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CONFORMALLY EUCLIDEAN METRICS ON $\mathbb{R}^n$ WITH ARBITRARY TOTAL Q-CURVATURE

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We study the existence of solution to the problem

$$(-\Delta)^{n/2}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \qquad \kappa := \int_{\mathbb{R}^n} Qe^{nu}\,dx < \infty,$$

where $Q \geq 0$, $\kappa \in (0,\infty)$ and $n \geq 3$. Using ODE techniques, Martinazzi (for $n = 6$) and Huang and Ye (for $n = 4m+2$) proved the existence of a solution to the above problem with $Q \equiv \text{constant} > 0$ and for every $\kappa \in (0,\infty)$. We extend these results in every dimension $n \geq 5$, thus completely answering the problem opened by Martinazzi. Our approach also extends to the case in which Q is nonconstant, and under some decay assumptions on Q we can also treat the cases $n = 3$ and $n = 4$.

1. Introduction

For a function $Q \in C^0(\mathbb{R}^n)$ we consider the problem

$$(-\Delta)^{n/2}u = Qe^{nu} \quad \text{in } \mathbb{R}^n, \qquad \kappa := \int_{\mathbb{R}^n} Qe^{nu}\,dx < \infty, \tag{1}$$

where for n odd the nonlocal operator $(-\Delta)^{n/2}$ is defined on page 639.

Geometrically if u is a smooth solution of (1) then the conformal metric $g_u := e^{2u}|dx|^2$ (here $|dx|^2$ is the Euclidean metric on $\mathbb{R}^n$) has the Q-curvature Q, at least when $n \geq 2$. Moreover, the total Q-curvature of the metric g_u is κ.

Solutions to (1) have been classified in terms of their asymptotic behavior at infinity. More precisely we have the following:

Theorem A [Chen and Li 1991; Da Lio et al. 2015; Lin 1998; Martinazzi 2009a; Jin et al. 2015; Hyder 2015; Xu 2005]. *Let $n \geq 1$. Let u be a solution of*

$$(-\Delta)^{n/2}u = (n-1)!\,e^{nu} \quad \textit{in } \mathbb{R}^n, \qquad \kappa := (n-1)!\int_{\mathbb{R}^n} e^{nu}\,dx < \infty. \tag{2}$$

Then

$$u(x) = \frac{(n-1)!}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right)e^{nu(y)}\,dy + P(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + o(\log|x|) \quad \textit{as } |x| \to \infty, \tag{3}$$

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where $\gamma_n := \frac{1}{2}(n-1)!|S^n|$, $\Lambda_1 := 2\gamma_n$, $o(\log|x|)/\log|x| \to 0$ *as* $|x| \to \infty$, P *is a polynomial of degree at most* $n-1$ *and* P *is bounded from above. If* $n \in \{3, 4\}$ *then* $\kappa \in (0, \Lambda_1]$ *and* $\kappa = \Lambda_1$ *if and only if* u *is a spherical solution, that is,*

$$u(x) = u_{\lambda,x_0}(x) := \log \frac{2\lambda}{1+\lambda^2|x-x_0|^2} \tag{4}$$

for some $x_0 \in \mathbb{R}^n$ *and* $\lambda > 0$. *Moreover* u *is spherical if and only if* P *is constant* (*which is always the case when* $n \in \{1, 2\}$).

Chang and Chen [2001] showed the existence of nonspherical solutions to (2) in even dimension $n \geq 4$ for every $\kappa \in (0, \Lambda_1)$.

A partial converse to Theorem A has been proven in dimension 4 by Wei and Ye [2008] and extended by Hyder and Martinazzi [2015] for $n \geq 4$ even and Hyder [2016] for $n \geq 3$.

Theorem B [Wei and Ye 2008; Hyder and Martinazzi 2015; Hyder 2016]. *Let* $n \geq 3$. *Then for every* $\kappa \in (0, \Lambda_1)$ *and for every polynomial* P *with*

$$\deg(P) \leq n-1 \quad \textit{and} \quad P(x) \xrightarrow{|x|\to\infty} -\infty,$$

there exists a solution u *to* (2) *having the asymptotic behavior given by* (3).

Although the assumption $\kappa \in (0, \Lambda_1]$ is a necessary condition for the existence of a solution to (2) for $n = 3, 4$, it is possible to have a solution for $\kappa > \Lambda_1$ arbitrarily large in higher dimension, as shown by Martinazzi [2013] for $n = 6$. Huang and Ye [2015] extended Martinazzi's result in arbitrary even dimension n of the form $n = 4m+2$ for some $m \geq 1$, proving that for every $\kappa \in (0, \infty)$ there exists a solution to (2). The case $n = 4m$ remained open.

The ideas in [Martinazzi 2013; Huang and Ye 2015] are based upon ODE theory. One considers only radial solutions so that the equation in (2) becomes an ODE, and the result is obtained by choosing suitable initial conditions and letting one of the parameters go to $+\infty$ (or $-\infty$). However, this technique does not work if the dimension n is a multiple of 4, and things get even worse in odd dimension since $(-\Delta)^{n/2}$ is nonlocal and ODE techniques cannot be used.

In this paper we extend the works of [Martinazzi 2013; Huang and Ye 2015] and completely solve the cases left open; namely we prove that when $n \geq 5$, problem (2) has a solution for every $\kappa \in (0, \infty)$. In fact we do not need to assume that Q is constant, but only that it is radially symmetric with growth at infinity suitably controlled, or not even radially symmetric. Moreover, we are able to prescribe the asymptotic behavior of the solution u, as in (3), up to a polynomial of degree 4 which cannot be prescribed and in particular cannot be required to vanish when $\kappa \geq \Lambda_1$. This in turn, together with Theorem A, is consistent with the requirement $n \geq 5$, because only when $n \geq 5$ does the asymptotic expansion of u at infinity admit polynomials of degree 4.

We prove the following two theorems.

Theorem 1.1. *Let* $n \geq 5$ *be an integer. Let* P *be a polynomial on* $\mathbb{R}^n$ *with degree at most* $n-1$. *Let* $Q \in C^0(\mathbb{R}^n)$ *be such that* $Q(0) > 0$, $Q \geq 0$, Qe^{nP} *is radially symmetric and*

$$\sup_{x\in\mathbb{R}^n} Q(x)e^{nP(x)} < \infty.$$

Then for every $\kappa > 0$ there exists a solution u to (1) such that

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + P(x) + c_1|x|^2 - c_2|x|^4 + C + o(1) \quad \text{as } |x| \to \infty$$

for some $c_1, c_2 > 0$ and $C \in \mathbb{R}$. In fact, there exists a radially symmetric function v on $\mathbb{R}^n$ and a constant c_v such that

$$v(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) + o(1) \quad \text{as } |x| \to \infty,$$

and

$$u = P + v + c_v - |x|^4, \quad x \in \mathbb{R}^n.$$

Taking $Q = (n-1)!$ and $P = 0$ in Theorem 1.1 one has the following corollary.

Corollary 1.2. *Let $n \geq 5$ and $\kappa \in (0, \infty)$. Then there exists a radially symmetric solution u to (2) such that*

$$u(x) = -\frac{2\kappa}{\Lambda_1}\log|x| + c_1|x|^2 - c_2|x|^4 + C + o(1) \quad \text{as } |x| \to \infty$$

for some $c_1, c_2 > 0$ and $C \in \mathbb{R}$.

