ANALYSIS & PDE

Volume 10 No. 5 2017

NORIYOSHI FUKAYA, MASAYUKI HAYASHI AND TAKAHISA INUI

A SUFFICIENT CONDITION FOR GLOBAL EXISTENCE OF SOLUTIONS TO A GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

msp

ANALYSIS AND PDE
Vol. 10, No. 5, 2017
dx.doi.org/10.2140/apde.2017.10.1149

A SUFFICIENT CONDITION FOR GLOBAL EXISTENCE OF SOLUTIONS TO A GENERALIZED DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

NORIYOSHI FUKAYA, MASAYUKI HAYASHI AND TAKAHISA INUI

We give a sufficient condition for global existence of the solutions to a generalized derivative nonlinear Schrödinger equation (gDNLS) by a variational argument. The variational argument is applicable to a cubic derivative nonlinear Schrödinger equation (DNLS). For (DNLS), Wu (2015) proved that the solution with the initial data u_0 is global if $\|u_0\|_{L^2}^2 < 4\pi$ by the sharp Gagliardo–Nirenberg inequality. The variational argument gives us another proof of the global existence for (DNLS). Moreover, by the variational argument, we can show that the solution to (DNLS) is global if the initial data u_0 satisfies $\|u_0\|_{L^2}^2 = 4\pi$ and the momentum $P(u_0)$ is negative.

1. Introduction 1149
2. Variational characterization of the solitary waves 1157
3. Global existence 1162
Appendix: Uniqueness and nonexistence 1164
Acknowledgements 1165
References 1166

1. Introduction

1A. *Background.* The following equation is known as a derivative nonlinear Schrödinger equation:

$$i\partial_t v + \partial_x^2 v + i\partial_x(|v|^2 v) = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{1-1}$$

This equation appears in plasma physics [Mio et al. 1976; Mjølhus 1976] and as a model for ultrashort optical pulses [Moses et al. 2007]. Using the gauge transformation

$$u(t, x) = v(t, x) \exp\left(\frac{i}{2}\int_{-\infty}^{x} |v(t, x)|^2 \, dx\right),$$

we get a Hamiltonian form of (1-1):

$$i\partial_t u + \partial_x^2 u + i|u|^2\partial_x u = 0, \qquad (t, x) \in \mathbb{R} \times \mathbb{R}. \tag{DNLS}$$

Namely, this equation can be written as $i\partial_t u = E'(u)$ (see below for the definition of the Hamiltonian E). The Cauchy problem for (DNLS) (or equivalently (1-1)) has been studied by many researchers. It is known that (DNLS) is locally well-posed in the energy space $H^1(\mathbb{R})$. See [Tsutsumi and Fukuda 1980; Hayashi

MSC2010: 35Q55.
Keywords: variational structure, generalized derivative nonlinear Schrödinger equation, global existence.

and Ozawa 1992; Hayashi 1993; Hayashi and Ozawa 1994a; 1994b]. Hayashi and Ozawa [1994a] proved that the solution is global if $\|u_0\|_{L^2}^2 < 2\pi$. See also [Ozawa 1996]. Wu [2013; 2015] proved that it holds if $\|u_0\|_{L^2}^2 < 4\pi$. Recently, Miao, Tang, and Xu obtained the global well-posedness by a variational argument (see the remark on page 1156). For the initial data with low regularity, there are also many references. Takaoka [1999] proved that (DNLS) is locally well-posed in $H^s(\mathbb{R})$ when $s \geq \frac{1}{2}$ by the Fourier restricted method. Biagioni and Linares [2001] proved that the solution map from $H^s(\mathbb{R})$ to $C([-T, T] : H^s(\mathbb{R}))$, where $T > 0$, for (DNLS) is not locally uniformly continuous when $s < \frac{1}{2}$. Colliander, Keel, Staffilani, Takaoka, and Tao [Colliander et al. 2002] proved that the H^s-solution is global if $\|u_0\|_{L^2}^2 < 2\pi$ when $s > \frac{1}{2}$ by the I-method (see also [Colliander et al. 2001; Takaoka 2001]). Recently, Miao, Wu, and Xu [Miao et al. 2011] showed that $H^{1/2}$-solution is global if $\|u_0\|_{L^2}^2 < 2\pi$. Guo and Wu [2017] improved their result; that is, they proved that $H^{1/2}$-solution is global if $\|u_0\|_{L^2}^2 < 4\pi$. The orbital stability of solitary waves has been also studied. It is known that (DNLS) has a two-parameter family of the solitary waves $u_{\omega,c}(t, x) = e^{i\omega t}\phi_{\omega,c}(x - ct)$, where (ω, c) satisfies $\omega > c^2/4$, or $\omega = c^2/4$ and $c > 0$ (see below for the explicit formula of $\phi_{\omega,c}$). Guo and Wu [1995] proved that the solitary waves $u_{\omega,c}$ are orbitally stable when $\omega > c^2/4$ and $c < 0$ by the abstract theory of Grillakis, Shatah, and Strauss [Grillakis et al. 1987; 1990] and the spectral analysis of the linearized operators. Colin and Ohta [2006] proved that the solitary waves $u_{\omega,c}$ are orbitally stable when $\omega > c^2/4$ by characterizing the solitary waves from the viewpoint of a variational structure. The case of $\omega = c^2/4$ and $c > 0$ was treated by Kwon and Wu [2016]. Recently, the stability of the multisolitons was studied by Miao, Tang, and Xu [Miao et al. 2017b] and Le Coz and Wu [2016].

To understand the structural properties of (DNLS), Liu, Simpson, and Sulem [Liu et al. 2013] introduced an extension of (DNLS) with general power nonlinearity. The generalized derivative nonlinear Schrödinger equation is

$$\begin{cases} i\partial_t u + \partial_x^2 u + i|u|^{2\sigma}\partial_x u = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ u(0, x) = u_0(x), & x \in \mathbb{R}, \end{cases} \tag{gDNLS}$$

where $\sigma > 0$. Equation (gDNLS) is invariant under the scaling transformation

$$u_\gamma(t, x) := \gamma^{1/(2\sigma)} u(\gamma^2 t, \gamma x), \qquad \gamma > 0.$$

This implies that its critical Sobolev exponent is $s_c = \frac{1}{2} - 1/(2\sigma)$. In particular, (DNLS) is L^2-critical. Liu et al. [2013] investigated the orbital stability of a two-parameter family of solitary waves

$$u_{\omega,c}(t, x) = e^{i\omega t}\phi_{\omega,c}(x - ct),$$

where (ω, c) satisfies $\omega > c^2/4$, or $\omega = c^2/4$ and $c > 0$, and

$$\phi_{\omega,c}(x) = \Phi_{\omega,c}(x) \exp\left(i\frac{c}{2}x - \frac{i}{2\sigma + 2}\int_0^x \Phi_{\omega,c}(y)^{2\sigma}\, dy\right), \tag{1-2}$$

$$\Phi_{\omega,c}(x) = \begin{cases} \left\{ \dfrac{(\sigma + 1)(4\omega - c^2)}{2\sqrt{\omega}\cosh(\sigma\sqrt{4\omega - c^2}x) - c} \right\}^{1/(2\sigma)} & \text{if } \omega > c^2/4, \\ \left\{ \dfrac{2(\sigma + 1)c}{\sigma^2(cx)^2 + 1} \right\}^{1/(2\sigma)} & \text{if } \omega = c^2/4 \text{ and } c > 0. \end{cases} \tag{1-3}$$

We note that $\Phi_{\omega,c}$ is the positive even solution of

$$-\Phi'' + (\omega - \tfrac{1}{4}c^2)\Phi + \tfrac{1}{2}c|\Phi|^{2\sigma}\Phi - \frac{2\sigma+1}{(2\sigma+2)^2}|\Phi|^{4\sigma}\Phi = 0, \quad x \in \mathbb{R}, \tag{1-4}$$

and then the complex-valued function $\phi_{\omega,c}$ satisfies

$$-\phi'' + \omega\phi + ic\phi' - i|\phi|^{2\sigma}\phi' = 0, \quad x \in \mathbb{R}.$$

Liu et al. [2013] proved that the solitary waves are orbitally stable if $-2\sqrt{\omega} < c < 2z_0\sqrt{\omega}$, and orbitally unstable if $2z_0\sqrt{\omega} < c < 2\sqrt{\omega}$ when $1 < \sigma < 2$, where the constant $z_0 = z_0(\sigma) \in (-1, 1)$ is the solution of

$$F_\sigma(z) := (\sigma-1)^2\left\{\int_0^\infty (\cosh y - z)^{-1/\sigma}\,dy\right\}^2 - \left\{\int_0^\infty (\cosh y - z)^{-1/\sigma-1}(z\cosh y - 1)\,dy\right\}^2 = 0.$$

Moreover, they also proved that the solitary waves for all $\omega > c^2/4$ are orbitally unstable when $\sigma \geq 2$ and orbitally stable when $0 < \sigma < 1$. Recently, Fukaya [2016] proved that the solitary waves are orbitally unstable if $c = 2z_0\sqrt{\omega}$ when $\frac{7}{6} < \sigma < 2$. More recently, Tang and Xu investigated stability of the sum of two solitary waves for (gDNLS) (see [Tang and Xu 2017] for more details). Before Liu et al. [2013], Hao [2007] considered (gDNLS) and proved the local well-posedness in $H^{1/2}(\mathbb{R})$ when $\sigma \geq \frac{5}{2}$. Santos [2015] proved the existence and uniqueness of a solution $u \in C([0, T]; H^{1/2}(\mathbb{R}))$ for sufficiently small initial data when $\sigma > 1$. Recently, Hayashi and Ozawa [2016] proved local well-posedness in $H^1(\mathbb{R})$ when $\sigma \geq 1$ and that the following quantities are conserved:

$$E(u) := \tfrac{1}{2}\|\partial_x u\|_{L^2}^2 - \frac{1}{2\sigma+2}\operatorname{Re}\int_{\mathbb{R}} i|u|^{2\sigma}\bar{u}\partial_x u\,dx, \tag{Energy}$$

$$M(u) := \|u\|_{L^2}^2, \tag{Mass}$$

$$P(u) := \operatorname{Re}\int_{\mathbb{R}} i\partial_x u\bar{u}\,dx. \tag{Momentum}$$

Moreover, they proved global well-posedness for small initial data. They also constructed global solutions for any initial data in $H^1(\mathbb{R})$ in the case $0 < \sigma < 1$ (L^2-subcritical case). However, in the case $\sigma \geq 1$ (L^2-critical or supercritical case), there has been no global existence result for large data. In the present paper, we investigate global well-posedness for (gDNLS) in the case $\sigma \geq 1$ by a variational argument. More precisely, we give a variational characterization of solitary waves and a sufficient condition for global existence of solutions to (gDNLS) by using the characterization. Such an argument was done for nonlinear hyperbolic partial differential equations by Sattinger [1968] (see also [Tsutsumi 1972; Payne and Sattinger 1975]). Our argument is also applicable to (DNLS). Indeed, the variational argument gives another proof of the result by Wu [2015]. Moreover, we prove that the solution of (DNLS) is global if the initial data u_0 satisfies $\|u_0\|_{L^2}^2 = 4\pi$ and $P(u_0) < 0$.

