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FOR THE DAMPED SCHRÖDINGER EQUATION

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We prove the local energy decay and the global smoothing effect for the damped Schrödinger equation on $\mathbb{R}^d$. The self-adjoint part is a Laplacian associated to a long-range perturbation of the flat metric. The proofs are based on uniform resolvent estimates obtained by the dissipative Mourre method. All the results depend on the strength of the dissipation that we consider.

1. Introduction

Let $d \geqslant 3$. Our purpose in this paper is to study on $\mathbb{R}^d$ the local energy decay and the Kato smoothing effect for the damped Schrödinger equation

$$\begin{cases} -i\,\partial_t u + Pu - i\,a(x)\langle D\rangle^\alpha a(x)u = 0, \\ u(0) = u_0. \end{cases} \tag{1-1}$$

The operator P is a Laplacian in divergence form associated to a long-range perturbation of the usual flat metric (see (1-2) below). For the dissipative part we have denoted by $\langle \cdot \rangle$ the function $(1 + |\cdot|^2)^{\frac{1}{2}}$ and by D the square root of the free Laplacian, so that $\langle D\rangle^\alpha$ stands for $(1 - \Delta)^{\frac{\alpha}{2}}$. The parameter α belongs to $[0, 2[$. The nonnegative-valued function a will be assumed to be of short range (see (1-3)), so that in terms of spacial decay, we have an absorption index $a(x)^2$ which decays at least like $\langle x\rangle^{-2-2\rho}$ for some $\rho > 0$.

It is known that the free Schrödinger equation ((1-1) with $P = -\Delta$ and $a = 0$) preserves the L^2-norm but satisfies the local energy decay: if u_0 is supported in the ball $B(R) = \{|x| \leqslant R\}$ of $\mathbb{R}^d$ for some $R > 0$ we have

$$\|e^{it\Delta}u_0\|_{L^2(B(R))} \lesssim C_R \langle t\rangle^{-\frac{d}{2}} \|u_0\|_{L^2(\mathbb{R}^d)}.$$

This means that the "mass" of the solution escapes at infinity. On the other hand, the Schrödinger equation has a regularizing effect:

$$\int_{\mathbb{R}} \|(1 - \Delta)^{\frac{1}{4}} e^{it\Delta} u_0\|^2_{L^2(B(R))}\, dt \leqslant C_R \|u_0\|^2_{L^2(\mathbb{R}^d)}.$$

There are many papers dealing with these properties for more and more general Schrödinger equations. Concerning the local energy decay for a self-adjoint Schrödinger equation, we only refer to [Rauch 1978] for the Schrödinger operator with an exponentially decaying potential, to [Tsutsumi 1984] for the free Schrödinger equation on an exterior domain, and to [Bouclet 2011; Bony and Häfner 2012] for a Laplacian associated to a long-range perturbation of the flat metric. For all these papers, the local energy

decays like $t^{-\frac{d}{2}}$ or like $t^{-\frac{d}{2}+\varepsilon}$ under a nontrapping assumption. There is also a huge literature for the closely related problem of the local energy decay for the wave equation (see [Lax et al. 1963; Ralston 1969; Morawetz et al. 1977; Burq 1998; Tataru 2013; Guillarmou et al. 2013]).

Concerning the smoothing effect we mention [Constantin and Saut 1988; Sjölin 1987] for the Laplacian on $\mathbb{R}^d$, [Ben-Artzi and Klainerman 1992] for the Schrödinger operator with a potential, [Burq et al. 2004] for the problem on an exterior domain and [Erdoğan et al. 2009] for the magnetic Schrödinger equation. We also refer to [Doi 1996; 2000; Burq 2004] for the necessity of the nontrapping condition.

In the dissipative context, the local energy decay for the damped Schrödinger equation in an exterior domain has been proved in [Aloui and Khenissi 2007]. In this context, the nontrapping condition can be replaced by the geometric control condition: there can be bounded classical trajectories but they have to go through the damping region (see [Rauch and Taylor 1974; Bardos et al. 1992] for the original geometric control condition, and [Aloui and Khenissi 2002] for the exterior geometric condition on an unbounded domain). Then the local energy decays like $t^{-\frac{d}{2}}$, as in the self-adjoint case under the nontrapping condition. A similar result has been obtained in [Aloui and Khenissi 2010] on an exterior domain with dissipation at the boundary, and in [Royer 2015] for the same problem on a wave guide (see also [D'Ancona and Racke 2012] for the undamped problem on a nonflat wave guide). In the latter case, the global energy decays exponentially and we have a smoothing effect in the unbounded directions. We also mention [Bortot and Cavalcanti 2014], where an exponential decay for the global energy is proved for the solution of the Schrödinger equation with a dissipation effective on a neighborhood of the infinity.

The dissipation by a potential ($\alpha = 0$ in our setting) is not strong enough to recover under the damping condition the same smoothing effect as under the nontrapping condition. However, it is known that this is the case for the so-called regularized Schrödinger equation ($\alpha = 1$). See [Aloui 2008a; 2008b] for the problem on a compact manifold and [Aloui et al. 2017] for the problem on an exterior domain. As in the self-adjoint case (see [Burq 2004]), we can recover a $H^{\frac{1}{2}-\varepsilon}$ smoothing effect if only a few classical trajectories fail to satisfy the assumption (see [Aloui et al. 2013]).

In these works, the problem is a compact perturbation of the free Schrödinger equation. Our purpose in this paper is to prove the local energy decay and the Kato smoothing effect for an asymptotically vanishing perturbation. In a similar context, the local energy decay has been studied for the dissipative wave equation in [Bouclet and Royer 2014].

We now describe more precisely the setting of our paper. We consider on $\mathbb{R}^d$ a long-range perturbation $G(x)$ of the identity: for some $\rho > 0$ there exist constants C_β for $\beta \in \mathbb{N}^d$ such that

$$\left| \partial^\beta \left(G(x) - I_d \right) \right| \leqslant C_\beta \langle x \rangle^{-\rho - |\beta|}. \tag{1-2}$$

Concerning the dissipative term, a is a smooth and nonnegative-valued function on $\mathbb{R}^d$. As already mentioned, it is of short range:

$$\left| \partial^\beta a(x) \right| \leqslant C_\beta \langle x \rangle^{-1-\rho - |\beta|}. \tag{1-3}$$

We will use the notation

$$B_\alpha = a(x) \langle D \rangle^\alpha a(x) \quad \text{and} \quad H = P - iB_\alpha. \tag{1-4}$$

We recall that $\alpha \in [0, 2[$ and we set

$$\tilde{\alpha} = \min(1, \alpha) \quad \text{and} \quad \kappa = \begin{cases} \frac{d}{2} & \text{if } d \text{ is even,} \\ \frac{d+1}{2} & \text{if } d \text{ is odd.} \end{cases} \tag{1-5}$$

Then $\tilde{\alpha} \in [0, 1]$ and $\kappa \geqslant 2$.

We will see that H is a maximal dissipative operator on L^2. In particular, for $u_0 \in \mathcal{D}(H) = H^2$ the problem (1-1) has a unique solution $t \mapsto e^{-itH} u_0$. The main purpose of this paper is to prove that this solution satisfies the local energy decay and the Kato smoothing effect as stated in the following two theorems:

Theorem 1.1 (local energy decay). *Let $\varepsilon > 0$. Let $\delta > \kappa + \frac{1}{2}$, $N \in \mathbb{N}$ and $\sigma \in [0, 2]$. Assume that*

  (i) *there are no bounded geodesics (see the nontrapping condition (1-8) below) or*

 (ii) *the bounded geodesics go through the damping region (see (1-9)), $N\tilde{\alpha} + \sigma \geqslant 2$ and $\delta > N - \frac{1}{2}$.*

Then there exists $C \geqslant 0$ such that for $u_0 \in H^{\sigma, \delta}$ and $t \geqslant 0$ we have

$$\|e^{-itH} u_0\|_{L^{2,-\delta}} \leqslant C t^{-\frac{d}{2} + \varepsilon} \|u_0\|_{H^{\sigma, \delta}}.$$

In this statement $L^{2,-\delta}$ denotes the weighted space $L^2(\langle x \rangle^{-2\delta} \, dx)$, while $\|u_0\|_{H^{\sigma, \delta}}$ is the L^2-norm of $\langle x \rangle^{\delta} \langle D \rangle^{\sigma} u_0$.

We remark that we have to take $\sigma = 2$ in the second case if $\alpha = 0$. This means that we have a loss of two derivatives. If $\alpha > 0$ we can take $\sigma = 0$ (no loss of derivative) as long as we choose δ large enough (if $\alpha \geqslant 1$ then we can take $N = 2$, and in this case the condition $\delta > N - \frac{1}{2}$ is weaker than $\delta > \kappa + \frac{1}{2}$). Under the nontrapping condition we can always take $\sigma = 0$.

In this setting, we obtain a decay which is almost as good as in the free case. We recall that for such a P, this is the best decay known even in the particular case $a = 0$ (see [Bouclet 2011]).

Theorem 1.2 (global smoothing effect). *Assume that the damping condition (1-9) holds. Then there exists $C \geqslant 0$ such that for all $u_0 \in L^2$ we have*

$$\int_0^{+\infty} \|\langle x \rangle^{-1} \langle D \rangle^{\frac{\tilde{\alpha}}{2}} e^{-itH} u_0\|_{L^2}^2 \, dt \leqslant C \|u_0\|_{L^2}^2.$$

Moreover, under the nontrapping condition (1-8), we can replace $\tilde{\alpha}$ by 1.

The last statement says that, despite the non-self-adjointness of H, we recover the same gain of regularity as in the self-adjoint case under the nontrapping assumption. However, the main result is that if the damping is strong enough, we have the same result for a trapping metric under the usual geometric condition. For a weaker damping we cannot reach the optimal result, but we still have some gain of regularity. As for the local energy decay above and for the resolvent estimates below, we can consider a very strong damping ($\alpha > 1$), but this does not improve the results (even if we allow trapped trajectories, there still are trajectories going to infinity, and their contributions are not controlled by the damping).

The proofs of Theorems 1.1 and 1.2 are based on uniform resolvent estimates. According to Proposition 2.2 below, the operator H is maximal dissipative, so for all z in

$$\mathbb{C}_+ := \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$$

we can consider in $\mathcal{L}(L^2)$ (the space of bounded operators on L^2) the resolvent

$$R(z) = (H - z)^{-1}. \tag{1-6}$$

After a Fourier transform, the solution u of (1-1) can be written as the integral over frequencies $\operatorname{Re}(z)$ of this resolvent when $\operatorname{Im}(z)$ goes to 0 (see Section 6). Thus the problem will be reduced to proving uniform estimates for $R(z)$ and its derivatives for $\operatorname{Im}(z)$ small, and then to controlling the dependence of these estimates with respect to $\operatorname{Re}(z)$. Since the self-adjoint part P of H is a nonnegative operator, the estimates for $\operatorname{Re}(z) < 0$ are easy: for $n \in \mathbb{N}$ and $z \in \mathbb{C}_+$ with $\operatorname{Re}(z) \leqslant -c_0 < 0$ we have

$$\|R^{n+1}(z)\|_{\mathcal{L}(L^2)} \leqslant \frac{C}{|\operatorname{Re}(z)|^{n+1}}. \tag{1-7}$$

Thus we will focus on $z \in \mathbb{C}_+$ with $\operatorname{Re}(z) \geqslant -c_0$, where $0 < c_0 \ll 1$. As usual, the difficulties will arise for low frequencies ($\operatorname{Re}(z)$ close to 0) and high frequencies ($\operatorname{Re}(z) \gg 1$). We first state the uniform resolvent estimates for intermediate frequencies:

Theorem 1.3 (intermediate-frequency estimates). *Let K be a compact subset of $\mathbb{C} \setminus \{0\}$. Let $n \in \mathbb{N}$ and $\delta > n + \frac{1}{2}$. Then there exists $C \geqslant 0$ such that for all $z \in K \cap \mathbb{C}_+$ we have*

$$\|\langle x \rangle^{-\delta} R^{n+1}(z) \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C.$$

We remark that compared to the resolvent for the dissipative wave equation (see [Bouclet and Royer 2014]), the derivatives of the resolvent correspond to its powers:

$$R^{(n)}(z) = n! \, R^{n+1}(z).$$

This will significantly simplify the discussion.

It is known that even for the free Laplacian, the estimates of Theorem 1.3 fail to hold uniformly when z goes to 0 if n is too large. This explains the restriction in the rate of decay in Theorem 1.1. For low frequencies we prove the following result:

Theorem 1.4 (low-frequency estimates). *Let $\varepsilon > 0$. Let $n \in \mathbb{N}$ and let δ be such that*

$$\delta > \begin{cases} n + \frac{1}{2} & \text{if } 2n + 1 \geqslant d, \\ n + 1 & \text{if } 2n + 1 < d. \end{cases}$$

Then there exist $C \geqslant 0$ and a neighborhood $\mathcal{U}$ of 0 in $\mathbb{C}$ such that for all $z \in \mathcal{U} \cap \mathbb{C}_+$ we have

$$\|\langle x \rangle^{-\delta} R^{n+1}(z) \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C(1 + |z|^{\frac{d}{2} - \varepsilon - 1 - n}).$$

In the self-adjoint case we can improve the estimate for a single resolvent. More precisely we can replace the weight $\langle x \rangle^{-\delta}$ for $\delta > 1$ by $\langle x \rangle^{-1}$. See [Bouclet and Royer 2015]. This is particularly interesting

for Theorem 1.2, which does not require estimates for the derivatives of the resolvent. This sharp resolvent estimate is also valid in our dissipative context:

Theorem 1.5 (sharp low-frequency estimate). *There exist $C \geqslant 0$ and a neighborhood $\mathcal{U}$ of 0 in $\mathbb{C}$ such that for all $z \in \mathcal{U} \cap \mathbb{C}_+$ we have*

$$\|\langle x \rangle^{-1} R(z) \langle x \rangle^{-1}\|_{\mathcal{L}(L^2)} \leqslant C.$$

The high-frequency properties of the problem are closely related to the corresponding classical problem. Here, the classical flow is the geodesic flow on $\mathbb{R}^{2d} \simeq T^*\mathbb{R}^d$ for the metric $G(x)^{-1}$. It is the Hamiltonian flow corresponding to the symbol

$$p(x, \xi) = \langle G(x)\xi, \xi \rangle.$$

We denote this flow by $\phi^t = (X(t), \Xi(t))$. Let

$$\Omega_b = \{w \in p^{-1}(\{1\}) : \sup_{t \in \mathbb{R}} |X(t, w)| < +\infty\}.$$

The assumptions used in the statements of Theorems 1.1 and 1.2 are the following. We say that the classical flow is nontrapping if there is no bounded geodesic:

$$\Omega_b = \varnothing. \tag{1-8}$$

We say that the damping condition on bounded geodesics (or geometric control condition) is satisfied if every bounded geodesic goes through the damping region $\{a(x) > 0\}$:

$$\forall w \in \Omega_b, \ \exists T \in \mathbb{R} \quad \text{such that} \quad a(X(T, w)) > 0. \tag{1-9}$$

Theorem 1.6 (high-frequency estimates). *Let $n \in \mathbb{N}$ and $\delta > n + \frac{1}{2}$.*

(i) *Assume that the nontrapping assumption* (1-8) *holds. Then there exists $C \geqslant 0$ such that for $z \in \mathbb{C}_+$ with $\operatorname{Re}(z) \geqslant C$ we have*

$$\|\langle x \rangle^{-\delta} R^{n+1}(z) \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C|z|^{-\frac{n+1}{2}}.$$

(ii) *Assume that the damping condition* (1-9) *holds. Then there exists $C \geqslant 0$ such that for $z \in \mathbb{C}_+$ with $\operatorname{Re}(z) \geqslant C$ we have*

$$\|\langle x \rangle^{-\delta} R^{n+1}(z) \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C|z|^{-\frac{(n+1)\tilde{\alpha}}{2}}$$

(we recall that $\tilde{\alpha}$ was defined in (1-5)).