Notice the polynomial part of the solution u in Theorem 1.1 is not exactly the prescribed polynomial P (compare to [Wei and Ye 2008; Hyder and Martinazzi 2015; Hyder 2016]). In general, without perturbing the polynomial part, it is not possible to find a solution for $\kappa \geq \Lambda_1$. For example, if P is nonincreasing and nonconstant then there is no solution u to (2) with $\kappa \geq \Lambda_1$ such that u has the asymptotic behavior (3) (see Lemma 3.6 below). This justifies the term $c_1|x|^2$ in Theorem 1.1. Then the additional term $-c_2|x|^4$ is also necessary to avoid that $u(x) \geq \frac{1}{2}c_1|x|^2$ for x large, which would contrast with the condition $\kappa < \infty$, at least if Q does not decay fast enough at infinity. In the latter case, the term $-c_2|x|^4$ can be avoided, and one obtains an existence result also in dimensions 3 and 4.

Theorem 1.3. *Let $n \geq 3$. Let $Q \in C^0_{\mathrm{rad}}(\mathbb{R}^n)$ be such that $Q \geq 0$, $Q(0) > 0$ and*

$$\int_{\mathbb{R}^n} Q(x)e^{\lambda|x|^2}\,dx < \infty \quad \text{for every } \lambda > 0, \qquad \int_{B_1(x)} \frac{Q(y)}{|x-y|^{n-1}}\,dy \xrightarrow{|x|\to\infty} 0.$$

Then for every $\kappa > 0$ there exists a radially symmetric solution u to (1).

The decay assumption on Q in Theorem 1.3 is sharp in the sense that if $Qe^{\lambda|x|^2} \notin L^1(\mathbb{R}^n)$ for some $\lambda > 0$, then problem (1) might not have a solution for every $\kappa > 0$. For instance, if $Q = e^{-\lambda|x|^2}$ for some $\lambda > 0$, then (1) with $n = 3, 4$ and $\kappa > \Lambda_1$ has no solution (see Lemma 3.5 below).

The proof of Theorem 1.1 is based on the Schauder fixed point theorem, and the main difficulty is to show that the "approximate solutions" are precompact (see in particular Lemma 2.2). We will do that using blow up analysis (see for instance [Adimurthi et al. 2006; Martinazzi 2009b; Robert 2006]). In general, if $\kappa \geq \Lambda_1$ one can expect blow up, but we will construct our approximate solutions carefully in a way that this does not happen. For instance in [Wei and Ye 2008; Hyder and Martinazzi 2015] one looks for solutions of the form $u = P + v + c_v$, where v satisfies the integral equation

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{nP(y)}e^{n(v(y)+c_v)}\,dy,$$

and c_v is a constant such that

$$\int_{\mathbb{R}^n} Qe^{n(P+v+c_v)}\,dx = \kappa.$$

With such a choice we would not be able to rule out blow up. Instead, by looking for solutions of the form

$$u = P + v + P_v + c_v,$$

where a posteriori $P_v = -|x|^4$, v satisfies

$$v(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n} \log\left(\frac{1}{|x-y|}\right) Q(y)e^{n(P(y)+P_v(y)+v(y)+c_v)}\,dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|, \tag{5}$$

and c_v is again a normalization constant, one can prove that the integral equation (5) enjoys sufficient compactness, essentially due to the term $\frac{1}{2n}|x|^2|\Delta v(0)|$ on the right-hand side. Indeed a sequence of (approximate) solutions v_k blowing up (for simplicity) at the origin, up to rescaling, leads to a sequence (η_k) of functions satisfying, for every $R > 0$,

$$\int_{B_R} |\Delta \eta_k - c_k|\,dx \le CR^{n-2} + o(1)R^{n+2}, \quad o(1) \xrightarrow{k\to\infty} 0,\ c_k > 0,$$

and converging to η_∞, solving (for simplicity here we ignore some cases)

$$(-\Delta)^{n/2}\eta_\infty = e^{n\eta_\infty} \quad \text{in } \mathbb{R}^n, \qquad \int_{\mathbb{R}^n} e^{n\eta_\infty}\,dx < \infty,$$

and

$$\int_{B_R} |\Delta\eta_\infty - c_\infty|\,dx \le CR^{n-2}, \quad c_\infty \ge 0, \tag{6}$$

where $c_\infty = 0$ corresponds to $\Delta\eta_\infty(0) = 0$ (see Subcase 1.1 in Lemma 2.2 with $x_k = 0$).

The estimate on $\|\Delta\eta_\infty\|_{L^1(B_R)}$ in (6) shows that the polynomial part P_∞ of η_∞, as in (3), has degree at most 2, and hence $\Delta P_\infty \le 0$ as P_∞ is bounded from above. Therefore, $c_\infty = 0 = \Delta P_\infty$, P_∞ is constant, and in particular η_∞ is a spherical solution by Theorem A, that is, $\eta_\infty = u_{\lambda,x_0}$ for some $\lambda > 0$ and $x_0 \in \mathbb{R}^n$, where u_{λ,x_0} is given by (4). This leads to a contradiction as $\Delta\eta_\infty(0) = 0$ and $\Delta u_{\lambda,x_0} < 0$ in $\mathbb{R}^n$.

In this work we focus only on the case $Q \ge 0$ because the negative case is relatively well understood. For instance by a simple application of maximum principle, one can show that problem (1) has no solution with $Q \equiv \text{constant} < 0$, $n = 2$ and $\kappa > -\infty$, but when Q is nonconstant, solutions do exist, as shown by Chanillo and Kiessling [2000] under suitable assumptions. Martinazzi [2008] proved that in higher even dimension $n = 2m \ge 4$, problem (1) with $Q \equiv \text{constant} < 0$ has solutions for some κ, and it has been shown in [Hyder and Martinazzi 2015] that actually for every $\kappa \in (-\infty, 0)$ and Q a negative constant, (1) has a solution. The same result has been recently extended to odd dimension $n \ge 3$ in [Hyder 2016].

2. Proof of Theorem 1.1

We consider the space

$$X := \{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric}, \|v\|_X < \infty\},$$

where

$$\|v\|_X := \sup_{x\in\mathbb{R}^n}\biggl(\sum_{|\alpha|\le 3}(1+|x|)^{|\alpha|-4}|D^\alpha v(x)| + \sum_{3<|\alpha|\le n-1}|D^\alpha v(x)|\biggr).$$

For $v \in X$ we set

$$A_v := \max\left\{0, \sup_{|x|\ge 10}\frac{v(x)-v(0)}{|x|^4}\right\}, \qquad P_v(x) := -|x|^4 - A_v|x|^4.$$

Then

$$v(x) + P_v(x) \le v(0) - |x|^4 \quad \text{for } |x| \ge 10.$$

Let c_v be the constant determined by

$$\int_{\mathbb{R}^n} K e^{n(v+c_v)}\,dx = \kappa, \qquad K := Qe^{nP}e^{nP_v},$$

where the functions Q and P satisfy the hypotheses in Theorem 1.1. Since $Q > 0$ in a neighborhood of the origin, by a dilation argument we can assume that $Q > 0$ on B_3. More precisely, if u is a solution to (1) then for any $\lambda > 0$, we know $u_\lambda(x) := u(\lambda x) + \log\lambda$ is also a solution to (1) with Q replaced by Q_λ, where $Q_\lambda(x) := Q(\lambda x)$. Now for a suitable choice of $\lambda > 0$, one has $Q_\lambda > 0$ on B_3.

