)}\|\dot{\Delta}_j w\|_{L^p}.$$

The last relation obviously implies (3-10). □

As a consequence of the above proposition and [Proposition 3.16](#) we get the following:

Proposition 3.21. *Let us consider a homogeneous function $A : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}$ of degree 0, $s \in \mathbb{R}$, $0 < \nu \leq 1$ and $p, r, r_1, r_2 \in [1, \infty]$ such that*

$$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

and

$$-1 - \min\left\{\frac{n}{p}, \frac{n}{p'}\right\} < s < \frac{n}{p} - \nu \quad \text{or} \quad s = \frac{n}{p} - \nu \quad \text{and} \quad r = r_2 = 1. \tag{3-11}$$

assume $w \in \dot{B}_{p,r_2}^{s+\nu}$ and $a \in L^\infty$ with $\nabla a \in \dot{B}_{\infty,r_1}^{-\nu}$. Then, the following estimate holds:

$$\| [A(D), a]w \|_{\dot{B}_{p,r}^{s+1}} \lesssim \| \nabla a \|_{\dot{B}_{p,r_1}^{n/p-\nu}} \| w \|_{\dot{B}_{p,r_2}^{s+\nu}}.$$

Properties of Lagrangian coordinates. The following results are gathered from [\[Danchin 2014\]](#) and [\[Danchin and Mucha 2012\]](#). More precisely, proofs of [Propositions 3.22, 3.23](#), the estimate [\(3-26\)](#) of [Proposition 3.27](#) and the estimate [\(3-31\)](#) of [Proposition 3.28](#) can be found in [\[Danchin 2014, pp. 782–786\]](#). [Propositions 3.27](#) and [3.28](#) can be found in the Appendix of [\[Danchin and Mucha 2012\]](#). [Proposition 3.26](#) is inspired by [\[Danchin and Mucha 2012\]](#).

Proposition 3.22. *Let X be a globally defined bi-Lipschitz diffeomorphism of \mathbb{R}^3 and $-\frac{3}{p'} < s \leq \frac{3}{p}$. Then $a \rightarrow a \circ X$ is a self-map over $\dot{B}_{p,1}^s$ whenever*

- (1) $s \in (0, 1)$;
- (2) $s \geq 1$ and $(DX - \text{Id}) \in \dot{B}_{p,1}^{3/p}$.

The following result interferes in a crucial manner in the proof of the well-posedness result for the inhomogeneous incompressible Navier–Stokes system.

Proposition 3.23. *Let m be a C^1 scalar function over \mathbb{R}^3 and $u \in \mathbb{R}^3$ a C^1 vector field. Let X be a C^1 diffeomorphism and we define $J := \det(DX)$. Suppose $J > 0$. Then, the following relations hold:*

$$(\nabla m) \circ X = J^{-1} \operatorname{div}(\operatorname{adj}(DX)m \circ X), \tag{3-12}$$

$$(\operatorname{div} u) \circ X = J^{-1} \operatorname{div}(\operatorname{adj}(DX)u \circ X). \tag{3-13}$$

Corollary 3.24. *Let m be a C^1 scalar function over \mathbb{R}^3 and $u \in \mathbb{R}^3$ be a C^1 vector field. Let X be a C^1 diffeomorphism and $J := \det(DX)$. Suppose $J > 0$. Then, we have*

$$J^{-1} \operatorname{div}(\operatorname{adj}(DX)u) = Du : (DX)^{-1}, \tag{3-14}$$

$$J^{-1} \operatorname{div}(\operatorname{adj}(DX)m) = [(DX)^{-1}]^T \nabla m. \tag{3-15}$$

