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We study the minimizers of an energy functional with a self-consistent magnetic field, which describes a quantum gas of almost-bosonic anyons in the average-field approximation. For the homogeneous gas we prove the existence of the thermodynamic limit of the energy at fixed effective statistics parameter, and the independence of such a limit from the shape of the domain. This result is then used in a local density approximation to derive an effective Thomas–Fermi-like model for the trapped anyon gas in the limit of a large effective statistics parameter (i.e., "less-bosonic" anyons).

1. Introduction

A convenient description of two-dimensional particles with exotic quantum statistics (different from Bose–Einstein and Fermi–Dirac) is via effective magnetic interactions. We are interested in a mean-field model for such particles, known as anyons. Indeed, in a certain scaling limit ("almost-bosonic anyons", see [Lundholm and Rougerie 2015]), a suitable magnetic nonlinear Schrödinger theory becomes appropriate. The corresponding energy functional is given by

$$\mathcal{E}_{\beta}^{\mathrm{af}}[u] := \int_{\mathbb{R}^2} \big(\big| (-i\nabla + \beta \boldsymbol{A}[|u|^2]) u \big|^2 + V |u|^2 \big), \tag{1-1}$$

acting on functions $u \in H^1(\mathbb{R}^2)$. Here $V \colon \mathbb{R}^2 \to \mathbb{R}^+$ is a trapping potential confining the particles, and the vector potential $\boldsymbol{A}[|u|^2] \colon \mathbb{R}^2 \to \mathbb{R}^2$ is defined through

$$\boldsymbol{A}[\varrho] := \nabla^{\perp} w_0 * \varrho, \quad w_0(\boldsymbol{x}) := \log |\boldsymbol{x}|, \tag{1-2}$$

for $\varrho = |u|^2 \in L^1(\mathbb{R}^2)$ and $\boldsymbol{x}^{\perp} = (x, y)^{\perp} := (-y, x)$. Thus, the self-consistent magnetic field, given by

$$\operatorname{curl} \boldsymbol{A}[\varrho](\boldsymbol{x}) = \Delta w_0 * \varrho(\boldsymbol{x}) = 2\pi \varrho(\boldsymbol{x}),$$

is proportional to the particles' density. The parameter $\beta \in \mathbb{R}$ then regulates the strength of the magnetic self-interactions and, for reasons explained below, we will call it the *scaled statistics parameter*. By symmetry of (1-1) under complex conjugation $u \mapsto \bar{u}$ we may and shall assume

$$\beta \geqslant 0$$

in the following. We will study the ground-state problem for (1-1), namely the minimization under the mass constraint

$$\int_{\mathbb{R}^2} |u|^2 = 1. \tag{1-3}$$

The functional $\mathcal{E}^{\mathrm{af}}$ bears some similarity with other mean-field models such as the Gross–Pitaevskii energy functional

$$\mathcal{E}^{\mathrm{GP}}[u] := \int_{\mathbb{R}^2} \left(|-i\nabla u + Au|^2 + V|u|^2 + g|u|^4 \right), \tag{1-4}$$

with *fixed* vector potential A. The above describes a gas of interacting bosons in a certain mean-field regime [Lieb et al. 2005; Lieb and Seiringer 2006; Nam et al. 2016; Rougerie 2014; 2015]: the quartic term originates from short-range pair interactions. The crucial difference between (1-1) and (1-4) is that, while the interactions of $\mathcal{E}^{\mathrm{GP}}$ are scalar (with interaction strength $g \in \mathbb{R}$), those of $\mathcal{E}^{\mathrm{af}}$ are purely magnetic and therefore involve mainly the phase of the function u. There is extensive literature dealing with (1-4) (see [Aftalion 2007; Correggi et al. 2011; 2012; Correggi and Rougerie 2013]) and with the related Ginzburg–Landau model of superconductivity [Bethuel et al. 1994; Fournais and Helffer 2010; Sandier and Serfaty 2007; Sigal 2015]. That the interactions are via the magnetic field in (1-1) poses however quite a few new difficulties in the asymptotic analysis of minimizers we initiate here. Note indeed (see the variational equation in Lemma A.2) that the nonlinearity consists in a quintic nonlocal semilinear term and a cubic quasilinear term (also nonlocal), both being critical when compared to the usual Laplacian.

The functional $\mathcal{E}^{\mathrm{af}}$ arises in a mean-field description[1] of a gas of particles whose many-body quantum wave function can change under particle exchange by a phase factor $e^{i\alpha\pi}$ (with $\alpha \in \mathbb{R}$ known as the statistics parameter). This is a generalization of the usual types of particles: bosons have $\alpha = 0$ (symmetric wave functions) and their mean-field description is via models of the form (1-4), and fermions have $\alpha = 1$ (antisymmetric wave functions) and appropriate models for them are Hartree–Fock functionals (see [Bach 1992; Lieb and Simon 1977; Lions 1987; 1988; Fournais et al. 2015]). For general α one speaks of anyons [Khare 2005; Myrheim 1999; Ouvry 2009; Wilczek 1990], which are believed to emerge as quasiparticle excitations of certain condensed-matter systems [Arovas et al. 1984; Haldane 1983; Halperin 1984; Zhang et al. 2014; Cooper and Simon 2015; Lundholm and Rougerie 2016].

Anyons can be modeled as bosons (respectively, fermions) but with a many-body magnetic interaction of coupling strength α (respectively, $\alpha - 1$). It was shown in [Lundholm and Rougerie 2015] that the ground-state energy per particle of such a system is correctly described by the minimum of (1-1) (and the ground states by the corresponding minimizers) in a limit where, as the number of particles N goes to ∞, one takes $\alpha = \beta/N \to 0$. We refer to this limit as that of *almost-bosonic* anyons, with β determining how far we are from usual bosons.

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[1] Usually referred to as an *average*-field description in this context.

In the following we treat the anyon gas as fully described by a one-body wave function $u \in H^1(\mathbb{R}^2)$ minimizing (1-1) under the mass constraint (1-3). We shall consider asymptotic regimes for this minimization problem. The limit $\beta \to 0$ is trivial and leads to a linear theory for noninteracting bosons (see [Lundholm and Rougerie 2015, Appendix A]). The limit $\beta \to \infty$ is more interesting and more physically relevant: in a physical situation, the statistics parameter α is fixed and finite and N large, so that taking $\beta \to \infty$ is the relevant regime, at least if one is allowed to exchange the two limits.

In an approximation that has been used frequently in the physics literature [Chitra and Sen 1992; Iengo and Lechner 1992; Li et al. 1992; Trugenberger 1992a; 1992b; Wen and Zee 1990; Westerberg 1993], the ground-state energy per particle of the N-particle anyon gas with statistics parameter α is given by

$$\frac{E_0(N)}{N} \approx \int_{\mathbb{R}^2} (2\pi |\alpha| N \varrho^2 + V \varrho). \tag{1-5}$$

This relies on assuming that each particle sees the others by their approximately constant *average* magnetic field $B(\boldsymbol{x}) \approx 2\pi \alpha N \varrho(\boldsymbol{x})$, with $\varrho(\boldsymbol{x}) \geqslant 0$ the local particle density (normalized to $\int_{\mathbb{R}^2} \varrho = 1$). In the ground state of this magnetic field (the lowest Landau level) this leads to a magnetic energy $|B| \approx 2\pi |\alpha| N \varrho$ per particle.[2]

In this work we prove that, for large β, the behavior of the functional (1-1) is captured at leading order by a Thomas–Fermi-type [Catto et al. 1998; Lieb 1981] energy functional of a form similar to the right-hand side of (1-5) with $|\alpha| N = \beta$. The coupling constant appearing in this functional is defined via the large-volume limit of the homogeneous anyon gas energy (i.e., the infimum of (1-1) confined to a bounded domain with $V = 0$). In particular we prove that this limit exists and is bounded from below by the value 2π predicted by (1-5). We do not know the exact value, but there are good reasons to believe that it is *not* equal to 2π, thus refining the simple approximations leading to (1-5).

We state our main theorems in Section 2 and present their proofs in Sections 3 and 4. The Appendix recalls a few facts concerning the minimizers of (1-1). In particular, although we do not need it for the proof of our main results, we derive the associated variational equation.

2. Main results

We now proceed to state our main theorems. We first discuss the large-volume limit for the homogeneous gas in Section 2A and then state our results about the trapped anyons functional (1-1) in Section 2B.

2A. *Thermodynamic limit for the homogeneous gas.* Let $\Omega \subset \mathbb{R}^2$ be a fixed bounded domain in $\mathbb{R}^2$, with the associated energy for almost-bosonic anyons confined to it:

$$\mathcal{E}_\Omega^{\mathrm{af}}[u] = \mathcal{E}_{\Omega,\beta}^{\mathrm{af}}[u] := \int_\Omega \big|(-i\nabla + \beta \boldsymbol{A}[|u|^2])u\big|^2, \tag{2-1}$$

with

$$\boldsymbol{A}[|u|^2](\boldsymbol{x}) = \int_\Omega \nabla^\perp w_0(\boldsymbol{x} - \boldsymbol{y})|u(\boldsymbol{y})|^2 \, \mathrm{d}\boldsymbol{y}. \tag{2-2}$$

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[2]Because of the periodicity of the exchange phase $e^{i\alpha\pi}$, it is known that such an approximation can only be valid for certain values of α and ϱ. See [Larson and Lundholm 2016; Lundholm 2016; Trugenberger 1992b] for further discussion.

We define two energies, with homogeneous Dirichlet boundary conditions

$$E_0(\Omega, \beta, M) := \inf\{\mathcal{E}^{\mathrm{af}}_{\Omega,\beta}[u] : u \in H^1_0(\Omega), \ \textstyle\int_\Omega |u|^2 = M\}, \tag{2-3}$$

and without boundary conditions,

$$E(\Omega, \beta, M) := \inf\{\mathcal{E}^{\mathrm{af}}_{\Omega,\beta}[u] : u \in H^1(\Omega), \ \textstyle\int_\Omega |u|^2 = M\}. \tag{2-4}$$

Of course, the last minimization leads to a magnetic Neumann boundary condition for the solutions. We are interested in the thermodynamic limit of these quantities, i.e., the scaling limit in which the size of the domain tends to ∞ with fixed density $\rho := M/|\Omega|$ and the normalization changes accordingly.

Theorem 2.1 (Thermodynamic limit for the homogeneous anyon gas).
Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with Lipschitz boundary, and let $\beta \geqslant 0$ and $\rho \geqslant 0$ be fixed parameters. Then, the limits

$$e(\beta, \rho) := \lim_{L \to \infty} \frac{E(L\Omega, \beta, \rho L^2|\Omega|)}{L^2|\Omega|} = \lim_{L \to \infty} \frac{E_0(L\Omega, \beta, \rho L^2|\Omega|)}{L^2|\Omega|} \tag{2-5}$$

exist, coincide and are independent of Ω. Moreover,

$$e(\beta, \rho) = \beta\rho^2 e(1, 1). \tag{2-6}$$

Remark 2.2 (Error estimate).
A close inspection of the proof reveals that we also have an estimate of the error appearing in (2-5), which coincides with the error appearing in the estimate of the difference between the Neumann and Dirichlet energies in a box (Lemma 3.8). Such a quantity is expected to be of the order of the box's side length L, which is subleading if compared to the total energy of order L^2. Our error estimate $O(L^{12/7+\varepsilon})$ (see (3-26)) is however much larger and far from being optimal.   ⋄

The above result defines the thermodynamic energy per unit area at scaled statistics parameter β and density ρ, denoted $e(\beta, \rho)$, and shows that it has a nice scaling property. The latter is responsible for the occurrence of a Thomas–Fermi-type functional in the trapped anyons case. The fact that $e(\beta, \rho)$ does not depend on boundary conditions is a crucial technical ingredient in our study of the trapped case. This is very different from the usual Schrödinger energy in a fixed external magnetic field, for example, a constant one, for which the type of boundary conditions do matter (see, e.g., [Fournais and Helffer 2010, Chapter 5]).

The constant $e(1, 1)$ will be used to define a corresponding coupling parameter below. One may observe that (see Lemma 3.7)

$$e(1, 1) \geqslant 2\pi, \tag{2-7}$$

and we conjecture that this inequality is actually *strict*, contrary to what might be expected when comparing to the coupling constant of the conventional (constant-field) average-field approximation (1-5). The reason for this is that the self-interaction encoded by the functional $\mathcal{E}^{\mathrm{af}}$ has not been fully incorporated in (1-5). In fact, the lower bound (2-7) is based on a magnetic L^4-bound (Lemma 3.2) which is saturated only for constant functions, and hence for constant densities, which certainly is compatible with (1-5) in the case

of homogeneous traps. On the other hand, in order to minimize the magnetic energy in (2-1) for large β, the function has to have a large phase circulation and therefore also a large vorticity. This suggests the formation of an approximately homogeneous vortex lattice, in some analogy to the Abrikosov lattice that arises in superconductivity and in rotating bosonic gases [Aftalion 2007; Correggi and Yngvason 2008; Sandier and Serfaty 2007]. Such a picture has already been hinted at in [Chen et al. 1989, p. 1012] for the almost-bosonic gas. However the implication that the actual coupling constant may then be larger than the one expected from (1-5) seems not to have been observed in the literature before.

One should note here that there is a certain abuse of language in using the term "thermodynamic limit". Indeed, we consider the large-volume behavior of a mean-field energy functional, and there is no guarantee that this rigorously approximates the true thermodynamic energy of the underlying many-body system.