The function $u = P + P_v + v + c_v$ satisfies

$$(-\Delta)^{n/2}u = Qe^{nu}, \qquad \kappa = \int_{\mathbb{R}^n} Qe^{nu}\,dx$$

if and only if v satisfies

$$(-\Delta)^{n/2}v = Ke^{n(v+c_v)}.$$

For odd integer n, the operator $(-\Delta)^{n/2}$ is defined as follows:

Definition. Let n be an odd integer. Let $f \in \mathcal{S}'(\mathbb{R}^n)$. We say that u is a solution of

$$(-\Delta)^{n/2}u = f \quad \text{in } \mathbb{R}^n$$

if $u \in W^{n-1,1}_{\mathrm{loc}}(\mathbb{R}^n)$ and $\Delta^{(n-1)/2}u \in L_{1/2}(\mathbb{R}^n)$ and for every test function $\varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}(-\Delta)^{(n-1)/2}u(-\Delta)^{1/2}\varphi\,dx = \langle f, \varphi\rangle.$$

Here $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space and the space $L_s(\mathbb{R}^n)$ is defined by

$$L_s(\mathbb{R}^n) := \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^n) : \|u\|_{L_s(\mathbb{R}^n)} := \int_{\mathbb{R}^n}\frac{|u(x)|}{1+|x|^{n+2s}}\,dx < \infty\right\}, \qquad s > 0.$$

For more details on the fractional Laplacian we refer the reader to [Di Nezza et al. 2012].

We define an operator $T : X \to X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n}\log\left(\frac{1}{|x-y|}\right)K(y)e^{n(v(y)+c_v)}\,dy + \frac{1}{2n}(|x|^2 - |x|^4)|\Delta v(0)|.$$

Lemma 2.1. *Let v solve $tT(v)=v$ for some $0<t\le 1$. Then*

$$v(x)=\frac{t}{\gamma_n}\int_{\mathbb{R}^n}\log\Big(\frac{1}{|x-y|}\Big)K(y)e^{n(v(y)+c_v)}\,dy+\frac{t}{2n}(|x|^2-|x|^4)|\Delta v(0)|, \tag{7}$$

$\Delta v(0)<0$, *and* $v(x)\to-\infty$ *as* $|x|\to\infty$. *Moreover,*

$$\sup_{x\in B_1^c} v(x)=v(1)=\inf_{x\in B_1} v(x),$$

and in particular $A_v=0$.

Proof. Since v satisfies $tT(v)=v$, equation (7) follows from the definition of T. Differentiating under the integral sign and observing that $\Delta\log(1/|\cdot-y|)<0$, from (7) one gets

$$\Delta v(x)<\frac{t}{2n}|\Delta v(0)|\Delta(|x|^2-|x|^4),\qquad x\in\mathbb{R}^n. \tag{8}$$

Taking $x=0$ in (8) we obtain $\Delta v(0)<t|\Delta v(0)|$, which implies that $\Delta v(0)<0$. Notice that the function

$$w(x):=v(x)+\frac{t}{2n}|\Delta v(0)|(|x|^4-|x|^2)$$

is monotone decreasing as $\Delta w<0$. This follows from (8) and the integral representation of radially symmetric functions given by

$$f(\xi)-f(\bar\xi)=\int_{\bar\xi}^{\xi}\frac{1}{\omega_{n-1}r^{n-1}}\int_{B_r}\Delta f(x)\,dx\,dr,\qquad 0\le\bar\xi<\xi,\ \ \omega_{n-1}:=|S^{n-1}|. \tag{9}$$

The monotonicity of w implies that $\sup_{x\in B_1^c}v(x)=v(1)=\inf_{x\in B_1}v(x)$, and hence $A_v=0$. Finally, together with $|\Delta v(0)|>0$, we conclude that $\lim_{|x|\to\infty}v(x)=-\infty$ as $\lim_{|x|\to\infty}w(x)\le w(1)$. □

Lemma 2.2. *Let $(v,t)\in X\times(0,1]$ satisfy $v=tT(v)$. Then there exists $C>0$ (independent of v and t) such that*

$$\sup_{B_{1/8}} w\le C,\qquad w:=v+c_v+\frac{1}{n}\log t.$$

Proof. Let us assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence $w_k=v_k+c_{v_k}+\frac{1}{n}\log t_k$ such that $\max_{\bar B_{1/8}}w_k=:w_k(\theta_k)\to\infty$.

If θ_k is a point of local maxima of w_k, we set $x_k=\theta_k$. Otherwise, we can choose $x_k\in B_{1/4}\setminus B_{1/8}$ such that x_k is a point of local maxima of w_k and $w_k(x_k)\ge w_k(x)$ for every $x\in B_{|x_k|}$. This follows from the fact that

$$\inf_{B_{1/4}\setminus B_{1/8}} w_k\not\to\infty,$$

which is a consequence of

$$\int_{\mathbb{R}^n}Ke^{nw_k}\,dx=t_k\kappa\le\kappa,\qquad K>0 \text{ on } B_3.$$

We set $\mu_k:=e^{-w_k(x_k)}$. We distinguish the following cases.

Case 1: Up to a subsequence, $t_k\mu_k^2|\Delta v_k(0)|\to c_0\in[0,\infty)$.

We set

$$\eta_k(x):=v_k(x_k+\mu_kx)-v_k(x_k)=w_k(x_k+\mu_kx)-w_k(x_k).$$

Notice that by (7) we have, for some dimensional constant C_1,

$$\Delta\eta_k(x)=\mu_k^2\Delta v_k(x_k+\mu_k x)=C_1\frac{\mu_k^2}{\gamma_n}\int_{\mathbb{R}^n}\frac{K(y)e^{nw_k(y)}}{|x_k+\mu_k x-y|^2}\,dy+t_k\mu_k^2\left(1-\frac{4(n+2)}{2n}|x_k+\mu_k x|^2\right)|\Delta v_k(0)|,$$

so that

$$\begin{aligned}\int_{B_R}&\left|\Delta\eta_k(x)-t_k\mu_k^2|\Delta v_k(0)|\left(1-\frac{2(n+2)}{n}|x_k|^2\right)\right|dx\\
&\le\frac{C_1}{\gamma_n}\int_{\mathbb{R}^n}K(y)e^{nw_k(y)}\int_{B_R}\frac{\mu_k^2\,dx}{|x_k+\mu_k x-y|^2}\,dy+Ct_k\mu_k^2|\Delta v_k(0)|\int_{B_R}(\mu_k|x_k\cdot x|+\mu_k^2|x|^2)\,dx\\
&\le\frac{C_1}{\gamma_n}t_k\kappa\int_{B_R}\frac{1}{|x|^2}\,dx+Ct_k\mu_k^2|\Delta v_k(0)|\int_{B_R}(\mu_k|x|+\mu_k^2|x|^2)\,dx\\
&\le C\kappa t_k R^{n-2}+Ct_k\mu_k^2|\Delta v_k(0)|(\mu_k R^{n+1}+\mu_k^2R^{n+2}).\end{aligned}\tag{10}$$

The function η_k satisfies

$$(-\Delta)^{n/2}\eta_k(x)=K(x_k+\mu_k x)e^{n\eta_k(x)}\quad\text{in }\mathbb{R}^n,\qquad\eta_k(0)=0.$$