To prove the uniform estimates of Theorems 1.3, 1.4 and 1.6 we use the commutators method of Mourre [1981] (see also [Amrein et al. 1996] for an overview of the subject). The method has been generalized to the dissipative setting in [Royer 2010], then in [Bouclet and Royer 2014] for the estimates of the derivatives of the resolvent and finally in [Royer 2016] for a dissipative perturbation in the sense of forms. Here the dissipative perturbation B_α is well defined as an operator on L^2 relatively bounded with respect to the self-adjoint part P. However, for $d \in \{3, 4\}$ the rescaled version of the dissipative part which we are going to use for low frequencies will be uniformly bounded as an operator in $\mathcal{L}(H^1, H^{-1})$ but not in $\mathcal{L}(H^2, L^2)$, so we will have to see H as a dissipative perturbation of P in the sense of forms. See Remark 4.7.

Let us come back to the statement of Theorem 1.2. To prove this theorem we will use in particular the resolvent estimates of Theorem 1.6, which in turn rely on the damping assumption (1-9). These estimates and hence the smoothing effect we obtain are optimal (in the sense that they are as good as in the self-adjoint case with the nontrapping condition) when $\alpha \geqslant 1$. However, with a weaker dissipation $(\alpha < 1)$ we can obtain (weaker) resolvent estimates and a (weaker) smoothing effect. Similarly, it is possible to prove high-frequency resolvent estimates weaker than those of Theorem 1.6 without the damping condition. We have already mentioned [Burq 2004] in the self-adjoint case and [Aloui et al. 2013] in the dissipative setting, where only a few hyperbolic classical trajectories deny the assumption (in these cases the high-frequency resolvent estimates are of size $\ln|z|/\sqrt{|z|}$, which gives a gain of $\frac{1}{2} - \varepsilon$ derivative). We do not prove resolvent estimates without damping condition in this paper, but we emphasize this fact with a more general version of Theorem 1.2 (for self-adjoint operators, we mention the result of [Thomann 2010], which gives a relation between the smoothing effect and the decay of the spectral projections).

Theorem 1.7. *Let $\gamma \in [0, 2]$. Assume that there exists $C \geqslant 0$ such that for all $z \in \mathbb{C}_+$ we have*

$$\|\langle x \rangle^{-1} R(z) \langle x \rangle^{-1}\|_{\mathcal{L}(L^2)} \leqslant C \langle z \rangle^{-\frac{\gamma}{2}}. \tag{1-10}$$

Then for all $u_0 \in L^2$ we have

$$\int_0^{+\infty} \|\langle x \rangle^{-1} \langle D \rangle^{\frac{\gamma}{2}} e^{-itH} u_0\|_{L^2}^2 \, dt \leqslant C \|u_0\|_{L^2}^2.$$

It is classical in the self-adjoint setting to prove the smoothing effect from resolvent estimates by means of the theory of relatively smooth operators in the sense of Kato [1966] (see also [Reed and Simon 1978]). Other ideas have been used for dissipative operators (see [Aloui et al. 2013; 2017]). However, the theory of Kato can also be used in this context (see [Royer 2010; 2015]). We will follow this idea to prove Theorem 1.7 and hence Theorem 1.2.

This paper is organized as follows. In Section 2 we recall all the abstract properties we need concerning dissipative operators (including the statement of the Mourre method). In Section 3 we prove Theorem 1.3. In Section 4 we deal with low frequencies. We first prove Theorem 1.4 for a small perturbation of the free Laplacian in Section 4A and then in the general setting in Section 4B. Theorem 1.5 is proved in Section 4C. In Section 5 we prove Theorem 1.6 concerning the high-frequency resolvent estimates. Finally, we turn to the time-dependent problem: we prove Theorem 1.1 in Section 6 and Theorems 1.7 and 1.2 in Section 7.

2. Abstract properties for dissipative operators

In this section we recall some general properties about dissipative operators. In particular we give the version of the Mourre's method that we use in this paper.

Let $\mathcal{H}$ be a Hilbert space. An operator H with domain $\mathcal{D}(H)$ on $\mathcal{H}$ is said to be dissipative (respectively accretive) if

$$\forall \varphi \in \mathcal{D}(H), \quad \mathrm{Im} \langle H\varphi, \varphi \rangle_{\mathcal{H}} \leqslant 0 \quad (\text{respectively } \mathrm{Re} \langle H\varphi, \varphi \rangle_{\mathcal{H}} \geqslant 0).$$

Moreover H is said to be maximal dissipative (respectively maximal accretive) if it has no other dissipative (respectively accretive) extension than itself. Notice that H is (maximal) dissipative if and only if iH is

(maximal) accretive. We recall that a dissipative operator H is maximal dissipative if and only if there exists $z \in \mathbb{C}_+$ such that the operator $(H - z)$ has a bounded inverse on $\mathcal{H}$. In this case any $z \in \mathbb{C}_+$ belongs to the resolvent set of H and

$$\|(H - z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{\operatorname{Im}(z)}. \tag{2-1}$$

According to the Hille–Yosida theorem this implies in particular that $-iH$ generates a contractions semigroup, and then for all $u_0 \in \mathcal{D}(H)$ the function $u : t \mapsto e^{-itH} u_0$ belongs to $C^0(\mathbb{R}_+, \mathcal{D}(H)) \cap C^1(\mathbb{R}_+, \mathcal{H})$ and is the unique solution for the problem

$$\begin{cases} -i\, \partial_t u + H u = 0 & \forall t > 0, \\ u(0) = u_0. \end{cases}$$

Moreover we have

$$\forall t \geqslant 0, \quad \|u(t)\|_{\mathcal{H}} \leqslant \|u_0\|_{\mathcal{H}}.$$

Remark 2.1. Assume that H is both dissipative and accretive. Then it is maximal dissipative if and only if it is maximal accretive. Indeed both properties are equivalent to the fact that $(H - (-1 + i))$ has a bounded inverse on $\mathcal{H}$. Moreover, for $z \in \mathbb{C}$ with $\operatorname{Im}(z) > 0$ or $\operatorname{Re}(z) < 0$ we have

$$\|(H - z)^{-1}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{1}{\max(\operatorname{Im}(z), -\operatorname{Re}(z))}.$$

Proposition 2.2. *The operator H defined by (1-4) is maximal dissipative and maximal accretive on L^2.*

Proof. The operators P and B_α are self-adjoint and nonnegative on L^2, so $H = P - iB_\alpha$ is dissipative and accretive. Let $\varphi \in \mathcal{D}(P) = H^2$. By interpolation there exists $C \geqslant 0$ (which only depends on a and α) such that for any $\varepsilon > 0$

$$\|B_\alpha \varphi\|_{L^2} \leqslant C \|\varphi\|_{H^\alpha} \leqslant C \|\varphi\|_{H^2}^{\frac{\alpha}{2}} \|\varphi\|_{L^2}^{1-\frac{\alpha}{2}} \leqslant \tfrac{1}{2}\alpha \varepsilon C \|\varphi\|_{H^2} + C \big(1 - \tfrac{1}{2}\alpha\big) \varepsilon^{-\frac{\alpha}{2-\alpha}} \|\varphi\|_{L^2}.$$

With $\varepsilon > 0$ small enough we obtain that the dissipative operator $-iB_\alpha$ is relatively bounded with respect to P with relative bound less than 1. According to [Royer 2010, Lemma 2.1], this proves that H is maximal dissipative in L^2. By Remark 2.1, H is also maximal accretive. $\qquad\square$

According to Proposition 2.2, the estimate of Remark 2.1 holds for H in $\mathcal{L}(L^2)$. As already mentioned, the difficulties in Theorems 1.3, 1.4 and 1.6 come from the behavior of the resolvent $R(z)$ when the spectral parameter $z \in \mathbb{C}_+$ approaches the nonnegative real axis. For this we are going to use a dissipative version of the Mourre method, which we recall now.

Let q_0 be a quadratic form closed, densely defined, symmetric and bounded from below on $\mathcal{H}$. We set $\mathcal{K} = \mathcal{D}(q_0)$. Let q_Θ be another symmetric form on $\mathcal{H}$, nonnegative and q_0-bounded. Let $q = q_0 - iq_\Theta$ and let H be the corresponding maximal dissipative operator (see Proposition 2.2 in [Royer 2016]). We denote by $\widetilde{H} : \mathcal{K} \to \mathcal{K}^*$ the operator which satisfies $q(\varphi, \psi) = \langle \widetilde{H}\varphi, \psi \rangle_{\mathcal{K}^*, \mathcal{K}}$ for all $\varphi, \psi \in \mathcal{K}$. Similarly, we denote by $\widetilde{H}_0$ and Θ the operators in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$ which correspond to the forms q_0 and q_Θ, respectively. By the Lax–Milgram theorem, the operator $(\widetilde{H} - z)$ has a bounded inverse in $\mathcal{L}(\mathcal{K}^*, \mathcal{K})$ for all $z \in \mathbb{C}_+$. Moreover for $\varphi \in \mathcal{H}$ we have $(H - z)^{-1}\varphi = (\widetilde{H} - z)^{-1}\varphi$.

Definition 2.3. Let A be a self-adjoint operator on $\mathcal{H}$ and $N \in \mathbb{N}^*$. We say that A is a conjugate operator (in the sense of forms) to H on the interval J, up to order N, and with bounds $\alpha_0 \in \,]0, 1]$, $\beta \geqslant 0$ and $\Upsilon_N \geqslant 0$ if the following conditions are satisfied:

  (i) The form domain $\mathcal{K}$ is left invariant by e^{-itA} for all $t \in \mathbb{R}$. We denote by $\mathcal{E}$ the domain of the generator of $e^{-itA}|_{\mathcal{K}}$.

 (ii) The commutators $\Lambda^0 = [\tilde{H}_0, iA]$ and $\Lambda_1 = [\tilde{H}, iA]$, a priori defined as operators in $\mathcal{L}(\mathcal{E}, \mathcal{E}^*)$, extend to operators in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$. Then for all $n \in [\![1, N]\!]$ the operator $[\Lambda_n, iA]$ defined (inductively) in $\mathcal{L}(\mathcal{E}, \mathcal{E}^*)$ extends to an operator in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$, which we denote by Λ_{n+1}.

(iii) We have

$$\|\Lambda_1\| \leqslant \sqrt{\alpha_0}\,\Upsilon_N, \quad \|\Lambda_1 + \beta\Theta\|\,\|\Lambda^0\| \leqslant \alpha_0\Upsilon_N, \quad \|[\Lambda_1, A]\| + \beta\|[\Theta, A]\| \leqslant \alpha_0\Upsilon_N$$

and

$$\sum_{n=2}^{N+1} \|\Lambda_n\|_{\mathcal{L}(\mathcal{K}, \mathcal{K}^*)} \leqslant \alpha_0\Upsilon_N,$$

where all the norms are in $\mathcal{L}(\mathcal{K}, \mathcal{K}^*)$.

(iv) We have

$$\mathbb{1}_J(H_0)(\Lambda^0 + \beta\Theta)\mathbb{1}_J(H_0) \geqslant \alpha_0\mathbb{1}_J(H_0). \tag{2-2}$$

Theorem 5.5 of [Royer 2016] in the particular case where all the inserted factors are equal to $\mathrm{Id}_{\mathcal{H}}$ gives the following abstract resolvent estimates:

Theorem 2.4. *Suppose the self-adjoint operator A is conjugate to the maximal dissipative operator H on J up to order $N \geqslant 2$ with bounds $(\alpha_0, \beta, \Upsilon_N)$. Let $n \in [\![1, N]\!]$. Let $I \subset \overset{\circ}{J}$ be a compact interval. Let $\delta > n - \frac{1}{2}$. Then there exists $c \geqslant 0$ which only depends on J, I, δ, β and Υ_N and such that for all $z \in \mathbb{C}_{I,+}$ we have*

$$\|\langle A \rangle^{-\delta}(H - z)^{-n}\langle A \rangle^{-\delta}\|_{\mathcal{L}(\mathcal{H})} \leqslant \frac{c}{\alpha_0^n}.$$

We finish this general section with the so-called quadratic estimates. The following result is a consequence of Proposition 4.4 in [Royer 2016]:

Proposition 2.5. *Let $T \in \mathcal{L}(\mathcal{K}, \mathcal{H})$ be such that $T^*T \leqslant q_\Theta$ in the sense of forms on $\mathcal{K}$. Let $Q \in \mathcal{L}(\mathcal{H}, \mathcal{K}^*)$. Then for all $z \in \mathbb{C}_+$ we have*

$$\|T(\tilde{H} - z)^{-1}Q\|_{\mathcal{L}(\mathcal{H})} \leqslant \|Q^*(\tilde{H} - z)^{-1}Q\|_{\mathcal{L}(\mathcal{H})}^{\frac{1}{2}}.$$

Applied with $Q = T^*$, this proposition gives the following particular case:

Corollary 2.6. *Let T be as in Proposition 2.5. Then for all $z \in \mathbb{C}_+$ we have*

$$\|T(\tilde{H} - z)^{-1}T^*\|_{\mathcal{L}(\mathcal{H})} \leqslant 1.$$

We are going to use all these results with the forms $q_0 : \varphi \mapsto \langle P\varphi, \varphi \rangle$ and $q_\Theta : \varphi \mapsto \langle B_\alpha\varphi, \varphi \rangle$ defined on $\mathcal{K} = H^1(\mathbb{R}^d)$.

