ANALYSIS & PDE

Volume 10 No. 3 2017

STEVE HOFMANN, PHI LE, JOSÉ MARÍA MARTELL AND KAJ NYSTRÖM

THE WEAK-A_∞ PROPERTY OF HARMONIC AND p-HARMONIC MEASURES IMPLIES UNIFORM RECTIFIABILITY

msp

ANALYSIS AND PDE
Vol. 10, No. 3, 2017
dx.doi.org/10.2140/apde.2017.10.513

THE WEAK-A_∞ PROPERTY OF HARMONIC AND p-HARMONIC MEASURES IMPLIES UNIFORM RECTIFIABILITY

STEVE HOFMANN, PHI LE, JOSÉ MARÍA MARTELL AND KAJ NYSTRÖM

Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an Ahlfors–David regular set of dimension n. We show that the weak-A_∞ property of harmonic measure, for the open set $\Omega := \mathbb{R}^{n+1} \setminus E$, implies uniform rectifiability of E. More generally, we establish a similar result for the Riesz measure, p-harmonic measure, associated to the p-Laplace operator, $1 < p < \infty$.

1. Introduction

In this paper we prove quantitative, scale invariant results of free boundary type, for harmonic measure and, more generally, for p-harmonic measure. More precisely, let $\Omega \subset \mathbb{R}^{n+1}$ be an open set (not necessarily connected nor bounded) satisfying an interior corkscrew condition, whose boundary is n-dimensional Ahlfors–David regular (ADR) (see Definition 2.1). Given these background hypotheses we prove that if ω, the harmonic measure for Ω, is absolutely continuous with respect to σ, and if the Poisson kernel $k = d\omega/d\sigma$ verifies an appropriate scale invariant higher integrability estimate (in particular, if ω belongs to weak-A_∞ with respect to σ), then $\partial\Omega$ is uniformly rectifiable in the sense of [David and Semmes 1991; 1993]; see Theorem 1.1 and Corollary 1.5 below. In particular, our background hypotheses hold in the case that $\Omega := \mathbb{R}^{n+1} \setminus E$ is the complement of an ADR set of codimension 1, as in that case it is well known that the corkscrew condition is verified automatically in Ω, i.e., in every ball $B = B(x, r)$ centered on E, there is some component of $\Omega \cap B$ that contains a point Y with $\operatorname{dist}(Y, E) \approx r$. Furthermore, our argument is general enough to allow us to establish a nonlinear version of Theorem 1.1 (see Theorem 1.12 below) involving the p-Laplace operator, p-harmonic functions, and p-harmonic measure.

Hofmann was supported by NSF grant DMS-1361701. Martell was supported by ICMAT Severo Ochoa project SEV-2015-0554. He also acknowledges that the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ERC agreement no. 615112 HAPDEGMT.
MSC2010: primary 31B05, 31B25, 35J08, 42B25, 42B37; secondary 28A75, 28A78.
Keywords: harmonic measure and p-harmonic measure, Poisson kernel, uniform rectifiability, Carleson measures, Green function, weak-A_∞.

To briefly outline previous work, in [Hofmann et al. 2014] the first and third authors, together with I. Uriarte-Tuero, proved the same result (cf. Theorem 1.1 and Corollary 1.5) under the additional strong hypothesis that Ω is a connected domain, satisfying an interior Harnack chain condition. In hindsight, under that extra assumption, one obtains the stronger conclusion that the exterior domain $\mathbb{R}^{n+1} \setminus \overline{\Omega}$ in fact also satisfies a corkscrew condition, and hence that Ω is an NTA domain in the sense of [Jerison and Kenig 1982]; see [Azzam et al. 2014] for the details. Compared to [Hofmann et al. 2014] the main new advances in the present paper are two. First, we remove any connectivity hypothesis; in particular, we avoid the Harnack chain condition. Second, we are able to establish a version of our results also in the nonlinear case $1 < p < \infty$. Our main results — Theorem 1.1, Corollary 1.5, and Theorem 1.12 — are new even in the linear case $p = 2$.

Our approach is decidedly influenced by prior work of Lewis and Vogel [2006; 2007]. In particular, a version of Theorem 1.12 and Theorem 1.1 was proved in [Lewis and Vogel 2007], under the stronger hypothesis that p-harmonic measure μ itself is an Ahlfors–David regular measure, which in the linear case $p = 2$ implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. However, to weaken the hypotheses on ω and μ, as we have done here, requires further considerations, which we discuss below in Section 1B.

To provide some additional context, we mention that out results here may be viewed as "large constant" analogues of results of Kenig and Toro [2003] in the linear case $p = 2$, and of J. Lewis and Nyström [2012], in the general p-harmonic case $1 < p < \infty$. These authors show that in the presence of a Reifenberg flatness condition and Ahlfors–David regularity, $\log k \in \mathrm{VMO}$ implies that the unit normal ν to the boundary belongs to VMO, where k is either the Poisson kernel with pole at some fixed point or the density of p-harmonic Riesz measure associated to a particular ball $B(x, r)$. Moreover, under the same background hypotheses, the condition $\nu \in \mathrm{VMO}$ is equivalent to a uniform rectifiability (UR) condition with vanishing trace. Thus $\log k \in \mathrm{VMO} \Longrightarrow$ vanishing UR, given sufficient Reifenberg flatness. On the other hand, our large constant version "almost" says "$\log k \in \mathrm{BMO} \Longrightarrow$ UR". Indeed, it is well known that the A_∞ condition, i.e., weak-A_∞ plus the doubling property, implies that $\log k \in \mathrm{BMO}$, while if $\log k \in \mathrm{BMO}$ with small norm, then $k \in A_\infty$. We further note that, in turn, the results of [Kenig and Toro 2003] may be viewed as an "endpoint" version of the free boundary results of [Alt and Caffarelli 1981; Jerison 1990], which establish, again in the presence of Reifenberg flatness, that Hölder continuity of $\log k$ implies that of the unit normal ν (and indeed, that $\partial\Omega$ is of class $C^{1,\alpha}$ for some $\alpha > 0$).

1A. ***Statement of main results.*** Given an open set $\Omega \subset \mathbb{R}^{n+1}$, and a Euclidean ball $B = B(x, r) \subset \mathbb{R}^{n+1}$ centered on $\partial\Omega$, we let $\Delta = \Delta(x, r) := B \cap \partial\Omega$ denote the corresponding surface ball. For $X \in \Omega$, let ω^X be harmonic measure for Ω, with pole at X. As mentioned above, all other terminology and notation will be defined below.

Concerning the Laplace operator and harmonic measure we prove the following results.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension n* (see Definition 2.1). *Suppose that there are positive constants C_0 and c_0, and an exponent $q > 1$, such* *that for every surface ball* $\Delta = \Delta(x, r)$, *with* $x \in \partial\Omega$ *and* $0 < r < \operatorname{diam}(\partial\Omega)$, *there exists* $X_\Delta \in B(x, r) \cap \Omega$, *with* $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r$, *satisfying*

($\star$) ***scale-invariant higher integrability***: $\omega^{X_\Delta} \ll \sigma$ *in* 2Δ, *and* $k^{X_\Delta} := d\omega^{X_\Delta}/d\sigma$ *satisfies*

$$\int_{2\Delta} k^{X_\Delta}(y)^q \, d\sigma(y) \leq C_0 \sigma(\Delta)^{1-q}. \tag{1.2}$$

Then $\partial\Omega$ *is uniformly rectifiable and moreover the "UR character"* (see Definition 2.4) *depends only on* n, *the ADR constants*, q, c_0, *and* C_0.

The point X_Δ in Theorem 1.1 is a "corkscrew point" for Ω, relative to Δ. An open set Ω for which there is such a point relative to every surface ball $\Delta(x, r)$, $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, with a uniform constant c_0, is said to satisfy the "corkscrew condition" (see Definition 2.5 below).

Remark 1.3. We note that, in lieu of absolute continuity and ($\star$), only the following apparently weaker condition is actually used in the proof of Theorem 1.1:

($\star\star$) **local nondegeneracy**: there exist uniform constants $\eta, \beta > 0$ such that if $A \subset \Delta$ is Borel measurable, then

$$\sigma(A) \geq (1-\eta)\sigma(\Delta) \implies \omega^{X_\Delta}(A) \geq \beta \omega^{X_\Delta}(\Delta).^{1} \tag{1.4}$$

Here $\Delta = \Delta(x, r)$ for $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and $X_\Delta \in B(x, r/2) \cap \Omega$ with $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r/2$.[2] We observe that there turns out to be some flexibility in the choice of X_Δ (see the discussion at the beginning of Section 4), and consequently it is not hard to see that ($\star$) implies ($\star\star$); see Lemma 4.3.

We also have the following easy corollary of Theorem 1.1 (we shall give the short proof of the corollary in Section 5D).

Corollary 1.5. *Let* $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, *be an open set satisfying the corkscrew condition, whose boundary is Ahlfors–David regular of dimension n. Suppose further that for every ball* $B = B(x, r)$ *with* $x \in \partial\Omega$ *and* $0 < r < \operatorname{diam}(\partial\Omega)$, *and every* $Y \in \Omega \setminus B(x, 2r)$, *harmonic measure* ω^Y *belongs to weak-*$A_\infty(\Delta(x, r))$, *i.e., there is a constant* $C_0 \geq 1$ *and an exponent* $q > 1$, *each of which is uniform with respect to* x, r, *and* Y, *such that* $\omega^Y \ll \sigma$ *in* $\Delta(x, r)$, *and* $k^Y = d\omega^Y/d\sigma$ *satisfies*

$$\left(\fint_{\Delta'} k^Y(z)^q \, d\sigma(z) \right)^{1/q} \leq C_0 \fint_{2\Delta'} k^Y(z) \, d\sigma(z) \tag{1.6}$$

for every surface ball centered on the boundary $\Delta' = B' \cap \partial\Omega$ *with* $2B' \subset B(x, r)$. *Then* $\partial\Omega$ *is uniformly rectifiable, and moreover, the "UR character"* (see Definition 2.4) *depends only on n, the ADR constant of* $\partial\Omega$, q, C_0, *and the corkscrew constant.*

Remark 1.7. As mentioned above, the corkscrew condition is automatically satisfied in the case that E is an n-dimensional ADR set (hence closed, see Definition 2.1 below), and $\Omega = \mathbb{R}^{n+1} \setminus E$ is its complement, with the corkscrew constant for Ω depending only on n and the ADR constant of E. Thus, in particular,

[1] This formulation is adapted from [Mourgoglou and Tolsa 2015]; see the discussion in Section 1D.

[2] For aesthetic reasons, and for convenience in the sequel, in contrast to condition ($\star$), we prefer to state condition ($\star\star$) in terms of Δ rather than 2Δ, and with $X_\Delta \in B(x, r/2)$ rather than $B(x, r)$.

Corollary 1.5 applies in that setting, so in the presence of the weak reverse Hölder condition (1.6), we deduce that E is uniformly rectifiable.

Combining Theorem 1.1 with the results in [Bortz and Hofmann 2015], we obtain as an immediate consequence a "big pieces" characterization of uniformly rectifiable sets of codimension 1, in terms of harmonic measure. Here and in the sequel, given an ADR set E, Q denotes a "dyadic cube" on E in the sense of [David and Semmes 1991; 1993; Christ 1990], and $\mathbb{D}(E)$ denotes the collection of all such cubes; see Lemma 2.6 below.

Theorem 1.8. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an n-dimensional ADR set. Let $\Omega := \mathbb{R}^{n+1} \setminus E$. Then E is uniformly rectifiable if and only if it has "big pieces of good harmonic measure estimates" in the following sense: for each $Q \in \mathbb{D}(E)$ there exists an open set $\widetilde{\Omega} = \widetilde{\Omega}_Q$ with the following properties, with uniform control of the various implicit constants:*

- *$\partial\widetilde{\Omega}$ is ADR;*
- *the interior corkscrew condition holds in $\widetilde{\Omega}$;*
- *$\partial\widetilde{\Omega}$ has a "big pieces" overlap with E, in the sense that $\sigma(Q \cap \partial\widetilde{\Omega}) \gtrsim \sigma(Q)$;*
- *for each surface ball $\Delta = \Delta(x, r) := B(x, r) \cap \partial\widetilde{\Omega}$ with $x \in \partial\widetilde{\Omega}$ and $r \in (0, \operatorname{diam}(\widetilde{\Omega}))$, there is an interior corkscrew point $X_\Delta \in \widetilde{\Omega}$ such that $\omega_{\widetilde{\Omega}}^{X_\Delta}$, the harmonic measure for $\widetilde{\Omega}$ with pole at X_Δ, satisfies $\omega_{\widetilde{\Omega}}^{X_\Delta}(\Delta) \gtrsim 1$, and belongs to weak-$A_\infty(\Delta)$.*

The "only if" direction is proved in [Bortz and Hofmann 2015], and the open sets $\widetilde{\Omega}$ constructed in [Bortz and Hofmann 2015] even satisfy a 2-sided corkscrew condition, and moreover, $\widetilde{\Omega} \subset \Omega$ with $\operatorname{diam}(\widetilde{\Omega}) \approx \operatorname{diam}(Q)$. To obtain the converse direction, we simply observe that by Theorem 1.1, the subdomains $\widetilde{\Omega}$ have uniformly rectifiable boundaries, with uniform control of the "UR character" of each $\partial\widetilde{\Omega}$, and thus, by [David and Semmes 1993], E is uniformly rectifiable.

To formulate our main result in the nonlinear setting we first need to introduce some notation. If $O \subset \mathbb{R}^{n+1}$ is an open set and $1 \leq p \leq \infty$, then by $W^{1,p}(O)$ we denote the space of equivalence classes of functions f with distributional gradient $\nabla f = (f_{x_1}, \dots, f_{x_{n+1}})$, both of which are q-th power integrable on O. Let $\|f\|_{1,p} = \|f\|_p + \||\nabla f|\|_p$ be the norm in $W^{1,p}(O)$, where $\|\cdot\|_q$ denotes the usual Lebesgue p norm in O. Next, let $C_0^\infty(O)$ be the set of infinitely differentiable functions with compact support in O, and let $W_0^{1,p}(O)$ be the closure of $C_0^\infty(O)$ in the norm of $W^{1,p}(O)$. We let $W_{\mathrm{loc}}^{1,p}(O)$ be the set of all functions u such that $u\Theta \in W_0^{1,p}(O)$ whenever $\Theta \in C_0^\infty(O)$.

Given an open set O and $1 < p < \infty$, we say that u is p-harmonic in O provided $u \in W_{\mathrm{loc}}^{1,p}(O)$ and

$$\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \Theta \, dX = 0, \quad \forall \Theta \in C_0^\infty(O). \tag{1.9}$$

Observe that if u is smooth and $\nabla u \neq 0$ in O, then

$$\nabla \cdot (|\nabla u|^{p-2} \nabla u) \equiv 0 \quad \text{in } O, \tag{1.10}$$

and u is a classical solution in O to the p-Laplace partial differential equation. Here, as in the sequel, $\nabla\cdot$ is the divergence operator.

Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set, not necessarily connected, with n-dimensional ADR boundary. Let $p \in (1, \infty)$. Given $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, let u be a nonnegative p-harmonic function in $\Omega \cap B(x, r)$ which vanishes continuously on $\Delta(x, r) := B(x, r) \cap \partial\Omega$. Extend u to all of $B(x, r)$ by putting $u \equiv 0$ on $B(x, r) \setminus \overline{\Omega}$. Then there exists a unique nonnegative finite Borel measure μ on $\mathbb{R}^{n+1}$, with support contained in $\Delta(x, r)$, such that

$$-\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2} \nabla u \cdot \nabla \phi \, dX = \int_{\partial\Omega} \phi \, d\mu, \quad \forall \phi \in C_0^\infty(B(x, r)); \tag{1.11}$$

see [Heinonen et al. 2006, Chapter 21] and Lemma 3.43 below. We refer to μ as the p-harmonic measure associated to u. In the case $p = 2$, and if u is the Green function for Ω with pole at $X \in \Omega$, then the measure μ coincides with harmonic measure at X, $\omega = \omega^X$.

Concerning the p-Laplace operator, p-harmonic functions, and p-harmonic measure, we prove the following theorem.

Theorem 1.12. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set whose boundary is Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Let C be a sufficiently large constant* (*to be specified*), *depending only on n and the ADR constant, and suppose that there exist $q > 1$ and a positive constant C_0 for which the following holds*: *for each $x \in \partial\Omega$ and each $0 < r < \operatorname{diam}(\partial\Omega)$, there is a nontrivial, nonnegative p-harmonic function $u = u_{x,r}$ in $\Omega \cap B(x, Cr)$, and corresponding p-harmonic measure $\mu = \mu_{x,r}$, such that $\mu \ll \sigma$ in $\Delta(x, Cr)$, and such that $k := d\mu/d\sigma$ satisfies*

$$\left(\fint_{\Delta(x,Cr)} k(y)^q \, d\sigma(y) \right)^{1/q} \leq C_0 \frac{\mu(\Delta(x, r))}{\sigma(\Delta(x, r))}. \tag{1.13}$$

Then $\partial\Omega$ is uniformly rectifiable, *and moreover the "UR character"* (see Definition 2.4) *depends only on n*, *the ADR constant*, *p*, *q*, *and C_0*.

Some remarks are in order concerning the hypotheses of Theorem 1.12. Let us observe that, in particular, Ahlfors–David regularity and (1.13) imply that

$$\mu(\Delta(x, Cr)) \leq C_1 \mu(\Delta(x, r)), \tag{1.14}$$

with $C_1 \approx C_0$. In the linear case, the latter estimate follows automatically, with $\mu = \omega^Y$ for some $Y \in B(x, r)$ such that $\operatorname{dist}(Y, E) \approx r$, and with C_1 depending only on n and the ADR constant, by Bourgain's Lemma 3.1 below, even though ω^Y need not be a doubling measure (i.e., (1.14) says nothing about points other than x nor about scales other than r). In the nonlinear case, it seems that we must impose condition (1.14) by hypothesis. We also observe that (1.13) holds in particular if $\mu \in$ weak-$A_\infty(\Delta(x, 2Cr))$ and satisfies (1.14) (with radius $2C$ in place of C). Of course, (1.14) holds trivially if μ is a doubling measure, but we do not assume doubling.

Remark 1.15. We note that, as in Remark 1.3, the proof of Theorem 1.12 will in fact use, in lieu of absolute continuity and (1.13), only the apparently weaker condition that there exist uniform constants $\eta, \beta \in (0, 1)$ such that for all $\Delta = \Delta(x, r)$, and for all Borel sets $A \subset \Delta$,

$$\sigma(A) \geq (1 - \eta)\sigma(\Delta) \implies \mu(A) \geq \beta\mu(\Delta). \tag{1.16}$$

1B. ***Brief outline of the proofs of the main results.*** As mentioned, the approach in the present paper is strongly influenced by prior work due to Lewis and Vogel [2006; 2007], who in the latter paper proved a version of Theorem 1.12, and Theorem 1.1, under the stronger hypothesis that p-harmonic measure μ itself is an Ahlfors–David regular measure. In the linear case $p = 2$, this implies that the Poisson kernel is a bounded, accretive function, i.e., $k \approx 1$. Assuming that p-harmonic measure μ is an Ahlfors–David regular measure, Lewis and Vogel were able to show that E satisfies the so-called weak exterior convexity (WEC) condition, which characterizes uniform rectifiability [David and Semmes 1993]. To weaken the hypotheses on ω and μ, as we have done here, requires two further considerations. The first is quite natural in this context: a stopping time argument, in the spirit of the proofs of the Kato square root conjecture [Hofmann and McIntosh 2002; Hofmann et al. 2002; Auscher et al. 2002a] (and of local Tb theorems [Christ 1990; Auscher et al. 2002b; Hofmann 2006]), by means of which we extract ample dyadic sawtooth regimes on which averages of harmonic measure and p-harmonic measure are bounded and accretive; see Lemma 4.12 below. This allows us to use the arguments of [Lewis and Vogel 2007] within these good sawtooth regions. The second new consideration is necessitated by the fact that in our setting, the doubling property may fail for harmonic and p-harmonic measure. In the absence of doubling, we are unable to obtain the WEC condition directly. Nonetheless, we are able to follow the arguments of [Lewis and Vogel 2007] very closely up to a point, to obtain a condition on $\partial\Omega$ which we call the "weak half space approximation" (WHSA) property (see Definition 2.19). Indeed, extracting the essence of the argument of [Lewis and Vogel 2007], while dispensing with the doubling property, one realizes that the WHSA is precisely what one obtains. In the sequel, we present the argument of [Lewis and Vogel 2007] as Lemma 5.10. Finally, having obtained that $\partial\Omega$ satisfies the WHSA property, we are able to prove the following proposition stating that WHSA implies uniform rectifiability.

Proposition 1.17. *An n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ is uniformly rectifiable if and only if it satisfies the WHSA property.*

While the WHSA condition, per se, is new, our proof of Proposition 1.17 is based on a modified version of part of the argument in [Lewis and Vogel 2007].

1C. ***Organization of the paper.*** The paper is organized as follows. In Section 2, we state several definitions, including definitions of ADR, UR, and dyadic grids, and introduce further notions and notation. In Section 3, we state, and either prove or give references for, the PDE estimates needed in the proofs of our main results. In Section 4, we begin the (simultaneous) proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. In Section 5, following [Lewis and Vogel 2006; 2007], we complete the proofs of Theorem 1.1 and Theorem 1.12, modulo Proposition 1.17. At the end of Section 5 we also give the (very short) proof of Corollary 1.5. In Section 6, we give the proof of Proposition 1.17, i.e., the proof of the fact that the WHSA condition implies uniform rectifiability.

1D. ***Discussion of recent related work.*** We note that some related work has recently appeared, or been carried out, while this manuscript was in preparation. In the setting of uniform domains with lower ADR boundary with locally finite n-dimensional Hausdorff measure, Mourgoglou [2015] has shown that rectifiability of the boundary implies absolute continuity of surface measure with respect to harmonic measure (for the Laplacian). Akman, Badger, Hofmann, and Martell [Akman et al. 2015], in the setting of uniform domains with ADR boundary, have characterized the rectifiability of the boundary in terms of the absolute continuity of harmonic measure and some elliptic measures and surface measure or in terms of some qualitative A_∞ condition. Also, Azzam, Mourgoglou, and Tolsa [Azzam et al. 2015] have obtained that absolute continuity of harmonic measure with respect to surface measure on an H^n-finite piece of the boundary implies that harmonic measure is rectifiable in that piece. The setting is very general as they only assume a "porosity" (i.e., corkscrew) condition in the complement of $\partial\Omega$. In [Hofmann et al. 2015], Hofmann, Martell, Mayboroda, Tolsa, and Volberg prove the same result removing the porosity assumption. Both [Azzam et al. 2015] and the follow-up version [Hofmann et al. 2015] (which will be combined in the forthcoming paper [Azzam et al. 2016]) rely on recent deep results of [Nazarov et al. 2014a; 2014b], concerning connections between rectifiability and the behavior of Riesz transforms.