Moreover, $\eta_k\le C(R)$ on B_R. This follows easily if $|x_k|\le\frac19$, as in that case $\eta_k\le 0$ on B_R for $k\ge k_0(R)$. On the other hand, for $\frac19<|x_k|\le\frac14$ one can use Lemma 2.4 (below). Therefore, by Lemma A.3 (and Lemmas 2.6, 2.7 if n is odd), up to a subsequence, $\eta_k\to\eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$, where η satisfies

$$(-\Delta)^{n/2}\eta=K(x_\infty)e^{n\eta}\quad\text{in }\mathbb{R}^n,\qquad K(x_\infty)\int_{\mathbb{R}^n}e^{n\eta}\,dx\le t_\infty\kappa<\infty,\qquad K(x_\infty)>0,$$

where (up to a subsequence) $t_k\to t_\infty$ and $x_k\to x_\infty$. Notice that $t_\infty\in(0,1]$, $x_\infty\in\bar B_{1/4}$ and for every $R>0$, by (10)

$$\int_{B_R}|\Delta\eta-c_0c_1|\,dx\le CR^{n-2},\qquad c_1=:1-\frac{2(n+2)}{n}|x_\infty|^2>0.\tag{11}$$

Hence by Theorem A we have

$$\eta(x)=P_0(x)-\alpha\log|x|+o(\log|x|)\quad\text{as }|x|\to\infty,$$

where P_0 is a polynomial of degree at most $n-1$, P_0 is bounded from above and α is a positive constant. In fact, by (11)

$$\int_{B_R}|\Delta P_0(x)-c_0c_1|\,dx\le CR^{n-2}\quad\text{for every }R>0.$$

Since $c_0, c_1\ge 0$, it follows that P_0 is a constant. This implies that η is a spherical solution and in particular $\Delta\eta<0$ on $\mathbb{R}^n$, and therefore, again by (11), we have $c_0=0$.

We consider the following subcases.

Subcase 1.1: There exists $M>0$ such that $|x_k|/\mu_k\le M$.

We set $y_k:=-x_k/\mu_k$. Then (up to a subsequence) $y_k\to y_\infty\in B_{M+1}$. Therefore,

$$\Delta\eta(y_\infty)=\lim_{k\to\infty}\Delta\eta_k(y_k)=\lim_{k\to\infty}\mu_k^2\Delta v_k(0)=\frac{c_0}{t_\infty}=0,$$

a contradiction as $\Delta\eta<0$ on $\mathbb{R}^n$.

Subcase 1.2: Up to a subsequence, $|x_k|/\mu_k \to \infty$.

For any $N \in \mathbb{N}$ we can choose $\xi_{1,k}, \dots, \xi_{N,k} \in \mathbb{R}^n$ such that $|\xi_{i,k}| = |x_k|$ for all $i = 1, \dots, N$ and the balls $B_{2\mu_k}(\xi_{i,k})$ are disjoint for k large enough. Since the v_k are radially symmetric, the functions $\eta_{i,k} := v_k(\xi_{i,k} + \mu_k x) - v_k(\xi_{i,k}) \to \eta_i = \eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$. Therefore,

$$\lim_{k\to\infty} \int_{B_1} e^{n(v_k + c_{v_k})}\, dx \ge N \lim_{k\to\infty} \int_{B_{\mu_k}(\xi_{1,k})} e^{n(v_k + c_{v_k})}\, dx = N \frac{1}{t_\infty} \int_{B_1} e^{n\eta}\, dx.$$

This contradicts the fact that

$$\int_{B_1} K e^{n(v_k + c_{v_k})}\, dx \le \kappa, \qquad K > 0 \text{ on } B_3.$$

Case 2: Up to a subsequence, $t_k \mu_k^2 |\Delta v_k(0)| \to \infty$.

We choose $\rho_k > 0$ such that $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| = 1$. We set

$$\psi_k(x) = v_k(x_k + \rho_k \mu_k x) - v_k(x_k).$$

Then one can get (similar to (10))

$$\int_{B_R} \left| \Delta\psi_k(x) - \left(1 - \frac{2(n+2)}{n}|x_k|^2\right)\right| dx$$
$$\le C_1 \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{B_R} \frac{\rho_k^2 \mu_k^2}{|x_k + \mu_k \rho_k x - y|^2}\, dx\, dy + C_2 \mu_k \rho_k \int_{B_R} (|x| + \mu_k \rho_k |x|^2)\, dx \xrightarrow{k\to\infty} 0,$$

thanks to Lemma 2.5 (below). Moreover, together with Lemma 2.4, ψ_k satisfies

$$(-\Delta)^{n/2} \psi_k = o(1) \quad \text{in } B_R, \qquad \psi_k(0) = 0, \qquad \psi_k \le C(R) \text{ on } B_R.$$

Hence, by Lemma A.3 (and Lemma 2.6 if n is odd), up to a subsequence, $\psi_k \to \psi$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$. Then ψ must satisfy

$$\int_{B_1} |\Delta\psi - c_0|\, dx = 0, \qquad c_0 := 1 - \frac{2(n+2)}{n}|x_\infty|^2 > 0,$$

where (up to a subsequence) $x_k \to x_\infty$. This shows that $\Delta\psi(0) = c_0 > 0$, which is a contradiction as

$$\Delta\psi(0) = \lim_{k\to\infty} \Delta\psi_k(0) = \lim_{k\to\infty} \rho_k^2 \mu_k^2 \Delta v_k(x_k) \le 0.$$

Here, $\Delta v_k(x_k) \le 0$ follows from the fact that x_k is a point of local maxima of v_k. □

A consequence of the local uniform upper bounds of w are the following global uniform upper bounds:

Lemma 2.3. *There exists a constant $C > 0$ such that for all $(v, t) \in X \times (0, 1]$ with $v = tT(v)$ we have $|\Delta v(0)| \le C$ and*

$$v(x) + c_v + \frac{1}{n}\log t \le C \quad \text{on } \mathbb{R}^n.$$

Proof. By Lemma 2.2 we have

$$\sup_{B_{1/8}} w := \sup_{B_{1/8}} \left(v + c_v + \frac{1}{n}\log t \right) \le C.$$

Differentiating under the integral sign from (7), and recalling that $\Delta v(0) < 0$, we obtain

$$|\Delta v(0)| \le C\int_{B_{1/8}} \frac{1}{|y|^2} K(y)e^{nw(y)}\,dy + C\int_{B_{1/8}^c} \frac{1}{|y|^2} K(y)e^{nw(y)}\,dy$$
$$\le C\sup_{B_{1/8}} K \int_{B_{1/8}} \frac{1}{|y|^2}\,dy + C\int_{B_{1/8}^c} Ke^{nw}\,dy \le C(\kappa, K).$$

By (8) we get

$$\Delta v(x) \le t|\Delta v(0)| \le C, \quad x \in \mathbb{R}^n,$$

and hence, together with (9)

$$v(x) = v(0) + \int_0^{|x|} \frac{1}{\omega_{n-1}r^{n-1}} \int_{B_r} \Delta v(y)\,dy\,dr \le v(0) + C|x|^2 \le C + v(0), \quad x \in B_2.$$

The lemma follows from Lemmas 2.1 and 2.2. □

Proof of Theorem 1.1. Let $v \in X$ be a solution of $v = tT(v)$ for some $0 < t \le 1$. Then $A_v = 0$ and $|\Delta v(0)| \le C$, thanks to Lemmas 2.1 and 2.3. Hence, for $0 \le |\beta| \le n-1$,

$$|D^\beta v(x)| \le C\int_{\mathbb{R}^n} \left|D^\beta \log\left(\frac{1}{|x-y|}\right)\right| K(y)e^{n(v(y)+c_v+(1/n)\log t)}\,dy + C|D^\beta(|x|^2 - |x|^4)|$$
$$\le C\int_{\mathbb{R}^n} \left|D^\beta \log\left(\frac{1}{|x-y|}\right)\right| e^{-|y|^4}\,dy + C|D^\beta(|x|^2 - |x|^4)|,$$

where in the second inequality we have used that

$$v(x) + c_v + \frac{1}{n}\log t \le C, \quad C \text{ is independent of } v \text{ and } t,$$

which follows from Lemma 2.3. Now as in Lemma 2.8 one can show that

$$\|v\|_X \le M,$$

and therefore, by Lemma A.1, the operator T has a fixed point (say) v. Then

$$u = P + v + c_v - |x|^4$$

is a solution to the problem (1) and u has the asymptotic behavior given by

$$u(x) = P(x) - \frac{2\kappa}{\Lambda_1}\log|x| + \frac{1}{2n}\Delta v(0)(|x|^4 - |x|^2) - |x|^4 + c_v + o(1) \quad \text{as } |x| \to \infty. \qquad \square$$

Now we give a proof of the technical lemmas used in the proof of Lemma 2.2.