2B. *Local density approximation for the trapped gas.* We now return to (1-1) and discuss the ground state problem

$$E_\beta^{\mathrm{af}} := \min\{\mathcal{E}_\beta^{\mathrm{af}}[u] : u \in H^1(\mathbb{R}^2),\ V|u|^2 \in L^1(\mathbb{R}^2),\ \textstyle\int_{\mathbb{R}^2} |u|^2 = 1\}. \tag{2-8}$$

We denote by u^{af} any associated minimizer. We refer to the Appendix (see also [Lundholm and Rougerie 2015, Appendix A]) for a discussion of the minimization domain as well as the existence of a minimizer. In the limit $\beta \to \infty$, the simpler Thomas–Fermi-like functional

$$\mathcal{E}^{\mathrm{TF}}[\varrho] = \mathcal{E}_\beta^{\mathrm{TF}}[\varrho] := \int_{\mathbb{R}^2} (\beta e(1,1)\varrho^2 + V\varrho) \tag{2-9}$$

emerges, whose ground-state energy we denote by

$$E_\beta^{\mathrm{TF}} := \min\{\mathcal{E}_\beta^{\mathrm{TF}}[\varrho] : \varrho \in L^2(\mathbb{R}^2; \mathbb{R}^+),\ V\varrho \in L^1(\mathbb{R}^2),\ \textstyle\int_{\mathbb{R}^2} \varrho = 1\}, \tag{2-10}$$

with associated (unique) minimizer $\varrho_\beta^{\mathrm{TF}}$. Here $e(1,1)$ is the fixed, universal constant defined by Theorem 2.1.

A typical potential one could have in mind for physical relevance is a harmonic trap, $V(\boldsymbol{x}) = c|\boldsymbol{x}|^2$, or an asymmetric trap, $V(x, y) = c_1 x^2 + c_2 y^2$. We shall work under the assumption that V is homogeneous of degree s and smooth:

$$V(\lambda \boldsymbol{x}) = \lambda^s V(\boldsymbol{x}), \quad V \in C^\infty(\mathbb{R}^2). \tag{2-11}$$

These conditions can be relaxed significantly; in particular we could extend the approach to asymptotically homogeneous potentials as defined in [Lieb et al. 2001, Definition 1.1]. We refrain from doing so to avoid lengthy technical discussions in the proofs. We shall always impose that V is trapping in the sense that it grows superlinearly at infinity, i.e., $s > 1$ and

$$\min_{|\boldsymbol{x}| \geqslant R} V(\boldsymbol{x}) \to \infty \quad \text{as } R \to \infty. \tag{2-12}$$

The Thomas–Fermi (TF) problem (2-10) has the merit of being exactly soluble. We obtain by scaling

$$E_\beta^{\mathrm{TF}} = \beta^{s/(s+2)} E_1^{\mathrm{TF}}, \quad \varrho_\beta^{\mathrm{TF}}(\boldsymbol{x}) = \beta^{-2/(2+s)} \varrho_1^{\mathrm{TF}}(\beta^{-1/(s+2)} \boldsymbol{x}), \tag{2-13}$$

and by an explicit computation

$$\varrho_1^{\mathrm{TF}}(\boldsymbol{x}) = \frac{1}{2e(1,1)}(\lambda_1^{\mathrm{TF}} - V(\boldsymbol{x}))_+, \tag{2-14}$$

with the chemical potential

$$\lambda_1^{\mathrm{TF}} = E_1^{\mathrm{TF}} + e(1,1)\int_{\mathbb{R}^2}(\varrho_1^{\mathrm{TF}})^2. \tag{2-15}$$

Clearly the above considerations imply

$$\operatorname{supp}(\varrho_\beta^{\mathrm{TF}}) \subset B_{C\beta^{1/(2+s)}}(0), \tag{2-16}$$

where $B_R(\boldsymbol{x})$ stands for a ball (disk) of radius R centered at $\boldsymbol{x}$, and the estimates

$$\|\varrho_\beta^{\mathrm{TF}}\|_{L^\infty(\mathbb{R}^2)} \leqslant C\beta^{-2/(2+s)}, \quad \|\nabla\varrho_\beta^{\mathrm{TF}}\|_{L^\infty(\mathbb{R}^2)} \leqslant C\beta^{-3/(2+s)} \tag{2-17}$$

for some fixed constant $C > 0$. Noticing that ϱ_1^{TF} vanishes along a level curve of the smooth homogeneous potential V, we also have the nondegeneracy

$$|\partial_{\boldsymbol{n}}V| \neq 0 \quad \text{a.e. on } \partial\operatorname{supp}(\varrho_1^{\mathrm{TF}}), \tag{2-18}$$

where $\boldsymbol{n}$ denotes the (say outward) normal vector to $\partial\operatorname{supp}(\varrho_1^{\mathrm{TF}})$.

We have the following result showing the accuracy of TF theory to determine the leading order of the minimization problem (2-8):

Theorem 2.3 (Local density approximation for the anyon gas).
Let V satisfy (2-11) and (2-12). In the limit $\beta \to \infty$ we have the energy convergence

$$\lim_{\beta\to+\infty}\frac{E_\beta^{\mathrm{af}}}{E_\beta^{\mathrm{TF}}} = 1. \tag{2-19}$$

Moreover, for any function u^{af} achieving the infimum (2-8), with $\varrho^{\mathrm{af}} := |u^{\mathrm{af}}|^2$, we have for any $R > 0$

$$\left\|\beta^{2/(2+s)}\varrho^{\mathrm{af}}(\beta^{1/(2+s)}\,\cdot\,) - \varrho_1^{\mathrm{TF}}\right\|_{W^{-1,1}(B_R(0))} \to 0 \quad \text{as } \beta \to 0, \tag{2-20}$$

where $W^{-1,1}(B_R(0))$ is the dual space of Lipschitz functions on the ball $B_R(0)$.

Remark 2.4 (Extension to more general potentials).
The result can be straightforwardly extended to asymptotically homogeneous potentials, i.e., functions $V(\boldsymbol{x})$ that satisfy the following property [Lieb et al. 2001, Definition 1.1]: there exists another function $\widetilde{V}$, nonvanishing for $\boldsymbol{x} \neq 0$, such that, for some $s > 0$,

$$\lim_{\lambda\to\infty}\frac{\lambda^{-s}V(\lambda\boldsymbol{x}) - \widetilde{V}(\boldsymbol{x})}{1 + |\widetilde{V}(\boldsymbol{x})|} = 0 \tag{2-21}$$

uniformly in $\boldsymbol{x} \in \mathbb{R}^2$. The function $\widetilde{V}$ is necessarily homogeneous of degree $s > 0$ and, if we denote by $\widetilde{\mathcal{E}}_\beta$ the TF functional (2-9) with $\widetilde{V}$ in place of V, we have

$$E_\beta^{\mathrm{TF}} = (1+o(1))\widetilde{E}_\beta^{\mathrm{TF}} \quad \text{and} \quad \widetilde{E}_\beta^{\mathrm{TF}} = \beta^{s/(s+2)}\widetilde{E}_1^{\mathrm{TF}} \quad \text{as } \beta \to \infty. \qquad \diamond$$

Remark 2.5 (Density approximation on finer length scales).

We conjecture that the estimate (2-20) can be improved to show that ϱ^{af} is close to $\varrho_\beta^{\mathrm{TF}}$ on finer scales. Namely (2-20) implies that they are close on length scales of order $\beta^{1/(2+s)}$, which is the extent of the support of $\varrho_\beta^{\mathrm{TF}}$, but we expect them to be close on scales $\gg \beta^{-s/(2(s+2))}$, which is the smallest length scale appearing in our proofs. We however believe that the density convergence *cannot* hold on scales smaller than $\beta^{-s/(2(s+2))}$, for we expect the latter to be the length scale of a vortex lattice developed by minimizers. ◇

Remark 2.6 (Large β limit for the homogeneous gas on bounded domains).

We can think of the homogeneous gas by formally taking the limit $s \to \infty$ of the homogeneous potentials we have considered so far, which naturally leads to the restriction of the functional $\mathcal{E}^{\mathrm{af}}$ in (1-1) to bounded domains Ω with $V = 0$ and Dirichlet boundary conditions, that is, (2-1)–(2-3). In fact, we have by the scaling laws discussed in Section 3B,

$$\lim_{\beta \to +\infty} \frac{E(\Omega, \beta, 1)}{\beta} = \lim_{\beta \to +\infty} \frac{E_0(\Omega, \beta, 1)}{\beta} = |\Omega|^{-1} e(1, 1) \tag{2-22}$$

for any bounded and simply connected Ω with Lipschitz boundary. Convergence of the density to the TF minimizer ϱ_1^{TF} holds true in the same form as in (2-20). In this case ϱ_1^{TF} is simply the constant function on the domain (confirming that the gas is indeed homogeneous). The shortest length scale on which we expect (but cannot prove) the density convergence is $\beta^{-1/2}$, which should be the typical length scale of the vortex structure. ◇

3. Proofs for the homogeneous gas

The basic ingredient of the proof for the inhomogeneous case is the understanding of the thermodynamic limit of the model where the trap is replaced by a finite domain with sharp walls. We discuss this here, proving Theorem 2.1 and defining the constant $e(1, 1)$ appearing in the TF functional (2-9). For the sake of concreteness we first set

$$e(\beta, \rho) := \liminf_{L \to \infty} \frac{E_0(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2 |\Omega|} \tag{3-1}$$

for Ω equal to a unit square and observe that such a quantity certainly exists and is nonnegative. At this stage it might as well be infinite but we are going to prove that actually the limit exists, is finite, and furthermore is independent of the domain shape.

We briefly outline here the plan for the proof: Section 3A contains basic technical estimates that we are going to use throughout the paper. Section 3B contains the proof of a crucial scaling property of the energy in the homogeneous case. In Section 3C we prove the existence of the thermodynamic limit for the case of squares, and then extend the result to general domains.

3A. *Toolbox.* Let us gather a few lemmas that will be used repeatedly in the sequel. We start with a variational a priori upper bound confirming that the energy scales like the area. The idea of the proof, relying deeply on the magnetic nature of the interaction, will be employed again several times.

Lemma 3.1 (Trial upper bound).

For any fixed bounded domain Ω, and β, $\rho \geqslant 0$, there exists a constant $C > 0$ such that

$$\frac{E(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2} \leqslant \frac{E_0(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2} \leqslant C, \quad \text{for all } L \geqslant 1.$$

Proof. Since $H_0^1(\Omega) \subseteq H^1(\Omega)$, it is trivial that the Dirichlet energy is an upper bound to the Neumann energy. Let us then prove the second inequality.

We fill the domain $L\Omega$ with $N \sim L^2$ subdomains on which we use fixed trial states with Dirichlet boundary conditions. The crucial observation is that the magnetic interactions between subdomains can be canceled by a suitable choice of phase (local gauge transformation). For concreteness we here take disks as our subdomains.

Let $f \in C_c^\infty(B_1(0); \mathbb{R}^+)$ be a radial function with $\int_{B_1(0)} |f|^2 = 1$, and let

$$u_j(\boldsymbol{x}) := \sqrt{\omega_N} f(\boldsymbol{x} - \boldsymbol{x}_j) \in C_c^\infty(B_j), \quad \omega_N := \rho L^2 |\Omega| / N.$$

Here the points $\boldsymbol{x}_j$, $j = 1, \ldots, N$, are distributed in $L\Omega$ in such a way that the disks $B_j := B_1(\boldsymbol{x}_j)$ are contained in $L\Omega$ and disjoint, with $N \sim c|L\Omega|$ as $L \to \infty$ for some $c > 0$. Hence

$$\lim_{N \to \infty} \omega_N = \rho/c.$$

Take then the trial state

$$u(\boldsymbol{x}) := \sum_{j=1}^N u_j(\boldsymbol{x}) e^{-i\beta \omega_N \sum_{k \neq j} \arg(\boldsymbol{x} - \boldsymbol{x}_k)} \in C_c^\infty(L\Omega).$$

Note that its phase is smooth on each piece B_j of its support and that

$$|u(\boldsymbol{x})|^2 = \sum_{j=1}^N |u_j(\boldsymbol{x})|^2 = \begin{cases} |u_j(\boldsymbol{x})|^2 & \text{for } \boldsymbol{x} \in B_j, \\ 0 & \text{otherwise,} \end{cases}$$

and hence

$$\int_{L\Omega} |u|^2 = N\omega_N = \rho L^2 |\Omega|.$$

Then

$$\mathcal{E}_{\Omega,\beta}^{\text{af}}[u] = \sum_{j=1}^N \int_{B_j} \left| \left(-i\nabla + \beta \sum_{k=1}^N \boldsymbol{A}[|u_k|^2] \right) e^{-i\beta \omega_N \sum_{k \neq j} \arg(\boldsymbol{x} - \boldsymbol{x}_k)} u_j(\boldsymbol{x}) \right|^2 d\boldsymbol{x}$$

$$= \sum_{j=1}^N \int_{B_j} \left| \left(-i\nabla + \beta \boldsymbol{A}[|u_j|^2] + \sum_{k \neq j} \left(\beta \boldsymbol{A}[|u_k|^2] - \beta \omega_N \nabla \arg(\boldsymbol{x} - \boldsymbol{x}_k) \right) \right) u_j(\boldsymbol{x}) \right|^2 d\boldsymbol{x}$$

$$= \sum_{j=1}^N \int_{B_j} \left| \left(-i\nabla + \beta \boldsymbol{A}[|u_j|^2] \right) u_j \right|^2 = N\omega_N \int_{B_1(0)} \left| \left(-i\nabla + \beta \omega_N \boldsymbol{A}[|f|^2] \right) f \right|^2,$$

where we used that by Newton's theorem [Lieb and Loss 2001, Theorem 9.7]

$$\boldsymbol{A}[|u_k|^2](\boldsymbol{x}) = \nabla^\perp \int_{B_k} \ln |\boldsymbol{x} - \boldsymbol{y}| \, |u_k(\boldsymbol{y})|^2 \, d\boldsymbol{y} = \nabla^\perp \ln |\boldsymbol{x} - \boldsymbol{x}_k| \int_{B_k} |u_k|^2 \, d\boldsymbol{y} = \omega_N \nabla \arg(\boldsymbol{x} - \boldsymbol{x}_k)$$

for $x \notin B_k$. It then follows that

$$E_0(L\Omega, \beta, \rho L^2 |\Omega|) \leqslant \mathcal{E}^{\mathrm{af}}_{\Omega, \beta}[u] \leqslant N\omega_N \big(\|\nabla f\|_{L^2} + \beta \omega_N \big\| A[|f|^2] f \big\|_{L^2}\big)^2 \leqslant CL^2$$

for some large enough constant $C > 0$ independent of N or L (but possibly depending on β, ρ and Ω). $\square$

The following well-known inequalities provide useful a priori bounds on the functional's minimizers:

Lemma 3.2 (Elementary magnetic inequalities).
Diamagnetic inequality: _for any $\beta \in \mathbb{R}$ and $u \in H^1(\Omega)$,_

$$\int_\Omega \big| (\nabla + i\beta A[|u|^2]) u \big|^2 \geqslant \int_\Omega \big| \nabla |u| \big|^2. \tag{3-2}$$

Magnetic L^4 bound: _for any $\beta \in \mathbb{R}$ and $u \in H^1_0(\Omega)$,_

$$\int_\Omega \big| (\nabla + i\beta A[|u|^2]) u \big|^2 \geqslant 2\pi |\beta| \int_\Omega |u|^4. \tag{3-3}$$

Proof. The diamagnetic inequality is, e.g., given in [Lieb and Loss 2001, Theorem 7.21], while the L^4 bound follows immediately from the well-known inequality

$$\int_\Omega \big| (\nabla + iA) u \big|^2 \geqslant \pm \int_\Omega \operatorname{curl} A \, |u|^2, \quad u \in H^1_0(\Omega); \tag{3-4}$$

see, e.g., [Fournais and Helffer 2010, Lemma 1.4.1].