Finally, we discuss two closely related papers treating the case $p = 2$. First, we mention that a preliminary version of our results, treating only the linear harmonic case (i.e., Theorem 1.1 of the present paper) under hypothesis ($\star$), appeared earlier in the unpublished preprint [Hofmann and Martell 2015]. That result, again in the case $p = 2$, was then essentially reproved, by a different method, in [Mourgoglou and Tolsa 2015], but assuming condition ($\star\star$) in place of ($\star$). While the present paper was in preparation, we learned of the work in [Mourgoglou and Tolsa 2015], and we realized that our arguments (and those of [Hofmann and Martell 2015]), almost unchanged, also allow ($\star$) to be replaced by ($\star\star$) or its p-harmonic equivalent. The current version of this manuscript incorporates this observation.[3] Let us mention also that the approach in [Mourgoglou and Tolsa 2015] is based on showing that ($\star\star$) for harmonic measure implies L^2-boundedness of the Riesz transforms, and thus it is a quantitative version of the method of [Azzam et al. 2016]. An interesting feature of the proof in [Mourgoglou and Tolsa 2015] is that it works even without the lower bound in the Ahlfors–David condition; in that case, one may deduce rectifiability, as opposed to uniform rectifiability, of the underlying measure on $\partial\Omega$. On the other hand, it seems difficult to generalize the approach of [Mourgoglou and Tolsa 2015] to the p-Laplace setting, since it is based on Riesz transforms, which are tied to the linear harmonic case.

2. ADR, UR, and dyadic grids

Definition 2.1 (Ahlfors–David regular (ADR)). We say that a set $E \subset \mathbb{R}^{n+1}$, of Hausdorff dimension n, is ADR if it is closed and if there is some uniform constant C such that

$$C^{-1} r^n \le \sigma(\Delta(x,r)) \le C r^n, \quad \forall r \in (0, \operatorname{diam}(E)),\ x \in E, \tag{2.2}$$

where $\operatorname{diam}(E)$ may be infinite. Here, $\Delta(x,r) := E \cap B(x,r)$ is the "surface ball" of radius r, and $\sigma := H^n|_E$ is the "surface measure" on E, where H^n denotes n-dimensional Hausdorff measure.

Definition 2.3 (uniformly rectifiable (UR)). An n-dimensional ADR (hence closed) set $E \subset \mathbb{R}^{n+1}$ is UR if and only if it contains "big pieces of Lipschitz images" of $\mathbb{R}^n$ (BPLI). This means that there are positive

[3] We thank Mourgoglou and Tolsa for making their preprint available to us while our manuscript was in preparation.

constants θ and M_0, such that for each $x \in E$ and each $r \in (0, \operatorname{diam}(E))$, there is a Lipschitz mapping $\rho = \rho_{x,r} : \mathbb{R}^n \to \mathbb{R}^{n+1}$, with Lipschitz constant no larger than M_0, such that

$$H^n\big(E \cap B(x,r) \cap \rho(\{z \in \mathbb{R}^n : |z| < r\})\big) \geq \theta r^n.$$

We recall that n-dimensional rectifiable sets are characterized by the property that they can be covered, up to a set of H^n measure 0, by a countable union of Lipschitz images of $\mathbb{R}^n$; we observe that BPLI is a quantitative version of this fact.

We remark that, at least among the class of ADR sets, the UR sets are precisely those for which all "sufficiently nice" singular integrals are L^2-bounded [David and Semmes 1991]. In fact, for n-dimensional ADR sets in $\mathbb{R}^{n+1}$, the L^2-boundedness of certain special singular integral operators (the "Riesz transforms") suffices to characterize uniform rectifiability (see [Mattila et al. 1996] for the case $n = 1$, and [Nazarov et al. 2014a] in general). We further remark that there exist sets that are ADR (and that even form the boundary of a domain satisfying interior corkscrew and Harnack chain conditions), but that are totally nonrectifiable (e.g., see the construction of Garnett's "4-corners Cantor set" in [David and Semmes 1993, Chapter1]). Finally, we mention that there are numerous other characterizations of UR sets (many of which remain valid in higher codimensions); see [David and Semmes 1991; 1993], and in particular Theorem 2.14 below. In this paper, we also present a new characterization of UR sets of codimension 1 (see Proposition 1.17 below), which will be very useful in the proof of Theorem 1.1.

Definition 2.4 (UR character). Given a UR set $E \subset \mathbb{R}^{n+1}$, its "UR character" is just the pair of constants (θ, M_0) involved in the definition of uniform rectifiability, along with the ADR constant; or equivalently, the quantitative bounds involved in any particular characterization of uniform rectifiability.

Definition 2.5 (corkscrew condition). Following [Jerison and Kenig 1982], we say that an open set $\Omega \subset \mathbb{R}^{n+1}$ satisfies the "corkscrew condition" if for some uniform constant $c_0 > 0$ and for every surface ball $\Delta := \Delta(x, r)$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, there is a point $X_\Delta \in B(x, r) \cap \Omega$ such that $\operatorname{dist}(X_\Delta, \partial\Omega) \geq c_0 r$. The point $X_\Delta \subset \Omega$ is called a "corkscrew point" relative to Δ.

Lemma 2.6 (existence and properties of the "dyadic grid" [David and Semmes 1991; 1993; Christ 1990]). *Suppose that $E \subset \mathbb{R}^{n+1}$ is a closed n-dimensional ADR set. Then there exist constants $a_0 > 0$, $\gamma > 0$, and $C_* < \infty$, depending only on n and the ADR constant, such that for each $k \in \mathbb{Z}$, there is a collection*

$$\mathbb{D}_k := \{Q_j^k \subset E : j \in \mathfrak{I}_k\}$$

of Borel sets ("cubes"), where $\mathfrak{I}_k$ denotes some (possibly finite) index set depending on k, satisfying

(i) $E = \bigcup_j Q_j^k$ *for each* $k \in \mathbb{Z}$;

(ii) *if* $m \geq k$ *then either* $Q_i^m \subset Q_j^k$ *or* $Q_i^m \cap Q_j^k = \varnothing$;

(iii) *for each* (j, k) *and each* $m < k$, *there is a unique* i *such that* $Q_j^k \subset Q_i^m$;

(iv) $\operatorname{diam}(Q_j^k) \leq C_* 2^{-k}$;

(v) *each* Q_j^k *contains some "surface ball"* $\Delta(x_j^k, a_0 2^{-k}) := B(x_j^k, a_0 2^{-k}) \cap E$;

(vi) $H^n\big(\{x \in Q_j^k : \operatorname{dist}(x, E \setminus Q_j^k) \leq \varrho\, 2^{-k}\}\big) \leq C_* \varrho^\gamma H^n(Q_j^k)$ *for all* k, j *and for all* $\varrho \in (0, a_0)$.

Let us make a few remarks concerning this lemma, and discuss some related notation and terminology.

- In the setting of a general space of homogeneous type, this lemma has been proved by Christ [1990], with the dyadic parameter $\frac{1}{2}$ replaced by some constant $\delta \in (0, 1)$. In fact, one may always take $\delta = \frac{1}{2}$ (cf. [Hofmann et al. 2017, Proof of Proposition 2.12]). In the presence of the Ahlfors–David property (2.2), the result already appears in [David and Semmes 1991; 1993].
- For our purposes, we may ignore those $k \in \mathbb{Z}$ such that $2^{-k} \gtrsim \operatorname{diam}(E)$, in the case that the latter is finite.
- We denote by $\mathbb{D} = \mathbb{D}(E)$ the collection of all relevant Q_j^k, i.e.,
$$\mathbb{D} := \bigcup_k \mathbb{D}_k,$$
where, if $\operatorname{diam}(E)$ is finite, the union runs over those k such that $2^{-k} \lesssim \operatorname{diam}(E)$.
- Properties (iv) and (v) imply that for each cube $Q \in \mathbb{D}_k$, there is a point $x_Q \in E$, a Euclidean ball $B(x_Q, r)$, and a surface ball $\Delta(x_Q, r) := B(x_Q, r) \cap E$ such that $r \approx 2^{-k} \approx \operatorname{diam}(Q)$ and
$$\Delta(x_Q, r) \subset Q \subset \Delta(x_Q, Cr) \tag{2.7}$$
for some uniform constant C. We denote this ball and surface ball by
$$B_Q := B(x_Q, r), \qquad \Delta_Q := \Delta(x_Q, r), \tag{2.8}$$
and we refer to the point x_Q as the "center" of Q.
- Given a dyadic cube $Q \in \mathbb{D}$, we define its "κ-dilate" by
$$\kappa Q := E \cap B(x_Q, \kappa \operatorname{diam}(Q)). \tag{2.9}$$
- For a dyadic cube $Q \in \mathbb{D}_k$, we set $\ell(Q) = 2^{-k}$, and we refer to this quantity as the "length" of Q. Clearly, $\ell(Q) \approx \operatorname{diam}(Q)$.
- For a dyadic cube $Q \in \mathbb{D}$, we let $k(Q)$ denote the "dyadic generation" to which Q belongs, i.e., we set $k = k(Q)$ if $Q \in \mathbb{D}_k$; thus, $\ell(Q) = 2^{-k(Q)}$.
- For any $Q \in \mathbb{D}(E)$, we set $\mathbb{D}_Q := \{Q' \in \mathbb{D} : Q' \subset Q\}$.
- Given $Q_0 \in \mathbb{D}(E)$ and a family $\mathcal{F} = \{Q_j\} \subset \mathbb{D}$ of pairwise disjoint cubes, we set
$$\mathbb{D}_{\mathcal{F},Q_0} := \{Q \in \mathbb{D}_{Q_0} : Q \text{ is not contained in any } Q_j \in \mathcal{F}\} = \mathbb{D}_{Q_0} \setminus \Big(\bigcup_{Q_j \in \mathcal{F}} \mathbb{D}_{Q_j} \Big). \tag{2.10}$$

Definition 2.11 (ε-local BAUP). Given $\varepsilon > 0$, we say that $Q \in \mathbb{D}(E)$ satisfies the *ε-local BAUP* condition if there is a family $\mathcal{P}$ of hyperplanes (depending on Q) such that every point in $10Q$ is at a distance at most $\varepsilon\ell(Q)$ from $\bigcup_{P \in \mathcal{P}} P$, and every point in $\big(\bigcup_{P \in \mathcal{P}} P\big) \cap B(x_Q, 10 \operatorname{diam}(Q))$ is at a distance at most $\varepsilon\ell(Q)$ from E.

Definition 2.12 (BAUP). We say that an n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the condition of *bilateral approximation by unions of planes* (BAUP) if for some $\varepsilon_0 > 0$, and for every positive $\varepsilon < \varepsilon_0$, there is a constant C_ε such that the set $\mathcal{B}$ of bad cubes in $\mathbb{D}(E)$, for which the ε-local BAUP condition fails, satisfies the packing condition

$$\sum_{Q' \subset Q,\, Q' \in \mathcal{B}} \sigma(Q') \leq C_\varepsilon \sigma(Q), \quad \forall\, Q \in \mathbb{D}(E). \tag{2.13}$$

For future reference, we recall the following result of David and Semmes.

Theorem 2.14 [David and Semmes 1993, Theorem I.2.18, p. 36]. *Let $E \subset \mathbb{R}^{n+1}$ be an n-dimensional ADR set. Then E is uniformly rectifiable if and only if it satisfies BAUP.*

We remark that the definition of BAUP in [David and Semmes 1993] is slightly different in superficial appearance, but it is not hard to verify that the dyadic version stated here is equivalent to their condition. We note that we shall not need the full strength of this equivalence here, but only the fact that our version of BAUP implies the version in [David and Semmes 1993], and hence implies UR.

We also require a new characterization of UR sets of codimension 1, which is related to the BAUP and its variants. For a sufficiently large constant K_0 to be chosen (see Lemma 4.24 below), we set

$$B_Q^* := B(x_Q, K_0^2 \ell(Q)), \qquad \Delta_Q^* := B_Q^* \cap E. \tag{2.15}$$

Given a small positive number ε, which we typically assume to be much smaller than K_0^{-6}, we also set

$$B_Q^{**} = B_Q^{**}(\varepsilon) := B(x_Q, \varepsilon^{-2}\ell(Q)), \qquad B_Q^{***} = B_Q^{***}(\varepsilon) := B(x_Q, \varepsilon^{-5}\ell(Q)). \tag{2.16}$$

Definition 2.17 (ε-local WHSA). Given $\varepsilon > 0$, we say that $Q \in \mathbb{D}(E)$ satisfies the *ε-local WHSA* condition (or more precisely, the "ε-local WHSA with parameter K_0") if there is a half-space $H = H(Q)$, a hyperplane $P = P(Q) = \partial H$, and a fixed positive number K_0 satisfying

(1) $\operatorname{dist}(Z, E) \leq \varepsilon \ell(Q)$ for every $Z \in P \cap B_Q^{**}(\varepsilon)$,

(2) $\operatorname{dist}(Q, P) \leq K_0^{3/2} \ell(Q)$, and

(3) $H \cap B_Q^{**}(\varepsilon) \cap E = \varnothing$.

Note that part (2) of the previous definition says that the hyperplane P has an "ample" intersection with the ball $B_Q^{**}(\varepsilon)$. Indeed,

$$\operatorname{dist}(x_Q, P) \lesssim K_0^{3/2} \ell(Q) \ll \varepsilon^{-2} \ell(Q). \tag{2.18}$$

Definition 2.19 (WHSA). We say that an n-dimensional ADR set $E \subset \mathbb{R}^{n+1}$ satisfies the *weak half-space approximation* property (WHSA) if for some pair of positive constants ε_0 and K_0, and for every positive $\varepsilon < \varepsilon_0$, there is a constant C_ε such that the set $\mathcal{B}$ of bad cubes in $\mathbb{D}(E)$, for which the ε-local WHSA condition with parameter K_0 fails, satisfies the packing condition

$$\sum_{Q \subset Q_0,\, Q \in \mathcal{B}} \sigma(Q) \leq C_\varepsilon \sigma(Q_0), \quad \forall\, Q_0 \in \mathbb{D}(E). \tag{2.20}$$

Next, we develop some further notation and terminology. Given a closed set E, set $\delta_E(Y) := \operatorname{dist}(Y, E)$, simply writing $\delta(Y)$ when the set has been fixed.

Let $\mathcal{W} = \mathcal{W}(\Omega)$ denote a collection of (closed) dyadic Whitney cubes of Ω, so that the cubes in $\mathcal{W}$ form a covering of Ω with nonoverlapping interiors, and which satisfy

$$4\,\mathrm{diam}(I) \le \mathrm{dist}(4I, \partial\Omega) \le \mathrm{dist}(I, \partial\Omega) \le 40\,\mathrm{diam}(I) \tag{2.21}$$

and

$$\mathrm{diam}(I_1) \approx \mathrm{diam}(I_2), \quad \text{whenever } I_1 \text{ and } I_2 \text{ touch.} \tag{2.22}$$

Assuming that $E = \partial\Omega$ is ADR and given $Q \in \mathbb{D}(E)$, for the same constant K_0 as in (2.15) we set

$$\mathcal{W}_Q := \{I \in \mathcal{W} : K_0^{-1}\ell(Q) \le \ell(I) \le K_0\ell(Q), \text{ and } \mathrm{dist}(I, Q) \le K_0\,\ell(Q)\}. \tag{2.23}$$

Fix a small, positive parameter τ, to be chosen momentarily, and given $I \in \mathcal{W}$, let

$$I^* = I^*(\tau) := (1+\tau)I \tag{2.24}$$

denote the corresponding "fattened" Whitney cube. We now choose τ sufficiently small that the cubes I^* retain the usual properties of Whitney cubes, in particular that

$$\mathrm{diam}(I) \approx \mathrm{diam}(I^*) \approx \mathrm{dist}(I^*, E) \approx \mathrm{dist}(I, E).$$

We then define Whitney regions with respect to Q by setting

$$U_Q := \bigcup_{I \in \mathcal{W}_Q} I^*. \tag{2.25}$$

We observe that these Whitney regions may have more than one connected component, but that the number of distinct components is uniformly bounded, depending only upon K_0 and dimension. We enumerate the components of U_Q as $\{U_Q^i\}_i$. Moreover, we enlarge the Whitney regions as follows.

Definition 2.26. For $\varepsilon > 0$, and given $Q \in \mathbb{D}(E)$, we write $X \approx_{\varepsilon,Q} Y$ if X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$. Given a sufficiently small parameter $\varepsilon > 0$, we then set

$$\widetilde{U}_Q^i := \{X \in \mathbb{R}^{n+1} \setminus E : X \approx_{\varepsilon,Q} Y, \text{ for some } Y \in U_Q^i\}. \tag{2.27}$$

Remark 2.28. Since $\widetilde{U}_Q^i$ is an enlarged version of U_Q, it may be that there are some $i \ne j$ for which $\widetilde{U}_Q^i$ meets $\widetilde{U}_Q^j$. This overlap will be harmless.

3. PDE estimates

In this section we recall several estimates for harmonic measure and harmonic functions, and also for p-harmonic measure and p-harmonic functions. Although some of the PDE results in the harmonic case $p = 2$ can be subsumed into the general p-harmonic theory, we choose to present some aspects of the harmonic theory separately, in part for the convenience of those readers who are more familiar with the case $p = 2$, and in part because the presence of the Green function is unique to that case.

3A. ***PDE estimates: the harmonic case.*** Next, we recall several facts concerning harmonic measure and Green's functions. Let Ω be an open set, not necessarily connected, and set $\delta(X) = \delta_{\partial\Omega}(X) = \operatorname{dist}(X, \partial\Omega)$.

Lemma 3.1 [Bourgain 1987]. *Suppose that $\partial\Omega$ is n-dimensional ADR. Then there are uniform constants $c \in (0, 1)$ and $C \in (1, \infty)$, depending only on n and ADR, such that for every $x \in \partial\Omega$ and every $r \in (0, \operatorname{diam}(\partial\Omega))$, if $Y \in \Omega \cap B(x, cr)$ then*

$$\omega^Y(\Delta(x, r)) \geq \frac{1}{C} > 0. \tag{3.2}$$

We refer the reader to [Bourgain 1987, Lemma 1] for the proof. We note for future reference that in particular, given $X \in \Omega$, if $\hat{x} \in \partial\Omega$ satisfies $|X - \hat{x}| = \delta(X)$ and $\Delta_X := \partial\Omega \cap B(\hat{x}, 10\delta(X))$, then for a slightly different uniform constant $C > 0$,

$$\omega^X(\Delta_X) \geq \frac{1}{C}. \tag{3.3}$$

Indeed, the latter bound follows immediately from (3.2), and the fact that we can form a Harnack chain connecting X to a point Y that lies on the line segment from X to $\hat{x}$ and satisfies $|Y - \hat{x}| = c\delta(X)$.

A proof of the next lemma may be found, e.g., in [Hofmann et al. $\geq$ 2017]. We note that, in particular, the ADR hypothesis implies that $\partial\Omega$ is Wiener regular at every point (see Lemma 3.27 below).

Lemma 3.4. *Let Ω be an open set with n-dimensional ADR boundary. There exist positive, finite constants C, depending only on dimension, and c_θ, depending on dimension and $\theta \in (0, 1)$, such that the Green function satisfies*

$$G(X, Y) \leq C|X - Y|^{1-n}; \tag{3.5}$$

$$c_\theta |X - Y|^{1-n} \leq G(X, Y), \quad \text{if } |X - Y| \leq \theta\delta(X),\ \theta \in (0, 1); \tag{3.6}$$

$$G(X, \cdot) \in C(\overline{\Omega} \setminus \{X\}) \quad \text{and} \quad G(X, \cdot)|_{\partial\Omega} \equiv 0, \quad \forall X \in \Omega; \tag{3.7}$$

$$G(X, Y) \geq 0, \quad \forall X, Y \in \Omega,\ X \neq Y; \tag{3.8}$$

$$G(X, Y) = G(Y, X), \quad \forall X, Y \in \Omega,\ X \neq Y; \tag{3.9}$$

and for every $\Phi \in C_0^\infty(\mathbb{R}^{n+1})$,

$$\int_{\partial\Omega} \Phi \, d\omega^X - \Phi(X) = -\iint_\Omega \nabla_Y G(Y, X) \cdot \nabla\Phi(Y) \, dY, \quad \forall X \in \Omega. \tag{3.10}$$

Next we present a version of one of the estimates obtained by Caffarelli, Fabes, Mortola, and Salsa in [Caffarelli et al. 1981], which remains true even in the absence of connectivity.

Lemma 3.11 ("CFMS" estimates). *Suppose that $\partial\Omega$ is n-dimensional ADR. For every $Y \in \Omega$ and $X \in \Omega$ such that $|X - Y| \geq \delta(Y)/2$, we have*

$$\frac{G(Y, X)}{\delta(Y)} \leq C \frac{\omega^X(\Delta_Y)}{\sigma(\Delta_Y)}, \tag{3.12}$$

where $\Delta_Y = B(\hat{y}, 10\delta(Y)) \cap E$, with $\hat{y} \in \partial\Omega$ such that $|Y - \hat{y}| = \delta(Y)$.

For future use, we note that as a consequence of (3.12), it follows directly that for every $Q \in \mathbb{D}(\partial\Omega)$, if $Y \in B(x_Q, C\ell(Q))$ with $\delta(Y) \geq c\ell(Q)$, then there exists $\kappa = \kappa(C, c)$ such that

$$\frac{G(Y,X)}{\ell(Q)} \lesssim \frac{\omega^X(\kappa Q)}{\sigma(Q)} \lesssim \kappa^n \left(\fint_Q (\mathcal{M}\omega^X)^{1/2}\, d\sigma \right)^2, \quad \forall X \notin B(x_Q, \kappa\ell(Q)), \tag{3.13}$$

where κQ is defined in (2.9), and $\mathcal{M}$ is the usual Hardy–Littlewood maximal operator on $\partial\Omega$.