3. Intermediate-frequency estimates

In this section we prove Theorem 1.3. For this, we will apply Theorem 2.4 with the generator of dilations as the conjugate operator. Let

$$A = -\tfrac{1}{2} i (x \cdot \nabla + \nabla \cdot x) = -i (x \cdot \nabla) - \tfrac{1}{2} i d.$$

We recall in the following proposition the main properties of A that we are going to use in this paper:

Proposition 3.1.   (i) *For $\theta \in \mathbb{R}$, $u \in \mathcal{S}$ and $x \in \mathbb{R}^d$ we have*

$$(e^{i\theta A} u)(x) = e^{\frac{d\theta}{2}} u(e^\theta x).$$

 (ii) *For $j \in [\![1, d]\!]$ and $\gamma \in C^\infty(\mathbb{R}^d)$ we have on $\mathcal{S}$*

$$[\partial_j, iA] = \partial_j \quad and \quad [\gamma, iA] = -(x \cdot \nabla)\gamma.$$

(iii) *For $p \in [1, +\infty]$, $\theta \in \mathbb{R}$ and $u \in \mathcal{S}$ we have*

$$\|e^{i\theta A} u\|_{L^p} = e^{\theta(\frac{d}{2} - \frac{d}{p})} \|u\|_{L^p}.$$

Now we give a proof of Theorem 1.3:

Proof of Theorem 1.3. Let $E > 0$. We check that the generator of dilations A is a conjugate operator for H on a neighborhood J of E in the sense of Definition 2.3. The form domain of H is the Sobolev space $H^1(\mathbb{R}^d)$. According to Proposition 3.1, it is left invariant by the dilation e^{-itA} for any $t \in \mathbb{R}$. By pseudodifferential calculus we can see that the commutators $[P, iA]$, $[[P, iA], iA]$, $[B_\alpha, iA]$ and $[[B_\alpha, iA], iA]$ define operators in $\mathcal{L}(H^2, L^2)$, hence in $\mathcal{L}(L^2, H^{-2})$ by duality, and in $\mathcal{L}(H^1, H^{-1})$ by interpolation.[1] Finally, we use the usual trick for the main assumption. For $\sigma > 0$ we set $J_\sigma = [E - \sigma, E + \sigma]$. We have

$$\mathbb{1}_{J_\sigma}(P)[P, iA]\mathbb{1}_{J_\sigma}(P) = \mathbb{1}_{J_\sigma}(P) \, 2P \, \mathbb{1}_{J_\sigma}(P) + W \mathbb{1}_{J_\sigma}(P)$$
$$\geqslant 2(E - \sigma)\mathbb{1}_{J_\sigma}(P) + W \mathbb{1}_{J_\sigma}(P),$$

where

$$W := \mathbb{1}_{J_\sigma}(P) \operatorname{div}\big((x \cdot \nabla)G(x)\big)\nabla$$

is a compact operator. Since $E > 0$ is not an eigenvalue of P (see [Koch and Tataru 2006]) the operator $\mathbb{1}_{J_\sigma}(P)$ goes strongly to 0 when σ goes to 0. Then for σ small enough we have

$$\mathbb{1}_{J_\sigma}(P)[P, iA]\mathbb{1}_{J_\sigma}(P) \geqslant E \, \mathbb{1}_{J_\sigma}(P).$$

Thus we can apply Theorem 2.4, which gives Theorem 1.3 for $\operatorname{Re}(z) \in J_\sigma$ and with weights $\langle A \rangle^{-\delta}$. By compactness of $K \subset \mathbb{C}^*$ and the easy estimate of Remark 2.1, we have a uniform estimate for all

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[1] In fact we can also compute these commutators explicitly with Proposition 3.1, except for the commutators of $\langle D \rangle^\alpha$ with A: for this we can write $\langle D \rangle^\alpha = (1 - \Delta)^2 \times (1 - \Delta)^{-2} \langle -\Delta \rangle^{\frac{\alpha}{2}}$ and use the Helffer–Sjöstrand formula for the second factor (see [Dimassi and Sjöstrand 1999; Davies 1995]).

$z \in \mathbb{C}_+ \cap K$. It remains to replace $\langle A \rangle^{-\delta}$ by $\langle x \rangle^{-\delta}$. For this, we use the resolvent identity

$$R(z) = R(i) + (z-i)R(i)R(z) = R(i) + (z-i)R(z)R(i).$$

It gives in particular, for $\nu \geqslant 2$,

$$R^\nu(z) = R(i)\big(R^\nu(z) + 2(z-i)R^{\nu-1}(z) + (z-i)^2 R^\nu(z)\big)R(i).$$

With these equalities in hand, we can prove by induction on $m \in \mathbb{N}^*$ that $R^{n+1}(z)$ can be written as a sum of terms of the form $(z-i)^\beta R^{n+1+\beta}(i)$ with $\beta \in \mathbb{N}$ or

$$(z-i)^{2m-n-1+\nu} R^m(i) R^\nu(z) R^m(i),$$

where $\max(1, n+1-2m) \leqslant \nu \leqslant n+1$. For any $\beta \in \mathbb{N}$, we know $R^{n+1+\beta}(i)$ is uniformly bounded in $\mathcal{L}(L^2)$. On the other hand,

$$\big\| \langle x \rangle^{-\delta} R^m(i) R^\nu(z) R^m(i) \langle x \rangle^{-\delta} \big\| \leqslant \big\| \langle x \rangle^{-\delta} R^m(i) \langle A \rangle^\delta \big\| \big\| \langle A \rangle^{-\delta} R^\nu(z) \langle A \rangle^{-\delta} \big\| \big\| \langle A \rangle^\delta R^m(i) \langle x \rangle^{-\delta} \big\|.$$

The first and third factors are bounded by pseudodifferential calculus if m is large enough and the second has been estimated uniformly by the Mourre method. This concludes the proof of Theorem 1.3. □

4. Low-frequency estimates

In this section we prove Theorems 1.4 and 1.5. As in [Bouclet 2011; Bouclet and Royer 2014], the proof of Theorem 1.4 is based on a scaling argument for a small perturbation of the free Laplacian (see Section 4A), and then on a perturbation argument to deal with the general case (see Section 4B). Theorem 1.5 is proved in Section 4C.

Let $\chi \in C_0^\infty(\mathbb{R}^d)$ be equal to 1 on a neighborhood of 0. For $\eta \in \,]0, 1]$ we set $\chi_\eta : x \mapsto \chi(\eta x)$. Then for $\eta_1 \in \,]0, 1]$ we set $G_{\eta_1}(x) = \chi_{\eta_1}(x)I_d + (1 - \chi_{\eta_1}(x))G(x)$,

$$P_{\eta_1} = -\operatorname{div} G_{\eta_1}(x)\nabla \quad \text{and} \quad P_{\eta_1,c} = P - P_{\eta_1} = -\operatorname{div}\big(\chi_{\eta_1}(x)(G(x) - I_d)\big)\nabla. \tag{4-1}$$

For the dissipative part we set

$$B^\alpha_{\eta_2} = a(1 - \chi_{\eta_2})\langle D \rangle^\alpha a + a\chi_{\eta_2}\langle D \rangle^\alpha a(1 - \chi_{\eta_2}) \tag{4-2}$$

and

$$B^\alpha_{\eta_2,c} = B_\alpha - B^\alpha_{\eta_2} = a\chi_{\eta_2}\langle D \rangle^\alpha a\chi_{\eta_2},$$

where $\eta_2 \in \,]0, 1]$. Finally, for the full operator we define

$$H_{\bar\eta} = P_{\eta_1} - iB^\alpha_{\eta_2} \quad \text{and} \quad R_{\bar\eta}(z) = (H_{\bar\eta} - z)^{-1},$$

where $\bar\eta = (\eta_1, \eta_2) \in \,]0, 1]^2$ and $z \in \mathbb{C}_+$. The fact that we can choose $\eta_1 \neq \eta_2$ will be important in the sequel (see Remark 4.11).

4A. *Low-frequency estimates for a small perturbation of the Laplacian.* In this paragraph we prove Theorem 1.4 with $R(z)$ replaced by $R_{\bar{\eta}}(z)$. Then in Section 4B we will add the contributions of $P_{\eta_1,c}$ and $B_{\eta_2,c}^{\alpha}$.

The proof relies on a scaling argument. To this purpose we use for $z \in \mathbb{C}^*$ the operator

$$\Theta_z = \exp\left(\tfrac{1}{2} i \ln|z| A\right).$$

For $u \in \mathcal{S}$ and $x \in \mathbb{R}^d$ we have $(\Theta_z u)(x) = |z|^{\frac{d}{4}} u(|z|^{\frac{1}{2}} x)$. According to Proposition 3.1 we have for $p \in [1, +\infty]$

$$\|\Theta_z\|_{\mathcal{L}(L^p)} = |z|^{\frac{d}{4} - \frac{d}{2p}}. \tag{4-3}$$

For a function u on $\mathbb{R}^d$ and $z \in \mathbb{C}^*$ we denote by u_z the function

$$u_z : x \mapsto u\left(\frac{x}{\sqrt{|z|}}\right).$$

Compared to the scaling for the wave equation we are using the parameter $\sqrt{|z|}$ instead of $|z|$.

Now we introduce the rescaled versions of our operators:

$$H_{\bar{\eta},z} = \frac{1}{|z|} \Theta_z^{-1} H_{\bar{\eta}} \Theta_z = P_{\eta_1,z} - i B_{\eta_2,z}^{\alpha},$$

where $P_{\eta_1,z} = -\operatorname{div} G_{\eta_1,z}(x) \nabla$ and

$$B_{\eta_2,z}^{\alpha} = \frac{1}{|z|}((1-\chi_{\eta_2})a)_z (1-|z|\Delta)^{\frac{\alpha}{2}} a_z + \frac{1}{|z|}(\chi_{\eta_2}a)_z (1-|z|\Delta)^{\frac{\alpha}{2}} ((1-\chi_{\eta_2})a)_z.$$

Then for $\zeta \in \mathbb{C}_+$ we set $R_{\bar{\eta},z}(\zeta) = (H_{\bar{\eta},z} - \zeta)^{-1}$, so that with the notation $\hat{z} = z/|z|$ we have for $z \in \mathbb{C}_+$

$$R_{\bar{\eta}}(z) = \frac{1}{|z|} \Theta_z R_{\bar{\eta},z}(\hat{z}) \Theta_z^{-1}.$$

Our analysis of the rescaled operators is based on the fact that if a function ϕ decays like $\langle x \rangle^{-\nu-\frac{\rho}{2}}$ (recall that $\rho > 0$ is fixed by (1-2) and (1-3)) then the multiplication by the rescaled function ϕ_λ behaves like a differential operator of order ν for low frequencies, in the sense that it is of size λ^ν as an operator from H^s to $H^{s-\nu}$. Since this observation relies on the Sobolev embeddings, there is however a restriction on the choices of ν and s. For $\sigma \in \mathbb{R}$, let $\mathcal{S}^{-\sigma}(\mathbb{R}^d)$ be the set of functions $\phi \in C^\infty(\mathbb{R}^d)$ such that

$$|\partial^\beta \phi(x)| \lesssim \langle x \rangle^{-\sigma-|\beta|}.$$

For $\nu \geq 0$, $N \in \mathbb{N}$ and $\phi \in \mathcal{S}^{-\nu-\frac{\rho}{2}}(\mathbb{R}^d)$ we set

$$\|\phi\|_{\nu,N} = \sup_{|\beta| \leq \kappa+1} \sum_{0 \leq m \leq N} \sup_{x \in \mathbb{R}^d} \left|\langle x \rangle^{\nu+\frac{\rho}{2}+|\beta|} (\partial^\beta (x \cdot \nabla)^m \phi)(x)\right|.$$

We recall that the integer κ was defined in (1-5). The following result is Proposition 7.2 in [Bouclet and Royer 2014]:

Proposition 4.1. *Let* $v \in \left[0, \frac{d}{2}\right[$ *and* $s \in \left]-\frac{d}{2}, \frac{d}{2}\right[$ *be such that* $s - v \in \left]-\frac{d}{2}, \frac{d}{2}\right[$. *Then there exists* $C \geqslant 0$ *such that for* $\phi \in \mathcal{S}^{-v-\frac{\rho}{2}}(\mathbb{R}^d)$, $u \in H^s$ *and* $\lambda > 0$ *we have*

$$\|\phi_\lambda u\|_{\dot{H}^{s-v}} \leqslant C\lambda^v \|\phi\|_{v,0} \|u\|_{\dot{H}^s},$$

$$\|\phi_\lambda u\|_{H^{s-v}} \leqslant C\lambda^v \|\phi\|_{v,0} \|u\|_{H^s}.$$

The reason for replacing $G(x)$ by $G_{\eta_1}(x)$ and a by $a(1 - \chi_{\eta_2})$ in the definition of $H_{\bar{\eta}}$ is that for all $N \in \mathbb{N}$ we have

$$\mathcal{N}_{\bar{\eta},N} := \sum_{j,k=1}^{d} \|G_{\eta_1,j,k}(x) - \delta_{j,k}\|_{0,N} + \|(1 - \chi_{\eta_2})a\|_{1,N} \big(\|a\|_{1,N} + \|\chi_{\eta_2}a\|_{1,N}\big) = \underset{\bar{\eta} \to 0}{O} \left(|\bar{\eta}|^{\rho/2}\right). \quad (4\text{-}4)$$

Thus this quantity is as small as we wish if we choose η_1 and η_2 small enough.

Given two operators T and S we set $\mathrm{ad}_T^0(S) = S$, $\mathrm{ad}_T(S) = \mathrm{ad}_T^1(S) = [S, T]$ and then, for $m \geqslant 2$, $\mathrm{ad}_T^m(S) = [\mathrm{ad}_T^{m-1}(S), S]$. For $\mu = (\mu_1, \ldots, \mu_d) \in \mathbb{N}^d$ we set

$$\mathrm{ad}_x^\mu := \mathrm{ad}_{x_1}^{\mu_1} \cdots \mathrm{ad}_{x_d}^{\mu_d}.$$

At the beginning of the section we said that $H_{\bar{\eta}}$ has to be close to the free Laplacian. What we need precisely is the following result:

Proposition 4.2. *Let* $\mu \in \mathbb{N}^d$, $m \in \mathbb{N}$, $\varepsilon_0 > 0$ *and* $s \in \mathbb{R}$. *There exists* $\eta_0 \in]0, 1]$ *such that for* $\bar{\eta} = (\eta_1, \eta_2) \in]0, \eta_0]^2$ *the following statements hold*:

(i) *If* $s \in \left]-\frac{d}{2}, \frac{d}{2}\right[$ *then for* $z \in \mathbb{C}_+$ *with* $|z| \leqslant 1$ *we have*

$$\|\mathrm{ad}_x^\mu \mathrm{ad}_A^m (P_{\eta_1,z}+)\|_{\mathcal{L}(H^{s+1}, H^{s-1})} \leqslant \varepsilon_0.$$

(ii) *If* $s \in \left]-\frac{d}{2} + 1, \frac{d}{2} - 1\right[$ *then we also have*

$$\|\mathrm{ad}_x^\mu \mathrm{ad}_A^m B_{\eta_2,z}^\alpha\|_{\mathcal{L}(H^{s+1}, H^{s-1})} \leqslant \varepsilon_0.$$

(iii) *For* $u \in H^2$ *we have*

$$\tfrac{1}{2}\|u\|_{\dot{H}^2} \leqslant \|P_{\eta_1}u\|_{L^2} \leqslant 2\|u\|_{\dot{H}^2}.$$

Proof. The first statement is the same as for the wave equation. See Proposition 7.6 in [Bouclet and Royer 2014]. In particular with $s = 1$, $|z| = 1$ and $\varepsilon_0 = \frac{1}{2}$ we obtain the last statement. It remains to prove (ii). Let $D_z = \sqrt{|z|}D$. We write

$$((1 - \chi_{\eta_2})a)_z \langle D_z \rangle^\alpha a_z = ((1 - \chi_{\eta_2})a)_z (-|z|\Delta + 1)\langle D_z \rangle^{\alpha-2} a_z.$$

