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We consider the free boundary problem arising from an energy functional which is the sum of a Dirichlet energy and a nonlinear function of either the classical or the fractional perimeter.

The main difference with the existing literature is that the total energy is here a nonlinear superposition of the either local or nonlocal surface tension effect with the elastic energy.

In sharp contrast with the linear case, the problem considered in this paper is unstable; namely a minimizer in a given domain is not necessarily a minimizer in a smaller domain.

We provide an explicit example for this instability. We also give a free boundary condition, which emphasizes the role played by the domain in the geometry of the free boundary. In addition, we provide density estimates for the free boundary and regularity results for the minimal solution.

As far as we know, this is the first case in which a nonlinear function of the perimeter is studied in this type of problem. Also, the results obtained in this nonlinear setting are new even in the case of the local perimeter, and indeed the instability feature is not a consequence of the possible nonlocality of the problem, but it is due to the nonlinear character of the energy functional.

1. Introduction

In this paper we consider a free boundary problem given by the superposition of a Dirichlet energy and an either classical or nonlocal perimeter functional. Differently from the existing literature, here we take into account the possibility that this energy superposition occurs in a nonlinear way; that is, the total energy functional is the sum of the Dirichlet energy plus a nonlinear function of the either local or nonlocal perimeter of the interface.

Unlike the cases already present in the literature, the nonlinear problem that we study may present a structural instability induced by the domain; namely a minimizer in a large domain may fail to be a minimizer in a small domain. This fact prevents the use of scaling arguments, which are frequently exploited in classical free boundary problems.

In this paper, after providing an explicit example of this type of structural instability, we describe the free boundary equation, which also underlines the striking role played by the total (either local or nonlocal) perimeter of the minimizing set in the domain, as modulated by the nonlinearity, in the local geometry of the interface. Then, we will present results concerning the Hölder regularity of the minimal solutions and the density of the interfaces in the one-phase problem.

The mathematical setting in which we work is the following. Given an (open, Lipschitz and bounded) domain $\Omega \subset \mathbb{R}^n$ and $\sigma \in (0, 1]$, we use the notation $\mathrm{Per}_\sigma(E, \Omega)$ for the classical perimeter of E in Ω when $\sigma = 1$ (which will be often denoted as $\mathrm{Per}(E, \Omega)$, see, e.g., [Ambrosio et al. 2000; Maggi 2012]) and the fractional perimeter of E in Ω when $\sigma \in (0, 1)$ (see [Caffarelli et al. 2010]). More explicitly, if $\sigma \in (0, 1)$, we have

$$\mathrm{Per}_\sigma(E, \Omega) := L(E \cap \Omega, E^c) + L(E^c \cap \Omega, E \cap \Omega^c), \tag{1-1}$$

where, for any measurable subsets $A, B \subseteq \mathbb{R}^n$ with $A \cap B$ of measure zero, we set

$$L(A, B) := \iint_{A \times B} \frac{dx\, dy}{|x - y|^{n + \sigma}}.$$

As customary, we are using here the superscript c for complementary set; i.e., $E^c := \mathbb{R}^n \setminus E$.

The notation used for Per_σ when $\sigma = 1$ is inspired by the fact that Per_σ, suitably rescaled, approaches the classical perimeter as $s \nearrow 1$; see, e.g., [Bourgain et al. 2001; Dávila 2002; Caffarelli and Valdinoci 2011; Ambrosio et al. 2011].

In our framework, the role played by the fractional perimeter is to allow long-range interaction to contribute to the energy arising from surface tension and phase segregation.

As a matter of fact, the fractional perimeter Per_σ naturally arises when one considers phase transition models with long-range particle interactions (see, e.g., [Savin and Valdinoci 2014]): roughly speaking, in this type of model, the remote interactions of the particles are sufficiently strong to persist even at a large scale, by possibly modifying the behavior of the phase separation.

The fractional perimeter Per_σ has also natural applications in motion by nonlocal mean curvatures, which in turn arises naturally in the study of cellular automata and in image digitization procedures (see, e.g., [Imbert 2009]).

It is also convenient[1] to fix $\Upsilon \in \left(0, \frac{1}{100}\right]$ and set

$$\Omega_\Upsilon := \bigcup_{p \in \Omega} B_\Upsilon(p) \quad \text{and} \quad \mathrm{Per}_\sigma^\star(E, \Omega) = \begin{cases} \mathrm{Per}(E, \Omega_\Upsilon) & \text{if } \sigma = 1, \\ \mathrm{Per}_\sigma(E, \Omega) & \text{if } \sigma \in (0, 1). \end{cases} \tag{1-2}$$

We consider a monotone nondecreasing and lower semicontinuous function $\Phi : [0, +\infty) \to [0, +\infty)$, with

$$\lim_{t \to +\infty} \Phi(t) = +\infty. \tag{1-3}$$

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[1]The explicit value of Υ plays no major role, since it can be fixed by an "initial scaling" of the problem, but we decided to require it to be less than $\frac{1}{100}$ to emphasize, from the psychological point of view, that Ω_Υ can be thought as a small enlargement of Ω.

The reason we introduced such an Υ is that, in the classical case, the interfaces inside Ω do not see the contributions that may come along $\partial\Omega$, since Ω is taken to be open (conversely, in the nonlocal case, these contributions are always counted). By enlarging the domain Ω by a small quantity Υ, we are able to count also the contributions on $\partial\Omega$ and this, roughly speaking, boils down to computing the classical perimeter in the closure of Ω.

For any measurable function $u : \mathbb{R}^n \to \mathbb{R}$ such that $|\nabla u| \in L^2(\Omega)$ and any measurable subset $E \subseteq \mathbb{R}^n$ such that $u \geqslant 0$ a.e. in E and $u \leqslant 0$ a.e. in E^c, we consider the energy functional

$$\mathcal{E}_\Omega(u, E) := \int_\Omega |\nabla u(x)|^2 \, dx + \Phi(\mathrm{Per}_\sigma^\star(E, \Omega)). \tag{1-4}$$

As usual, the notation ∇u stands for the distributional gradient.

When Φ is the identity, the functional in (1-4) provides a typical problem for (either local or nonlocal) free boundary problems; see [Athanasopoulos et al. 2001; Caffarelli et al. 2015].

The goal of this paper is to study the minimizers of the functional in (1-4). For this, we say that (u, E) is an admissible pair if

- $u : \mathbb{R}^n \to \mathbb{R}$ is a measurable function such that $u \in H^1(\Omega)$,

- $E \subseteq \mathbb{R}^n$ is a measurable set with $\mathrm{Per}_\sigma^\star(E, \Omega) < +\infty$, and

- $u \geqslant 0$ a.e. in E and $u \leqslant 0$ a.e. in E^c.

Then, we say that (u, E) is a minimal pair in Ω if

- (u, E) is an admissible pair,

- $\mathcal{E}_\Omega(u, E) < +\infty$, and

- for any admissible pair (v, F) such that $v - u \in H_0^1(\Omega)$ and $F \setminus \Omega = E \setminus \Omega$ up to sets of measure zero, we have

$$\mathcal{E}_\Omega(u, E) \leqslant \mathcal{E}_\Omega(v, F).$$

The existence[2] of minimal pairs for fixed domains and fixed conditions outside the domain follows from the direct methods in the calculus of variations (see Lemma 2.3 below for details).

A natural question in this framework is whether or not this minimization procedure is "stable" with respect to the choice of the domain, i.e., whether or not a minimal pair in a domain Ω is also a minimal pair in any subdomain $\Omega' \subset \Omega$. This stability property is indeed typical for "linear" free boundary problems, i.e., when Φ is the identity, see [Athanasopoulos et al. 2001; Caffarelli et al. 2015], and it often plays a crucial role in many arguments based on scaling and blow-up analysis.

In the "nonlinear" case, i.e., when Φ is not the identity, this stability property is lost, and we will provide a concrete example for that. In further detail, we consider the planar case of $\mathbb{R}^2$, we take coordinates $X := (x, y) \in \mathbb{R}^2$ and we set

$$\tilde{u}(x, y) := xy \tag{1-5}$$

and

$$
\begin{aligned}
\widetilde{E} &:= \{(x, y) \in \mathbb{R}^2 : xy > 0\} \\
&= \{(x, y) \in \mathbb{R}^2 : x > 0 \text{ and } y > 0\} \cup \{(x, y) \in \mathbb{R}^2 : x < 0 \text{ and } y < 0\}.
\end{aligned} \tag{1-6}
$$

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[2]As a technical remark, we point out that the definition in (1-2) is useful to make sense of nontrivial versions of this minimization problem when $\sigma = 1$ and $u \geqslant 0$. Indeed, in this case, the setting in (1-2) "forces" the sets to interact with the boundary data. This expedient is not necessary when $\sigma = 0$ since, in this case, the nonlocal effect produces the nontrivial interactions.

In this setting, we show that:

Theorem 1.1 (an explicit counterexample). *There exists $K_o > 2$ such that the following statement is true. Let $n = 2$. Assume*

$$\Phi(t) = t^\gamma \quad \text{for any } t \in [0, 1]$$

for some

$$\gamma \in \left(0, \frac{4}{2-\sigma}\right),$$

and

$$\Phi(t) = 1 \quad \text{for any } t \in [2, K_o]. \tag{1-7}$$

Then, there exist $R_o > r_o > 0$ such that $(\tilde{u}, \tilde{E})$ is a minimal pair in B_{R_o} and is not a minimal pair in B_r for any $r \in (0, r_o]$.

The heuristic idea underlying Theorem 1.1 is, roughly speaking, that the nonlinear energy term Φ weights differently the fractional perimeter with respect to the Dirichlet energy in different energy regimes, so it may favor a minimal pair (u, E) to be either "close to a harmonic function" in the u or "close to a fractional minimal surface" in the E, depending on the minimal energy level reached in a given domain.

It is worth stressing that, in other circumstances, rather surprising instability features in interface problems arise as a consequence of the fractional behavior of the energy; see, for instance, [Dipierro et al. 2017]. Differently from these cases, the unstable free boundaries presented in Theorem 1.1 are not caused by the existence of possibly nonlocal features, and indeed Theorem 1.1 holds true (and is new) even in the case of the local perimeter.

The instability phenomenon pointed out by Theorem 1.1 in a concrete case is also quite general, as it can be understood also in the light of the associated equation on the free boundary. Indeed, the free boundary equation takes into account a "global" term of the type $\Phi'(\text{Per}_\sigma^\star(E, \Omega))$, which varies in dependence of the domain Ω. To clarify this point, we denote by H_σ^E the (either classical or fractional) mean curvature of ∂E (see [Caffarelli et al. 2010; Abatangelo and Valdinoci 2014] for the case $\sigma \in (0, 1)$). Namely, if $\sigma = 1$ the above notation stands for the classical mean curvature, while for $\sigma \in (0, 1)$, if $x \in \partial E$, we set

$$H_\sigma^E(x) := \limsup_{\delta \to 0} \int_{\mathbb{R}^n \setminus B_\delta(x)} \frac{\chi_{E^c}(y) - \chi_E(y)}{|x - y|^{n+\sigma}} \, dy.$$

In this setting, we have:

Theorem 1.2 (free boundary equation). *Let $\Phi \in C^{1,\alpha}(0, +\infty)$ for some $\alpha \in (0, 1)$. Assume (u, E) is a minimal pair in Ω. Assume*

$$(\partial E) \cap \Omega \text{ is of class } C^{1,\tau} \text{ with } \tau \in (\sigma, 1) \text{ when } \sigma \in (0, 1) \text{ and of class } C^2 \text{ when } \sigma = 1. \tag{1-8}$$

Suppose also

$$u > 0 \text{ in the interior of } E \cap \Omega, \quad u < 0 \text{ in the interior of } E^c \cap \Omega, \tag{1-9}$$

and

$$u \in C^1(\overline{\{u > 0\} \cap \Omega}) \cap C^1(\overline{\{u < 0\} \cap \Omega}). \tag{1-10}$$

Let also ν be the exterior normal of E, and for any $x \in (\partial E) \cap \Omega$ let

$$\partial_\nu^+ u(x) := \lim_{t \to 0} \frac{u(x - t\nu) - u(x)}{t} \quad and \quad \partial_\nu^- u(x) := \lim_{t \to 0} \frac{u(x + t\nu) - u(x)}{t}. \tag{1-11}$$

Then, for any $x \in (\partial E) \cap \Omega$, we have

$$(\partial_\nu^+ u(x))^2 - (\partial_\nu^- u(x))^2 = H_\sigma^E(x) \, \Phi'(\mathrm{Per}_\sigma^\star(E, \Omega)). \tag{1-12}$$

We remark that (1-12) has a simple geometric consequence when $\Phi' > 0$ and we consider the one-phase problem in which $u \geqslant 0$: indeed, in this case, we have $\partial_\nu^- u = 0$ and therefore formula (1-12) reduces to

$$(\partial_\nu^+ u(x))^2 = H_\sigma^E(x) \, \Phi'(\mathrm{Per}_\sigma^\star(E, \Omega)).$$

In particular, we get that $H_\sigma^E \geqslant 0$; namely, in this case, the (either classical or fractional) mean curvature of the free boundary is nonnegative.

In order to better understand the structure of the solution and of the free boundary points, we now focus, for the sake of simplicity, on the one-phase case; i.e., we suppose that $u \geqslant 0$ to start with. In this setting, we investigate the Hölder regularity of the function u by obtaining uniform bounds and uniform growth conditions from the free boundary. For this, it is also convenient to introduce the auxiliary set

$$\mathcal{U}_0 := \big\{ x \in \Omega : \text{there exists a sequence } x_k \in \Omega : x_k \to x \text{ with } u(x_k) \to 0 \text{ as } k \to +\infty \big\}. \tag{1-13}$$

Notice that $\{u = 0\}$ lies in $\mathcal{U}_0$ (just taking a constant sequence in the definition above). Also, if $u \geqslant 0$, then ∂E lies in $\mathcal{U}_0$ (since in this case u must vanish in the complement of E).

