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CONICAL MAXIMAL REGULARITY FOR ELLIPTIC OPERATORS VIA HARDY SPACES

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We give a technically simple approach to the maximal regularity problem in parabolic tent spaces for second-order, divergence-form, complex-valued elliptic operators. By using the associated Hardy space theory combined with certain L^2-L^2 off-diagonal estimates, we reduce the tent space boundedness in the upper half-space to the reverse Riesz inequalities in the boundary space. This way, we also improve recent results obtained by P. Auscher et al.

1. Introduction

Let $\mathbb{R}^{1+n}_+$ be the upper half-space $\mathbb{R}_+ \times \mathbb{R}^n$ with $\mathbb{R}_+ = (0, \infty)$ and $n \in \mathbb{N}_+ = \{1, 2, \dots\}$. Define the tent space T^p_{par}, $n/(n+1) < p < \infty$, as the space of all locally square-integrable functions on $\mathbb{R}^{1+n}_+$ such that

$$\|F\|_{T^p_{\mathrm{par}}} := \left(\int_{\mathbb{R}^n} \left(\iint_{\mathbb{R}^{1+n}_+} \frac{\mathbf{1}_{B(x,\, t^{1/2})}(y)}{t^{n/2}} |F(t, y)|^2 \, dt \, dy \right)^{p/2} dx \right)^{1/p} < \infty. \tag{1}$$

The scale T^p_{par}, $n/(n+1) < p < \infty$, is a parabolic analogue of the tent spaces introduced by R. R. Coifman, Y. Meyer and E. M. Stein [Coifman et al. 1985].

Let $A = A(x)$ be an $n \times n$ matrix of complex L^∞ coefficients, defined on $\mathbb{R}^n$, and satisfying the ellipticity (or "accretivity") condition

$$\lambda |\xi|^2 \le \operatorname{Re} A\xi \cdot \bar{\xi} \quad \text{and} \quad |A\xi \cdot \bar{\zeta}| \le \Lambda |\xi| |\zeta| \tag{2}$$

for $\xi, \zeta \in \mathbb{C}^n$ and for some λ and Λ such that $0 < \lambda \le \Lambda < \infty$. Let

$$L := -\operatorname{div} A \nabla$$

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(its precise definition will be recalled in next section). Consider the associated forward maximal regularity operator $\boldsymbol{M}_L^+$ given by

$$\boldsymbol{M}_L^+(F)_t := \int_0^t Le^{-(t-s)L} F_s \, ds, \tag{3}$$

originally defined on $F \in L^2(\mathbb{R}_+; \boldsymbol{D}(L))$. Here $\boldsymbol{D}(L)$ is the domain of L in $L^2(\mathbb{R}^n)$ and $F_s = F(s, \cdot)$. By a classical result of L. de Simon [1964], $\boldsymbol{M}_L^+$ extends to a bounded operator on $L^2(\mathbb{R}_+; L^2(\mathbb{R}^n))$. By Fubini's theorem,

$$T^2_{\text{par}}(\mathbb{R}^{1+n}_+) \simeq L^2(\mathbb{R}^n; L^2(\mathbb{R}_+)). \tag{4}$$

For p different from 2, the analogous equivalence of (4) between $T^p_{\text{par}}(\mathbb{R}^{1+n}_+)$ and $L^p(\mathbb{R}^n; L^2(\mathbb{R}_+))$ breaks down. We shall refer to the maximal regularity (namely, the boundedness of $\boldsymbol{M}_L^+$) in T^p_{par} as *conical* maximal regularity for the reason that (parabolic) *cones* are involved in defining tent spaces in (1).