Then $\mathrm{ad}_x^\mu \mathrm{ad}_A^m \big(((1 - \chi_{\eta_2})a)_z \langle D_z \rangle^\alpha a_z\big)$ can be written as a sum of terms of the form

$$\mathrm{ad}_x^{\mu_1} \mathrm{ad}_A^{m_1} \big(((1 - \chi_{\eta_2})a)_z\big) \, \mathrm{ad}_x^{\mu_2} \mathrm{ad}_A^{m_2} (-|z|\Delta + 1) \, \mathrm{ad}_x^{\mu_3} \mathrm{ad}_A^{m_3} (\langle D_z \rangle^{\alpha-2}) \, \mathrm{ad}_x^{\mu_4} \mathrm{ad}_A^{m_4} (a_z),$$

where $\mu_1, \mu_2, \mu_3, \mu_4 \in \mathbb{N}^d$ and $m_1, m_2, m_3, m_4 \in [\![0, m]\!]$ are such that $\mu_1 + \mu_2 + \mu_3 + \mu_4 = \mu$ and $m_1 + m_2 + m_3 + m_4 = m$. Let $\gamma \in [0, 1]$. According to Proposition 4.1 we have for $z \in \mathbb{C}_+$

$$\|\mathrm{ad}_x^{\mu_1} \mathrm{ad}_A^{m_1} ((1 - \chi_{\eta_2})a)_z\|_{\mathcal{L}(H^{s-1+\gamma}, H^{s-1})} \lesssim \eta_2^{1+\frac{\rho}{2}-\gamma} |z|^{\frac{\gamma}{2}} \quad (4\text{-}5)$$

and

$$\|\mathrm{ad}_x^{\mu_4}\mathrm{ad}_A^{m_4}a_z\|_{\mathcal{L}(H^{s+1},H^{s+1-\gamma})} \lesssim |z|^{\frac{\gamma}{2}}. \tag{4-6}$$

To estimate $\mathrm{ad}_x^{\mu_3}\mathrm{ad}_A^{m_3}(\langle D_z\rangle^{\alpha-2})$, we use the Helffer–Sjöstrand formula (see [Dimassi and Sjöstrand 1999; Davies 1995]). We can check that for $\zeta \in \mathbb{C} \setminus \mathbb{R}$ we have

$$\|\mathrm{ad}_x^{\mu_3}\mathrm{ad}_A^{m_3}(-|z|\Delta - \zeta)^{-1}\|_{\mathcal{L}(H^{s+1-\gamma})} \lesssim \frac{\langle\zeta\rangle^{|\mu_3|+m_3}}{|\mathrm{Im}(\zeta)|^{|\mu_3|+m_3+1}}.$$

Let $f : \tau \mapsto (\tau+1)^{\frac{\alpha-2}{2}}$. Let $\phi \in C_0^\infty(\mathbb{R}, [0,1])$ be supported in $[-2,2]$ and equal to 1 on $[-1,1]$. For $M > |\mu_3| + m_3 + 1$ and $\zeta = x + iy$ we set

$$\tilde{f}_M(\zeta) = \phi\left(\frac{y}{\langle x\rangle}\right)\sum_{k=0}^M f^{(k)}(x)\frac{(iy)^k}{k!}.$$

We have

$$\left|\frac{\partial\tilde{f}_M}{\partial\bar{\zeta}}(\zeta)\right| \lesssim \mathbb{1}_{\{\langle x\rangle \le |y| \le 2\langle x\rangle\}}(\zeta)\langle x\rangle^{-1+\frac{\alpha-2}{2}} + \mathbb{1}_{\{|y| \le 2\langle x\rangle\}}(\zeta)|y|^M\langle x\rangle^{-M-1+\frac{\alpha-2}{2}},$$

so we can write

$$(-|z|\Delta + 1)^{\frac{\alpha-2}{2}} = \frac{1}{\pi}\int_{\zeta=x+iy\in\mathbb{C}}\frac{\partial\tilde{f}_M}{\partial\bar{z}}(\zeta)(-|z|\Delta - \zeta)^{-1}\,dx\,dy.$$

Then we can check that

$$\|\mathrm{ad}_x^{\mu_3}\mathrm{ad}_A^{m_3}\langle D_z\rangle^{\alpha-2}\|_{\mathcal{L}(H^{s+1-\gamma})} \lesssim 1. \tag{4-7}$$

It remains to estimate

$$\mathrm{ad}_x^{\mu_2}\mathrm{ad}_A^{m_2}(\langle D_z\rangle^2) = -|z|\mathrm{ad}_x^{\mu_2}\mathrm{ad}_A^{m_2}\Delta + \mathrm{ad}_x^{\mu_2}\mathrm{ad}_A^{m_2}(1). \tag{4-8}$$

We have $\||z|\mathrm{ad}_x^{\mu_2}\mathrm{ad}_A^{m_2}\Delta\| \lesssim |z|$ in $\mathcal{L}(H^{s+1},H^{s-1})$ so with (4-5), (4-6) and (4-7) applied with $\gamma = 0$ we obtain in $\mathcal{L}(H^{s+1},H^{s-1})$

$$\left\|\mathrm{ad}_x^{\mu_1}\mathrm{ad}_A^{m_1}\big(((1-\chi_{\eta_2})a)_z\big)\,\mathrm{ad}_x^{\mu_2}\mathrm{ad}_A^{m_2}(-|z|\Delta)\,\mathrm{ad}_x^{\mu_3}\mathrm{ad}_A^{m_3}(\langle D_z\rangle^{\alpha-2})\,\mathrm{ad}_x^{\mu_4}\mathrm{ad}_A^{m_4}(a_z)\right\| \lesssim |z|\eta_2^{1+\frac{\rho}{2}}. \tag{4-9}$$

If $|\mu_2| = m_2 = 0$ we also have to consider the second term in (4-8). For this we apply (4-5), (4-6) and (4-7) with $\gamma = 1$, which gives

$$\left\|\mathrm{ad}_x^{\mu_1}\mathrm{ad}_A^{m_1}\big(((1-\chi_{\eta_2})a)_z\big)\,\mathrm{ad}_x^{\mu_3}\mathrm{ad}_A^{m_3}(\langle D_z\rangle^{\alpha-2})\,\mathrm{ad}_x^{\mu_4}\mathrm{ad}_A^{m_4}(a_z)\right\|_{\mathcal{L}(H^{s+1},H^{s-1})} \lesssim |z|\eta_2^{\frac{\rho}{2}}.$$

Thus we have proved that $\mathrm{ad}_x^\mu\mathrm{ad}_A^m\big(((1-\chi_{\eta_2})a)_z\langle D_z\rangle^\alpha a_z\big)$ is of size $O(|z|\eta_2^{\frac{\rho}{2}})$ in $\mathcal{L}(H^{s+1},H^{s-1})$. We proceed similarly for $\mathrm{ad}_x^\mu\mathrm{ad}_A^m\big((\chi_{\eta_2}a)_z\langle D_z\rangle^\alpha((1-\chi_{\eta_2})a)_z\big)$, and the statement follows. $\qquad\square$

Remark 4.3. If $d \geq 5$ we can replace P_{η_1} by $H_{\bar{\eta}}$ in the last statement of Proposition 4.2. This is not the case for $d \in \{3, 4\}$. This is due to the fact that $s = 1$ does not belong to $\left]-\frac{d}{2}+1, \frac{d}{2}-1\right[$ and hence $B_{\eta_2,z}^\alpha$ is not small in $\mathcal{L}(\dot{H}^2, L^2)$ in these cases.

Proposition 4.4. *Let $\mu \in \mathbb{N}^d$, $m \in \mathbb{N}$ and $s \in \left]-\frac{d}{2}+1, \frac{d}{2}-1\right[$. There exists $\eta_0 \in]0,1]$ such that the operator*

$$\operatorname{ad}_x^{\mu} \operatorname{ad}_A^m R_{\bar{\eta},z}(-1)$$

is bounded as an operator from H^{s-1} to H^{s+1} uniformly in $z \in \mathbb{C}_+$ with $|z| \leqslant 1$ and $\bar{\eta} = (\eta_1, \eta_2) \in]0, \eta_0]^2$.

Proof. The idea of the proof is the same as the proof of Proposition 7.9 in [Bouclet and Royer 2014]. We only have to be careful with the fact that the dissipative term has to be seen as an operator of order 2. However, with the smallness assumption on $a(1 - \chi_{\eta_2})$, it is still a small perturbation of $-\Delta$, and we can proceed as for the wave resolvent. We also have to be careful with the restriction on s, which is stronger than for the wave equation. This is due to the analogous restriction in the second statement of Proposition 4.2. We omit the details. $\qquad\square$

Proposition 4.5. (i) *Let $s \in \left[0, \frac{d}{2}\right[$, $\delta > s$ and $m \in \mathbb{N}$ be such that $m \geqslant s$. Then there exist $\eta_0 \in]0, 1]$ and $C \geqslant 0$ such that for $z \in \mathbb{C}_+$ with $|z| \leqslant 1$ and $\bar{\eta} \in]0, \eta_0]^2$ we have*

$$\|\langle x \rangle^{-\delta} \Theta_z R_{\bar{\eta},z}^m(-1) \Theta_z^{-1} \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C |z|^s.$$

(ii) *Let $s \in \left[0, \frac{d}{2}\right[$, $\delta > s$, and $m \in \mathbb{N}$ large enough $\left(\text{say } m \geqslant \delta + \frac{s}{2} + 1\right)$. Then there exist $\eta_0 \in]0, 1]$ and $C \geqslant 0$ such that for $z \in \mathbb{C}_+$ with $|z| \leqslant 1$ and $\bar{\eta} \in]0, \eta_0]^2$ we have*

$$\|\langle x \rangle^{-\delta} \Theta_z R_{\bar{\eta},z}^m(-1) \langle A \rangle^{\delta}\|_{\mathcal{L}(L^2)} \leqslant C |z|^{\frac{s}{2}},$$

$$\|\langle A \rangle^{\delta} R_{\bar{\eta},z}^m(-1) \Theta_z^{-1} \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C |z|^{\frac{s}{2}}.$$

Proof. According to Proposition 4.4 the operator $R_{\bar{\eta},z}^m(-1)$ is bounded in $\mathcal{L}(H^{-s}, H^s)$ uniformly in $z \in \mathbb{C}_+$ with $|z| \leqslant 1$ and $\bar{\eta}$ close to $(0,0)$. On the other hand, according to the Sobolev embedding $H^s \subset L^p$ for $p = \frac{2d}{d-2s}$, the fact that $\langle x \rangle^{-\delta}$ belongs to $\mathcal{L}(L^p, L^2)$ and (4-3) we have

$$\|\langle x \rangle^{-\delta} \Theta_z\|_{\mathcal{L}(H^s, L^2)} \lesssim \|\Theta_z\|_{\mathcal{L}(L^p)} \lesssim |z|^{\frac{s}{2}}.$$

We similarly have

$$\|\Theta_z^{-1} \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2, H^{-s})} \lesssim |z|^{\frac{s}{2}},$$

and the first statement follows. For the second statement we use the same idea as in the proof of Proposition 7.11 in [Bouclet and Royer 2014]. We only prove the first estimate. For this we first remark that

$$\|\langle x \rangle^{-\delta} \Theta_z (1 + |x|^{\delta})\|_{\mathcal{L}(H^s, L^2)} \lesssim \|\langle x \rangle^{-\delta} \Theta_z\|_{\mathcal{L}(H^s, L^2)} + \|\langle x \rangle^{-\delta} \Theta_z |x|^{\delta}\|_{\mathcal{L}(L^2)}$$

$$\lesssim \|\Theta_z\|_{\mathcal{L}(L^p)} + |z|^{\frac{\delta}{2}} \|\langle x \rangle^{-\delta} |x|^{\delta} \Theta_z\|_{\mathcal{L}(L^2)} \lesssim |z|^{\frac{s}{2}},$$

where, again, p stands for $\frac{2d}{d-2s}$. Then it remains to prove that for all $\delta \geqslant 0$ (we no longer need the assumption that $\delta > s$), $m \geqslant \delta + \frac{s}{2} + 1$ and $\mu \in \mathbb{N}^d$ the operator

$$\langle x \rangle^{-\delta} \operatorname{ad}_x^{\mu} \left(R_{\bar{\eta},z}^m(-1) \langle A \rangle^{\delta} \right)$$

is bounded in $\mathcal{L}(L^2, H^s)$ uniformly in $z \in \mathbb{C}_+$. With $\mu = 0$ this will conclude the proof. By interpolation it is enough to consider the case where δ is an integer and $m \geqslant \delta + \frac{s}{2}$ (we do not mean to be sharp with

this assumption). We proceed by induction. The statement for $\delta = 0$ is given by Proposition 4.4. Now let $\delta \in \mathbb{N}^*$. We have

$$R_{\bar{\eta},z}^m(-1)A^\delta = \sum_{k=0}^\delta C_\delta^k R_{\bar{\eta},z}^{m-1}(-1)A^{\delta-k}\operatorname{ad}_A^k(R_{\bar{\eta},z}(-1)).$$

When $k \neq 0$ we can apply the inductive assumption to $R_{\bar{\eta},z}^{m-1}(-1)A^{\delta-k}$. With Proposition 4.4 we obtain that the contributions of the corresponding terms are uniformly bounded in $\mathcal{L}(L^2, H^s)$ as expected. It remains to consider the term corresponding to $k = 0$. It is enough to consider

$$R_{\bar{\eta},z}^{m-1}(-1)A^{\delta-1}x_j\,D_j\,R_{\bar{\eta},z}(-1)$$

for some $j \in [\![1, d]\!]$. The operator $D_j R_{\bar{\eta},z}(-1)$ and its commutators with powers of x are uniformly bounded operators on L^2, and

$$R_{\bar{\eta},z}^{m-1}(-1)A^{\delta-1}x_j = x_j R_{\bar{\eta},z}^{m-1}(-1)A^{\delta-1} + \operatorname{ad}_{x_j}\left(R_{\bar{\eta},z}^{m-1}(-1)A^{\delta-1}\right).$$

We conclude with the inductive assumption. $\qquad\square$

Proposition 4.6. *Let $k \in \mathbb{N}$ and $\delta > k + \frac{1}{2}$. Then there exist $\eta_0 \in]0, 1]$ and $C \geqslant 0$ such that for $z \in \mathbb{C}_+$ with $|z| \leqslant 1$ and $\bar{\eta} = (\eta_1, \eta_2) \in]0, \eta_0]^2$ we have*

$$\|\langle A\rangle^{-\delta} R_{\bar{\eta},z}^{k+1}(\hat{z})\langle A\rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C.$$

Proof. The estimate is clear when $\hat{z}$ is outside some neighborhood of 1. For $\hat{z}$ close to 1 we apply Theorem 2.4 uniformly in z with A as a conjugate operator. We have already said that e^{-itA} leaves H^1 invariant for all $t \in \mathbb{R}$. The assumptions (ii) and (iii) of Definition 2.3 with $\alpha_0 = \frac{1}{2}$ and $\beta = 0$ are consequences of Proposition 4.2 applied with $s = 0$, $\mu = 0$ and $m \in \mathbb{N}^*$. For $m \in \{0, 1\}$, $z \in \mathbb{C}_+$ and $u \in \mathcal{S}$ we have

$$\left|\langle \operatorname{ad}_{iA}^m(P_{\eta_1,z})u, u\rangle_{L^2} - 2^m\langle -\Delta u, u\rangle_{L^2}\right| \leqslant \sum_{j,k=1}^d \left|\langle(2 - x\cdot\nabla)^m(G_{\eta_1,z,j,k} - \delta_{j,k})D_j u, D_k u\rangle_{L^2}\right|$$

$$\lesssim O(\eta_1^{\rho/2})\|\nabla u\|_{L^2}^2,$$

and hence

$$[P_{\eta_1,z}, iA] \geqslant (2 - O(\eta_1^{\rho/2}))(-\Delta) \geqslant (2 - O(\eta_1^{\rho/2}))P_{\eta_1,z}.$$

Let $J = \left]\frac{1}{2}, \frac{3}{2}\right[$. After conjugation by $\mathbb{1}_J(P_{\eta_1,z})$ we obtain that if η_0 is small enough then for all $\eta_1 \in]0, \eta_0]$ and $z \in \mathbb{C}_+$ we have

$$\mathbb{1}_J(P_{\eta_1,z})[P_{\eta_1,z}, iA]\mathbb{1}_J(P_{\eta_1,z}) \geqslant \tfrac{1}{2}\mathbb{1}_J(P_{\eta_1,z}).$$

Then Proposition 4.6 follows from Theorem 2.4. $\qquad\square$

Remark 4.7. It is important to notice that we have estimated $[B_{\eta_2}^\alpha, iA]$ and $[[B_{\eta_2}^\alpha, iA], iA]$ in $\mathcal{L}(H^1, H^{-1})$ and not in $\mathcal{L}(H^2, L^2)$. By pseudodifferential calculus, these two commutators define operators in

$\mathcal{L}(H^2, L^2)$. But in low dimensions ($d \in \{3, 4\}$) they can be estimated uniformly by Proposition 4.2 only in the sense of forms. This is why we need a form version of the dissipative Mourre method here.