Proof. In order to ease reading, we define $F|_x := F(x)$. Writing u as $u \circ X \circ X^{-1}$, using the chain rule and Einstein convention over repeated index, we write

$$\begin{aligned} (\operatorname{div} u)|_x &= \partial_k (u^i \circ X)|_{X^{-1}(x)} \partial_i (X^{-1})^k|_x \\ &= D(u \circ X)|_{X^{-1}(x)} : D(X^{-1})|_x \\ &= D(u \circ X)|_{X^{-1}(x)} : (DX)^{-1}|_{X^{-1}(x)}, \end{aligned}$$

and thus, we get

$$(\operatorname{div} u) \circ X = D(u \circ X) : (DX)^{-1}. \quad (3-16)$$

Then, using (3-13) and (3-16) we get

$$\begin{aligned} J^{-1} \operatorname{div}(\operatorname{adj}(DX)u) &= J^{-1} \operatorname{div}(\operatorname{adj}(DX)u \circ X^{-1} \circ X) = (\operatorname{div} u \circ X^{-1}) \circ X \\ &= D(u \circ X^{-1} \circ X) : (DX)^{-1} = Du : (DX)^{-1}. \end{aligned}$$

In a similar manner we prove (3-15). \square

For any \bar{v} a time-dependent vector field we set

$$X_{\bar{v}}(t, x) = x + \int_0^t \bar{v}(\tau, x) d\tau \quad (3-17)$$

and we define

$$A_{\bar{v}} = (DX_{\bar{v}})^{-1}. \quad (3-18)$$

It is crucial to know (in order to pass back in Eulerian coordinates, for instance) when $X_{\bar{v}}$ is a global diffeomorphism. In order to achieve this, we will use the following theorem due to Hadamard:

Theorem 3.25 (Hadamard). *Let $X : \mathbb{R}^n \rightarrow \mathbb{R}^n$ a function of class C^1 . Then, the following are equivalent:*

- (1) X is a local diffeomorphism and $\lim_{|x| \rightarrow \infty} |X(x)| = \infty$.
- (2) X is a global C^1 -diffeomorphism over \mathbb{R}^n .

For a proof of this result one can consult, for instance, [Katriel 1994].

Proposition 3.26. *Let us consider $\bar{v} \in C_b([0, T], \dot{B}_{p,1}^{3/p-1})$ with $\partial_t \bar{v}, \nabla^2 \bar{v} \in L_T^1(\dot{B}_{p,1}^{3/p-1})$. Then, there exists a positive α such that if*

$$\|\nabla \bar{v}\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \leq \alpha, \quad (3-19)$$

then, for any $t \in [0, T]$, we have $X_{\bar{v}}(t, \cdot)$ introduced in (3-17) is a global C^1 -diffeomorphism over \mathbb{R}^3 and $\det(DX_{\bar{v}}) > 0$. Moreover, if

$$\operatorname{div}(\operatorname{adj}(DX_{\bar{v}})\bar{v}) = 0 \quad (3-20)$$

then, $X_{\bar{v}}$ is measure-preserving, i.e.,

$$\det DX_{\bar{v}} = 1. \quad (3-21)$$

Proof. Differentiating $X_{\bar{v}}$, we obtain

$$DX_{\bar{v}}(t, \cdot) = \operatorname{Id} + \int_0^t D\bar{v}(\tau, \cdot) d\tau$$

and because of the embedding of $\dot{B}_{p,1}^{3/p}$ into the space of continuous functions, see Proposition 3.7, we conclude $X_{\bar{v}} \in C^1([0, T] \times \mathbb{R}^3)$. We observe that

$$\begin{aligned} \|DX_{\bar{v}}(t, \cdot) - \operatorname{Id}\|_{L^\infty(\mathbb{R}^3)} &\leq \int_0^t \|D\bar{v}(\tau, \cdot)\|_{L^\infty} d\tau \\ &\leq C \|\nabla \bar{v}\|_{L_t^1(\dot{B}_{p,1}^{3/p})} \leq \alpha C. \end{aligned}$$

Thus choosing α sufficiently small ensures that $X_{\bar{v}}(t, \cdot)$ is a local C^1 -diffeomorphism over \mathbb{R}^3 . The second condition of Hadamard’s theorem is verified in the following lines. Using the triangle inequality we get

$$\begin{aligned} |X_{\bar{v}}(t, x)| &\geq |x| - \int_0^t |\bar{v}(\tau, x)| d\tau \geq |x| - \int_0^t \|\bar{v}(\tau, \cdot)\|_{L^\infty} d\tau \\ &\geq |x| - C \int_0^t \|\bar{v}(\tau, \cdot)\|_{\dot{B}_{p,1}^{3/p}} d\tau \\ &\geq |x| - C \sqrt{t} \|\bar{v}\|_{L_t^2(\dot{B}_{p,1}^{3/p})}. \end{aligned}$$