Of course, when u is continuous, such a set lies in the zero level set of u, but since we do not have this information a priori, it is useful to consider explicitly this set, and prove the following result:

Theorem 1.3 (growth from the free boundary). *Let R_o, $Q > 0$. Assume*

$$\Phi \text{ is Lipschitz continuous in } [0, Q], \text{ with Lipschitz constant bounded by } L_Q. \tag{1-14}$$

Assume (u, E) is a minimal pair in Ω, with $B_{R_o} \Subset \Omega$,

$$0 \in \mathcal{U}_0 \tag{1-15}$$

and $u \geqslant 0$ in $\mathbb{R}^n \setminus \Omega$. Suppose $R \in (0, R_o]$ and

$$\mathrm{Per}_\sigma^\star(E, \Omega) + R^{n-\sigma} \, \mathrm{Per}_\sigma(B_1, \mathbb{R}^n) \leqslant Q. \tag{1-16}$$

Then, there exists $C > 0$, possibly depending on R_o, n and σ such that, for any $x \in B_{R/2}$,

$$u(x) \leqslant C \sqrt{L_Q} \, |x|^{1-\sigma/2}.$$

We observe that condition (1-14) is always satisfied if Φ is globally Lipschitz, but the statement of Theorem 1.3 is more general, since it may take into account a locally Lipschitz Φ, provided that the domain is small enough to satisfy (1-16) (indeed, small domains satisfy this condition for locally Lipschitz Φ, as remarked in the forthcoming Lemma 2.8).

We also point out that (1-16) may be equivalently written

$$\text{Per}_\sigma^\star(E, \Omega) + \text{Per}_\sigma(B_R, \mathbb{R}^n) \leqslant Q. \tag{1-17}$$

One natural way to interpret (1-16), or (1-17), is that once $\text{Per}_\sigma^\star(E, \Omega)$ is strictly less than Q (i.e., strictly less than the size of the interval in which Φ is Lipschitz), then (1-16), and thus (1-17), holds true as long as R is sufficiently small.

The growth result in Theorem 1.3 implies, as a byproduct, an interior Hölder regularity result:

Corollary 1.4. *Let $Q > 0$ and assume Φ is Lipschitz continuous in $[0, Q]$, with Lipschitz constant bounded by L_Q.*

Assume (u, E) is a minimal pair in Ω, with $B_R \Subset \Omega$ and $u \geqslant 0$ in $\mathbb{R}^n \setminus \Omega$.

Suppose that $\text{Per}_\sigma^\star(E, \Omega) + R^{n-\sigma} \text{Per}_\sigma(B_1, \mathbb{R}^n) \leqslant Q$ and that $u \leqslant M$ on $\partial\Omega$.

Then $u \in C^{1-\sigma/2}(B_{R/4})$, with

$$\|u\|_{C^{1-\sigma/2}(B_{R/4})} \leqslant C\left(\sqrt{L_Q} + \frac{M}{R^{1-\sigma/2}}\right),$$

for some $C > 0$, possibly depending on n and σ.

When Φ is linear, the result in Corollary 1.4 was obtained in Theorem 3.1 of [Athanasopoulos et al. 2001] if $\sigma = 1$ and in Theorem 1.1 of [Caffarelli et al. 2015] if $\sigma \in (0, 1)$. Differently than in our framework, in both papers mentioned above, scaling arguments are available, since scaling is compatible with the minimization procedure.

Now we investigate the structure of the free boundary points in terms of local densities of the phases. Indeed, we show that the free boundary points always have uniform density from outside E, according to the following result:

Theorem 1.5 (density estimate from the null side). *Assume (u, E) is a minimal pair in Ω, with $B_R \subseteq \Omega$, $0 \in \partial E$ and $u \geqslant 0$ in $\mathbb{R}^n \setminus \Omega$. Set*

$$P = P(E, \Omega, R) := \text{Per}_\sigma^\star(E, \Omega) + R^{n-\sigma} \text{Per}_\sigma(B_1, \mathbb{R}^n) \tag{1-18}$$

and assume

$$\Phi \text{ is strictly increasing in the interval } (0, P). \tag{1-19}$$

Then there exists $\delta > 0$, possibly depending on n and σ such that, for any $r \in \left(0, \frac{1}{2}R\right)$,

$$|B_r \setminus E| \geqslant \delta r^n.$$

We point out that condition (1-19) is always satisfied if Φ is strictly increasing in the whole of $[0, +\infty)$, but Theorem 1.5 is also general enough to take into consideration the case in which Φ is strictly increasing only in a subinterval, provided that the energy domain is sufficiently small to make the perimeter values lie in the strict monotonicity interval of Φ (as a matter of fact, the perimeter contributions in small domains are small, as we will point out in the forthcoming Lemma 2.8).

The investigation of the density properties of the free boundary is also completed by the following counterpart of Theorem 1.5, which proves the positive density of the set E:

Theorem 1.6 (density estimate from the positive side). *Let $Q > 0$ and assume*

$$\Phi \text{ is Lipschitz continuous in } [0, Q], \text{ with Lipschitz constant bounded by } L_Q, \tag{1-20}$$

and

$$\Phi' \geqslant c_o \quad a.e. \text{ in } [0, Q] \tag{1-21}$$

for some $c_o > 0$.

Assume (u, E) is a minimal pair in Ω, with $B_R \Subset \Omega$, $0 \in \partial E$ and $u \geqslant 0$ in $\mathbb{R}^n \setminus \Omega$. Suppose

$$\operatorname{Per}_\sigma^\star(E, \Omega) + R^{n-\sigma} \operatorname{Per}_\sigma(B_1, \mathbb{R}^n) \leqslant Q. \tag{1-22}$$

Then there exists $\delta_* > 0$, possibly depending on n, σ, c_o and L_Q, such that, for any $r \in \left(0, \tfrac{1}{2}R\right)$,

$$|B_r \cap E| \geqslant \delta_* r^n.$$

More explicitly, such δ_* can be taken to be of the form

$$\delta_* := \delta_o \min\left\{1, \left(\frac{c_o}{L_Q}\right)^{n/\sigma}\right\} \tag{1-23}$$

for some $\delta_o > 0$, possibly depending on n and σ.

We remark that the results obtained in this paper are new even in the local case in which $\sigma = 1$. Also, we think it is an interesting point of this paper that all the cases $\sigma \in (0, 1)$ and $\sigma = 1$ are treated simultaneously in a unified fashion. The methods presented are also general enough to treat the case $\sigma = 0$, which would correspond to a volume term (see, e.g., [Maz'ya and Shaposhnikova 2002; Dipierro et al. 2013]). This case is in fact rich in results and so we will discuss it in detail in a forthcoming paper.

The rest of the paper is organized as follows. In Section 2 we show some preliminary properties of the minimal pair, such as existence, harmonicity and subharmonicity properties, and a comparison principle. We also prove a "locality" property for the (either classical or fractional) perimeter and provide a uniform bound on the (classical or fractional) perimeter of the set in the minimal pair.

Section 3 is devoted to the construction of the counterexample in Theorem 1.1. In Section 4 we provide the free boundary equation and prove Theorem 1.2.

Then we deal with the regularity of the function u in the minimal pair in the one-phase case, and we prove Theorem 1.3 and Corollary 1.4 in Sections 5 and 6, respectively. Finally, Sections 7 and 8 are devoted to the proofs of the density estimates from both sides provided by Theorems 1.5 and 1.6, respectively.

Since we hope that the paper may be of interest for different communities (such as scientists working in free boundary problems, variational methods, partial differential equations, geometric measure theory and fractional problems), we made an effort to give the details of the arguments involved in the proofs in a clear and widely accessible way.

2. Preliminaries

We start with a useful observation about the positivity sets of sequences of admissible pairs:

Lemma 2.1. *Let (u_j, E_j) be a sequence of admissible pairs. Assume $u_j \to u$ a.e. in $\mathbb{R}^n$ and $\chi_{E_j} \to \chi_E$ a.e. in $\mathbb{R}^n$ for some u and E. Then $u \geqslant 0$ a.e. in E and $u \leqslant 0$ a.e. in E^c.*

Proof. We show that $u \geqslant 0$ a.e. in E (the other claim being analogous). For this, we write $\mathbb{R}^n = X \cup Z$, with $|Z| = 0$ and such that for any $x \in X$ we have

$$\lim_{j \to +\infty} u_j(x) = u(x) \quad \text{and} \quad \lim_{j \to +\infty} \chi_{E_j}(x) = \chi_E(x).$$

Let now $x \in E \cap X$. Then

$$\lim_{j \to +\infty} \chi_{E_j}(x) = \chi_E(x) = 1,$$

and so there exists $j_x \in \mathbb{N}$ such that $\chi_{E_j}(x) \geqslant \frac{1}{2}$ for any $j \geqslant j_x$. Since the image of a characteristic function lies in $\{0, 1\}$, this implies $\chi_{E_j}(x) = 1$ for any $j \geqslant j_x$, and therefore $u_j(x) \geqslant 0$ for any $j \geqslant j_x$. Taking the limit, we obtain $u(x) \geqslant 0$. Since this is valid for any $x \in E \cap X$ and $E \cap X^c \subseteq Z$, which has null measure, we have obtained the desired result. $\square$

Now we recall a useful auxiliary identity for the (classical or fractional) perimeter:

Lemma 2.2 ("clean cut" lemma). *Let $\Omega' \Subset \Omega$. Assume $\mathrm{Per}_\sigma(E, \Omega) < +\infty$ and $\mathrm{Per}_\sigma(F, \Omega) < +\infty$. Suppose also that*

$$E \setminus \overline{\Omega}' = F \setminus \overline{\Omega}'. \tag{2-1}$$

Then

$$\mathrm{Per}_\sigma(E, \Omega) - \mathrm{Per}_\sigma(F, \Omega) = \mathrm{Per}_\sigma(E, \overline{\Omega}') - \mathrm{Per}_\sigma(F, \overline{\Omega}'). \tag{2-2}$$

If in addition $\mathrm{Per}_\sigma^\star(E, \Omega) < +\infty$ and $\mathrm{Per}_\sigma^\star(F, \Omega) < +\infty$, then

$$\mathrm{Per}_\sigma^\star(E, \Omega) - \mathrm{Per}_\sigma^\star(F, \Omega) = \mathrm{Per}_\sigma(E, \overline{\Omega}') - \mathrm{Per}_\sigma(F, \overline{\Omega}'). \tag{2-3}$$

Proof. For completeness, we distinguish the cases $\sigma = 1$ and $\sigma \in (0, 1)$. If $\sigma = 1$, we write the perimeter of E in term of the Gauss–Green measure μ_E (see Remark 12.2 in [Maggi 2012]); namely

$$\mathrm{Per}(E, \Omega) = |\mu_E|(\Omega).$$

So we define

$$U := \Omega \setminus \overline{\Omega}'. \tag{2-4}$$

We remark that U is open and $\Omega = \overline{\Omega}' \cup U$, with disjoint union. Thus we obtain

$$\mathrm{Per}(E, \Omega) - \mathrm{Per}(F, \Omega) - \mathrm{Per}(E, \overline{\Omega}') + \mathrm{Per}(F, \overline{\Omega}')$$
$$= |\mu_E|(\Omega) - |\mu_F|(\Omega) - |\mu_E|(\overline{\Omega}') + |\mu_F|(\overline{\Omega}')$$
$$= |\mu_E|(\overline{\Omega}' \cup U) - |\mu_F|(\overline{\Omega}' \cup U) - |\mu_E|(\overline{\Omega}') + |\mu_F|(\overline{\Omega}')$$
$$= |\mu_E|(\overline{\Omega}') + |\mu_E|(U) - |\mu_F|(\overline{\Omega}') - |\mu_F|(U) - |\mu_E|(\overline{\Omega}') + |\mu_F|(\overline{\Omega}')$$
$$= |\mu_E|(U) - |\mu_F|(U) = \mathrm{Per}(E, U) - \mathrm{Per}(F, U). \tag{2-5}$$

Now we observe that

$$E \cap U = E \cap (\Omega \setminus \overline{\Omega}') = E \cap \Omega \cap (\overline{\Omega}')^c = (E \setminus \overline{\Omega}') \cap \Omega,$$

and a similar set identity holds for F. Thus, by (2-1), it follows that $E \cap U = F \cap U$. Therefore, by the locality of the classical perimeter (see, e.g., Proposition 3.38(c) in [Ambrosio et al. 2000]), we obtain

$$\text{Per}(E, U) = \text{Per}(F, U).$$

If one inserts this into (2-5), then one obtains (2-2) when $\sigma = 1$.

Now we deal with the case $\sigma \in (0, 1)$. For this we use (1-1) and (2-4) and we get

$$
\begin{aligned}
\text{Per}_\sigma(E, \Omega) - \text{Per}_\sigma(E, \overline{\Omega}') &= L(E \cap \Omega, E^c) + L(E^c \cap \Omega, E \setminus \Omega) - L(E \cap \overline{\Omega}', E^c) - L(E^c \cap \overline{\Omega}', E \setminus \overline{\Omega}') \\
&= L(E \cap \overline{\Omega}', E^c) + L(E \cap U, E^c) + L(E^c \cap \overline{\Omega}', E \setminus \Omega) + L(E^c \cap U, E \setminus \Omega) \\
&\quad - L(E \cap \overline{\Omega}', E^c) - L(E^c \cap \overline{\Omega}', E \setminus \Omega) - L(E^c \cap \overline{\Omega}', E \cap U) \\
&= L(E \cap U, E^c) + L(E^c \cap U, E \setminus \Omega) - L(E^c \cap \overline{\Omega}', E \cap U) \\
&= L(E \cap U, E^c \setminus \overline{\Omega}') + L(E^c \cap U, E \setminus \Omega),
\end{aligned}
$$

and a similar formula holds for F replacing E. Now, from (2-1), we see that

$$E \cap U = F \cap U, \quad E^c \cap U = F^c \cap U, \quad E^c \setminus \overline{\Omega}' = F^c \setminus \overline{\Omega}' \quad \text{and} \quad E \setminus \Omega = F \setminus \Omega;$$

thus we obtain (2-2) when $\sigma \in (0, 1)$.