The maximal regularity operator $\boldsymbol{M}_L^+$ is a typical example of singular integral operators with operator-valued kernels. Let $1 \le p \le 2$. Let

$$\operatorname{dist}(E, E') := \inf\{|x - y| : x \in E, \ y \in E'\}.$$

We shall say that a class of uniformly $L^2 = L^2(\mathbb{R}^n)$ bounded kernels $\{T(t)\}_{t>0}$ satisfies the L^p-L^2 off-diagonal decay with some order $M \in \mathbb{N}_+$ if we have

$$\|\mathbf{1}_{E'} T(t) \mathbf{1}_E f\|_{L^2} \lesssim t^{-(n/2)(1/p-1/2)} \left(1 + \frac{\operatorname{dist}(E, E')^2}{t}\right)^{-M} \|\mathbf{1}_E f\|_{L^p} \tag{5}$$

for all Borel sets $E, E' \subset \mathbb{R}^n$, all $t > 0$ and all $f \in L^p \cap L^2$. We shall say $\{T(t)\}_{t>0}$ satisfies the L^p-L^2 off-diagonal decay if it satisfies the L^p-L^2 off-diagonal decay with any order $M \in \mathbb{N}_+$. Denote by $p_- = p_-(L)$ the infimum of p for which the heat semigroup $\{e^{-tL}\}_{t>0}$ satisfies the L^p-L^2 off-diagonal decay. Define the index

$$(p_-)_* := \frac{np_-}{n + p_-}. \tag{6}$$

For $L = -\Delta = -\operatorname{div} \nabla$, one has $p_- = 1$ and $1_* = n/(n+1)$.

Our main result in this letter reads as follows.

Theorem 1.1. *Let $L = -\operatorname{div} A\nabla$ with A satisfying* (2) *and p_- defined as in* (6)*. Then for $p \in ((p_-)_*, 2]$, the maximal regularity operator $\boldsymbol{M}_L^+$ defined as in* (3) *extends to a bounded operator on T^p_{par}.*

We end the introduction with several remarks.

Remark 1.2. Under the assumption $(p_-)_* < 1$, Theorem 1.1 was first proved by Auscher et al. [2012a, Theorem 3.1] (with $m = 2$, $\beta = 0$ and q close to p_- in their statement). Indeed, we note that $(p_-)_* < 1$ is equivalent to $(p_-)' > n$, where $(p_-)'$ is the dual exponent of p_-. A threshold condition essentially the same as $(p_-)' > n$ is used in [Auscher et al. 2012a].

A general framework of singular integral operators on tent spaces is also presented by Auscher et al. [2012a]. Their method is heavily based on the L^p-L^2 off-diagonal decay of the family $\{tLe^{-tL}\}_{t>0}$

for $p \in (p_-, 2)$. Note that they already improved the previous result in [Auscher et al. 2012b], the T^p_{par}-boundedness of $\boldsymbol{M}_L^+$ for $p \in (2_*, 2]$, which assumes L^2-L^2 off-diagonal decay only.

Here we shall give a technically simple approach to Theorem 1.1 by using the well-established L-associated Hardy space theory combined (mainly) with L^2-L^2 off-diagonal decay of $\{tLe^{-tL}\}_{t>0}$.

Remark 1.3. The motivation of the reduction scheme

$$\text{(operator theory on tent spaces)} \to \text{(Hardy space theory)},$$

which is involved in our proof of Theorem 1.1, comes from the study of conical maximal regularity (in elliptic tent spaces) for first-order perturbed Dirac operators [Huang 2015, Chapter 5]. Furthermore, the motivation of considering such conical (elliptic) maximal regularity estimates is suggested by their applications to boundary-value elliptic problems (see [Auscher and Axelsson 2011] for example). In the parabolic case, the conical maximal regularity results have already proven to be useful in various settings (see for example [Auscher et al. 2014; Auscher and Frey 2015]).

Remark 1.4. Though the singularity of the integral operator $\boldsymbol{M}_L^+$ is at $s = t$, the most involved part turns out to be the estimation of tent space norms when $s \to 0$. For more explanations concerning the "singularity" pertaining to singular integral operators and maximal regularity operators on tent spaces, see [Auscher et al. 2012a, Remark 3.6; Auscher and Frey 2015, Remark 5.23].