The conclusion is that $X_{\bar{v}}(t, \cdot)$ is a global C^1 -diffeomorphism over \mathbb{R}^3 . Let us define $J_{\bar{v}} := \det DX_{\bar{v}} \neq 0$. Using Jacobi’s formula we get

$$J_{\bar{v}}(t, x) = 1 + \int_0^t \text{tr}(D\bar{v}(\tau, x) \text{adj}(DX_{\bar{v}})(\tau, x)) d\tau.$$

Recall that according to [Danchin and Mucha 2012, Lemma A.4] we may write

$$\text{Id} - \text{adj}(DX_{\bar{v}}) = \int_0^t (D\bar{v} - \text{div } \bar{v} \text{Id}) d\tau + P_2 \left(\int_0^t D\bar{v} d\tau \right),$$

where the coefficients of the matrix $P_2 : \mathcal{M}_n(\mathbb{R}) \rightarrow \mathcal{M}_n(\mathbb{R})$ are at least quadratic polynomial functions of degree $n - 1$. Using this identity combined with the embedding $L^\infty \hookrightarrow \dot{B}_{p,1}^{3/p}$ and the smallness condition (3-19) we get $J_{\bar{v}} > 0$. In order to prove the second part of the proposition, let us define

$$v(t, x) := \bar{v}(t, X_{\bar{v}}^{-1}(t, x)).$$

Using relation (3-20) combined with (3-13) we get

$$0 = J_{\bar{v}}^{-1} \text{div}(\text{adj}(DX_{\bar{v}})\bar{v}) = \text{div}(\bar{v} \circ X_{\bar{v}}^{-1}) \circ X_{\bar{v}},$$

which implies

$$\text{div } v = \text{div}(\bar{v} \circ X_{\bar{v}}^{-1}) = 0.$$

Since $X_{\bar{v}}$ can be viewed as being the flow of v , using Jacobi’s formula we can conclude the validity of (3-21). Indeed, we have

$$\begin{aligned} X_{\bar{v}}(t, x) &= x + \int_0^t \bar{v}(\tau, x) d\tau \\ &= x + \int_0^t \bar{v}(\tau, X_{\bar{v}}^{-1}(\tau, X_{\bar{v}}(\tau, x))) d\tau \\ &= x + \int_0^t v(\tau, X_{\bar{v}}(\tau, x)) d\tau. \end{aligned}$$

Then, Jacobi’s formula implies

$$\det(DX_{\bar{v}})(t, x) = \exp\left(\int_0^t (\text{div } v)(\tau, X_{\bar{v}}(\tau, x))\right) = 1. \quad \square$$

Proposition 3.27. Consider $\bar{v} \in C_b([0, T], \dot{B}_{p,1}^{3/p-1})$ with $\partial_t \bar{v}, \nabla^2 \bar{v} \in L_T^1(\dot{B}_{p,1}^{3/p-1})$ satisfying the smallness condition (3-3). Let $X_{\bar{v}}$ be defined by (3-17) and $J_{\bar{v}} = \det DX_{\bar{v}}$. Then for all $t \in [0, T]$,

$$\|\text{Id} - A_{\bar{v}}(t)\|_{\dot{B}_{p,1}^{3/p}} \lesssim \|\nabla \bar{v}\|_{L_t^1(\dot{B}_{p,1}^{3/p})}, \quad (3-22)$$

$$\|\text{Id} - \text{adj}(DX_{\bar{v}})(t)\|_{\dot{B}_{p,1}^{3/p}} \lesssim \|\nabla \bar{v}\|_{L_t^1(\dot{B}_{p,1}^{3/p})}, \quad (3-23)$$