Now, to prove (2-3), we can focus on the case $\sigma = 1$ (since $\text{Per}_\sigma^\star = \text{Per}_\sigma$ when $\sigma \in (0, 1)$, in this case we return simply to (2-2)). To this end, we observe that $\Omega' \Subset \Omega_\Upsilon$ (recall formula (1-2)), so we can apply (2-2) to the sets Ω' and Ω_Υ and obtain, when $\sigma = 1$,

$$\text{Per}_\sigma^\star(E, \Omega) - \text{Per}_\sigma^\star(F, \Omega) = \text{Per}(E, \Omega_\Upsilon) - \text{Per}(F, \Omega_\Upsilon) = \text{Per}(E, \overline{\Omega}') - \text{Per}(F, \overline{\Omega}').$$

This completes the proof of (2-3). □

Now we state the basic existence result for the minimizers of the functional in (1-4):

Lemma 2.3 (existence of minimal pairs). *Fix an admissible pair (u_o, E_o) such that $\mathcal{E}_\Omega(u_o, E_o) < +\infty$. Then there exists a minimal pair (u, E) in Ω such that $u - u_o \in H_0^1(\Omega)$ and $E \setminus \Omega$ coincides with $E_o \setminus \Omega$ up to sets of measure zero.*

Proof. Let (u_j, E_j) be a minimizing sequence, namely

$$\lim_{j \to +\infty} \mathcal{E}_\Omega(u_j, E_j) = \inf_{X_\Omega(u_o, E_o)} \mathcal{E}_\Omega, \tag{2-6}$$

where $X_\Omega(u_o, E_o)$ denotes the family of all admissible pairs (v, F) in Ω such that $v - u_o \in H_0^1(\Omega)$ and $F \setminus \Omega$ coincides with $E_o \setminus \Omega$ up to sets of measure zero.

We stress that

$$\sup_{j \in \mathbb{N}} \Phi(\text{Per}_\sigma^\star(E_j, \Omega)) < +\infty,$$

thanks to (2-6). By this and (1-3), we obtain

$$\sup_{j\in\mathbb{N}} \operatorname{Per}_\sigma(E_j, \Omega) < +\infty.$$

Using this and (2-6), by compactness (see, e.g., Corollary 3.49 in [Ambrosio et al. 2000] for the case $\sigma = 1$ or Theorem 7.1 in [Di Nezza et al. 2012] for the case $\sigma \in (0, 1)$), we obtain that, up to subsequences, u_j converges to some u weakly in $H^1(\Omega)$ and strongly in $L^2(\Omega)$, and χ_{E_j} converges to some χ_E strongly in $L^1(\Omega)$ as $j \to +\infty$. By Lemma 2.1, we have that (u, E) is an admissible pair, and so by construction

$$(u, E) \in X_\Omega(u_o, E_o). \tag{2-7}$$

Also, by the lower semicontinuity (or Fatou's lemma; see, e.g., Proposition 3.38(b) in [Ambrosio et al. 2000] for the case $\sigma = 1$) we have

$$\liminf_{j\to+\infty} \int_\Omega |\nabla u_j(x)|^2 \, dx \geqslant \int_\Omega |\nabla u(x)|^2 \, dx \quad \text{and} \quad \liminf_{j\to+\infty} \operatorname{Per}_\sigma^\star(E_j, \Omega) \geqslant \operatorname{Per}_\sigma^\star(E, \Omega),$$

and so, using also the monotonicity and the lower semicontinuity of Φ,

$$\liminf_{j\to+\infty} \Phi(\operatorname{Per}_\sigma^\star(E_j, \Omega)) \geqslant \Phi\big(\liminf_{j\to+\infty} \operatorname{Per}_\sigma^\star(E_j, \Omega)\big) \geqslant \Phi(\operatorname{Per}_\sigma^\star(E, \Omega)).$$

These inequalities and (2-6) give that

$$\mathcal{E}_\Omega(u, E) \leqslant \inf_{X_\Omega(u_o, E_o)} \mathcal{E}_\Omega,$$

and then equality holds in the formula above, thanks to (2-7). □

As it often happens in free boundary problems (see, e.g., [Alt and Caffarelli 1981; Athanasopoulos et al. 2001; Caffarelli et al. 2015]), the solutions are harmonic in the positivity or negativity sets. This happens also in our case, as clarified by the following observation:

Lemma 2.4. *Let (u, E) be a minimal pair in Ω. Let U be an open set. Assume that either $\inf_U u > 0$ or $\sup_U u < 0$. Then u is harmonic in U.*

Proof. The proof is standard, but we give the details to assist the reader. We suppose

$$\inf_U u > 0, \tag{2-8}$$

the other case being similar. Let $x_o \in U$. Since U is open, there exists $r > 0$ such that $B_r(x_o) \subset U$. Let $\psi \in C_0^\infty(B_{r/2}(x_o))$. Let also $u_\epsilon := u + \epsilon\psi$ and

$$m := \inf_{B_{r/2}(x_o)} u.$$

By (2-8), we know $m > 0$. Thus, if $\epsilon \in \mathbb{R}$, with $|\epsilon| < (1 + \|\psi\|_{L^\infty(\mathbb{R}^n)})^{-1} m$, we have $u_\epsilon \geqslant u - \epsilon \|\psi\|_{L^\infty(\mathbb{R}^n)} \geqslant 0$ in $B_{r/2}(x_o)$. This and the fact that ψ vanishes outside $B_{r/2}(x_o)$ give that (u_ϵ, E) is an admissible pair. Thus, the minimality of (u, E) gives

$$0 \leqslant \mathcal{E}_\Omega(u_\epsilon, E) - \mathcal{E}_\Omega(u, E) = \int_\Omega \big(|\nabla u(x) + \epsilon\nabla\psi(x)|^2 - |\nabla u(x)|^2\big) \, dx,$$

from which the desired result easily follows. □

As often happens in free boundary problems, the minimizers satisfy the following subharmonicity property:

Lemma 2.5. *Let (u, E) be a minimal pair in Ω and $u^+ := \max\{u, 0\}$ and $u^- := u^+ - u = -\min\{u, 0\}$. Then both u^+ and u^- are subharmonic in Ω in the sense that*

$$\int_\Omega \nabla u^\pm(x) \cdot \nabla \psi(x)\, dx \leqslant 0$$

for any $\psi \in H_0^1(\Omega)$, with $\psi \geqslant 0$ a.e. in Ω.

Proof. The proof is a modification of the one in Lemma 2.7 in [Athanasopoulos et al. 2001], where this result was proved for the case in which Φ is the identity and $\sigma = 1$. We give the details to assist the reader. We argue for u^+, since a similar reasoning works for u^-. We define $v^\star$ to be the harmonic replacement of u^+ in Ω which vanishes in E^c, that is, the minimizer of the Dirichlet energy in Ω among all the functions v in $H^1(\Omega)$ such that $v - u^+ \in H_0^1(\Omega)$ and $v = 0$ a.e. in E^c. For the existence and the uniqueness of the harmonic replacement, see, e.g., Section 2 in [Athanasopoulos et al. 2001] or Lemma 2.1 in [Dipierro and Valdinoci 2015]. In particular, the uniqueness result gives that

if v in $H^1(\Omega)$ is such that $v - u^+ \in H_0^1(\Omega)$, $v = 0$ a.e. in E^c

$$\text{and} \quad \int_\Omega |\nabla v(x)|^2\, dx \leqslant \int_\Omega |\nabla v^\star(x)|^2\, dx, \text{ then } v = v^\star \text{ a.e. in } \mathbb{R}^n. \quad (2\text{-}9)$$

Moreover, by Lemma 2.3 in [Athanasopoulos et al. 2001], we have

$$v^\star \text{ is subharmonic.} \quad (2\text{-}10)$$

We also notice that $v^\star \geqslant 0$ by the classical maximum principle and therefore $(v^\star, E)$ is an admissible pair. Then, the minimality of (u, E) implies

$$0 \geqslant \mathcal{E}_\Omega(u, E) - \mathcal{E}_\Omega(v^\star, E) = \int_\Omega |\nabla u(x)|^2\, dx - \int_\Omega |\nabla v^\star(x)|^2\, dx \geqslant \int_\Omega |\nabla u^+(x)|^2\, dx - \int_\Omega |\nabla v^\star(x)|^2\, dx.$$

This implies that u^+ coincides with $v^\star$, thanks to (2-9), and so it is subharmonic, in light of (2-10). $\quad\square$

Remark 2.6. In light of Lemma 2.5, we have (see, e.g., Proposition 2.2 in [Giaquinta 1983]) that the map

$$R \to \frac{1}{|B_R|} \int_{B_R(p)} u^+(x)\, dx$$

is monotone nondecreasing; therefore, up to changing u^+ in a set of measure zero, we can (and implicitly do from now on) suppose

$$u(p) = \lim_{\epsilon \searrow 0} \frac{1}{|B_\epsilon|} \int_{B_\epsilon(p)} u^+(x)\, dx.$$

Another simple and interesting property of the solution is given by the following maximum principle:

Lemma 2.7. *Assume*

$$\Phi(0) < \Phi(t) \quad \text{for any } t > 0. \quad (2\text{-}11)$$

Let (u, E) be a minimal pair in Ω and let $a \in \mathbb{R}$. If $u \leqslant a$ in Ω^c, then $u \leqslant a$ in the whole of $\mathbb{R}^n$.

Similarly, if $u \geqslant a$ in Ω^c, then $u \geqslant a$ in the whole of $\mathbb{R}^n$.

Proof. We suppose

$$u \geqslant a \quad \text{in } \Omega^c, \tag{2-12}$$

the other case being analogous.

We need to distinguish the cases $a \leqslant 0$ and $a > 0$.

If $a \leqslant 0$, we take $u^\star := \max\{u, a\}$. Notice that $(u^\star, E)$ is an admissible pair: indeed, a.e. in E we have $0 \leqslant u \leqslant u^\star$, while a.e. in E^c we have $u \leqslant 0$ and so $u^\star \leqslant 0$. Also, by (2-12), we have $u \geqslant a$ in Ω^c, and so $u^\star = u$ in Ω^c. As a consequence, the minimality of (u, E) gives

$$0 \leqslant \mathcal{E}_\Omega(u^\star, E) - \mathcal{E}_\Omega(u, E) = \int_\Omega \left(|\nabla u^\star(x)|^2 - |\nabla u(x)|^2 \right) dx = - \int_{\Omega \cap \{u < a\}} |\nabla u(x)|^2 \, dx,$$

which implies $u \geqslant a$, as desired.

Now suppose $a > 0$. We take $u^\sharp$ to be the minimizer of the Dirichlet energy in Ω with trace datum u along $\partial \Omega$ (and thus we set $u^\sharp := u$ outside Ω); then we have

$$\Gamma := \int_\Omega |\nabla u(x)|^2 \, dx - \int_\Omega |\nabla u^\sharp(x)|^2 \, dx \geqslant 0. \tag{2-13}$$

Moreover, by (2-12) and the classical maximum principle, we know

$$u^\sharp \geqslant a \quad \text{in the whole of } \mathbb{R}^n. \tag{2-14}$$

Thus, $u^\sharp > 0$ and so $(u^\sharp, \mathbb{R}^n)$ is an admissible pair. Accordingly, the minimality of (u, E) and (2-13) give

$$\begin{aligned}
0 &\leqslant \mathcal{E}_\Omega(u^\sharp, \mathbb{R}^n) - \mathcal{E}_\Omega(u, E) \\
&= \int_\Omega |\nabla u^\sharp(x)|^2 \, dx + \Phi(0) - \int_\Omega |\nabla u(x)|^2 \, dx - \Phi(\text{Per}_\sigma^\star(E, \Omega)) \\
&= -\Gamma + \Phi(0) - \Phi(\text{Per}_\sigma^\star(E, \Omega)).
\end{aligned} \tag{2-15}$$

As a consequence,

$$\Phi(\text{Per}_\sigma^\star(E, \Omega)) \leqslant -\Gamma + \Phi(0) \leqslant \Phi(0);$$

hence, exploiting (2-11), we see that $\text{Per}_\sigma^\star(E, \Omega) = 0$. Plugging this information into (2-15), we obtain that $0 \leqslant -\Gamma$ and thus, recalling (2-13), we conclude that $\Gamma = 0$. By the uniqueness of the minimizer of the Dirichlet energy, this implies that $u^\sharp$ coincides with u. In light of this and of (2-14), we have $u = u^\sharp \geqslant a$, as desired. $\qquad \square$

Now we give a uniform bound on the (classical or fractional) perimeter of the sets in the minimal pairs:

Lemma 2.8. *Suppose Ω is strictly star-shaped, i.e., $t\overline{\Omega} \subseteq \Omega$ for any $t \in (0, 1)$, and that*

$$\Phi \text{ is strictly monotone.} \tag{2-16}$$

Let (u, E) be a minimal pair in Ω. Assume $u \geqslant 0$. Then, for any $\Omega' \subseteq \Omega$, with Ω' open, Lipschitz and bounded, we have

$$\text{Per}_\sigma(E, \Omega') \leqslant 2 \text{Per}_\sigma(\Omega', \mathbb{R}^n). \tag{2-17}$$

In particular, if $\Omega \supseteq B_R$, then, for any $r \in (0, R]$,

$$\text{Per}_\sigma(E, B_r) \leqslant Cr^{n-\sigma} \tag{2-18}$$

for some $C > 0$ possibly depending on n and σ.

Proof. We observe that (2-18) follows from (2-17) by taking $\Omega' := B_r$, so we focus on the proof of (2-17). For this, first we suppose that $\Omega' \Subset \Omega$ (the general case in which $\Omega' \subseteq \Omega$ will be considered at the end of the proof, by a limit procedure). Let $F := E \cup \Omega'$. Notice that $F \setminus \overline{\Omega}' = E \cup \Omega' \cap (\overline{\Omega}')^c = E \setminus \overline{\Omega}'$. Thus, by formula (2-3) in Lemma 2.2, we get

$$\text{Per}_\sigma^\star(E, \Omega) - \text{Per}_\sigma^\star(F, \Omega) = \text{Per}_\sigma(E, \overline{\Omega}') - \text{Per}_\sigma(F, \overline{\Omega}'). \tag{2-19}$$

Now, let v be the minimizer of the Dirichlet energy in Ω' with trace datum u along $\partial\Omega'$ (then take $v := u$ outside Ω'). Since $u \geqslant 0$, so is v. Hence, the pair (v, F) is admissible. Therefore, the minimality of (u, E) implies

$$0 \leqslant \mathcal{E}_\Omega(v, F) - \mathcal{E}_\Omega(u, E)$$

$$= \int_{\Omega'} |\nabla v(x)|^2 \, dx - \int_{\Omega'} |\nabla u(x)|^2 \, dx + \Phi(\text{Per}_\sigma^\star(F, \Omega)) - \Phi(\text{Per}_\sigma^\star(E, \Omega))$$

$$\leqslant 0 + \Phi(\text{Per}_\sigma^\star(F, \Omega)) - \Phi(\text{Per}_\sigma^\star(E, \Omega)).$$

Hence, by (2-16), we have $\text{Per}_\sigma^\star(E, \Omega) \leqslant \text{Per}_\sigma^\star(F, \Omega)$ and so, by (2-19),

$$\text{Per}_\sigma(E, \overline{\Omega}') - \text{Per}_\sigma(F, \overline{\Omega}') = \text{Per}_\sigma^\star(E, \Omega) - \text{Per}_\sigma^\star(F, \Omega) \leqslant 0. \tag{2-20}$$

In addition, we have

$$\text{Per}_\sigma(F, \overline{\Omega}') = \text{Per}_\sigma(E \cup \Omega', \overline{\Omega}') \leqslant 2\text{Per}_\sigma(\Omega', \mathbb{R}^n),$$

where the last formula follows using (1-1) if $\sigma \in (0, 1)$ and, for instance, formula (16.12) in [Maggi 2012] when $\sigma = 1$.