Proposition 4.8. *Let* $\varepsilon > 0$ *and* $n \in \mathbb{N}$. *Let* δ *be as in the statement of Theorem 1.4. Then there exist* $\eta_0 \in \,]0, 1]$, $C \geqslant 0$ *and a neighborhood* $\mathcal{U}$ *of* 0 *in* $\mathbb{C}$ *such that for* $\bar{\eta} = (\eta_1, \eta_2) \in \,]0, \eta_0]^2$ *and* $\beta_l, \beta_r \in \mathbb{R}_+$ *with* $\beta_l + \beta_r \leqslant 2$ *we have*

$$\left\| \langle x \rangle^{-\delta} \langle D \rangle^{\beta_l} R_{\bar{\eta}}^{n+1}(z) \langle D \rangle^{\beta_r} \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2)} \leqslant C (1 + |z|^{\frac{d}{2} - \varepsilon - 1 - n}).$$

Remark 4.9. Compared to the analogous result for the wave equation (see Theorem 1.3 in [Bouclet and Royer 2014]) there is no gain when we add a derivative. This is a consequence of the restriction on the Sobolev index s in Proposition 4.4, which is stronger than in Proposition 7.9 in [Bouclet and Royer 2014].

Proof of Proposition 4.8. First assume that $n \geqslant 1$. By the resolvent identity we have

$$\begin{aligned}
\langle x \rangle^{-\delta} \langle D \rangle^{\beta_l} R_{\bar{\eta}}^{n+1}(z) \langle D \rangle^{\beta_r} \langle x \rangle^{-\delta} &= \langle x \rangle^{-\delta} \langle D \rangle^{\beta_l} R_{\bar{\eta}}(-1) \langle x \rangle^{\delta} \\
&\quad \times \langle x \rangle^{-\delta} \big(R_{\bar{\eta}}^{n-1}(z) + 2(1+z) R_{\bar{\eta}}^n(z) + (1+z)^2 R^{n+1}(z) \big) \langle x \rangle^{-\delta} \\
&\quad \times \langle x \rangle^{\delta} R_{\bar{\eta}}(-1) \langle D \rangle^{\beta_r} \langle x \rangle^{-\delta}.
\end{aligned}$$

The first and last factors are bounded on L^2 uniformly in $\bar{\eta} \in \,]0, 1]^2$ by pseudodifferential calculus, so it is enough to prove the statement without additional derivatives if $n \geqslant 1$. Since $\beta_l + \beta_r \leqslant 2$ we have a similar argument for $n = 0$.

We have

$$\langle x \rangle^{-\delta} R_{\bar{\eta}}^{n+1}(z) \langle x \rangle^{-\delta} = |z|^{-(n+1)} \langle x \rangle^{-\delta} \Theta_z R_{\bar{\eta}, z}^{n+1}(\hat{z}) \Theta_z^{-1} \langle x \rangle^{-\delta}.$$

As in the proof of Theorem 1.3 in Section 3 we can prove by induction on $m \in \mathbb{N}^*$ that $R_{\bar{\eta}, z}^{n+1}(\hat{z})$ can be written as a sum of terms of the form

$$(1 + \hat{z})^{\beta} R_{\bar{\eta}, z}^{n+1+\beta}(-1) \quad \text{or} \quad (1 + \hat{z})^{2m-n-1+\nu} R_{\bar{\eta}, z}^m(-1) R_{\bar{\eta}, z}^\nu(\hat{z}) R_{\bar{\eta}, z}^m(-1), \tag{4-10}$$

where $\max(1, n + 1 - 2m) \leqslant \nu \leqslant n + 1$ and $\beta \in \mathbb{N}$. Let $s = \min\left(n + 1, \frac{d}{2} - \varepsilon\right)$. For $\beta \in \mathbb{N}$ we have $s \in \left[0, \frac{d}{2}\right[$, $n + 1 + \beta \geqslant s$ and $\delta > s$, so according to the first statement of Proposition 4.5 we have

$$|z|^{-(n+1)} \left\| \langle x \rangle^{-\delta} R_{\bar{\eta}, z}^{n+1+\beta}(-1) \langle x \rangle^{-\delta} \right\|_{\mathcal{L}(L^2)} \lesssim |z|^{s-(n+1)} \lesssim 1 + |z|^{\frac{d}{2} - \varepsilon - n - 1}.$$

Now we consider the contributions of terms of the second kind in (4-10). We can assume that m is large enough to apply the second statement of Proposition 4.5. We have $\delta > \nu - \frac{1}{2}$, so with Proposition 4.6 we get

$$\begin{aligned}
|z|^{-(n+1)} &\left\| \langle x \rangle^{-\delta} \Theta_z R_{\bar{\eta}, z}^m(-1) R_{\bar{\eta}, z}^\nu(\hat{z}) R_{\bar{\eta}, z}^m(-1) \Theta_z^{-1} \langle x \rangle^{-\delta} \right\| \\
&\leqslant |z|^{-(n+1)} \left\| \langle x \rangle^{-\delta} \Theta_z R_{\bar{\eta}, z}^m(-1) \langle A \rangle^{\delta} \right\| \left\| \langle A \rangle^{-\delta} R_{\bar{\eta}, z}^\nu(\hat{z}) \langle A \rangle^{-\delta} \right\| \left\| \langle A \rangle^{\delta} R_{\bar{\eta}, z}^m(-1) \Theta_z^{-1} \langle x \rangle^{-\delta} \right\| \\
&\lesssim |z|^{s-(n+1)} \lesssim 1 + |z|^{\frac{d}{2} - \varepsilon - n - 1}. \qquad \square
\end{aligned}$$

4B. *Low-frequency estimates for a general perturbation of the Laplacian.* In this paragraph we use the estimates on $R_{\bar\eta}(z)$ to prove the same estimates for $R(z)$. To this purpose we have to add the contribution of $P_{\eta_1,c}$ in the self-adjoint part and the contribution of $B_{\eta_2,c}^\alpha$ in the dissipative part.

For $\eta_0, \eta_2 \in\,]0,1]$ and $\bar\eta = (\eta_0, \eta_2)$ we set, for $\psi \in C_0^\infty(\mathbb{R})$,

$$S_{\psi,\bar\eta}(z) = P_{\eta_0,c}\, R_{\bar\eta}(z)\psi(P).$$

From Proposition 4.8 we obtain the following result:

Proposition 4.10. *Let $\varepsilon > 0$, $n \in \mathbb{N}$ and $M \in \mathbb{R}$. Let $\psi \in C_0^\infty(\mathbb{R})$. Let δ be as in the statement of Theorem 1.4. Let $\eta_0 \in\,]0,1]$ be given by Proposition 4.8. Then there exist $C \geqslant 0$ and a neighborhood $\mathcal{U}$ of 0 in $\mathbb{C}$ such that for $\eta_2 \in\,]0,\eta_0]$ and $z \in \mathcal{U} \cap \mathbb{C}_+$ we have*

$$\big\|\langle x\rangle^M S_{\psi,\bar\eta}^{(n)}(z)\langle x\rangle^{-\delta}\big\|_{\mathcal{L}(L^2)} \leqslant C\big(1 + |z|^{\frac{d}{2}-\varepsilon-1-n}\big),$$

where $\bar\eta = (\eta_0, \eta_2)$.

Proof. The proposition is a consequence of Proposition 4.8, the boundedness of $\psi(P)$ in $L^{2,\delta}$ and the boundedness of $\langle x\rangle^M P_{\eta_0,c}(1-\Delta)^{-1}\langle x\rangle^\delta$. $\qquad\square$

Remark 4.11. Until now we had not used the distinction between η_1 and η_2. However, the size of $\langle x\rangle^M P_{\eta_1,c}$ depends on η_1, so η_1 has to be fixed in order to obtain uniform estimates in Proposition 4.10 and in Proposition 4.12 below. On the other hand, we have to keep the possibility to take η_2 small. More precisely, the choice of the cut-off function ψ in Proposition 4.12 (and hence in the proof of Proposition 4.13) will depend on η_1, and then the choice of η_2 will in turn depend on ψ. This is why we could not simply take $\eta_1 = \eta_2$ in the definition of $H_{\bar\eta}$.

Proposition 4.12. *Let $\eta_0 \in\,]0,1]$ be given by Proposition 4.8. Let $\varepsilon_1 > 0$, $\sigma > 2$ and $M \geqslant 0$. Then there exist a bounded neighborhood $\mathcal{U}$ of 0 in $\mathbb{C}$, $\psi \in C_0^\infty(\mathbb{R})$ equal to 1 on a neighborhood of 0 and $\tilde\eta \in\,]0,\eta_0]$ such that for $\eta_2 \in\,]0,\tilde\eta]$ and $z \in \mathcal{U} \cap \mathbb{C}_+$ we have*

$$\big\|\langle x\rangle^M S_{\psi,\bar\eta}(z)\langle x\rangle^{-\sigma}\big\|_{\mathcal{L}(L^2)} \leqslant \varepsilon_1,$$

where $\bar\eta = (\eta_0, \eta_2)$.

Proof. According to the Hardy inequality we have for $u \in \mathcal{S}$

$$\big\|\langle x\rangle^M P_{\eta_0,c}u\big\|_{L^2} \lesssim \sum_{j,k=1}^d \big\|\langle x\rangle^M (D_j(\chi_{\eta_0}G_{j,k}))D_k u\big\|_{L^2} + \sum_{j,k=1}^d \big\|\langle x\rangle^M \chi_{\eta_0}(G_{j,k}-\delta_{j,k})D_j D_k u\big\|_{L^2} \lesssim \|u\|_{\dot H^2}.$$

According to the third statement of Proposition 4.2 we obtain for $\mu > 0$

$$\big\|\langle x\rangle^M P_{\eta_0,c} R_{\bar\eta}(i\mu)\psi(P)\langle x\rangle^{-\sigma} u\big\|_{L^2}$$
$$\lesssim \big\|R_{\bar\eta}(i\mu)\psi(P)\langle x\rangle^{-\sigma} u\big\|_{\dot H^2}$$
$$\lesssim \big\|P_{\eta_0} R_{\bar\eta}(i\mu)\psi(P)\langle x\rangle^{-\sigma} u\big\|_{L^2}$$
$$\lesssim \big\|\psi(P)\langle x\rangle^{-\sigma} u\big\|_{L^2} + \mu\big\|R_{\bar\eta}(i\mu)\psi(P)\langle x\rangle^{-\sigma} u\big\|_{L^2} + \big\|B_{\eta_2}^\alpha R_{\bar\eta}(i\mu)\psi(P)\langle x\rangle^{-\sigma} u\big\|$$
$$\lesssim \big\|\psi(P)\langle x\rangle^{-\sigma} u\big\|_{L^2} + O(\eta_2^{1+\rho})\big\|\langle x\rangle^{-1-\rho}(-\Delta+1)R_{\bar\eta}(i\mu)\psi(P)\langle x\rangle^{-\sigma} u\big\|_{L^2}.$$

The term with the factor μ is estimated by the analog of (2-1) for $H_{\bar{\eta}}$. For the term involving $B_{\eta_2}^{\alpha}$ we have used the fact that

$$\|B_{\eta_2}^{\alpha}(-\Delta + 1)^{-1}\langle x \rangle^{1+\rho}\|_{\mathcal{L}(L^2)} = \underset{\eta_2 \to 0}{O}(\eta_2^{1+\rho}).$$

Let $\psi_1 \in C_0^{\infty}(\mathbb{R})$ be equal to 1 on $[-1, 1]$. For $\psi \in C_0^{\infty}(\mathbb{R})$ supported in $]-1, 1[$ we have $\psi(P)\langle x \rangle^{-\sigma} = \psi(P)\psi_1(P)\langle x \rangle^{-\sigma}$. The operator $\psi_1(P)\langle x \rangle^{-\sigma}$ is compact. On the other hand, since 0 is not an eigenvalue of P, the operator $\psi(P)$ goes weakly to 0 when the support of ψ shrinks to $\{0\}$. Thus we can find ψ equal to 1 on a neighborhood of 0 such that for $\mu > 0$ and η_2 small enough we have

$$\|\langle x \rangle^M S_{\psi, \bar{\eta}}(i\mu)\langle x \rangle^{-\sigma}\|_{\mathcal{L}(L^2)} \leqslant \tfrac{1}{2}\varepsilon_1.$$

Now let $\tau \in \mathbb{R}$ and $\mu > 0$. We have

$$\|\langle x \rangle^M P_{\eta_0, c}\big(R_{\bar{\eta}}(\tau + i\mu) - R_{\bar{\eta}}(i\mu)\big)\psi(P)\langle x \rangle^{-\sigma}\|$$
$$\leqslant \|\langle x \rangle^M P_{\eta_0, c}(-\Delta + 1)^{-1}\langle x \rangle^{\sigma}\| \times \int_0^{\tau} \|\langle x \rangle^{-\sigma}(-\Delta + 1)R_{\bar{\eta}}^2(\theta + i\mu)\langle x \rangle^{-\sigma}\| \, d\theta.$$

The first factor is bounded by pseudodifferential calculus, and the second factor is of size $O(|\tau|)$ by Proposition 4.8. Thus this norm is not greater that $\tfrac{1}{2}\varepsilon_1$ if τ is small enough, and the proposition is proved. $\qquad\square$

For $z \in \mathbb{C}_+$ and $\eta_2 \in \,]0, 1]$ we set

$$R_0(z) = (P - z)^{-1} \tag{4-11}$$

and

$$\widetilde{R}_{\eta_2}(z) = (P - iB_{\eta_2}^{\alpha} - z)^{-1}.$$

In the following proposition we prove the resolvent estimates for $\widetilde{R}_{\eta_2}(z)$. Then we will add the contribution of $B_{\eta_2, c}^{\alpha}$ in the dissipative part to conclude the proof of Theorem 1.4.