Remark 1.5. Theorem 1.1 also extends to higher order elliptic operators. Then one changes correspondingly the homogeneity of tent spaces and off-diagonal decay in (5). We leave this issue to the interested reader.

2. Elliptic operators and Hardy spaces

We give some preliminary materials needed in the proof of Theorem 1.1.

Let A satisfy (2). We define the divergence-form elliptic operator

$$Lf := -\operatorname{div}(A\nabla f),$$

which we interpret in the sense of maximal-accretive operators via a sesquilinear form. That is, $\boldsymbol{D}(L)$ is the largest subspace contained in $W^{1,2}$ for which

$$\left| \int_{\mathbb{R}^n} A\nabla f \cdot \nabla g \right| \le C\|g\|_2$$

for all $g \in W^{1,2}$, and we set Lf by

$$\langle Lf, g\rangle = \int_{\mathbb{R}^n} A\nabla f \cdot \overline{\nabla g}$$

for $f \in \boldsymbol{D}(L)$ and $g \in W^{1,2}$. Thus defined, L is a maximal-accretive operator on L^2 and $\boldsymbol{D}(L)$ is dense in $W^{1,2}$. Furthermore, L has a square root, denoted by $L^{1/2}$ and defined as the unique maximal-accretive operator such that

$$L^{1/2}L^{1/2} = L \tag{7}$$

as unbounded operators [Kato 1976, p. 281].

For L as formulated above, the development of an L-associated Hardy space theory was taken in [Hofmann and Mayboroda 2009] (and independently in [Auscher et al. 2008] in a different geometric setting), in which the authors considered the model case $H^1_L(\mathbb{R}^n)$. In presence of pointwise heat kernel bounds, see [Duong and Yan 2005]. The definition of H^1_L given in [Hofmann and Mayboroda 2009; Auscher et al. 2008] can be extended immediately to $n/(n+1) < p \le 2$ [Hofmann et al. 2011]. To this end, consider the (conical) square function associated with the heat semigroup generated by L

$$\boldsymbol{S}_L(f)(x) := \left(\iint_{\Gamma(x)} |t^2 L e^{-t^2 L} f(y)|^2 \,\frac{dt\,dy}{t^{1+n}}\right)^{1/2}, \quad x \in \mathbb{R}^n,$$

where, as usual,

$$\Gamma(x) = \{(t, y) \in \mathbb{R}^{1+n}_+ : |x - y| < t\}$$

is a nontangential cone with vertex at $x \in \mathbb{R}^n$. As in [Hofmann and Mayboroda 2009; Hofmann et al. 2011], we define $H^p_L(\mathbb{R}^n)$ for $n/(n+1) < p \le 2$ as the completion of

$$\{f \in L^2(\mathbb{R}^n) : \boldsymbol{S}_L(f) \in L^p(\mathbb{R}^n)\}$$

in the quasinorm

$$\|f\|_{H^p_L(\mathbb{R}^n)} := \|\boldsymbol{S}_L(f)\|_{L^p(\mathbb{R}^n)}.$$

We will not get into the dual side ($p > 2$) of the Hardy space theory.

For L^2-L^2 off-diagonal decay related to $\{e^{-sL}, sLe^{-sL}, \sqrt{s}\nabla e^{-sL}\}_{s>0}$, and other holomorphic functions of L (for example $(I - e^{-sL})^\sigma$ with $\sigma > 0$), we refer to Chapter 2 of the memoir [Auscher 2007].

3. Proof of Theorem 1.1

Note that the extension of $\boldsymbol{M}^+_L$ will be divided into two steps: first from $F \in L^2(\mathbb{R}_+; \boldsymbol{D}(L))$ to T^2_{par} and then for $n/(n+1) < p < 2$ from $T^2_{\text{par}} \cap T^p_{\text{par}}$ to T^p_{par}.