The latter inequality and (2-20) give

$$\text{Per}_\sigma(E, \Omega') \leqslant \text{Per}_\sigma(E, \overline{\Omega}') \leqslant \text{Per}_\sigma(F, \overline{\Omega}') \leqslant 2\text{Per}_\sigma(\Omega', \mathbb{R}^n).$$

This proves the desired result when $\Omega' \Subset \Omega$. Let us now deal with the case $\Omega' \subseteq \Omega$. For this, we set $\Omega'_\epsilon := (1 - \epsilon)\Omega'$. Since Ω is strictly star-shaped, we have $\overline{\Omega}'_\epsilon = (1 - \epsilon)\overline{\Omega}' \subseteq (1 - \epsilon)\overline{\Omega} \subseteq \Omega$ for any $\epsilon \in (0, 1)$, so we can use the result already proved and we get

$$\text{Per}_\sigma(E, \Omega'_\epsilon) \leqslant 2\text{Per}_\sigma(\Omega'_\epsilon, \mathbb{R}^n). \tag{2-21}$$

Moreover,

$$\text{Per}_\sigma(\Omega'_\epsilon, \mathbb{R}^n) = (1 - \epsilon)^{n-\sigma} \text{Per}_\sigma(\Omega', \mathbb{R}^n). \tag{2-22}$$

Also, we claim that

$$\lim_{\epsilon \searrow 0} \text{Per}_\sigma(E, \Omega'_\epsilon) = \text{Per}_\sigma(E, \Omega'). \tag{2-23}$$

To prove it, we distinguish the cases $\sigma = 1$ and $\sigma \in (0, 1)$. If $\sigma = 1$, we use the representation of the perimeter of E in term of the Gauss–Green measure μ_E (see Remark 12.2 in [Maggi 2012]) and the

monotone convergence theorem (applied to the monotone sequence of sets Ω'_ϵ, see, e.g., Theorem 1.26(a) in [Yeh 2006]): in this way, we have

$$\lim_{\epsilon \searrow 0} \operatorname{Per}(E, \Omega'_\epsilon) = \lim_{\epsilon \searrow 0} |\mu_E|(\Omega'_\epsilon) = |\mu_E|(\Omega') = \operatorname{Per}(E, \Omega').$$

This proves (2-23) when $\sigma = 1$. If instead $\sigma \in (0, 1)$, we first observe that $\operatorname{Per}_\sigma(E, \Omega'_\epsilon) \leqslant \operatorname{Per}_\sigma(E, \Omega')$ and then

$$\limsup_{\epsilon \searrow 0} \operatorname{Per}_\sigma(E, \Omega'_\epsilon) \leqslant \operatorname{Per}_\sigma(E, \Omega'). \tag{2-24}$$

Conversely, we use (1-1) to write

$$\operatorname{Per}_\sigma(E, \Omega'_\epsilon) = L(E \cap \Omega'_\epsilon, E^c) + L(E^c \cap \Omega'_\epsilon, E \cap (\Omega'_\epsilon)^c)$$
$$\geqslant L(E \cap \Omega'_\epsilon, E^c) + L(E^c \cap \Omega'_\epsilon, E \cap (\Omega')^c).$$

Consequently, by taking the limit of the inequality above and using Fatou's lemma,

$$\liminf_{\epsilon \searrow 0} \operatorname{Per}_\sigma(E, \Omega'_\epsilon) \geqslant L(E \cap \Omega', E^c) + L(E^c \cap \Omega', E \cap (\Omega')^c) = \operatorname{Per}_\sigma(E, \Omega').$$

This, together with (2-24), establishes (2-23).

Now, combining (2-21)–(2-23), we obtain (2-17) by taking a limit in ϵ. □

3. Proof of Theorem 1.1

Now we prove Theorem 1.1. The idea of the proof is that, on the one hand, for large balls, we obtain a large contribution of the perimeter, which makes the energy functional simply the Dirichlet energy plus a constant, due to the special form of Φ. On the other hand, for small balls, both the Dirichlet energy and the perimeter give a small contribution, and in this range the contribution of the perimeter becomes predominant. This dichotomy of the energy behavior makes the minimal pair change accordingly; namely, in large balls, harmonic functions are favored, somehow independently of their level sets, while, conversely, for small balls the sets which minimize the perimeter are favored, somehow independently on the Dirichlet energy of the function that they support. That is, in the end, the core of the counterexample is, roughly speaking, that being a minimal surface is something rather different than being the level set of a harmonic function.

Of course, some computations are needed to justify the above heuristic arguments and we present now all the details of the proof.

Estimates on $\operatorname{Per}_\sigma(E, B_R)$ from below. Here we obtain bounds from below for the (either classical or fractional) perimeter of a set E in B_R, once E is "suitably fixed" outside[3] the ball $B_R \subset \mathbb{R}^2$. For this scope, we recall the notation in (1-5) and (1-6), and we have:

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[3]For simplicity, we state and prove all the results of this part only in $\mathbb{R}^2$, though some of the arguments would also be valid in higher dimensions.

Lemma 3.1. *Let $c_o > 0$. Let (u, E) be an admissible pair in $\mathbb{R}^2$. Assume $u - \tilde{u} \in H_0^1(B_1)$ and*

$$\int_{B_1} |\nabla u(X)|^2 \, dX \leqslant c_o.$$

Then there exists $c > 0$, possibly depending on c_o, such that

$$\operatorname{Per}_\sigma(E, B_1) \geqslant c. \tag{3-1}$$

Proof. We argue by contradiction. If the thesis in (3-1) were false, there would exist a sequence of admissible pairs (u_j, E_j) such that $u_j - \tilde{u} \in H_0^1(B_1)$,

$$\int_{B_1} |\nabla u_j(X)|^2 \, dX \leqslant c_o$$

and

$$\operatorname{Per}_\sigma(E_j, B_1) \leqslant \frac{1}{j}. \tag{3-2}$$

Thus, by compactness, (see, e.g., Corollary 3.49 in [Ambrosio et al. 2000] for the case $\sigma = 1$ or Theorem 7.1 in [Di Nezza et al. 2012] for the case $\sigma \in (0, 1)$), we conclude that, up to subsequences, u_j converges to some u_∞ weakly in $H^1(B_1)$ and strongly in $L^2(B_1)$, with

$$u_\infty - \tilde{u} \in H_0^1(B_1), \tag{3-3}$$

and χ_{E_j} converges to some χ_{E_∞} strongly in $L^1(B_1)$ as $j \to +\infty$. Accordingly, by the lower semicontinuity of the (either classical or fractional) perimeter (or by Fatou's lemma; see, e.g., Proposition 3.38(b) in [Ambrosio et al. 2000] for the case $\sigma = 1$) we deduce from (3-2) that

$$\operatorname{Per}_\sigma(E_\infty, B_1) = 0.$$

Hence, from the relative isoperimetric inequality (see, e.g., Lemma 2.5 in [Di Castro et al. 2015] when $\sigma \in (0, 1)$ and formula (12.46) in [Maggi 2012] when $\sigma = 1$),

$$\min\{|B_1 \cap E_\infty|^{(2-\sigma)/2}, |B_1 \setminus E_\infty|^{(2-\sigma)/2}\} \leqslant \widehat{C} \operatorname{Per}_\sigma(E_\infty, B_1) = 0$$

for some $\widehat{C} > 0$. Thus, we can suppose

$$|B_1 \cap E_\infty| = 0, \tag{3-4}$$

the case $|B_1 \setminus E_\infty| = 0$ being similar. Also, by virtue of Lemma 2.1, we have $u_\infty \geqslant 0$ a.e. in E_∞ and $u_\infty \leqslant 0$ a.e. in E_∞^c. Thus, by (3-4), we obtain that $u_\infty \leqslant 0$ a.e. in B_1. Looking at a neighborhood of ∂B_1 in the first quadrant, we obtain that this is in contradiction with (3-3), thus proving the desired result. $\square$

By scaling Lemma 3.1, we obtain:

Lemma 3.2. *Let $c_o > 0$ and $R > 0$. Let (u, E) be an admissible pair in $\mathbb{R}^2$. Assume $u - \tilde{u} \in H_0^1(B_R)$ and*

$$\int_{B_R} |\nabla u(X)|^2 \, dX \leqslant c_o R^4. \tag{3-5}$$

Then there exists $c > 0$, possibly depending on c_o, such that

$$\operatorname{Per}_\sigma(E, B_R) \geqslant c R^{2-\sigma}.$$

Proof. We set

$$u_*(X) := R^{-2} u(RX) \quad \text{and} \quad E_* := \frac{E}{R} := \left\{ \frac{X}{R} : X \in E \right\}.$$

Notice that $R^{-2} \tilde{u}(RX) = R^{-2}(Rx)(Ry) = \tilde{u}(X)$; therefore $u_* - \tilde{u} \in H_0^1(B_1)$. Also, (u_*, E_*) is an admissible pair. In addition,

$$\int_{B_1} |\nabla u_*(X)|^2 \, dX = R^{-2} \int_{B_1} |\nabla u(RX)|^2 \, dX = R^{-4} \int_{B_R} |\nabla u(Y)|^2 \, dY \leqslant c_o,$$

thanks to (3-5). As a consequence, we are in a position to apply Lemma 3.1 to the pair (u_*, E_*) and thus we obtain

$$c \leqslant \text{Per}_\sigma(E_*, B_1) = \text{Per}_\sigma \left(\frac{E}{R}, \frac{B_R}{R} \right) = \frac{1}{R^{2-\sigma}} \text{Per}_\sigma(E, B_R),$$

as desired. $\square$

Analysis of minimizers in large balls. Now we give a concrete example of a minimizer in $B_R \subset \mathbb{R}^2$ for R large enough. To this end, we consider a monotone nondecreasing and lower semicontinuous function $\widetilde{\Phi} : [0, +\infty) \to [0, +\infty)$, with

$$\widetilde{\Phi}(t) = 1 \quad \text{for any } t \in [2, +\infty). \tag{3-6}$$

We let

$$\widetilde{\mathcal{E}}_\Omega(u, E) := \int_\Omega |\nabla u(X)|^2 \, dX + \widetilde{\Phi}(\text{Per}_\sigma^\star(E, \Omega)).$$

We remark that, in principle, the minimization procedure in Lemma 2.3 fails for this functional, since the coercivity assumption (1-3) is not satisfied by $\widetilde{\Phi}$. Nevertheless, we will be able to construct explicitly a minimizer for large balls of $\widetilde{\mathcal{E}}$. Then, we will modify $\widetilde{\Phi}$ at infinity and we will obtain from it a minimizer for a functional of the type in (1-4), with a coercive Φ. The details are as follows.

Proposition 3.3. *Let $n = 2$. Let $\tilde{u}$ and $\widetilde{E}$ be as in (1-5) and (1-6).*

Then, there exists $R_o > 0$, only depending on n and σ, such that if $R \geqslant R_o$ then

$$\widetilde{\mathcal{E}}_{B_R}(\tilde{u}, \widetilde{E}) \leqslant \widetilde{\mathcal{E}}_{B_R}(v, F) \tag{3-7}$$

for any admissible pair (v, F) such that $v - \tilde{u} \in H_0^1(B_R)$ and $F \setminus B_R = \widetilde{E} \setminus B_R$, up to sets of measure zero.

Proof. We observe that $\nabla \tilde{u}(x, y) = (y, x)$, and so

$$\int_{B_R} |\nabla \tilde{u}(X)|^2 \, dX = \int_{B_R} |X|^2 \, dX \leqslant C_1 R^4 \tag{3-8}$$

for some $C_1 > 0$. Moreover, since $\widetilde{E}$ is a cone, we have $\widetilde{E} = R\widetilde{E}$; thus

$$\text{Per}_\sigma(\widetilde{E}, B_R) = \text{Per}_\sigma(R\widetilde{E}, RB_1) = C_2 R^{2-\sigma}$$

for some $C_2 > 0$. In particular, if $R \geqslant (2/C_2)^{1/(2-\sigma)}$, we have

$$\text{Per}_\sigma^\star(\widetilde{E}, B_R) \geqslant \text{Per}_\sigma(\widetilde{E}, B_R) \geqslant 2,$$

and then, by (3-6),

$$\widetilde{\Phi}(\mathrm{Per}_\sigma^\star(\widetilde{E}, B_R)) = 1. \tag{3-9}$$

This and (3-8) imply that

$$\widetilde{\mathcal{E}}_{B_R}(\tilde{u}, \widetilde{E}) \leqslant C_1 R^4 + 1 \leqslant 2C_1 R^4 \tag{3-10}$$

if R is large enough.