Lemma 2.4. *Let $\varepsilon > 0$. Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (7) or (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1 \setminus B_\varepsilon$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. Then*

$$v_k(x_k + x) - v_k(x_k) \le C(n, \varepsilon)|x|^2 t_k|\Delta v_k(0)|, \quad x \in B_1.$$

Proof. If $|x_k + x| \le |x_k|$ then $v_k(x_k + x) - v_k(x_k) \le 0$ as $v_k(x_k) \ge v_k(y)$ for every $y \in B_{|x_k|}$. For $|x_k| < |x_k + x|$, setting $a = a(k, x) := x_k + x$, and together with (9) we obtain

$$\begin{aligned} v_k(x_k + x) - v_k(x_k) &= \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_r \setminus B_{|x_k|}} \Delta v_k(x)\, dx\, dr \\ &\le \int_{|x_k|}^{|a|} \frac{1}{\omega_{n-1} r^{n-1}} \int_{B_{|a|} \setminus B_{|x_k|}} t_k |\Delta v_k(0)|\, dx\, d\rho \\ &\le C(n) t_k |\Delta v_k(0)| (|B_{|a|}| - |B_{|x_k|}|) \left(\frac{1}{|x_k|^{n-2}} - \frac{1}{|a|^{n-2}} \right) \\ &\le C(n, \varepsilon) t_k |x|^2 |\Delta v_k(0)|, \end{aligned}$$

where in the first equality we have used that

$$0 = v_k'(x_k) = \frac{1}{\omega_{n-1} |x_k|^{n-1}} \int_{B_{|x_k|}} \Delta v_k\, dx.$$

Hence we have the lemma. □

Lemma 2.5. *Let* $(v_k, t_k) \in X \times (0, 1]$ *satisfy* (7) *for all* $k \in \mathbb{N}$. *Let* $x_k \in B_1$ *be a point of maxima of* v_k *on* $\bar{B}_{|x_k|}$ *and* $v_k'(x_k) = 0$. *We set* $w_k = v_k + c_{v_k} + \frac{1}{n} \log t_k$ *and* $\mu_k = e^{-w_k(x_k)}$. *Let* $\rho_k > 0$ *be such that* $t_k \rho_k^2 \mu_k^2 |\Delta v_k(0)| \le C$ *and* $\rho_k \mu_k \to 0$. *Then for any* $R_0 > 0$,

$$\lim_{k\to\infty} \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{B_{R_0}} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2}\, dx\, dy =: \lim_{k\to\infty} I_k = 0.$$

Proof. In order to prove the lemma we fix $R > 0$ (large). We split B_{R_0} into

$$A_1(R, y) := \left\{ x \in B_{R_0} : |x_k + \rho_k \mu_k x - y| > R \rho_k \mu_k \right\}, \quad A_2(R, y) := B_{R_0} \setminus A_1(R, y).$$

Then we can write $I_k = I_{1,k} + I_{2,k}$, where

$$I_{i,k} := \int_{\mathbb{R}^n} K(y) e^{n w_k(y)} \int_{A_i(R,y)} \frac{\rho_k^2 \mu_k^2}{|x_k + \rho_k \mu_k x - y|^2}\, dx\, dy, \quad i = 1, 2.$$

Changing the variable $y \mapsto x_k + \rho_k \mu_k y$ and by Fubini's theorem, one gets

$$\begin{aligned} I_{2,k} &= \rho_k^n \int_{B_{R_0}} \int_{\mathbb{R}^n} K(x_k + \rho_k \mu_k y) e^{n \eta_k(y)} \frac{1}{|x - y|^2} \chi_{|x-y| \le R}\, dy\, dx \\ &\le \rho_k^n \int_{B_{R_0}} \int_{B_{R+R_0}} K(x_k + \rho_k \mu_k y) e^{n \eta_k(y)} \frac{1}{|x - y|^2}\, dy\, dx \\ &\le C(n, \varepsilon) (\sup\nolimits_{B_{R+R_0+1}} K e^{n \eta_k}) (R + R_0)^n R_0^{n-2} \rho_k^n, \end{aligned}$$

where $\eta_k(y) := w_k(x_k + \rho_k \mu_k y) - w_k(x_k)$. If $x_k \to 0$ then $\eta_k \le 0$ on B_{R+R_0+1} for k large. Otherwise, for k large, $\rho_k \mu_k y \in B_1$ for every $y \in B_{R+R_0+1}$ and hence, by Lemma 2.4

$$\eta_k(y) = v_k(x_k + \rho_k \mu_k y) - v_k(x_k) \le C |\rho_k \mu_k y|^2 t_k |\Delta v_k(0)| \le C(R, R_0).$$

Therefore,

$$\lim_{k\to\infty} I_{2,k} = 0.$$

Using the definition of c_v we bound

$$I_{1,k} \le \frac{|B_{R_0}|}{R^2} \int_{\mathbb{R}^n} K(y) e^{n w_k(y)}\, dy \le C(n, \kappa, R_0) \frac{1}{R^2}.$$

Since $R > 0$ is arbitrary, we conclude the lemma. $\square$

We need the following two lemmas only for n odd.

Lemma 2.6. *Let $n \ge 5$. Let v be given by (7). For any $r > 0$ and $\xi \in \mathbb{R}^n$ we set*

$$w(x) = v(rx + \xi), \quad x \in \mathbb{R}^n.$$

Then there exists $C > 0$ (independent of v, t, r, ξ) such that for every multi-index $\alpha \in \mathbb{N}^n$ with $|\alpha| = n-1$ we have $\|D^\alpha w\|_{L_{1/2}(\mathbb{R}^n)} \le Ct(1 + r^4|\Delta v(0)|)$. Moreover, for any $\varepsilon > 0$ there exists $R > 0$ (independent of r, ξ and t) such that

$$\int_{B_R^c} \frac{|D^\alpha w(x)|}{1+|x|^{n+1}}\, dx < \varepsilon t(1 + r^4|\Delta v(0)|), \quad |\alpha| = n-1.$$

Proof. Differentiating under the integral sign we obtain

$$|D^\alpha w(x)| \le Ct \int_{\mathbb{R}^n} \frac{r^{n-1}}{|rx + \xi - y|^{n-1}} f(y)\, dy + Ctr^4|\Delta v(0)|, \quad f(y) := K(y) e^{n(v(y)+c_v)}.$$