Proof of Lemma 3.11. We follow the well known argument of [Caffarelli et al. 1981] (see also [Kenig 1994, Lemma 1.3.3]). Fix $Y \in \Omega$ and write $B^Y = \overline{B(Y, \delta(Y)/2)}$. Consider the open set $\widehat{\Omega} = \Omega \setminus B^Y$ for which clearly $\partial\widehat{\Omega} = \partial\Omega \cup \partial B^Y$. Set

$$u(X) := G(Y,X)/\delta(Y), \qquad v(X) := \omega^X(\Delta_Y)/\sigma(\Delta_Y),$$

for every $X \in \widehat{\Omega}$. Note that both u and v are nonnegative harmonic functions in $\widehat{\Omega}$. If $X \in \partial\Omega$ then $u(X) = 0 \leq v(X)$. Take now $X \in \partial B^Y$, so that $u(X) \lesssim \delta(Y)^{-n}$ by (3.5). On the other hand, if we fix $X_0 \in \partial B^Y$ with X_0 on the line segment that joints Y and $\hat{y}$, then $2\Delta_{X_0} = \Delta_Y$, so that $v(X_0) \gtrsim \delta(Y)^{-n}$, by (3.3). By Harnack's inequality, we then obtain $v(X) \gtrsim \delta(Y)^{-n}$ for all $X \in \partial B^Y$. Thus, $u \lesssim v$ in $\partial\widehat{\Omega}$ and by the maximum principle this immediately extends to $\widehat{\Omega}$ as desired. □

Lemma 3.14. *Let $\partial\Omega$ be n-dimensional ADR. Let $B = B(x,r)$ with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$, and set $\Delta = B \cap \partial\Omega$. There exist constants $\kappa_0 > 2$, $C > 1$, and $M_1 > 1$, depending only on n and the ADR constant of $\partial\Omega$, such that for $X \in \Omega \setminus B(x, \kappa_0 r)$, we have*

$$\sup_{\frac{1}{2}B} G(\cdot, X) \lesssim \frac{1}{|B|} \iint_B G(Y,X)\, dY \leq Cr \frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}. \tag{3.15}$$

Moreover, for each $\gamma \in (0,1]$,

$$\frac{1}{|B|} \iint_{B \cap \{Y : \delta(Y) < \gamma r\}} G(Y,X)\, dY \leq C\gamma^2 r \frac{\omega^X(\Delta(x, M_1 r))}{\sigma(\Delta)}, \tag{3.16}$$

where C depends on n and the ADR constant of $\partial\Omega$.

We note that in the previous estimates it is implicitly understood that $G(\cdot, X)$ is extended to be 0 outside of Ω.

Proof. Extending $G(\cdot, X)$ to be 0 outside of Ω, we obtain a subharmonic function in B. The first inequality in (3.15) follows immediately. The second inequality in (3.15) is just the special case $\gamma = 1$ of (3.16), so it suffices to prove the latter. Set $\Sigma_\gamma = \{I \in \mathcal{W} : I \cap B \neq \varnothing, \operatorname{dist}(I, \partial\Omega) < \gamma r\}$, and note that if $I \in \Sigma_\gamma$ then by (2.21),

$$40^{-1} \operatorname{dist}(I, \partial\Omega) \leq \operatorname{diam}(I) \leq \operatorname{dist}(I, \partial\Omega) < \gamma r \leq r, \qquad \operatorname{dist}(I, x) \leq r.$$

In particular, $I \subset B(x, 2r)$. Furthermore, we can find κ_0, depending only on dimension, such that $\operatorname{dist}(X, 4I) \geq 4r$ for every $I \in \Sigma_\gamma$ and $X \in \Omega \setminus B(x, \kappa_0 r)$. Let $Q_I \in \mathbb{D}$ be such that $\ell(Q_I) = \ell(I)$ and $\operatorname{dist}(I, \partial\Omega) = \operatorname{dist}(I, Q_I)$. Then $\ell(Q_I) \leq \gamma r$, and $Y(I)$, the center of I, satisfies $Y(I) \in B(x_{Q_I}, C\ell(Q_I))$

and $\delta(Y(I)) \approx \ell(I) = \ell(Q_I)$. Hence we can invoke (3.13) (taking κ_0 larger if needed) and obtain that for every $Y \in I$,

$$G(Y, X) \approx G(Y(I), X) \lesssim \ell(I) \frac{\omega^X(\kappa Q_I)}{\sigma(Q_I)},$$

where the first estimate uses Harnack's inequality in $2I \subset \Omega$. Hence,

$$\begin{aligned}\iint_{B\cap\{Y:\delta(Y)<\gamma r\}} G(Y, X)\, dY &\le \sum_{I\in\Sigma_\gamma} \iint_I G(Y, X)\, dY \lesssim \sum_{I\in\Sigma_\gamma} \ell(I)^2 \omega^X(\kappa Q_I) \\ &\le \sum_{k:2^{-k}\lesssim\gamma r} 2^{-2k} \sum_{I\in\Sigma_\gamma:\ell(I)=2^{-k}} \omega^X(\kappa Q_I) \lesssim (\gamma r)^2 \omega^X(\Delta(x, M_1 r)),\end{aligned}$$

where in the last step we have used that for each fixed k, the cubes κQ_I with $\ell(I) = 2^{-k}$ have uniformly bounded overlaps, and are all contained in $\Delta(x, M_1 r)$ for M_1 large enough. Dividing by $|B| \approx r^{n+1}$ and using the ADR property, we obtain the desired estimate. □

3B. ***PDE estimates: the p-harmonic case.*** We now recall several fundamental estimates for p-harmonic functions and p-harmonic measure, some of which generalize certain of the preceding estimates that we have stated in the harmonic case. We ask the reader to forgive a moderate amount of redundancy. Given a closed set E, as above we set $\delta(Y) := \operatorname{dist}(Y, E)$.

Lemma 3.17. *Let p, $1 < p < \infty$, be given. Let u be a positive p-harmonic function in $B(X, 2r)$. Then*

$$\left(\frac{1}{|B(X, r/2)|} \iint_{B(X,r/2)} |\nabla u|^p\, dy\right)^{1/p} \le \frac{C}{r} \max_{B(X,r)} u, \tag{3.18}$$

$$\max_{B(X,r)} u \le C \min_{B(X,r)} u. \tag{3.19}$$

Furthermore, there exists $\alpha = \alpha(p, n) \in (0, 1)$ such that if $Y, Y' \in B(X, r)$, then

$$|u(Y) - u(Y')| \le C \left(\frac{|Y - Y'|}{r}\right)^\alpha \max_{B(X,2r)} u. \tag{3.20}$$

Proof. The inequality (3.18) is a standard energy estimate, (3.19) is the well known Harnack inequality for positive solutions to the p-Laplace operator, and (3.20) is a well known interior Hölder continuity estimate for solutions to equations of p-Laplace type. We refer to [Serrin 1964] for these results. □

Definition 3.21. Let $O \subset \mathbb{R}^{n+1}$ be open and let K be a compact subset of O. Given p, $1 < p < \infty$, we let

$$\operatorname{Cap}_p(K, O) = \inf\left\{\iint_O |\nabla\phi|^p\, dY : \phi \in C_0^\infty(O),\ \phi \ge 1 \text{ in } K\right\}.$$

$\operatorname{Cap}_p(K, O)$ is referred to as the p-capacity of K relative to O. The p-capacity of an arbitrary set $E \subset O$ is defined by

$$\operatorname{Cap}_p(E, O) = \inf_{\substack{E\subset G\subset O\\ G \text{ open}}} \sup_{\substack{K\subset G\\ K \text{ compact}}} \operatorname{Cap}_p(K, O). \tag{3.22}$$

Definition 3.23. Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \operatorname{diam}(E)$. Given p, $1 < p < \infty$, we say that $E \cap B(x, 4r)$ is p-thick if for every $x \in E \cap B(x, 4r)$ there exists $r_x > 0$ such that

$$\int_0^{r_x} \left[\frac{\operatorname{Cap}_p(E \cap B(x, \rho), B(x, 2\rho))}{\operatorname{Cap}_p(B(x, \rho), B(x, 2\rho))} \right]^{1/(p-1)} \frac{d\rho}{\rho} = \infty.$$

We note that this definition is just the Wiener criterion in the p-harmonic case. As it can be seen in [Heinonen et al. 2006, Chapter 6], p-thickness implies that all points on $E \cap B(x, 4r)$ are regular for the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$.

Definition 3.24. Let $E \subset \mathbb{R}^{n+1}$ be a closed set and let $x \in E$, $0 < r < \operatorname{diam}(E)$. Given p, $1 < p < \infty$, and $\eta > 0$ we say that $E \cap B(x, 4r)$ is uniformly p-thick with constant η if

$$\frac{\operatorname{Cap}_p(E \cap B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))}{\operatorname{Cap}_p(B(\hat{x}, \hat{r}), B(\hat{x}, 2\hat{r}))} \geq \eta \tag{3.25}$$

whenever $\hat{x} \in E \cap B(x, 4r)$ and $B(\hat{x}, 2\hat{r}) \subset B(x, 4r)$.

Remark 3.26. In the case $p = 2$, the condition defined in Definition 3.24 is sometimes called the capacity density condition (CDC); see for instance [Aikawa 2004]. Note that uniform p-thickness is a strong quantitative version of the p-thickness defined above and hence of the Wiener regularity for the Laplace and the p-Laplace operator.

Lemma 3.27. *Let $E \subset \mathbb{R}^{n+1}$, $n \geq 2$, be Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Then $E \cap B(x, 4r)$ is uniformly p-thick for some constant η, depending only on p, n, and the ADR constant, whenever $x \in E$, $0 < r < \frac{1}{4} \operatorname{diam} E$.*

Proof. We first observe that since the ADR condition is scale-invariant we may translate and rescale to prove (3.25) only for $\hat{x} = 0$ and $\hat{r} = 1$ (we would also need to rescale E, but abusing the notation we still call it E). Write $B = B(0, 1)$ and observe that, for every $1 < p < \infty$, [Heinonen et al. 2006, Example 2.12] gives

$$\operatorname{Cap}_p(B, 2B) = C(n, p). \tag{3.28}$$

The desired bound from below follows at once if $p > n + 1$ from the estimate in [Heinonen et al. 2006, Example 2.12]:

$$\operatorname{Cap}_p(E \cap B, 2B) \geq \operatorname{Cap}_p(\{0\}, 2B) = C(n, p)'.$$

Let us now consider the case $1 < p \leq n + 1$. Write $K = E \cap \overline{\tfrac{1}{2} B}$. Combining [Heinonen et al. 2006, Theorem 2.38; Adams and Hedberg 1999, Theorems 2.2.7 and 4.5.2] we have that

$$\operatorname{Cap}_p(E \cap B, 2B) \gtrsim \widetilde{\operatorname{Cap}}_p(K) \gtrsim \sup_\mu \left(\frac{\mu(K)}{\| W_p(\mu) \|_{L^1(\mu)}^{1/p'}} \right)^p. \tag{3.29}$$

In the previous expression the implicit constants depend only on p and n; $\widetilde{\operatorname{Cap}}_p$ stands for the inhomogeneous p-capacity, that is,

$$\widetilde{\operatorname{Cap}}_p(K) = \inf \left\{ \iint_{\mathbb{R}^{n+1}} (|\phi|^p + |\nabla \phi|^p) \, dY : \phi \in C_0^\infty(\mathbb{R}), \ \phi \geq 1 \text{ in } K \right\};$$

the sup runs over all Radon positive measures supported on K; and

$$W_p(\mu)(y) := \int_0^1 \left(\frac{\mu(B(y,t))}{t^{n+1-p}}\right)^{p'-1} \frac{dt}{t}, \quad x \in \operatorname{supp}\mu.$$

We choose $\mu = H^n|_K$ and observe that, if $y \in \operatorname{supp}\mu \subset K \subset E$ and $0 < t < 1$, then, by ADR, $\mu(B(y,t)) = \sigma(B(y,t) \cap B(0, \frac{1}{2}) \lesssim t^n$. This easily gives $W_p(\mu)(y) \lesssim 1$ for every $y \in \operatorname{supp}\mu$ and, by ADR,

$$\int_K W_p(\mu)(y)\, d\mu(y) \le \mu(K) \le \sigma(B) \lesssim 1.$$

We can now use (3.29) and ADR again to conclude that

$$\operatorname{Cap}_p(E \cap B, 2B) \gtrsim \mu(K) \ge \sigma\big(B\big(0, \tfrac{1}{2}\big)\big)^p \gtrsim 1.$$

Combining this with (3.28) we readily obtain (3.25). □

Lemma 3.30. *Let $E \subset \mathbb{R}^{n+1}$, $n \ge 2$, be Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in E$ and $0 < r < \operatorname{diam}(E)$. Then, given $f \in W^{1,p}(B(x,4r))$ there exists a unique p-harmonic function $u \in W^{1,p}(B(x,4r) \setminus E)$ such that $u - f \in W_0^{1,p}(B(x,4r) \setminus E)$. Furthermore, let $u, v \in W_{\mathrm{loc}}^{1,p}(B(x,4r) \setminus E)$ be a p-superharmonic function and a p-subharmonic function in Ω, respectively. If $\inf\{u-v, 0\} \in W_0^{1,p}(B(x,4r) \setminus E)$, then $u \ge v$ a.e. in $B(x,4r) \setminus E$. Finally, every point $\hat{x} \in E \cap B(x,4r)$ is regular for the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$.*

Proof. The first part of the lemma is a standard maximum principle. The fact that every $\hat{x} \in E \cap B(x,4r)$ is regular in the continuous Dirichlet problem for $\nabla \cdot (|\nabla u|^{p-2} \nabla u) = 0$ follows from the fact that Lemma 3.27 implies that $E \cap B(x,4r)$ is uniformly p-thick for every $1 < p < \infty$, and hence we can invoke [Heinonen et al. 2006, Chapter 6]. □

Lemma 3.31. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be an open set whose boundary is Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that u is nonnegative and p-harmonic in $B(x,4r) \cap \Omega$, continuous on $B(x,4r) \cap \overline{\Omega}$, and that $u = 0$ on $\partial\Omega \cap B(x,4r)$. Then, extending u to be 0 in $B(x,4r) \setminus \overline{\Omega}$, we have*

$$\left(\frac{1}{|B(x,r/2)|} \iint_{B(x,r/2)} |\nabla u|^p \, dy\right)^{1/p} \le \frac{C}{r} \left(\frac{1}{|B(x,r)|} \iint_{B(x,r)} u^{p-1}\right)^{1/(p-1)}. \tag{3.32}$$

Furthermore, there exists $\alpha \in (0,1)$, depending only on p, n, and the ADR constant, such that if $Y, Y' \in B(x,r)$, then

$$|u(Y) - u(Y')| \le C \left(\frac{|Y - Y'|}{r}\right)^{\alpha} \max_{B(x,2r)} u. \tag{3.33}$$

Proof. Since u, extended as above to all of $B(x,4r)$, is a nonnegative p-subsolution in $B(x,4r)$, (3.32) is just a standard energy or Caccioppoli estimate plus a standard interior estimate. Thus, we only prove (3.33). Since $E \cap B(x,4r)$ is uniformly p-thick as seen in Lemma 3.27, we can invoke [Heinonen et al. 2006, Theorem 6.38] to obtain that there exist $C \ge 1$ and $\alpha = \alpha \in (0,1)$, depending only on n, p, and the ADR constant, such that

$$\max_{B(x,\rho)} u \le C\Big(\frac{\rho}{r}\Big)^{\alpha} \max_{B(x,r)} u, \quad \text{whenever } 0 < \rho \le r. \tag{3.34}$$

This, the triangle inequality, and elementary arguments give (3.33). □

Lemma 3.35. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be an open set whose boundary is Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that u is nonnegative and p-harmonic in $B(x, 4r) \cap \Omega$, continuous on $B(x, 4r) \cap \overline{\Omega}$, and that $u = 0$ on $\partial\Omega \cap B(x, 4r)$. Then, extending u to be 0 in $B(x, 4r) \setminus \overline{\Omega}$, there exists $\alpha > 0$ such that*

$$u(Y) \le C\Big(\frac{\delta(Y)}{r}\Big)^{\alpha}\Big(\frac{1}{|B(x, 2r)|}\iint_{B(x,2r)} u^{p-1}(Z)\, dZ\Big)^{1/(p-1)} \tag{3.36}$$

for all $Y \in B(x, r)$, where the constants C and α depend only on n, p, and the ADR constant of $\partial\Omega$.

Proof. This follows from Lemma 3.31 and standard estimates for p-subsolutions. Let us note that in the linear case (i.e, $p = 2$) one can give an alternative proof based on Bourgain's Lemma 3.1 and an iteration argument (see [Hofmann et al. ≥ 2017] for details). □

Lemma 3.37. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \ge 2$, be an open set whose boundary is Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$ and consider $0 < r < \operatorname{diam}(\partial\Omega)$. Assume also that u is nonnegative and p-harmonic in $B(x, 4r) \cap \Omega$, continuous on $B(x, 4r) \cap \overline{\Omega}$, that $u = 0$ on $\partial\Omega \cap B(x, 4r)$, and that u is extended to be 0 in $B(x, 4r) \setminus \overline{\Omega}$. Then u has a representative in $W^{1,p}(B(x, 4r))$ with Hölder continuous partial derivatives in $B(x, 4r) \setminus \partial\Omega$. Furthermore, there exists $\beta \in (0, 1]$ such that if $Y, Y' \in B(X, \hat r/2)$, with $B(X, 4\hat r) \subset B(x, 4r) \setminus \partial\Omega$, then*

$$|\nabla u(Y) - \nabla u(Y')| \lesssim \Big(\frac{|Y - Y'|}{\hat r}\Big)^{\beta} \max_{B(X,\hat r)} |\nabla u| \lesssim \frac{1}{\hat r}\Big(\frac{|Y - Y'|}{\hat r}\Big)^{\beta} \max_{B(X,2\hat r)} u\,, \tag{3.38}$$

where β and the implicit constants depend only on p and n. Furthermore, if

$$\frac{u(Y)}{\delta(Y)} \approx |\nabla u(Y)|, \quad Y \in B(X, 3\hat r), \tag{3.39}$$

then u has continuous second derivatives in $B(X, 3\hat r)$, and there exists $C \ge 1$, depending only on n, p, and the implicit constants in (3.39), *such that*

$$\max_{B(X,\hat r/2)} |\nabla^2 u| \le C\Big(\frac{1}{|B(X, \hat r)|}\iint_{B(X,\hat r)} |\nabla^2 u(Y)|^2\, dY\Big)^{1/2} \le C^2 \frac{u(X)}{\delta(X)^2}. \tag{3.40}$$

Proof. For (3.38) we refer, for example, to [Tolksdorf 1984]; (3.40) is a consequence of (3.38), (3.39), and Schauder type estimates, see [Gilbarg and Trudinger 1983]. For a more detailed proof of (3.40), see [Lewis and Vogel 2006, Lemma 2.4(d)] for example. □

Remark 3.41. We note that the second inequality in (3.38) and (3.19) give

$$|\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)}, \quad Y \in B(x, 2r) \setminus \partial\Omega. \tag{3.42}$$

Lemma 3.43. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and suppose that u is nonnegative and p-harmonic in $B(x, 4r) \cap \Omega$, vanishing continuously on $B(x, 4r) \cap \Omega$ (hence u is continuous in $B(x, 4r)$ after being extended by 0 in $B(x, 4r) \setminus \overline{\Omega}$). There exists a unique finite positive Borel measure μ on $\mathbb{R}^{n+1}$, with support in $\partial\Omega \cap B(x, 4r)$, such that*

$$-\iint_{\mathbb{R}^{n+1}} |\nabla u|^{p-2}\nabla u \cdot \nabla\phi \, dY = \int \phi \, d\mu \tag{3.44}$$

whenever $\phi \in C_0^\infty(B(x, 4r))$. Furthermore, there exists $C < \infty$, depending only on p, n, and the ADR constant, such that

$$\left(\frac{\max_{B(x,r)} u}{r}\right)^{p-1} \leq C\frac{\mu(\Delta(x, 2r))}{\sigma(\Delta(x, 2r))}. \tag{3.45}$$

Note that (3.45) is the p-harmonic analogue of Lemma 3.11.

Proof. For the proof of (3.44), see [Heinonen et al. 2006, Chapter 21]. Using Lemma 3.27 and Lemma 3.31, (3.45) follows directly from [Kilpeläinen and Zhong 2003, Lemma 3.1]; see also [Eremenko and Lewis 1991]. □

The following lemma generalizes Lemma 3.14 to the case $1 < p < \infty$.

Lemma 3.46. *Let $\Omega \subset \mathbb{R}^{n+1}$, $n \geq 2$, be an open set and assume that $\partial\Omega$ is Ahlfors–David regular of dimension n. Let p, $1 < p < \infty$, be given. Let $x \in \partial\Omega$, $0 < r < \operatorname{diam}(\partial\Omega)$, and suppose that u and μ are as in Lemma 3.43. Then there exist constants C and M_1, depending only on n and the ADR constant, such that if $B(y, M_1 s) \subset B(x, 2r)$ with $y \in \partial\Omega$, then*

$$\max_{B(y,s/2)} u^{p-1} \lesssim \frac{1}{|B(y, s)|} \iint_{B(y,s)} u^{p-1}(Z) \, dZ \leq C s^{p-1} \frac{\mu(\Delta(y, M_1 s))}{\sigma(\Delta(y, s))}.$$

Moreover, for all $\gamma \in (0, 1]$,

$$\frac{1}{|B(y, s)|} \iint_{B(y,s)\cap\{Y:\delta(Y)\leq\gamma s\}} u^{p-1}(Z) \, dZ \leq C\gamma^p s^{p-1} \frac{\mu(\Delta(y, M_1 s))}{\sigma(\Delta(y, s))}.$$

We note that in the previous estimates it is implicitly understood that u is extended to be 0 on $B(x, 4r) \setminus \overline{\Omega}$.