Now suppose, by contradiction, that (3-7) is violated, i.e.,

$$\widetilde{\mathcal{E}}_{B_R}(\tilde{u}, \widetilde{E}) > \widetilde{\mathcal{E}}_{B_R}(v, F) \tag{3-11}$$

for some competitor (v, F). In particular, by (3-10),

$$\int_{B_R} |\nabla v(X)|^2 \, dX \leqslant \widetilde{\mathcal{E}}_{B_R}(v, F) \leqslant \widetilde{\mathcal{E}}_{B_R}(\tilde{u}, \widetilde{E}) \leqslant 2C_1 R^4. \tag{3-12}$$

This says that formula (3-5) is satisfied by the pair (v, F) with $c_o := 2C_1$, and so Lemma 3.2 gives

$$\mathrm{Per}_\sigma^\star(F, B_R) \geqslant \mathrm{Per}_\sigma(F, B_R) \geqslant cR^{2-\sigma}$$

for some $c > 0$. In particular, for large R, we have

$$\widetilde{\Phi}(\mathrm{Per}_\sigma^\star(F, B_R)) = 1$$

and therefore

$$\widetilde{\mathcal{E}}_{B_R}(v, F) = \int_{B_R} |\nabla v(X)|^2 \, dX + 1. \tag{3-13}$$

On the other hand, since $\tilde{u}$ is harmonic,

$$\int_{B_R} |\nabla v(X)|^2 \, dX \geqslant \int_{B_R} |\nabla \tilde{u}(X)|^2 \, dX;$$

hence (3-13) and (3-9) give

$$\widetilde{\mathcal{E}}_{B_R}(v, F) \geqslant \int_{B_R} |\nabla \tilde{u}(X)|^2 \, dX + 1 = \widetilde{\mathcal{E}}_{B_R}(\tilde{u}, \widetilde{E}).$$

This is in contradiction with (3-11) and so the desired result is established. $\square$

Corollary 3.4. *Let $n = 2$. Let $\tilde{u}$ and $\widetilde{E}$ be as in (1-5) and (1-6). There exists $K_o > 2$ such that the following statement is true. Assume*

$$\Phi(t) = 1 \quad \text{for any } t \in [2, K_o]. \tag{3-14}$$

Then, there exists $R_o > 0$ such that $(\tilde{u}, \widetilde{E})$ is a minimal pair in B_{R_o}.

Proof. We define

$$\widetilde{\Phi}(t) := \begin{cases} \Phi(t) & \text{if } t \in [0, 2], \\ 1 & \text{if } t \in (2, +\infty). \end{cases}$$

Then we are in the setting of Proposition 3.3 and we obtain that there exists $R_o > 0$, only depending on n and σ, such that $(\tilde{u}, \widetilde{E})$ is a minimal pair for $\widetilde{\mathcal{E}}_{B_{R_o}}$. So we define

$$K_o := \mathrm{Per}_\sigma^\star(\widetilde{E}, B_{R_o}) + 3.$$

Notice that K_o only depends on n and σ, since does R_o also, and $\tilde{u}$ and $\widetilde{E}$ are fixed.

To complete the proof of the desired claim, we need to show that $(\tilde{u}, \widetilde{E})$ is a minimal pair for $\mathcal{E}_{B_{R_o}}$, as long as (3-14) is satisfied. For this, we remark that, since Φ is monotone, we have $\Phi(t) \geqslant \Phi(2) = 1$ for any $t \geqslant 2$. As a consequence, we get $\Phi(t) \geqslant \widetilde{\Phi}(t)$ for any $t \geqslant 0$. Therefore, if (v, F) is a competitor for $(\tilde{u}, \widetilde{E})$, we deduce from (3-7) that

$$\widetilde{\mathcal{E}}_{B_{R_o}}(\tilde{u}, \widetilde{E}) \leqslant \widetilde{\mathcal{E}}_{B_{R_o}}(v, F) \leqslant \mathcal{E}_{B_{R_o}}(v, F). \tag{3-15}$$

On the other hand,

$$\mathrm{Per}_\sigma^\star(\widetilde{E}, B_{R_o}) \leqslant K_o. \tag{3-16}$$

Moreover, we have $\widetilde{\Phi}(t) = 1 = \Phi(t)$ if $t \in (2, K_o]$. Therefore, we get $\widetilde{\Phi} = \Phi$ in $[0, K_o]$ and thus, by (3-16),

$$\widetilde{\Phi}(\mathrm{Per}_\sigma^\star(\widetilde{E}, B_{R_o})) = \Phi(\mathrm{Per}_\sigma^\star(\widetilde{E}, B_{R_o})).$$

By plugging this into (3-15), we conclude that

$$\mathcal{E}_{B_{R_o}}(\tilde{u}, \widetilde{E}) = \widetilde{\mathcal{E}}_{B_{R_o}}(\tilde{u}, \widetilde{E}) \leqslant \mathcal{E}_{B_{R_o}}(v, F),$$

as desired. $\square$

Estimates in small balls. Here, we show that the minimal pair constructed in Corollary 3.4 in large balls does not remain minimal in small balls.

Proposition 3.5. *Let $n = 2$. Assume*

$$\Phi(t) = t^\gamma \text{ for any } t \in [0, 1] \tag{3-17}$$

for some

$$\gamma \in \left(0, \frac{4}{2-\sigma}\right). \tag{3-18}$$

Let $\tilde{u}$ and $\widetilde{E}$ be as in (1-5) *and* (1-6).

Then there exists $r_o > 0$ such that if $r \in (0, r_o]$ then the pair $(\tilde{u}, \widetilde{E})$ is not minimal in B_r.

Proof. We suppose, by contradiction, that $(\tilde{u}, \widetilde{E})$ is minimal in B_r, with r sufficiently small.

We observe that $\widetilde{E}$ is not a minimizer of the perimeter in $\bar{B}_{1/2}$ (see [Savin and Valdinoci 2013] for the case $\sigma \in (0, 1)$). Therefore there exists a perturbation $E_\sharp$ of $\widetilde{E}$ inside $\bar{B}_{1/2}$ for which

$$\mathrm{Per}_\sigma(E_\sharp, \bar{B}_{1/2}) \leqslant \mathrm{Per}_\sigma(\widetilde{E}, \bar{B}_{1/2}) - a$$

for some (small, but fixed) $a > 0$. As a consequence, recalling Lemma 2.2,

$$\mathrm{Per}_\sigma(E_\sharp, B_1) - \mathrm{Per}_\sigma(\widetilde{E}, B_1) = \mathrm{Per}_\sigma(E_\sharp, \bar{B}_{1/2}) - \mathrm{Per}_\sigma(\widetilde{E}, \bar{B}_{1/2}) \leqslant -a. \tag{3-19}$$

Now we take $\psi \in C^\infty(\mathbb{R}^2, [0, 1])$ such that $\psi(X) = 0$ for any $X \in B_{3/4}$ and $\psi(X) = 1$ for any $X \in B_{9/10}^c$. We define

$$u_\sharp(X) = u_\sharp(x, y) := \tilde{u}(X)\,\psi(X) = xy\,\psi(x, y).$$

We claim that

$$u_\sharp \geqslant 0 \quad \text{a.e. in } E_\sharp \qquad \text{and} \qquad u_\sharp \leqslant 0 \quad \text{a.e. in } E_\sharp^c. \tag{3-20}$$

To check this, we observe that $u_\sharp = 0$ in $B_{3/4}$, so it is enough to prove (3-20) for points outside $B_{3/4}$. Then, we also remark that $E_\sharp \setminus B_{3/4} = \tilde{E} \setminus B_{3/4}$, and, as a consequence, we get that $\tilde{u} \geqslant 0$ a.e. in $E_\sharp \setminus B_{3/4}$ and $\tilde{u} \leqslant 0$ a.e. in $E_\sharp^c \setminus B_{3/4}$. Hence, since $\psi \geqslant 0$, we obtain that $u_\sharp \geqslant 0$ a.e. in $E_\sharp \setminus B_{3/4}$ and $u_\sharp \leqslant 0$ a.e. in $E_\sharp^c \setminus B_{3/4}$. These observations complete the proof of (3-20).

Now we define

$$u_r(X) := r^2 u_\sharp\left(\frac{X}{r}\right) = xy\,\psi\left(\frac{X}{r}\right) = \tilde{u}(X)\,\psi\left(\frac{X}{r}\right)$$

and

$$E_r := r E_\sharp.$$

From (3-20), we obtain that $u_r \geqslant 0$ a.e. in E_r and $u_r \leqslant 0$ a.e. in E_r^c, and thus (u_r, E_r) is an admissible pair.

Now we check that the data of (u_r, E_r) coincide with $(\tilde{u}, \tilde{E})$ outside B_r. First of all, we have that $\psi = 1$ in $B_{9/10}^c$; thus, if $X \in B_{9r/10}^c$ we have $u_r(X) = \tilde{u}(X)$. This shows that

$$u_r - \tilde{u} \in H_0^1(B_r). \tag{3-21}$$

Moreover,

$$E_r \setminus B_r = \{X \in B_r^c : r^{-1}X \in E_\sharp\} = \{X = rY : Y \in E_\sharp \setminus B_1\} = \{X = rY : Y \in \tilde{E} \setminus B_1\}.$$

Now, since $\tilde{E}$ is a cone, we have $Y \in \tilde{E}$ if and only if $rY \in \tilde{E}$, and so, as a consequence,

$$E_r \setminus B_r = \{X = rY \in \tilde{E} : Y \in B_1^c\} = \tilde{E} \setminus B_r.$$

Using this and (3-21), we obtain that, if $(\tilde{u}, \tilde{E})$ is minimal in B_r, then

$$\mathcal{E}_{B_r}(\tilde{u}, \tilde{E}) \leqslant \mathcal{E}_{B_r}(u_r, E_r). \tag{3-22}$$

Now we remark that, since $\tilde{E}$ is a cone,

$$\text{Per}_\sigma(\tilde{E}, B_r) = \text{Per}_\sigma(r\tilde{E}, rB_1) = r^{2-\sigma}\,\text{Per}_\sigma(\tilde{E}, B_1). \tag{3-23}$$

Now we define

$$\vartheta := \begin{cases} 4\Upsilon & \text{if } \sigma = 1, \\ 0 & \text{if } \sigma \in (0, 1), \end{cases}$$

and we claim that

$$\text{Per}_\sigma^\star(\tilde{E}, B_r) = r^{2-\sigma}\,\text{Per}_\sigma(\tilde{E}, B_1) + \vartheta. \tag{3-24}$$

Indeed, if $\sigma \in (0, 1)$, then (3-24) boils down to (3-23). If instead $\sigma = 1$, we use (3-23) in the following computation:

$$\text{Per}_\sigma^\star(\tilde{E}, B_r) = \text{Per}(\tilde{E}, B_{r+\Upsilon}) = \text{Per}(\tilde{E}, B_r) + \text{Per}(\tilde{E}, B_{r+\Upsilon} \setminus B_r) = r^{2-\sigma}\,\text{Per}_\sigma(\tilde{E}, B_1) + 4\Upsilon.$$

This proves (3-24).

From (3-24) we obtain that

$$\mathcal{E}_{B_r}(\tilde{u}, \widetilde{E}) \geqslant \Phi\big(r^{2-\sigma} \operatorname{Per}_\sigma(\widetilde{E}, B_1) + \vartheta\big). \tag{3-25}$$

On the other hand, recalling (3-19), we have

$$\operatorname{Per}_\sigma(E_r, B_r) = \operatorname{Per}_\sigma(r E_\sharp, B_r) = r^{2-\sigma} \operatorname{Per}_\sigma(E_\sharp, B_1) \leqslant r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a). \tag{3-26}$$

Now we claim that

$$\operatorname{Per}_\sigma^\star(E_r, B_r) \leqslant r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a) + \vartheta. \tag{3-27}$$

Indeed, if $\sigma \in (0, 1)$ then (3-27) reduces to (3-26). If instead $\sigma = 1$, we use the fact that E_r coincides with $\widetilde{E}$ outside B_r and (3-26) to see that

$$\operatorname{Per}_\sigma^\star(E_r, B_r) = \operatorname{Per}(E_r, B_{r+\Upsilon}) = \operatorname{Per}(E_r, B_r) + \operatorname{Per}(E_r, B_{r+\Upsilon} \setminus B_r) \leqslant r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a) + 4\Upsilon.$$

This establishes (3-27).

Then, the monotonicity of Φ and (3-27) give

$$\Phi(\operatorname{Per}_\sigma^\star(E_r, B_r)) \leqslant \Phi\big(r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a) + \vartheta\big) \tag{3-28}$$

Now we remark that

$$|\nabla u_r(X)| \leqslant |\nabla \tilde{u}(X)\, \psi(X/r)| + r^{-1}|\tilde{u}(X)\, \nabla\psi(X/r)| \leqslant |X| + Cr^{-1}|X|^2,$$

for some $C > 0$. As a consequence of this, and possibly renaming $C > 0$, we obtain

$$\int_{B_r} |\nabla u_r(X)|^2\, dX \leqslant C \int_{B_r} \big(|X|^2 + r^{-2}|X|^4\big)\, dX \leqslant Cr^4.$$

This and (3-28) give

$$\mathcal{E}_{B_r}(u_r, E_r) \leqslant Cr^4 + \Phi\big(r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a) + \vartheta\big).$$

Putting together this, (3-22) and (3-25), we conclude that

$$\Phi\big(r^{2-\sigma} \operatorname{Per}_\sigma(\widetilde{E}, B_1) + \vartheta\big) \leqslant Cr^4 + \Phi\big(r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a) + \vartheta\big).$$

Thus, if $r^{2-\sigma} \operatorname{Per}_\sigma(\widetilde{E}, B_1) \leqslant \frac{1}{2}$, and so $\operatorname{Per}_\sigma(\widetilde{E}, B_1) + \vartheta \leqslant 1$, we can use (3-17) and obtain

$$\big[r^{2-\sigma} \operatorname{Per}_\sigma(\widetilde{E}, B_1) + \vartheta\big]^\gamma \leqslant Cr^4 + \big[r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a) + \vartheta\big]^\gamma. \tag{3-29}$$

Now we distinguish the cases $\sigma \in (0, 1)$ and $\sigma = 1$. When $\sigma \in (0, 1)$, we have $\vartheta = 0$ and so (3-29) becomes

$$r^{(2-\sigma)\gamma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1))^\gamma \leqslant Cr^4 + r^{(2-\sigma)\gamma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a)^\gamma.$$

So we multiply by $r^{(\sigma-2)\gamma}$ and we get

$$a_* := (\operatorname{Per}_\sigma(\widetilde{E}, B_1))^\gamma - (\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a)^\gamma \leqslant Cr^{4+(\sigma-2)\gamma}.$$

Notice that $a_* > 0$ since $a > 0$, and therefore the latter inequality gives a contradiction if r is small enough, thanks to (3-18). This concludes the case in which $\sigma \in (0, 1)$.