A proof of (3-4) is to integrate the identity

$$|(\nabla + iA) u|^2 = \big| ((\partial_1 + iA_1) \pm i(\partial_2 + iA_2)) u \big|^2 \pm \operatorname{curl} J[u] \pm A \cdot \nabla^\perp |u|^2,$$

with

$$J[u] := \frac{i}{2} (u\nabla\bar{u} - \bar{u}\nabla u).$$

Thanks to the Dirichlet boundary conditions, the integral of the next-to-last term vanishes, while the last one can be integrated by parts yielding

$$\mp \int_\Omega \operatorname{curl} A \, |u|^2.$$

Again, no boundary terms are present because of the vanishing of u on $\partial\Omega$. Dirichlet boundary conditions are necessary since the bound (3-4) (resp. (3-3)) is otherwise invalid as $A \to 0$ (resp. $\beta \to 0$), as can be seen by taking the trial state $u \equiv 1$. $\square$

In order to perform energy localizations we shall also need an IMS-type inequality,[3] i.e., a suitable generalization of the well-known localization formula [Cycon et al. 1987, Theorem 3.2]:

$$|\nabla u|^2 = |\nabla(\chi u)|^2 + |\nabla(\eta u)|^2 - \big(|\nabla \chi|^2 + |\nabla \eta|^2 \big) |u|^2, \tag{3-5}$$

where χ^2, η^2 form a partition of unity.

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[3]The initials IMS may refer either to Israel Michael Sigal or to Ismagilov–Morgan–Simon.

Lemma 3.3 (IMS formula).

Let $\Omega \subseteq \mathbb{R}^2$ be a domain with Lipschitz boundary and $\chi^2 + \eta^2 = 1$ be a partition of unity such that $\chi \in C_c^\infty(\Omega)$ and supp χ is simply connected. Then, for any $u \in H^1(\Omega)$ and $\beta \in \mathbb{R}$,

$$\mathcal{E}^{\mathrm{af}}_{\Omega,\beta}[u] = \int_\Omega \left|(\nabla + i\beta A[|u|^2])(\chi u)\right|^2 + \int_\Omega \left|(\nabla + i\beta A[|u|^2])(\eta u)\right|^2 - \int_\Omega \left(|\nabla\chi|^2 + |\nabla\eta|^2\right)|u|^2, \quad (3\text{-}6)$$

where

$$\int_\Omega \left|(\nabla + i\beta A[|u|^2])(\eta u)\right|^2 \geqslant \int_\Omega \left|\nabla|\eta u|\right|^2 \tag{3-7}$$

and

$$\int_\Omega \left|(\nabla + i\beta A[|u|^2])(\chi u)\right|^2 \geqslant \begin{cases} \displaystyle\int_\Omega \left|\nabla|\chi u|\right|^2, \\[2mm] \displaystyle 2\pi|\beta| \int_\Omega \chi^2 |u|^4, \\[2mm] \displaystyle (1-\varepsilon)\mathcal{E}^{\mathrm{af}}_{\Omega,\beta}[\psi] - (\varepsilon^{-1} - 1)\beta^2 \int_\Omega \left|A[|\eta u|^2 \mathbb{1}_K]\right|^2 |\chi u|^2, \end{cases} \tag{3-8}$$

with $\varepsilon \in (0,1)$ arbitrary, $K := \mathrm{supp}\,\chi \cap \mathrm{supp}\,\eta$, and $\psi = e^{i\beta\phi}\chi u \in H_0^1(\mathrm{supp}\,\chi)$ for some harmonic function $\phi \in C^2(\mathrm{supp}\,\chi)$.

Proof. We expand

$$\mathcal{E}^{\mathrm{af}}_{\Omega,\beta}[u] = \int_\Omega |\nabla u|^2 + 2\beta \int_\Omega A[|u|^2] \cdot J[u] + \beta^2 \int_\Omega \left|A[|u|^2]\right|^2 |u|^2.$$

For the first term we use the standard IMS formula (3-5), while for the term involving J we have

$$\frac{2}{i}\left(J[\chi u] + J[\eta u]\right) = u\chi\nabla(\chi\bar{u}) + u\eta\nabla(\eta\bar{u}) - \bar{u}\chi\nabla(\chi u) - \bar{u}\eta\nabla(\eta u)$$

$$= u(\chi^2 + \eta^2)\nabla\bar{u} - \bar{u}(\chi^2 + \eta^2)\nabla u = \frac{2}{i}J[u].$$

We can then recollect the terms to obtain (3-6). Equation (3-7) and the first version of (3-8) follow from the diamagnetic inequality (3-2), while the second version of (3-8) follows from the magnetic bound (3-3) with Dirichlet boundary conditions. For the third version we write

$$\int_\Omega \left|(\nabla + i\beta A[|u|^2])(\chi u)\right|^2 = \int_\Omega \left|(\nabla + i\beta A[|\chi u|^2] + i\beta A[|\eta u|^2 \mathbb{1}_K] + i\beta\left(A[|\eta u|^2 \mathbb{1}_{K^c}] - \nabla\phi\right))(e^{i\beta\phi}\chi u)\right|^2,$$

where the last magnetic term vanishes by taking the gauge choice

$$\phi(\boldsymbol{x}) := \int_{K^c} \arg(\boldsymbol{x} - \boldsymbol{y})|\eta u(\boldsymbol{y})|^2 \, d\boldsymbol{y}, \quad \boldsymbol{x} \in \mathrm{supp}\,\chi.$$

Thus, noting that $|\chi u|^2 = |\psi|^2$,

$$\int_\Omega \left|(\nabla + i\beta A[|u|^2])(\chi u)\right|^2 = \int_\Omega \left|(\nabla + i\beta A[|\psi|^2])\psi + i\beta A[|\eta u|^2 \mathbb{1}_K]\psi\right|^2,$$

and we can conclude by expanding the square and bounding the cross-term using Cauchy–Schwarz. $\square$

3B. *Scaling laws.* In fact the large β and large volume limits are equivalent, as follows from the simple observation:

Lemma 3.4 (Scaling laws for the homogeneous gas).
For any domain $\Omega \subset \mathbb{R}^2$ and $\lambda, \mu > 0$ we have

$$E(\Omega, \beta, M) = \frac{1}{\lambda^2} E\left(\mu\Omega, x\frac{\beta}{\lambda^2\mu^2}, \lambda^2\mu^2 M\right), \tag{3-9}$$

and an identical scaling relation holds true for $E_0(\Omega, \beta, M)$.

Proof. Given any $u \in H^1(\Omega)$ we may set

$$u_{\lambda,\mu}(x) := \lambda u(x/\mu), \tag{3-10}$$

and observe that $u_{\lambda,\mu} \in H^1(\mu\Omega)$,

$$\int_{\mu\Omega} |u_{\lambda,\mu}|^2 = \lambda^2\mu^2 \int_{\Omega} |u|^2 \quad \text{and} \quad \mathcal{E}^{\mathrm{af}}_{\mu\Omega,\beta}[u_{\lambda,\mu}] = \lambda^2 \mathcal{E}^{\mathrm{af}}_{\Omega,\beta\lambda^2\mu^2}[u].$$

Namely, using $\nabla^\perp w_0(x) = x^{-\perp} := x^\perp/|x|^2$ and

$$\begin{aligned}
A_{\mu\Omega}[|u_{\lambda,\mu}|^2](x) &= \int_{\mu\Omega} (x-y)^{-\perp}|u_{\lambda,\mu}(y)|^2 \, dy = \lambda^2 \int_{\mu\Omega} (x-y)^{-\perp}|u(y/\mu)|^2 \, dy \\
&= \lambda^2\mu \int_{\Omega} (x/\mu - z)^{-\perp}|u(z)|^2 \, dz = \lambda^2\mu A_\Omega[|u|^2](x/\mu),
\end{aligned}$$

we have

$$\begin{aligned}
\mathcal{E}^{\mathrm{af}}_{\mu\Omega,\beta}[u_{\lambda,\mu}] &= \int_{\mu\Omega} \left|\nabla u_{\lambda,\mu}(x) + i\beta A_{\mu\Omega}[|u_{\lambda,\mu}|^2](x)u_{\lambda,\mu}(x)\right|^2 dx \\
&= \int_{\mu\Omega} \left|\lambda\mu^{-1}(\nabla u)(x/\mu) + i\beta\lambda^3\mu A_\Omega[|u|^2](x/\mu)u(x/\mu)\right|^2 dx \\
&= \lambda^2\mu^{-2} \int_{\mu\Omega} \left|(\nabla u)(x/\mu) + i\beta\lambda^2\mu^2 A_\Omega[|u|^2](x/\mu)u(x/\mu)\right|^2 dx \\
&= \lambda^2 \int_{\Omega} \left|\nabla u(z) + i\beta\lambda^2\mu^2 A_\Omega[|u|^2](z)u(z)\right|^2 dz = \lambda^2 \mathcal{E}^{\mathrm{af}}_{\Omega,\beta\lambda^2\mu^2}[u].
\end{aligned}$$

Hence, we may take as a trial state for $\mathcal{E}^{\mathrm{af}}_{\mu\Omega,\beta\lambda^{-2}\mu^{-2}}$ the function $u_{\lambda,\mu}$, where u is the minimizer (or minimizing sequence) of $\mathcal{E}^{\mathrm{af}}_{\Omega,\beta}$, and vice versa. Moreover, if $u \in H^1_0$ then so is $u_{\lambda,\mu}$. $\qquad\square$

It follows immediately from the above that the thermodynamic energy has a very simple dependence on its parameters, which justifies (2-6) and the way it appears in (2-9).

Corollary 3.5 (Scaling laws for $e(\beta, \rho)$).
For any $\rho \geqslant 0$ and bounded $\Omega \subset \mathbb{R}^2$, with $e(\beta, \rho)$ defined as in (3-1), we have

$$e(1, \rho) = |\Omega| \liminf_{\beta\to\infty} \frac{E_0(\Omega, \beta, \rho)}{\beta}, \tag{3-11}$$

and for any $\beta, \rho \geqslant 0$,

$$e(\beta, \rho) = \beta\rho^2 e(1, 1). \tag{3-12}$$

Remark 3.6. At the moment each shape of the domain Ω may give rise to a different limit $e(\beta, \rho)$ in (3-1), and this corollary and proof apply in such a situation. However, it will be shown below in the case of Lipschitz regular domains that the limit is independent of the shape, and one may therefore without loss of generality take the unit square $\Omega = Q$ as a reference domain.

Proof. A first consequence of the scaling property (3-9) is that taking the thermodynamic limit as described in (2-5) or (3-1) is equivalent to taking the limit $\beta \to \infty$ at a fixed size of the domain, i.e.,

$$e(c, \rho) = \liminf_{L \to \infty} \frac{E_0(L\Omega, c, \rho |\Omega| L^2)}{L^2 |\Omega|} = \liminf_{L \to \infty} \frac{E_0(\Omega, cL^2 |\Omega|, \rho)}{L^2},$$

where we have applied (3-9) with $\mu = L$, $\lambda = |\Omega|^{1/2}$ and $M = \rho$. Now if, for any $c > 0$, we set $\beta = cL^2 |\Omega| \to \infty$, the above expression becomes

$$e(c, \rho) = c|\Omega| \liminf_{\beta \to \infty} \frac{E_0(\Omega, \beta, \rho)}{\beta}, \tag{3-13}$$

which proves the first claim, and also implies

$$e(c, \rho) = c\, e(1, \rho). \tag{3-14}$$

Next we take $\mu = 1$ in (3-9) and obtain

$$E_0(\Omega, \beta, M) = \lambda^{-2} E_0(\Omega, \beta \lambda^{-2}, \lambda^2 M).$$

Taking $M = |\Omega|$, dividing by $|\Omega|$ and taking the limit $|\Omega| \to \infty$, we deduce

$$e(\beta, 1) = \lambda^{-2} e(\beta \lambda^{-2}, \lambda^2) = \lambda^{-4} e(\beta, \lambda^2),$$

where we used (3-14) in the last equality. This yields

$$e(\beta, \rho) = \rho^2 e(\beta, 1) \tag{3-15}$$

for all $\beta, \rho \geqslant 0$. Combining (3-14) and (3-15) yields the result (3-12). $\qquad\square$

3C. *Proof of Theorem 2.1.* We split the proof in three lemmas:

Lemma 3.7 (Thermodynamic limit for the Dirichlet energy in a square).
Let Q be a unit square, and $\rho \geqslant 0$ and $\beta \geqslant 0$ be fixed constants. The limit

$$e(\beta, \rho) = \lim_{L \to +\infty} \frac{E_0(LQ, \beta, \rho L^2)}{L^2}$$

exists, is finite, and satisfies $e(\beta, \rho) \geqslant 2\pi\beta\rho^2$.