Proof. Using (3.45), the proof of Lemma 3.46 is the same mutatis mutandi as that of Lemma 3.14. We omit further details. □

4. Proofs of Theorem 1.1 and Theorem 1.12: preliminary arguments

We start the proofs of Theorem 1.1 and Theorem 1.12 by giving some preliminary arguments. We first show that (1.2) implies (1.4). To this end, we claim that, without loss of generality, we may suppose that for a surface ball $\Delta = \Delta(x, r)$, the point X_Δ in the statement of Theorem 1.1 satisfies (3.2), i.e., there is some $c_1 = c_1(n, \text{ADR}) > 0$ such that

$$\omega^{X_\Delta}(\Delta) \geq c_1. \tag{4.1}$$

The only price to be paid is that the constants c_0, C_0 may now be slightly different (depending only on n and ADR), and that (1.2) now holds with Δ in place of 2Δ, i.e., for the (possibly) new point X_Δ, we have

$$\int_\Delta k^{X_\Delta}(y)^q \, d\sigma(y) \le C_0 \sigma(\Delta)^{1-q}. \tag{4.2}$$

Indeed, set $\Delta' := \Delta(x, r/2)$, and let $X' := X_{\Delta'} \in B(x, r/2) \cap \Omega$ be the point such that (1.2) holds for Δ'. Fix $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$. Suppose first that $\delta(X') \le r/4$, in which case $\Delta(\hat{x}, r/4) \subset \Delta$. Thus, if in addition $\delta(X') < cr/4$, where $c \in (0, 1)$ is the constant in Lemma 3.1, then we set $X_\Delta := X'$, and (4.1) holds by Lemma 3.1. On the other hand, if $cr/4 \le \delta(X_\Delta) \le r/4$, we select X_Δ along the line segment joining X' to $\hat{x}$, such that $\delta(X_\Delta) = |X_\Delta - \hat{x}| = cr/8$, and (4.1) holds exactly as before. Moreover, (4.2) holds for this new X_Δ, in the first case, immediately by (1.2) applied to $X' = X_{\Delta'}$, and in the second case, by moving from X' to X_Δ via Harnack's inequality (which may be used within the touching ball $B(X', \delta(X'))$). Let us finally consider the case $\delta(X') > r/4$. Then we can use Harnack within the ball $B(X', r/4)$ to pass to a point X'' on the line segment joining X' to x such that $|X' - X''| = r/8$, and consequently $\delta(X'') \le |X'' - x| < 3r/8$ (since $X' \in B(x, r/2)$). Hence (1.2) holds (with different constant) for Δ' with X'' in place of $X_{\Delta'}$. Now take $\hat{x} \in \partial\Omega$ such that $\delta(X'') = |X'' - \hat{x}|$ and note that $\Delta(\hat{x}, r/4) \subset \Delta$. We can now repeat the previous argument with X'' in place of X'. Details are left to the interested reader.

Similarly, if (1.4) holds for $\Delta = \Delta(x, r)$, with $X_\Delta \in B(x, r/2) \cap \Omega$, then again without loss of generality we may suppose that (4.1) holds, possibly for a new $X_\Delta \in B(x, r) \cap \Omega$. Indeed if we let $X' \in B(x, r/2) \cap \Omega$ be the original point X_Δ for which (1.4) holds, we may then follow the argument in the previous paragraph, mutatis mutandi. We choose $\hat{x} \in \partial\Omega$ such that $\delta(X') = |X' - \hat{x}|$ and suppose first that $\delta(X') \le r/4$, so that $\Delta(\hat{x}, r/4) \subset \Delta$. Considering the same two cases as before we pick X_Δ and in either case (4.1) holds by Lemma 3.1 applied to the surface ball $\Delta(\hat{x}, r/4)$. Note that in the second case, (1.4) continues to hold for X_Δ, with a different but still uniform β, using Harnack's inequality within the touching ball $B(X', \delta(X'))$ to move from X' to X_Δ. When $r/4 < \delta(X')$ we choose X'' as before, and by Harnack's inequality, (1.4) holds with X'' in place of X', for a different but still uniform β. Again, if we let $\hat{x} \in \partial\Omega$ with $\delta(X'') = |X'' - \hat{x}|$, then $\Delta(\hat{x}, r/4) \subset \Delta$, and we may now repeat the previous argument with X'' in place of X'.

We are now ready to show that (1.2) implies (1.4).

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^{n+1}$ be an open set with n-dimensional ADR boundary, and let $\Delta = \Delta(x, r)$ be a surface ball on $\partial\Omega$. Let μ be a measure on $\partial\Omega$ such that $\mu|_\Delta \ll \sigma$, and such that for some $q > 1$ and $\Lambda < \infty$,*

$$\fint_\Delta k^q \, d\sigma \le \Lambda, \tag{4.4}$$

where $k := d\mu/d\sigma$ on Δ. Suppose also that

$$\frac{\mu(\Delta)}{\sigma(\Delta)} \ge 1. \tag{4.5}$$

Then there are constants $\eta, \beta \in (0, 1)$, depending only on n, q, Λ, and ADR, such that for any Borel set $A \subset \Delta$,

$$\sigma(A) \ge (1 - \eta)\sigma(\Delta) \implies \mu(A) \ge \beta\mu(\Delta). \tag{4.6}$$

Remark 4.7. Let k be a normalized version of harmonic measure: $k = c_1^{-1}\sigma(\Delta)k^{X_\Delta}$, with X_Δ a point for which (4.1) and (4.2) hold. Then clearly (4.4) and (4.5) hold for k, and the conclusion (4.6) is just a reformulation of (1.4). We note that in the sequel, we actually use only (4.6) or (1.4), rather than condition (4.4) or (4.2). Thus, Theorem 1.1 could just as well have been stated with condition ($\star\star$) (see Remark 1.3) in place of ($\star$).

Proof of Lemma 4.3. Set $F := \Delta \setminus A$, so $\sigma(F) \le \eta\sigma(\Delta)$. Then

$$\begin{aligned}\mu(F) = \int_F k\, d\sigma &\le \sigma(F)^{1/q'}\biggl(\int_\Delta k^q\, d\sigma\biggr)^{1/q} \\ &\le \Lambda^{1/q}\sigma(F)^{1/q'}\sigma(\Delta)^{1/q} \le \Lambda^{1/q}\eta^{1/q'}\sigma(\Delta) \le \Lambda^{1/q}\eta^{1/q'}\mu(\Delta),\end{aligned}$$

where in the last step we have used (4.5). Thus,

$$\mu(A) \ge \bigl(1 - \Lambda^{1/q}\eta^{1/q'}\bigr)\mu(\Delta) \ge \tfrac{1}{2}\mu(\Delta)$$

for η small enough. This completes the proof. □

Fix $Q_0 \in \mathbb{D}(\partial\Omega)$. As in (2.8), we set $B_{Q_0} = B(x_{Q_0}, r_0)$, with $r_0 := r_{Q_0} \approx \ell(Q_0)$, so that $\Delta_{Q_0} = B_{Q_0} \cap \partial\Omega \subset Q_0$.

Proceeding first in the setting of Theorem 1.1, let $X_0 := X_{\Delta_{Q_0}}$ be the point relative to $\Delta = \Delta_{Q_0}$ such that (4.1) and (4.2) hold. Note that (4.1) trivially implies that

$$\omega^{X_0}(Q_0) \ge c_1.$$

With the pole X_0 fixed, we define the normalized harmonic measure and the normalized Green's function, respectively, by

$$\mu := \frac{1}{c_1}\sigma(Q_0)\omega^{X_0}, \quad u(Y) := \frac{1}{c_1}\sigma(Q_0)G(X_0, Y). \tag{4.8}$$

Then under this normalization, setting $\|\mu\| = \mu(\partial\Omega)$, we have

$$1 \le \frac{\mu(Q_0)}{\sigma(Q_0)} \le \frac{\|\mu\|}{\sigma(Q_0)} \le C_1, \tag{4.9}$$

with $C_1 = 1/c_1$. Furthermore, we may apply Lemma 4.3 (using (4.1) and with $\Lambda \approx C_0/c_1$) to obtain (4.6) for μ, with $\Delta = \Delta_{Q_0}$. In turn, the latter bound, in conjunction with (4.1) and ADR, clearly implies an analogous estimate for Q_0, namely that there are constants that we again call $\eta, \beta \in (0, 1)$ such that for any Borel set $A \subset Q_0$,

$$\sigma(A) \ge (1-\eta)\sigma(Q_0) \implies \mu(A) \ge \beta\mu(Q_0). \tag{4.10}$$

Here, of course, we may have different values of the parameters η and β, but these have the same dependence as the original values, so for convenience we maintain the same notation.

In the p-harmonic case, proceeding under the setup of Theorem 1.12, we let u and μ be the p-harmonic function and its associated p-harmonic measure, corresponding to the point $x = x_{Q_0}$ and the radius $r = Cr_0 := Cr_{Q_0}$, satisfying the hypotheses of Theorem 1.12, where we choose the constant C depending only on n and ADR, such that $Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ (thus, in particular, μ is defined on Q_0). Since we assume that u is nontrivial and nonnegative, we can apply Lemma 3.43 in $B(x_{Q_0}, Cr_0)$ and use (1.14) to conclude that $\mu(\Delta_{Q_0}) > 0$. We can therefore normalize u and μ (abusing the notation we call the normalizations u and μ) so that $\mu(\Delta_{Q_0})/\sigma(Q_0) = 1$, and since $\Delta_{Q_0} \subset Q_0 \subset \Delta(x_{Q_0}, Cr_0)$ by (1.14), we also have $\mu(\Delta(x_{Q_0}, Cr_0))/\sigma(\Delta(x_{Q_0}, Cr_0)) \approx \mu(Q_0)/\sigma(Q_0) \approx 1$. Set $k := d\mu/d\sigma$. As above, by (1.13) and (1.14), we may then use Lemma 4.3 to see that again μ satisfies both (4.9), now with $\|\mu\| := \mu(\Delta(x_{Q_0}, Cr_0))$, and (4.10). The constants C_1, η, and β depend on C, n, the ADR constant, C_0, and q.

Remark 4.11. Under the assumptions of Theorems 1.1 and 1.12 and throughout this section and Section 6, for $Q_0 \in \mathbb{D}(E)$ fixed, u and μ will continue to denote the normalized Green function and harmonic measure or the normalized nonnegative p-harmonic solution and p-harmonic Riesz measure, as defined above. In particular, (4.9) and (4.10) hold for all $1 < p < \infty$.

As above, let $\mathcal{M}$ denote the usual Hardy–Littlewood maximal operator on $\partial\Omega$ and recall the definition of $\mathbb{D}_{\mathcal{F},Q_0}$ in (2.10).

Lemma 4.12. *Let $Q_0 \in \mathbb{D}$, and suppose that μ satisfies* (4.9) *and* (4.10). *Then there is a pairwise disjoint family $\mathcal{F} = \{Q_j\}_{j\geq 1} \subset \mathbb{D}_{Q_0}$ such that*

$$\sigma\big(Q_0 \setminus \big(\bigcup_j Q_j\big)\big) \geq \frac{1}{C}\,\sigma(Q_0) \tag{4.13}$$

and

$$\frac{\beta}{2} \leq \frac{\mu(Q)}{\sigma(Q)} \leq \bigg(\fint_Q (\mathcal{M}\mu)^{1/2}\, d\sigma \bigg)^2 \leq C, \quad \forall\, Q \in \mathbb{D}_{\mathcal{F},Q_0}, \tag{4.14}$$

where $C > 1$ depends only on η, β, C_1, n, and ADR.

Proof. The proof is based on a stopping time argument similar to those used in the proof of the Kato square root conjecture [Hofmann and McIntosh 2002; Hofmann et al. 2002; Auscher et al. 2002a], and in local Tb theorems. We begin by noting that

$$\|\mathcal{M}\mu\|_{L^{1,\infty}(\sigma)} := \sup_{\lambda>0} \lambda\sigma\{\mathcal{M}\mu > \lambda\} \lesssim \|\mu\| \lesssim \sigma(Q_0) \tag{4.15}$$

by the Hardy–Littlewood theorem and (4.9). Consequently, by Kolmogorov's criterion,

$$\fint_{Q_0} (\mathcal{M}\mu)^{1/2}\, d\sigma \leq C = C(n, \text{ADR}, C_1). \tag{4.16}$$

We now perform a stopping time argument to extract a family $\mathcal{F} = \{Q_j\}$ of dyadic subcubes of Q_0 that are maximal with respect to the property that either

$$\frac{\mu(Q_j)}{\sigma(Q_j)} < \frac{\beta}{2} \tag{4.17}$$

and/or

$$\fint_{Q_j} (\mathcal{M}\mu)^{1/2}\, d\sigma > K, \tag{4.18}$$

where $K \geq 1$ is a sufficiently large number to be chosen momentarily. Note that $Q_0 \notin \mathcal{F}$, by (4.9) and (4.16). We say that Q_j is of "type I" if (4.17) holds, and of "type II" if (4.18) holds but (4.17) does not. Set $A := Q_0 \setminus \big(\bigcup_j Q_j\big)$, and $F := \bigcup_{Q_j \text{ type II}} Q_j$. Then by (4.9),

$$\sigma(Q_0) \leq \mu(Q_0) = \sum_{Q_j \text{ type I}} \mu(Q_j) + \mu(F) + \mu(A). \tag{4.19}$$

By definition of the type I cubes,

$$\sum_{Q_j \text{ type I}} \mu(Q_j) \leq \frac{\beta}{2} \sum_j \sigma(Q_j) \leq \frac{\beta}{2} \sigma(Q_0). \tag{4.20}$$

To handle the remaining terms, observe that

$$\begin{aligned} \sigma(F) = \sum_{Q_j \text{ type II}} \sigma(Q_j) &\leq \frac{1}{K} \sum_j \int_{Q_j} (\mathcal{M}\mu)^{1/2}\, d\sigma \\ &\leq \frac{1}{K} \int_{Q_0} (\mathcal{M}\mu)^{1/2}\, d\sigma \leq \eta\, \sigma(Q_0), \end{aligned} \tag{4.21}$$

by the definition of the type II cubes, (4.16), and the choice of $K = C\eta^{-1}$. By (4.10) and complementation, we therefore find that

$$\mu(F) \leq (1-\beta)\mu(Q_0). \tag{4.22}$$

Next, if $x \in A$, then every $Q \in \mathbb{D}_{Q_0}$ that contains x must satisfy the opposite inequality to (4.18), and therefore, by Lebesgue's differentiation theorem,

$$\mathcal{M}\mu(x) \leq K^2, \quad \text{for } \sigma\text{-a.e. } x \in A.$$

Thus $\mu|_A \ll \sigma$, with $d\mu|_A/d\sigma \leq K^2$, and thus,

$$\mu(A) \leq K^2 \sigma(A).$$

Combining the latter estimate with (4.19), (4.20), and (4.22), we obtain

$$\beta\mu(Q_0) \leq \frac{\beta}{2}\sigma(Q_0) + K^2\sigma(A).$$

Using (4.9), we then find that

$$\beta\sigma(Q_0) \leq \beta\mu(Q_0) \leq \frac{\beta}{2}\sigma(Q_0) + K^2\sigma(A).$$

The conclusion of the lemma now follows readily. □

For future reference, let us note an easy consequence of the last inequality in (4.14) and the ADR property: for all $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, and for any constant $b > 1$, we have

$$\mu\big(\Delta(x_Q, b\operatorname{diam}(Q))\big) \lesssim b^n \sigma(Q) \bigg(\fint_Q (\mathcal{M}\mu)^{1/2}\, d\sigma \bigg)^2 \lesssim b^n \sigma(Q). \tag{4.23}$$

Recall that the ball B_Q^* and surface ball Δ_Q^* are defined in (2.15).

Lemma 4.24. *Let u, μ be as in Remark 4.11. If the constant K_0 in* (2.15) *and* (2.23) *is chosen sufficiently large, then for each $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ with $\ell(Q) \le K_0^{-1}\ell(Q_0)$, there exists $Y_Q \in U_Q$ with*

$$\delta(Y_Q) \le |Y_Q - x_Q| \lesssim \ell(Q),$$

where the implicit constant is independent of K_0, such that

$$\frac{\mu(Q)}{\sigma(Q)} \le C|\nabla u(Y_Q)|^{p-1}, \tag{4.25}$$

where C depends on K_0 and the implicit constants in the hypotheses of Theorems 1.1 and 1.12.

Remark 4.26. Recalling the construction at the beginning of Section 4, and the fact that we have defined $X_0 := X_{\Delta_{Q_0}}$, we see that $\ell(Q_0) \approx \delta(X_0) \ge K_0^{-1/2}\ell(Q_0)$, for K_0 chosen large enough. We note further that the point Y_Q whose existence is guaranteed by Lemma 4.24 is essentially a corkscrew point relative to Q. Indeed, $\delta(Y_Q) \gtrsim K_0^{-1}\ell(Q)$ (since $Y \in U_Q$), and also $|Y_Q - x_Q| \lesssim \ell(Q)$ (with constant independent of K_0). With a slight abuse of terminology, we shall refer to Y_Q as a corkscrew point relative to Q, with corkscrew constant depending on K_0.

Proof of Lemma 4.24. Fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le K_0^{-1}\ell(Q_0)$, where, as in Remark 4.26, we have chosen K_0 large enough that $\ell(Q_0) \approx \delta(X_0) \ge K_0^{-1/2}\ell(Q_0)$. Recall (2.7) and (2.8), and set $\hat{B}_Q = B(x_Q, \hat{r}_Q)$, $\hat{\Delta}_Q = \hat{B}_Q \cap \partial\Omega$, with $\hat{r}_Q \approx \ell(Q)$ and $Q \subset \hat{\Delta}_Q$. Let $0 \le \phi_Q \in C_0^\infty(2\hat{B}_Q)$, such that $\phi_Q \equiv 1$ in $\hat{B}_Q$ and $\|\nabla\phi_Q\| \lesssim \ell(Q)^{-1}$. Note that

$$K_0^{1/2}\ell(Q) \le K_0^{-1/2}\ell(Q_0) \le \delta(X_0) \le |X_0 - x_Q|,$$

which implies that $X_0 \notin 4\hat{B}_Q$ provided K_0 is large enough. Thus, by (3.10) in the linear case, or (3.44) in general,

$$\begin{aligned} \ell(Q)\mu(Q) \le \ell(Q)\int_{\partial\Omega}\phi_Q\,d\mu &\lesssim \iint_{\hat{B}_Q\cap\Omega}|\nabla u(Y)|^{p-1}\,dY \\ &\le \iint_{\hat{B}_Q\cap U_Q}|\nabla u(Y)|^{p-1}\,dY + \iint_{(\hat{B}_Q\cap\Omega)\setminus U_Q}|\nabla u(Y)|^{p-1}\,dY \\ &=: \mathcal{I} + \mathcal{II}. \end{aligned} \tag{4.27}$$

Notice that by construction,

$$(\hat{B}_Q \cap \Omega) \setminus U_Q \subset \{Y \in \hat{B}_Q : \delta(Y) \le CK_0^{-1}\ell(Q)\}.$$

We may therefore cover the latter region by a family of balls $\{B_k\}_k$, centered on $\partial\Omega$, of radius $CK_0^{-1}\ell(Q)$, such that their doubles $\{2B_k\}$ have bounded overlaps and satisfy

$$\bigcup_k 2B_k \subset \{Y \in 2\hat{B}_Q : \delta(Y) \le 2CK_0^{-1}\ell(Q)\} =: \Sigma(K_0).$$

By the boundary Cacciopoli estimate in Lemma 3.31, plus Hölder's inequality, we obtain

$$\begin{aligned}\mathcal{II} &\leq \sum_k \iint_{B_k} |\nabla u(Y)|^{p-1}\,dY \lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} \sum_k \iint_{2B_k} |u(Y)|^{p-1}\,dY \\ &\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} \iint_{\Sigma(K_0)} |u(Y)|^{p-1}\,dY \\ &\lesssim \left(\frac{K_0}{\ell(Q)}\right)^{p-1} K_0^{-p}\ell(Q)^p \mu(\Delta(x_Q, 2M_1\hat{r}_Q)) \\ &\lesssim K_0^{-1}\ell(Q)\sigma(Q) \leq \frac{1}{2}\ell(Q)\mu(Q),\end{aligned}$$

where in the last three steps we have used (3.16) (when $p = 2$) or Lemma 3.46 ($1 < p < \infty$), (4.23), and finally the choice of K_0 large enough. We can then hide this term on the left-hand side of (4.27), so that

$$\begin{aligned}\ell(Q)\mu(Q) \lesssim \mathcal{I} = \iint_{\hat{B}_Q\cap U_Q} |\nabla u(Y)|^{p-1}\,dY &= \sum_i \iint_{\hat{B}_Q\cap U_Q^i} |\nabla u(Y)|^{p-1}\,dY \\ &\lesssim \ell(Q)^{n+1} \max_i \sup_{Y\in\hat{B}_Q\cap U_Q^i} |\nabla u(Y)|^{p-1} \\ &\approx \ell(Q)\sigma(Q) \max_i \sup_{Y\in\hat{B}_Q\cap U_Q^i} |\nabla u(Y)|^{p-1},\end{aligned}$$

and we recall that $\{U_Q^i\}_i$ is an enumeration of the connected components of U_Q, and that the number of these components is uniformly bounded. Thus, for some i, there is a point $Y_Q \in \hat{B}_Q \cap U_Q^i$ such that $\mu(Q)/\sigma(Q) \lesssim |\nabla u(Y_Q)|^{p-1}$. To complete the proof, we simply observe that by construction, $\delta(Y_Q) \leq |Y_Q - x_Q| \leq \hat{r}_Q \lesssim \ell(Q)$. □

5. Proof of Theorem 1.1, Corollary 1.5, and Theorem 1.12

In this section we complete the proofs of Theorem 1.1 and Theorem 1.12 by proving that $E := \partial\Omega$ satisfies WHSA, and hence, by Proposition 1.17, E is UR. The proof of Corollary 1.5 follows almost immediately from Theorem 1.1 and we supply the proof at the end of the section. Our approach to the proofs of Theorems 1.1 and 1.12 is a refinement and extension of the arguments in [Lewis and Vogel 2007], who, as mentioned in the introduction, treated the special case that $k \approx 1$.