If instead $\sigma = 1$, then we have $\vartheta > 0$ and so, for small t, we have

$$(t + \vartheta)^\gamma = \vartheta^\gamma + \gamma\vartheta^{\gamma-1}t + O(t^2).$$

Therefore, we infer from (3-29) that

$$\vartheta^\gamma + \gamma\vartheta^{\gamma-1}r^{2-\sigma} \operatorname{Per}_\sigma(\widetilde{E}, B_1) \leqslant \vartheta^\gamma + \gamma\vartheta^{\gamma-1}r^{2-\sigma}(\operatorname{Per}_\sigma(\widetilde{E}, B_1) - a) + O(r^{4-2\sigma}).$$

Hence we simplify some terms and we divide by $r^{2-\sigma}$ to obtain

$$a \leqslant O(r^{2-\sigma}),$$

which gives a contradiction for small $r > 0$. This completes the case $\sigma = 1$. $\square$

Completion of the proof of Theorem 1.1. The claim in Theorem 1.1 now follows plainly by combining Corollary 3.4 and Proposition 3.5.

4. Proof of Theorem 1.2

The argument is a combination of a classical domain variation (see, e.g., [Alt and Caffarelli 1981]) with an expansion of the (classical or fractional) perimeter. Some similar perturbative methods appear, in the classical case, for instance, in [Garofalo and Lin 1986; Caffarelli et al. 2009]. Since the arguments involved here use both standard and nonstandard observations, we give all the details to assist the reader. First, we observe that

the function $\Xi := (\partial_\nu^+ u(x))^2 - (\partial_\nu^- u(x))^2 - H_\sigma^E(x) \, \Phi'(\mathrm{Per}_\sigma^\star(E, \Omega))$ belongs to $C(\partial E \cap \Omega)$,   (4-1)

thanks to (1-8), (1-10) and Proposition 6.3 in [Figalli et al. 2015] (to be used when $\sigma \in (0, 1)$).

Also, given a vector field $V \in C^\infty(\mathbb{R}^n, \mathbb{R}^n)$ such that

$$V(x) = 0 \quad \text{for any } x \in \Omega^c,$$ (4-2)

for small $t \in \mathbb{R}$ we consider the ODE flow $y = y(t; x)$ given by the Cauchy problem

$$\begin{cases} \partial_t y(t; x) = V(y(t; x)), \\ y(0; x) = x. \end{cases}$$ (4-3)

We remark that, for small $t \in \mathbb{R}$,

$$y(t; x) = x + t V(y(t; x)) + o(t) = x + t V(x) + o(t).$$ (4-4)

Accordingly,

$$D_x y(t; x) = I + t \, DV(x) + o(t) = I + t \, DV(y(t; x)) + o(t),$$ (4-5)

where I denotes the n-dimensional identity matrix.

Also, the map $\mathbb{R}^n \ni x \mapsto y(t; x)$ is invertible for small t; i.e., we can consider the inverse diffeomorphism $x(t; y)$. In this way,

$$x(t; y(t; x)) = x \quad \text{and} \quad y(t; x(t; x)) = y.$$ (4-6)

By (4-4), we know

$$x(t; y) = y(t; x(t; y)) - t V(y(t; x(t; y))) + o(t) = y - t V(y) + o(t),$$ (4-7)

and therefore

$$D_y x(t; y) = I - t \, DV(y) + o(t).$$

In particular,

$$\det D_y x(t; y) = 1 - t \operatorname{div} V(y) + o(t). \tag{4-8}$$

Now, given a minimal pair (u, E) as in the statement of Theorem 1.2, we define

$$u_t(y) := u(x(t; y)).$$

We remark that the subscript t above does not represent a time derivative. By (4-6), we can write $u(x) = u_t(y(t; x))$ and thus, recalling (4-5),

$$\nabla u(x) = D_x y(t; x) \, \nabla u_t(y(t; x)) = \nabla u_t(y(t; x)) + t \, DV(y(t; x)) \, \nabla u_t(y(t; x)) + o(t). \tag{4-9}$$

Also, we consider the image of the set E under the diffeomorphism $y(t; \cdot)$; i.e., we define

$$E_t := y(t; E).$$

We claim that

$$\text{the pair } (u_t, E_t) \text{ is admissible.} \tag{4-10}$$

To check this, let $y \in E_t$ (resp., $y \in E_t^c$). Then there exists

$$x \in E \quad (\text{resp. } x \in E^c) \tag{4-11}$$

such that $y = y(t; x)$. Then, by (4-6), we have

$$x(t; y) = x(t; y(t; x)) = x.$$

This identity and (4-11) imply

$$0 \leqslant u(x) = u(x(t; y)) = u_t(y) \quad (\text{resp. } 0 \geqslant u_t(y)).$$

From this, we obtain (4-10).

In addition, we recall that

$$y(t; x) = x \quad \text{for any } x \in \Omega^c, \tag{4-12}$$

thanks to (4-2) and (4-3). Therefore, we have

$$y(t; \Omega) = \Omega. \tag{4-13}$$

Moreover, as a consequence of (4-12) and of (4-10), and using the minimality of (u, E), we have

$$0 \leqslant \mathcal{E}_\Omega(u_t, E_t) - \mathcal{E}_\Omega(u, E). \tag{4-14}$$

Now we compute the first order in t of the right-hand side of (4-14). For this scope, using, for instance, formula (6.3) (when $\sigma = 1$) or formula (6.12) (when $\sigma \in (0, 1)$) in [Figalli et al. 2015], and recalling that V vanishes outside Ω, one obtains that

$$\operatorname{Per}_\sigma^\star(E_t, \Omega) = \operatorname{Per}_\sigma^\star(E, \Omega) + t \int_{(\partial E) \cap \Omega} H_\sigma^E(x) \, V(x) \cdot \nu(x) \, d\mathcal{H}^{n-1}(x) + o(t). \tag{4-15}$$

Above, we denote by ν the exterior normal of E and by $\mathcal{H}^{n-1}$ the $(n-1)$-dimensional Hausdorff measure.

From (4-15), we obtain that

$$\Phi(\mathrm{Per}_\sigma^\star(E_t,\Omega)) = \Phi\left(\mathrm{Per}_\sigma^\star(E,\Omega)+t\int_{(\partial E)\cap\Omega} H_\sigma^E(x)\,V(x)\cdot\nu(x)\,d\mathcal{H}^{n-1}(x)+o(t)\right)$$

$$= \Phi(\mathrm{Per}_\sigma^\star(E,\Omega))+t\,\Phi'(\mathrm{Per}_\sigma^\star(E,\Omega))\int_{(\partial E)\cap\Omega} H_\sigma^E(x)\,V(x)\cdot\nu(x)\,d\mathcal{H}^{n-1}(x)+o(t). \quad (4\text{-}16)$$

Moreover, by (4-9),

$$|\nabla u(x)|^2 = |\nabla u_t(y(t;x))|^2 + 2t\,\nabla u_t(y(t;x))\cdot\big(DV(y(t;x))\,\nabla u_t(y(t;x))\big)+o(t).$$

Now we integrate this equation in x over Ω and we use the change of variable $y := y(t;x)$. In this way, recalling (4-8) and (4-13), we see that

$$\int_\Omega |\nabla u(x)|^2\,dx = \int_\Omega \big[|\nabla u_t(y(t;x))|^2 + 2t\,\nabla u_t(y(t;x))\cdot\big(DV(y(t;x))\,\nabla u_t(y(t;x))\big)\big]\,dx + o(t)$$

$$= \int_\Omega \big[|\nabla u_t(y)|^2 + 2t\,\nabla u_t(y)\cdot(DV(y)\,\nabla u_t(y))\big]\,|\det D_y x(t;y)|\,dy + o(t)$$

$$= \int_\Omega \big[|\nabla u_t(y)|^2 + 2t\,\nabla u_t(y)\cdot(DV(y)\,\nabla u_t(y))\big]\,[1 - t\,\mathrm{div}\,V(y)]\,dy + o(t)$$

$$= \int_\Omega \big[|\nabla u_t(y)|^2 + 2t\,\nabla u_t(y)\cdot(DV(y)\,\nabla u_t(y)) - t\,|\nabla u_t(y)|^2\,\mathrm{div}\,V(y)\big]\,dy + o(t).$$

We write this formula as

$$\int_\Omega |\nabla u_t(y)|^2\,dy$$

$$= \int_\Omega |\nabla u(x)|^2\,dx + t\int_\Omega \big[|\nabla u_t(y)|^2\,\mathrm{div}\,V(y) - 2\,\nabla u_t(y)\cdot(DV(y)\,\nabla u_t(y))\big]\,dy + o(t). \quad (4\text{-}17)$$

Also, by (4-9),

$$\nabla u(x) = \nabla u_t(y(t;x)) + O(t),$$

and so, evaluating this expression at $x := x(t;y)$ and using (4-7), we get

$$\nabla u_t(y) = \nabla u_t(y(t;x(t;y))) = \nabla u(x(t;y)) + O(t) = \nabla u(y) + O(t).$$

We can substitute this into (4-17), thus obtaining

$$\int_\Omega |\nabla u_t(y)|^2\,dy$$

$$= \int_\Omega |\nabla u(x)|^2\,dx + t\int_\Omega \big[|\nabla u(y)|^2\,\mathrm{div}\,V(y) - 2\,\nabla u(y)\cdot(DV(y)\,\nabla u(y))\big]\,dy + o(t). \quad (4\text{-}18)$$

Now we define $\Omega_1 := \Omega\cap\{u>0\}$ and $\Omega_2 := \Omega\cap\{u<0\}$. Notice that $\Delta u = 0$ in Ω_1 and in Ω_2, thanks to Lemma 2.4. Accordingly, in both Ω_1 and Ω_2 we have

$$\mathrm{div}(|\nabla u|^2\,V) = |\nabla u|^2\,\mathrm{div}\,V + 2V\cdot(D^2u\,\nabla u) \quad (4\text{-}19)$$

and

$$\mathrm{div}((V\cdot\nabla u)\nabla u) = \nabla(V\cdot\nabla u)\cdot\nabla u = \nabla u\cdot(DV\nabla u) + V\cdot(D^2u\,\nabla u). \quad (4\text{-}20)$$

So, we take the quantity in (4-19) and we subtract twice the quantity in (4-20); in this way we see that, in both Ω_1 and Ω_2,

$$\mathrm{div}(|\nabla u|^2\, V) - 2\,\mathrm{div}((V \cdot \nabla u)\nabla u) = |\nabla u|^2\,\mathrm{div}\,V + 2V \cdot (D^2 u\,\nabla u) - 2\big[\nabla u \cdot (DV\nabla u) + V \cdot (D^2 u\,\nabla u)\big]$$
$$= |\nabla u|^2\,\mathrm{div}\,V - 2\nabla u \cdot (DV\nabla u).$$

We remark that the last expression is exactly the quantity appearing in one integrand of (4-18); therefore we can write (4-18) as

$$\int_\Omega |\nabla u_t(y)|^2\,dy$$
$$= \int_\Omega |\nabla u(x)|^2\,dx + t \sum_{i \in \{1,2\}} \int_{\Omega_i} \big[\mathrm{div}(|\nabla u(y)|^2\, V(y)) - 2\,\mathrm{div}((V(y) \cdot \nabla u(y))\nabla u(y))\big]\,dy + o(t). \quad (4\text{-}21)$$

Now we recall (1-9) and we notice that the exterior normal ν_1 of Ω_1 coincides with ν, while the exterior normal ν_2 of Ω_2 coincides with $-\nu$. Furthermore, by (1-11), we see that $\nu_1 = -\nabla u/|\nabla u| = -\nabla u/|\partial_\nu^+ u|$ coming from Ω_1 and $\nu_2 = \nabla u/|\nabla u| = \nabla u/|\partial_\nu^- u|$ coming from Ω_2. Accordingly, coming from Ω_1, we have

$$\partial_{\nu_1} u = \nu_1 \cdot \nabla u = -\frac{\nabla u}{|\nabla u|} \cdot \nabla u = -|\partial_\nu^+ u|.$$

Similarly, coming from Ω_2,

$$\partial_{\nu_2} u = \nu_2 \cdot \nabla u = \frac{\nabla u}{|\nabla u|} \cdot \nabla u = |\partial_\nu^- u|.$$

Therefore, coming from Ω_1,

$$\nabla u\,\partial_{\nu_1} u = -|\nabla u|\,\partial_{\nu_1} u\,\nu_1 = |\partial_\nu^+ u|^2\,\nu,$$

and coming from Ω_2,

$$\nabla u\,\partial_{\nu_2} u = |\nabla u|\,\partial_{\nu_2} u\,\nu_2 = -|\partial_\nu^- u|^2\,\nu.$$

Consequently, coming from Ω_1 we have

$$|\nabla u|^2\, V \cdot \nu_1 - 2(V \cdot \nabla u)\partial_{\nu_1} u = |\partial_\nu^+ u|^2 V \cdot \nu - 2(V \cdot \nu)\,|\partial_\nu^+ u|^2 = -|\partial_\nu^+ u|^2 V \cdot \nu,$$

while, coming from Ω_2,

$$|\nabla u|^2\, V \cdot \nu_2 - 2(V \cdot \nabla u)\partial_{\nu_2} u = -|\partial_\nu^- u|^2 V \cdot \nu + 2(V \cdot \nu)\,|\partial_\nu^- u|^2 = |\partial_\nu^- u|^2 V \cdot \nu.$$

Hence, if we apply the divergence theorem in (4-21), we obtain

$$\int_\Omega |\nabla u_t(y)|^2\,dy - \int_\Omega |\nabla u(x)|^2\,dx$$
$$= t \sum_{i \in \{1,2\}} \int_{\partial\Omega_i} \big[|\nabla u(y)|^2\, V(y) \cdot \nu_i(y) - 2(V(y) \cdot \nabla u(y))\partial_{\nu_i} u(y)\big]\,d\mathcal{H}^{n-1}(y) + o(t)$$
$$= -t \int_{(\partial E)\cap\Omega} |\partial_\nu^+ u(y)|^2\, V(y) \cdot \nu(y)\,d\mathcal{H}^{n-1}(y) + t \int_{(\partial E)\cap\Omega} |\partial_\nu^- u(y)|^2\, V(y) \cdot \nu(y)\,d\mathcal{H}^{n-1}(y) + o(t). \quad (4\text{-}22)$$

Using this and (4-16), and also recalling the definition in (4-1), we conclude that

$$
\begin{aligned}
\mathcal{E}_\Omega(u_t, E_t) - \mathcal{E}_\Omega(u, E) &= \int_\Omega |\nabla u_t(y)|^2 \, dy - \int_\Omega |\nabla u(x)|^2 \, dx + \Phi(\mathrm{Per}^\star_\sigma(E_t, \Omega)) - \Phi(\mathrm{Per}^\star_\sigma(E, \Omega)) \\
&= t \int_{(\partial E) \cap \Omega} \left(|\partial^-_\nu u(y)|^2 - |\partial^+_\nu u(y)|^2 \right) V(y) \cdot \nu(y) \, d\mathcal{H}^{n-1}(y) \\
&\qquad + t\, \Phi'(\mathrm{Per}^\star_\sigma(E, \Omega)) \int_{(\partial E) \cap \Omega} H^E_\sigma(x) \, V(x) \cdot \nu(x) \, d\mathcal{H}^{n-1}(x) + o(t) \\
&= -t \int_{(\partial E) \cap \Omega} \Xi(x) \, V(x) \cdot \nu(x) \, d\mathcal{H}^{n-1}(x) + o(t).
\end{aligned}
$$

This and (4-14) imply

$$
\int_{(\partial E) \cap \Omega} \Xi(x) \, V(x) \cdot \nu(x) \, d\mathcal{H}^{n-1}(x) = 0.
$$

Since V is arbitrary, the latter identity and (4-1) imply that Ξ vanishes in the whole of $\partial E \cap \Omega$, which completes the proof of Theorem 1.2.