If $n > 5$ then the above inequality is true without the term $Ctr^4|\Delta v(0)|$. Using a change of variable $y \mapsto \xi + ry$, we get

$$\int_\Omega \frac{|D^\alpha w(x)|}{1+|x|^{n+1}}\, dx \le Ctr^n \int_{\mathbb{R}^n} f(\xi + ry) \int_\Omega \frac{1}{|x-y|^{n-1}} \frac{1}{1+|x|^{n+1}}\, dx\, dy + Ctr^4|\Delta v(0)| \int_\Omega \frac{dx}{1+|x|^{n+1}}.$$

The lemma follows by taking $\Omega = \mathbb{R}^n$ or B_R^c. $\square$

Lemma 2.7. *Let $\eta_k \to \eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$. We assume that for every $\varepsilon > 0$ there exists $R > 0$ such that*

$$\int_{B_R^c} \frac{|\Delta^{(n-1)/2}\eta_k(x)|}{1+|x|^{n+1}}\, dx < \varepsilon \quad \text{for } k = 1, 2, \ldots. \tag{12}$$

We further assume that

$$(-\Delta)^{n/2}\eta_k = K(x_k + \mu_k x) e^{n\eta_k} \quad \text{in } \mathbb{R}^n, \qquad \int_{\mathbb{R}^n} |K(x_k + \mu_k x)| e^{n\eta_k(x)}\, dx \le C,$$

where $x_k \to x_\infty$, $\mu_k \to 0$, K is a continuous function and $K(x_\infty) > 0$. Then $e^{n\eta} \in L^1(\mathbb{R}^n)$ and η satisfies

$$(-\Delta)^{n/2}\eta = K(x_\infty) e^{n\eta} \quad \text{in } \mathbb{R}^n.$$

Proof. First notice that $\Delta^{(n-1)/2}\eta_k \to \Delta^{(n-1)/2}\eta$ in $L_{1/2}(\mathbb{R}^n)$, thanks to (12) and the convergence $\eta_k \to \eta$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$.

We claim that η satisfies $(-\Delta)^{n/2}\eta = K(x_\infty)e^{n\eta}$ in $\mathbb{R}^n$ in the sense of distribution.

In order to prove the claim we let $\varphi \in C^\infty_c(\mathbb{R}^n)$. Then

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} K(x_k+\mu_k x)e^{n\eta_k(x)}\varphi(x)\,dx = \int_{\mathbb{R}^n} K(x_\infty)e^{n\eta(x)}\varphi(x)\,dx,$$

and

$$\lim_{k\to\infty}\int_{\mathbb{R}^n}(-\Delta)^{(n-1)/2}\eta_k(-\Delta)^{1/2}\varphi\,dx = \int_{\mathbb{R}^n}(-\Delta)^{(n-1)/2}\eta(-\Delta)^{1/2}\varphi\,dx.$$

We conclude the claim.

To complete the lemma first notice that $e^{n\eta} \in L^1(\mathbb{R}^n)$, which follows from the fact that for any $R > 0$

$$\int_{B_R} e^{n\eta}\,dx = \lim_{k\to\infty}\int_{B_R} e^{n\eta_k}\,dx = \lim_{k\to\infty}\int_{B_R}\frac{K(x_k+\mu_k x)}{K(x_\infty)}e^{n\eta_k(x)}\,dx \le \frac{C}{K(x_\infty)}.$$

We fix a function $\psi \in C^\infty_c(B_2)$ such that $\psi = 1$ on B_1. For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ we set $\varphi_k(x) = \varphi(x)\psi(x/k)$. The lemma follows by taking $k \to \infty$, thanks to the previous claim. □

Lemma 2.8. *The operator $T : X \to X$ is compact.*

Proof. Let v_k be a bounded sequence in X. Then (up to a subsequence) $\{v_k(0)\}$, $\{\Delta v_k(0)\}$, $\{A_{v_k}\}$ and $\{c_{v_k}\}$ are convergent sequences. Therefore, $|\Delta v_k(0)|(|x|^2-|x|^4)$ converges to some function in X. To conclude the lemma, it is sufficient to show that up to a subsequence $\{f_k\}$ converges in X, where f_k is defined by

$$f_k(x) = \int_{\mathbb{R}^n}\log\left(\frac{1}{|x-y|}\right)Q(y)e^{nP(y)}e^{nP_{v_k}(y)}e^{n(v_k(y)+c_{v_k})}\,dy.$$

Differentiating under the integral sign, for $0 < |\beta| \le n-1$, one gets

$$|D^\beta f_k(x)| \le C\int_{\mathbb{R}^n}\frac{1}{|x-y|^{|\beta|}}Q(y)e^{nP(y)}e^{nP_{v_k}(y)}e^{n(v_k(y)+c_{v_k})}\,dy \le C\int_{\mathbb{R}^n}\frac{1}{|x-y|^{|\beta|}}e^{-|y|^4}\,dy \le C,$$

where the second inequality follows from the uniform bounds

$$|v_k(0)| \le C, \quad |c_{v_k}| \le C, \quad Qe^{nP} \le C, \quad \text{and} \quad v_k(x)+P_{v_k}(x) \le v_k(0)-|x|^4. \tag{13}$$

Indeed, for $0 < |\beta| \le n-1$

$$\lim_{R\to\infty}\sup_k\sup_{x\in B_R^c}|D^\beta f_k(x)| = 0,$$

and for every $0 < s < 1$ we have $\|D^{n-1}f_k\|_{C^{0,s}(B_R)} \le C(R,s)$. Finally, using (13) we have the bound

$$|f_k(x)| \le C\int_{\mathbb{R}^n}\big|\log|x-y|\big|e^{-|y|^4}\,dy \le C\log(2+|x|).$$

Thus, by Ascoli's theorem, up to a subsequence, $f_k \to f$ in $C^{n-1}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $f \in C^{n-1}(\mathbb{R}^n)$, and the global uniform estimates of f_k and $D^\beta f_k$ would imply that $f_k \to f$ in X. □

3. Proof of Theorem 1.3

We consider the space

$$X := \{v \in C^{n-1}(\mathbb{R}^n) : v \text{ is radially symmetric}, \|v\|_X < \infty\},$$

where

$$\|v\|_X := \sup_{x\in\mathbb{R}^n}\bigg(\sum_{|\alpha|\le 1}(1+|x|)^{|\alpha|-2}|D^\alpha v(x)| + \sum_{1<|\alpha|\le n-1}|D^\alpha v(x)|\bigg).$$

For $v \in X$, let c_v be the constant determined by

$$\int_{\mathbb{R}^n} Qe^{n(v+c_v)}\,dy = \kappa,$$

where Q satisfies the hypothesis in Theorem 1.3. Again by a dilation argument we can assume $Q > 0$ on B_3.

We define an operator $T : X \to X$ given by $T(v) = \bar{v}$, where

$$\bar{v}(x) = \frac{1}{\gamma_n}\int_{\mathbb{R}^n}\log\Big(\frac{1}{|x-y|}\Big)Q(y)e^{n(v(y)+c_v)}\,dy + \frac{1}{2n}|\Delta v(0)||x|^2.$$

As in Lemma 2.8 one can show that the operator T is compact.

The proofs of the following two lemmas are similar to those of Lemmas 2.1 and 2.5 respectively.