Lemma 3.8 (Neumann–Dirichlet comparison).
Let Ω be a bounded simply connected domain with Lipschitz boundary. Then for any fixed ρ and β positive, as $L \to \infty$

$$\frac{E_0(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2 |\Omega|} \geqslant \frac{E(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2 |\Omega|} \geqslant \frac{E_0(L\Omega, \beta, \rho L^2 |\Omega|)}{L^2 |\Omega|} - o(1).$$
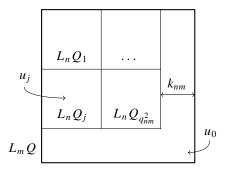
Figure 1. Filling the square $L_m Q$ with smaller squares $L_n Q_j$.

Lemma 3.9 (Thermodynamic limit for the Dirichlet energy in a general domain).
Let $\Omega \subset \mathbb{R}^2$ be a bounded simply connected domain with Lipschitz boundary, then

$$\lim_{L \to +\infty} \frac{E_0(L\Omega, \beta, \rho L^2|\Omega|)}{L^2|\Omega|} = e(\beta, \rho). \tag{3-16}$$

Theorem 2.1 immediately follows from these three results: combining Lemma 3.7 with Lemma 3.8 one obtains the existence of the thermodynamic limit for squares. In order to derive the result for general domains, one then uses Lemma 3.9 together with Lemma 3.8. Notice that the proof of Lemma 3.9 requires only Lemmas 3.7 and 3.8 for squares as key ingredients.

Proof of Lemma 3.7. From Lemma 3.1 we know that the sequence of energies per unit area has both an upper and lower limit. We denote by $(L_n)_{n\in\mathbb{N}}$ and $(L_m)_{m\in\mathbb{N}}$ two increasing sequences of positive real numbers such that $L_n \to \infty$, $L_m \to \infty$ and

$$\frac{E_0(L_n Q, \beta, \rho L_n^2)}{L_n^2} \to \liminf_{L \to \infty} \frac{E_0(L Q, \beta, \rho L^2)}{L^2} \quad \text{as } n \to \infty,$$

$$\frac{E_0(L_m Q, \beta, \rho L_m^2)}{L_m^2} \to \limsup_{L \to \infty} \frac{E_0(L Q, \beta, \rho L^2)}{L^2} \quad \text{as } m \to \infty.$$

For each n, there must exist a sequence of integers

$$q_{nm} \to +\infty \quad \text{as } m \to \infty$$

such that, for m large enough, e.g., $m \gg n$,

$$L_m = q_{nm} L_n + k_{nm}, \quad 0 \leqslant k_{nm} < L_n.$$

We then build a trial state for $E_0(L_m Q, \beta, \rho L_m^2)$ as follows (see Figure 1). The square $L_m Q$ must contain q_{nm}^2 disjoint squares of side length L_n that we denote by $L_n Q_j$, $j = 1, \ldots, q_{nm}^2$. Then we pick u_j a minimizer of $E_0(L_n Q_j, \beta, \rho L_n^2)$ and remark that by definition,

$$\sum_{k=1, k \neq j}^{q_{nm}^2} \operatorname{curl} A[|u_k|^2] = 0 \quad \text{in } L_n Q_j.$$

Thus there exists a gauge phase ϕ_j on the simply connected domain $L_n Q_j$ such that

$$\sum_{k=1, k \neq j}^{q_{nm}^2} A[|u_k|^2] = \nabla \phi_j \quad \text{in } L_n Q_j.$$

Similarly, there exists ϕ_0 on the remaining part of the domain (which can be arranged to be simply connected as well, as in Figure 1) such that

$$\sum_{k=1}^{q_{nm}^2} A[|u_k|^2] = \nabla \phi_0 \quad \text{on } L_m Q \setminus \bigcup_{j=1}^{q_{nm}^2} L_n Q_j.$$

We define the trial state as (see the proof of Lemma 3.1)

$$u := \sum_{j=1}^{q_{nm}^2} u_j e^{-i\beta\phi_j} + u_0 e^{-i\beta\phi_0},$$

where u_0 is a function with compact support in $L_m Q \setminus \bigcup_{j=1}^{q_{nm}^2} L_n Q_j$ satisfying

$$\int_{L_m Q} |u_0|^2 = \rho L_m^2 - q_{nm}^2 \rho L_n^2.$$

By Lemma 3.1, we can construct u_0 such that

$$\int_{L_m Q} \left|(\nabla + i\beta A[|u_0|^2])u_0\right|^2 \leqslant C(L_m^2 - q_{nm}^2 L_n^2) \leqslant 2C L_m k_{nm}$$

(where $C > 0$ may depend on β and ρ). The function u is an admissible trial state on $L_m Q$ because it is in H^1 on each subdomain, and continuous across boundaries due to the Dirichlet boundary conditions satisfied by each u_j. Computing the energy, we have

$$\mathcal{E}_{L_m Q, \beta}^{\text{af}}[u] = \sum_{j=0}^{q_{nm}^2} \int_{L_m Q} \left|e^{-i\phi_j}\left(\nabla + i\beta A[|u|^2] - i\beta\nabla\phi_j\right)u_j\right|^2 = \sum_{j=0}^{q_{nm}^2} \int_{L_m Q} \left|(\nabla + i\beta A[|u_j|^2])u_j\right|^2$$

$$= \sum_{j=1}^{q_{nm}^2} \mathcal{E}_{L_n Q, \beta}^{\text{af}}[u_j] + \int_{L_m Q} \left|(\nabla + i\beta A[|u_0|^2])u_0\right|^2 = q_{nm}^2 E_0(L_n Q, \beta, \rho L_n^2) + O(L_m k_{nm}),$$

with

$$q_{nm}^2 = \frac{L_m^2}{L_n^2}\left(1 - \frac{k_{nm}}{L_m}\right)^2.$$

Since u has by definition mass ρL_m^2, it follows from the variational principle that

$$\frac{E_0(L_m Q, \beta, \rho L_m^2)}{L_m^2} \leqslant \frac{E_0(L_n Q, \beta, \rho L_n^2)}{L_n^2}\left(1 + O\left(\frac{k_{nm}}{L_m}\right)\right) + O\left(\frac{k_{nm}}{L_m}\right).$$
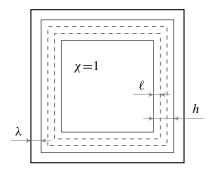
Figure 2. Localizing on thin shells for the square LQ.

Passing to the limit $m \to \infty$ first and then $n \to \infty$ yields

$$\limsup_{L \to \infty} \frac{E_0(LQ, \beta, \rho L^2)}{L^2} \leqslant \liminf_{L \to \infty} \frac{E_0(LQ, \beta, \rho L^2)}{L^2},$$

and thus the limit exists.

Additionally, we have by the bound (3-3),

$$\frac{1}{L^2} \mathcal{E}_{LQ,\beta}^{\mathrm{af}}[u] \geqslant \frac{2\pi\beta}{L^2} \int_{LQ} |u|^4 \geqslant \frac{2\pi\beta}{L^4} \left(\int_{LQ} |u|^2 \right)^2$$

for any $u \in H_0^1(LQ)$, proving that $e(\beta, \rho) \geqslant 2\pi\beta\rho^2$. □

Proof of Lemma 3.8. Since $H_0^1(\Omega) \subseteq H^1(\Omega)$, we obviously have

$$E_0(\Omega, \beta, M) \geqslant E(\Omega, \beta, M).$$

Only the second inequality in the statement requires some work. Let $u \in H^1(L\Omega)$ denote the minimizer of $\mathcal{E}_{L\Omega,\beta}^{\mathrm{af}}[u]$ (see Proposition A.1 of the Appendix) with mass

$$\int_{L\Omega} |u|^2 = \rho L^2 |\Omega|$$

and no further constraint (thus satisfying Neumann boundary conditions). In the sequel we take $\beta = 1$ and $|\Omega| = 1$ to simplify the notation.

We will need to make an IMS localization on a small enough region, and therefore consider a division of $L\Omega$ into a bulk region surrounded by thin shells close to the boundary, where we will be using several different length scales $L^{-1/3} \lesssim \lambda \ll 1 \ll L$ and $L^{-1} \ll \ell \ll h \ll L$ (see Figure 2 for the case of $\Omega = Q$ a square).

We shall use Lemma 3.3 a first time at distance λ from the boundary to deduce some useful a priori bounds. Next, using a mean-value argument we show that, within a window of thickness h further from the boundary, there must exist one particular shell of thickness ℓ where we have a good control on the mass and energy. Finally we perform a second IMS localization with the truncation located in this particular shell. This yields a lower bound in terms of the Dirichlet energy in the bulk region, plus error terms that

we can control using the a priori bounds and in particular the good control on mass and energy in the second localization shell.

Step 1: a priori bounds. Let $\delta_\Omega(\boldsymbol{x}) := \mathrm{dist}(\boldsymbol{x}, \partial(L\Omega))$ denote the distance function to the boundary, which is Lipschitz and satisfies $|\nabla \delta_\Omega| \leqslant 1$ a.e. We make a first partition of unity

$$\tilde{\chi}^2 + \tilde{\eta}^2 = 1$$

such that $\tilde{\chi}$ varies smoothly from 1 to 0 on a shell K_λ of width λ closest to the boundary of $L\Omega$, i.e., $K_\lambda := \{\boldsymbol{x} \in L\Omega : \delta_\Omega(\boldsymbol{x}) < \lambda\}$. One may note that it is possible to construct these functions so as to satisfy

$$|\nabla \tilde{\chi}| \leqslant c\lambda^{-1} \tilde{\chi}^{1-\mu}, \quad |\nabla \tilde{\eta}| \leqslant c\lambda^{-1} \tilde{\chi}^{1-\mu}$$

for some arbitrarily small $\mu > 0$, independent of λ, e.g., by taking $\tilde{\chi} = f^a$ and $\tilde{\eta} = \sqrt{1 - \tilde{\chi}^2}$ in $\mathrm{supp}\, \tilde{\chi} \cap \mathrm{supp}\, \tilde{\eta}$ for a large and some smooth function $0 \leqslant f \leqslant 1$ varying on the right length scale and reflection symmetric. Then, by Lemmas 3.1 and 3.3,

$$CL^2 \geqslant \mathcal{E}_{L\Omega,1}^{\mathrm{af}}[u] \geqslant \int_{L\Omega} \left(2\pi \tilde{\chi}^2 |u|^4 + \big|\nabla|\tilde{\eta}u|\big|^2 - (|\nabla \tilde{\chi}|^2 + |\nabla \tilde{\eta}|^2)|u|^2\right)$$

$$\geqslant \int_{L\Omega} \left(2\pi \tilde{\chi}^2 |u|^4 + \big|\nabla|\tilde{\eta}u|\big|^2 - C\lambda^{-2}\mathbb{1}_{K_\lambda}\tilde{\chi}^{2-2\mu}|u|^2\right). \tag{3-17}$$

We bound the unwanted negative term as follows:

$$\lambda^{-2}\int_{L\Omega} \mathbb{1}_{K_\lambda}\tilde{\chi}^{2-2\mu}|u|^2 \leqslant \lambda^{-2}\left(\int_{K_\lambda}\tilde{\chi}^{2-4\mu}\right)^{1/2}\left(\int_{K_\lambda}\tilde{\chi}^2|u|^4\right)^{1/2}$$

$$\leqslant C\lambda^{-3/2}L^{1/2}\left(\int_{K_\lambda}\tilde{\chi}^2|u|^4\right)^{1/2} \leqslant C\delta L\lambda^{-3} + C\delta^{-1}\int_{L\Omega}\tilde{\chi}^2|u|^4,$$

with δ a fixed, large enough, constant. Combining with (3-17) we deduce

$$\int_{L\Omega}\left(2\pi \tilde{\chi}^2|u|^4 + \big|\nabla|\tilde{\eta}u|\big|^2\right) \leqslant CL^2 + CL\lambda^{-3} \leqslant CL^2 \tag{3-18}$$

since we have chosen $\lambda \gtrsim L^{-1/3}$. We note that this bound implies for the mass in a shell K_ℓ of thickness ℓ in $L\Omega \setminus K_\lambda$

$$\int_{K_\ell}|u|^2 \leqslant |K_\ell|^{1/2}\left(\int_{K_\ell}\tilde{\chi}^2|u|^4\right)^{1/2} \lesssim \ell^{1/2}L^{3/2}. \tag{3-19}$$

Step 2: finding a good shell. We now select a region where the bounds (3-18) and (3-19) can be improved. Consider dividing $L\Omega \setminus K_\lambda$ into shells of thickness ℓ that form a layer closest to the shell K_λ of total thickness $h \sim L^{1-\varepsilon} \gg \ell$ (again, see Figure 2). Hence, we have

$$N_s := h/\ell \gg 1$$

such shells in the layer. Denote by N_M the number of such shells K_ℓ with $\int_{K_\ell}|u|^4 \geqslant M$. If $N_M < N_s$, there must exist a shell K_ℓ with $\int_{K_\ell}|u|^4 \leqslant M$. But, using (3-18) and the fact that all the shells are included

in the region where $\tilde{\chi} = 1$, we have

$$MN_M \leqslant \int_{L\Omega} \tilde{\chi}^2 |u|^4 \leqslant CL^2.$$

We can thus ensure that $N_M < N_s$ by setting

$$N_s = h/\ell \sim L^{1-\varepsilon}\ell^{-1} \sim L^2/M,$$

i.e., taking $M \sim \ell L^{1+\varepsilon}$. Hence we have found a shell K_ℓ with

$$\int_{K_\ell} |u|^4 \leqslant C\ell L^{1+\varepsilon}, \tag{3-20}$$

and thus

$$\int_{K_\ell} |u|^2 \leqslant C(\ell L)^{1/2}(\ell L^{1+\varepsilon})^{1/2} = C\ell L^{1+\varepsilon/2}, \tag{3-21}$$

improving (3-19).