$$\|\partial_t \text{adj}(DX_{\bar{v}})(t)\|_{\dot{B}_{p,1}^{3/p-1}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}_{p,1}^{3/p-1}}, \quad \text{if } p < 6, \quad (3-24)$$

$$\|\partial_t \text{adj}(DX_{\bar{v}})(t)\|_{\dot{B}_{p,1}^{3/p}} \lesssim \|\nabla \bar{v}(t)\|_{\dot{B}_{p,1}^{3/p}}, \quad (3-25)$$

$$\|J_{\bar{v}}^{\pm}(t) - 1\|_{\dot{B}_{p,1}^{3/p}} \lesssim \|\nabla \bar{v}\|_{L_t^1(\dot{B}_{p,1}^{3/p})}. \quad (3-26)$$

In order to establish stability estimates we use the following:

Proposition 3.28. Let $\bar{v}_1, \bar{v}_2 \in C_b([0, T], \dot{B}_{p,1}^{3/p-1})$ with $\partial_t \bar{v}_1, \partial_t \bar{v}_2, \nabla^2 \bar{v}_1, \nabla^2 \bar{v}_2 \in L_T^1(\dot{B}_{p,1}^{3/p-1})$, both satisfying the smallness condition (3-19) and $\delta v = \bar{v}_2 - \bar{v}_1$. Then we have

$$\|A_{\bar{v}_1} - A_{\bar{v}_2}\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})} \lesssim \|\nabla \delta v\|_{L_T^1(\dot{B}_{p,1}^{3/p})}, \quad (3-27)$$

$$\|\text{adj}(DX_{\bar{v}_1}) - \text{adj}(DX_{\bar{v}_2})\|_{L_T^\infty(\dot{B}_{p,1}^{3/p})} \lesssim \|\nabla \delta v\|_{L_T^1(\dot{B}_{p,1}^{3/p})}, \quad (3-28)$$

$$\|\partial_t \text{adj}(DX_{\bar{v}_1}) - \partial_t \text{adj}(DX_{\bar{v}_2})\|_{L_T^1(\dot{B}_{p,1}^{3/p})} \lesssim \|\nabla \delta v\|_{L_T^1(\dot{B}_{p,1}^{3/p})}, \quad (3-29)$$

$$\|\partial_t \text{adj}(DX_{\bar{v}_1}) - \partial_t \text{adj}(DX_{\bar{v}_2})\|_{L_T^2(\dot{B}_{p,1}^{3/p-1})} \lesssim \|\nabla \delta v\|_{L_T^2(\dot{B}_{p,1}^{3/p-1})}, \quad \text{if } p < 6, \quad (3-30)$$

$$\|J_{\bar{v}_1}^{\pm}(t) - J_{\bar{v}_2}^{\pm}(t)\|_{\dot{B}_{p,1}^{3/p}} \lesssim \|\nabla \delta v\|_{L_t^1(\dot{B}_{p,1}^{3/p})}. \quad (3-31)$$

Acknowledgements

After submitting this work, we were informed of the existence of [Zhai and Yin 2017], which contains similar results obtained using another method.

The author is grateful to the anonymous referee for his comments and suggestions that have considerably improved the quality of this work.

References

- [Abidi 2007] H. Abidi, “Équation de Navier–Stokes avec densité et viscosité variables dans l’espace critique”, *Rev. Mat. Iberoam.* **23**:2 (2007), 537–586. [MR](#) [Zbl](#)
- [Abidi and Paicu 2007] H. Abidi and M. Paicu, “Existence globale pour un fluide inhomogène”, *Ann. Inst. Fourier (Grenoble)* **57**:3 (2007), 883–917. [MR](#) [Zbl](#)
- [Abidi et al. 2012] H. Abidi, G. Gui, and P. Zhang, “On the wellposedness of three-dimensional inhomogeneous Navier–Stokes equations in the critical spaces”, *Arch. Ration. Mech. Anal.* **204**:1 (2012), 189–230. [MR](#) [Zbl](#)
- [Abidi et al. 2013] H. Abidi, G. Gui, and P. Zhang, “Well-posedness of 3-D inhomogeneous Navier–Stokes equations with highly oscillatory initial velocity field”, *J. Math. Pures Appl.* (9) **100**:2 (2013), 166–203. [MR](#) [Zbl](#)
- [Antontsev et al. 1990] S. N. Antontsev, A. V. Kazhikhov, and V. N. Monakhov, *Boundary value problems in mechanics of nonhomogeneous fluids*, Studies in Mathematics and its Applications **22**, North-Holland, Amsterdam, 1990. [MR](#) [Zbl](#)