We fix $Q_0 \in \mathbb{D}(E)$ and we let u and μ be as in Remark 4.11. We recall that by (4.9),

$$\frac{\mu(Q_0)}{\sigma(Q_0)} \approx 1. \tag{5.1}$$

Let $\mathcal{F} = \{Q_j\}_j$ be the family of maximal stopping time cubes constructed in Lemma 4.12. Combining (4.25) and (4.14), we see that

$$|\nabla u(Y_Q)| \gtrsim 1, \quad \forall Q \in \mathbb{D}^*_{\mathcal{F},Q_0} := \{Q \in \mathbb{D}_{\mathcal{F},Q_0} : \ell(Q) \leq K_0^{-1}\ell(Q_0)\}, \tag{5.2}$$

where $Y_Q \in U_Q$ is the point constructed in Lemma 4.24. We recall that the Whitney region U_Q has a uniformly bounded number of connected components, which we have enumerated as $\{U_Q^i\}_i$. We now fix the particular i such that $Y_Q \in U_Q^i \subset \widetilde{U}_Q^i$, where the latter is the enlarged Whitney region constructed in Definition 2.26.

For a suitably small ε_0, say $\varepsilon_0 \ll K_0^{-6}$, we fix an arbitrary positive $\varepsilon < \varepsilon_0$, and we fix also a large positive number M to be chosen. For each point $Y \in \Omega$, we set

$$B_Y := \overline{B(Y, (1-\varepsilon^{2M/\alpha})\delta(Y))}, \qquad \widetilde{B}_Y := \overline{B(Y, \delta(Y))}, \tag{5.3}$$

where $0 < \alpha < 1$ is the exponent appearing in Lemma 3.35. For $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, we consider three cases.

Case 0: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) > \varepsilon^{10}\ell(Q_0)$.

Case 1: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le \varepsilon^{10}\ell(Q_0)$ and

$$\sup_{X\in\widetilde{U}_Q^i} \sup_{Z\in B_X} |\nabla u(Z) - \nabla u(Y_Q)| > \varepsilon^{2M}. \tag{5.4}$$

Case 2: $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, with $\ell(Q) \le \varepsilon^{10}\ell(Q_0)$ and

$$\sup_{X\in\widetilde{U}_Q^i} \sup_{Z\in B_X} |\nabla u(Z) - \nabla u(Y_Q)| \le \varepsilon^{2M}. \tag{5.5}$$

We trivially see that the cubes in Case 0 satisfy a packing condition:

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0} \\ \text{Case 0 holds}}} \sigma(Q) \le \sum_{\substack{Q\in\mathbb{D}_{Q_0} \\ \ell(Q)>\varepsilon^{10}\ell(Q_0)}} \sigma(Q) \lesssim (\log\varepsilon^{-1})\sigma(Q_0). \tag{5.6}$$

Note that in Case 1 and Case 2 we have $Q \in \mathbb{D}^*_{\mathcal{F},Q_0}$ (see (5.2)). Furthermore, if $\ell(Q) \le \varepsilon^{10}\ell(Q_0)$, then by (5.2), (3.42), and either (3.13) (which we apply in the case $p = 2$, with $X = X_0$, since $\ell(Q) \ll \ell(Q_0)$) or (3.45) (for general p, $1 < p < \infty$), and (4.14), we have

$$1 \lesssim |\nabla u(Y_Q)| \lesssim \frac{u(Y_Q)}{\delta(Y_Q)} \lesssim 1. \tag{5.7}$$

Regarding Case 1 we obtain the following packing condition.

Lemma 5.8. *Under the previous assumptions, the following packing condition holds*:

$$\frac{1}{\sigma(Q_0)} \sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \le C(\varepsilon, K_0, M, \eta). \tag{5.9}$$

On the other hand, we show that the cubes in Case 2 satisfy the ε-local WHSA property. Given $\varepsilon > 0$, recall that $B_Q^{***}(\varepsilon) = B(x_Q, \varepsilon^{-5}\ell(Q))$ (see (2.16)). We also introduce

$$B_Q^{\text{big}} = B_Q^{\text{big}}(\varepsilon) := B(x_Q, \varepsilon^{-8}\ell(Q)), \qquad \Delta_Q^{\text{big}} := B_Q^{\text{big}} \cap E.$$

Lemma 5.10. *Fix $\varepsilon \in (0, K_0^{-6})$, and let $1 < p < \infty$. Suppose that u is nonnegative and p-harmonic in $\Omega_Q := \Omega \cap B_Q^{\text{big}}$, $u \in C(\overline{\Omega_Q})$, $u \equiv 0$ on Δ_Q^{big}. Suppose also that for some i, there exists a point $Y_Q \in U_Q^i$ such that*

$$|\nabla u(Y_Q)| \approx 1, \tag{5.11}$$

and furthermore, that

$$\sup_{B_Q^{***}} u \lesssim \varepsilon^{-5}\ell(Q) \tag{5.12}$$

and

$$\sup_{X,Y\in \widetilde{U}_Q^i} \ \sup_{Z_1\in B_Y,\, Z_2\in B_X} |\nabla u(Z_1)-\nabla u(Z_2)| \le 2\varepsilon^{2M}. \tag{5.13}$$

Then Q satisfies the ε-local WHSA, provided that M is large enough, depending only on dimension and on the implicit constants in the stated hypotheses.

Assuming these results momentarily, we can complete the proofs of Theorem 1.1 and Theorem 1.12 as follows. First we see that we can apply Lemma 5.10 to the cubes in Case 2. Indeed, let Q be a cube such that $Q\in\mathbb{D}_{\mathcal{F},Q_0}$, $\ell(Q)\le\varepsilon^{10}\ell(Q_0)$, and (5.5) holds. Hence (5.11) follows by virtue of (5.7), while (5.12) holds by Lemma 3.14 applied with $B=2B_Q^{***}$ (or Lemma 3.46, with $B(y,s)=2B_Q^{***}$), and (4.23). Moreover, (5.13) follows trivially from (5.5). Thus, the hypotheses of Lemma 5.10 are all verified and hence Q satisfies the ε-local WHSA condition. In particular, the cubes $Q\in\mathbb{D}_{\mathcal{F},Q_0}$, which belong to the bad collection $\mathcal{B}$ of cubes in $\mathbb{D}(E)$ for which the ε-local WHSA condition fails, must be as in Case 0 or Case 1. By (5.6) and (5.9) these cubes satisfy the packing estimate

$$\sum_{Q\in\mathcal{B}\cap\mathbb{D}_{\mathcal{F},Q_0}} \sigma(Q)\le C_\varepsilon\,\sigma(Q_0). \tag{5.14}$$

For each $Q_0\in\mathbb{D}(E)$, there is a family $\mathcal{F}\subset\mathbb{D}_{Q_0}$ for which (5.14), and also the “ampleness” condition (4.13), hold uniformly. We may therefore invoke a well known lemma of John–Nirenberg type to deduce that (2.20) holds for all $\varepsilon\in(0,\varepsilon_0)$, and therefore to conclude that E satisfies the WHSA condition, Definition 2.19. Hence E is UR by Proposition 1.17.

The rest of the section is devoted to the proof of Lemmas 5.8 and 5.10. We shall first prove Lemma 5.8 in the relatively simpler linear case $p=2$ (see Section 5A). The proof of Lemma 5.8 in the general case $1<p<\infty$ is a bit more delicate and given in Section 5B. Lemma 5.10 is proved in Section 5C. Finally, the proof of Corollary 1.5 is given in Section 5D.

Before passing to the subsections we first introduce some additional notation to be used in the sequel. We augment $\widetilde{U}_Q^i$ as follows. Set

$$\mathcal{W}_Q^{i,*} := \Big\{ I\in\mathcal{W} : I^* \text{ meets } B_Y \text{ for some } Y\in\Big(\bigcup_{X\in\widetilde{U}_Q^i} B_X\Big)\Big\} \tag{5.15}$$

(and define $\mathcal{W}_Q^{j,*}$ analogously for all other $\widetilde{U}_Q^j$), and set

$$U_Q^{i,*} := \bigcup_{I\in\mathcal{W}_Q^{i,*}} I^{**}, \qquad U_Q^* := \bigcup_j U_Q^{j,*}, \tag{5.16}$$

where $I^{**}=(1+2\tau)I$ is a suitably fattened Whitney cube, with τ fixed as above. By construction,

$$\widetilde{U}_Q^i \subset \bigcup_{X\in\widetilde{U}_Q^i} B_X \subset \bigcup_{Y\in\bigcup_{X\in\widetilde{U}_Q^i} B_X} B_Y \subset U_Q^{i,*},$$

and for all $Y \in U_Q^{i,*}$, we have that $\delta(Y) \approx \ell(Q)$ (depending of course on ε). Moreover, also by construction, there is a Harnack path connecting any pair of points in $U_Q^{i,*}$ (depending again on ε), and furthermore, for every $I \in \mathcal{W}_Q^{i,*}$ (or for that matter for every $I \in \mathcal{W}_Q^{j,*}$, $j \neq i$),

$$\varepsilon^s \ell(Q) \lesssim \ell(I) \lesssim \varepsilon^{-3}\ell(Q), \qquad \operatorname{dist}(I, Q) \lesssim \varepsilon^{-4}\ell(Q),$$

where $0 < s = s(M, \alpha)$. Thus, by Harnack's inequality and (5.7),

$$C^{-1}\delta(Y) \le u(Y) \le C\delta(Y), \quad \forall Y \in U_Q^{i,*}, \tag{5.17}$$

with $C = C(K_0, \varepsilon, M)$. Moreover, for future reference, we note that the upper bound for u holds in all of U_Q^*, i.e.,

$$u(Y) \le C\delta(Y), \quad \forall Y \in U_Q^*, \tag{5.18}$$

by (3.12) or (3.45) and (4.14), where again $C = C(K_0, \varepsilon, M)$.

5A. *Proof of Lemma 5.8 in the linear case* ($p = 2$). Here we complete the proof of estimate (5.9) in the relatively simpler linear case $p = 2$. To start the proof of (5.9), we fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ so that Case 1 holds. We see that if we choose Z as in (5.4), and use the mean value property of harmonic functions, then

$$\varepsilon^{2M} \le C_\varepsilon(\ell(Q))^{-(n+1)} \iint_{B_Z \cup B_{Y_Q}} |\nabla u(Y) - \vec{\beta}|\, dY,$$

where $\vec{\beta}$ is a constant vector at our disposal. By Poincaré's inequality (see, e.g., [Hofmann and Martell 2014, Section 4] in this context), we obtain that

$$\sigma(Q) \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 \delta(Y)\, dY \lesssim \iint_{U_Q^{i,*}} |\nabla^2 u(Y)|^2 u(Y)\, dY,$$

where the implicit constants depend on ε, and in the last step we have used (5.17). Consequently,

$$\sum_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \text{Case 1 holds}}} \sigma(Q) \lesssim \sum_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \ell(Q) \le \varepsilon^{10}\ell(Q_0)}} \iint_{U_Q^*} |\nabla^2 u(Y)|^2 u(Y)\, dY \lesssim \iint_{\Omega^*_{\mathcal{F},Q_0}} |\nabla^2 u(Y)|^2 u(Y)\, dY, \tag{5.19}$$

where

$$\Omega^*_{\mathcal{F},Q_0} := \operatorname{int}\Bigg(\bigcup_{\substack{Q \in \mathbb{D}_{\mathcal{F},Q_0} \\ \ell(Q) \le \varepsilon^{10}\ell(Q_0)}} U_Q^* \Bigg), \tag{5.20}$$

and where we have used that the enlarged Whitney regions U_Q^* have bounded overlaps.

Take an arbitrary $N > 1/\varepsilon$ (eventually $N \to \infty$), and augment $\mathcal{F}$ by adding to it all subcubes $Q \subset Q_0$ with $\ell(Q) \le 2^{-N}\ell(Q_0)$. Let $\mathcal{F}_N \subset \mathbb{D}_{Q_0}$ denote the collection of maximal cubes of this augmented family. Thus, $Q \in \mathbb{D}_{\mathcal{F}_N,Q_0}$ if and only if $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ and $\ell(Q) > 2^{-N}\ell(Q_0)$. Clearly, $\mathbb{D}_{\mathcal{F}_N,Q_0} \subset \mathbb{D}_{\mathcal{F}_{N'},Q_0}$ if $N \le N'$, and therefore $\Omega^*_{\mathcal{F}_N,Q_0} \subset \Omega^*_{\mathcal{F}_{N'},Q_0}$ (where $\Omega^*_{\mathcal{F}_N,Q_0}$ is defined as in (5.20) with $\mathcal{F}_N$ replacing $\mathcal{F}$).

By monotone convergence and (5.19), we have that

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 1 holds}}} \sigma(Q) \lesssim \limsup_{N\to\infty} \iint_{\Omega^*_{\mathcal{F}_N,Q_0}} |\nabla^2 u(Y)|^2 u(Y)\,dY. \tag{5.21}$$

It therefore suffices to establish bounds for the latter integral that are uniform in N, with N large.

Let us then fix $N > 1/\varepsilon$. Since $\Omega^*_{\mathcal{F}_N,Q_0}$ is a finite union of fattened Whitney boxes, we may now integrate by parts, using the identity $2|\nabla\partial_k u|^2 = \operatorname{div}\nabla(\partial_k u)^2$ for harmonic functions, to obtain that

$$\iint_{\Omega^*_{\mathcal{F}_N,Q_0}} |\nabla^2 u(Y)|^2 u(Y)\,dY \lesssim \int_{\partial\Omega^*_{\mathcal{F}_N,Q_0}} (|\nabla^2 u||\nabla u|u + |\nabla u|^3)\,dH^n \le C_\varepsilon H^n(\partial\Omega^*_{\mathcal{F}_N,Q_0}), \tag{5.22}$$

where in the second inequality we have used the standard estimates

$$\delta(Y)|\nabla^2 u(Y)|,\ |\nabla u(Y)| \lesssim \frac{u(Y)}{\delta(Y)},$$

along with (5.18). We observe that $\Omega^*_{\mathcal{F}_N,Q_0}$ is a sawtooth domain in the sense of [Hofmann et al. 2016], or to be more precise, it is a union of a bounded number, depending on ε, of such sawtooths, one for each maximal subcube of Q_0 with length on the order of $\varepsilon^{10}\ell(Q_0)$. By [Hofmann et al. 2016, Appendix A] each of the previous sawtooth domains is ADR uniformly in N. Hence, its union is upper ADR uniformly in N with constant depending on the number of sawtooth domains in the union, which ultimately depends on ε. Therefore,

$$H^n(\partial\Omega^*_{\mathcal{F}_N,Q_0}) \le C_\varepsilon(\operatorname{diam}(\partial\Omega^*_{\mathcal{F}_N,Q_0}))^n \le C_\varepsilon\sigma(Q_0).$$

Combining the latter estimate with (5.21) and (5.22), we obtain (5.9), as desired, in the case $p = 2$.

5B. *Proof of Lemma 5.8 in the general case* ($1 < p < \infty$). Here we prove (5.9) for general p, $1 < p < \infty$, by proceeding along the lines of the proof of Lemma 2.5 in [Lewis and Vogel 2006]. We fix $Q \in \mathbb{D}_{\mathcal{F},Q_0}$ so that Case 1, and hence (5.4), holds. Let us recall that we have verified estimates (5.7), (5.17), and (5.18) for all p, $1 < p < \infty$.

Recall that if $X \in \widetilde{U}^i_Q$, then by definition X can be connected to some $\tilde{Y} \in U^i_Q$, and then to $Y_Q \in U^i_Q$, by a chain of at most $C\varepsilon^{-1}$ balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$. Note that using the triangle inequality and the definition of $\widetilde{U}^i_Q$, we may suppose that $Y_{k+1} \in B(Y_k, 3\delta(Y_k)/4) \subset B_{Y_k}$; otherwise we increase the chain by introducing some intermediate points and the new chain will have essentially the same length. Fix now Q, a cube in Case 1, and by (5.4) we can pick $X \in \widetilde{U}^i_Q$ so that

$$\sup_{Y\in B_X} |\nabla u(Y) - \nabla u(Y_Q)| > \varepsilon^{2M}.$$

As observed previously, we can form a Harnack chain connecting X and Y_Q so that $Y_1 = Y_Q$ and $Y_l = X$ and $l \le C\varepsilon^{-1}$. Then the previous expression can be written as

$$\sup_{Y\in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_1)| > \varepsilon^{2M}. \tag{5.23}$$

Obviously we may assume that

$$\sup_{Y \in B_{Y_j}} |\nabla u(Y) - \nabla u(Y_1)| \le \varepsilon^{2M} \tag{5.24}$$

whenever $1 < j \le l-1$, and $l > 1$, since otherwise we shorten the chain (and work with the first Y_j for which (5.23) holds). This and the fact that $Y_{j+1} \in B_{Y_j}$ for every $1 \le j \le l-1$ imply that

$$|\nabla u(Y_j)| \ge |\nabla u(Y_1)| - \varepsilon^{2M}, \quad \text{for } 1 \le j \le l. \tag{5.25}$$

Furthermore, using the triangle inequality,

$$\varepsilon^{2M} \le \sup_{Y \in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_l)| + \sum_{j=1}^{l-1} |\nabla u(Y_{j+1}) - \nabla u(Y_j)|. \tag{5.26}$$

Hence, using this and the fact that $l \lesssim \varepsilon^{-1}$ we have that either

$$\begin{aligned} &\text{(i)} \quad \sup_{Y \in B_{Y_l}} |\nabla u(Y) - \nabla u(Y_l)| \ge \varepsilon^{2M+2}, \quad \text{or} \\ &\text{(ii)} \quad |\nabla u(Y_{j+1}) - \nabla u(Y_j)| \ge \varepsilon^{2M+2}, \quad \text{for some } 1 \le j \le l-1. \end{aligned} \tag{5.27}$$

By (5.18) and (3.42) we have

$$|\nabla u(Y)| \le C_\varepsilon, \quad \forall Y \in U_Q^*. \tag{5.28}$$

In scenario (i) of (5.27) we take Y, a point where the sup is attained. This choice, (5.28), and the first inequality in (3.38) imply that $|Y - Y_l| \approx_\varepsilon \ell(Q)$. We then construct $\Gamma_0(Q)$ a (possibly rotated) rectangle as follows. The base and the top are two n-dimensional cubes of side length $c_\varepsilon \ell(Q)$, with c_ε chosen sufficiently small, centered respectively at the points Y and Y_l, and lying in the two parallel hyperplanes passing through the points Y and Y_l and perpendicular to the vector joining these two points. Note that for this rectangle, all side lengths are of the order of $\ell(Q)$ with implicit constants possibly depending on ε. In scenario (ii) of (5.27) we do the same construction with Y_{j+1} and Y_j in place of Y and Y_l and define $\Gamma_0(Q)$ which verifies the same properties. Note that in either case, (5.28) and the first inequality in (3.38) give the property that

$$|\nabla u(Y) - \nabla u(W)| \ge \varepsilon^{2M+4} \tag{5.29}$$

whenever W and Y are in the base and top of the parallelepiped, respectively. By construction, at least the top, which we denote by $t(Q)$, is centered on Y_j, for some $1 \le j \le l$. We observe that by (5.25) and (5.7), since $Y_1 := Y_Q$, and since ε is very small, we have for each Y_j, $1 \le j \le l$,

$$|\nabla u(Y_j)| \ge a, \tag{5.30}$$

for some uniform constant a independent of ε. Therefore, by (3.38), we also have

$$|\nabla u(Y)| \ge \frac{a}{2}, \quad \forall Y \in t(Q), \tag{5.31}$$

provided that we take c_ε small enough, since $\operatorname{diam}(t(Q)) \approx c_\varepsilon \ell(Q)$. Moving downward, that is, from top to base, through $\Gamma_0(Q)$ along slices parallel to $t(Q)$, we stop the first time that we reach a slice $b(Q)$

which contains a point Z with $|\nabla u(Z)| \le a/4$. If there is such a slice, we form a new rectangle $\Gamma(Q)$ with base $b(Q)$ and top $t(Q)$; otherwise, we set $\Gamma(Q) := \Gamma_0(Q)$, and let $b(Q)$ denote the base in this case as well. In either case, $\operatorname{dist}(b(Q), t(Q)) \approx \ell(Q)$, with implicit constants possibly depending on ε, by (3.38) and (5.31). Note that by construction and the continuity of ∇u,

$$|\nabla u(Y)| \ge \frac{a}{4}, \quad \forall Y \in \Gamma(Q), \tag{5.32}$$

and that $|\Gamma(Q)| \approx \ell(Q)^{n+1}$, again with implicit constants which may depend on ε. Furthermore, if $\Gamma(Q) = \Gamma_0(Q)$, then (5.29) holds for all $W \in b(Q)$ and $Y \in t(Q)$. Otherwise, if $\Gamma(Q)$ is strictly contained in $\Gamma_0(Q)$, then, since $\operatorname{diam}(b(Q)) \approx c_\varepsilon \ell(Q)$ with c_ε small, and since by construction $b(Q)$ contains a point Z with $|\nabla u(Z)| = a/4$, it follows that $|\nabla u(W)| \le 3a/8$ for all $W \in b(Q)$, by (3.38). Hence, in either situation, since $a/8 \gg \varepsilon^{2M+4}$, we have

$$|\nabla u(Y) - \nabla u(W)| \ge \varepsilon^{2M+4}, \quad \forall W \in b(Q),\ Y \in t(Q). \tag{5.33}$$

We let $\gamma = a/8$ and set

$$F_\gamma(|\nabla u|) := \max(|\nabla u|^2 - \gamma^2, 0).$$

Then by (5.32) we see that

$$F_\gamma(|\nabla u|) \ge \frac{a^2}{64}, \quad \forall Y \in \Gamma(Q). \tag{5.34}$$

Furthermore, by (5.33), the fundamental theorem of calculus, (5.17), (5.32), and (5.34), we have

$$\ell(Q)^n \lesssim \iint_{\Gamma(Q)} u|\nabla^2 u|^2\, dX \lesssim \iint_{\Gamma(Q)} u F_\gamma(|\nabla u|)\,|\nabla u|^{p-2}|\nabla^2 u|^2\, dY,$$

where the implicit constants depend on ε. In particular, since $\Gamma(Q) \subset U_Q^{i,*} \subset U_Q^*$, by ADR we obtain

$$\sigma(Q) \lesssim \iint_{U_Q^*} u F_\gamma(|\nabla u|)\,|\nabla u|^{p-2}|\nabla^2 u|^2\, dY,$$

where the implicit constants still depend on ε, and this estimate holds for all cubes $Q \in \mathbb{D}_{\mathcal{F},Q_0}$, so that Case 1 holds. Hence,

$$\sum_{\substack{Q\in\mathbb{D}_{\mathcal{F},Q_0}\\ \text{Case 1 holds}}} \sigma(Q) \lesssim \iint_{\Omega^*_{\mathcal{F},Q_0}} u F_\gamma(|\nabla u|)\,|\nabla u|^{p-2}|\nabla^2 u|^2\, dY, \tag{5.35}$$

where $\Omega^*_{\mathcal{F},Q_0}$ was defined in (5.20) and where we have used that the enlarged Whitney regions U_Q^* have bounded overlaps. To prove (5.9) in the general case $1 < p < \infty$, it therefore suffices to establish the local square function bound

$$\iint_{\Omega^*_{\mathcal{F},Q_0}} u F_\gamma(|\nabla u|)\,|\nabla u|^{p-2}|\nabla^2 u|^2\, dY \lesssim \sigma(Q_0), \tag{5.36}$$

where, as we recall, u is a nonnegative p-harmonic function in the open set $\Omega_0 := \Omega \cap B(x_{Q_0}, Cr_{Q_0})$, vanishing on $\Delta(x_{Q_0}, Cr_{Q_0})$.