1B. ***Main results.*** To state our main results, we introduce some notations. Let (ω, c) satisfy

$$\omega > c^2/4 \quad \text{or} \quad \omega = c^2/4 \text{ and } c > 0. \tag{1-5}$$

For (ω, c) satisfying (1-5), we define

$$S_{\omega,c}(\varphi) := E(\varphi) + \tfrac{1}{2}\omega M(\varphi) + \tfrac{1}{2}cP(\varphi).$$

We denote the nonlinear term by

$$N(\varphi) := \mathrm{Re}\int_{\mathbb{R}} i|\varphi|^{2\sigma}\bar{\varphi}\partial_x\varphi\,dx.$$

We define

$$\widetilde{S}_{\omega,c}(\psi) := \tfrac{1}{2}\|\partial_x\psi\|_{L^2}^2 + \tfrac{1}{2}(\omega - \tfrac{1}{4}c^2)\|\psi\|_{L^2}^2 + \frac{c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - \frac{1}{2\sigma+2}N(\psi).$$

Then, we have $S_{\omega,c}(\varphi) = \widetilde{S}_{\omega,c}(e^{-(c/2)ix}\varphi)$ by using the identities

$$cP(\varphi) = -\|\partial_x\varphi\|_{L^2}^2 - \tfrac{1}{4}c^2\|\varphi\|_{L^2}^2 + \|\partial_x(e^{-(c/2)ix}\varphi)\|_{L^2}^2, \tag{1-6}$$

$$N(\varphi) = -\tfrac{1}{2}c\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2} + N(e^{-(c/2)ix}\varphi). \tag{1-7}$$

We denote the scaling transformation by $f_\lambda^{\alpha,\beta}(x) := e^{\alpha\lambda}f(e^{-\beta\lambda}x)$ for $(\alpha,\beta)\in\mathbb{R}^2$ and any function f. For $(\alpha,\beta)\in\mathbb{R}^2$, we define

$$\widetilde{K}_{\omega,c}^{\alpha,\beta}(\psi) := \partial_\lambda\widetilde{S}_{\omega,c}(\psi_\lambda^{\alpha,\beta})|_{\lambda=0},$$
$$K_{\omega,c}^{\alpha,\beta}(\varphi) := \widetilde{K}_{\omega,c}^{\alpha,\beta}(e^{-(c/2)ix}\varphi).$$

By a direct calculation, we have the explicit formulae

$$\begin{aligned}
\widetilde{K}_{\omega,c}^{\alpha,\beta}(\psi) &= \langle \widetilde{S}'_{\omega,c}(\psi), \alpha\psi - \beta x\partial_x\psi\rangle \\
&= \frac{2\alpha-\beta}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{2\alpha+\beta}{2}\left(\omega - \frac{c^2}{4}\right)\|\psi\|_{L^2}^2 + \frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - \alpha N(\psi), \\
K_{\omega,c}^{\alpha,\beta}(\varphi) &= \langle \widetilde{S}'_{\omega,c}(e^{-(c/2)ix}\varphi), \alpha e^{-(c/2)ix}\varphi - \beta x\partial_x(e^{-(c/2)ix}\varphi)\rangle \\
&= \langle S'_{\omega,c}(\varphi), \alpha\varphi + \tfrac{1}{2}ci\beta x\varphi - \beta x\partial_x\varphi\rangle \\
&= \frac{2\alpha-\beta}{2}\|\partial_x\varphi\|_{L^2}^2 + \left(\frac{2\alpha+\beta}{2}\omega - \frac{c^2}{4}\beta\right)\|\varphi\|_{L^2}^2 + \frac{2\alpha-\beta}{2}cP(\varphi) + \frac{\beta c}{2(2\sigma+2)}\|\varphi\|_{L^{2\sigma+2}}^{2\sigma+2} - \alpha N(\varphi),
\end{aligned}$$

where we have used (1-6) and (1-7).

Remark. (1) If $\beta \neq 0$, then $K_{\omega,c}^{\alpha,\beta}$ is different from $I_{\omega,c}^{\alpha,\beta}(\varphi) := \partial_\lambda S_{\omega,c}(\varphi_\lambda^{\alpha,\beta})|_{\lambda=0}$. Indeed, the explicit formula of $I_{\omega,c}^{\alpha,\beta}$ is

$$I_{\omega,c}^{\alpha,\beta}(\varphi) = \frac{2\alpha-\beta}{2}\|\partial_x\varphi\|_{L^2}^2 + \frac{2\alpha+\beta}{2}\omega\|\varphi\|_{L^2}^2 + c\alpha P(\varphi) - \alpha N(\varphi).$$

We note that $K_{\omega,c}^{\alpha,0}$ coincides with $I_{\omega,c}^{\alpha,0}$, and especially $K_{\omega,c}^{1,0} = I_{\omega,c}^{1,0}$ is nothing but the Nehari functional.

(2) It is not clear whether the momentum P is positive or not. That is why we introduce $\widetilde{S}_{\omega,c}$ by using (1-6). Such an argument can be seen in [Bellazzini et al. 2014b] (see (14) therein for the details).

(3) The functional $K_{\omega,c}^{\alpha,\beta}$ is more useful to obtain the characterization of the solitary waves when $\omega = c^2/4$ and $c > 0$ than $I_{\omega,c}^{\alpha,\beta}$ since $K_{\omega,c}^{\alpha,\beta}$ contains the $L^{2\sigma+2}$-norm (see the proof in Section 2B).

(4) $\widetilde{S}_{\omega,c}$ and $\widetilde{K}^{\alpha,\beta}_{\omega,c}$ are relevant to the elliptic equation

$$-\psi'' + (\omega - \tfrac{1}{4}c^2)\psi + \tfrac{1}{2}c|\psi|^{2\sigma}\psi - i|\psi|^{2\sigma}\psi' = 0, \quad x \in \mathbb{R}.$$

We define the following function space for (ω, c) satisfying (1-5):

$$X_{\omega,c} := \begin{cases} H^1(\mathbb{R}) & \text{if } \omega > c^2/4, \\ \dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R}) & \text{if } \omega = c^2/4 \text{ and } c > 0. \end{cases}$$

We consider the following minimization problem:

$$\begin{aligned} \mu^{\alpha,\beta}_{\omega,c} &:= \inf\{S_{\omega,c}(\varphi) : e^{-(c/2)ix}\varphi \in X_{\omega,c} \setminus \{0\},\ K^{\alpha,\beta}_{\omega,c}(\varphi) = 0\} \\ &= \inf\{\widetilde{S}_{\omega,c}(\psi) : \psi \in X_{\omega,c} \setminus \{0\},\ \widetilde{K}^{\alpha,\beta}_{\omega,c}(\psi) = 0\}. \end{aligned}$$

Remark. (1) We note that the solitary waves $\phi_{c^2/4,c}$ do not belong to $L^2(\mathbb{R})$ when $\sigma \geq 2$. Therefore, we define $X_{c^2/4,c} := \dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R})$ to characterize the solitary waves $\phi_{c^2/4,c}$ (cf. [Kwon and Wu 2016]).

(2) $S_{c^2/4,c}$ seems meaningless on the function space $\{\varphi : e^{-(c/2)ix}\varphi \in X_{c^2/4,c}\}$ since $S_{c^2/4,c}$ contains L^2-norm. However, in fact, $S_{c^2/4,c}$ is well-defined on the function space since $\widetilde{S}_{c^2/4,c}$ is defined on $\dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R})$ and the equality $S_{c^2/4,c}(\varphi) = \widetilde{S}_{c^2/4,c}(e^{-(c/2)ix}\varphi)$ holds. Similarly, $K^{\alpha,\beta}_{c^2/4,c}$ is well-defined on this function space.

(3) Since $\varphi \in H^1(\mathbb{R})$ if and only if $e^{-(c/2)ix}\varphi \in H^1(\mathbb{R})$, when $\omega > c^2/4$, we have

$$\mu^{\alpha,\beta}_{\omega,c} = \inf\{S_{\omega,c}(\varphi) : \varphi \in H^1(\mathbb{R}) \setminus \{0\},\ K^{\alpha,\beta}_{\omega,c}(\varphi) = 0\}.$$

However, when $\omega = c^2/4$ and $c > 0$, the above equality does not hold.

We assume that $(\alpha, \beta) \in \mathbb{R}^2$ satisfies

$$\begin{cases} 2\alpha - \beta > 0,\ 2\alpha + \beta > 0, \text{ and } \beta c \leq 0 & \text{when } \omega > c^2/4, \\ 2\alpha - \beta > 0,\ 2\alpha + \beta > 0, \text{ and } \beta < 0 & \text{when } \omega = c^2/4 \text{ and } c > 0. \end{cases} \tag{1-8}$$

We define some function spaces:

$$\begin{aligned} \mathscr{M}^{\alpha,\beta}_{\omega,c} &:= \{\varphi : e^{-(c/2)ix}\varphi \in X_{\omega,c} \setminus \{0\},\ S_{\omega,c}(\varphi) = \mu^{\alpha,\beta}_{\omega,c},\ K^{\alpha,\beta}_{\omega,c}(\varphi) = 0\}, \\ \mathscr{G}_{\omega,c} &:= \{\varphi : e^{-(c/2)ix}\varphi \in X_{\omega,c} \setminus \{0\},\ S'_{\omega,c}(\varphi) = 0\}. \end{aligned}$$

We give the following characterization of the solitary waves.

Theorem 1.1. *Let $\sigma \geq 1$, (ω, c) satisfy (1-5), and (α, β) satisfy (1-8). Then,*

$$\mathscr{M}^{\alpha,\beta}_{\omega,c} = \mathscr{G}_{\omega,c} = \{e^{i\theta_0}\phi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi),\ x_0 \in \mathbb{R}\}.$$

Theorem 1.1 also means that $\mu^{\alpha,\beta}_{\omega,c}$ and $\mathscr{M}^{\alpha,\beta}_{\omega,c}$ are independent of (α, β) and $\mathscr{M}^{\alpha,\beta}_{\omega,c}$ is not empty. Thus, we denote $\mu^{\alpha,\beta}_{\omega,c}$ by $\mu_{\omega,c}$.

We define

$$\begin{aligned} \mathscr{K}^{\alpha,\beta,+}_{\omega,c} &:= \{\varphi \in H^1(\mathbb{R}) : S_{\omega,c}(\varphi) \leq \mu_{\omega,c},\ K^{\alpha,\beta}_{\omega,c}(\varphi) \geq 0\}, \\ \mathscr{K}^{\alpha,\beta,-}_{\omega,c} &:= \{\varphi \in H^1(\mathbb{R}) : S_{\omega,c}(\varphi) \leq \mu_{\omega,c},\ K^{\alpha,\beta}_{\omega,c}(\varphi) < 0\}. \end{aligned}$$

The characterization by Theorem 1.1 gives us the following sufficient condition for global existence.