- [Bahouri et al. 2011] H. Bahouri, J.-Y. Chemin, and R. Danchin, *Fourier analysis and nonlinear partial differential equations*, Grundlehren der Mathematischen Wissenschaften **343**, Springer, Heidelberg, 2011. [MR](#) [Zbl](#)
- [Bony 1981] J.-M. Bony, “Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires”, *Ann. Sci. École Norm. Sup. (4)* **14**:2 (1981), 209–246. [MR](#) [Zbl](#)
- [Choe and Kim 2003] H. J. Choe and H. Kim, “Strong solutions of the Navier–Stokes equations for nonhomogeneous incompressible fluids”, *Comm. Partial Differential Equations* **28**:5–6 (2003), 1183–1201. [MR](#) [Zbl](#)
- [Danchin 2003] R. Danchin, “Density-dependent incompressible viscous fluids in critical spaces”, *Proc. Roy. Soc. Edinburgh Sect. A* **133**:6 (2003), 1311–1334. [MR](#) [Zbl](#)
- [Danchin 2004] R. Danchin, “Local and global well-posedness results for flows of inhomogeneous viscous fluids”, *Adv. Differential Equations* **9**:3–4 (2004), 353–386. [MR](#) [Zbl](#)
- [Danchin 2007] R. Danchin, “Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density”, *Comm. Partial Differential Equations* **32**:7–9 (2007), 1373–1397. [MR](#) [Zbl](#)
- [Danchin 2010] R. Danchin, “On the well-posedness of the incompressible density-dependent Euler equations in the L^p framework”, *J. Differential Equations* **248**:8 (2010), 2130–2170. [MR](#) [Zbl](#)
- [Danchin 2014] R. Danchin, “A Lagrangian approach for the compressible Navier–Stokes equations”, *Ann. Inst. Fourier (Grenoble)* **64**:2 (2014), 753–791. [MR](#) [Zbl](#)
- [Danchin and Mucha 2009] R. Danchin and P. B. Mucha, “A critical functional framework for the inhomogeneous Navier–Stokes equations in the half-space”, *J. Funct. Anal.* **256**:3 (2009), 881–927. [MR](#) [Zbl](#)
- [Danchin and Mucha 2012] R. Danchin and P. B. Mucha, “A Lagrangian approach for the incompressible Navier–Stokes equations with variable density”, *Comm. Pure Appl. Math.* **65**:10 (2012), 1458–1480. [MR](#) [Zbl](#)
- [Danchin and Mucha 2013] R. Danchin and P. B. Mucha, “Incompressible flows with piecewise constant density”, *Arch. Ration. Mech. Anal.* **207**:3 (2013), 991–1023. [MR](#) [Zbl](#)
- [Danchin and Mucha 2015] R. Danchin and P. B. Mucha, *Critical functional framework and maximal regularity in action on systems of incompressible flows*, *Mém. Soc. Math. Fr. (N.S.)* **143**, Société Mathématique de France, Paris, 2015. [MR](#) [Zbl](#)
- [Desjardins 1997] B. t. Desjardins, “Regularity results for two-dimensional flows of multiphase viscous fluids”, *Arch. Rational Mech. Anal.* **137**:2 (1997), 135–158. [MR](#) [Zbl](#)
- [Haspot 2012] B. Haspot, “Well-posedness for density-dependent incompressible fluids with non-Lipschitz velocity”, *Ann. Inst. Fourier (Grenoble)* **62**:5 (2012), 1717–1763. [MR](#) [Zbl](#)
- [Huang et al. 2013a] J. Huang, M. Paicu, and P. Zhang, “Global solutions to 2-D inhomogeneous Navier–Stokes system with general velocity”, *J. Math. Pures Appl. (9)* **100**:6 (2013), 806–831. [MR](#) [Zbl](#)
- [Huang et al. 2013b] J. Huang, M. Paicu, and P. Zhang, “Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system with rough density”, pp. 159–180 in *Studies in phase space analysis with applications to PDEs*, edited by M. Cicognani et al., *Progr. Nonlinear Differential Equations Appl.* **84**, Springer, 2013. [MR](#) [Zbl](#)
- [Huang et al. 2013c] J. Huang, M. Paicu, and P. Zhang, “Global well-posedness of incompressible inhomogeneous fluid systems with bounded density or non-Lipschitz velocity”, *Arch. Ration. Mech. Anal.* **209**:2 (2013), 631–682. [MR](#) [Zbl](#)
- [Katriel 1994] G. Katriel, “Mountain pass theorems and global homeomorphism theorems”, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **11**:2 (1994), 189–209. [MR](#) [Zbl](#)
- [Kažihov 1974] A. V. Kažihov, “Solvability of the initial-boundary value problem for the equations of the motion of an inhomogeneous viscous incompressible fluid”, *Dokl. Akad. Nauk SSSR* **216** (1974), 1008–1010. [MR](#)
- [Krylov 2008] N. V. Krylov, *Lectures on elliptic and parabolic equations in Sobolev spaces*, Graduate Studies in Mathematics **96**, American Mathematical Society, Providence, RI, 2008. [MR](#) [Zbl](#)
- [Ladyzhenskaya and Solonnikov 1975] O. A. Ladyzhenskaya and V. A. Solonnikov, “The unique solvability of an initial-boundary value problem for viscous incompressible inhomogeneous fluids”, *Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI)* **52** (1975), 52–109. In Russian; translated in *J. Soviet Math.* **9**:5 (1978), 697–749. [MR](#) [Zbl](#)
- [Lions 1996] P.-L. Lions, *Mathematical topics in fluid mechanics, I: Incompressible models*, Oxford Lecture Series in Mathematics and its Applications **3**, Oxford University Press, New York, 1996. [MR](#) [Zbl](#)