First we split the operator $\boldsymbol{M}^+_L$: for $\ell \in \mathbb{N}_+$ large, set

$$\boldsymbol{R}^\ell_L := \boldsymbol{M}^+_L - \boldsymbol{V}^\ell_L, \tag{8}$$

where for $F \in L^2(\mathbb{R}_+; \boldsymbol{D}(L))$ the singular part $\boldsymbol{R}^\ell_L$ is given formally by

$$\boldsymbol{R}^\ell_L(F)_t = \int_0^t L e^{-(t-s)L}(I - e^{-2sL})^\ell F_s\,ds \tag{9}$$

and the regular part is defined by

$$\boldsymbol{V}^\ell_L = \sum_{k=1}^{\ell} \binom{\ell}{k} \boldsymbol{V}_{L,k}$$

with

$$\boldsymbol{V}_{L,k}(F)_t := \int_0^t L e^{-(t+(2k-1)s)L} F_s\,ds, \quad t \in \mathbb{R}_+.$$

For the above binomial sum $\boldsymbol{V}^\ell_L$, it suffices to consider $\boldsymbol{V}_L := \boldsymbol{V}_{L,1}$.

Let $2\mathbb{N}_+ = \{2, 4, \dots\}$. We make the following observation.

Lemma 3.1. *For $\ell \in 2\mathbb{N}_+$ and $\frac{1}{2}\ell > \frac{1}{2} + \frac{1}{4}n$, the operator $\boldsymbol{R}_L^\ell$, as given in* (9) *through* (8), *extends to a bounded operator on T^p_{par} for any $n/(n+1) < p \le 2$.*

Proof. The T^2_{par}-boundedness is de Simon's theorem plus the uniform L^2-boundedness of $\{(I-e^{-2sL})^\ell\}_{s>0}$. By interpolation it suffices to consider $n/(n+1) < p \le 1$, and this follows from Lemmata 3.4 and 3.5 of [Auscher et al. 2012a] in the particular case $m = 2$, $\beta = 0$ and $q = 2$.[1] Indeed, first we can decompose the operator $\boldsymbol{R}_L^\ell$ as in [Auscher et al. 2012a] in the way

$$\boldsymbol{R}_L^\ell(F)_t = \int_{t/2}^t Le^{-(t-s)L}(I-e^{-2sL})^\ell F_s\,ds + \int_0^{t/2} Le^{-(t-s)L}(I-e^{-2sL})^\ell F_s\,ds =: \mathrm{I} + \mathrm{II}.$$

Here we view $\mathcal{T}_1 = \{(I-e^{-2sL})^\ell\}_{s>0}$ as an operator on T^p_{par} given by

$$\mathcal{T}_1 : F \mapsto \mathcal{T}_1(F)_s := (I-e^{-2sL})^\ell F_s,$$

with the similar interpretation for $\mathcal{T}_2 = \{(I-e^{-2sL})^\ell/(sL)^{\ell/2}\}_{s>0}$ in

$$Le^{-(t-s)L}(I-e^{-2sL})^\ell = \left(\frac{s}{t-s}\right)^{\ell/2} L((t-s)L)^{\ell/2} e^{-(t-s)L} \frac{(I-e^{-2sL})^\ell}{(sL)^{\ell/2}}.$$

Note that $t-s \sim t$ when $s < t/2$. Therefore, to obtain the T^p_{par}-boundedness of $\boldsymbol{R}_L^\ell$ for $n/(n+1) < p \le 1$, we can use Lemma 3.4 of [Auscher et al. 2012a] together with the T^p_{par}-boundedness of $\mathcal{T}_1$ to estimate I and use Lemma 3.5 of [Auscher et al. 2012a] together with the T^p_{par}-boundedness of $\mathcal{T}_2$ to estimate II. The latter tent space boundedness results on $\mathcal{T}_i$, $i = 1, 2$, are implied by their L^2-L^2 off-diagonal decay with order at least $\frac{1}{2}\ell$, which satisfies the condition

$$\frac{\ell}{2} > \frac{1}{2} + \frac{n}{4} = \frac{n}{2}\left(\frac{1}{n/(n+1)} - \frac{1}{2}\right).$$