Lemma 3.1. *Let v solve $tT(v) = v$ for some $0 < t \le 1$. Then $\Delta v(0) < 0$, and*

$$v(x) = \frac{t}{\gamma_n}\int_{\mathbb{R}^n}\log\Big(\frac{1}{|x-y|}\Big)Q(y)e^{n(v(y)+c_v)}\,dy + \frac{t}{2n}|\Delta v(0)||x|^2. \tag{14}$$

Lemma 3.2. *Let $(v_k, t_k) \in X \times (0, 1]$ satisfy (14) for all $k \in \mathbb{N}$. Let $x_k \in B_1$ be a point of maxima of v_k on $\bar{B}_{|x_k|}$ and $v_k'(x_k) = 0$. We set $w_k = v_k + c_{v_k} + \frac{1}{n}\log t_k$ and $\mu_k = e^{-w_k(x_k)}$. Let $\rho_k > 0$ be such that $\rho_k^2 t_k\mu_k^2|\Delta v_k(0)| \le C$ and $\rho_k\mu_k \to 0$. Then for any $R_0 > 0$*

$$\lim_{k\to\infty}\int_{\mathbb{R}^n} Q(y)e^{nw_k(y)}\int_{B_{R_0}}\frac{\rho_k^2\mu_k^2}{|x_k+\rho_k\mu_k x - y|^2}\,dx\,dy = 0.$$

Now we prove similar local uniform upper bounds to those in Lemma 2.2.

Lemma 3.3. *Let $(v, t) \in X \times (0, 1]$ satisfy (14). Then there exists $C > 0$ (independent of v and t) such that*

$$\sup_{B_{1/8}} w \le C, \qquad w := v + c_v + \frac{1}{n}\log t.$$

Proof. The proof is very similar to that of Lemma 2.2. Here we briefly sketch it.

We assume by contradiction that the conclusion of the lemma is false. Then there exists a sequence of (v_k, t_k) and a sequence of points x_k in $B_{1/4}$ such that

$$w_k(x_k) \to \infty, \quad w_k \le w_k(x_k) \text{ on } B_{|x_k|}, \quad x_k \text{ is a point of local maxima of } v_k.$$

We set $\mu_k := e^{-w_k(x_k)}$ and we distinguish following cases.

Case 1: Up to a subsequence, $t_k\mu_k^2|\Delta v_k(0)| \to c_0 \in [0, \infty)$.

We set $\eta_k(x) := v_k(x_k + \mu_k x) - v_k(x_k)$. Then we have

$$\int_{B_R} \big|\Delta\eta_k - t_k\mu_k^2|\Delta v_k(0)|\big|\,dx \le C t_k R^{n-2}.$$

Now one can proceed exactly as in Case 1 in Lemma 2.2.

Case 2: Up to a subsequence, $t_k\mu_k^2|\Delta v_k(0)| \to \infty$.

We set $\psi_k(x) = v_k(x_k + \rho_k\mu_k x) - v_k(x_k)$, where ρ_k is determined by $t_k\rho_k^2\mu_k^2|\Delta v_k(0)| = 1$. Then by Lemma 3.2

$$\int_{B_R} |\Delta\psi_k - 1|\,dx = o(1) \quad \text{as } k \to \infty.$$

Similar to Case 2 in Lemma 2.2 one can get a contradiction. □

With the help of Lemma 3.3 we prove:

Lemma 3.4. *There exists a constant $M > 0$ such that for all $(v, t) \in X \times (0, 1]$ satisfying (14) we have $\|v\| \le M$.*

Proof. Let $(v, t) \in X \times (0, 1]$ satisfy (14). We set $w := v + c_v + \frac{1}{n}\log t$.

First we show that $|\Delta v(0)| \le C$ for some $C > 0$ independent of v and t. Indeed, differentiating under the integral sign, from (14), and together with Lemma 3.3, we get

$$\begin{aligned}
|\Delta v(0)|(1+t) &\le C\int_{\mathbb{R}^n} \frac{1}{|y|^2} Q(y)e^{nw(y)}\,dy \\
&= C\int_{B_{1/8}} \frac{1}{|y|^2} Q(y)e^{nw(y)}\,dy + C\int_{B_{1/8}^c} \frac{1}{|y|^2} Q(y)e^{nw(y)}\,dy \\
&\le C\int_{B_{1/8}} \frac{1}{|y|^2} Q(y)\,dy + C\kappa \le C.
\end{aligned}$$

Hence $|\Delta v(0)| \le C$.

We define a function $\xi(x) := v(x) - (t/2n)|\Delta v(0)||x|^2$. Then ξ is monotone decreasing on $(0, \infty)$, which follows from the fact that $\Delta\xi \le 0$. Therefore,

$$\begin{aligned}
w(x) &= \xi(x) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2 \\
&\le \xi\big(\tfrac{1}{8}\big) + c_v + \frac{1}{n}\log t + \frac{t}{2n}|\Delta v(0)||x|^2 \\
&\le w\big(\tfrac{1}{8}\big) + \frac{t}{2n}|\Delta v(0)||x|^2.
\end{aligned}$$

Hence, $w(x) \le \lambda(1 + |x|^2)$ on $\mathbb{R}^n$ for some $\lambda > 0$ independent of v and t. Using this in (14) one can show

$$|v(x)| \le C\log(2 + |x|) + C|x|^2,$$

and differentiating under the integral sign, from (14)

$$|D^\beta v(x)| \le C\int_{\mathbb{R}^n} \frac{1}{|x-y|^{|\beta|}} Q(y)e^{\lambda(1+|y|^2)}\,dy + C\big|D^\beta|x|^2\big|, \quad 0 < |\beta| \le n-1.$$

The lemma follows easily. □

Proof of Theorem 1.3. By the Schauder fixed point theorem (see Lemma A.1), the operator T has a fixed point, thanks to Lemma 3.4. Let v be a fixed point of T. Then $u = v + c_v$ is a solution of (1). □

Now we prove the nonexistence results stated in the Introduction.

Lemma 3.5. *Let $n \in \{3, 4\}$. If $Q(x) = e^{-\lambda|x|^2}$ for some $\lambda > 0$ then there is no solution to (1) with $\kappa > \Lambda_1$. If $Q \in C^1_{\mathrm{rad}}(\mathbb{R}^n)$ is of the form $Q = e^{\xi}$ and it satisfies*

$$Q' \le 0, \quad |x \cdot \nabla Q(x)| \le C, \quad \frac{\xi(x)}{|x|^2} \xrightarrow{|x|\to\infty} 0,$$

then there is no radially symmetric solution to (1) with $\kappa > \Lambda_1$.

Proof. First we consider the case when $Q = e^{-\lambda|x|^2}$. Let u be a solution to (1) with $Q = e^{-\lambda|x|^2}$. Then the function $w(x) := u - (\lambda/n)|x|^2$ satisfies

$$(-\Delta)^{n/2} w = e^{nw}, \quad \kappa = \int_{\mathbb{R}^n} Q e^{nu}\, dx = \int_{\mathbb{R}^n} e^{nw}\, dx < \infty.$$

It follows from [Lin 1998; Jin et al. 2015] that $\kappa \le \Lambda_1$.

In order to prove the lemma for $Q = e^{\xi}$, we assume by contradiction that there is a solution u to (1) with $\kappa > \Lambda_1$. We set

$$v(x) := \frac{1}{\gamma_n} \int_{\mathbb{R}^n} \log\left(\frac{|y|}{|x-y|}\right) Q(y) e^{nu(y)}\, dy, \quad h := u - v.$$

Then $v(x) = -(2\kappa/\Lambda_1)\log|x| + o(\log|x|)$ as $|x| \to \infty$. Notice that h is radially symmetric and $(-\Delta)^{n/2} h = 0$ on $\mathbb{R}^n$. Therefore, $h(x) = c_1 + c_2|x|^2$ for some $c_1, c_2 \in \mathbb{R}$. This follows easily if $n = 4$. For $n = 3$, first notice that $\Delta h \in L_{1/2}(\mathbb{R}^3)$. Hence, by [Jin et al. 2015, Lemma 15] $\Delta h \equiv$ constant. Now radial symmetry of h implies that $h(x) = c_1 + c_2|x|^2$.