- [Paicu and Zhang 2012] M. Paicu and P. Zhang, “Global solutions to the 3-D incompressible inhomogeneous Navier–Stokes system”, *J. Funct. Anal.* **262**:8 (2012), 3556–3584. [MR](#) [Zbl](#)
- [Paicu et al. 2013] M. Paicu, P. Zhang, and Z. Zhang, “Global unique solvability of inhomogeneous Navier–Stokes equations with bounded density”, *Comm. Partial Differential Equations* **38**:7 (2013), 1208–1234. [MR](#) [Zbl](#)
- [Poulon 2015] E. Poulon, “Well-posedness for density-dependent incompressible viscous fluids on the torus \mathbb{T}^3 ”, preprint, 2015. [arXiv](#)
- [Xu et al. 2016] H. Xu, Y. Li, and X. Zhai, “On the well-posedness of 2-D incompressible Navier–Stokes equations with variable viscosity in critical spaces”, *J. Differential Equations* **260**:8 (2016), 6604–6637. [MR](#) [Zbl](#)
- [Zhai and Yin 2017] X. Zhai and Z. Yin, “Global well-posedness for the 3D incompressible inhomogeneous Navier–Stokes equations and MHD equations”, *J. Differential Equations* **262**:3 (2017), 1359–1412. [MR](#) [Zbl](#)

Received 26 Jul 2016. Revised 18 Oct 2016. Accepted 28 Nov 2016.