Theorem 1.2. *Let $\sigma \geq 1$, (ω, c) satisfy (1-5), and (α, β) satisfy (1-8). Then, $\mathscr{K}^{\alpha,\beta,\pm}_{\omega,c}$ are invariant under the flow of (gDNLS). Namely, if the initial data u_0 belongs to $\mathscr{K}^{\alpha,\beta,\pm}_{\omega,c}$, then the solution $u(t)$ of (gDNLS) also belongs to $\mathscr{K}^{\alpha,\beta,\pm}_{\omega,c}$ for all $t \in I_{\max}$, where $I_{\max}$ denotes the maximal existence time.*

Moreover, if the initial data u_0 belongs to $\mathscr{K}^{\alpha,\beta,+}_{\omega,c}$ for some (ω, c) satisfying (1-5) and (α, β) satisfying (1-8), then the corresponding solution u of (gDNLS) exists globally in time and

$$\|u\|_{L^\infty(\mathbb{R}:H^1(\mathbb{R}))} \leq C(\|u_0\|_{H^1}),$$

where $C : [0, \infty) \to \mathbb{R}$ is continuous.

Recently, Miao et al. [2017a] independently obtained the results similar to Theorems 1.1 and 1.2 when $\sigma = 1$. We will compare their method with our argument in the remark on page 1156.

We show that Theorem 1.2 gives us some interesting corollaries for (DNLS).

Corollary 1.3. *Let $\sigma = 1$. If the initial data $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0\|^2_{L^2} < 4\pi$, then the solution of (DNLS) is global.*

Two proofs have been known for Corollary 1.3. One was obtained by Wu [2015] and another one by Guo and Wu [2017]. We give another proof by Theorem 1.2. We compare the methods of [Wu 2015; Guo and Wu 2017], which depend on the sharp Gagliardo–Nirenberg-type inequality, with our variational argument. Using the gauge transformation to the solution of (DNLS)

$$u(t, x) = w(t, x) \exp\left(-\frac{i}{4} \int_{-\infty}^{x} |w(t, x)|^2 \, dx\right), \tag{1-9}$$

then w satisfies the equation

$$\begin{cases} i\partial_t w + \partial_x^2 w + \frac{1}{2} i |w|^2 \partial_x w - \frac{1}{2} i w^2 \partial_x \overline{w} + \frac{3}{16} |w|^4 w = 0, & (t, x) \in \mathbb{R} \times \mathbb{R}, \\ w(0, x) = w_0(x), & x \in \mathbb{R}. \end{cases} \tag{1-10}$$

The energy and the momentum are transformed as

$$\begin{aligned} \mathcal{E}(w) &= \tfrac{1}{2}\|\partial_x w\|^2_{L^2} - \tfrac{1}{32}\|w\|^6_{L^6}, \\ \mathcal{P}(w) &= \operatorname{Re} \int_{\mathbb{R}} i \partial_x w \overline{w} \, dx + \tfrac{1}{4}\|w\|^4_{L^4}. \end{aligned}$$

Hayashi and Ozawa [1992] used the sharp Gagliardo–Nirenberg inequality

$$\|f\|^6_{L^6} \leq \frac{4}{\pi^2} \|f\|^4_{L^2} \|\partial_x f\|^2_{L^2} \tag{1-11}$$

in order to obtain an a priori estimate in $\dot{H}^1(\mathbb{R})$. We note that the optimizer for the inequality (1-11) is given by $Q := \Phi_{1,0}$ and Q satisfies the elliptic equation

$$-Q'' + Q - \tfrac{3}{16} Q^5 = 0. \tag{1-12}$$

Hayashi and Ozawa [1992] proved the H^1-solution of (DNLS) is global if the initial data u_0 satisfies $\|u_0\|^2_{L^2} = \|w_0\|^2_{L^2} < \|Q\|^2_{L^2} = 2\pi$ (see also [Weinstein 1982]). Wu [2015] used not only the energy but also the momentum and the sharp Gagliardo–Nirenberg inequality

$$\|f\|_{L^6}^6 \le 3(2\pi)^{-2/3}\|f\|_{L^4}^{16/3}\|\partial_x f\|_{L^2}^{2/3}. \tag{1-13}$$

We note that the optimizer for the inequality (1-13) is given by $W := \Phi_{1/4,1}$ and W satisfies the elliptic equation

$$-W'' + \tfrac{1}{2}W^3 - \tfrac{3}{16}W^5 = 0. \tag{1-14}$$

Wu [2015] proved that the H^1-solution of (DNLS) is global if the initial data u_0 satisfies $\|u_0\|_{L^2}^2 = \|w_0\|_{L^2}^2 < \|W\|_{L^2}^2 = 4\pi$. His proof depends on a contradiction argument. Supposing that there exists a time sequence $\{t_n\}_{n\in\mathbb{N}}$ with $t_n \to T_{\max}$ or $-T_{\min}$ such that $\|\partial_x w(t_n)\|_{L^2} \to \infty$ as $n \to \infty$, where $(-T_{\min}, T_{\max})$ is the maximal time interval, he mainly proved that $X = \|w(t_n)\|_{L^4}^8/\|w(t_n)\|_{L^6}^6$ satisfies $X^3 - \|w\|_{L^2}^2 X^2 + 16\{3(2\pi)^{-2/3}\}^{-3}\|w\|_{L^2}^2 < 0$, but this does not hold when $\|w\|_{L^2}^2 < 4\pi$. On the other hand, Guo and Wu [2017] gave an a priori estimate directly for (1-10) by the sharp Gagliardo–Nirenberg inequality (1-13). More precisely, they showed in [Guo and Wu 2017, Lemma 2.1] the inequality

$$\mathcal{P}(w) \ge \tfrac{1}{4}\|w\|_{L^4}^4\left(1 - \frac{\|w\|_{L^2}}{2\sqrt{\pi}}\right) - \frac{8\sqrt{\pi}\,\mathcal{E}(w)\|w\|_{L^2}}{\|w\|_{L^4}^4}, \tag{1-15}$$

and thus, $\|\partial_x w\|_{L^2}^2$ is bounded by $\mathcal{P}$ and $\mathcal{E}$ if $\|w\|_{L^2}^2 < 4\pi$ [Guo and Wu 2017, Lemma 2.2]. In our variational argument, we do not use a contradiction argument, the gauge transformation like (1-9), or any sharp Gagliardo–Nirenberg inequality.

We give the global existence result in the threshold case by Theorem 1.2.

Corollary 1.4. *Let $\sigma = 1$. We assume that the initial data $u_0 \in H^1(\mathbb{R})$ satisfies $\|u_0\|_{L^2}^2 = 4\pi$. If $P(u_0) < 0$, then the solution of* (DNLS) *is global.*

After submitting the present paper, Guo pointed out that Corollary 1.4 can be obtained by (1-15). We also give the proof by (1-15) for the reader's convenience.

The following corollary means that there exist global solutions with any large mass.

Corollary 1.5. *Let $\sigma \ge 1$. Given $\psi \in H^1(\mathbb{R})$, set the initial data as $u_{0,c} = e^{(c/2)ix}\psi$. Then there exists $c_0 > 0$ such that, if $c \ge c_0$, then the corresponding solution u_c of* (gDNLS) *is global.*

Remark. The existence of blow-up solutions in finite time is still an open problem. It might be a very interesting problem whether finite-time blow-up occurs when the initial data u_0 satisfies $\|u_0\|_{L^2}^2 = 4\pi$ and $P(u_0) > 0$.

1C. ***Compare DNLS with mass-critical NLS.*** Equation (DNLS) is L^2-critical in the sense that the equation and L^2-norm are invariant under the scaling transformation

$$u_\gamma(t,x) := \gamma^{1/2}u(\gamma^2 t, \gamma x), \quad \gamma > 0.$$

The same invariance holds for the quintic nonlinear Schrödinger equation in one-dimensional space:

$$i\partial_t u + \partial_x^2 u + \tfrac{3}{16}|u|^4 u = 0, \quad (t,x) \in \mathbb{R}\times\mathbb{R}. \tag{1-16}$$

This equation has the same energy as (1-10). It is known that (1-16) is locally well-posed in the energy space $H^1(\mathbb{R})$ and the solution is global if the initial data u_0 satisfies $\|u_0\|_{L^2}^2 < \|Q\|_{L^2}^2$, where Q is the ground state of the same elliptic equation (1-12). The condition $\|u_0\|_{L^2}^2 < \|Q\|_{L^2}^2$ is equivalent to the condition obtained by the variational argument. In this argument, the momentum is not essential since (1-16) is invariant under the Galilean transformation, and thus, we may assume that the momentum is zero. On the other hand, (DNLS) is not invariant under the Galilean transformation. Therefore, the condition by the variational argument is better than the assumption $\|u_0\|_{L^2}^2 < \|W\|_{L^2}^2 = 4\pi$. Indeed, the momentum and the parameter c play important roles in Corollaries 1.4 and 1.5.

1D. ***Idea of proofs.*** The proof of Theorem 1.1 is based on the method of Colin and Ohta [2006] (concentration compactness method). They characterized the solitary waves for $\omega > c^2/4$ when $\sigma = 1$ by the Nehari functional $I_{\omega,c}^{1,0}$. However, in the case $\omega = c^2/4$ and $c > 0$, we cannot apply their argument directly since the L^2-norm in $I_{\omega,c}^{1,0}$ disappears by (1-6). Therefore, we introduce the new functional $K_{\omega,c}^{\alpha,\beta}$ for (α, β) satisfying (1-8). We can use the $L^{2\sigma+2}$-norm instead of the L^2-norm by using $K_{\omega,c}^{\alpha,\beta}$. That is why we introduce the function space $X_{\omega,c}$ as $\dot{H}^1 \cap L^{2\sigma+2}$ in the massless case (i.e., $\omega = c^2/4$ and $c > 0$). Noting that the solitary waves $\phi_{c^2/4,c}$ do not belong to $L^2(\mathbb{R})$ when $\sigma \geq 2$, the function space $X_{\omega,c}$ is essential to obtain the characterization of the solitary waves $\phi_{c^2/4,c}$. Based on the argument of Colin and Ohta [2006], we characterize the solitary waves $\phi_{c^2/4,c}$ by $K_{\omega,c}^{\alpha,\beta}$. By the conservation laws and the characterization of the solitary waves, we get an a priori estimate and thus obtain Theorem 1.2. The corollaries follow from Theorem 1.2. In their proofs, the parameter c plays an important role. More precisely, taking $c > 0$ large, we get the corollaries. At last, we emphasize that we do not use the sharp Gagliardo–Nirenberg inequality and we do not apply the gauge transformation to (gDNLS) since the equation after applying the transformation is complicated unlike (DNLS).