To start the proof of (5.36), for each $Q \in \mathbb{D}(E)$, we define a further fattening of U_Q^* as follows. Set

$$U_Q^{i,**} := \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{***}, \qquad U_Q^{**} := \bigcup_i U_Q^{i,**},$$

$$U_Q^{i,***} := \bigcup_{I \in \mathcal{W}_Q^{i,*}} I^{****}, \qquad U_Q^{***} := \bigcup_i U_Q^{i,***},$$

where $I^{***} = (1+3\tau)I$ and $I^{****} = (1+4\tau)I$ are fattened Whitney regions, for some fixed small τ as above; see (5.15)–(5.16). Notice that $I^{**} \subset I^{***} \subset I^{****}$. We observe that the fattened Whitney regions U_Q^{***} have bounded overlaps, say

$$\sum_{Q \in \mathbb{D}(E)} 1_{U_Q^{***}}(Y) \le M_0, \tag{5.37}$$

where $M_0 < \infty$ is a uniform constant depending on K_0, ε, τ, and n. Next, let $\{\eta_Q\}_Q$ be a partition of unity adapted to U_Q^{**}. That is,

(1) $\sum_Q \eta_Q(Y) \equiv 1$ whenever $Y \in \Omega$,

(2) $\operatorname{supp} \eta_Q \subset U_Q^{**}$, and

(3) $\eta_Q \in C_0^\infty(\mathbb{R}^{n+1})$ with $0 \le \eta_Q \le 1$, $\eta_Q \ge c$ on U_Q^*, and $|\nabla \eta_Q| \le C\ell(Q)^{-1}$.

Set

$$\mathbb{D}_{\mathcal{F},Q_0,\varepsilon} := \{Q \in \mathbb{D}_{\mathcal{F},Q_0} : \ell(Q) \le \varepsilon^{10}\ell(Q_0)\},$$

and recall from (5.20) that

$$\Omega_{\mathcal{F},Q_0}^* := \operatorname{int}\Bigg(\bigcup_{Q \in \mathbb{D}_{\mathcal{F},Q_0,\varepsilon}} U_Q^*\Bigg).$$

Given a large number $N \gg \varepsilon^{-10}$, set

$$\Lambda = \Lambda(N) = \{Q \in \mathbb{D}(E) : U_Q^{**} \cap \Omega_{\mathcal{F},Q_0}^* \neq \varnothing \text{ and } \ell(Q) \ge N^{-1}\ell(Q_0)\}.$$

Eventually, we shall let $N \to \infty$. Let

$$I_1(N) := \sum_{Q \in \Lambda(N)} \iint u F_\gamma(|\nabla u|) \Bigg(\sum_{i,j=1}^{n+1} u_{y_i y_j}^2\Bigg) \eta_Q \, dY$$

and note, by positivity of u and the properties of η_Q, that we then have

$$\iint_{\Omega_{\mathcal{F},Q_0}^*} u F_\gamma(|\nabla u|) \, |\nabla^2 u|^2 \, dY \lesssim \lim_{N \to \infty} I_1(N).$$

We now fix N. We intend to perform integration by parts and in this argument, we exploit that $|\nabla u|^2$ is a subsolution to a certain linear PDE defined based on u. To describe this in detail, let $Q \in \Lambda(N)$ be such that $F_\gamma(|\nabla u(Y)|) \neq 0$ for some $Y \in U_Q^{**}$. Then $|\nabla u(Y)| \ge \gamma$ and there exists $C = C(\gamma) \ge 1$ such that

$$C^{-1} \le |\nabla u(X)| \lesssim 1 \quad \text{whenever } X \in B(Y, \delta(Y)/C), \tag{5.38}$$

and where the upper bound follows from (5.18) and the lower bound uses also (3.38). Let $\zeta = \nabla u \cdot \xi$, for some $\xi \in \mathbb{R}^{n+1}$. Then ζ satisfies, at $X \in B(Y, \delta(Y)/C)$, the partial differential equation

$$L\zeta = \nabla \cdot \big[(p-2)|\nabla u|^{p-4}(\nabla u \cdot \nabla \zeta)\nabla u + |\nabla u|^{p-2}\nabla \zeta\big] = 0, \tag{5.39}$$

as is seen by a straightforward calculation from differentiating the p-Laplace partial differential equation for u with respect to ξ. Note that (5.39) can be written in the form

$$L\zeta = \sum_{i,j=1}^{n+1} \frac{\partial}{\partial y_i}[b_{ij}(\,\cdot\,)\zeta_{y_j}(\,\cdot\,)] = 0, \tag{5.40}$$

where

$$b_{ij}(Y) = |\nabla u|^{p-4}[(p-2)u_{y_i}u_{y_j} + \delta_{ij}|\nabla u|^2](Y), \quad 1 \le i, j \le n+1, \tag{5.41}$$

and δ_{ij} is the Kronecker δ. Clearly we also have

$$Lu(Y) = (p-1)\nabla \cdot [|\nabla u|^{p-2}\nabla u](Y) = 0. \tag{5.42}$$

In particular, u and $(\nabla u \cdot \xi)$ for each $\xi \in \mathbb{R}^{n+1}$ all satisfy the divergence form partial differential equation (5.40).

It is easy to see that $(b_{ij})_{ij}$ satisfies the following degenerate ellipticity condition: for every $\xi \in \mathbb{R}^{n+1}$ one has

$$\begin{aligned}\sum_{i,j=1}^{n+1} b_{ij}\xi_i\xi_j &= (p-2)\,|\nabla u|^{p-4}\sum_{i,j=1}^{n+1} u_i u_j \xi_i \xi_j + |\nabla u|^{p-2}\sum_{i,j=1}^{n+1}\delta_{ij}\xi_i\xi_j \\ &= (p-2)\,|\nabla u|^{p-4}(\nabla u \cdot \xi)^2 + |\nabla u|^{p-2}\,|\xi|^2 \ge \min\{1, p-1\}\,|\nabla u|^{p-2}\,|\xi|^2,\end{aligned} \tag{5.43}$$

where the last inequality is immediate when $p \ge 2$ and uses the Cauchy–Schwarz inequality when $1 < p < 2$. Hence, $|\nabla u|^2$ is a subsolution to the PDE defined in (5.40), (5.41), as seen from the calculation

$$L(|\nabla u|^2) = 2\sum_{i,j,k=1}^{n+1} b_{ij}u_{y_i y_k}u_{y_j y_k} \gtrsim |\nabla u|^{p-2}\bigg(\sum_{i,j=1}^{n+1} u^2_{y_i y_j}\bigg). \tag{5.44}$$

Now, using (5.44) and the fact that (5.38) holds for every Y such that $F_\gamma(|\nabla u(Y)|) \neq 0$, we see that $I_1(N) \lesssim J_1(N)$, where

$$J_1(N) := \sum_{Q \in \Lambda(N)} \iint u F_\gamma(|\nabla u|) L(|\nabla u|^2)\eta_Q \, dY.$$

Hence it suffices to establish bounds for the integral $J_1 := J_1(N)$ that are uniform in N, with N large. In the following we let $v = F_\gamma(|\nabla u|)$ and note that $\nabla v = \nabla(|\nabla u|^2)$ whenever $v > 0$. Using this and integration by parts we see that

$$J_1 = -J_2 - J_3 - J_4,$$

where

$$J_2 = \sum_{Q\in\Lambda(N)} \iint v \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} v_{y_j} \eta_Q \, dY,$$
$$J_3 = \sum_{Q\in\Lambda(N)} \iint u \sum_{i,j=1}^{n+1} b_{ij} v_{y_i} v_{y_j} \eta_Q \, dY,$$
$$J_4 = \sum_{Q\in\Lambda(N)} \iint uv \sum_{i,j=1}^{n+1} b_{ij} v_{y_j} (\eta_Q)_{y_i} \, dY.$$

We estimate J_4 first. Set $\Lambda_1 = \Lambda_{11} \cup \Lambda_{12}$, where

$$\Lambda_{11} := \{Q \in \Lambda : U_Q^{**} \text{ meets } \Omega \setminus \Omega_{\mathcal{F},Q_0}\},$$

and

$$\Lambda_{12} := \{Q \in \Lambda : U_Q^{**} \text{ meets } U_{Q'}^{**} \text{ such that } \ell(Q') < N^{-1}\ell(Q_0)\}.$$

From the definition of η_Q, we obtain

$$|J_4| \lesssim \sum_{Q\in\Lambda_{11}} \iint uv \sum_{i,j=1}^{n+1} |u_{ij}||u_i||(\eta_Q)_j| \, dY + \sum_{Q\in\Lambda_{11}} \iint uv \sum_{i,j=1}^{n+1} |u_{ij}||u_i||(\eta_Q)_j| \, dY =: J_{51} + J_{52}.$$

Notice that, equivalently, Λ_{11} is the subcollection of $Q \in \Lambda_1$ such that U_Q^{**} meets $\partial\Omega^*_{\mathcal{F},Q_0}$. We start with J_{51}. Note that by (3.38), (5.18), and Harnack's inequality,

$$\delta(Y)|\nabla u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q) \tag{5.45}$$

whenever $Y \in U_Q^{***}$. Furthermore, if $v \neq 0$ for some $Y \in U_Q^{***}$, then using (5.38) and (3.40), we also have

$$(\delta(Y))^2|\nabla^2 u(Y)| \lesssim u(Y) \lesssim \delta(Y) \approx \ell(Q). \tag{5.46}$$

In particular, $u|\nabla\eta_Q| \lesssim 1$ by construction of η_Q, $|\nabla u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$, and $\delta(Y)|\nabla^2 u(Y)| \lesssim 1$ whenever $Y \in U_Q^{***}$ and $v \neq 0$. Thus,

$$J_{51} \lesssim \sum_{Q\in\Lambda_{11}} \ell(Q)^n \lesssim \sum_{Q\in\Lambda_{11}} H^n(U_Q^{***} \cap \partial\Omega^*_{\mathcal{F},Q_0}) \lesssim \sum_{Q\in\Lambda_{11}} H^n(\partial\Omega^*_{\mathcal{F},Q_0}) \lesssim \sigma(Q_0),$$

where we have used that $\partial\Omega^*_{\mathcal{F},Q_0}$ is ADR (see [Hofmann et al. 2016]), and the bounded overlap property (5.37). To estimate J_{52}, observe that for each $Q \in \Lambda_{12}$, we have $\ell(Q) \approx N^{-1}\ell(Q_0)$ by properties of Whitney regions. Hence, by a slightly simpler version of the argument used for J_{51}, we obtain

$$J_{52} \lesssim \sum_{Q\in\Lambda_{12}} \sigma(Q) \lesssim \sigma(Q_0).$$

Therefore, $|J_4| \lesssim J_{51} + J_{52} \lesssim \sigma(Q_0)$.

To handle J_2 we use the fact that u is a solution to (5.40). Indeed, by integration by parts, using the identity $2vv_{y_j} = (v^2)_{y_j}$ we see that

$$2J_2 = \sum_{Q\in\Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} (v^2)_{y_j} \eta_Q \, dY = -\sum_{Q\in\Lambda(N)} \iint \sum_{i,j=1}^{n+1} b_{ij} u_{y_i} v^2 (\eta_Q)_{y_j} \, dY,$$

and by the same argument as in the estimate of J_4 we obtain $|J_2| \lesssim \sigma(Q_0)$.

To conclude, we collect the estimates for J_2 and J_4, and use the fact that J_3 is nonnegative by (5.43) to obtain $J_1(N) \lesssim \sigma(Q_0)$, with constants independent of N. The proof of (5.9) in the general case $1 < p < \infty$ is then complete.

5C. *Proof of Lemma 5.10.* To prove Lemma 5.10, we follow the corresponding argument in [Lewis and Vogel 2007] closely, but with some modifications due to the fact that in contrast to the situation in that paper, our solution u need not be Lipschitz up to the boundary, and our harmonic/p-harmonic measures need not be doubling. It is the latter obstacle that has forced us to introduce the WHSA condition, rather than to work with the weak exterior convexity condition used by Lewis and Vogel. Lemma 5.10 is essentially a distillation of the main argument of the corresponding part of [Lewis and Vogel 2007], but with the doubling hypothesis removed.

In the remainder of this section, for convenience we use the notational convention that implicit and generic constants are allowed to depend upon K_0, but not on ε or M. Dependence on the latter is stated explicitly. We first prove the following lemma. Recall that the balls B_Y and $\widetilde{B}_Y$ are defined in (5.3).

Lemma 5.47. *Let $Y \in U_Q^i$, $X \in \widetilde{U}_Q^i$. Suppose first that $w \in \partial\widetilde{B}_Y \cap E$, and let W be the radial projection of w onto ∂B_Y. Then*

$$u(W) \lesssim \varepsilon^{2M-5}\delta(Y). \tag{5.48}$$

If $w \in \partial\widetilde{B}_X \cap E$, and W now is the radial projection of w onto ∂B_X, then

$$u(W) \lesssim \varepsilon^{2M-5}\ell(Q). \tag{5.49}$$

Proof. Since $K_0^{-1}\ell(Q) \lesssim \delta(Y) \lesssim K_0\ell(Q)$ for $Y \in U_Q^i$, it is enough to prove (5.49). To prove (5.49), we first note that

$$|W-w| = \varepsilon^{2M/\alpha}\delta(X) \lesssim \varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q),$$

by definition of B_X, $\widetilde{B}_X$ and the fact that by construction of $\widetilde{U}_Q^i$,

$$\varepsilon^3\ell(Q) \lesssim \delta(X) \lesssim \varepsilon^{-3}\ell(Q), \quad \forall X \in \widetilde{U}_Q^i. \tag{5.50}$$

In addition, again by construction of $\widetilde{U}_Q^i$,

$$\operatorname{diam}(\widetilde{U}_Q^i) \lesssim \varepsilon^{-4}\ell(Q). \tag{5.51}$$

Consequently, $W \in \frac{1}{2}B_Q^{***} = B\big(x_Q, \frac{1}{2}\varepsilon^{-5}\ell(Q)\big)$, so by Lemma 3.35 and (5.12),

$$u(W) \lesssim \left(\frac{\varepsilon^{2M/\alpha}\varepsilon^{-3}\ell(Q)}{\varepsilon^{-5}\ell(Q)}\right)^{\alpha} \frac{1}{|B_Q^{***}|} \iint_{B_Q^{***}} u \lesssim \varepsilon^{2M+2\alpha-5}\ell(Q) \le \varepsilon^{2M-5}\ell(Q). \qquad \square$$

Claim 5.52. *Let $Y \in U_Q^i$. For all $W \in B_Y$,*

$$|u(W) - u(Y) - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M} \delta(Y). \tag{5.53}$$

Proof of Claim 5.52. Let $W \in B_Y$. Then for some $\widetilde{W} \in B_Y$,

$$u(W) - u(Y) = \nabla u(\widetilde{W}) \cdot (W - Y).$$

We may then invoke (5.13), with $X = Y$, $Z_1 = \widetilde{W}$, and $Z_2 = Y$, to obtain (5.53). □

Claim 5.54. *Let $Y \in U_Q^i$. Suppose that $w \in \partial \widetilde{B}_Y \cap E$. Then*

$$|u(Y) - \nabla u(Y) \cdot (Y - w)| = |u(w) - u(Y) - \nabla u(Y) \cdot (w - Y)| \lesssim \varepsilon^{2M-5} \delta(Y). \tag{5.55}$$

Proof of Claim 5.54. Given $w \in \partial \widetilde{B}_Y \cap E$, let W be the radial projection of w onto ∂B_Y, so that $|W - w| = \varepsilon^{2M/\alpha} \delta(Y)$. Since $u(w) = 0$, by (5.48) we have

$$|u(W) - u(w)| = u(W) \lesssim \varepsilon^{2M-5} \delta(Y).$$

Since (5.53) holds for W, we obtain (5.55) by (5.11) and (5.13). □

To simplify notation, we now set $Y := Y_Q$, the point in U_Q^i satisfying (5.11). By (5.11) and (5.13), for $\varepsilon < \frac{1}{2}$, and M chosen large enough, we have that

$$|\nabla u(Z)| \approx 1, \quad \forall Z \in \widetilde{U}_Q^i. \tag{5.56}$$

By translation and rotation, we assume that $0 \in \partial \widetilde{B}_Y \cap E$ and that $Y = \delta(Y) e_{n+1}$, where as usual $e_{n+1} := (0, \dots, 0, 1)$.

Claim 5.57. *We claim that*

$$\big|\nabla u(Y) \cdot e_{n+1} - |\nabla u(Y)|\big| \lesssim \varepsilon^{2M-5}. \tag{5.58}$$

Proof of Claim 5.57. We apply (5.55), with $w = 0$, to obtain

$$|u(Y) - \nabla u(Y) \cdot Y| \lesssim \varepsilon^{2M-5} \delta(Y).$$

Combining the latter bound with (5.53), we find that

$$|u(W) - \nabla u(Y) \cdot W| = |u(W) - \nabla u(Y) \cdot Y - \nabla u(Y) \cdot (W - Y)| \lesssim \varepsilon^{2M-5} \delta(Y), \quad \forall W \in B_Y. \tag{5.59}$$

Fix $W \in \partial B_Y$ so that $\nabla u(Y) \cdot \dfrac{W - Y}{|W - Y|} = -|\nabla u(Y)|$. Since $|W - Y| = (1 - \varepsilon^{2M/\alpha}) \delta(Y)$, and since $u \geq 0$, we have

$$\begin{aligned}
0 \leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} &\leq |\nabla u(Y)| - \nabla u(Y) \cdot e_{n+1} + \frac{u(W)}{\delta(Y)} \\
&\leq \frac{1}{\delta(Y)} \left(-\nabla u(Y) \cdot \frac{(W - Y)}{1 - \varepsilon^{2M/\alpha}} - \nabla u(Y) \cdot Y + u(W) \right) \\
&\lesssim (\varepsilon^{2M-5} + \varepsilon^{2M/\alpha}) \approx \varepsilon^{2M-5},
\end{aligned} \tag{5.60}$$

by (5.59) and (5.11). □

Claim 5.61. *Suppose that $M > 5$. Then*

$$\big||\nabla u(Y)|e_{n+1} - \nabla u(Y)\big| \lesssim \varepsilon^{M-3}. \tag{5.62}$$

Proof of Claim 5.61. By Claim 5.57,

$$\big||\nabla u(Y)|e_{n+1} - (\nabla u(Y)\cdot e_{n+1})e_{n+1}\big| \lesssim \varepsilon^{2M-5}.$$

Therefore, it is enough to consider $\nabla_{\|} u := \nabla u - (\nabla u(Y)\cdot e_{n+1})e_{n+1}$. Observe that

$$\begin{aligned}|\nabla_{\|} u(Y)|^2 &= |\nabla u(Y)|^2 - (\nabla u(Y)\cdot e_{n+1})^2\\ &= (|\nabla u(Y)| - \nabla u(Y)\cdot e_{n+1})(|\nabla u(Y)| + \nabla u(Y)\cdot e_{n+1}) \lesssim \varepsilon^{2M-5},\end{aligned}$$

by (5.58) and (5.11). □

Now for $Y = \delta(Y)e_{n+1} \in U_Q^i$ fixed as above, we consider another point $X \in \widetilde{U}_Q^i$. By definition of $\widetilde{U}_Q^i$, there is a polygonal path in $\widetilde{U}_Q^i$, joining Y to X, with vertices

$$Y_0 := Y, Y_1, Y_2, \dots, Y_N := X, \quad N \lesssim \varepsilon^{-4},$$

such that $Y_{k+1} \in B_{Y_k} \cap B(Y_k, \ell(Q))$, $0 \le k \le N-1$, and such that the distance between consecutive vertices is at most $C\ell(Q)$. Indeed, by definition of $\widetilde{U}_Q^i$, we may connect Y to X by a polygonal path connecting the centers of at most ε^{-1} balls, such that the distance between consecutive vertices is between $\varepsilon^3\ell(Q)/2$ and $\varepsilon^{-3}\ell(Q)/2$. If any such distance is greater than $\ell(Q)$, we take at most $C\varepsilon^{-3}$ intermediate vertices with distances on the order of $\ell(Q)$. The total length of the path is thus on the order of $N\ell(Q)$ with $N \lesssim \varepsilon^{-4}$. Furthermore, by (5.13) and (5.62),

$$\begin{aligned}\big|\nabla u(W) - |\nabla u(Y)|e_{n+1}\big| &\le |\nabla u(W) - \nabla u(Y)| + \big|\nabla u(Y) - |\nabla u(Y)|e_{n+1}\big|\\ &\lesssim \varepsilon^{2M} + \varepsilon^{M-3} \lesssim \varepsilon^{M-3}, \quad \forall W \in B_Z,\ \forall Z \in \widetilde{U}_Q^i.\end{aligned} \tag{5.63}$$

Claim 5.64. *Assume $M > 7$. Then for each $k = 1, 2, \dots, N$,*

$$\big|u(Y_k) - |\nabla u(Y)|Y_k\cdot e_{n+1}\big| \lesssim k\varepsilon^{M-3}\ell(Q). \tag{5.65}$$

Moreover,

$$\big|u(W) - |\nabla u(Y)|W_{n+1}\big| \lesssim \varepsilon^{M-7}\ell(Q), \quad \forall W \in B_X,\ \forall X \in \widetilde{U}_Q^i. \tag{5.66}$$