<u>Step 3</u>: IMS localization in the good shell. We now perform a new magnetic localization on this K_ℓ. We pick a partition $\chi^2 + \eta^2 = 1$, such that χ varies smoothly from 1 to 0 outwards on K_ℓ, so that $\chi = 1$ (resp. $\eta = 1$) on the inner (resp. outer) component of K_ℓ^c. Then, using Lemma 3.3, we have

$$\mathcal{E}_{L\Omega,1}^{\mathrm{af}}[u] \geqslant (1-\delta)\mathcal{E}_{L\Omega,1}^{\mathrm{af}}[\psi] - (\delta^{-1}-1)\int_{L\Omega}\left|A[|\eta u|^2 \mathbb{1}_K]\right|^2 |\chi u|^2 - \int_{K_\ell}(|\nabla\chi|^2 + |\nabla\eta|^2)|u|^2 \tag{3-22}$$

for any $\delta \in (0,1)$, where we let $\psi = \chi e^{i\phi}u$ and $K = \operatorname{supp}\chi \cap \operatorname{supp}\eta \subseteq K_\ell$. Since ψ is compactly supported in $L\Omega$, we have for the first term

$$\mathcal{E}_{L\Omega,1}^{\mathrm{af}}[\psi] \geqslant E_0(L\Omega, 1, \|\psi\|_{L^2(L\Omega)}^2) = E_0(L\Omega, 1, \|\chi u\|_{L^2(L\Omega)}^2).$$

Recalling the scaling relation (3-9) (taking $\mu = \lambda^{-1} = \widetilde{L}/L$) and defining

$$M = \int_{L\Omega} \chi^2 |u|^2, \quad \widetilde{L} = \sqrt{M/\rho},$$

we have

$$E_0(L\Omega, 1, M) = \frac{M}{\rho L^2} E_0(\widetilde{L}\Omega, 1, \rho\widetilde{L}^2). \tag{3-23}$$

We need to estimate the deviation of the mass M of $\chi^2|u|^2$ from $\rho L^2 = \int_{L\Omega}|u|^2$:

$$\begin{aligned}
\left|\rho L^2 - \int_{L\Omega}\chi^2|u|^2\right| &= \int_{L\Omega}\eta^2|u|^2 = \int_{L\Omega}\tilde{\eta}^2|u|^2 + \int_{L\Omega}\tilde{\chi}^2\eta^2|u|^2 \\
&\leqslant C\lambda^2\int_{K_\lambda}\left|\nabla|\tilde{\eta}u|\right|^2 + \left(\int_{L\Omega}\eta^2\tilde{\chi}^2\right)^{1/2}\left(\int_{L\Omega}\tilde{\chi}^2|u|^4\right)^{1/2} \\
&\leqslant C\lambda^2 L^2 + Ch^{1/2}L^{3/2} \ll L^2. \tag{3-24}
\end{aligned}$$

Here we have used a Poincaré inequality to control the $\tilde{\eta}^2|u|^2$ term, making use of the fact that this function vanishes at the inner boundary of K_λ. It is not difficult (see the proof methods of [Evans 1998, Theorems 1 and 2 in Section 5.8.1] and [Lieb and Loss 2001, Theorem 8.11]) to realize that the constant

involved in this inequality applied on the set K_λ can be taken to be proportional to λ^2. Note that $\widetilde{L} \to \infty$, if $L \to \infty$, thanks to (3-24). Hence, inserting the above estimate in (3-23), we get

$$\frac{\mathcal{E}^{\mathrm{af}}_{L\Omega,1}[\psi]}{L^2} \geqslant \frac{E_0(L\Omega, 1, M)}{L^2} = \frac{M^2}{(\rho L^2)^2} \frac{E_0(\widetilde{L}\Omega, 1, \rho\widetilde{L}^2)}{\widetilde{L}^2} = (1+o(1))\frac{E_0(\widetilde{L}\Omega, 1, \rho\widetilde{L}^2)}{\widetilde{L}^2}. \qquad (3\text{-}25)$$

Then, there only remains to control the error terms in (3-22): Using the Hölder and generalized Young inequalities ($\|\cdot\|_{p,w}$ denotes the weak-L^p norm [Lieb and Loss 2001, Theorem 4.3, Remarks]),

$$\int_{L\Omega} \big|A[|\eta u|^2 \mathbb{1}_K]\big|^2 |\chi u|^2 \leqslant \big\|\nabla w_0 * |\eta u|^2 \mathbb{1}_K\big\|_{2p}^2 \|\chi u\|_{2q}^2 \leqslant c\|\nabla w_0\|_{2,w}^2 \|\eta u \mathbb{1}_K\|_{2r}^4 \|\chi u\|_{2q}^2$$

$$\leqslant C\left(\int_{K_\ell} |\eta u|^{4q/(2q-1)}\right)^{(2q-1)/q} \left(\int_{L\Omega} |\chi u|^{2q}\right)^{1/q},$$

where

$$\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad 1 + \frac{1}{2p} = \frac{1}{2} + \frac{1}{r},$$

that is,

$$r = \frac{2q}{2q-1} \in (1, 2) \quad \text{with } q \in (1, \infty).$$

We can take $q = 2$ and insert (3-18)–(3-20) to obtain

$$\left(\int_{K_\ell} |\eta u|^{8/3}\right)^{3/2} \left(\int_{L\Omega} |\chi u|^4\right)^{1/2} \leqslant |K_\ell|^{1/2} \int_{K_\ell} |\eta u|^4 \left(\int_{L\Omega} |\chi u|^4\right)^{1/2} \lesssim (\ell L)^{1/2} \ell L^{1+\varepsilon} (L^2)^{1/2} = \ell^{3/2} L^{5/2+\varepsilon}.$$

The last term in (3-22) is, using (3-21), bounded by

$$c\ell^{-2} \int_{K_\ell} |u|^2 \lesssim \ell^{-1} L^{1+\varepsilon/2}.$$

There only remains to optimize the error terms in (3-22):

$$\delta E_0(L\Omega, 1, \|\psi\|_{L^2(L\Omega)}^2) + c_1(\delta^{-1} - 1)\ell^{3/2} L^{5/2+\varepsilon} + c_2\ell^{-1} L^{1+\varepsilon/2} \leqslant c_3\delta L^2 + c_4\delta^{-2/5} L^{8/5+7\varepsilon/10},$$

where we have picked $\ell = L^{-3/5-\varepsilon/5}\delta^{2/5}$, assuming that $\delta \ll 1$, as it will be. Thus, optimizing now over δ, i.e., taking $\delta \sim L^{-2/7+\varepsilon/2}$, we have the bounds

$$\frac{E_0(L\Omega, 1, \rho L^2)}{L^2} \geqslant \frac{E(L\Omega, 1, \rho L^2)}{L^2} \geqslant \frac{E_0(L\Omega, 1, \|\psi\|_{L^2(L\Omega)}^2)}{L^2} - cL^{-2/7+\varepsilon/2}. \qquad (3\text{-}26)$$

Combining with (3-25) and passing to the liminf completes the proof. $\qquad \square$

Proof of Lemma 3.9. The result is proven as usual by comparing suitable upper and lower bounds to the energy.

<u>Step 1</u>: upper bound. We first cover $L\Omega$ with squares Q_j, $j = 1, \ldots, N_\ell$, of side length $\ell = L^\eta$, $0 < \eta < 1$, retaining only the squares Q_j completely contained in $L\Omega$. One can estimate the area not covered by

such squares as

$$\left| \Omega \setminus \left(\bigcup_{j=1}^{N_\ell} Q_j \right) \right| \leqslant C\ell L = o(L^2). \tag{3-27}$$

Then we define the trial state

$$u(\boldsymbol{x}) := \sum_{j=1}^{N_\ell} u_j e^{-i\beta\phi_j}, \tag{3-28}$$

where

$$u_j(\boldsymbol{x}) := u_0(\boldsymbol{x} - \boldsymbol{x}_j) \mathbb{1}_{Q_j}, \tag{3-29}$$

with u_0 a minimizer of the Dirichlet problem with mass $\rho L^2 |\Omega|/N_\ell$ in a square Q with side length ℓ centered at the origin, and $\boldsymbol{x}_j$ the center point of Q_j. The phases ϕ_j are chosen in such a way that (see the proof of Lemma 3.1 again)

$$\sum_{k=1, k \neq j}^{N_\ell} \boldsymbol{A}[|u_k|^2] = \nabla\phi_j \quad \text{in } Q_j.$$

The existence of such phases is indeed guaranteed by the fact that

$$\sum_{k=1, k \neq j}^{N_\ell} \operatorname{curl} \boldsymbol{A}[|u_k|^2] = 0 \quad \text{in } Q_j.$$

Hence

$$\mathcal{E}_{L\Omega,\beta}^{\text{af}}[u] = \sum_{j=1}^{N_\ell} \mathcal{E}_{Q_j,\beta}^{\text{af}}[u_j] = \sum_{j=1}^{N_\ell} E_0(\ell Q, \beta, \rho L^2 |\Omega| N_\ell^{-1}),$$

which implies

$$\frac{E_0(L\Omega, \beta, \rho L^2)}{L^2 |\Omega|} \leqslant \frac{1}{L^2 |\Omega|} \sum_{j=1}^{N_\ell} E_0(\ell Q, \beta, \rho L^2 |\Omega| N_\ell^{-1})$$

$$= \frac{\ell^2}{L^2 |\Omega|} \sum_{j=1}^{N_\ell} E_0(\ell Q, \beta, (1 + o(1))\rho\ell^2)/\ell^2 = (1 + o(1))e(\beta, \rho), \tag{3-30}$$

where we have estimated

$$N_\ell = \frac{\left| \bigcup_j Q_j \right|}{|Q_j|} = \frac{(1 + o(1))L^2 |\Omega|}{\ell^2}, \tag{3-31}$$

and used Lemma 3.7. Notice that, thanks to the assumption on η, we have $\ell \to \infty$, which is crucial in order to apply Lemma 3.7.

<u>Step 2</u>: lower bound. We again cover $L\Omega$ with squares Q_j, $j = 1, \ldots, N_\ell$, this time keeping the full covering but still having $\ell^2 N_\ell /|L\Omega| \to 1$ as $L \to \infty$. We pick a minimizer $u^{\text{af}} = u_L^{\text{af}} \in H_0^1(L\Omega)$ of $\mathcal{E}_{L\Omega,\beta}^{\text{af}}$, with mass $\rho L^2 |\Omega|$, and set

$$u_j^{\text{af}} := u^{\text{af}} \mathbb{1}_{Q_j}, \quad \rho_j := \fint_{Q_j} |u^{\text{af}}(\boldsymbol{x})|^2 \, d\boldsymbol{x}. \tag{3-32}$$

The idea of the proof is reminiscent of that in the upper bound part: we gauge away the magnetic interaction between the cells, and this leads to a lower bound in terms of the Neumann energy of the cells.

Note that $u_j^{\mathrm{af}} \in H^1(Q_j)$ for each j, and

$$\sum_{j=1}^{N_\ell} \rho_j \ell^2 = \rho L^2 |\Omega|.$$

Before estimating the energy, we need to distinguish between squares with sufficient mass and squares which will not contribute to the energy to leading order. We thus set

$$\mathcal{Q}_L := \left\{ Q_j, \, j \in \{1, \ldots, N_\ell\} \, : \, \rho_j \geqslant L^{-2\eta+\delta} \right\} \tag{3-33}$$

for some $0 < \delta < 2\eta$. Note that the mass concentrated outside cells $\mathcal{Q}_L$ is relatively small:

$$\sum_{Q_j \notin \mathcal{Q}_L} \rho_j \ell^2 \leqslant C \ell^2 N_\ell L^{-2\eta+\delta} = o(L^2). \tag{3-34}$$