5. Proof of Theorem 1.3

Energy of the harmonic replacement of a minimal solution. We start with a computation on the harmonic replacement:

Lemma 5.1. *Assume that* (1-14) *holds true. Let* (u, E) *be a minimal pair in* Ω, *with* $u \geqslant 0$ *a.e. in* Ω^c *and* $B_{R_o} \Subset \Omega$. *Let* $R \in (0, R_o]$ *and* u_R *be the function minimizing the Dirichlet energy in* B_R *among all the functions* v *such that* $v - u \in H^1_0(B_R)$. *Then*

$$
\int_{B_R} |\nabla u(x) - \nabla u_R(x)|^2 \, dx \leqslant C \, L_Q \, R^{n-\sigma}
$$

for some $C > 0$, *possibly depending on* R_o, *n and* σ, *and* L_Q *is the one introduced in* (1-14).

Proof. We observe that $u \geqslant 0$ a.e. in $\mathbb{R}^n$, thanks to Lemma 2.7. Hence $u_R \geqslant 0$ a.e., by the classical maximum principle, and therefore, taking $u_R := u$ in B_R^c, we see that $(u_R, E \cup B_R)$ is an admissible pair, and an admissible competitor against (u, E). Therefore, by the minimality of (u, E),

$$
\begin{aligned}
0 &\leqslant \mathcal{E}_\Omega(u_R, E \cup B_R) - \mathcal{E}_\Omega(u, E) \\
&= \int_{B_R} \left(|\nabla u_R(x)|^2 - |\nabla u(x)|^2 \right) dx + \Phi(\mathrm{Per}^\star_\sigma(E \cup B_R, \Omega)) - \Phi(\mathrm{Per}^\star_\sigma(E, \Omega)). \quad (5\text{-}1)
\end{aligned}
$$

Now we use the subadditivity of the (either classical or fractional) perimeter (see, e.g., Proposition 3.38(d) in [Ambrosio et al. 2000] when $\sigma = 1$ and formula (3.1) in [Dipierro et al. 2013] when $\sigma \in (0, 1)$) and we remark that

$$
\begin{aligned}
\mathrm{Per}^\star_\sigma(E \cup B_R, \Omega) &\leqslant \mathrm{Per}^\star_\sigma(E, \Omega) + \mathrm{Per}^\star_\sigma(B_R, \Omega) \leqslant \mathrm{Per}^\star_\sigma(E, \Omega) + \mathrm{Per}_\sigma(B_R, \mathbb{R}^n) \\
&= \mathrm{Per}^\star_\sigma(E, \Omega) + R^{n-\sigma} \, \mathrm{Per}_\sigma(B_1, \mathbb{R}^n) \leqslant Q, \quad (5\text{-}2)
\end{aligned}
$$

in light of (1-16).

Now we claim that

$$\Phi(\mathrm{Per}^\star_\sigma(E \cup B_R, \Omega)) - \Phi(\mathrm{Per}^\star_\sigma(E, \Omega)) \leqslant C L_Q R^{n-\sigma}. \tag{5-3}$$

To prove it, we observe that if $\mathrm{Per}^\star_\sigma(E \cup B_R, \Omega) \leqslant \mathrm{Per}^\star_\sigma(E, \Omega)$ then, by the monotonicity of Φ it follows that $\Phi(\mathrm{Per}^\star_\sigma(E \cup B_R, \Omega)) \leqslant \Phi(\mathrm{Per}^\star_\sigma(E, \Omega))$, which implies (5-3). Therefore, we can assume that $\mathrm{Per}^\star_\sigma(E \cup B_R, \Omega) > \mathrm{Per}^\star_\sigma(E, \Omega)$. Then, by (1-14), which can be utilized here in view of (5-2), and using again the subadditivity of the (either classical or fractional) perimeter,

$$\Phi(\mathrm{Per}^\star_\sigma(E \cup B_R, \Omega)) - \Phi(\mathrm{Per}^\star_\sigma(E, \Omega)) \leqslant L_Q \left| \mathrm{Per}^\star_\sigma(E \cup B_R, \Omega) - \mathrm{Per}^\star_\sigma(E, \Omega) \right|$$

$$\leqslant L_Q \mathrm{Per}^\star_\sigma(B_R, \Omega) \leqslant L_Q \mathrm{Per}_\sigma(B_R, \mathbb{R}^n) \leqslant C L_Q R^{n-\sigma}.$$

This proves (5-3).

By (5-3) and (5-1) we obtain

$$\begin{aligned}
C L_Q R^{n-\sigma} &\geqslant \int_{B_R} \left(|\nabla u(x)|^2 - |\nabla u_R(x)|^2 \right) dx \\
&= \int_{B_R} (\nabla u(x) + \nabla u_R(x)) \cdot (\nabla u(x) - \nabla u_R(x)) \, dx \\
&= \int_{B_R} \left(\nabla u(x) - \nabla u_R(x) + 2\nabla u_R(x) \right) \cdot (\nabla u(x) - \nabla u_R(x)) \, dx \\
&= \int_{B_R} |\nabla u(x) - \nabla u_R(x)|^2 \, dx + 2 \int_{B_R} \nabla u_R(x) \cdot (\nabla u(x) - \nabla u_R(x)) \, dx \\
&= \int_{B_R} |\nabla u(x) - \nabla u_R(x)|^2 \, dx,
\end{aligned}$$

where the last equality follows from the fact that u_R is harmonic in B_R. The desired result is thus established. □

Remark 5.2. From Lemma 5.1 it follows that the gradient of the minimizers locally belongs to the Campanato space $\mathcal{L}^{p,\lambda}$, with $p := 2$ and $\lambda := n - \sigma$, and thus to the Morrey space $L^{2,n-\sigma}$. This and the Poincaré inequality would give that the minimizers belong to the Campanato space $\mathcal{L}^{2,n+2-\sigma}$, and thus to the Hölder space of continuous functions with exponents $\frac{1}{2}((n+2-\sigma)-n) = 1 - \frac{1}{2}\sigma$. In any case, in the forthcoming Section 6 we will provide an alternate approach to continuity results.

Estimate on the average of minimal solutions. Now we estimate the average in balls for minimal solutions:

Lemma 5.3. *Assume that* (1-14) *holds true. Let* (u, E) *be a minimal pair in* Ω, *with* $u \geqslant 0$ *a.e. in* Ω^c *and* $B_{R_o}(p) \Subset \Omega$. *Assume* $R \in (0, R_o]$ *and* $p \in \mathcal{U}_0$. *Then*

$$\frac{1}{|B_R(p)|} \int_{B_R(p)} u(x) \, dx \leqslant C \sqrt{L_Q} \, R^{1-\sigma/2}$$

for some $C > 0$, *possibly depending on* R_o, n *and* σ, *and* L_Q *is the one introduced in* (1-14).

Proof. By (1-13), we can take a sequence p_k with

$$\lim_{k \to +\infty} u(p_k) = 0. \tag{5-4}$$

For any $r \in (0, R]$ and for any $k \in \mathbb{N}$, we define

$$\psi(r) := r^{-n} \int_{B_r(p)} u(x)\, dx \quad \text{and} \quad \psi_k(r) := r^{-n} \int_{B_r(p_k)} u(x)\, dx.$$

We observe that

$$\lim_{k \to +\infty} \psi_k(r) = \psi(r). \tag{5-5}$$

To check this, we let $\bar{R} > R_o$, with $B_{\bar{R}}(p) \Subset \Omega$ and we consider a continuous approximation of u in $L^1(B_{\bar{R}}(p))$. That is, we take continuous functions u_ϵ such that

$$\lim_{\epsilon \searrow 0} \int_{B_{\bar{R}}(p)} |u(x) - u_\epsilon(x)|\, dx = 0. \tag{5-6}$$

For large k, we have $B_r(p_k) \subseteq B_{\bar{R}}(p)$, and so

$$r^n |\psi_k(r) - \psi(r)| = \left| \int_{B_r(p_k)} u(x)\, dx - \int_{B_r(p)} u(x)\, dx \right|$$

$$\leqslant \left| \int_{B_r(p_k)} u_\epsilon(x)\, dx - \int_{B_r(p)} u_\epsilon(x)\, dx \right| + 2 \int_{B_{\bar{R}}(p)} |u(x) - u_\epsilon(x)|\, dx$$

$$= \left| \int_{B_r} \big(u_\epsilon(x + p_k) - u_\epsilon(x + p) \big)\, dx \right| + 2 \int_{B_{\bar{R}}(p)} |u(x) - u_\epsilon(x)|\, dx.$$

Hence, taking the limit in k and using the dominated convergence theorem, we get

$$\lim_{k \to +\infty} r^n |\psi_k(r) - \psi(r)| \leqslant 2 \int_{B_{\bar{R}}(p)} |u(x) - u_\epsilon(x)|\, dx.$$

Then, we take the limit in ϵ and we obtain (5-5) from (5-6), as desired.

Now, we recall that $u \geqslant 0$ a.e. in $\mathbb{R}^n$, thanks to Lemma 2.7. Thus, by Remark 2.6,

$$\psi_k(0) := \lim_{r \searrow 0} \psi_k(r) = u(p_k). \tag{5-7}$$

Furthermore, using polar coordinates,

$$\psi_k'(r) = \frac{d}{dr} \int_{B_1} u(p_k + ry)\, dy = \int_{B_1} \nabla u(p_k + ry) \cdot y\, dy$$

$$= \int_0^1 \left[t^n \int_{S^{n-1}} \nabla u(p_k + rt\omega) \cdot \omega\, d\mathcal{H}^{n-1}(\omega) \right] dt = \int_0^1 \left[t^n \int_{\partial B_1} \partial_\nu u(p_k + rt\omega)\, d\mathcal{H}^{n-1}(\omega) \right] dt, \tag{5-8}$$

where ν is the exterior normal of B_1.

Now, for a fixed $k \in \mathbb{N}$, we use the notation of Lemma 5.1 for the harmonic replacement u_r in $B_r(p_k) \Subset \Omega$. For $\rho \in (0, r]$, we define $v_r(x) := u_r(p_k + \rho x)$ and we observe that, for any $x \in B_1$, we have $\Delta v_r(x) =$

$\rho^2 \Delta u_r(p_k + \rho x) = 0$, and so

$$0 = \int_{B_1} \Delta v_r(x)\, dx = \int_{\partial B_1} \partial_\nu v_r(\omega)\, d\mathcal{H}^{n-1}(\omega) = \rho \int_{\partial B_1} \partial_\nu u_r(p_k + \rho \omega)\, d\mathcal{H}^{n-1}(\omega).$$

We take $\rho := rt$ and we insert this into (5-8). In this way, we obtain

$$\psi_k'(r) = \int_0^1 \left[t^n \int_{\partial B_1} \big(\partial_\nu u(p_k + rt\omega) - \partial_\nu u_r(p_k + rt\omega) \big)\, d\mathcal{H}^{n-1}(\omega) \right] dt.$$

That is, switching from polar to Cartesian coordinates and making the change of variable $y := p_k + rx$,

$$\psi_k'(r) = \int_{B_1} x \cdot \big(\nabla u(p_k + rx) - \nabla u_r(p_k + rx) \big)\, dx = r^{-(n+1)} \int_{B_r(p_k)} (y - p_k) \cdot (\nabla u(y) - \nabla u_r(y))\, dy.$$

Hence, using the Hölder inequality and Lemma 5.1,

$$\psi_k'(r) \leqslant r^{-n} \int_{B_r(p_k)} |\nabla u(y) - \nabla u_r(y)|\, dy \leqslant C r^{-n/2} \left(\int_{B_r(p_k)} |\nabla u(y) - \nabla u_r(y)|^2\, dy \right)^{1/2} \leqslant C \sqrt{L_Q}\, r^{-\sigma/2}$$

for some $C > 0$. This and (5-7) give

$$\psi_k(R) - u(p_k) = \psi_k(R) - \psi_k(0) = \int_0^R \psi_k'(r)\, dr \leqslant C \sqrt{L_Q} \int_0^R r^{-\sigma/2} \leqslant C \sqrt{L_Q}\, R^{1-\sigma/2},$$

up to renaming constants. Hence, making use of (5-4) and (5-5), we find that

$$\psi(R) \leqslant C \sqrt{L_Q}\, R^{1-\sigma/2},$$

which is the desired claim. $\square$

Completion of the proof of Theorem 1.3. We recall that $u \geqslant 0$ a.e. in $\mathbb{R}^n$, thanks to Lemma 2.7. In particular, u is subharmonic, thanks to Lemma 2.5, and thus

$$\frac{1}{|B_\rho|} \int_{B_\rho(x)} u(y)\, dy \geqslant u(x) \tag{5-9}$$

for small $\rho > 0$. Now we take $x \in \Omega$, with $|x|$ suitably small, and we define $R := |x|$. Notice that $B_R(x) \subseteq B_{2R}$ and therefore, since $u \geqslant 0$,

$$\int_{B_R(x)} u(y)\, dy \leqslant \int_{B_{2R}} u(y)\, dy. \tag{5-10}$$

In addition, by applying Lemma 5.3 in B_{2R}, we find that

$$\frac{1}{R^n} \int_{B_{2R}} u(y)\, dy \leqslant C \sqrt{L_Q}\, R^{1-\sigma/2}.$$

As a result, exploiting (5-9) and (5-10),

$$u(x) \leqslant \frac{C}{R^n} \int_{B_R(x)} u(y)\, dy \leqslant \frac{C}{R^n} \int_{B_{2R}} u(y)\, dy \leqslant C \sqrt{L_Q}\, R^{1-\sigma/2} = C \sqrt{L_Q}\, |x|^{1-\sigma/2},$$

up to renaming constants. This proves Theorem 1.3.