Remark. Miao et al. [2017a] treated the case of $\sigma = 1$. They considered (1-10) by using the gauge transformation and defined the action by $\mathcal{S}_{\omega,c} := \mathcal{E} + \omega M/2 + c\mathcal{P}/2$. They applied a concentration compactness argument to give the variational characterization of the solitary waves. Then, they use the Nehari functional $\mathcal{K}_{\omega,c}$ derived from the action $\mathcal{S}_{\omega,c}$. The explicit formula of $\mathcal{K}_{\omega,c}$ is

$$\mathcal{K}_{\omega,c}(w) := \|\partial_x w\|_{L^2}^2 - \tfrac{3}{16}\|w\|_{L^6}^6 + \omega\|w\|_{L^2}^2 + c\,\mathrm{Re}\int_{\mathbb{R}} i\partial_x w\overline{w}\,dx + \tfrac{1}{2}c\|w\|_{L^4}^4.$$

They defined

$$\mathscr{A}_{\omega,c}^{\pm} := \{\varphi \in H^1(\mathbb{R}) : \mathcal{S}_{\omega,c}(\varphi) \leq \mathcal{S}_{\omega,c}(\phi_{\omega,c}),\ \mathcal{K}_{\omega,c}(\varphi) \gtreqless 0\},$$

and they also showed that $\mathscr{A}_{\omega,c}^{\pm}$ are invariant under the flow of (1-10) and the solution to (1-10) is global if $w_0 \in \mathscr{A}_{\omega,c}^{+}$ for some (ω, c). The functional $\mathcal{K}_{\omega,c}$ is useful to characterize the solitary waves $\phi_{c^2/4,c}$ since it contains L^4-norm. Namely, one can use the Nehari functional by the gauge transformation. On the other hand, we cannot use the Nehari functional when we do not apply the gauge transformation, and thus, we introduce the new functionals $K_{\omega,c}^{\alpha,\beta}$.

The rest of the present paper is as follows. In Section 2A, we prepare some lemmas to obtain the characterization of the solitary waves and prove the a priori estimate (see (2-2)). In Section 2B, we give the characterization of the solitary waves $\phi_{c^2/4,c}$. We remark that the characterization of the solitary waves $\phi_{\omega,c}$ for $\omega > c^2/4$ can be obtained in the same manner as in [Colin and Ohta 2006], and then we omit the proof. Section 3 is devoted to the proof of Theorem 1.2 and the corollaries. In the Appendix, we show that there is no nontrivial solution of the nonlinear elliptic equation (1-4) if $\omega < c^2/4$, or $\omega = c^2/4$ and $c \le 0$.

2. Variational characterization of the solitary waves

2A. *Preliminaries.* We define function spaces

$$\widetilde{\mathcal{M}}_{\omega,c}^{\alpha,\beta} := \{\psi \in X_{\omega,c} \setminus \{0\} : \widetilde{S}_{\omega,c}(\psi) = \mu_{\omega,c}^{\alpha,\beta},\ \widetilde{K}_{\omega,c}^{\alpha,\beta}(\psi) = 0\},$$
$$\widetilde{\mathcal{G}}_{\omega,c} := \{\psi \in X_{\omega,c} \setminus \{0\} : \widetilde{S}'_{\omega,c}(\psi) = 0\}.$$

In this section, we prove the following proposition, which gives Theorem 1.1.

Proposition 2.1. *Let* (ω, c) *satisfy* (1-5) *and* (α, β) *satisfy* (1-8). *Then*

$$\widetilde{\mathcal{M}}_{\omega,c}^{\alpha,\beta} = \widetilde{\mathcal{G}}_{\omega,c} = \{e^{i\theta} e^{-(c/2)ix} \phi_{\omega,c}(\cdot - y) : \theta \in [0, 2\pi),\ y \in \mathbb{R}\}.$$

Indeed, Theorem 1.1 follows from Proposition 2.1 and the following properties:

$$\varphi \in \mathcal{M}_{\omega,c}^{\alpha,\beta} \iff e^{-(c/2)ix}\varphi \in \widetilde{\mathcal{M}}_{\omega,c}^{\alpha,\beta},$$
$$\varphi \in \mathcal{G}_{\omega,c} \iff e^{-(c/2)ix}\varphi \in \widetilde{\mathcal{G}}_{\omega,c},$$

where we note that $\widetilde{S}'_{\omega,c}(e^{-(c/2)ix}\varphi) = e^{-(c/2)ix} S'_{\omega,c}(\varphi)$ holds.

To prove Proposition 2.1, we prepare some basic lemmas. We have the Gagliardo–Nirenberg-type inequality.

Lemma 2.2. *Let* $p \ge 1$. *We have the estimate*

$$\|f\|_{L^\infty}^{2p} \le 2p \|f\|_{L^{4p-2}}^{2p-1} \|\partial_x f\|_{L^2}. \tag{2-1}$$

Proof. By the Hölder inequality,

$$\begin{aligned}
|f(x)|^{2p} &= \int_{-\infty}^{x} \frac{d}{dx}(|f(y)|^{2p})\, dy \\
&= \int_{-\infty}^{x} 2p|f(y)|^{2p-2} \operatorname{Re}(\overline{f(y)}(\partial_x f)(y))\, dy \\
&\le 2p \||f|^{2p-1}\|_{L^2} \|\partial_x f\|_{L^2} \\
&= 2p \|f\|_{L^{4p-2}}^{2p-1} \|\partial_x f\|_{L^2}.
\end{aligned}$$

Taking the supremum, we obtain (2-1). □

We have the Lieb compactness lemma. See [Lieb 1983] for $p = 2$ and [Bellazzini et al. 2014a, Lemma 2.1] for more general setting.

Lemma 2.3. *Let* $p \ge 2$ *and* $d \in \mathbb{N}$. *Let* $\{f_n\}$ *be a bounded sequence in* $\dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$. *Assume that there exists* $q \in (p, 2^*)$ *such that* $\limsup_{n\to\infty} \|f_n\|_{L^q} > 0$. *Then there exist* $\{y_n\}$ *and* $f \in \dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d) \setminus \{0\}$ *such that* $\{f_n(\cdot - y_n)\}$ *has a subsequence that converges to* f *weakly in* $\dot{H}^1(\mathbb{R}^d) \cap L^p(\mathbb{R}^d)$.

We have the Brézis–Lieb lemma [1983].

Lemma 2.4. *Let $d \in \mathbb{N}$ and $1 < p < \infty$. Let $\{f_n\}$ be a bounded sequence in $L^p(\mathbb{R}^d)$ and $f_n \to f$ a.e. in $\mathbb{R}^d$. Then*

$$\|f_n\|_{L^p}^p - \|f_n - f\|_{L^p}^p - \|f\|_{L^p}^p \to 0.$$

If $\{f_n\}$ is a bounded sequence in $L^2(\mathbb{R}^d)$ and f_n converges to f weakly in $L^2(\mathbb{R}^d)$, then the statement with $p = 2$ holds.

A direct calculation gives us the following relation.

Lemma 2.5. *We have*

$$\alpha(2\sigma+2)\widetilde{S}_{\omega,c}(\psi) = \widetilde{K}_{\omega,c}^{\alpha,\beta}(\psi) + \frac{2\sigma\alpha+\beta}{2}\|\partial_x\psi\|_{L^2}^2 + (\omega - \tfrac{1}{4}c^2)\frac{2\sigma\alpha-\beta}{2}\|\psi\|_{L^2}^2 - \frac{\beta c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2}. \tag{2-2}$$

We denote the difference $\alpha(2\sigma+2)\widetilde{S}_{\omega,c}(\psi) - \widetilde{K}_{\omega,c}^{\alpha,\beta}(\psi)$ by

$$\tilde{J}_{\omega,c}^{\alpha,\beta}(\psi) := \frac{2\sigma\alpha+\beta}{2}\|\partial_x\psi\|_{L^2}^2 + (\omega - \tfrac{1}{4}c^2)\frac{2\sigma\alpha-\beta}{2}\|\psi\|_{L^2}^2 - \frac{\beta c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2}.$$

2B. *Variational characterization.* First we consider the case of $\omega = c^2/4$ and $c > 0$. Then (α, β) satisfies

$$2\alpha - \beta > 0, \qquad 2\alpha + \beta > 0, \qquad \beta < 0. \tag{2-3}$$

Hereafter, we often omit the indices ω, c, α, and β for simplicity.

Lemma 2.6. *The following equality holds*:

$$\widetilde{\mathscr{G}}_{\omega,c} = \{e^{i\theta_0}e^{-(c/2)ix}\phi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi),\ x_0 \in \mathbb{R}\}.$$

Proof. Since $e^{-(c/2)ix}\phi_{\omega,c}$ satisfies $\widetilde{S}'_{\omega,c}(e^{-(c/2)ix}\phi_{\omega,c}) = e^{-(c/2)ix}S'_{\omega,c}(\phi_{\omega,c}) = 0$, we have $\widetilde{\mathscr{G}}_{\omega,c} \supset \{e^{i\theta_0}e^{-(c/2)ix}\phi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi), x_0 \in \mathbb{R}\}$. We prove $\widetilde{\mathscr{G}}_{\omega,c} \subset \{e^{i\theta_0}e^{-(c/2)ix}\phi_{\omega,c}(\cdot - x_0) : \theta_0 \in [0, 2\pi), x_0 \in \mathbb{R}\}$. Letting $\psi \in \widetilde{\mathscr{G}}_{\omega,c}$ and

$$\psi(x) = \Phi(x)\exp\biggl(-\frac{i}{2\sigma+2}\int_0^x |\Phi(y)|^{2\sigma}\,dy\biggr),$$

then Φ is a solution of

$$-\Phi'' + \tfrac{1}{2}c|\Phi|^{2\sigma}\Phi - \frac{2\sigma+1}{(2\sigma+2)^2}|\Phi|^{4\sigma}\Phi + \frac{\sigma}{\sigma+1}|\Phi|^{2\sigma-2}\operatorname{Im}(\bar{\Phi}\Phi')\Phi = 0.$$

Setting $A(\Phi) := \frac{1}{2}c|\Phi|^{2\sigma} - ((2\sigma+1)/(2\sigma+2)^2)|\Phi|^{4\sigma} + (\sigma/(\sigma+1))|\Phi|^{2\sigma-2}\operatorname{Im}(\bar{\Phi}\Phi')$, $f := \operatorname{Re}\Phi$, and $g := \operatorname{Im}\Phi$,

$$f'' = A(\Phi)f, \qquad g'' = A(\Phi)g.$$

Therefore,

$$(fg' - gf')' = fg'' - gf'' = fA(\Phi)g - gA(\Phi)f = A(\Phi)fg - A(\Phi)fg = 0.$$

Since $f, g \in \dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R})$, we obtain $fg' - gf' = 0$. On the other hand, $fg' - gf' = \operatorname{Re}\Phi \operatorname{Im}\Phi' - \operatorname{Im}\Phi \operatorname{Re}\Phi' = \operatorname{Im}(\bar{\Phi}\Phi')$. Thus, $\operatorname{Im}(\bar{\Phi}\Phi') = 0$ for any $x \in \mathbb{R}$. Therefore, Φ satisfies

$$-\Phi'' + \tfrac{1}{2}c|\Phi|^{2\sigma}\Phi - \frac{2\sigma+1}{(2\sigma+2)^2}|\Phi|^{4\sigma}\Phi = 0. \tag{2-4}$$

Therefore, there exist θ_0 and x_0 such that $\Phi = e^{i\theta_0}\Phi_{\omega,c}(\cdot - x_0)$ since $\Phi_{\omega,c}$ is the unique solution of (2-4) up to translation and phase (see the Appendix). This implies $\psi(x) = e^{i\theta}e^{-(c/2)ix}\phi_{\omega,c}(x - x_0)$. □

Remark. According to [Colin and Ohta 2006], it looks natural to take the integral on the infinite interval $(-\infty, x]$ in the gauge transformation as

$$\psi(x) = \Phi(x)\exp\biggl(-\frac{i}{2\sigma+2}\int_{-\infty}^{x}|\Phi(y)|^{2\sigma}\,dy\biggr).$$

However, in the massless case, it is not clear whether $\psi \in \widetilde{\mathscr{G}}_{\omega,c}$ belongs to $L^{2\sigma}(\mathbb{R})$. This is why we take the integral on the finite interval $[0, x]$ instead of $(-\infty, x]$.