We can now estimate, using the gauge covariance of the functional on each Q_j,

$$
\begin{aligned}
E_0(L\Omega, \beta, \rho L^2|\Omega|) = \mathcal{E}_{L\Omega,\beta}^{\mathrm{af}}[u^{\mathrm{af}}] &\geqslant \sum_{j=1}^{N_\ell} \int_{Q_j} \left| \left(-i\nabla + \beta A[|u^{\mathrm{af}}|^2]\right) u^{\mathrm{af}} \right|^2 \\
&= \sum_{j=1}^{N_\ell} \int_{Q_j} \left| \left(-i\nabla + \beta A[|u_j^{\mathrm{af}} e^{i\beta\phi_j}|^2]\right) u_j^{\mathrm{af}} e^{i\beta\phi_j} \right|^2 \\
&\geqslant \sum_{j=1}^{N_\ell} \rho_j \ell^2 \frac{E(\ell Q, \beta, \rho_j \ell^2)}{\rho_j \ell^2} \geqslant \sum_{j:Q_j \in \mathcal{Q}_L} \rho_j^2 \ell^2 \frac{E(\ell_j Q, \beta, \ell_j^2)}{\ell_j^2}, \tag{3-35}
\end{aligned}
$$

where ϕ_j satisfies (observe that the left-hand side is curl-free on Q_j)

$$\sum_{k=1, k \neq j}^{N_\ell} A[|u_k^{\mathrm{af}}|^2] = \nabla\phi_j \quad \text{in } Q_j,$$

and in the last step we used the scaling law (3-9) with $\mu = 1/\lambda = \sqrt{\rho_j}$. Also,

$$\ell_j := \sqrt{\rho_j}\ell \geqslant L^{\delta/2} \to +\infty \quad \text{as } L \to \infty$$

uniformly in j for cells $Q_j \in \mathcal{Q}_L$, and we thus conclude by Lemmas 3.7 and 3.8 that

$$\frac{1}{L^2|\Omega|} E_0(L\Omega, \beta, \rho L^2|\Omega|) \geqslant (1 - o(1)) \frac{e(\beta, 1)}{L^2|\Omega|} \sum_{j:Q_j \in \mathcal{Q}_L} \rho_j^2 \ell^2 = (1 - o(1)) \frac{e(\beta, 1)}{L^2|\Omega|} \int_{\mathcal{Q}} \bar{\varrho}^2, \tag{3-36}$$

where we consider here the step function $\bar{\varrho} := \sum_{j:Q_j \in \mathcal{Q}_L} \rho_j \mathbb{1}_{Q_j}$ and denote by $\mathcal{Q}$ the union of the cells $\mathcal{Q}_L$. It remains then to observe that the constrained minimum

$$B = \min\left\{ \int_{\mathcal{Q}} \varrho^2 \, : \, 0 \leqslant \varrho \in L^2(\mathcal{Q}), \int_{\mathcal{Q}} \varrho = (1 - o(1))\rho L^2|\Omega| \right\}$$

is achieved by ϱ constant and thus

$$\int_{\mathcal{Q}} \bar{\varrho}^2 \geqslant B = \left((1 - o(1))\rho L^2 |\Omega|\right)^2 |\mathcal{Q}|^{-1} \geqslant (1 - o(1))\rho^2 L^2 |\Omega|.$$

Inserting this in (3-36) and using $\rho^2 e(\beta, 1) = e(\beta, \rho)$ leads to the desired energy lower bound. $\qquad\square$

4. Proofs for the trapped gas

4A. *Local density approximation: energy upper bound.* Here we prove the upper bound corresponding to (2-19):

$$E_\beta^{\mathrm{af}} \leqslant E_\beta^{\mathrm{TF}}(1 + o(1)) \quad \text{as } \beta \to \infty. \tag{4-1}$$

We start by covering the support of $\varrho_\beta^{\mathrm{TF}}$ with squares Q_j, $j = 1, \ldots, N_\beta$, centered at points x_j and of side length L with

$$L = \beta^\eta, \quad -\frac{s}{2(s+2)} < \eta < \frac{1}{s+2}. \tag{4-2}$$

We choose the tiling in such a way that for any $j = 1, \ldots, N_\beta$, we have $Q_j \cap \mathrm{supp}(\varrho_\beta^{\mathrm{TF}}) \neq \varnothing$. The upper bound on L indicates that the length scale of the tiling is much smaller than the size of the TF support. The lower bound ensures that it is much larger than the scale on which we expect the fine structure of the minimizer to live.

Our trial state is defined much as in the proof of Lemma 3.9:

$$u^{\mathrm{test}} := \sum_{j=1}^{N_\beta} u_j e^{-i\beta\phi_j}, \tag{4-3}$$

where u_j realizes the Dirichlet infimum

$$E_0(Q_j, \beta, M_j) := \min\left\{\mathcal{E}_j^{\mathrm{af}}[u] : u \in H_0^1(Q_j), \int_{Q_j} |u|^2 = M_j\right\},$$

where of course

$$\mathcal{E}_j^{\mathrm{af}}[u] = \mathcal{E}_{Q_j, \beta}^{\mathrm{af}}[u] = \int_{Q_j} \left|(-i\nabla + \beta A[|u|^2])u\right|^2$$

and we set

$$M_j = \int_{Q_j} |u_j|^2 := \int_{Q_j} \varrho_\beta^{\mathrm{TF}}, \quad \rho_j := M_j/L^2 = \fint_{Q_j} \varrho_\beta^{\mathrm{TF}}. \tag{4-4}$$

The phase factors in (4-3) are again defined so as to gauge away the interaction between cells, i.e.,

$$\sum_{k=1, k\neq j}^{N_\beta} A[|u_k|^2] = \nabla\phi_j \quad \text{in } Q_j.$$

This construction yields an admissible trial state since u^{test} is locally in H^1, continuous across cells by being zero on the boundaries, and clearly

$$\int_{\mathbb{R}^2} |u^{\text{test}}|^2 = \sum_{j=1}^{N_\beta} \int_{Q_j} |u_j|^2 = \sum_{j=1}^{N_\beta} \int_{Q_j} \varrho_\beta^{\text{TF}} = 1.$$

Much as in the proofs of Lemmas 3.1 and 3.9 we thus obtain

$$E_\beta^{\text{af}} \leqslant \mathcal{E}_\beta^{\text{af}}[u^{\text{test}}] = \sum_{j=1}^{N_\beta} \mathcal{E}_j^{\text{af}}[u_j] + \int_{\mathbb{R}^2} V |u^{\text{test}}|^2 = \sum_{j=1}^{N_\beta} E_0(Q_j, \beta, M_j) + \int_{\mathbb{R}^2} V |u^{\text{test}}|^2. \tag{4-5}$$

Our task is then to estimate the right-hand side.

We denote, for some $\varepsilon > 0$ small enough

$$S_\varepsilon = \left\{ x \in \text{supp}(\varrho_\beta^{\text{TF}}) : \varrho_\beta^{\text{TF}}(x) \geqslant \beta^{-2/(s+2)-\varepsilon} \right\}$$

and split the above sum into two parts, distinguishing between cells fully included in S_ε and the others. Using (2-13), it is clear that

$$\left| \text{supp}(\varrho_\beta^{\text{TF}}) \setminus S_\varepsilon \right| \leqslant C \beta^{1/(s+2)} \cdot \beta^{1/(s+2)-\varepsilon},$$

where the first factor comes from the dilation transforming ϱ_1^{TF} into $\varrho_\beta^{\text{TF}}$ and the second one is an estimate of the thickness of S_ε based on (2-16)–(2-18).

By a simple estimate of the potential V in the vicinity of S_ε, we obtain

$$\sum_{j:Q_j \not\subseteq S_\varepsilon} \int_{Q_j} V |u_j|^2 \leqslant C \beta^{s/(s+2)} \cdot \beta^{2/(s+2)-\varepsilon} \cdot \beta^{-2/(s+2)-\varepsilon} = C \beta^{s/(s+2)-2\varepsilon} \ll E_\beta^{\text{TF}},$$

where the factor $\beta^{s/(s+2)}$ accounts for the supremum of V, the factor $\beta^{2/(s+2)-\varepsilon}$ for the volume of the integration domain and the factor $\beta^{-2/(s+2)-\varepsilon}$ for the typical value of $|u_j|^2$ on this domain. Also, using in addition Lemmas 3.4 and 3.1, we deduce

$$\sum_{j:Q_j \not\subseteq S_\varepsilon} E_0(Q_j, \beta, M_j) = \sum_{j:Q_j \not\subseteq S_\varepsilon} E_0(\beta^\eta Q, \beta, \beta^{2\eta} \rho_j) \ll E_\beta^{\text{TF}}.$$

For the main part of the sum in (4-5) we use the scaling law (take $\lambda = \sqrt{\rho_j}$ and $\mu = \sqrt{\beta \rho_j}$ in Lemma 3.4) to write

$$E_0(Q_j, \beta, M_j) = \rho_j E_0(L \sqrt{\beta \rho_j} Q, 1, L^2 \beta \rho_j),$$

with Q the unit square. Then

$$\sum_{j:Q_j \subseteq S_\varepsilon} E_0(Q_j, \beta, M_j) = \sum_{j:Q_j \subseteq S_\varepsilon} L^2 \beta \rho_j^2 e(1, 1) + \sum_{j:Q_j \subseteq S_\varepsilon} L^2 \beta \rho_j^2 \left(\frac{E_0(L_j Q, 1, L_j^2)}{L_j^2} - e(1, 1) \right)$$

with, provided ε is suitably small and in view of the lower bound in (4-2) and the fact that we sum over squares included in S_ε,

$$L_j := L \sqrt{\beta \rho_j} \geqslant \beta^{\eta + s/(2(s+2)) - \varepsilon/2} \to +\infty, \quad \text{uniformly with respect to } j = 1, \ldots, N_\beta.$$

We thus obtain (recall the definition of the thermodynamic energy in (2-5))

$$\frac{E_0(L_j Q, 1, L_j^2)}{L_j^2} \to e(1, 1) \quad \text{as } L_j \to \infty$$

uniformly in j, and deduce that

$$\sum_{j:Q_j \subset S_\varepsilon} E_0(Q_j, \beta, M_j) = (1 + o(1))\beta e(1, 1) \sum_{j:Q_j \subset S_\varepsilon} \rho_j^2 L^2.$$

Recalling that

$$\rho_j = \fint_{Q_j} \varrho_\beta^{\text{TF}}(x) \, dx,$$

we recognize a Riemann sum in the above. Using (2-17) and the upper bound in (4-2) we may approximate $\varrho_\beta^{\text{TF}}$ by a constant in each square (this is most easily seen by rescaling to ϱ_1^{TF} and observing that the size of squares then tends to zero), and bound the part of the integral located in the complement of S_ε in the same way as above to conclude that

$$\sum_{j:Q_j \subset S_\varepsilon} E_0(Q_j, \beta, M_j) = (1 + o(1))\beta e(1, 1) \int_{\mathbb{R}^2} (\varrho_\beta^{\text{TF}})^2.$$

Using (2-11) and (2-16) we obtain

$$|\nabla V(x)| \leqslant C\beta^{(s-1)/(s+2)}$$

for any $x \in S_\varepsilon$. Combining with (4-2) we deduce as above that

$$\sum_{j:Q_j \subset S_\varepsilon} \int_{Q_j} V|u_j|^2 = (1 + o(1)) \int_{\mathbb{R}^2} V\varrho_\beta^{\text{TF}}$$

and this completes the proof of (4-1).

4B. *Local density approximation: energy lower bound.* Let us now complement (4-1) by proving the lower bound

$$E_\beta^{\text{af}} \geqslant E_\beta^{\text{TF}}(1 + o(1)), \tag{4-6}$$

thus completing the proof of (2-19). We again tile the plane with squares Q_j, $j = 1, \dots, N_\beta$, of side length

$$L = \beta^\eta$$

satisfying (4-2), and taken to cover the finite disk $B_{\beta^t}(0)$ with

$$t := \frac{1}{2+s} + \varepsilon$$

for some $\varepsilon > 0$ to be chosen small enough. We also define

$$\mathcal{Q}_\beta := \left\{ Q_j \subset B_{\beta^t}(0) : L\sqrt{\rho_j \beta} \geqslant \beta^\mu \right\}, \tag{4-7}$$

where $u^{\text{af}} = u_\beta^{\text{af}}$ is a minimizer for $\mathcal{E}_\beta^{\text{af}}$ with unit mass and

$$\rho_j := \fint_{Q_j} |u^{\text{af}}(\boldsymbol{x})|^2 \, \mathrm{d}\boldsymbol{x}.$$

Define the piecewise constant function

$$\bar{\varrho}^{\text{af}}(\boldsymbol{x}) := \sum_{Q_j \in \mathcal{Q}_\beta} \rho_j \mathbb{1}_{Q_j}(\boldsymbol{x}). \tag{4-8}$$

We claim that one may find some $\mu > 0$ in (4-7) such that

$$M := \int_{\mathbb{R}^2} \bar{\varrho}^{\text{af}} \to 1 \quad \text{as } \beta \to \infty. \tag{4-9}$$

Indeed, using (2-11) and (2-12) we get that for any $\boldsymbol{x} \in B_{\beta^t}^c(0)$

$$V(\boldsymbol{x}) \geqslant C\beta^{st} \min_{B_1^c(0)} V \geqslant C\beta^{st}$$

for β large enough. Thus, using the energy upper bound (4-1) and dropping some positive terms we obtain

$$\beta^{st} \int_{B_{\beta^t}^c(0)} |u^{\text{af}}|^2 \leqslant \int_{\mathbb{R}^2} V |u^{\text{af}}|^2 \leqslant \mathcal{E}_\beta^{\text{af}}[u^{\text{af}}] \leqslant C\beta^{s/(s+2)}$$

and thus

$$\int_{B_{\beta^t}^c(0)} |u^{\text{af}}|^2 \leqslant C\beta^{-s\varepsilon}. \tag{4-10}$$

On the other hand, by the definition of $\mathcal{Q}_\beta$,

$$\sum_{Q_j \notin \mathcal{Q}_\beta} \int_{Q_j} |u^{\text{af}}|^2 \leqslant N_\beta \beta^{2\mu-1},$$

where N_β is the total number of squares needed to tile $B_{\beta^t}(0)$. Clearly, we may estimate $N_\beta \leqslant C\beta^{2t} L^{-2} = C\beta^{2(t-\eta)}$ and then

$$\sum_{Q_j \notin \mathcal{Q}_\beta} \int_{Q_j} |u^{\text{af}}|^2 \leqslant C\beta^{2t-2\eta+2\mu-1} \ll 1 \tag{4-11}$$

because of (4-2), which implies $-s/(s+2) - 2\eta < 0$, and provided we take ε and μ positive and small enough, e.g. (recall that $L = \beta^\eta$ is the side length of the tiling squares),

$$0 < \varepsilon \leqslant \frac{1}{4}\left(\frac{s}{s+2} + 2\eta\right), \quad 0 < \mu \leqslant \varepsilon. \tag{4-12}$$

Combining (4-10) and (4-11) and recalling that u^{af} is L^2-normalized proves (4-9).