This implication can be easily verified via the extrapolation method on tent spaces through atomic decompositions. Note that we also need the condition $\frac{1}{2}\ell > \frac{1}{2} + \frac{1}{4}n$ in $(s/(t-s))^{\ell/2} \sim (s/t)^{\ell/2}$ when applying Lemma 3.5 of [Auscher et al. 2012a]. □

Next we rewrite the operator $\boldsymbol{V}_L$ in the following way:

$$\boldsymbol{V}_L(F)_t = -\widetilde{\boldsymbol{V}}_L(F)_t + \boldsymbol{I}_L(F)_t, \quad t \in \mathbb{R}_+, \tag{10}$$

where for $F \in L^2(\mathbb{R}_+; \boldsymbol{D}(L))$ the backward part $\widetilde{\boldsymbol{V}}_L$ is defined by

$$\widetilde{\boldsymbol{V}}_L(F)_t := \int_t^\infty Le^{-(t+s)L} F_s\,ds, \quad t \in \mathbb{R}_+, \tag{11}$$

and the trace part $\boldsymbol{I}_L$ is defined by

$$\boldsymbol{I}_L(F)_t := \int_0^\infty Le^{-(t+s)L} F_s\,ds = \sqrt{L}e^{-tL} \int_0^\infty \sqrt{L}e^{-sL} F_s\,ds.$$

We used the square root property $\sqrt{L}\sqrt{L} = L$ recalled in (7).

[1] We point out that one can also prove this lemma by adapting directly the arguments for Lemma 3.4 of [Auscher et al. 2012a] (see [Huang 2015] for details).

Lemma 3.2. *The integral operator* $\widetilde{\boldsymbol{V}}_L$ *as given in* (11) *extends to a bounded operator on* T^p_{par} *for any* $n/(n+1) < p \le 2$.[2]

Proof. This is a consequence of a more general claim by Auscher et al. [2012a, Proposition 3.7], again corresponding to the case $m = 2$, $\beta = 0$ and $q = 2$. Indeed, [Auscher et al. 2012a, Proposition 3.7] deals with a counterpart to $\boldsymbol{M}^+_L$, namely the backward maximal regularity operator

$$\boldsymbol{M}^-_L(F)_t := \int_t^\infty Le^{-(s-t)L} F_s\, ds,$$

where $F \in L^2(\mathbb{R}_+; \boldsymbol{D}(L))$, and they use the splitting

$$\boldsymbol{M}^-_L(F)_t = \int_t^{2t} Le^{-(s-t)L} F_s\, ds + \int_{2t}^\infty Le^{-(s-t)L} F_s\, ds =: \text{III} + \text{IV}.$$

We only need to use those arguments in proving [Auscher et al. 2012a, Proposition 3.7] with IV involved since $s - t \sim s$ when $s > 2t$, which is equivalent to $s + t \sim s$ when $s > t$ in our setting. We omit the details. □

Now we use the L-associated Hardy spaces, which we recalled in Section 2, to treat the trace part $\boldsymbol{I}_L$. First, from the conical square function estimates [Hofmann et al. 2011, Proposition 4.9], one has, for $n/(n+1) < p \le 2$,

$$\left\| \sqrt{L}e^{-tL} \int_0^\infty \sqrt{L}e^{-sL} F_s\, ds \right\|_{T^p_{\text{par}}} \lesssim \left\| \int_0^\infty \sqrt{L}e^{-sL} F_s\, ds \right\|_{H^p_L}$$

for $F \in L^2(\mathbb{R}_+; \boldsymbol{D}(L))$. Next, from the reverse Riesz inequalities [Hofmann et al. 2011, Proposition 5.17], one has, for $p \in ((p_-)_*, 2]$,

$$\|\sqrt{L} f\|_{H^p_L} \lesssim \|\nabla f\|_{H^p}$$

for $f \in L^2$; hence, one further has, for $p \in ((p_-)_*, 2]$,

$$\left\| \int_0^\infty \sqrt{L}e^{-sL} F_s\, ds \right\|_{H^p_L} \lesssim \left\| \int_0^\infty \nabla e^{-sL} F_s\, ds \right\|_{H^p}.$$