Proposition 4.13. *Let $\varepsilon > 0$ and $n \in \mathbb{N}$. Let δ be as in Theorem 1.4. Then there exist $\eta_2, C \geqslant 0$ and a neighborhood $\mathcal{U}$ of 0 in $\mathbb{C}$ such that for $z \in \mathcal{U} \cap \mathbb{C}_+$ and $\beta_l, \beta_r \in \mathbb{R}_+$ with $\beta_l + \beta_r \leqslant 2$ we have*

$$\big\|\langle x \rangle^{-\delta}\langle D \rangle^{\beta_l} \widetilde{R}_{\eta_2}^{n+1}(z)\langle D \rangle^{\beta_r}\langle x \rangle^{-\delta}\big\|_{\mathcal{L}(L^2)} \leqslant C(1 + |z|^{\frac{d}{2}-\varepsilon-1-n}).$$

Proof. As for Proposition 4.8 we see that it is enough to consider the case $\beta_l = \beta_r = 0$. Let $\sigma = \max(\delta, 3)$. Let $\varepsilon_1 \in \,]0, \tfrac{1}{4}]$ and consider $\psi \in C_0^{\infty}(\mathbb{R})$ as given by Proposition 4.12 for $M = \sigma$. We set $\mathcal{B}_{\psi}(z) = R_0(z)(1 - \psi)(P)$. For any $\gamma \in \mathbb{R}$, this operator and its derivatives are uniformly bounded on $L^{2,\gamma}$ for $z \in \mathbb{C}_+$ close to 0. Let η_0 be given by Proposition 4.8. Given $\eta_2 \in \,]0, \eta_0]$ we write $\bar{\eta}$ for (η_0, η_2). We have

$$\widetilde{R}_{\eta_2}(z) = R_{\bar{\eta}}(z)\psi(P) - \widetilde{R}_{\eta_2}(z)S_{\psi, \bar{\eta}}(z) + \mathcal{B}_{\psi}(z) + i\widetilde{R}_{\eta_2}(z)B_{\eta_2}^{\alpha}\mathcal{B}_{\psi}(z),$$

and hence for $n \in \mathbb{N}$

$$\widetilde{R}_{\eta_2}^{(n)}(z) = R_{\bar{\eta}}^{(n)}(z)\psi(P) + \widetilde{R}_{\eta_2}^{(n)}(z)\big(-S_{\psi, \bar{\eta}}(z) + iB_{\eta_2}^{\alpha}\mathcal{B}_{\psi}(z)\big) + \mathcal{B}_{\psi}^{(n)}(z)$$
$$+ i\sum_{j=0}^{n-1} C_j^n \widetilde{R}_{\eta_2}^{(j)}(z)\big(-S_{\psi, \bar{\eta}}^{(n-j)}(z) + iB_{\eta_2}^{\alpha}\mathcal{B}_{\psi}^{(n-j)}(z)\big). \tag{4-12}$$

We prove by induction on $n \in \mathbb{N}$ that

$$\| \langle x \rangle^{-\delta} \widetilde{R}_{\eta_2}^{(n)}(z) \langle x \rangle^{-\sigma} \| \lesssim 1 + |z|^{\frac{d}{2} - \varepsilon - n - 1}. \tag{4-13}$$

According to Propositions 4.8, 4.10 and 4.12, the fact that $\psi(P)$ is uniformly bounded on $L^{2,\sigma}$ and the inductive assumption for the sum in (4-12) (it vanishes if $n = 0$), there exists $C \geqslant 0$ such that for $z \in \mathbb{C}_+$ close to 0 we have

$$\| \langle x \rangle^{-\delta} \widetilde{R}_{\eta_2}^{(n)}(z) \langle x \rangle^{-\sigma} \| \left(1 - \varepsilon_1 - \| \langle x \rangle^{\sigma} B_{\eta_2}^{\alpha} \mathcal{B}_{\psi}(z) \langle x \rangle^{-\sigma} \| \right) \leqslant C (1 + |z|^{\frac{d}{2} - \varepsilon - n - 1}).$$

By pseudodifferential calculus we see that the norm of $\langle x \rangle^{\sigma} B_{\eta_2}^{\alpha} \mathcal{B}_{\psi}(z) \langle x \rangle^{-\sigma}$ goes to 0 when η_2 goes to 0. Thus if η_2 is small enough we have

$$1 - \varepsilon_1 - \| \langle x \rangle^{\sigma} B_{\eta_2}^{\alpha} \mathcal{B}_{\psi}(z) \langle x \rangle^{-\sigma} \| \geqslant \tfrac{1}{2},$$

which concludes the proof of (4-13). In order to replace σ by δ we use (4-12) again and, estimating the second term with (4-13) and Proposition 4.10 instead of Proposition 4.12 we obtain

$$\| \langle x \rangle^{-\delta} \widetilde{R}_{\eta_2}^{(n)}(z) \langle x \rangle^{-\delta} \| \left(1 - \| \langle x \rangle^{\delta} B_{\eta_2}^{\alpha} \mathcal{B}_{\psi}(z) \langle x \rangle^{-\delta} \| \right) \leqslant C (1 + |z|^{\frac{d}{2} - \varepsilon - n - 1}),$$

and we conclude similarly. $\qquad\square$

In order to prove Theorem 1.4 it remains to add the dissipative part with compactly supported absorption index. We begin with a lemma:

Lemma 4.14. *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{R}_0, \mathcal{R}_1 \in \mathcal{L}(\mathcal{H})$ and let $\mathcal{B}$ be such that*

$$\mathcal{R}_1 = \mathcal{R}_0 - \mathcal{R}_0 \mathcal{B} \mathcal{R}_1 = \mathcal{R}_0 - \mathcal{R}_1 \mathcal{B} \mathcal{R}_0.$$

Then for all $m \in \mathbb{N}$ we can write $\mathcal{R}_1^{m+1}$ as a linear combination of terms of the form

$$\mathcal{R}_0^{m_1 + 1} \mathcal{B} \mathcal{R}_{j_2}^{m_2 + 1} \mathcal{B} \cdots \mathcal{B} \mathcal{R}_{j_{k-1}}^{m_k + 1} \mathcal{B} \mathcal{R}_0^{m_k + 1}, \tag{4-14}$$

where $k \in \mathbb{N}^*$, $j_1, \ldots, j_{k-1} \in \{0, 1\}$ and $m_1, \ldots, m_k \in \mathbb{N}$ are such that

$$\sum_{l=1}^{k} m_l \leqslant m \quad and \quad m_l = 0 \quad if \ j_l = 1.$$

Proof. Using both of the identities above involving $\mathcal{R}_1$ and $\mathcal{R}_0$ we obtain

$$\mathcal{R}_1(z) = \mathcal{R}_0(z) - \mathcal{R}_0(z) \mathcal{B} \mathcal{R}_0(z) + \mathcal{R}_0(z) \mathcal{B} \mathcal{R}_1(z) \mathcal{B} \mathcal{R}_0(z).$$

Then the result is proved by induction on m. $\qquad\square$

Now we can finish the proof of Theorem 1.4:

Proof of Theorem 1.4. Let η_2 be given by Proposition 4.13. Let $T = \langle D \rangle^{\frac{\alpha}{2}} a \chi_{\eta_2} \in \mathcal{L}(H^1, L^2)$. We have $T^* T = B_{\eta_2, c}^{\alpha} \lesssim B_{\alpha}$, so according to Corollary 2.6 we have

$$\| T R(z) T^* \|_{\mathcal{L}(L^2)} \lesssim 1.$$

Let $M \geqslant 0$ and $T_M = \langle x \rangle^{-M} \langle D \rangle^{\frac{\alpha}{2}}$. We can write $B_{\eta_2, c}^{\alpha} = T_M^* \mathcal{B}_1 T = T^* \mathcal{B}_2 T_M = T_M^* \mathcal{B}_3 T_M$, where $\mathcal{B}_1$, $\mathcal{B}_2$ and $\mathcal{B}_3$ are bounded on L^2. According to Lemma 4.14 applied with

$$\mathcal{R}_0 = \widetilde{R}_{\eta_2}(z), \quad \mathcal{R}_1 = R(z) \quad \text{and} \quad \mathcal{B} = B_{\eta_2, c}^{\alpha},$$

we can write $R^{m+1}(z)$ as a sum of terms of the form

$$\mathcal{T} = \mathcal{R}_0^{m_1 + 1}(z) \mathcal{B} \mathcal{R}_{j_2}^{m_2 + 1}(z) \mathcal{B} \cdots \mathcal{B} \mathcal{R}_{j_{k-1}}^{m_k + 1}(z) \mathcal{B} \mathcal{R}_0^{m_k + 1}(z), \tag{4-15}$$

where $k \in \mathbb{N}^*$, $j_1, \dots, j_{k-1} \in \{0, 1\}$ and $m_1, \dots, m_k \in \mathbb{N}$ are such that $\sum_{l=1}^k m_l \lesssim m$ and $m_l = 0$ if $j_l = 1$. If $k \geqslant 3$ and M is large enough we obtain for such a term

$$\| \langle x \rangle^{-\delta} \mathcal{T} \langle x \rangle^{-\delta} \|$$
$$\lesssim \| \langle x \rangle^{-\delta} \widetilde{R}_{\eta_2}^{m_1 + 1}(z) T_M^* \| \times \prod_{\substack{l=2 \\ j_l = 0}}^{k-1} \| T_M \widetilde{R}_{\eta_2}^{m_l + 1}(z) T_M^* \| \times \prod_{\substack{l=2 \\ j_l = 1}}^{k-1} \| T R(z) T^* \| \times \| T_M \widetilde{R}_{\eta_2}^{m_k + 1}(z) \langle x \rangle^{-\delta} \|$$
$$\lesssim \prod_{l=1}^k (1 + |z|^{\frac{d}{2} - 1 - m_l - \varepsilon}) \lesssim (1 + |z|^{\frac{d}{2} - 1 - m - \varepsilon}).$$

The cases $k = 1$ and $k = 2$ are estimated similarly. This concludes the proof of Theorem 1.4. $\qquad \square$

4C. *Sharp low-frequency resolvent estimate.* We finish this section with the proof of Theorem 1.5. The result follows from the self-adjoint analog by a simple perturbation argument, using the quadratic estimates and the spatial decay of the dissipative term:

Proof of Theorem 1.5. According to the resolvent identity, Proposition 2.5 and Theorem 1.1 in [Bouclet and Royer 2015] we have

$$\| \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} \| = \| \langle x \rangle^{-1} R_0(z) \langle x \rangle^{-1} \| + \| \langle x \rangle^{-1} R_0(z) \sqrt{B_\alpha} \| \| \sqrt{B_\alpha} R(z) \langle x \rangle^{-1} \|$$
$$\lesssim 1 + \| \langle x \rangle^{-1} R_0(z) \sqrt{B_\alpha} \| \| \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} \|^{\frac{1}{2}}.$$

Moreover,

$$\| \langle x \rangle^{-1} R_0(z) \sqrt{B_\alpha} \| \leqslant \| \langle x \rangle^{-1} R_0(i) \sqrt{B_\alpha} \| + |z - i| \| \langle x \rangle^{-1} R_0(z) \langle x \rangle^{-1} \| \| \langle x \rangle R_0(i) \sqrt{B_\alpha} \| \lesssim 1.$$

For the norms involving $R_0(i)$ we have used the fact that $\langle x \rangle^\sigma R_0(i) \sqrt{B_\alpha}$ extends to a bounded operator since for $\sigma \leqslant 1$ and $u \in \mathcal{S}$ we have by pseudodifferential calculus

$$\| \sqrt{B_\alpha} R_0(i) \langle x \rangle^\sigma u \|_{L^2}^2 \leqslant \langle \langle x \rangle^\sigma R_0(-i) B_\alpha R_0(i) \langle x \rangle^\sigma u, u \rangle \lesssim \| u \|_{L^2}^2.$$

This gives

$$\| \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} \| \lesssim 1 + \| \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} \|^{\frac{1}{2}},$$

from which the conclusion follows. $\qquad \square$

5. High-frequency estimates

In this section we prove Theorem 1.6. To this purpose we use semiclassical analysis (see for instance [Zworski 2012]). For $h > 0$ and $\zeta \in \mathbb{C}_+$ we set $H_h = h^2 H$, $P_h = h^2 P$ and $R_h(\zeta) = (H_h - \zeta)^{-1}$. Then for $n \in \mathbb{N}$, $z \in \mathbb{C}_+$ and $h = |z|^{-\frac{1}{2}}$ we have

$$R(z)^{n+1} = \frac{1}{|z|^{n+1}} R_h^{n+1}(\hat{z}) = h^{2(n+1)} R_h^{n+1}(\hat{z}) \tag{5-1}$$

(we recall that $\hat{z} = z/|z|$).

In order to prove uniform estimates for the resolvent $R_h(z)$ we use again the Mourre method. For high frequencies and in a dissipative context we follow [Royer 2010; Bouclet and Royer 2014]. Here we have to be careful with the form of the dissipative part $h^2 B_\alpha$.

Let $\chi_\alpha \in C_0^\infty(\mathbb{R})$ be positive in a neighborhood of 1 and such that $0 \leqslant \chi_\alpha(r) \leqslant r^{\frac{\alpha}{2}}$ for all $r \in \mathbb{R}_+$. For $h \in \,]0, 1]$ we set

$$B_h^\alpha = a(x) \chi_\alpha(-h^2 \Delta) a(x).$$

Then we have

$$0 \leqslant h^{2-\tilde{\alpha}} B_h^\alpha \leqslant h^{2-\alpha} B_h^\alpha \leqslant h^2 a(x)(-\Delta)^{\frac{\alpha}{2}} a(x) \leqslant h^2 B_\alpha, \tag{5-2}$$

in the sense that for all $\varphi \in H^{\alpha/2}(\mathbb{R}^d)$ we have

$$0 \leqslant h^{2-\tilde{\alpha}} \langle B_h^\alpha \varphi, \varphi \rangle_{L^2(\mathbb{R}^d)} \leqslant h^2 \langle B_\alpha \varphi, \varphi \rangle_{L^2(\mathbb{R}^d)}. \tag{5-3}$$

The operator B_h^α is a bounded pseudodifferential operator on L^2. Its principal symbol is

$$b(x, \xi) = a(x)^2 \chi_\alpha(|\xi|^2).$$

The damping assumption (1-9) on bounded trajectories is satisfied with b instead of a:

$$\forall w \in \Omega_b, \ \exists T \in \mathbb{R} \quad \text{such that} \quad b(\phi^T(w)) > 0.$$

Set

$$f_0(x, \xi) = x \cdot \xi.$$

As in [Bouclet and Royer 2014] (see Proposition 8.1), we can prove that there exist an open neighborhood $\tilde{J}$ of 1, $f_c \in C_0^\infty(\mathbb{R}^{2n}, \mathbb{R})$, $\beta \geqslant 0$ and $c_0 > 0$ such that on $p^{-1}(\tilde{J})$ we have

$$\{p, f_0 + f_c\} + \beta b \geqslant 3c_0, \tag{5-4}$$

where $\{p, q\}$ is the Poisson bracket $\nabla_\xi p \cdot \nabla_x q - \nabla_x p \cdot \nabla_\xi q$. The fact that the symbol of the dissipative part depends on ξ does not change anything in the proof of this statement. We set

$$F_h = \mathrm{Op}_h^w(f_0 + f_c),$$

where Op_h^w is the Weyl quantization:

$$\mathrm{Op}_h^w(q)u(x) = \frac{1}{(2\pi h)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{\frac{i}{h}\langle x-y, \xi \rangle} q\left(\frac{x+y}{2}, \xi\right) u(y) \, dy \, d\xi.$$

Let J be a neighborhood of 1 and a compact subset of $\widetilde{J}$. Let $\chi \in C_0^\infty(\widetilde{J}, [0,1])$ be equal to 1 on a neighborhood of J. After multiplication by $(\chi \circ p)^2$, the (easy) Gårding inequality (Theorem 4.26 in [Zworski 2012]) gives for $h > 0$ small enough

$$\mathrm{Op}_h\big((\chi \circ p)^2 \{p, f_0 + f_c\} + \beta b(\chi \circ p)^2 + 3c_0(1 - (\chi \circ p)^2)\big) \geqslant 3c_0 - O(h) \geqslant 2c_0.$$

After multiplication by $h^{2-\tilde{\alpha}}$ we obtain

$$\chi(P_h)\big([P_h, ih^{1-\tilde{\alpha}} F_h] + \beta h^{2-\tilde{\alpha}} B_h^\alpha\big)\chi(P_h) + 3c_0(1 - \chi^2)(P_h) \geqslant 2c_0 h^{2-\tilde{\alpha}} - O(h^{3-\tilde{\alpha}}).$$

After conjugation by $\mathbb{1}_J(P_h)$ we obtain for h small enough

$$\mathbb{1}_J(P_h)\big([P_h, ih^{1-\tilde{\alpha}} F_h] + \beta h^{2-\tilde{\alpha}} B_h^\alpha\big)\mathbb{1}_J(P_h) \geqslant c_0 h^{2-\tilde{\alpha}} \mathbb{1}_J(P_h).$$

According to (5-2) this finally gives

$$\mathbb{1}_J(P_h)\big([P_h, ih^{1-\tilde{\alpha}} F_h] + \beta h^2 B_\alpha\big)\mathbb{1}_J(P_h) \geqslant c_0 h^{2-\tilde{\alpha}} \mathbb{1}_J(P_h), \tag{5-5}$$

which is the main assumption of Definition 2.3 with βh^2 instead of β and $\alpha = c_0 h^{2-\tilde{\alpha}}$.