Lemma 2.7. *We have $\widetilde{\mathscr{G}}_{\omega,c} \supset \widetilde{\mathscr{M}}^{\alpha,\beta}_{\omega,c}$.*

Proof. This is obvious if $\widetilde{\mathscr{M}} = \varnothing$. We consider the case of $\widetilde{\mathscr{M}} \neq \varnothing$. Let $\psi \in \widetilde{\mathscr{M}}$. Since ψ is a minimizer, there exists a Lagrange multiplier $\eta \in \mathbb{R}$ such that $\widetilde{S}'(\psi) = \eta\widetilde{K}'(\psi)$. Then

$$0 = \widetilde{K}(\psi) = \langle \widetilde{S}'(\psi), \partial_\lambda\psi^{\alpha,\beta}_\lambda|_{\lambda=0}\rangle = \eta\langle \widetilde{K}'(\psi), \partial_\lambda\psi^{\alpha,\beta}_\lambda|_{\lambda=0}\rangle = \eta\partial_\lambda\widetilde{K}(\psi^{\alpha,\beta}_\lambda)|_{\lambda=0},$$

where we remark that this is justified by a density argument. By a direct calculation, we obtain

$$\begin{aligned}\partial_\lambda\widetilde{K}(\psi^{\alpha,\beta}_\lambda)|_{\lambda=0} &= \frac{(2\alpha-\beta)^2}{2}\|\partial_x\psi\|^2_{L^2} - \frac{\{(2\sigma+2)\alpha+\beta\}^2}{2(2\sigma+2)}\|\psi\|^{2\sigma+2}_{L^{2\sigma+2}} - \frac{\{(2\sigma+2)\alpha\}^2}{2\sigma+2}N(\psi)\\ &= \frac{-(2\alpha-\beta)(2\sigma\alpha+\beta)}{2}\|\partial_x\psi\|^2_{L^2} + \frac{\{(2\sigma+2)\alpha+\beta\}\beta c}{2(2\sigma+2)}\|\psi\|^{2\sigma+2}_{L^{2\sigma+2}} + (2\sigma+2)\alpha\widetilde{K}(\psi)\\ &< 0,\end{aligned}$$

where in the last inequality we use

$$2\alpha-\beta > 0, \qquad 2\alpha+\beta > 0, \qquad \beta < 0, \qquad \widetilde{K}(\psi) = 0.$$

Therefore, $\eta = 0$. This implies $\widetilde{S}'_{\omega,c}(\psi) = 0$ and then $\psi \in \widetilde{\mathscr{G}}_{\omega,c}$. □

Lemma 2.8. *We have $\widetilde{\mathscr{G}}_{\omega,c} \subset \widetilde{\mathscr{M}}^{\alpha,\beta}_{\omega,c}$ if $\widetilde{\mathscr{M}}^{\alpha,\beta}_{\omega,c} \neq \varnothing$.*

Proof. Let $\psi \in \widetilde{\mathscr{G}}$. Then there exist $\theta_0 \in [0, 2\pi)$ and $x_0 \in \mathbb{R}$ such that $\psi = e^{i\theta_0}e^{-(c/2)ix}\phi_{\omega,c}(\cdot - x_0)$ by Lemma 2.6. If $\widetilde{\mathscr{M}} \neq \varnothing$, then we can take $\varphi \in \widetilde{\mathscr{M}}$. By Lemmas 2.6 and 2.7, there exist $\theta_1 \in [0, 2\pi)$ and $x_1 \in \mathbb{R}$ such that $\varphi = e^{i\theta_1}e^{-(c/2)ix}\phi_{\omega,c}(\cdot - x_1)$. Thus, $\widetilde{S}_{\omega,c}(\psi) = \widetilde{S}_{\omega,c}(\phi_{\omega,c}) = \widetilde{S}_{\omega,c}(\varphi) = \mu_{\omega,c}$. Moreover, we have $\widetilde{K}(\psi) = \langle \widetilde{S}'_{\omega,c}(\psi), \partial_\lambda\psi^{\alpha,\beta}_\lambda|_{\lambda=0}\rangle = 0$. □

Lemma 2.9. *We have $\widetilde{\mathscr{M}}^{\alpha,\beta}_{\omega,c} \neq \varnothing$.*

To prove this lemma, we show the following proposition.

Proposition 2.10. *Let $\{\psi_n\}_{n\in\mathbb{N}} \subset X_{\omega,c}$ satisfy*

$$\widetilde{S}_{\omega,c}(\psi_n) \to \mu^{\alpha,\beta}_{\omega,c} \quad \text{and} \quad \widetilde{K}^{\alpha,\beta}_{\omega,c}(\psi_n) \to 0.$$

Then there exist $\{y_n\} \subset \mathbb{R}$ and $\psi \in \widetilde{\mathscr{M}}^{\alpha,\beta}_{\omega,c}$ such that $\{\psi_n(\cdot - y_n)\}$ has a subsequence which converges to ψ strongly in $X_{\omega,c}$.

To prove this proposition, first, we prove the following lemma.

Lemma 2.11. *We have $\mu^{\alpha,\beta}_{\omega,c} > 0$.*

Proof. We recall that $\mu^{\alpha,\beta}_{\omega,c} = \inf\{\widetilde{S}_{\omega,c}(\psi) : \psi \in X_{\omega,c} \setminus \{0\},\ \widetilde{K}^{\alpha,\beta}_{\omega,c}(\psi) = 0\}$. By (2-2), it is trivial that $\mu \geq 0$. We prove $\mu > 0$ by contradiction. We assume that $\mu = 0$. Taking the minimizing sequence $\{\psi_n\} \subset X_{\omega,c}$, i.e., $\widetilde{S}(\psi_n) \to \mu = 0$ and $\widetilde{K}(\psi_n) = 0$, we have $\|\partial_x \psi_n\|^2_{L^2} \to 0$ and $\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} \to 0$ by (2-2) and (2-3). Then, by using (2-1) with $p = (\sigma+2)/2$, we get $\|\psi_n\|_{L^\infty} \to 0$ as $n \to \infty$. By using

$$-N(\psi) = -\|\partial_x\psi\|^2_{L^2} - \tfrac{1}{4}\|\psi\|^{4\sigma+2}_{L^{4\sigma+2}} + \|\partial_x\psi + \tfrac{1}{2}i|\psi|^{2\sigma}\psi\|^2_{L^2},$$

we obtain

$$\begin{aligned}
\widetilde{K}(\psi_n) &= \frac{2\alpha-\beta}{2}\|\partial_x\psi_n\|^2_{L^2} + \frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} - \alpha N(\psi_n) \\
&= -\tfrac{1}{2}\beta\|\partial_x\psi_n\|^2_{L^2} + \frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} - \tfrac{1}{4}\alpha\|\psi_n\|^{4\sigma+2}_{L^{4\sigma+2}} + \alpha\|\partial_x\psi_n + \tfrac{1}{2}i|\psi_n|^{2\sigma}\psi_n\|^2_{L^2} \\
&\geq \frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} - \tfrac{1}{4}\alpha\|\psi_n\|^{4\sigma+2}_{L^{4\sigma+2}} \\
&\geq \frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} - \tfrac{1}{4}\alpha\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}}\|\psi_n\|^{2\sigma}_{L^\infty} \\
&\geq \left(\frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)} - \tfrac{1}{4}\alpha\|\psi_n\|^{2\sigma}_{L^\infty}\right)\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} \\
&> 0,
\end{aligned}$$

for large $n \in \mathbb{N}$ since $\|\psi_n\|_{L^\infty} \to 0$ as $n \to \infty$. However, this contradicts $\widetilde{K}(\psi_n) = 0$ for all $n \in \mathbb{N}$. □

Proof of Proposition 2.10. We take $\{\psi_n\} \subset X_{\omega,c}$ such that $\widetilde{S}_{\omega,c}(\psi_n) \to \mu^{\alpha,\beta}_{\omega,c}$ and $\widetilde{K}^{\alpha,\beta}_{\omega,c}(\psi_n) \to 0$. Then, $\{\psi_n\}$ is a bounded sequence in $X_{\omega,c}$ by (2-2).

Step 1. We prove $\limsup_{n\to\infty}\|\psi_n\|_{L^{4\sigma+2}} > 0$ by contradiction. We suppose that $\limsup_{n\to\infty}\|\psi_n\|_{L^{4\sigma+2}} = 0$. Since

$$0 \leftarrow \widetilde{K}(\psi_n) \geq -\tfrac{1}{2}\beta\|\partial_x\psi_n\|^2_{L^2} + \frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} - \tfrac{1}{4}\alpha\|\psi_n\|^{4\sigma+2}_{L^{4\sigma+2}},$$

we obtain $\|\partial_x\psi_n\|^2_{L^2} \to 0$ and $\|\psi_n\|^{2\sigma+2}_{L^{2\sigma+2}} \to 0$ as $n \to \infty$. By (2-2), we get $\widetilde{S}(\psi_n) \to 0$. This contradicts $\mu > 0$.

Step 2. Since $\{\psi_n\}$ is bounded in $X_{\omega,c} = \dot{H}^1(\mathbb{R}) \cap L^{2\sigma+2}(\mathbb{R})$ and $\limsup_{n\to\infty}\|\psi_n\|_{L^{4\sigma+2}} > 0$, by applying Lemma 2.3 with $f_n = \psi_n$, $d = 1$, and $p = 2\sigma+2$, there exist $\{y_n\}$ and $v \in X_{\omega,c} \setminus \{0\}$ such that $\{\psi_n(\cdot - y_n)\}$ (we denote this by v_n) has a subsequence that converges to v weakly in $X_{\omega,c}$.