Proof of Claim 5.64. By (5.59) and (5.62), we have

$$\begin{aligned}\big|u(W) - |\nabla u(Y)|W_{n+1}\big| &\lesssim |u(W) - \nabla u(Y)\cdot W| + \big|(\nabla u(Y) - |\nabla u(Y)|e_{n+1})\cdot W\big|\\ &\lesssim \varepsilon^{2M-5}\delta(Y) + \varepsilon^{M-3}|W| \lesssim \varepsilon^{M-3}\ell(Q), \quad \forall W \in B_Y,\end{aligned} \tag{5.67}$$

since $\delta(Z) \approx \ell(Q)$, for all $Z \in U_Q^i$ (so in particular, for $Z = Y$), and since $|W| \le 2\delta(Y) \lesssim \ell(Q)$, for all $W \in B_Y$. Thus, (5.65) holds with $k = 1$, since $Y_1 \in B_Y$, by construction. Now suppose that (5.65) holds for all $1 \le i \le k$, with $k \le N$. Let $W \in B_{Y_k}$, so that W may be joined to Y_k by a line segment of length less than $\delta(Y_k) \lesssim \varepsilon^{-3}\ell(Q)$ (the latter bound holds by (5.50)). We note also that if $k \le N-1$, and if $W = Y_{k+1}$, then this line segment has length at most $\ell(Q)$, by construction. Then

$$\begin{aligned}\big|u(W) - |\nabla u(Y)|W_{n+1}\big| &\le \big|u(W) - u(Y_k) + |\nabla u(Y)|(Y_k - W)\cdot e_{n+1}\big| + \big|u(Y_k) - |\nabla u(Y)|Y_k \cdot e_{n+1}\big| \\ &= \big|(W - Y_k)\cdot \nabla u(W_1) + |\nabla u(Y)|(Y_k - W)\cdot e_{n+1}\big| + O(k\varepsilon^{M-3}\ell(Q)),\end{aligned}$$

where W_1 is an appropriate point on the line segment joining W and Y_k, and where we have used that Y_k satisfies (5.65). By (5.63), applied to W_1, we find in turn that

$$\big|u(W) - |\nabla u(Y)|W_{n+1}\big| \lesssim \varepsilon^{M-3}|W - Y_k| + k\varepsilon^{M-3}\ell(Q), \tag{5.68}$$

which, by our previous observations, is bounded by $C(k+1)\varepsilon^{M-3}\ell(Q)$ if $W = Y_{k+1}$, or by $(\varepsilon^{M-6} + k\varepsilon^{M-3})\ell(Q)$ in general. In the former case, we find that (5.65) holds for all $k = 1, 2, \dots, N$, and in the latter case, taking $k = N \lesssim \varepsilon^{-4}$, we obtain (5.66). □

Claim 5.69. *Let $X \in \widetilde{U}^i_Q$, and let $w \in E \cap \partial\widetilde{B}_X$. Then*

$$|\nabla u(Y)||w_{n+1}| \lesssim \varepsilon^{M/2}\ell(Q). \tag{5.70}$$

Proof of Claim 5.69. Let W be the radial projection of w onto ∂B_X, so that

$$|W - w| = \varepsilon^{2M/\alpha}\delta(X) \lesssim \varepsilon^{(2M/\alpha)-3}\ell(Q), \tag{5.71}$$

by (5.50). We write

$$|\nabla u(Y)||w_{n+1}| \le |\nabla u(Y)||W - w| + \big|u(W) - |\nabla u(Y)|W_{n+1}\big| + u(W) =: I + II + u(W).$$

Note that $I \lesssim \varepsilon^{(2M/\alpha)-3}\ell(Q)$ by (5.71) and (5.11) (recall that $Y = Y_Q$), and that $II \lesssim \varepsilon^{M-7}\ell(Q)$ by (5.66). Furthermore, $u(W) \lesssim \varepsilon^{2M-5}\ell(Q)$, by (5.49). For M chosen large enough, we obtain (5.70). □

We note that since we have fixed $Y = Y_Q$, it then follows from (5.70) and (5.11) that

$$|w_{n+1}| \lesssim \varepsilon^{M/2}\ell(Q), \quad \forall w \in E \cap \partial\widetilde{B}_X,\ \forall X \in \widetilde{U}^i_Q. \tag{5.72}$$

Recall that x_Q denotes the "center" of Q (see (2.7)–(2.8)). Set

$$\mathcal{O} := B(x_Q, 2\varepsilon^{-2}\ell(Q)) \cap \{W : W_{n+1} > \varepsilon^2\ell(Q)\}. \tag{5.73}$$

Claim 5.74. *For every point $X \in O$, we have $X \approx_{\varepsilon,Q} Y$ (see Definition 2.26). Thus, in particular, $O \subset \widetilde{U}^i_Q$.*

Proof of Claim 5.74. Let $X \in O$. We need to show that X may be connected to Y by a chain of at most ε^{-1} balls of the form $B(Y_k, \delta(Y_k)/2)$, with $\varepsilon^3\ell(Q) \le \delta(Y_k) \le \varepsilon^{-3}\ell(Q)$ (for convenience, we shall refer to such balls as "admissible"). We first observe that if $X = te_{n+1}$, with $\varepsilon^3\ell(Q) \le t \le \varepsilon^{-3}\ell(Q)$, then by an iteration argument using (5.72) (with M chosen large enough), we may join X to Y by at most $C\log(1/\varepsilon)$ admissible balls. The point $(2\varepsilon)^{-3}\ell(Q)e_{n+1}$ may then be joined to any point of the form $(X', (2\varepsilon)^{-3}\ell(Q))$ by a chain of at most C admissible balls, whenever $X' \in \mathbb{R}^n$ with $|X'| \le \varepsilon^{-3}\ell(Q)$. In turn, the latter point may then be joined to $(X', \varepsilon^3\ell(Q))$ by at most $C\log(1/\varepsilon)$ admissible balls. □

We note that Claim 5.74 implies that

$$E \cap O = \varnothing. \tag{5.75}$$

Indeed, $O \subset \widetilde{U}_Q^i \subset \Omega$. Let P_0 denote the hyperplane

$$P_0 := \{Z : Z_{n+1} = 0\}.$$

Claim 5.76. *If $Z \in P_0$, with $|Z - x_Q| \le \frac{3}{2}\varepsilon^{-2}\ell(Q)$, then*

$$\delta(Z) = \operatorname{dist}(Z, E) \le 16\varepsilon^2 \ell(Q). \tag{5.77}$$

Proof of Claim 5.76. Observe that $B(Z, 2\varepsilon^2\ell(Q))$ meets O. Then by Claim 5.74, there is a point $X \in \widetilde{U}_Q^i \cap B(Z, 2\varepsilon^2\ell(Q))$. Suppose that (5.77) is false, which in particular implies that $\delta(X) \ge 14\varepsilon^2\ell(Q)$. Then $B(Z, 4\varepsilon^2\ell(Q)) \subset B_X$, so by (5.66), we have

$$\big|u(W) - |\nabla u(Y)| W_{n+1}\big| \le C\varepsilon^{M-7}\ell(Q), \quad \forall\, W \in B(Z, 4\varepsilon^2\ell(Q)). \tag{5.78}$$

In particular, since $Z_{n+1} = 0$, we may choose W such that $W_{n+1} = -\varepsilon^2\ell(Q)$, to obtain that

$$|\nabla u(Y)|\varepsilon^2\ell(Q) \le C\varepsilon^{M-7}\ell(Q),$$

since $u \ge 0$. But for $\varepsilon < \frac{1}{2}$, and M large enough, this is a contradiction, by (5.11) (recall that we have fixed $Y = Y_Q$). □

It now follows by Definition 2.17 that Q satisfies the ε-local WHSA condition, with

$$P = P(Q) := \{Z : Z_{n+1} = \varepsilon^2\ell(Q)\}, \qquad H = H(Q) := \{Z : Z_{n+1} > \varepsilon^2\ell(Q)\}.$$

This concludes the proof of Lemma 5.10.

5D. *Proof of Corollary 1.5.* Now Corollary 1.5 follows almost immediately from Theorem 1.1. Let $B = B(x, r)$ and $\Delta = B \cap \partial\Omega$, with $x \in \partial\Omega$ and $0 < r < \operatorname{diam}(\partial\Omega)$. Let c be the constant in Lemma 3.1. By hypothesis, there is a point $X_\Delta \in B \cap \Omega$ which is a corkscrew point relative to Δ, that is, there is a uniform constant $c_0 > 0$ such that $\delta(X_\Delta) \ge c_0 r$. Thus, to apply Theorem 1.1, it remains only to verify hypothesis ($\star$). For a sufficiently large constant C_1, set $\Delta^{\text{fat}} = \Delta(x, C_1 r)$. Cover Δ^{fat} by a collection of surface balls $\{\Delta_i\}_{i=1}^N$ with $\Delta_i = B_i \cap \partial\Omega$ and $B_i := B(x_i, c_0 r/4)$, where $x_i \in \Delta^{\text{fat}}$ and where N is uniformly bounded, depending only on n, c_0, C_1, and ADR. By construction, $X_\Delta \in \Omega \setminus 4B_i$, so by hypothesis, $\omega^{X_\Delta} \in$ weak-$A_\infty(2\Delta_i)$. Hence, $\omega^{X_\Delta} \ll \sigma$ in $2\Delta_i$, and (1.6) holds with $Y = X_\Delta$, and with $\Delta' = \Delta_i$. Consequently, $\omega^{X_\Delta} \ll \sigma$ in Δ^{fat}, and if we write $k^{X_\Delta} = d\omega^{X_\Delta}/d\sigma$, we obtain

$$\begin{aligned}\int_{\Delta^{\text{fat}}} k^{X_\Delta}(z)^q\, d\sigma(z) &\le \sum_{i=1}^N \int_{\Delta_i} k^{X_\Delta}(z)^q\, d\sigma(z) \lesssim \sum_{i=1}^N \sigma(\Delta_i)\bigg(\fint_{2\Delta_i} k^{X_\Delta}(z)\, d\sigma(z)\bigg)^q \\ &\lesssim \sum_{i=1}^N \sigma(2\Delta_i)^{1-q}\omega^{X_\Delta}(2\Delta_i) \lesssim \sigma(\Delta^{\text{fat}})^{1-q},\end{aligned}$$

where in the last estimate we have used the ADR property, the uniform boundedness of N, and the fact that $\omega^{X_\Delta}(2\Delta_i) \le 1$. By Theorem 1.1, it then follows that $\partial\Omega$ is UR as desired. □

6. Proof of Proposition 1.17

Here we prove Proposition 1.17. We first observe that if E is UR then it satisfies the so-called "bilateral weak geometric lemma" (BWGL); see [David and Semmes 1991, Theorem I.2.4, p. 32]. In turn, in [David and Semmes 1991, Section II.2.1, p. 97], one can find a dyadic formulation of the BWGL as follows. Given ε small enough and $k > 1$ large to be chosen, $\mathbb{D}(E)$ can be split in two collections, one of "bad cubes" and another of "good cubes", so that the "bad cubes" satisfy a packing condition and each "good cube" Q verifies the following: there is a hyperplane $P = P(Q)$ such that $\operatorname{dist}(Z, E) \le \varepsilon\ell(Q)$ for every $Z \in P \cap B(x_Q, k\ell(Q))$, and $\operatorname{dist}(Z, P) \le \varepsilon\ell(Q)$ for every $Z \in B(x_Q, k\ell(Q)) \cap E$. In turn, this implies that $B(x_Q, k\ell(Q)) \cap E$ is sandwiched between two planes parallel to P at distance $\varepsilon\ell(Q)$. Hence, at that scale, we have a half-space (indeed we have two) free of E, and clearly the 2ε-local WHSA holds provided K is taken of the order of ε^{-2} or larger. Further details are left to the interested reader. Thus we obtain the easy implication UR $\Longrightarrow$ WHSA.

The main part of the proof is to establish the opposite implication. To this end, we assume that E satisfies the WHSA property and show that E is UR. Given a positive $\varepsilon < \varepsilon_0 \ll K_0^{-6}$, we let $\mathcal{B}_0$ denote the collection of bad cubes for which ε-local WHSA fails. By Definition 2.19, $\mathcal{B}_0$ satisfies the Carleson packing condition (2.20). We now introduce a variant of the packing measure for $\mathcal{B}_0$. We recall that $B_Q^* = B(x_Q, K_0^2\ell(Q))$, and given $Q \in \mathbb{D}(E)$, we set

$$\mathbb{D}_\varepsilon(Q) := \{Q' \in \mathbb{D}(E) : \varepsilon^{3/2}\ell(Q) \le \ell(Q') \le \ell(Q),\ Q' \text{ meets } B_Q^*\}. \tag{6.1}$$

Set

$$\alpha_Q := \begin{cases} \sigma(Q) & \text{if } \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \ne \varnothing, \\ 0 & \text{otherwise,} \end{cases} \tag{6.2}$$

and define

$$\mathfrak{m}(\mathbb{D}') := \sum_{Q \in \mathbb{D}'} \alpha_Q, \quad \mathbb{D}' \subset \mathbb{D}(E). \tag{6.3}$$

Then $\mathfrak{m}$ is a discrete Carleson measure, with

$$\mathfrak{m}(\mathbb{D}_{Q_0}) = \sum_{Q \subset Q_0} \alpha_Q \le C_\varepsilon \sigma(Q_0), \quad Q_0 \in \mathbb{D}(E). \tag{6.4}$$

Indeed, note that for any Q', the cardinality of $\{Q : Q' \in \mathbb{D}_\varepsilon(Q)\}$ is uniformly bounded, depending on n, ε, and ADR, and that $\sigma(Q) \le C_\varepsilon \sigma(Q')$ if $Q' \in \mathbb{D}_\varepsilon(Q)$. Then given any $Q_0 \in \mathbb{D}(E)$,

$$\begin{aligned} \mathfrak{m}(\mathbb{D}_{Q_0}) &= \sum_{Q \subset Q_0 : \mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) \ne \varnothing} \sigma(Q) \le \sum_{Q' \in \mathcal{B}_0} \sum_{Q \subset Q_0 : Q' \in \mathbb{D}_\varepsilon(Q)} \sigma(Q) \\ &\le C_\varepsilon \sum_{Q' \in \mathcal{B}_0 :\, Q' \subset 2B_{Q_0}^*} \sigma(Q') \le \mathcal{C}_\varepsilon \sigma(Q_0), \end{aligned}$$

by (2.20) and ADR.

To prove Proposition 1.17, we are required to show that the collection $\mathcal{B}$ of bad cubes for which the $\sqrt{\varepsilon}$-local BAUP condition fails satisfies a packing condition. That is, we establish the discrete Carleson measure estimate

$$\widetilde{\mathfrak{m}}(\mathbb{D}_{Q_0}) = \sum_{Q\subset Q_0:Q\in\mathcal{B}} \sigma(Q) \le C_\varepsilon \sigma(Q_0), \quad Q_0 \in \mathbb{D}(E). \tag{6.5}$$

To this end, by (6.4), it suffices to show that if $Q \in \mathcal{B}$, then $\alpha_Q \neq 0$ (and thus $\alpha_Q = \sigma(Q)$, by definition). In fact, we prove the contrapositive statement.

Claim 6.6. *Suppose that $\alpha_Q = 0$. Then the $\sqrt{\varepsilon}$-local BAUP condition holds for Q.*

Proof of Claim 6.6. We first note that since $\alpha_Q = 0$, then by definition of α_Q,

$$\mathcal{B}_0 \cap \mathbb{D}_\varepsilon(Q) = \varnothing. \tag{6.7}$$

Thus, the ε-local WHSA condition (Definition 2.17) holds for every $Q' \in \mathbb{D}_\varepsilon(Q)$ (in particular, for Q itself). By rotation and translation, we may suppose that the hyperplane $P = P(Q)$ in Definition 2.17 is

$$P = \{Z \in \mathbb{R}^{n+1} : Z_{n+1} = 0\},$$

and that the half-space $H = H(Q)$ is the upper half-space $\mathbb{R}^{n+1}_+ = \{Z : Z_{n+1} > 0\}$. We recall that by Definition 2.17, P and H satisfy

$$\operatorname{dist}(Z, E) \le \varepsilon\ell(Q), \quad \forall Z \in P \cap B^{**}_Q(\varepsilon), \tag{6.8}$$

$$\operatorname{dist}(P, Q) \le K_0^{3/2}\ell(Q), \tag{6.9}$$

and

$$H \cap B^{**}_Q(\varepsilon) \cap E = \varnothing. \tag{6.10}$$

The proof now follows by a construction similar to that in [Lewis and Vogel 2007], used to establish the weak exterior convexity condition. By (6.10), there are two cases.

Case 1: $10Q \subset \{Z : -\sqrt{\varepsilon}\ell(Q) \le Z_{n+1} \le 0\}$. In this case, the $\sqrt{\varepsilon}$-local BAUP condition holds trivially for Q, with $\mathcal{P} = \{P\}$.

Case 2: There is a point $x \in 10Q$ such that $x_{n+1} < -\sqrt{\varepsilon}\ell(Q)$. In this case, we choose $Q' \ni x$ with $\varepsilon^{3/4}\ell(Q) \le \ell(Q') < 2\varepsilon^{3/4}\ell(Q)$. Thus,

$$Q' \subset \left\{Z : Z_{n+1} \le -\tfrac{1}{2}\sqrt{\varepsilon}\ell(Q)\right\}. \tag{6.11}$$

Moreover, $Q' \in \mathbb{D}_\varepsilon(Q)$, so by (6.7), $Q' \notin \mathcal{B}_0$, i.e., Q' satisfies the ε-local WHSA. Let $P' = P(Q')$ and $H' = H(Q')$ denote the hyperplane and half-space corresponding to Q' in Definition 2.17, so that

$$\operatorname{dist}(Z, E) \le \varepsilon\ell(Q') \le 2\varepsilon^{7/4}\ell(Q), \quad \forall Z \in P' \cap B^{**}_{Q'}(\varepsilon), \tag{6.12}$$

$$\operatorname{dist}(P', Q') \le K_0^{3/2}\ell(Q') \approx K_0^{3/2}\varepsilon^{3/4}\ell(Q) \ll \varepsilon^{1/2}\ell(Q) \tag{6.13}$$

(where the last inequality holds since $\varepsilon \ll K_0^{-6}$), and

$$H' \cap B^{**}_{Q'}(\varepsilon) \cap E = \varnothing, \tag{6.14}$$

where we recall that $B^{**}_{Q'}(\varepsilon) := B(x_{Q'}, \varepsilon^{-2}\ell(Q'))$ (see (2.16)). We note that

$$B^*_Q \subset \widetilde{B}_Q(\varepsilon) := B(x_Q, \varepsilon^{-1}\ell(Q)) \subset B^{**}_{Q'}(\varepsilon) \cap B^{**}_Q(\varepsilon), \tag{6.15}$$

by construction, since $\varepsilon \ll K_0^{-6}$. Let ν' denote the unit normal vector to P', pointing into H'. Note that by (6.10), (6.12), and the definition of H,

$$P' \cap \widetilde{B}_Q(\varepsilon) \cap \{Z : Z_{n+1} > 2\varepsilon^{7/4}\ell(Q)\} = \varnothing. \tag{6.16}$$

Moreover, ν' points "downward", i.e., $\nu' \cdot e_{n+1} < 0$, as otherwise, $H' \cap \widetilde{B}_Q(\varepsilon)$ would meet E by (6.8), (6.11), and (6.13). More precisely, we have the following.

Claim 6.17. *The angle θ between ν' and $-e_{n+1}$ satisfies $0 \le \theta \approx \sin\theta \lesssim \varepsilon$.*

Indeed, since Q' meets $10Q$, (6.9) and (6.13) imply that $\operatorname{dist}(P, P') \lesssim K_0^{3/2}\ell(Q)$, and that the latter estimate is attained near Q. By (6.16) and a trigonometric argument, one then obtains Claim 6.17 (more precisely, one obtains $\theta \lesssim K_0^{3/2}\varepsilon$, but in this section, we continue to use the notational convention that implicit constants may depend upon K_0, but K_0 is fixed, and $\varepsilon \ll K_0^{-6}$). The interested reader could probably supply the remaining details of the argument that we have just sketched, but for the sake of completeness, we give the full proof at the end of this section.

We therefore take Claim 6.17 for granted, and proceed with the argument. We note first that every point in $(P \cup P') \cap B^*_Q$ is at a distance at most $\varepsilon\ell(Q)$ from E by (6.8), (6.12), and (6.15). To complete the proof of Claim 6.6, it therefore remains only to verify the following. As with the previous claim, we provide a condensed proof immediately, and present a more detailed argument at the end of the section.

Claim 6.18. *Every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P \cup P'$.*

Suppose not. We could then repeat the previous argument, to construct a cube Q'', a hyperplane P'', a unit vector ν'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'', with the same properties as Q', P', ν', and H'. In particular, we have the respective analogues of (6.13) and (6.14), namely

$$\operatorname{dist}(P'', Q'') \le K_0^{3/2}\ell(Q') \approx K_0^{3/2}\varepsilon^{3/4}\ell(Q) \ll \varepsilon^{1/2}\ell(Q) \tag{6.19}$$

and

$$H'' \cap B^{**}_{Q''}(\varepsilon) \cap E = \varnothing, \tag{6.20}$$

Also, we have the analogue of (6.11), with Q'', P' in place of Q', P. Thus

$$\operatorname{dist}(Q'', P') \ge \tfrac{1}{2}\sqrt{\varepsilon}\ell(Q) \quad \text{and} \quad Q'' \cap H' = \varnothing. \tag{6.21}$$

In addition, as in (6.15), we also have $B^*_Q \subset B^{**}_{Q''}(\varepsilon)$. On the other hand, the angle between ν' and ν'' is very small. Thus, combining (6.12), (6.19), and (6.21), we see that $H'' \cap B^*_Q$ captures points in E, which contradicts (6.20).

Claim 6.6 therefore holds (in fact, with a union of at most 2 planes), and thus we obtain the conclusion of Proposition 1.17. □

We now provide detailed proofs of Claims 6.17 and 6.18.