It remains to check the other assumptions of Definition 2.3. The first is proved as in [Bouclet and Royer 2014] (except that we look at the norm in the form domain H^1 instead of the domain H^2), and the commutator properties are proved using (standard) pseudodifferential calculus, considering h as a parameter (for the dissipative part we cannot use $h^{2-\alpha} B_h^\alpha$ as above, so we have to control directly the commutators of $h^2 B_\alpha$ with $h^{1-\tilde{\alpha}} F_h$).

Thus we have proved that for $h \in \,]0, h_0]$ the operator $h^{1-\tilde{\alpha}} F_h$ is a conjugate operator to H_h on a neighborhood J of 1 with lower bounds $h^{2-\tilde{\alpha}} c_0$ for some $c_0 > 0$. According to Theorem 2.4 we have proved the following result with $\langle F_h \rangle^{-\delta}$ instead of $\langle x \rangle^{-\delta}$:

Proposition 5.1. *Let $n \in \mathbb{N}$ and $\delta > n + \frac{1}{2}$. There exists a neighborhood J of 1, $h_0 > 0$ and $C \geqslant 0$ such that for all $\zeta \in \mathbb{C}_+$ with $\mathrm{Re}(\zeta) \in J$ we have*

$$\|\langle x \rangle^{-\delta} R_h^{n+1}(\zeta) \langle x \rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant \frac{C}{h^{(2-\tilde{\alpha})(n+1)}}.$$

In order to have the estimate with $\langle x \rangle^{-\delta}$ we proceed as usual (see the end of Section 3 for intermediate frequencies or [Royer 2010] in the semiclassical context). With (5-1) and Proposition 5.1 we obtain the second statement of Theorem 1.6. For the first statement, we observe that under the nontrapping condition we can proceed as above with $\beta = 0$ and with $\tilde{\alpha}$ replaced by 1 in (5-5).

6. Local energy decay

In this section we use Theorems 1.3, 1.4 and 1.6 to prove Theorem 1.1. Let $u_0 \in \mathcal{S}$. We denote by u the solution of (1-1). Let $\mu > 0$. For $t \in \mathbb{R}$ we set

$$u_\mu(t) = \mathbb{1}_{\mathbb{R}_+}(t) u(t) e^{-t\mu}.$$

Then for $\tau \in \mathbb{R}$ we set

$$\check{u}_\mu(\tau) = \int_{\mathbb{R}} e^{it\tau} u_\mu(t)\, dt = \int_0^{+\infty} e^{it(\tau+i\mu)} u(t)\, dt, \qquad (6\text{-}1)$$

so that for all $n \in \mathbb{N}$ and $\tau \in \mathbb{R}$ we have

$$\check{u}_\mu^{(n)}(\tau) = \int_{\mathbb{R}} (it)^n e^{it\tau} u_\mu(t)\, dt. \qquad (6\text{-}2)$$

We multiply (1-1) by $e^{it(\tau+i\mu)}$ and integrate over $\mathbb{R}_+$. This yields

$$(H - (\tau + i\mu))\check{u}_\mu(\tau) = -iu_0$$

and hence, for all $n \in \mathbb{N}$

$$\check{u}_\mu^{(n)}(\tau) = -i\, n!\, R^{n+1}(\tau + i\mu)u_0. \qquad (6\text{-}3)$$

Lemma 6.1. *For all $n \in \mathbb{N}^*$ and $\mu > 0$ the map $\tau \mapsto R^{n+1}(\tau + i\mu)u_0$ belongs to $L^1(\mathbb{R}, L^2(\mathbb{R}^d))$.*

Proof. Let $\chi_0 \in C_0^\infty(\mathbb{R}, [0, 1])$ be equal to 1 on a neighborhood of 0. According to (1-7) the map $\tau \mapsto R^{n+1}(\tau + i\mu)u_0$ is bounded, so it is enough to prove that $\tau \mapsto (1 - \chi_0)(\tau)R^{n+1}(\tau + i\mu)u_0$ belongs to $L^1(\mathbb{R})$. Let $z \in \mathbb{C}_+$. Using twice the identity

$$R(z) = \frac{R(z)(H + 1) - 1}{z + 1},$$

we get

$$R(z)u_0 = \frac{1}{(z+1)^2} R(z)(H+1)^2 u_0 - \frac{1}{(z+1)^2}(H+1)u_0 - \frac{1}{z+1}u_0.$$

The result follows after at least one differentiation with respect to z. $\qquad\square$

This lemma does not provide any uniform estimate, but now we can take the Fourier transform of (6-2). With (6-3) this gives for all $t \geqslant 0$

$$(it)^n e^{-t\mu} u(t) = -\frac{i\, n!}{2\pi} \int_{\tau \in \mathbb{R}} e^{-it\tau} R^{n+1}(\tau + i\mu)u_0\, d\tau. \qquad (6\text{-}4)$$

We consider $\chi_-, \chi_0, \chi \in C^\infty(\mathbb{R}, [0, 1])$ such that χ_- is supported in $]-\infty, 0[$, χ_0 is compactly supported and equal to 1 on a neighborhood of 0, χ is compactly supported in $]0, +\infty[$ and

$$\chi_- + \chi_0 + \sum_{j \in \mathbb{N}^*} \chi_j = 1 \quad \text{on } \mathbb{R},$$

where for $j \in \mathbb{N}^*$ and $\tau \in \mathbb{R}$ we have set $\chi_j(\tau) = \chi(\tau/2^{j-1})$. We set $\chi_+ = \sum_{j \in \mathbb{N}^*} \chi_j$. Starting from (6-4) applied with $n = \kappa - 1$ (κ was defined in (1-5)) we can write

$$u_\mu(t) = -\frac{i\, n!}{2\pi (it)^{\kappa-1}} \big(v_-(t) + v_0(t) + v_+(t)\big), \qquad (6\text{-}5)$$

where for $* \in \{-, 0, +\}$ we have set

$$v_*(t) = \int_{\tau \in \mathbb{R}} \chi_*(\tau) e^{-it\tau} R^\kappa(\tau + i\mu)u_0\, d\tau. \qquad (6\text{-}6)$$

To simplify the notation we forget the dependence on μ. From now on, all the quantities depend on $\mu > 0$ but the estimates are uniform in μ.

Proposition 6.2. *Let $k \in \mathbb{N}$. There exists $C \geqslant 0$ which does not depend on $u_0 \in \mathcal{S}$ such that for all $\mu > 0$ and $t \geqslant 0$ we have*

$$\|v_-(t)\|_{L^2} \leqslant C \langle t \rangle^{-k} \|u_0\|_{L^2}.$$

This implies that the corresponding contribution for $u(t)$ decays like any power of t in L^2.

Proof. After k partial integrations in (6-6) we get

$$(it)^k v_-(t) = \int_{\mathbb{R}} e^{-it\tau} \frac{d^k}{d\tau^k} (\chi_-(\tau) R(\tau + i\mu)^\kappa) u_0 \, d\tau.$$

According to Remark 2.1 we have

$$\left\| \frac{d^k}{d\tau^k} (\chi_-(\tau) R(\tau + i\mu)^\kappa) \right\|_{\mathcal{L}(L^2)} \lesssim \langle \tau \rangle^{-(\kappa+k)},$$

and the result follows. $\qquad\square$

We now deal with v_0. The following result is (a slightly modified version of) Lemma 4.3 in [Bouclet and Royer 2014]:

Lemma 6.3. *Let $\mathcal{H}$ be a Hilbert space. Let $f \in C^1(\mathbb{R}^*, \mathcal{H})$ be equal to 0 outside a compact subset of $\mathbb{R}$. Assume that for some $\gamma \in]0, 1[$ and $M_f \geqslant 0$ we have*

$$\forall \tau \in \mathbb{R}^*, \quad \|f(\tau)\|_{\mathcal{H}} \leqslant M_f |\tau|^{-\gamma} \quad and \quad \|f'(\tau)\|_{\mathcal{H}} \leqslant M_f |\tau|^{-1-\gamma}.$$

Let $\beta \in [0, 1[$. Then there exists $C \geqslant 0$ which does not depend on f and such that for all $t \in \mathbb{R}$ we have

$$\|\hat{f}(t)\|_{\mathcal{H}} \leqslant C \, M_f \, \langle t \rangle^{\beta(\gamma-1)}.$$

Proof. Following the proof of [Bouclet and Royer 2014] we set $f_t(\tau) = \int_{-1}^{1} \phi(s) f\left(\tau - \frac{s}{t}\right) ds$, where $\phi \in C_0^\infty(]-1, 1[, \mathbb{R})$ satisfies $\int_{\mathbb{R}} \phi = 1$ and we write for $|t| \geqslant 1$

$$|\hat{f}(t)| \leqslant \int_{|\tau| \leqslant t^{-\beta}} \|f(\tau)\| \, d\tau + \int_{|\tau| \geqslant t^{-\beta}} \|f(\tau) - f_t(\tau)\| \, d\tau + \left\| \int_{|\tau| \geqslant t^{-\beta}} e^{-it\tau} f_t(\tau) \, d\tau \right\|$$

$$\lesssim |t|^{-\beta(1-\gamma)} + |t|^{\gamma\beta-1} + \frac{1}{t} \left(\|f_t(t^{-\beta})\| + \|f_t(-t^{-\beta})\| + \left\| \int_{|\tau| \geqslant t^{-\beta}} e^{-it\tau} f_t'(\tau) \, d\tau \right\| \right) \lesssim |t|^{-\beta(1-\gamma)}.$$

We omit the details. $\qquad\square$

Proposition 6.4. *Let $\varepsilon \in]0, \frac{1}{2}[$ and $\delta > \kappa + \frac{1}{2}$. Then there exists $C \geqslant 0$ which does not depend on $u_0 \in \mathcal{S}$ and such that for all $\mu > 0$ and $t \geqslant 0$ we have*

$$\|v_0(t)\|_{L^{2,-\delta}} \leqslant \langle t \rangle^{\kappa-1-\frac{d}{2}+\varepsilon} \|u_0\|_{L^{2,\delta}}.$$

Proof. According to Theorem 1.4 applied with $\frac{\varepsilon}{2}$ instead of ε and Theorem 1.3 there exists $C \geqslant 0$ (which does not depend on u_0) such that for $\mu > 0$, $\tau \in \mathbb{R}$ and $z = \tau + i\mu$ we have

$$\|\chi_0(\tau)R^\kappa(z)u_0\|_{L^{2,-\delta}} \leqslant C|z|^{\frac{d}{2}-\kappa-\frac{\varepsilon}{2}}\|u_0\|_{L^{2,\delta}},$$

$$\left\|\frac{d}{d\tau}\big(\chi_0(\tau)R^\kappa(z)\big)u_0\right\|_{L^{2,-\delta}} \leqslant C|z|^{\frac{d}{2}-\kappa-1-\frac{\varepsilon}{2}}\|u_0\|_{L^{2,\delta}}.$$

Then the statement follows from Lemma 6.3 applied with $\beta \in]0,1[$ so close to 1 that

$$\beta\left(\kappa - \frac{d}{2} - 1 + \frac{\varepsilon}{2}\right) \leqslant \kappa - \frac{d}{2} - 1 + \varepsilon. \qquad \square$$

To finish the proof of Theorem 1.1 we have to estimate $v_+(t)$. As for $v_-(t)$ above, k partial integrations yield

$$(it)^k v_+(t) = \int_{\mathbb{R}} e^{-it\tau} \sum_{j=1}^{k} C_k^j \chi_+^{(j)}(\tau) R^{\kappa+k-j}(\tau+i\mu)u_0 \, d\tau + \int_{\mathbb{R}} e^{-it\tau}\chi_+(\tau)R^{\kappa+k}(\tau+i\mu)u_0 \, d\tau$$

$$=: v_{+,k}^0(t) + w_k(t).$$

The following proposition proves that the contribution of $v_+(t)$ in (6-5) decays like any power of t. However, there may be a loss of two derivatives when $\alpha = 0$ if the nontrapping assumption does not hold. We apply the following result with $k \geqslant 1$ to conclude the proof of Theorem 1.1.

Proposition 6.5. *Let $k \in \mathbb{N}^*$ and $\delta > \kappa + k - \frac{1}{2}$. Let $\sigma \in [0,2]$.*

(i) *There exists $C \geqslant 0$ which does not depend on u_0 and such that for all $\mu > 0$ and $t \geqslant 1$ we have*

$$\|\langle x \rangle^{-\delta} v_{+,k}^0(t)\|_{L^2} \leqslant C\|u_0\|_{L^{2,\delta}}.$$

(ii) *Assume that the nontrapping assumption (1-8) holds or that we have the damping condition (1-9) together with $(\kappa+k)\tilde{\alpha} + \sigma \geqslant 2$. Then there exists $C \geqslant 0$ which does not depend on u_0 such that for all $\mu > 0$ and $t \geqslant 1$ we have*

$$\|\langle x \rangle^{-\delta} w_k(t)\|_{L^2} \leqslant C\|u_0\|_{H^{\sigma,\delta}}.$$

Proof. Statement (i) follows from Theorem 1.3 and the fact that $\chi_+^{(j)}$ is compactly supported in $]0,+\infty[$ for all $j \geqslant 1$. We turn to the proof of (ii).