With this in hand we turn to the energy lower bound per se. Let us again set

$$u_j^{\text{af}} = u^{\text{af}} \mathbb{1}_{Q_j}, \quad M_j = \rho_j L^2 = \int_{Q_j} |u^{\text{af}}|^2.$$

Dropping some positive terms we get

$$
\begin{aligned}
E_\beta^{\mathrm{af}} = \mathcal{E}_\beta^{\mathrm{af}}[u^{\mathrm{af}}] &\geqslant \sum_{Q_j \in \mathcal{Q}_\beta} \int_{Q_j} \left\{ \left| (-i\nabla + \beta \boldsymbol{A}[|u^{\mathrm{af}}|^2]) u^{\mathrm{af}} \right|^2 + V|u^{\mathrm{af}}|^2 \right\} \\
&= \sum_{Q_j \in \mathcal{Q}_\beta} \int_{Q_j} \left\{ \left| (-i\nabla + \beta \boldsymbol{A}[|u_j^{\mathrm{af}} e^{i\beta\phi_j}|^2]) u_j^{\mathrm{af}} e^{i\beta\phi_j} \right|^2 + V|u_j^{\mathrm{af}}|^2 \right\} \\
&\geqslant \sum_{Q_j \in \mathcal{Q}_\beta} \left\{ E(Q_j, \beta, M_j) + \int_{Q_j} V|u_j^{\mathrm{af}}|^2 \right\} \\
&\geqslant \sum_{Q_j \in \mathcal{Q}_\beta} \left\{ \rho_j E\big(L\sqrt{\beta\rho_j}\,Q, 1, (L\sqrt{\beta\rho_j})^2\big) + \int_{Q_j} V|u_j^{\mathrm{af}}|^2 \right\},
\end{aligned}
\tag{4-13}
$$

where the local gauge phase factors are defined as in previous arguments by demanding that (this is again possible because the left-hand side is curl-free in the simply connected domain Q_j)

$$
\sum_{k=1, k\neq j}^{N_\beta} \boldsymbol{A}[|u_k^{\mathrm{af}}|^2] = \nabla\phi_j \quad \text{in } Q_j.
$$

The minimum (Neumann) energy $E(Q_j, \beta, M_j)$ in the square Q_j is defined as in (2-4) and we used the scaling laws following from Lemma 3.4 as previously to obtain

$$
E(Q_j, \beta, M_j) = \rho_j E\big(L\sqrt{\beta\rho_j}\,Q, 1, (L\sqrt{\beta\rho_j})^2\big),
$$

with Q the unit square. Next, we note that (4-2) and (4-7) imply, using (4-12),

$$
L_j = L\sqrt{\beta\rho_j} \geqslant \beta^\mu \to \infty
$$

uniformly in j for all j such that $Q_j \in \mathcal{Q}_\beta$. Then, by Theorem 2.1,

$$
\begin{aligned}
\sum_{Q_j \in \mathcal{Q}_\beta} \rho_j E\big(L\sqrt{\beta\rho_j}\,Q, 1, (L\sqrt{\beta\rho_j})^2\big) &= \sum_{Q_j \in \mathcal{Q}_\beta} \beta L^2 \rho_j^2 \, E(L_j Q, 1, L_j^2)/L_j^2 \\
&= (1+o(1))\beta e(1,1) \sum_{Q_j \in \mathcal{Q}_\beta} L^2 \rho_j^2 = (1+o(1))\beta e(1,1) \int_{\mathbb{R}^2} (\bar{\varrho}^{\mathrm{af}})^2.
\end{aligned}
$$

On the other hand, it follows from (2-11) that, on all the squares of $\mathcal{Q}_\beta$,

$$
|\nabla V| \leqslant C\beta^{(s-1)/(s+2)+\varepsilon(s-1)},
$$

and thus if

$$
\widetilde{V}(\boldsymbol{x}) := \sum_{Q_j \in \mathcal{Q}_\beta} V(\boldsymbol{x}_j) \mathbb{1}_{Q_j}(\boldsymbol{x}),
\tag{4-14}
$$

we have

$$
|V(\boldsymbol{x}) - \widetilde{V}(\boldsymbol{x})| \leqslant CL\beta^{(s-1)/(s+2)+\varepsilon(s-1)} = o(E_\beta^{\mathrm{TF}}) \quad \text{for any } \boldsymbol{x} \in \mathcal{Q}_\beta.
$$

Recalling (4-8) and (4-9) we then have

$$\sum_{Q_j \in \mathcal{Q}_\beta} \int_{Q_j} V \, |u_j^{\mathrm{af}}|^2 = \int_{\mathbb{R}^2} \widetilde{V} \, \bar{\varrho}^{\mathrm{af}} + O(L\beta^{(s-1)/(s+2)+\varepsilon(s-1)}) = \int_{\mathbb{R}^2} \widetilde{V} \, \bar{\varrho}^{\mathrm{af}} + o(E_\beta^{\mathrm{TF}}). \tag{4-15}$$

The last assertion follows from (2-13) and (4-2), provided we take ε small enough; e.g., for $s > 1$ (recall that the tiling squares have side length $L = \beta^\eta$),

$$\varepsilon \leqslant \frac{1}{2(s-1)}\left(\frac{s-1}{s+2} + \eta\right). \tag{4-16}$$

In the very same way however we can put back V in place of $\widetilde{V}$, obtaining

$$\sum_{Q_j \in \mathcal{Q}_\beta} \int_{Q_j} V \, |u_j^{\mathrm{af}}|^2 = \int_{\mathbb{R}^2} \widetilde{V} \, \bar{\varrho}^{\mathrm{af}} + o(E_\beta^{\mathrm{TF}}) = \int_{\mathbb{R}^2} V \, \bar{\varrho}^{\mathrm{af}} + o(E_\beta^{\mathrm{TF}}). \tag{4-17}$$

Combining (4-13), (4-15) and (4-17) yields

$$E_\beta^{\mathrm{af}} \geqslant \int_{\mathbb{R}^2} V \, \bar{\varrho}^{\mathrm{af}} + (1+o(1))\beta e(1,1) \int_{\mathbb{R}^2} (\bar{\varrho}^{\mathrm{af}})^2 + o(E_\beta^{\mathrm{TF}})$$
$$\geqslant (1+o(1))\mathcal{E}_\beta^{\mathrm{TF}}[\bar{\varrho}^{\mathrm{af}}] + o(E_\beta^{\mathrm{TF}}) \geqslant (1+o(1))E_\beta^{\mathrm{TF}}(M) + o(E_\beta^{\mathrm{TF}}), \tag{4-18}$$

where the latter energy denotes the ground state energy of the TF functional (2-9) minimized under the constraint that the L^1-norm be equal to M. Inserting (4-9) and using explicit expressions as in (2-13) and (2-14), one obtains

$$E_\beta^{\mathrm{TF}}(M) = (1+o(1))E_\beta^{\mathrm{TF}}$$

in the limit $\beta \to \infty$, thus completing the proof of (4-6).

4C. *Density convergence.* The lower bound in (4-6) together with the energy upper bound (4-1) implies that $\bar{\varrho}^{\mathrm{af}}$, the piecewise constant approximation of ϱ^{af} on scale $L = \beta^\eta$, is close in strong L^2 sense to $\varrho_\beta^{\mathrm{TF}}$. We will deduce (2-20) from the following.

Lemma 4.1 (Convergence of the piecewise approximation).
Let $\bar{\varrho}^{\mathrm{af}}$ be defined as in (4-8) and $\varrho_\beta^{\mathrm{TF}}$ be the minimizer of (2-9). Then

$$\|\bar{\varrho}^{\mathrm{af}} - \varrho_\beta^{\mathrm{TF}}\|_{L^2(\mathbb{R}^2)} = o(\beta^{-1/(s+2)}) \tag{4-19}$$

in the limit $\beta \to \infty$.

Proof. Combining (4-1) and (4-18) we have

$$\mathcal{E}_\beta^{\mathrm{TF}}[\bar{\varrho}^{\mathrm{af}}] \leqslant E_\beta^{\mathrm{af}} + o(1)\beta^{s/(s+2)} \leqslant E_\beta^{\mathrm{TF}} + o(1)\beta^{s/(s+2)}. \tag{4-20}$$

The variational equation for $\varrho_\beta^{\mathrm{TF}}$ takes the form

$$2\beta e(1,1)\varrho_\beta^{\mathrm{TF}} + V = \lambda_\beta^{\mathrm{TF}} = E_\beta^{\mathrm{TF}} + \beta e(1,1)\int_{\mathbb{R}^2} (\varrho_\beta^{\mathrm{TF}})^2$$

on the support of $\varrho_\beta^{\mathrm{TF}}$ (recall (2-14) and (2-15)). Thus,

$$
\begin{aligned}
\int_{\mathbb{R}^2} (\bar{\varrho}^{\mathrm{af}} - \varrho_\beta^{\mathrm{TF}})^2 &= \int_{\mathbb{R}^2} \left((\bar{\varrho}^{\mathrm{af}})^2 + (\varrho_\beta^{\mathrm{TF}})^2 \right) - 2 \int_{\mathbb{R}^2} \bar{\varrho}^{\mathrm{af}} \varrho_\beta^{\mathrm{TF}} \\
&= \int_{\mathbb{R}^2} (\bar{\varrho}^{\mathrm{af}})^2 + \int_{\mathbb{R}^2} (\varrho_\beta^{\mathrm{TF}})^2 - \frac{1}{\beta e(1,1)} \int_{\mathbb{R}^2} \bar{\varrho}^{\mathrm{af}} (\lambda_\beta^{\mathrm{TF}} - V)_+ \\
&\leqslant \frac{1}{\beta e(1,1)} \left[\mathcal{E}_\beta^{\mathrm{TF}}[\bar{\varrho}^{\mathrm{af}}] - \lambda_\beta^{\mathrm{TF}} + \beta e(1,1) \int_{\mathbb{R}^2} (\varrho_\beta^{\mathrm{TF}})^2 \right] \\
&= \frac{1}{\beta e(1,1)} [\mathcal{E}_\beta^{\mathrm{TF}}[\bar{\varrho}^{\mathrm{af}}] - E_\beta^{\mathrm{TF}}] = o(\beta^{-2/(s+2)}),
\end{aligned}
$$

where we used (4-20) in the last step. $\qquad\square$

By the definition (4-8) of $\bar{\varrho}^{\mathrm{af}}$ we also have, for any Lipschitz function ϕ with compact support,

$$
\begin{aligned}
\int_{\mathbb{R}^2} \phi(\beta^{-1/(s+2)} \boldsymbol{x}) \bar{\varrho}^{\mathrm{af}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} &= \sum_{j=1}^{N_\beta} \int_{Q_j} \phi(\beta^{-1/(s+2)} \boldsymbol{x}) \, \bar{\varrho}^{\mathrm{af}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} \\
&= \sum_{j=1}^{N_\beta} \phi(\beta^{-1/(s+2)} \boldsymbol{x}_j) \int_{Q_j} \varrho^{\mathrm{af}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + O(\beta^{\eta-1/(s+2)} \|\phi\|_{\mathrm{Lip}}) \\
&= \int_{\mathbb{R}^2} \phi(\beta^{-1/(s+2)} \boldsymbol{x}) \, \varrho^{\mathrm{af}}(\boldsymbol{x}) \, \mathrm{d}\boldsymbol{x} + O(\beta^{\eta-1/(s+2)} \|\phi\|_{\mathrm{Lip}}),
\end{aligned}
$$

using the normalization of ϱ^{af}. Furthermore, by Cauchy–Schwarz and Lemma 4.1 we obtain

$$
\int_{\mathbb{R}^2} \phi(\beta^{-1/(s+2)} \boldsymbol{x}) \left(\bar{\varrho}^{\mathrm{af}}(\boldsymbol{x}) - \varrho_\beta^{\mathrm{TF}}(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x} = o(1) \|\phi\|_{L^2(\mathbb{R}^2)}.
$$

Since the above estimates are uniform with respect to the Lipschitz norm of ϕ, we can take $\eta < 1/(s+2)$, change scales in the above and recall (2-13) to deduce

$$
\sup_{\substack{\phi \in C_0(B_R(0)) \\ \|\phi\|_{\mathrm{Lip}} \leqslant 1}} \left| \int_{\mathbb{R}^2} \phi(\boldsymbol{x}) \left(\beta^{2/(s+2)} \varrho^{\mathrm{af}}(\beta^{1/(s+2)} \boldsymbol{x}) - \varrho_1^{\mathrm{TF}}(\boldsymbol{x}) \right) \mathrm{d}\boldsymbol{x} \right| = o(1), \quad \beta \to \infty,
$$

for fixed $R > 0$, and hence (2-20).

Appendix: Properties of minimizers

In this appendix we recall a few fundamental properties of the average-field functional (1-1) in a trap V, respectively (2-1) on a domain Ω, as well as their minimizers.

As discussed in [Lundholm and Rougerie 2015, Appendix], the natural, maximal domain of $\mathcal{E}^{\mathrm{af}}$ is

$$
\mathscr{D}^{\mathrm{af}} := \left\{ u \in H^1(\Omega) : \int_{\mathbb{R}^2} V|u|^2 < \infty \right\},
$$

and one may also use that the space $C_c^\infty(\mathbb{R}^2)$ is dense in this form domain with respect to $\mathcal{E}^{\mathrm{af}}$. Furthermore, [Lundholm and Rougerie 2015, Appendix: Proposition 3.7] ensures the existence of a minimizer $u^{\mathrm{af}} \in \mathscr{D}^{\mathrm{af}}$

of $\mathcal{E}_{\beta}^{\mathrm{af}}$ for any value of $\beta \in \mathbb{R}$ for confining potentials V, and by a similar proof and the compactness of the embedding $H_0^1(\Omega) \subset H^1(\Omega) \hookrightarrow L^p(\Omega)$, $1 \leqslant p < \infty$, the same holds for $\mathcal{E}_{\Omega}^{\mathrm{af}}$ for any bounded Ω with Lipschitz boundary:

Proposition A.1 (Existence of minimizers).
Let $\beta \in \mathbb{R}$ be arbitrary. Given any $V : \mathbb{R}^2 \to \mathbb{R}^+$ such that $-\Delta + V$ has compact resolvent, there exists $u^{\mathrm{af}} \in \mathscr{D}^{\mathrm{af}}$ with $\int_{\mathbb{R}^2} |u^{\mathrm{af}}|^2 = 1$ and $\mathcal{E}_{\beta}^{\mathrm{af}}[u^{\mathrm{af}}] = E^{\mathrm{af}}$. Moreover, if $M \geqslant 0$ and $\Omega \subset \mathbb{R}^2$ is bounded with Lipschitz boundary then there exists $u^{\mathrm{af}} \in H_{(0)}^1(\Omega)$ with $\int_{\Omega} |u^{\mathrm{af}}|^2 = M$ and $\mathcal{E}_{\Omega, \beta}^{\mathrm{af}}[u^{\mathrm{af}}] = E_{(0)}(\Omega, \beta, M)$.