COSMIN BURTEA: cosmin.burtea@u-pec.fr

Université Paris-Est Créteil, LAMA - CNRS UMR 8050, 61 Avenue du Général de Gaulle, 94010 Créteil, France

Analysis & PDE

msp.org/apde

EDITORS

EDITOR-IN-CHIEF

Patrick Gérard

patrick.gerard@math.u-psud.fr

Université Paris Sud XI

Orsay, France

BOARD OF EDITORS

Nicolas Burq	Université Paris-Sud 11, France nicolas.burq@math.u-psud.fr	Werner Müller	Universität Bonn, Germany mueller@math.uni-bonn.de
Massimiliano Berti	Scuola Intern. Sup. di Studi Avanzati, Italy berti@sissa.it	Gilles Pisier	Texas A&M University, and Paris 6 pisier@math.tamu.edu
Sun-Yung Alice Chang	Princeton University, USA chang@math.princeton.edu	Tristan Rivière	ETH, Switzerland riviere@math.ethz.ch
Michael Christ	University of California, Berkeley, USA mchrist@math.berkeley.edu	Igor Rodnianski	Princeton University, USA irod@math.princeton.edu
Charles Fefferman	Princeton University, USA cf@math.princeton.edu	Wilhelm Schlag	University of Chicago, USA schlag@math.uchicago.edu
Ursula Hamenstaedt	Universität Bonn, Germany ursula@math.uni-bonn.de	Sylvia Serfaty	New York University, USA serfaty@cims.nyu.edu
Vaughan Jones	U.C. Berkeley & Vanderbilt University vaughan.f.jones@vanderbilt.edu	Yum-Tong Siu	Harvard University, USA siu@math.harvard.edu
Vadim Kaloshin	University of Maryland, USA vadim.kaloshin@gmail.com	Terence Tao	University of California, Los Angeles, USA tao@math.ucla.edu
Herbert Koch	Universität Bonn, Germany koch@math.uni-bonn.de	Michael E. Taylor	Univ. of North Carolina, Chapel Hill, USA met@math.unc.edu
Izabella Laba	University of British Columbia, Canada ilaba@math.ubc.ca	Gunther Uhlmann	University of Washington, USA gunther@math.washington.edu
Gilles Lebeau	Université de Nice Sophia Antipolis, France lebeau@unice.fr	András Vasy	Stanford University, USA andras@math.stanford.edu
Richard B. Melrose	Massachusetts Inst. of Tech., USA rbb@math.mit.edu	Dan Virgil Voiculescu	University of California, Berkeley, USA dvv@math.berkeley.edu
Frank Merle	Université de Cergy-Pontoise, France Frank.Merle@u-cergy.fr	Steven Zelditch	Northwestern University, USA zelditch@math.northwestern.edu
William Minicozzi II	Johns Hopkins University, USA minicozz@math.jhu.edu	Maciej Zworski	University of California, Berkeley, USA zvorski@math.berkeley.edu
Clément Mouhot	Cambridge University, UK c.mouhot@dpms.cam.ac.uk		

PRODUCTION

production@msp.org

Silvio Levy, Scientific Editor


See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2017 is US \$265/year for the electronic version, and \$470/year (+\$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscribers address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2017 Mathematical Sciences Publishers

ANALYSIS & PDE

Volume 10 No. 2 2017

Some energy inequalities involving fractional GJMS operators JEFFREY S. CASE	253
Exact controllability for quasilinear perturbations of KdV PIETRO BALDI, GIUSEPPE FLORIDIA and EMANUELE HAUS	281
Operators of subprincipal type NILS DENCKER	323
Anisotropic Ornstein nonequalities KRYSTIAN KAZANIECKI, DMITRIY M. STOLYAROV and MICHAŁ WOJCIECHOWSKI	351
A note on stability shifting for the Muskat problem, II: From stable to unstable and back to stable DIEGO CÓRDOBA, JAVIER GÓMEZ-SERRANO and ANDREJ ZLATOŠ	367
Derivation of an effective evolution equation for a strongly coupled polaron RUPERT L. FRANK and ZHOU GANG	379
Time-weighted estimates in Lorentz spaces and self-similarity for wave equations with singular potentials MARCELO F. DE ALMEIDA and LUCAS C. F. FERREIRA	423
Optimal well-posedness for the inhomogeneous incompressible Navier–Stokes system with general viscosity COSMIN BURTEA	439
Global dynamics below the standing waves for the focusing semilinear Schrödinger equation with a repulsive Dirac delta potential MASAHIRO IKEDA and TAKAHISA INUI	481