• For $j \in \mathbb{N}^*$ we set

$$w_{k,j}(t) = \int_{\tau \in \mathbb{R}} \chi_j(\tau)e^{-it\tau}R^{\kappa+k}(\tau+i\mu)u_0 \, d\tau.$$

Let $\tilde{\chi} \in C_0^\infty(\mathbb{R}_+^*, [0,1])$ be equal to 1 on a neighborhood of supp χ. For $\tau \in \mathbb{R}$ and $j \in \mathbb{N}^*$ we set $\tilde{\chi}_j(\tau) = \tilde{\chi}(\tau/2^{j-1})$. Let

$$I_{k,j}(t) = \int_{\tau \in \mathbb{R}} \chi_j(\tau)e^{-it\tau}\langle x \rangle^{-\delta} R^{\kappa+k}(\tau+i\mu)\langle x \rangle^{-\delta} \, d\tau \in \mathcal{L}(L^2).$$

We have

$$\langle x \rangle^{-\delta} w_{k,j}(t) = w_{k,j}^1(t) + w_{k,j}^2(t) + w_{k,j}^3(t),$$

where

$$w_{k,j}^1(t) = \tilde{\chi}_j(P)I_{k,j}(t)\tilde{\chi}_j(P)\langle x\rangle^\delta u_0,$$
$$w_{k,j}^2(t) = (1-\tilde{\chi}_j)(P)I_{k,j}(t)\tilde{\chi}_j(P)\langle x\rangle^\delta u_0,$$
$$w_{k,j}^3(t) = I_{k,j}(t)(1-\tilde{\chi}_j)(P)\langle x\rangle^\delta u_0.$$

• By almost orthogonality, Theorem 1.6 and almost orthogonality again we have

$$\left\|\sum_{j\in\mathbb{N}^*} w_{k,j}^1(t)\right\|^2 \lesssim \sum_{j\in\mathbb{N}^*} \|w_{k,j}^1(t)\|^2$$

$$\lesssim \sup_{j\in\mathbb{N}^*}\left(\int_{\tau\in\mathbb{R}}\chi_j(\tau)\|\langle x\rangle^{-\delta}R^{\kappa+k}(\tau+i\mu)\langle x\rangle^{-\delta}\|\,d\tau\right)^2 \times \sum_{j\in\mathbb{N}^*}\|\tilde{\chi}_j(P)\langle P\rangle^{-\frac{\sigma}{2}}\langle P\rangle^{\frac{\sigma}{2}}\langle x\rangle^\delta u_0\|^2$$

$$\lesssim \sup_{j\in\mathbb{N}^*} 2^{2j}2^{-j(\kappa+k)\alpha}2^{-j\sigma}\|\langle P\rangle^{\frac{\sigma}{2}}\langle x\rangle^\delta u_0\|^2$$

$$\lesssim \|u_0\|_{H^{\sigma,\delta}}^2.$$

It remains to prove that

$$\|w_{k,j}^2(t)\| + \|w_{k,j}^3(t)\| \lesssim 2^{-j}\|u_0\|_{L^{2,\delta}}. \tag{6-7}$$

• For the contribution of $w_{k,j}^2(t)$ we prove that there exists $C \geqslant 0$ such that for $j \in \mathbb{N}^*$ and $\tau \in \mathrm{supp}(\chi_j)$ we have

$$\|(1-\tilde{\chi}_j)(P)\langle x\rangle^{-\delta}R^{\kappa+k}(\tau+i\mu)\langle x\rangle^{-\delta}\|_{\mathcal{L}(L^2)} \leqslant C2^{-2j}. \tag{6-8}$$

For this, we prove by induction on $m \in \mathbb{N}^*$ and then on $\ell \in \mathbb{N}$ that for $\delta > m - \frac{1}{2}$ and $\phi \in C_0^\infty(\mathbb{R}_+^*, [0,1])$ equal to 1 on a neighborhood of $\mathrm{supp}(\chi)$ there exists $C \geqslant 0$ such that for all $j \in \mathbb{N}^*$, $z = \tau + i\mu$ with $\tau \in \mathrm{supp}(\chi)$ and $\mu > 0$

$$\|\langle x\rangle^{-\delta}(1-\phi_j)(P)R^m(z)\langle x\rangle^{-\delta}\| \leqslant C2^{-j\min\left(m,\ell\left(1-\frac{\alpha}{2}\right)\right)}, \tag{6-9}$$

where for $j \in \mathbb{N}^*$ we have set $\phi_j = \phi(\cdot/2^{j-1})$. Let $m \in \mathbb{N}^*$. If $m \geqslant 2$, we assume that the estimate is proved up to order $m-1$ (for all $\ell \in \mathbb{N}$). Note that we will not use any inductive assumption on m for $m = 1$. Then we prove the estimate by induction on $\ell \in \mathbb{N}$. For $\ell = 0$ it follows from Theorem 1.6 and the boundedness of $(1-\phi_j)(P)$ in weighted spaces. Assume that (6-9) is proved up to order $\ell-1$ for some $\ell \in \mathbb{N}^*$. Let $\tilde{\phi} \in C_0^\infty(\mathbb{R}_+^*)$ be equal to 1 on a neighborhood of $\mathrm{supp}(\chi)$ and such that $\phi = 1$ on a neighborhood of $\mathrm{supp}(\tilde{\phi})$. For $j \in \mathbb{N}^*$ we set $\tilde{\phi}_j = \tilde{\phi}(\cdot/2^{j-1})$. We recall that for $z \in \mathbb{C}_+$ we have set $R_0(z) = (P-z)^{-1}$. By the resolvent identity we have

$$R^m(z) = R_0(z)R^{m-1}(z) + iR_0(z)B_\alpha R^m(z). \tag{6-10}$$

If $m = 1$, the first term is just $R_0(z)$ and we have

$$\|\langle x\rangle^{-\delta}(1-\phi_j)(P)R_0(z)\langle x\rangle^{-\delta}\| \lesssim 2^{-j}.$$

If $m \geqslant 2$, the contribution of the first term is estimated as follows:

$$\left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)R^{m-1}(z)\langle x \rangle^{-\delta} \right\|$$
$$\leqslant \left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)\tilde{\phi}_j(P)\langle x \rangle^{\delta} \right\| \left\| \langle x \rangle^{-\delta} R^{m-1}(z)\langle x \rangle^{-\delta} \right\|$$
$$\leqslant \left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)\langle x \rangle^{\delta} \right\| \left\| \langle x \rangle^{-\delta}(1-\tilde{\phi}_j(P))R^{m-1}(z)\langle x \rangle^{-\delta} \right\|.$$

Using Theorem 8.7 in [Dimassi and Sjöstrand 1999] about functions of a self-adjoint semiclassical pseudodifferential operator (with $h = 2^{-\frac{j-1}{2}}$) we see that for any $M \geqslant 0$ we have

$$\left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)\langle x \rangle^{\delta}\tilde{\phi}_j(P) \right\|_{\mathcal{L}(L^2)} \lesssim 2^{-jM}.$$

We also have

$$\left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)\langle x \rangle^{\delta} \right\| \lesssim 2^{-j}.$$

With Theorem 1.6 and the inductive assumption (for $\tilde{\phi}$ instead of ϕ) we obtain (6-9) with $R^m(z)$ replaced by $R_0(z)R^{m-1}(z)$. For the contribution of the second term in (6-10) we similarly write

$$\left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)B_\alpha R^m(z)\langle x \rangle^{-\delta} \right\|$$
$$\leqslant \left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)B_\alpha \tilde{\phi}_j(P)\langle x \rangle^{\delta} \right\| \left\| \langle x \rangle^{-\delta} R^m(z)\langle x \rangle^{-\delta} \right\|$$
$$\leqslant \left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)B_\alpha \langle x \rangle^{\delta} \right\| \left\| \langle x \rangle^{-\delta}(1-\tilde{\phi}_j(P))R^m(z)\langle x \rangle^{-\delta} \right\|.$$

Here we only have

$$\left\| \langle x \rangle^{-\delta}(1-\phi_j)(P)R_0(z)B_\alpha \langle x \rangle^{\delta} \right\| \lesssim 2^{-j\left(1-\frac{\alpha}{2}\right)},$$

but with the inductive assumption (on ℓ), we still can conclude. Thus, (6-9) is proved for all $m, \ell \in \mathbb{N}$. With $m = \kappa + k$ and ℓ large enough (we recall that $\alpha < 2$), this gives (6-8). After integration over $\tau \in \operatorname{supp}(\chi_j)$, this gives (6-7) for $w_{k,j}^2(t)$. The contribution of $w_{k,j}^3(t)$ is estimated similarly, and the proof is complete. $\qquad \square$

7. Smoothing effect

In this section we prove Theorem 1.7. With Theorems 1.3, 1.5 and 1.6 it implies Theorem 1.2. For this we use a dissipative version of the theory of relatively smooth operators in the sense of Kato.

Proposition 7.1. *Under the assumption of Theorem 1.7 there exists $C \geqslant 0$ such that for all $z \in \mathbb{C}_+$ we have*

$$\|\langle x \rangle^{-1}\langle P \rangle^{\frac{\gamma}{4}}R(z)\langle P \rangle^{\frac{\gamma}{4}}\langle x \rangle^{-1}\|_{\mathcal{L}(L^2)} \leqslant C.$$

Proof. • Let K be a compact subset of $\mathbb{C}$. Using the resolvent identity

$$R(z) = R(i) + (z-i)R(i)^2 + (z-i)^2 R(i)R(z)R(i),$$

we obtain for $z \in \mathbb{C}_+ \cap K$

$$\|\langle x \rangle^{-1}\langle P \rangle^{\frac{\gamma}{4}}R(z)\langle P \rangle^{\frac{\gamma}{4}}\langle x \rangle^{-1}\| \lesssim 1 + \left\|\langle x \rangle^{-1}\langle P \rangle^{\frac{\gamma}{4}}R(i)\langle x \rangle\right\| \left\|\langle x \rangle^{-1}R(z)\langle x \rangle^{-1}\right\| \left\|\langle x \rangle R(i)\langle P \rangle^{\frac{\gamma}{4}}\langle x \rangle^{-1}\right\|.$$

By pseudodifferential calculus the operators $\langle P \rangle^{\frac{\gamma}{4}} R(i)$ and $R(i)\langle P \rangle^{\frac{\gamma}{4}}$ are bounded on $L^{2,-1}$ and $L^{2,1}$, respectively. For the second factor on the right-hand side we use (1-10), and the conclusion follows for $z \in \mathbb{C}_+ \cap K$.

• It remains to prove the result for $|z| \gg 1$. Let $\chi \in C_0^\infty(\mathbb{R}, [0, 1])$ be supported on $[-3, 3]$ and equal to 1 on $[-2, 2]$. For $z \in \mathbb{C}_+$ we define $\chi_z : \lambda \mapsto \chi(\lambda/|z|)$. The operator $\varepsilon^{\frac{\gamma}{4}} \langle P \rangle^{\frac{\gamma}{4}} \langle \varepsilon P \rangle^{-\frac{\gamma}{4}}$ is a pseudodifferential operator whose symbol has bounded derivatives uniformly in $\varepsilon \in \,]0, 1]$, so the operator

$$|z|^{-\frac{\gamma}{4}} \langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} \left\langle \frac{P}{|z|} \right\rangle^{-\frac{\gamma}{4}} \langle x \rangle \tag{7-1}$$

extends to a bounded operator on L^2 uniformly in z with $|z| \geqslant 1$. The operator

$$\langle x \rangle^{-1} \left\langle \frac{P}{|z|} \right\rangle^{\frac{\gamma}{4}} \chi \left(\frac{P}{|z|} \right) \langle x \rangle \tag{7-2}$$

is also bounded on L^2 uniformly in z with $|z| \geqslant 1$, and we have similar estimates for the adjoint operators of (7-1) and (7-2). Thus

$$\left\| \langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} \chi_z(P) R(z) \chi_z(P) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \right\| \lesssim |z|^{\frac{\gamma}{2}} \left\| \langle x \rangle^{-1} \langle |z|^{-1} P \rangle^{\frac{\gamma}{4}} \chi_z(P) R(z) \chi_z(P) \langle |z|^{-1} P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \right\|$$

$$\lesssim |z|^{\frac{\gamma}{2}} \left\| \langle x \rangle^{-1} R(z) \langle x \rangle^{-1} \right\| \lesssim 1.$$

• With $R_0(z) = (P - z)^{-1}$ we have the resolvent identity

$$R(z) = R_0(z) + i R(z) B_\alpha R_0(z).$$

We have

$$\left\| \langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} \chi_z(P) R_0(z) (1 - \chi_z)(P) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \right\| \leqslant \left\| \langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} \chi_z(P) \right\| \left\| R_0(z) (1 - \chi_z)(P) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \right\|$$

$$\lesssim \langle z \rangle^{\frac{\gamma}{4}} \langle z \rangle^{\frac{\gamma}{4} - 1} \lesssim 1.$$

We have estimated the first factor as above and the second by the spectral theorem. On the other hand, since the operator $\sqrt{B_\alpha} \langle P \rangle^{-\frac{1}{2}}$ is bounded we also have by Proposition 2.5

$$\left\| \langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} \chi_z(P) R(z) B_\alpha R_0(z) (1 - \chi_z)(P) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \right\|$$

$$\leqslant \left\| \langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} \chi_z(P) \langle x \rangle \right\| \left\| \langle x \rangle^{-1} R(z) \sqrt{B_\alpha} \right\| \left\| \langle P \rangle^{\frac{1}{2}} R_0(z) (1 - \chi_z)(P) \langle P \rangle^{\frac{\gamma}{4}} \right\|$$

$$\lesssim \langle z \rangle^{\frac{\gamma}{4}} \langle z \rangle^{-\frac{\gamma}{4}} \langle z \rangle^{\frac{1}{2} + \frac{\gamma}{4} - 1} \lesssim 1.$$

This proves that

$$\left\| \langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} \chi_z(P) R(z) (1 - \chi_z)(P) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \right\| \lesssim 1.$$

• The operator

$$\langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} (1 - \chi_z)(P) R(z) \chi_z(P) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1}$$

is estimated similarly. Finally for

$$\langle x \rangle^{-1} \langle P \rangle^{\frac{\gamma}{4}} (1 - \chi_z)(P) R(z) (1 - \chi_z)(P) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1}$$

we only have to use twice the resolvent identity

$$R(z) = R_0(z) + i R_0(z) B_\alpha R_0(z) - R_0(z) B_\alpha R(z) B_\alpha R_0(z).$$

Then we apply the same idea as above, using Corollary 2.6 to estimate $\sqrt{B_\alpha} R(z) \sqrt{B_\alpha}$. $\qquad\square$

Taking the adjoint in the estimate of Proposition 7.1 we obtain the same estimate with $R(z)$ replaced by $R(z)^* = (P + i B_\alpha - \bar{z})^{-1}$ (the same is true for the estimates of Theorems 1.3, 1.5 and 1.6). In particular we obtain the following result:

Corollary 7.2. *Then there exists $C \geqslant 0$ such that for all $z \in \mathbb{C}_+$ and $\varphi \in \mathcal{S}$ we have*

$$\left| \left\langle \left((H - z)^{-1} - (H^* - \bar{z})^{-1} \right) \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \varphi, \langle P \rangle^{\frac{\gamma}{4}} \langle x \rangle^{-1} \varphi \right\rangle_{L^2} \right| \leqslant C \|\varphi\|_{L^2}^2.$$

It is known that such an estimate on the resolvent implies Theorem 1.7. This comes from the dissipative version of the theory of relatively smooth operators. The self-adjoint theory can be found in [Reed and Simon 1978, §XIII.7]. The dissipative version uses the theory of self-adjoint dilations for a dissipative operator described in [Sz.-Nagy et al. 2010]. All this has been combined in Proposition 6.2 in [Royer 2016], from which Theorem 1.7 follows.

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