Proof of Claim 6.17. By (6.13) we can pick $x' \in Q'$, $y' \in P'$ such that $|y' - x'| \ll \varepsilon^{1/2}\ell(Q)$, and therefore $y' \in 11Q$. Also, from (6.9) and (6.10) we can find $\bar{x} \in Q$ such that $-K_0^{3/2}\ell(Q) < \bar{x}_{n+1} \leq 0$. This and (6.11) yield

$$-2K_0^{3/2}\ell(Q) < y'_{n+1} < -\tfrac{1}{4}\sqrt{\varepsilon}\ell(Q). \tag{6.22}$$

Let π be the orthogonal projection onto P. Let $Z \in P$ (i.e., $Z_{n+1} = 0$) be such that $|Z - \pi(y')| \leq K_0^{3/2}\ell(Q)$. Then $Z \in B(x_Q, 4K_0^{3/2}\ell(Q)) \subset B_Q^*$. Hence $Z \in P \cap B_Q^{**}(\varepsilon)$ and by (6.8), $\operatorname{dist}(Z, E) \leq \varepsilon\ell(Q)$. Then there exists $x_Z \in E$ with $|Z - x_Z| \leq \varepsilon\ell(Q)$, which in turn implies that $|(x_Z)_{n+1}| \leq \varepsilon\ell(Q)$. Note that $x_Z \in B(x_Q, 5K_0^{3/2}\ell(Q)) \subset B_Q^*$ and by (6.15), $x_Z \in E \cap B_Q^{**}(\varepsilon) \cap B_{Q'}^{**}(\varepsilon)$. This, (6.10), and (6.14) imply that $x_Z \notin H \cup H'$. Hence, $(x_Z)_{n+1} \leq 0$ and $(x_Z - y') \cdot \nu' \leq 0$, since $y' \in P'$ and ν' denote the unit normal vector to P' pointing into H'. Using (6.22) we observe that

$$\tfrac{1}{8}\sqrt{\varepsilon}\ell(Q) < -\varepsilon\ell(Q) + \tfrac{1}{4}\sqrt{\varepsilon}\ell(Q) < (x_Z - y')_{n+1} < 2K_0^{3/2}\ell(Q), \tag{6.23}$$

and that

$$\begin{aligned}(x_Z - y')_{n+1}\nu'_{n+1} &\leq -\pi(x_Z - y')\cdot\pi(\nu') \\ &\leq |x_Z - z| - \pi(Z - y')\cdot\pi(\nu') \leq \varepsilon\ell(Q) - \pi(Z - y')\cdot\pi(\nu').\end{aligned} \tag{6.24}$$

We prove that $\nu'_{n+1} < -\frac{1}{8} < 0$ by considering two cases.

Case 1: $|\pi(\nu')| \geq \frac{1}{2}$. We pick

$$Z_1 = \pi(y') + K_0^{3/2}\ell(Q)\frac{\pi(\nu')}{|\pi(\nu')|}.$$

By construction, $Z_1 \in P$ and $|Z_1 - \pi(y')| \leq K_0^{3/2}\ell(Q)$. Hence, we can use (6.24) with Z_1:

$$\begin{aligned}(x_{Z_1} - y')_{n+1}\,\nu'_{n+1} &\leq \varepsilon\ell(Q) - \pi(Z_1 - y')\cdot\pi(\nu') \\ &= \varepsilon\ell(Q) - K_0^{3/2}\ell(Q)\,|\pi(\nu')| \leq -\tfrac{1}{4}K_0^{3/2}\ell(Q).\end{aligned}$$

This together with (6.23) give that $\nu'_{n+1} < -\frac{1}{8} < 0$.

Case 2: $|\pi(\nu')| < \frac{1}{2}$. This case is much simpler. Note first that $|\nu'_{n+1}|^2 = 1 - |\pi(\nu')|^2 > \frac{3}{4}$, and thus either $\nu'_{n+1} < -\frac{1}{2}\sqrt{3}$ or $\nu'_{n+1} > \frac{1}{2}\sqrt{3}$. We see that the second scenario leads to a contradiction. Assume then that $\nu'_{n+1} > \frac{1}{2}\sqrt{3}$. We take $Z_2 = \pi(y') \in P$, which clearly satisfies $|Z_2 - \pi(y')| \leq K_0^{3/2}\ell(Q)$. Again (6.24) and (6.23) are applicable with Z_2:

$$\frac{1}{8}\sqrt{\varepsilon}\ell(Q)\frac{\sqrt{3}}{2} < (x_{Z_2} - y')_{n+1}\nu'_{n+1} \leq \varepsilon\ell(Q) \ll \sqrt{\varepsilon}\,\ell(Q),$$

and we get a contradiction. Hence necessarily $\nu'_{n+1} \leq -\frac{1}{2}\sqrt{3} < -\frac{1}{8} < 0$.

Having proved that $\nu'_{n+1} < -\frac{1}{8} < 0$, we estimate θ, the angle between ν' and $-e_{n+1}$. Note first $\cos\theta = -\nu'_{n+1} > \frac{1}{8}$. If $\cos\theta = 1$ (which occurs if $\nu' = -e_{n+1}$), then $\theta = \sin\theta = 0$ and the proof is complete. Assume then that $\cos\theta \neq 1$, in which case $\frac{1}{8} < -\nu'_{n+1} < 1$ and hence $|\pi(\nu')| \neq 0$. Pick

$$Z_3 = y' + \frac{\ell(Q)}{2\varepsilon}\hat{\nu}', \qquad \hat{\nu}' = \frac{e_{n+1} - \nu'_{n+1}\nu'}{|\pi(\nu')|}.$$

Then $\hat{\nu}'\cdot\nu'=0$ and hence $Z_3\in P'$ as $y'\in P'$. Also, $|\hat{\nu}'|=1$ and therefore $|Z_3-y'|=\ell(Q)/(2\varepsilon)$. This in turn gives that $Z_3\in\widetilde{B}_Q(\varepsilon)$. We have obtained that $Z_3\in P'\cap\widetilde{B}_Q(\varepsilon)$, and hence $(Z_3)_{n+1}\le 2\varepsilon^{7/4}\ell(Q)$ by (6.16). This and (6.23) applied to Z_3 easily give

$$\begin{aligned}4K_0^{3/2}\ell(Q)\ge 2\varepsilon^{7/4}\ell(Q)\ge (Z_3)_{n+1}&=y'_{n+1}+\frac{\ell(Q)}{2\varepsilon}\frac{1-(\nu'_{n+1})^2}{|\pi(\nu')|}\\&=y'_{n+1}+\frac{\ell(Q)}{2\varepsilon}|\pi(\nu')|\ge -2\,K_0^{3/2}\ell(Q)+\frac{\ell(Q)}{2\varepsilon}|\pi(\nu')|.\end{aligned}$$

This readily yields $|\sin\theta|=|\pi(\nu')|\le 8K_0^{3/2}\varepsilon$, and the proof is complete. □

Proof of Claim 6.18. We want to prove that every point in $10Q$ lies within $\sqrt{\varepsilon}\ell(Q)$ of a point in $P\cup P'$. We argue by contradiction and hence we assume that there exists $x'\in 10Q$ with $\operatorname{dist}(x',P\cup P')>\sqrt{\varepsilon}\ell(Q)$. In particular, $x'_{n+1}<-\sqrt{\varepsilon}\ell(Q)$, and as observed above, we may repeat the previous argument to construct a cube Q'', a hyperplane P'', a unit vector ν'' forming a small angle with $-e_{n+1}$, and a half-space H'' with boundary P'', with the same properties as Q', P', ν', and H', namely (6.19), (6.21), and (6.20). Also,

$$\sqrt{\varepsilon}\ell(Q)\le\operatorname{dist}(x',P')\le\operatorname{diam}(Q'')+\operatorname{dist}(Q'',P')\le\tfrac12\sqrt{\varepsilon}\,\ell(Q)+\operatorname{dist}(Q'',P'),$$

and, in addition, as in (6.15), we have $B_Q^*\subset B_{Q''}^{**}(\varepsilon)$.

By (6.19) there is $y''\in Q''$ and $z''\in P''$ such that $|y''-z''|\ll\varepsilon^{1/2}\ell(Q)$. By (6.20) $y''\notin H'$. Write π' to denote the orthogonal projection onto P' and note that (6.21) gives $\operatorname{dist}(y'',P')=|y''-\pi'(y'')|\ge\frac12\sqrt{\varepsilon}\,\ell(Q)$. Note also that

$$\begin{aligned}|y''-\pi'(y'')|&=\operatorname{dist}(y'',P')\\&\le|y''-x'|+|x'-x|+\operatorname{diam}(Q')+\operatorname{dist}(Q',P')\le 11\operatorname{diam}(Q)\end{aligned}$$

and that

$$|\pi'(y'')-x_Q|\le|\pi'(y'')-y''|+|y''-x'|+|x'-x_Q|<22\operatorname{diam}(Q)<K_0^2\ell(Q).$$

Hence $\pi'(y'')\in B_Q^*\subset\widetilde{B}_Q(\varepsilon)$, and since $\pi'(y'')\in P'$, (6.12) gives $\tilde{y}\in E$ with $|\pi'(y'')-\tilde{y}|\le 2\varepsilon^{7/4}\ell(Q)$. Then $\tilde{y}\in 23Q\subset B_Q^*\cap E$ and $|\tilde{y}-z''|<12\operatorname{diam}(Q)$. To complete our proof we just need to show that $\tilde{y}\in H''$, which contradicts (6.20).

Write ν'' to denote the unit normal vector to P'' pointing into H'', and let us momentarily assume that

$$|\nu'-\nu''|\le 16\sqrt{2}\,K_0^{2/3}\varepsilon. \tag{6.25}$$

Recalling that $y''\notin H'$, we then obtain that

$$\begin{aligned}\tfrac12\sqrt{\varepsilon}\,\ell(Q)&\le|y''-\pi'(y'')|=(\pi'(y'')-y'')\cdot\nu'\\&\le|\pi'(y'')-\tilde{y}|+|\tilde{y}-z''|\,|\nu'-\nu''|+(\tilde{y}-z'')\cdot\nu''+|z''-y''|\\&<\tfrac14\sqrt{\varepsilon}\,\ell(Q)+(\tilde{y}-z'')\cdot\nu''.\end{aligned}$$

This immediately gives that $(\tilde{y}-z'')\cdot\nu'' > \frac{1}{4}\sqrt{\varepsilon}\,\ell(Q) > 0$, and hence $\tilde{y}\in H''$ as desired. Thus, to complete the proof we have to prove (6.25). We first note that if $|\alpha| < \frac{\pi}{4}$, then

$$1-\cos\alpha = 1-\sqrt{1-\sin^2\alpha} \le \sin^2\alpha.$$

In particular, we can apply this to θ (resp. θ'), which is the angle between ν' (resp. ν'') and $-e_{n+1}$, and as we showed that $|\sin\theta|$, $|\sin\theta'| \le 8K_0^{3/2}\varepsilon$, we see that

$$\sqrt{1-\cos\theta}+\sqrt{1-\cos\theta'} \le 16K_0^{3/2}\varepsilon.$$

Using the trivial formula

$$|a-b|^2 = 2(1-a\dot{b}), \quad \forall a,b\in\mathbb{R}^{n+1},\ |a|=|b|=1,$$

we conclude that

$$\begin{aligned}|\nu'-\nu''| &\le |\nu'-(-e_{n+1})| + |(-e_{n+1})-\nu''|\\ &= \sqrt{2(1+\nu'\,e_{n+1})}+\sqrt{2(1+\nu''e_{n+1})}\\ &= \sqrt{2(1-\cos\theta)}+\sqrt{2(1-\cos\theta')} \le 16\sqrt{2}K_0^{3/2}\varepsilon.\end{aligned}$$

This proves (6.25), and hence the proof of Claim 6.18 is complete. □

References

[Adams and Hedberg 1999] D. R. Adams and L. I. Hedberg, *Function spaces and potential theory*, corrected 2nd printing, Grundlehren der Mathematischen Wissenschaften **314**, Springer, Berlin, 1999. MR Zbl

[Aikawa 2004] H. Aikawa, "Potential-theoretic characterizations of nonsmooth domains", *Bull. London Math. Soc.* **36**:4 (2004), 469–482. MR Zbl

[Akman et al. 2015] M. Akman, M. Badger, S. Hofmann, and J. M. Martell, "Rectifiability and elliptic measures on 1-sided NTA domains with Ahlfors–David regular boundaries", preprint, 2015. To appear in *Trans. Amer. Math. Soc.* arXiv

[Alt and Caffarelli 1981] H. W. Alt and L. A. Caffarelli, "Existence and regularity for a minimum problem with free boundary", *J. Reine Angew. Math.* **325** (1981), 105–144. MR Zbl

[Auscher et al. 2002a] P. Auscher, S. Hofmann, M. Lacey, A. McIntosh, and P. Tchamitchian, "The solution of the Kato square root problem for second order elliptic operators on $\mathbb{R}^n$", *Ann. of Math.* (2) **156**:2 (2002), 633–654. MR Zbl

[Auscher et al. 2002b] P. Auscher, S. Hofmann, C. Muscalu, T. Tao, and C. Thiele, "Carleson measures, trees, extrapolation, and $T(b)$ theorems", *Publ. Mat.* **46**:2 (2002), 257–325. MR Zbl

[Azzam et al. 2014] J. Azzam, S. Hofmann, J. M. Martell, K. Nyström, and T. Toro, "A new characterization of chord-arc domains", preprint, 2014. To appear in *J. Eur. Math. Soc. (JEMS)*. arXiv

[Azzam et al. 2015] J. Azzam, M. Mourgoglou, and X. Tolsa, "Rectifiability of harmonic measure in domains with porous boundaries", preprint, 2015. arXiv

[Azzam et al. 2016] J. Azzam, S. Hofmann, J. M. Martell, S. Mayboroda, M. Mourgoglou, X. Tolsa, and A. Volberg, "Rectifiability of harmonic measure", *Geom. Funct. Anal.* **26**:3 (2016), 703–728. MR Zbl

[Bortz and Hofmann 2015] S. Bortz and S. Hofmann, "Harmonic measure and approximation of uniformly rectifiable sets", preprint, 2015. To appear in *Rev. Mat. Iberoamericana*. arXiv

[Bourgain 1987] J. Bourgain, "On the Hausdorff dimension of harmonic measure in higher dimension", *Invent. Math.* **87**:3 (1987), 477–483. MR Zbl

[Caffarelli et al. 1981] L. Caffarelli, E. Fabes, S. Mortola, and S. Salsa, "Boundary behavior of nonnegative solutions of elliptic operators in divergence form", *Indiana Univ. Math. J.* **30**:4 (1981), 621–640. MR Zbl

[Christ 1990] M. Christ, "A $T(b)$ theorem with remarks on analytic capacity and the Cauchy integral", *Colloq. Math.* **60/61**:2 (1990), 601–628. MR Zbl

[David and Semmes 1991] G. David and S. Semmes, *Singular integrals and rectifiable sets in* $\mathbf{R}^n$: *Beyond Lipschitz graphs*, Astérisque **193**, Société Mathématique de France, Paris, 1991. MR Zbl

[David and Semmes 1993] G. David and S. Semmes, *Analysis of and on uniformly rectifiable sets*, Mathematical Surveys and Monographs **38**, American Mathematical Society, Providence, RI, 1993. MR Zbl

[Eremenko and Lewis 1991] A. Eremenko and J. L. Lewis, "Uniform limits of certain A-harmonic functions with applications to quasiregular mappings", *Ann. Acad. Sci. Fenn. Ser. A I Math.* **16**:2 (1991), 361–375. MR Zbl

[Gilbarg and Trudinger 1983] D. Gilbarg and N. S. Trudinger, *Elliptic partial differential equations of second order*, 2nd ed., Grundlehren der Mathematischen Wissenschaften **224**, Springer, 1983. MR Zbl

[Heinonen et al. 2006] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, 2nd ed., Dover Publications, Mineola, NY, 2006. MR Zbl

[Hofmann 2006] S. Hofmann, "Local Tb theorems and applications in PDE", pp. 1375–1392 in *International Congress of Mathematicians*, vol. 2, edited by M. Sanz-Solé et al., European Mathematical Society, Zürich, 2006. MR Zbl

[Hofmann and Martell 2014] S. Hofmann and J. M. Martell, "Uniform rectifiability and harmonic measure, I: Uniform rectifiability implies Poisson kernels in L^p", *Ann. Sci. Éc. Norm. Supér.* (4) **47**:3 (2014), 577–654. MR Zbl

[Hofmann and Martell 2015] S. Hofmann and J. M. Martell, "Uniform Rectifiability and harmonic measure, IV: Ahlfors regularity plus Poisson kernels in L^p implies uniform rectifiability", preprint, 2015. arXiv

[Hofmann and McIntosh 2002] S. Hofmann and A. McIntosh, "The solution of the Kato problem in two dimensions", pp. 143–160 in *Proceedings of the 6th International Conference on Harmonic Analysis and Partial Differential Equations* (El Escorial, 2000), edited by P. Cifuentes et al., 2002. MR Zbl

[Hofmann et al. 2002] S. Hofmann, M. Lacey, and A. McIntosh, "The solution of the Kato problem for divergence form elliptic operators with Gaussian heat kernel bounds", *Ann. of Math.* (2) **156**:2 (2002), 623–631. MR Zbl

[Hofmann et al. 2014] S. Hofmann, J. M. Martell, and I. Uriarte-Tuero, "Uniform rectifiability and harmonic measure, II: Poisson kernels in L^p imply uniform rectifiability", *Duke Math. J.* **163**:8 (2014), 1601–1654. MR Zbl

[Hofmann et al. 2015] S. Hofmann, J. M. Martell, S. Mayboroda, X. Tolsa, and A. Volberg, "Absolute continuity between the surface measure and harmonic measure implies rectifiability", preprint, 2015. arXiv

[Hofmann et al. 2016] S. Hofmann, J. M. Martell, and S. Mayboroda, "Uniform rectifiability, Carleson measure estimates, and approximation of harmonic functions", *Duke Math. J.* **165**:12 (2016), 2331–2389. MR Zbl

[Hofmann et al. 2017] S. Hofmann, D. Mitrea, M. Mitrea, and A. Morris, *L^p-square function estimates on spaces of homogeneous type and on uniformly rectifiable sets*, Mem. Amer. Math. Soc. **1159**, American Mathematical Society, Providence, RI, 2017.

[Hofmann et al. $\geq$ 2017] S. Hofmann, J. M. Martell, and T. Toro, "General divergence form elliptic operators on domains with ADR boundaries, and on 1-sided NTA domains", work in progress.

[Jerison 1990] D. Jerison, "Regularity of the Poisson kernel and free boundary problems", *Colloq. Math.* **60/61**:2 (1990), 547–568. MR Zbl

[Jerison and Kenig 1982] D. S. Jerison and C. E. Kenig, "Boundary behavior of harmonic functions in nontangentially accessible domains", *Adv. in Math.* **46**:1 (1982), 80–147. MR Zbl

[Kenig 1994] C. E. Kenig, *Harmonic analysis techniques for second order elliptic boundary value problems*, CBMS Regional Conference Series in Mathematics **83**, American Mathematical Society, Providence, RI, 1994. MR Zbl

[Kenig and Toro 2003] C. E. Kenig and T. Toro, "Poisson kernel characterization of Reifenberg flat chord arc domains", *Ann. Sci. École Norm. Sup.* (4) **36**:3 (2003), 323–401. MR Zbl

[Kilpeläinen and Zhong 2003] T. Kilpeläinen and X. Zhong, "Growth of entire $\mathcal{A}$-subharmonic functions", *Ann. Acad. Sci. Fenn. Math.* **28**:1 (2003), 181–192. MR Zbl

[Lewis and Nyström 2012] J. L. Lewis and K. Nyström, "Regularity and free boundary regularity for the p-Laplace operator in Reifenberg flat and Ahlfors regular domains", *J. Amer. Math. Soc.* **25**:3 (2012), 827–862. MR Zbl

[Lewis and Vogel 2006] J. L. Lewis and A. L. Vogel, "Uniqueness in a free boundary problem", *Comm. Partial Differential Equations* **31**:10-12 (2006), 1591–1614. MR Zbl

[Lewis and Vogel 2007] J. L. Lewis and A. L. Vogel, "Symmetry theorems and uniform rectifiability", *Bound. Value Probl.* (2007), art. id. 30190. MR Zbl

[Mattila et al. 1996] P. Mattila, M. S. Melnikov, and J. Verdera, "The Cauchy integral, analytic capacity, and uniform rectifiability", *Ann. of Math.* (2) **144**:1 (1996), 127–136. MR Zbl

[Mourgoglou 2015] M. Mourgoglou, "Uniform domains with rectifiable boundaries and harmonic measure", preprint, 2015. arXiv

[Mourgoglou and Tolsa 2015] M. Mourgoglou and X. Tolsa, "Harmonic measure and Riesz transform in uniform and general domains", preprint, 2015. arXiv

[Nazarov et al. 2014a] F. Nazarov, X. Tolsa, and A. Volberg, "On the uniform rectifiability of AD-regular measures with bounded Riesz transform operator: the case of codimension 1", *Acta Math.* **213**:2 (2014), 237–321. MR Zbl

[Nazarov et al. 2014b] F. Nazarov, X. Tolsa, and A. Volberg, "The Riesz transform, rectifiability, and removability for Lipschitz harmonic functions", *Publ. Mat.* **58**:2 (2014), 517–532. MR Zbl

[Serrin 1964] J. Serrin, "Local behavior of solutions of quasi-linear equations", *Acta Math.* **111** (1964), 247–302. MR Zbl

[Tolksdorf 1984] P. Tolksdorf, "Regularity for a more general class of quasilinear elliptic equations", *J. Differential Equations* **51**:1 (1984), 126–150. MR Zbl

Received 12 Feb 2016. Accepted 12 Nov 2016.

STEVE HOFMANN: hofmanns@missouri.edu
Department of Mathematics, University of Missouri, Columbia, MO 65211, United States

PHI LE: ple101@syr.edu
Mathematics Department, Syracuse University, 215 Carnegie Building, Syracuse, NY 13244, United States

JOSÉ MARÍA MARTELL: chema.martell@icmat.es
Instituto de Ciencias Matemáticas CSIC-UAM-UC3M-UCM, Consejo Superior de Investigaciones Científicas, C/ Nicolás Cabrera, 13-15, E-28049 Madrid, Spain

KAJ NYSTRÖM: kaj.nystrom@math.uu.se
Department of Mathematics, Uppsala University, SE-751 06 Uppsala, Sweden

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2017 is US $265/year for the electronic version, and $470/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 10 No. 3 2017

2157-5045(2017)10:3;1-V