Step 3. We show

$$\widetilde{K}(v_n)-\widetilde{K}(v-v_n)-\widetilde{K}(v)\to 0 \quad \text{as } n\to\infty. \tag{2-5}$$

We note that

$$\widetilde{K}(\psi)=-\tfrac{1}{2}\beta\|\partial_x\psi\|_{L^2}^2+\frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2}-\tfrac{1}{4}\alpha\|\psi\|_{L^{4\sigma+2}}^{4\sigma+2}+\alpha\|\partial_x\psi+\tfrac{1}{2}i|\psi|^{2\sigma}\psi\|_{L^2}^2, \tag{2-6}$$

for any $\psi\in X_{\omega,c}$. Since v_n converges to v weakly in $X_{\omega,c}$, we have $v_n\to v$ a.e. in $\mathbb{R}$. Therefore, by Lemma 2.4, we have $\|v_n\|_{L^p}^p-\|v_n-v\|_{L^p}^p-\|v\|_{L^p}^p\to 0$ for $2\sigma+2\le p<\infty$. Moreover, setting

$$w_n:=\partial_x v_n+\tfrac{1}{2}i|v_n|^{2\sigma}v_n \quad \text{and} \quad w=\partial_x v+\tfrac{1}{2}i|v|^{2\sigma}v,$$

w_n converges to w weakly in $L^2(\mathbb{R})$. Indeed, it is obvious that $\partial_x v_n\rightharpoonup\partial_x v$ in $L^2(\mathbb{R})$ and we have, for any $f\in C_0^\infty(\mathbb{R})$,

$$\begin{aligned}\left|\int_{\mathbb{R}} f(x)(|v_n(x)|^{2\sigma}v_n(x)-|v(x)|^{2\sigma}v(x))\,dx\right| &\lesssim \int_{\operatorname{supp} f}|f(x)|(|v_n(x)|^{2\sigma}+|v(x)|^{2\sigma})|v_n(x)-v(x)|\,dx\\ &\lesssim \int_{\operatorname{supp} f}|v_n(x)-v(x)|\,dx\to 0,\end{aligned}$$

where we use the Hölder inequality, the fact that $\{v_n\}$ is bounded in $L^\infty(\mathbb{R})$, and the compactness of the embedding $\dot{H}^1(\Omega)\cap L^{2\sigma+2}(\Omega)\hookrightarrow H^1(\Omega)\hookrightarrow L^p(\Omega)$ for a bounded domain $\Omega\subset\mathbb{R}$ and $1\le p\le\infty$. Thus, w_n converges to w weakly in $L^2(\mathbb{R})$. Therefore, by (2-6), we get (2-5).

Step 4. We prove $\alpha(2\sigma+2)\mu<\tilde{J}(\psi)$ if $\widetilde{K}(\psi)<0$. By the definition of μ,

$$\mu_{\omega,c}^{\alpha,\beta}=\frac{1}{\alpha(2\sigma+2)}\inf\{\tilde{J}_{\omega,c}^{\alpha,\beta}(\psi):\psi\in X_{\omega,c}\setminus\{0\},\ \widetilde{K}_{\omega,c}^{\alpha,\beta}(\psi)=0\}. \tag{2-7}$$

If $\psi\in X_{\omega,c}$ satisfies $\widetilde{K}(\psi)<0$, then there exists $\lambda_0\in(0,1)$ such that $\widetilde{K}(\lambda_0\psi)=0$ since $\widetilde{K}(\lambda\psi)>0$ for small $\lambda\in(0,1)$. Therefore, we have $\alpha(2\sigma+2)\mu\le\tilde{J}(\lambda_0\psi)<\tilde{J}(\psi)$.

Step 5. We prove $\widetilde{K}(v)\le 0$ by contradiction. We suppose $\widetilde{K}(v)>0$. Since $\widetilde{K}(v_n)\to 0$ and (2-5) hold,

$$\widetilde{K}(v-v_n)\to-\widetilde{K}(v)<0.$$

This implies that $\widetilde{K}(v-v_n)<0$ for large $n\in\mathbb{N}$. Therefore, by Step 4, we get $\alpha(2\sigma+2)\mu<\tilde{J}(v-v_n)$ for large $n\in\mathbb{N}$. By the same argument as in Step 3,

$$\tilde{J}(v_n)-\tilde{J}(v-v_n)-\tilde{J}(v)\to 0 \quad \text{as } n\to\infty.$$

Therefore, we get $\tilde{J}(v)=\lim_{n\to\infty}(\tilde{J}(v_n)-\tilde{J}(v-v_n))\le 0$ since we have $\tilde{J}(v_n)\to\alpha(2\sigma+2)\mu$ by the definition of $\tilde{J}$ and $\widetilde{K}(v_n)\to 0$. By Step 2, we have $v\ne 0$ and then $\tilde{J}(v)>0$. This is a contradiction.

Step 6. We prove that v belongs to $\tilde{\mathscr{M}}$. By (2-7) and the weakly lower semicontinuity of $\tilde{J}$, we obtain

$$\alpha(2\sigma+2)\mu\le\tilde{J}(v)\le\liminf_{n\to\infty}\tilde{J}(v_n)=\alpha(2\sigma+2)\mu.$$

Thus, $\tilde{J}(v)=\alpha(2\sigma+2)\mu$ and v_n converges to v strongly in $X_{\omega,c}$. Therefore, we get $\widetilde{S}(v)=\mu$ and $\widetilde{K}(v)=0$ by Steps 4 and 5. □

Therefore, we obtain Proposition 2.1 when $\omega = c^2/4$ and $c > 0$.

The case of $\omega > c^2/4$ is much easier. Indeed, we can obtain Proposition 2.1 by the same argument as in the case $\omega = c^2/4$ and $c > 0$ by using $L^2(\mathbb{R})$ instead of $L^{2\sigma+2}(\mathbb{R})$. See also [Colin and Ohta 2006], where the statement only for the Nehari functional $K^{1,0}_{\omega,c}$ is obtained. Thus, we omit the proof.

3. Global existence

In this section, we show Theorem 1.2.

Proof of Theorem 1.2. Let u_0 belong to $\mathscr{K}^{\alpha,\beta,+}_{\omega,c}$. First, we consider the case that $K^{\alpha,\beta}_{\omega,c}(u_0) = 0$. Then, $u_0 = 0$ or $u_0 = e^{i\theta_0}\phi_{\omega,c}(\,\cdot\, - x_0)$ by Theorem 1.1. By the uniqueness of solution to (gDNLS), we have $u(t) = 0$ or $u(t) = e^{i\theta_0}e^{i\omega t}\phi_{\omega,c}(x - ct - x_0)$, respectively. This implies that $K^{\alpha,\beta}_{\omega,c}(u(t)) = 0$ for all time. This means that $u(t) \in \mathscr{K}^{\alpha,\beta,+}_{\omega,c}$ for all time. Next, we consider the case that $K^{\alpha,\beta}_{\omega,c}(u_0) > 0$. We suppose that there exists a time t such that $K^{\alpha,\beta}_{\omega,c}(u(t)) \le 0$. Then there exists t_* such that $K^{\alpha,\beta}_{\omega,c}(u(t_*)) = 0$ by the continuity of the flow. By the above argument, $K^{\alpha,\beta}_{\omega,c}(u(t)) = 0$ for all time. This is a contradiction. Thus, $u(t)$ belongs to $\mathscr{K}^{\alpha,\beta,+}_{\omega,c}$ for all time. When u_0 belongs to $\mathscr{K}^{\alpha,\beta,-}_{\omega,c}$, the same argument implies that $u(t)$ belongs to $\mathscr{K}^{\alpha,\beta,-}_{\omega,c}$ for all time. Next, we prove that the solution is global if $u_0 \in \mathscr{K}^{\alpha,\beta,+}_{\omega,c}$. Then, since

$$\alpha(2\sigma+2)S_{\omega,c}(\varphi) = K^{\alpha,\beta}_{\omega,c}(\varphi) + \frac{2\sigma\alpha+\beta}{2}\|\partial_x\varphi - \tfrac{1}{2}ci\varphi\|^2_{L^2} + (\omega - \tfrac{1}{4}c^2)\frac{2\sigma\alpha-\beta}{2}\|\varphi\|^2_{L^2} - \frac{\beta c}{2(2\sigma+2)}\|\varphi\|^{2\sigma+2}_{L^{2\sigma+2}} \tag{3-1}$$

and $K^{\alpha,\beta}_{\omega,c}(u(t)) > 0$ for all time t, we have that $\|\partial_x u(t) - \frac{1}{2}ciu(t)\|^2_{L^2}$ is uniformly bounded. Therefore,

$$\|\partial_x u(t)\|_{L^2} \le \|\partial_x u(t) - \tfrac{1}{2}ciu(t)\|_{L^2} + \tfrac{1}{2}|c|\|u(t)\|_{L^2} < C + \tfrac{1}{2}|c|\|u_0\|_{L^2},$$

for some positive constant C independent of t. This boundedness and the conservation law of the L^2-norm imply that u is global in time. □

We give proofs of Corollaries 1.3, 1.4, and 1.5. Direct calculations imply the following lemma (see [Colin and Ohta 2006] for the details).

Lemma 3.1. *Let $\sigma = 1$ and (ω, c) satisfy (1-5). Then, we have the relations*

$$M(\phi_{\omega,c}) = 8\tan^{-1}\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}},$$
$$P(\phi_{\omega,c}) = 2\sqrt{4\omega - c^2},$$
$$E(\phi_{\omega,c}) = -\tfrac{1}{2}c\sqrt{4\omega - c^2}.$$

In particular,

$$S_{\omega,c}(\phi_{\omega,c}) = 4\omega\tan^{-1}\sqrt{\frac{2\sqrt{\omega}+c}{2\sqrt{\omega}-c}} + \tfrac{1}{2}c\sqrt{4\omega - c^2}.$$

Remark. When $\sigma = 1$, we have $M(\phi_{c^2/4,c}) = 4\pi$, $P(\phi_{c^2/4,c}) = 0$, and $E(\phi_{c^2/4,c}) = 0$ for all $c > 0$ by Lemma 3.1. On the other hand, if $M(\phi) = 4\pi$, $P(\phi) = 0$, and $E(\phi) \le 0$, then $\phi(x) = e^{i\theta_0}\phi_{c_0^2/4,c_0}(x - x_0)$

for some $\theta_0 \in \mathbb{R}$, $x_0 \in \mathbb{R}$, and $c_0 > 0$. Indeed, $M(\phi) = 4\pi$, $P(\phi) = 0$, and $E(\phi) \le 0$ imply that

$$K^{\alpha,\beta}_{c^2/4,c}(\phi) \le -\frac{2\alpha+\beta}{2}\|\partial_x\phi\|^2_{L^2} + \frac{2\alpha-\beta}{2}c^2\pi + \frac{\beta c}{8}\|\phi\|^4_{L^4}.$$

Since $K^{\alpha,\beta}_{c^2/4,c}(\phi) < 0$ for small $c > 0$ and $K^{\alpha,\beta}_{c^2/4,c}(\phi) \to +\infty$ as $c \to \infty$, there exists $c_0 > 0$ such that $K^{\alpha,\beta}_{c_0^2/4,c_0}(\phi) = 0$. Therefore, Theorem 1.1 implies that $\phi(x) = e^{i\theta_0}\phi_{c_0^2/4,c_0}(x - x_0)$. Note that this means that there is no function satisfying $M(\phi) = 4\pi$, $P(\phi) = 0$, and $E(\phi) < 0$.

First, we prove Corollary 1.3.