Proof. The first part is [Lundholm and Rougerie 2015, Appendix: Proposition 3.7]. For $\Omega \subset \mathbb{R}^2$ we have by the Hölder, weak Young, and Sobolev inequalities, as well as Lemma 3.2, that

$$\left\| \boldsymbol{A}[|u|^2]u \right\|_{L^2(\Omega)} \leqslant \left\| \boldsymbol{A}[|u|^2] \right\|_{L^4(\Omega)} \|u\|_{L^4(\Omega)}$$
$$\leqslant C \left\| |u|^2 \right\|_{L^{4/3}(\Omega)} \|\nabla w_0\|_{L^{2,w}(\mathbb{R}^2)} \|u\|_{L^4(\Omega)} \leqslant C' \left\| |u| \right\|_{H^1(\Omega)}^3 \leqslant C'(M + \mathcal{E}_{\Omega}^{\mathrm{af}}[u])^{3/2},$$

and therefore

$$\|\nabla u\|_{L^2(\Omega)} = \left\| \nabla u + i\beta \boldsymbol{A}[|u|^2]u - i\beta \boldsymbol{A}[|u|^2]u \right\|_{L^2(\Omega)} \leqslant \mathcal{E}^{\mathrm{af}}[u]^{1/2} + C'|\beta|(M + \mathcal{E}_{\Omega}^{\mathrm{af}}[u])^{3/2}.$$

Hence, given a minimizing sequence

$$(u_n)_{n \to \infty} \subset H_{(0)}^1(\Omega), \quad \|u_n\|_{L^2(\Omega)}^2 = M, \quad \lim_{n \to \infty} \mathcal{E}_{\Omega}^{\mathrm{af}}[u_n] = E_{(0)}(\Omega, \beta, M),$$

by uniform boundedness and the Rellich–Kondrachov theorem (see, for example, [Lieb and Loss 2001, Theorem 8.9]) there exists a convergent subsequence (again denoted u_n) and a limit $u^{\mathrm{af}} \in H_{(0)}^1(\Omega)$ such that

$$u_n \to u^{\mathrm{af}} \quad \text{in } L^p(\Omega), \ 1 \leqslant p < \infty, \qquad \nabla u_n \rightharpoonup \nabla u^{\mathrm{af}} \quad \text{in } L^2(\Omega).$$

Furthermore, by estimating

$$\left\| \boldsymbol{A}[|u_n|^2]u_n - \boldsymbol{A}[|u^{\mathrm{af}}|^2]u^{\mathrm{af}} \right\|_2 \leqslant \left\| \boldsymbol{A}[|u_n|^2 - |u^{\mathrm{af}}|^2]u_n \right\|_2 + \left\| \boldsymbol{A}[|u^{\mathrm{af}}|^2](u_n - u) \right\|_2$$

as above and using the strong convergence in $L^p(\Omega)$ for any $1 \leqslant p < \infty$, we have

$$\boldsymbol{A}[|u_n|^2]u_n \to \boldsymbol{A}[|u^{\mathrm{af}}|^2]u^{\mathrm{af}} \text{ in } L^2(\Omega).$$

Hence,

$$\left\| \left(\nabla + i\beta \boldsymbol{A}[|u^{\mathrm{af}}|^2] \right) u^{\mathrm{af}} \right\|_2 = \sup_{\|v\|=1} \left| \langle \nabla u^{\mathrm{af}} + i\beta \boldsymbol{A}[|u^{\mathrm{af}}|^2]u^{\mathrm{af}}, v \rangle \right| = \sup_{\|v\|=1} \lim_{n \to \infty} \left| \langle \nabla u_n + i\beta \boldsymbol{A}[|u_n|^2]u_n, v \rangle \right|$$
$$\leqslant \liminf_{n \to \infty} \sup_{\|v\|=1} \left| \langle \nabla u_n + i\beta \boldsymbol{A}[|u_n|^2]u_n, v \rangle \right| = \liminf_{n \to \infty} \left\| \left(\nabla + i\beta \boldsymbol{A}[|u_n|^2] \right) u_n \right\|_2,$$

that is, $E_{(0)}(\Omega, \beta, M) \leqslant \mathcal{E}_{\Omega}^{\mathrm{af}}[u^{\mathrm{af}}] \leqslant \liminf_{n \to \infty} \mathcal{E}_{\Omega}^{\mathrm{af}}[u_n] = E_{(0)}(\Omega, \beta, M)$, and furthermore $\int_{\Omega} |u^{\mathrm{af}}|^2 = \lim_{n \to \infty} \int_{\Omega} |u_n|^2 = M$. $\qquad \square$

For completeness, we finish with a derivation of the variational equation associated to the minimization of the energy functional (1-1). Let us define

$$\boldsymbol{J}[u] := \frac{i}{2}(u\nabla\bar{u} - \bar{u}\nabla u)$$

and for two vector functions $F, G : \mathbb{R}^2 \to \mathbb{R}^2$, their convolution

$$F * G(x) := \int_{\mathbb{R}^2} F(x - y) \cdot G(y) \, dy.$$

Lemma A.2 (Variational equation).
Let $u = u^{\mathrm{af}}$ be a solution to (2-8). Then

$$\left[\left(-i\nabla + \beta A[|u|^2]\right)^2 + V - 2\beta \nabla^{\perp} w_0 * \left(\beta A[|u|^2]|u|^2 + J[u]\right)\right]u = \lambda u, \tag{A-1}$$

where

$$\lambda = \mathcal{E}^{\mathrm{af}}[u] + \int_{\mathbb{R}^2} \left(2\beta A[|u|^2] \cdot J[u] + 2\beta^2 \left|A[|u|^2]\right|^2 |u|^2\right)$$

$$= \int_{\mathbb{R}^2} \left(1(|\nabla u|^2 + V|u|^2) + 2 \cdot 2\beta A[|u|^2] \cdot J[u] + 3\beta^2 \left|A[|u|^2]\right|^2 |u|^2\right). \tag{A-2}$$

(Note that the factors 1, 2, and 3 correspond to the total degree of $|u|^2$ in each term.)

Proof. Let

$$\mathcal{F}[u, \bar{u}, \lambda] := \mathcal{E}^{\mathrm{af}}[u, \bar{u}] + \lambda(1 - \int |u|^2)$$

$$= \int \left(|\nabla u|^2 + (V - \lambda)|u|^2 + \beta^2 \left|A[|u|^2]\right|^2 |u|^2 + 2\beta A[|u|^2] \cdot J[u]\right) + \lambda,$$

$$\mathcal{E}_1[u, \bar{u}] := \int |A[u\bar{u}]|^2 u\bar{u} = \iiint \nabla^{\perp} w_0(x - y) \cdot \nabla^{\perp} w_0(x - z) \, u\bar{u}(x) \, u\bar{u}(y) \, u\bar{u}(z) \, dx \, dy \, dz,$$

$$\mathcal{E}_2[u, \bar{u}] := \int A[u\bar{u}] \cdot i(u\nabla\bar{u} - \bar{u}\nabla u) = \iint \nabla^{\perp} w_0(x - y) \, u\bar{u}(y) \cdot i(u\nabla\bar{u} - \bar{u}\nabla u)(x) \, dx \, dy.$$

We have

$$\mathcal{E}_1[u, \bar{u} + \varepsilon v] = \mathcal{E}_1[u, \bar{u}] + \varepsilon \iiint \left(\nabla^{\perp} w_0(x - y) \cdot \nabla^{\perp} w_0(x - z)\left(v(x)u(x)|u(y)|^2 |u(z)|^2\right.\right.$$

$$\left.\left. + |u(x)|^2 u(y)v(y)|u(z)|^2 + |u(x)|^2 |u(y)|^2 u(z)v(z)\right)\right) dx \, dy \, dz + O(\varepsilon^2).$$

Hence at $O(\varepsilon)$,

$$\int_x v(x)u(x) A[|u|^2]^2 \, dx - \int_y v(y)u(y) \int_x \nabla^{\perp} w_0(y - x)|u(x)|^2 \cdot \int_z \nabla^{\perp} w_0(x - z)|u(z)|^2 \, dz \, dx \, dy$$

$$- \int_z v(z)u(z) \int_x \nabla^{\perp} w_0(z - x)|u(x)|^2 \cdot \int_y \nabla^{\perp} w_0(x - y)|u(y)|^2 \, dy \, dx \, dz$$

$$= \int vu A[|u|^2]^2 - 2\int vu \nabla^{\perp} w_0 * |u|^2 A[|u|^2].$$

Also

$$\mathcal{E}_2[u, \bar{u} + \varepsilon v] = \mathcal{E}_2[u, \bar{u}] + \varepsilon \iint \left(\nabla^{\perp} w_0(x - y)u(y)v(y) \cdot i(u\nabla\bar{u} - \bar{u}\nabla u)(x)\right.$$

$$\left. + \nabla^{\perp} w_0(x - y)|u(y)|^2 \cdot i\left(u(x)\nabla v(x) - v(x)\nabla u(x)\right)\right) dx \, dy + O(\varepsilon^2),$$

hence at $O(\varepsilon)$ and using $\nabla \cdot \boldsymbol{A} = 0$,

$$-\int_y v(\boldsymbol{y})u(\boldsymbol{y})\int_x \nabla^\perp w_0(\boldsymbol{y}-\boldsymbol{x}) \cdot 2\boldsymbol{J}[u](\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} - i\int v(\boldsymbol{x})\nabla u(\boldsymbol{x}) \cdot \boldsymbol{A}[|u|^2](\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}$$

$$+\underbrace{i\int u(\boldsymbol{x})\boldsymbol{A}[|u|^2](\boldsymbol{x}) \cdot \nabla v(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}}_{=\{PI\}=-i\int \nabla u \cdot \boldsymbol{A}v - i\int u(\nabla\cdot\boldsymbol{A})v} = -2\int vu\nabla^\perp w_0 * \boldsymbol{J}[u] - 2i\int v\nabla u \cdot \boldsymbol{A}[|u|^2].$$

Thus

$$\mathcal{F}[u,\bar{u}+\varepsilon v,\lambda] = \mathcal{F}[u,\bar{u},\lambda] + \varepsilon\int v\Big[(-\Delta+V-\lambda)u + \beta^2\big|\boldsymbol{A}[|u|^2]\big|^2 u - 2\beta^2\nabla^\perp w_0 * |u|^2\boldsymbol{A}[|u|^2]u$$

$$- 2\beta\nabla^\perp w_0 * \boldsymbol{J}[u]u - 2i\beta\boldsymbol{A}[|u|^2] \cdot \nabla u\Big] + O(\varepsilon^2),$$

and using

$$\big(-i\nabla+\beta\boldsymbol{A}[|u|^2]\big)^2 u = -\Delta u - 2i\beta\boldsymbol{A}[|u|^2] \cdot \nabla u + \beta^2\boldsymbol{A}[|u|^2]^2 u,$$

we arrive at (A-1).

For (A-2) we use $\int |u|^2 = 1$ by multiplying (A-1) with $\bar{u}$ and integrating:

$$\lambda = \mathcal{E}^{\mathrm{af}}[u] - 2\beta\int |u|^2\nabla^\perp w_0 * \big(\beta\boldsymbol{A}[|u|^2]|u|^2 + \boldsymbol{J}[u]\big).$$

We then use that

$$\int |u|^2\nabla^\perp w_0 * \boldsymbol{A}[|u|^2]|u|^2 = \iiint |u(\boldsymbol{x})|^2\nabla^\perp w_0(\boldsymbol{x}-\boldsymbol{y}) \cdot \nabla^\perp w_0(\boldsymbol{y}-\boldsymbol{z})|u(\boldsymbol{z})|^2|u(\boldsymbol{y})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}\,\mathrm{d}\boldsymbol{z}$$

$$= -\iiint \nabla^\perp w_0(\boldsymbol{y}-\boldsymbol{x}) \cdot \nabla^\perp w_0(\boldsymbol{y}-\boldsymbol{z})|u(\boldsymbol{x})|^2|u(\boldsymbol{z})|^2|u(\boldsymbol{y})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{z}\,\mathrm{d}\boldsymbol{y}$$

$$= -\int \boldsymbol{A}[|u|^2]^2|u|^2$$

and

$$2\int |u|^2\nabla^\perp w_0 * \boldsymbol{J}[u] = \iint |u(\boldsymbol{x})|^2\nabla^\perp w_0(\boldsymbol{x}-\boldsymbol{y}) \cdot i\big(u(\boldsymbol{y})\nabla\bar{u}(\boldsymbol{y}) - \bar{u}(\boldsymbol{y})\nabla u(\boldsymbol{y})\big)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}$$

$$= -\int_y i(u\nabla\bar{u} - \bar{u}\nabla u)(\boldsymbol{y}) \cdot \int_x \nabla^\perp w_0(\boldsymbol{y}-\boldsymbol{x})|u(\boldsymbol{x})|^2\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} = -2\int \boldsymbol{J}[u] \cdot \boldsymbol{A}[|u|^2]$$

to arrive at (A-2). $\qquad\square$

Acknowledgments

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