Here, as usual, we use the convention $H^p = L^p$ for $p > 1$.[3]

For $F \in T^2_{\text{par}}$, consider the sweeping operator

$$\pi_L(F) := \int_0^\infty \nabla e^{-sL} F_s\, ds.$$

An equivalent formulation of the Kato square root estimate for L^* [Auscher et al. 2002] is the square function estimate

$$\iint_{\mathbb{R}^{1+n}_+} |e^{-tL^*} \operatorname{div} \vec{F}(y)|^2\, dt\, dy \lesssim \|\vec{F}\|_2^2$$

[2] As we will see in the proof, the lemma also holds for any $0 < p \le 2$. But that does not help in proving Theorem 1.1.

[3] We remark that in [Auscher and Frey 2015, Lemma 5.21] a variant of $\boldsymbol{I}_L$ is treated in a similar way, with informative connections to the Hardy space theory associated with the first-order perturbed Dirac operators as alluded to in Remark 1.3.

for all $\vec{F} \in L^2(\mathbb{R}^n; \mathbb{C}^n)$; hence, the mapping given by

$$\mathbb{Q}_{L^*} : \vec{F} \mapsto \mathbb{Q}_{L^*}(\vec{F})(t, y) := (e^{-tL^*} \operatorname{div} \vec{F})(y)$$

is bounded from $L^2(\mathbb{R}^n; \mathbb{C}^n)$ to T^2_{par}. Thereby, we see that $\pi_L : T^2_{\text{par}} \to L^2$ is a bounded operator by duality with $\mathbb{Q}_{L^*}$.

Recall that a T^p_{par}-atom A supported in the parabolic Carleson cylinder

$$\text{Cyl}(B) := (0, r_B^2) \times B$$

for some ball $B \subset \mathbb{R}^n$ (with radius r_B) satisfies the size estimate

$$\|A\|_{T^2_{\text{par}}} \leq |B|^{-(1/p-1/2)}. \tag{12}$$

We have the following result on π_L.

Lemma 3.3. *For any $n/(n+1) < p \leq 1$ and any T^p_{par}-atom A with* supp $A \subset \text{Cyl}(B)$ *for some ball $B \subset \mathbb{R}^n$ (with radius r_B),*

$$m := \pi_L(A) = \int_0^{r_B^2} \nabla e^{-sL} A_s \, ds$$

satisfies the uniform estimate

$$\|m\|_{H^p} \lesssim 1. \tag{13}$$

Hence, π_L extends to a bounded operator from T^p_{par} to H^p for $n/(n+1) < p \leq 2$.

Proof. For $m = \pi_L(A)$ with A being T^p_{par}-atoms, $n/(n+1) < p \leq 1$, and by adapting [Coifman et al. 1983, Théorème 3; 1985, Theorem 6], (13) follows from the L^2-L^2 off-diagonal decay for the heat semigroup $\{e^{-sL}\}_{s>0}$ and the gradient family $\{\sqrt{s}\nabla e^{-sL}\}_{s>0}$, the size estimate (12) and the Coifman–Weiss molecular theory for H^p. Then for $n/(n+1) < p \leq 1$, π_L extends to a bounded operator from T^p_{par} to H^p, and by interpolation, π_L extends to a bounded operator from T^p_{par} to H^p for $n/(n+1) < p \leq 2$. □

With the splittings (8) and (10), together with the conditions $\ell \in 2\mathbb{N}_+$ and $\frac{1}{2}\ell > \frac{1}{2} + \frac{1}{4}n$, and using Lemmata 3.1, 3.2 and 3.3 in order, the proof of Theorem 1.1 (with $p \in ((p_-)_*, 2])$ is then concluded.

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