Proof of Corollary 1.3. Let u_0 satisfy $\|u_0\|^2_{L^2} < 4\pi$. The statement is trivial if $u_0 = 0$. We assume that $u_0 \neq 0$. Since $\|u_0\|^2_{L^2} < 4\pi$,

$$S_{c^2/4,c}(u_0) = E(u_0) + \tfrac{1}{8}c^2\|u_0\|^2_{L^2} + \tfrac{1}{2}cP(u_0) < c^2\pi/2,$$

for sufficiently large $c > 0$. Moreover, since $\|u_0\|^2_{L^2} \neq 0$,

$$\begin{aligned} K^{\alpha,\beta}_{c^2/4,c}(u_0) &= \frac{2\alpha-\beta}{2}\|\partial_x u_0\|^2_{L^2} + \frac{2\alpha-\beta}{2}\frac{c^2}{4}\|u_0\|^2_{L^2} + \frac{2\alpha-\beta}{2}cP(u_0) + \frac{\beta c}{8}\|u_0\|^4_{L^4} - \alpha N(u_0) \\ &\to \infty \quad \text{as } c \to \infty, \end{aligned} \tag{3-2}$$

for any (α, β) satisfying (1-8). Thus, $K^{\alpha,\beta}_{c^2/4,c}(u_0) > 0$ for large $c > 0$. Thus, there exists $c > 0$ such that $K^{\alpha,\beta}_{c^2/4,c}(u_0) > 0$ and $S_{c^2/4,c}(u_0) < c^2\pi/2$, where we note that $\mu_{c^2/4,c} = c^2\pi/2$ by Lemma 3.1 when $\sigma = 1$. By Theorem 1.2, the solution u is global. □

Secondly, we give a proof of Corollary 1.4 by Theorem 1.2.

Proof of Corollary 1.4. Let u_0 satisfy $\|u_0\|^2_{L^2} = 4\pi$ and $P(u_0) < 0$. We recall that $\mu_{c^2/4,c} = c^2\pi/2$ by Lemma 3.1 when $\sigma = 1$. Since $P(u_0) < 0$, we have, for large $c > 0$,

$$S_{c^2/4,c}(u_0) = E(u_0) + \tfrac{1}{2}c^2\pi + \tfrac{1}{2}cP(u_0) \le \mu_{c^2/4,c}.$$

On the other hand, because $2\alpha - \beta > 0$ and $\|u_0\|^2_{L^2} \neq 0$, we obtain (3-2). Thus, $K^{\alpha,\beta}_{c^2/4,c}(u_0) > 0$ for large $c > 0$. This means that the assumption in Theorem 1.2 holds for sufficiently large c. This implies that u is global. □

We give another proof. This is due to [Guo and Wu 2017].

Another proof of Corollary 1.4. We have

$$P(u) \ge \tfrac{1}{4}\|u\|^4_{L^4}\left(1 - \frac{\|u\|_{L^2}}{2\sqrt{\pi}}\right) - \frac{8\sqrt{\pi}E(u)\|u\|_{L^2}}{\|u\|^4_{L^4}},$$

applying the gauge transformation $u = w\exp(-\frac{1}{4}i\int_{-\infty}^{x}|w(y)|^2\,dy)$ to (1-15). See [Guo and Wu 2017, Lemma 2.1] for the proof of (1-15). When $\|u_0\|^2_{L^2} = 4\pi$ and $P(u_0) < 0$, we get

$$\|u(t)\|^4_{L^4} \le \frac{8\sqrt{\pi}E(u_0)\|u_0\|_{L^2}}{|P(u_0)|}. \tag{3-3}$$

Therefore, by the Hölder inequality, the Gagliardo–Nirenberg inequality, and the Young inequality,

$$\begin{aligned}\|\partial_x u(t)\|_{L^2}^2 &= 2E(u_0) + \tfrac{1}{2}\operatorname{Re}\int_{\mathbb{R}} i|u(t,x)|^2\overline{u(t,x)}\partial_x u(t,x)\,dx \\ &\le 2E(u_0) + \tfrac{1}{2}\|u(t)\|_{L^6}^3\|\partial_x u(t)\|_{L^2} \\ &\le 2E(u_0) + C\|u(t)\|_{L^4}^{8/3}\|\partial_x u(t)\|_{L^2}^{4/3} \\ &\le 2E(u_0) + C\|u(t)\|_{L^4}^{8} + \tfrac{1}{2}\|\partial_x u(t)\|_{L^2}^2.\end{aligned}$$

This inequality and (3-3) give an a priori estimate, and thus, the solution is global. □

At last, we prove Corollary 1.5.

Proof of Corollary 1.5. Let $\sigma \ge 1$. Since $u_{0,c} = e^{(c/2)ix}\psi$,

$$\begin{aligned}S_{c^2/4,c}(u_{0,c}) &= \widetilde{S}_{c^2/4,c}(\psi) \\ &= \tfrac{1}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - \frac{1}{2\sigma+2}N(\psi) \\ &\le c^{1+1/\sigma}S_{1/4,1}(\phi_{1/4,1}) = S_{c^2/4,c}(\phi_{c^2/4,c}), \\ K^{\alpha,\beta}_{c^2/4,c}(u_{0,c}) &= \widetilde{K}^{\alpha,\beta}_{c^2/4,c}(\psi) \\ &= \frac{2\alpha-\beta}{2}\|\partial_x\psi\|_{L^2}^2 + \frac{\{(2\sigma+2)\alpha+\beta\}c}{2(2\sigma+2)}\|\psi\|_{L^{2\sigma+2}}^{2\sigma+2} - \alpha N(\psi) \\ &\ge 0,\end{aligned}$$

for large $c > 0$. By Theorem 1.2, therefore, the solution u_c with the initial data $u_{0,c}$ is global for large $c > 0$. □

Appendix: Uniqueness and nonexistence

We prove the uniqueness of the massless elliptic equation.

Proposition A.1. *Let $1 < p < q < \infty$, $a > 0$, and $b > 0$. Assume there exists a nontrivial solution in $\dot{H}^1(\mathbb{R}) \cap L^{p+1}(\mathbb{R})$ of the equation*

$$-\varphi'' + a|\varphi|^{p-1}\varphi - b|\varphi|^{q-1}\varphi = 0 \tag{A-1}$$

in the distribution sense. Then there exist $\theta_0 \in [0, 2\pi)$ and $x_0 \in \mathbb{R}$ such that $\varphi = e^{i\theta_0}\psi(\cdot - x_0)$, where ψ is the unique positive, even, and decreasing function which satisfies (A-1).

Proof. Since $a|\varphi|^{p-1}\varphi - b|\varphi|^{q-1}\varphi$ belongs to $L^2(\mathbb{R})$, we obtain $\varphi \in \dot{H}^2(\mathbb{R})$. A bootstrap argument gives us that $\varphi \in \dot{H}^3(\mathbb{R})$. By the Sobolev embedding, $\varphi \in C^2(\mathbb{R})$ and φ satisfies the equation in the classical sense. Multiplying the equation by φ' and integrating on $(-\infty, x)$, we obtain

$$-\tfrac{1}{2}|\varphi'(x)|^2 + \frac{a}{p+1}|\varphi(x)|^{p+1} - \frac{b}{q+1}|\varphi(x)|^{q+1} = 0. \tag{A-2}$$

We write $\varphi = \rho e^{i\theta}$, where $\rho > 0$ and $\rho, \theta \in C^2(\mathbb{R})$. It is easily seen that $\theta \equiv \theta_0$ for some $\theta_0 \in [0, 2\pi)$. Since $\rho \in L^{p+1}(\mathbb{R})$, there must exist $x_0 \in \mathbb{R}$ such that $\rho'(x_0) = 0$. By (A-2), $\rho(x_0) = c$, where $c^{q-p} =$

$(a(q+1))/(b(p+1))$. Let ψ be the real-valued solution of (A-1) such that $\psi(0)=c$ and $\psi'(0)=0$. Using the uniqueness of the ordinary differential equation, we can deduce that $\varphi = e^{i\theta_0}\psi(\cdot - x_0)$. □

We prove the nonexistence of a nontrivial solution to the nonlinear elliptic equation (1-4) in the case $\omega < c^2/4$, or $\omega = c^2/4$ and $c \le 0$. See [Berestycki and Lions 1983, Theorem 5] for the necessary and sufficient condition for the existence of nontrivial solutions to more general second-order ordinary differential equations.

Proposition A.2. *Let $1 < p, q < \infty$. If $\varphi \in H^1(\mathbb{R})$ satisfies*

$$-\varphi'' + \omega\varphi + a|\varphi|^{p-1}\varphi - b|\varphi|^{q-1}\varphi = 0 \quad \text{in the distribution sense},$$

where $a, b \in \mathbb{R}$ and $\omega < 0$, then we have $\varphi = 0$.

Proof. By a usual bootstrap argument [Cazenave 2003, §8], we have $\varphi \in H^3(\mathbb{R})$. We get $\varphi \in C^2(\mathbb{R})$ by the Sobolev embedding. Therefore, $\varphi'(x) \to 0$ and $\varphi(x) \to 0$ as $|x| \to \infty$. Multiplying the equation by φ' and integrating on $(-\infty, x)$, we obtain

$$-\tfrac{1}{2}|\varphi'(x)|^2 + \tfrac{1}{2}\omega|\varphi(x)|^2 + \frac{a}{p+1}|\varphi(x)|^{p+1} - \frac{b}{q+1}|\varphi(x)|^{q+1} = 0. \tag{A-3}$$

Since $\varphi(x) \to 0$ as $|x| \to \infty$, we get

$$\tfrac{1}{2}\omega|\varphi(x)|^2 + \frac{a}{p+1}|\varphi(x)|^{p+1} - \frac{b}{q+1}|\varphi(x)|^{q+1} < 0 \quad \text{for some } x$$

or

$$|\varphi(x)| = 0 \quad \text{for some } x.$$

In the former case, we obtain $|\varphi'(x)| < 0$ by (A-3). This is a contradiction. In the latter case, we obtain $|\varphi'(x)| = 0$ by (A-3). By the uniqueness of the ordinary differential equation, we get $\varphi = 0$. □

By the same argument as in the proof of Proposition A.2, we obtain the nonexistence of a nontrivial solution to the nonlinear elliptic equation (1-4) when $\omega = c^2/4$ and $c \le 0$ as follows.

Proposition A.3. *Let $1 < p, q < \infty$. If $\varphi \in \dot{H}^1(\mathbb{R}) \cap L^{p+1}(\mathbb{R})$ satisfies*

$$-\varphi'' - a|\varphi|^{p-1}\varphi - b|\varphi|^{q-1}\varphi = 0 \quad \text{in the distribution sense},$$

where $a \ge 0$ and $b > 0$, then we have $\varphi = 0$.

Acknowledgements

The authors would like to express deep appreciation to Professor Kenji Nakanishi for constant encouragement, Professor Masahito Ohta for many useful suggestions, and Professor Tohru Ozawa for advice on notations. Inui is supported by Grant-in-Aid for JSPS Research Fellow 15J02570. The authors also would like to thank Guixiang Xu for introducing their works, Zihua Guo for a suggestion about Corollary 1.4, and the anonymous referee for his valuable comments.

References

[Bellazzini et al. 2014a] J. Bellazzini, R. L. Frank, and N. Visciglia, "Maximizers for Gagliardo–Nirenberg inequalities and related non-local problems", *Math. Ann.* **360**:3–4 (2014), 653–673.