From a Pohozaev-type identity in [Xu 2005, Theorem 2.1], we get

$$\frac{\kappa}{\gamma_n}\left(\frac{\kappa}{\gamma_n} - 2\right) = \frac{1}{\gamma_n} \int_{\mathbb{R}^n} (x \cdot \nabla K(x)) e^{nv(x)}\, dx, \quad K := Q e^{nh}. \tag{15}$$

Since $\kappa > \Lambda_1 = 2\gamma_n$, from (15) we deduce that $x \cdot \nabla K(x) > 0$ for some $x \in \mathbb{R}^n$. Using that $Q e^{nu} \in L^1(\mathbb{R}^n)$ and that $\xi(x) = o(|x|^2)$ at infinity, one has $c_2 \le 0$. Therefore, $x \cdot \nabla K(x) \le 0$ in $\mathbb{R}^n$, a contradiction. □

The proof of the following lemma is similar to that of Lemma 3.5.

Lemma 3.6. *Let $\kappa \ge \Lambda_1$. Let P be a nonconstant and nonincreasing radially symmetric polynomial of degree at most $n - 1$. Then there is no solution u to (2) (with $n \ge 3$) such that u has the asymptotic behavior given by*

$$u(x) = -\frac{2\kappa}{\Lambda_1} \log|x| + P(x) + o(\log|x|) \quad \textit{as } |x| \to \infty.$$

Appendix

Lemma A.1 [Gilbarg and Trudinger 1998, Theorem 11.3]. *Let T be a compact mapping of a Banach space X into itself, and suppose that there exists a constant M such that*

$$\|x\|_X < M$$

for all $x \in X$ and $t \in (0, 1]$ satisfying $tTx = x$. Then T has a fixed point.

The following identity (16) is due to Pizzetti [1909]. Simple proofs of (16) and (17) can be found in Lemma 3 and Proposition 4, respectively, of [Martinazzi 2009a].

Lemma A.2 [Pizzetti 1909; Martinazzi 2009a]. *Let $\Delta^m h = 0$ in $B_{4R} \subset \mathbb{R}^n$. For any $x \in B_R$ and $0 < r < R - |x|$ we have*

$$\frac{1}{|B_r|}\int_{B_r(x)} h(z)\,dz = \sum_{i=0}^{m-1} c_i r^{2i} \Delta^i h(x), \tag{16}$$

where

$$c_0 = 1, \qquad c_i = c(i, n) > 0 \quad \text{for } i \geq 1.$$

Moreover, for every $k \geq 0$ there exists $C = C(k, R) > 0$ such that

$$\|h\|_{C^k(B_R)} \leq C\|h\|_{L^1(B_{4R})}. \tag{17}$$

Lemma A.3. *Let $R > 0$ and $B_R \subset \mathbb{R}^n$. Let $u_k \in C^{n-1,\alpha}(\mathbb{R}^n)$ for some $\alpha \in \left(\frac{1}{2}, 1\right)$ be such that*

$$u_k(0) = 0, \quad \|u_k^+\|_{L^\infty(B_R)} \leq C, \quad \|(-\Delta)^{n/2} u_k\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta u_k|\,dx \leq C.$$

If n is an odd integer, we also assume that $\|\Delta^{(n-1)/2} u_k\|_{L_{1/2}(\mathbb{R}^n)} \leq C$. Then (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{R/8})$.

Proof. First we prove the lemma for n even.

We write $u_k = w_k + h_k$, where

$$\begin{cases} (-\Delta)^{n/2} w_k = (-\Delta)^{n/2} u_k & \text{in } B_R, \\ \Delta^j w_k = 0 & \text{on } \partial B_R,\ j = 0, 1, \dots, \frac{1}{2}(n-2). \end{cases}$$

Then by standard elliptic estimates, the w_k are uniformly bounded in $C^{n-1,\beta}(B_R)$. Therefore,

$$|h_k(0)| \leq C, \quad \|h_k^+\|_{L^\infty(B_R)} \leq C, \quad \int_{B_R} |\Delta h_k|\,dx \leq C.$$

Since the h_k are $\frac{n}{2}$-harmonic, the Δh_k are $\left(\frac{n}{2}-1\right)$-harmonic in B_R, and by (17) we obtain

$$\|\Delta h_k\|_{C^n(B_{R/4})} \leq C\|\Delta h_k\|_{L^1(B_R)} \leq C.$$

Using the identity (16) we have the bound

$$\begin{aligned}\frac{1}{|B_R|}\int_{B_R(0)} h_k^-(z)\,dz &= \frac{1}{|B_R|}\int_{B_R(0)} h_k^+(z)\,dz - \frac{1}{|B_R|}\int_{B_R(0)} h_k(z)\,dz \\ &= \frac{1}{|B_R|}\int_{B_R(0)} h_k^+(z)\,dz - h_k(0) - \sum_{i=1}^{n/2-1} c_i R^{2i}\Delta^i h_k(0) \le C,\end{aligned}$$

and hence

$$\int_{B_R} |h_k(z)|\,dz = \int_{B_R} h_k^+(z)\,dz + \int_{B_R} h_k^-(z)\,dz \le C.$$

Again by (17) we obtain

$$\|h_k\|_{C^n(B_{R/4})} \le C\|h_k\|_{L^1(B_R)} \le C.$$

Thus, the u_k are uniformly bounded in $C^{n-1,\beta}(B_{R/4})$ and (up to a subsequence) $u_k \to u$ in $C^{n-1}(B_{R/4})$ for some $u \in C^{n-1}(B_{R/4})$.

It remains to prove the lemma for n odd.

If n is odd then $\frac{1}{2}(n-1)$ is an integer. We split $\Delta^{(n-1)/2}u_k = w_k + h_k$, where

$$\begin{cases}(-\Delta)^{1/2}w_k = (-\Delta)^{1/2}\Delta^{(n-1)/2}u_k & \text{in } B_R,\\ w_k = 0 & \text{in } B_R^c.\end{cases}$$

Then by Lemmas A.4 and A.5 one has $\|\Delta^{(n-1)/2}u_k\|_{C^{1/2}(B_{R/2})} \le C$. Now one can proceed as in the case of even integer. □

Lemma A.4 [Jin et al. 2015, Proposition 22]. *Let $u \in L_\sigma(\mathbb{R}^n)$ for some $\sigma \in (0,1)$ and $(-\Delta)^\sigma u = 0$ in B_{2R}. Then for every $k \in \mathbb{N}$,*

$$\|\nabla^k u\|_{C^0(B_R)} \le C(n,\sigma,k)\frac{1}{R^k}\left(R^{2\sigma}\int_{\mathbb{R}^n\setminus B_{2R}} \frac{|u(x)|}{|x|^{n+2\sigma}}\,dx + \frac{\|u\|_{L^1(B_{2R})}}{R^n}\right),$$

where $\alpha \in (0,1)$ and k is a nonnegative integer.

Lemma A.5 [Ros-Oton and Serra 2014, Proposition 1.1]. *Let $\sigma \in (0,1)$. Let u be a solution of*

$$\begin{cases}(-\Delta)^\sigma u = f & \text{in } B_R,\\ u = 0 & \text{in } B_R^c.\end{cases}$$

Then

$$\|u\|_{C^\sigma(\mathbb{R}^n)} \le C(R,\sigma)\|f\|_{L^\infty(B_R)}.$$

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