[Bellazzini et al. 2014b] J. Bellazzini, M. Ghimenti, and S. Le Coz, "Multi-solitary waves for the nonlinear Klein–Gordon equation", *Comm. Partial Differential Equations* **39**:8 (2014), 1479–1522.

[Berestycki and Lions 1983] H. Berestycki and P.-L. Lions, "Nonlinear scalar field equations, I: Existence of a ground state", *Arch. Rational Mech. Anal.* **82**:4 (1983), 313–345.

[Biagioni and Linares 2001] H. A. Biagioni and F. Linares, "Ill-posedness for the derivative Schrödinger and generalized Benjamin–Ono equations", *Trans. Amer. Math. Soc.* **353**:9 (2001), 3649–3659.

[Brézis and Lieb 1983] H. m. Brézis and E. Lieb, "A relation between pointwise convergence of functions and convergence of functionals", *Proc. Amer. Math. Soc.* **88**:3 (1983), 486–490.

[Cazenave 2003] T. Cazenave, *Semilinear Schrödinger equations*, Courant Lecture Notes in Mathematics **10**, American Mathematical Society, Providence, RI, 2003.

[Colin and Ohta 2006] M. Colin and M. Ohta, "Stability of solitary waves for derivative nonlinear Schrödinger equation", *Ann. Inst. H. Poincaré Anal. Non Linéaire* **23**:5 (2006), 753–764.

[Colliander et al. 2001] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "Global well-posedness for Schrödinger equations with derivative", *SIAM J. Math. Anal.* **33**:3 (2001), 649–669.

[Colliander et al. 2002] J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao, "A refined global well-posedness result for Schrödinger equations with derivative", *SIAM J. Math. Anal.* **34**:1 (2002), 64–86.

[Fukaya 2016] N. Fukaya, "Instability of solitary waves for a generalized derivative nonlinear Schrödinger equation in a borderline case", preprint, 2016. To appear in *Kodai Math. J.* arXiv

[Grillakis et al. 1987] M. Grillakis, J. Shatah, and W. Strauss, "Stability theory of solitary waves in the presence of symmetry, I", *J. Funct. Anal.* **74**:1 (1987), 160–197.

[Grillakis et al. 1990] M. Grillakis, J. Shatah, and W. Strauss, "Stability theory of solitary waves in the presence of symmetry, II", *J. Funct. Anal.* **94**:2 (1990), 308–348.

[Guo and Wu 1995] B. L. Guo and Y. P. Wu, "Orbital stability of solitary waves for the nonlinear derivative Schrödinger equation", *J. Differential Equations* **123**:1 (1995), 35–55.

[Guo and Wu 2017] Z. Guo and Y. Wu, "Global well-posedness for the derivative nonlinear Schrödinger equation in $H^{\frac{1}{2}}(\mathbb{R})$", *Discrete Contin. Dyn. Syst.* **37**:1 (2017), 257–264.

[Hao 2007] C. Hao, "Well-posedness for one-dimensional derivative nonlinear Schrödinger equations", *Commun. Pure Appl. Anal.* **6**:4 (2007), 997–1021.

[Hayashi 1993] N. Hayashi, "The initial value problem for the derivative nonlinear Schrödinger equation in the energy space", *Nonlinear Anal.* **20**:7 (1993), 823–833.

[Hayashi and Ozawa 1992] N. Hayashi and T. Ozawa, "On the derivative nonlinear Schrödinger equation", *Phys. D* **55**:1–2 (1992), 14–36.

[Hayashi and Ozawa 1994a] N. Hayashi and T. Ozawa, "Finite energy solutions of nonlinear Schrödinger equations of derivative type", *SIAM J. Math. Anal.* **25**:6 (1994), 1488–1503.

[Hayashi and Ozawa 1994b] N. Hayashi and T. Ozawa, "Remarks on nonlinear Schrödinger equations in one space dimension", *Differential Integral Equations* **7**:2 (1994), 453–461.

[Hayashi and Ozawa 2016] M. Hayashi and T. Ozawa, "Well-posedness for a generalized derivative nonlinear Schrödinger equation", *J. Differential Equations* **261**:10 (2016), 5424–5445.

[Kwon and Wu 2016] S. Kwon and Y. Wu, "Orbital stability of solitary waves for derivative nonlinear Schrödinger equation", preprint, 2016. arXiv

[Le Coz and Wu 2016] S. Le Coz and Y. Wu, "Stability of multi-solitons for the derivative nonlinear Schrödinger equation", preprint, 2016. arXiv

[Lieb 1983] E. H. Lieb, "On the lowest eigenvalue of the Laplacian for the intersection of two domains", *Invent. Math.* **74**:3 (1983), 441–448.

[Liu et al. 2013] X. Liu, G. Simpson, and C. Sulem, "Stability of solitary waves for a generalized derivative nonlinear Schrödinger equation", *J. Nonlinear Sci.* **23**:4 (2013), 557–583.

[Miao et al. 2011] C. Miao, Y. Wu, and G. Xu, "Global well-posedness for Schrödinger equation with derivative in $H^{\frac{1}{2}}(\mathbb{R})$", *J. Differential Equations* **251**:8 (2011), 2164–2195.

[Miao et al. 2017a] C. Miao, X. Tang, and G. Xu, "Solitary waves for nonlinear Schrödinger equation with derivative", preprint, 2017. arXiv

[Miao et al. 2017b] C. Miao, X. Tang, and G. Xu, "Stability of the traveling waves for the derivative Schrödinger equation in the energy space", *Calc. Var. Partial Differential Equations* **56**:2 (2017), 56:45.

[Mio et al. 1976] K. Mio, T. Ogino, K. Minami, and S. Takeda, "Modified nonlinear Schrödinger equation for Alfvén waves propagating along the magnetic field in cold plasmas", *J. Phys. Soc. Japan* **41**:1 (1976), 265–271.

[Mjølhus 1976] E. Mjølhus, "On the modulational instability of hydromagnetic waves parallel to the magnetic field", *J. Plasma Phys.* **16**:3 (1976), 321–334.

[Moses et al. 2007] J. Moses, B. A. Malomed, and F. W. Wise, "Self-steepening of ultrashort optical pulses without self-phase-modulation", *Phys. Rev. A* **76**:2 (2007), 021802(R).

[Ozawa 1996] T. Ozawa, "On the nonlinear Schrödinger equations of derivative type", *Indiana Univ. Math. J.* **45**:1 (1996), 137–163.

[Payne and Sattinger 1975] L. E. Payne and D. H. Sattinger, "Saddle points and instability of nonlinear hyperbolic equations", *Israel J. Math.* **22**:3-4 (1975), 273–303.

[Santos 2015] G. d. N. Santos, "Existence and uniqueness of solution for a generalized nonlinear derivative Schrödinger equation", *J. Differential Equations* **259**:5 (2015), 2030–2060.

[Sattinger 1968] D. H. Sattinger, "On global solution of nonlinear hyperbolic equations", *Arch. Rational Mech. Anal.* **30** (1968), 148–172.

[Takaoka 1999] H. Takaoka, "Well-posedness for the one-dimensional nonlinear Schrödinger equation with the derivative nonlinearity", *Adv. Differential Equations* **4**:4 (1999), 561–580.

[Takaoka 2001] H. Takaoka, "Global well-posedness for Schrödinger equations with derivative in a nonlinear term and data in low-order Sobolev spaces", *Electron. J. Differential Equations* **2001**:42 (2001).

[Tang and Xu 2017] X. Tang and G. Xu, "Stability of the sum of two solitary waves for (gDNLS) in the energy space", preprint, 2017. arXiv

[Tsutsumi 1972] M. Tsutsumi, "On solutions of semilinear differential equations in a Hilbert space", *Math. Japon.* **17** (1972), 173–193.

[Tsutsumi and Fukuda 1980] M. Tsutsumi and I. Fukuda, "On solutions of the derivative nonlinear Schrödinger equation: existence and uniqueness theorem", *Funkcial. Ekvac.* **23**:3 (1980), 259–277.

[Weinstein 1982] M. I. Weinstein, "Nonlinear Schrödinger equations and sharp interpolation estimates", *Comm. Math. Phys.* **87**:4 (1982), 567–576.

[Wu 2013] Y. Wu, "Global well-posedness for the nonlinear Schrödinger equation with derivative in energy space", *Anal. PDE* **6**:8 (2013), 1989–2002.

[Wu 2015] Y. Wu, "Global well-posedness on the derivative nonlinear Schrödinger equation", *Anal. PDE* **8**:5 (2015), 1101–1112.

Received 8 Oct 2016. Revised 28 Feb 2017. Accepted 3 Apr 2017.

NORIYOSHI FUKAYA: 1116702@ed.tus.ac.jp
Department of Mathematics, Graduate School of Science, Tokyo University of Science, Shinjuku, Tokyo 162-8601, Japan

MASAYUKI HAYASHI: masayuki-884@fuji.waseda.jp
Department of Applied Physics, Waseda University, Shinjuku, Tokyo 169-8555, Japan

TAKAHISA INUI: inui@math.kyoto-u.ac.jp
Department of Mathematics, Graduate School of Science, Kyoto University, Kyoto City, Kyoto 606-8502, Japan

mathematical sciences publishers

Analysis & PDE
msp.org/apde

See inside back cover or msp.org/apde for submission instructions.

The subscription price for 2017 is US $265/year for the electronic version, and $470/year (+$55, if shipping outside the US) for print and electronic. Subscriptions, requests for back issues from the last three years and changes of subscriber address should be sent to MSP.

Analysis & PDE (ISSN 1948-206X electronic, 2157-5045 printed) at Mathematical Sciences Publishers, 798 Evans Hall #3840, c/o University of California, Berkeley, CA 94720-3840, is published continuously online. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices.

APDE peer review and production are managed by EditFlow® from MSP.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

Analysis & PDE

Volume 10 No. 5 2017

Hardy-singular boundary mass and Sobolev-critical variational problems 1017
NASSIF GHOUSSOUB and FRÉDÉRIC ROBERT

Conical maximal regularity for elliptic operators via Hardy spaces 1081
YI HUANG

Local exponential stabilization for a class of Korteweg–de Vries equations by means of time-varying feedback laws 1089
JEAN-MICHEL CORON, IVONNE RIVAS and SHENGQUAN XIANG

On the growth of Sobolev norms for NLS on 2- and 3-dimensional manifolds 1123
FABRICE PLANCHON, NIKOLAY TZVETKOV and NICOLA VISCIGLIA

A sufficient condition for global existence of solutions to a generalized derivative nonlinear Schrödinger equation 1149
NORIYOSHI FUKAYA, MASAYUKI HAYASHI and TAKAHISA INUI

Local density approximation for the almost-bosonic anyon gas 1169
MICHELE CORREGGI, DOUGLAS LUNDHOLM and NICOLAS ROUGERIE

Regularity of velocity averages for transport equations on random discrete velocity grids 1201
NATHALIE AYI and THIERRY GOUDON

Perron's method for nonlocal fully nonlinear equations 1227
CHENCHEN MOU

A sparse domination principle for rough singular integrals 1255
JOSÉ M. CONDE-ALONSO, AMALIA CULIUC, FRANCESCO DI PLINIO and YUMENG OU