6. Proof of Corollary 1.4

To prove Corollary 1.4, it is useful to point out a strengthening of Lemma 2.4 in which one replaces the condition on the infimum with a pointwise condition (this refinement is possible by virtue of Theorem 1.3):

Lemma 6.1. *Let the assumptions of Corollary 1.4 hold true. Let (u, E) be a minimal pair in Ω, with $u \geqslant 0$. Let $U \Subset \Omega$ be an open set with $u > 0$ in U. Then u is harmonic in U.*

Proof. Let $U' \Subset U$ be open. The claim is proved if we show that u is harmonic in U'. To this aim, we claim that

$$\inf_{U'} u > 0. \tag{6-1}$$

We argue for a contradiction, assuming that this infimum is equal to 0. Then, recalling (1-13), we have that there exists $x_\star \in \overline{U}' \cap \mathcal{U}_0$. In particular, since $x_\star \in \overline{U}' \subset U$, we know that

$$u(x_\star) > 0. \tag{6-2}$$

On the other hand, by Theorem 1.3, for small y,

$$u(x_\star + y) \leqslant C \sqrt{L_Q}\, |y|^{1-\sigma/2}.$$

As a result, recalling Remark 2.6,

$$u(x_\star) = u^+(x_\star) = \lim_{\epsilon \searrow 0} \frac{1}{|B_\epsilon|} \int_{B_\epsilon} u^+(x_\star + y)\, dy \leqslant C \sqrt{L_Q} \lim_{\epsilon \searrow 0} \frac{1}{|B_\epsilon|} \int_{B_\epsilon} |y|^{1-\sigma/2}\, dy = 0.$$

This is in contradiction with (6-2) and so we have proved (6-1).

Then, in light of (6-1), we fall under the assumptions of Lemma 2.4, which in turn implies the desired claim. □

First we recall that $u \geqslant 0$ a.e. in $\mathbb{R}^n$, thanks to Lemma 2.7. Also we know that u is subharmonic in Ω (recall Lemma 2.5) and therefore, by the classical maximum principle,

$$u(x) \leqslant M \tag{6-3}$$

for any $x \in \Omega$. Also, we may suppose that

$$\text{there exists } q_o \in B_{3R/10} \text{ such that } u(q_o) = 0. \tag{6-4}$$

Indeed, if this does not hold, then u is harmonic in $B_{3R/10}$, due to Lemma 6.1, and thus

$$\sup_{B_{R/4}} |\nabla u| \leqslant \frac{C}{R} \sup_{B_{3R/10}} u \leqslant \frac{CM}{R}$$

for some $C > 0$, where we also used (6-3) in the latter inequality. This implies

$$|u(x) - u(y)| \leqslant \frac{CM}{R} |x - y| \leqslant \frac{CM}{R^{1-\sigma/2}} |x - y|^{1-\sigma/2},$$

which gives the desired result in this case.

Hence, from now on, we can suppose that (6-4) holds true. We fix $x \neq y \in B_{R/4}$ and we define $d(x)$ to be the distance from x to the set $\{u = 0\}$; we define $d(y)$ analogously. By (6-4), we know that $d(x)$, $d(y) \in \left[0, \frac{3}{5}R\right]$. We distinguish two cases:

Case 1: $|x - y| \geqslant \frac{1}{2} \max\{d(x),\, d(y)\}$.

Case 2: $|x - y| < \frac{1}{2} \max\{d(x),\, d(y)\}$.

First, we deal with Case 1. In this case, we use Theorem 1.3 and we have

$$|u(x)| \leqslant C \sqrt{L_Q}\, (d(x))^{1-\sigma/2} \quad \text{and} \quad |u(y)| \leqslant C \sqrt{L_Q}\, (d(y))^{1-\sigma/2}.$$

As a consequence,

$$|u(x) - u(y)| \leqslant |u(x)| + |u(y)| \leqslant C \sqrt{L_Q}\, \big((d(x))^{1-\sigma/2} + (d(y))^{1-\sigma/2}\big).$$

Then, the assumption of Case 1 implies

$$|u(x) - u(y)| \leqslant C \sqrt{L_Q}\, |x - y|^{1-\sigma/2},$$

up to renaming constants, which gives the desired result in this case.

Now we consider Case 2. In this case, up to exchanging x and y, we have

$$0 \leqslant 2|x - y| < d(x) = \max\{d(x),\, d(y)\} \tag{6-5}$$

and $u > 0$ in $B_{d(x)}(x)$. Then, by Lemma 6.1, we know that u is harmonic in $B_{d(x)}(x)$ and thus

$$\sup_{B_{9d(x)/10}(x)} |\nabla u| \leqslant \frac{C}{d(x)} \sup_{B_{d(x)}(x)} u \tag{6-6}$$

for some $C > 0$.

Now, we prove

$$\sup_{B_{d(x)}(x)} u \leqslant C \sqrt{L_Q}\, (d(x))^{1-\sigma/2} \tag{6-7}$$

for some $C > 0$. For this, take $\eta \in B_{d(x)}(x)$. By construction, there exists $\zeta \in \overline{B_{d(x)}(x)}$ such that $u(\zeta) = 0$. Accordingly, we have $|\eta - \zeta| \leqslant |\eta - x| + |x - \zeta| \leqslant 2d(x)$, and then, by Theorem 1.3,

$$u(\eta) \leqslant C \sqrt{L_Q}\, |\eta - \zeta|^{1-\sigma/2} \leqslant C \sqrt{L_Q}\, (d(x))^{1-\sigma/2},$$

up to renaming $C > 0$, and this establishes (6-7).

Thus, exploiting (6-6) and (6-7), and possibly renaming constants, we obtain that

$$\sup_{B_{9d(x)/10}(x)} |\nabla u| \leqslant C \sqrt{L_Q}\, (d(x))^{-\sigma/2}.$$

Notice now that $y \in B_{d(x)/2}(x) \subset B_{9d(x)/10}(x)$, thanks to (6-5); therefore

$$|u(x) - u(y)| \leqslant C \sqrt{L_Q}\, (d(x))^{-\sigma/2}\, |x - y| \leqslant C \sqrt{L_Q}\, |x - y|^{1-\sigma/2},$$

up to renaming constants. This establishes the desired result also in Case 2 and so the proof of Corollary 1.4 is now completed.

7. Proof of Theorem 1.5

The proof is based on a measure theoretic argument that was used, in different forms, in [Caffarelli et al. 2015; Dipierro and Valdinoci 2016], but unlike the proof in the existing literature, we cannot use here the scaling properties of the functional: namely, the existing proofs can always reduce to the unit ball, since the rescaled minimal pair is a minimal pair for the rescaled functional, whereas this procedure fails in our case (as stressed for instance by Theorem 1.1). For this reason, we need to perform a measure-theoretic argument which works at every scale. To this end, for any $r \in (0, R)$ we define

$$V(r) := |B_r \setminus E| \quad \text{and} \quad a(r) := \mathcal{H}^{n-1}((\partial B_r) \setminus E)$$

and we observe that

$$V(r) = \int_0^r a(t) \, dt; \tag{7-1}$$

see, e.g., formula (13.3) in [Maggi 2012].

The proof of Theorem 1.5 is by contradiction: we suppose that, for some $r_o \in \left(0, \frac{1}{2}R\right)$, we have

$$V(r_o) = |B_{r_o} \setminus E| \leqslant \delta r_o^n \tag{7-2}$$

and we derive a contradiction if $\delta > 0$ is sufficiently small. We recall that $u \geqslant 0$ a.e. in $\mathbb{R}^n$, due to Lemma 2.7, and we define

$$A := B_r \setminus E.$$

We observe that $(u, E \cup A)$ is admissible, since $(E \cup A)^c = E^c \cap A^c \subseteq E^c$. Then, by the minimality of (u, E), we obtain

$$0 \leqslant \mathcal{E}_\Omega(u, E \cup A) - \mathcal{E}_\Omega(u, E) = \Phi(\mathrm{Per}_\sigma^\star(E \cup A, \Omega)) - \Phi(\mathrm{Per}_\sigma^\star(E, \Omega)). \tag{7-3}$$

Now, by the subadditivity of the (either classical or fractional) perimeter (see, e.g., Proposition 3.38(d) in [Ambrosio et al. 2000] when $\sigma = 1$ and formula (3.1) in [Dipierro et al. 2013] when $\sigma \in (0, 1)$), we have

$$\mathrm{Per}_\sigma^\star(E \cup A, \Omega) = \mathrm{Per}_\sigma^\star(E \cup B_r, \Omega) \leqslant \mathrm{Per}_\sigma^\star(E, \Omega) + \mathrm{Per}_\sigma^\star(B_r, \Omega)$$

$$\leqslant \mathrm{Per}_\sigma^\star(E, \Omega) + \mathrm{Per}_\sigma(B_r, \mathbb{R}^n) \leqslant \mathrm{Per}_\sigma^\star(E, \Omega) + R^{n-\sigma} \mathrm{Per}_\sigma(B_1, \mathbb{R}^n).$$

Then, both $\mathrm{Per}_\sigma^\star(E, \Omega)$ and $\mathrm{Per}_\sigma^\star(E \cup A, \Omega)$ are bounded by P, as defined in (1-18), and so they lie in the invertibility range of Φ, as prescribed by (1-19). This observation and (7-3) imply

$$\mathrm{Per}_\sigma^\star(E, \Omega) \leqslant \mathrm{Per}_\sigma^\star(E \cup A, \Omega). \tag{7-4}$$

Now we claim that

$$\mathrm{Per}_\sigma(E, \Omega) \leqslant \mathrm{Per}_\sigma(E \cup A, \Omega). \tag{7-5}$$

Indeed, if $\sigma \in (0, 1)$, then (7-5) is simply (7-4). If instead $\sigma = 1$, we notice that $E \setminus \bar{B}_r = (E \cup A) \setminus \bar{B}_r$ and so we use (2-2), (2-3) and (7-4) to obtain

$$0 \leqslant \mathrm{Per}_\sigma^\star(E \cup A, \Omega) - \mathrm{Per}_\sigma^\star(E, \Omega)$$

$$= \mathrm{Per}_\sigma(E \cup A, \bar{B}_r) - \mathrm{Per}_\sigma(E, \bar{B}_r) = \mathrm{Per}_\sigma(E \cup A, \Omega) - \mathrm{Per}_\sigma(E, \Omega),$$

which establishes (7-5).

Now we use the (either classical or fractional) isoperimetric inequality in the whole of $\mathbb{R}^n$ (see, e.g., Theorem 3.46 in [Ambrosio et al. 2000] when $\sigma = 1$, and [Frank et al. 2008], or Corollary 25 in [Caffarelli and Valdinoci 2011] when $\sigma \in (0, 1)$); in this way, we have

$$(V(r))^{(n-\sigma)/n} = |B_r \setminus E|^{(n-\sigma)/n} = |A|^{(n-\sigma)/n} \leqslant C \operatorname{Per}_\sigma(A, \mathbb{R}^n) \tag{7-6}$$

for some $C > 0$.

Now we claim that, for a.e. $r \in (0, R)$,

$$\operatorname{Per}_\sigma(A, \mathbb{R}^n) \leqslant \begin{cases} C\, a(r) & \text{if } \sigma = 1, \\ C \int_0^r a(\rho)(r-\rho)^{-\sigma}\, d\rho & \text{if } \sigma \in (0, 1) \end{cases} \tag{7-7}$$

for some $C > 0$ (up to renaming C). First we prove (7-7) when $\sigma = 1$. For this, we write the perimeter of E in term of the Gauss–Green measure μ_E (see Remark 12.2 in [Maggi 2012]), we use the additivity of the measures on disjoint sets and we obtain

$$\operatorname{Per}(E, B_r) + \operatorname{Per}(E, \Omega \setminus \bar{B}_r) = |\mu_E|(B_r) + |\mu_E|(\Omega \setminus \bar{B}_r)$$
$$\leqslant |\mu_E|(B_r) + |\mu_E|(\Omega \setminus B_r) = |\mu_E|(\Omega) = \operatorname{Per}(E, \Omega). \tag{7-8}$$

Now we prove that, for a.e. $r \in (0, R)$, we have

$$\mathcal{H}^{n-1}((\partial B_r) \setminus E) = \operatorname{Per}(B_r \setminus E, \Omega) - \operatorname{Per}(E, B_r). \tag{7-9}$$

For this scope, we make use of the property of the Gauss–Green measure with respect to the intersection with balls (see formula (15.14) in Lemma 15.12 of [Maggi 2012], applied here to the complement of E). In this way, we see

$$\mathcal{H}^{n-1}((\partial B_r) \setminus E) = \mathcal{H}^{n-1}((\partial B_r) \cap E^c \cap \Omega) = \mathcal{H}^{n-1}\big|_{E^c \cap (\partial B_r)}(\Omega)$$
$$= |\mu_{E^c \cap B_r}|(\Omega) - |\mu_{E^c}|\big|_{B_r}(\Omega)$$
$$= \operatorname{Per}(E^c \cap B_r, \Omega) - |\mu_{E^c}|(B_r \cap \Omega)$$
$$= \operatorname{Per}(E^c \cap B_r, \Omega) - |\mu_{E^c}|(B_r)$$
$$= \operatorname{Per}(E^c \cap B_r, \Omega) - \operatorname{Per}(E^c, B_r).$$

From this and the fact that $\operatorname{Per}(E^c, B_r) = \operatorname{Per}(E, B_r)$ (see, for instance, Proposition 3.38(d) in [Ambrosio et al. 2000]), we obtain that (7-9) holds true.

Now we claim that, for a.e. $r \in (0, R)$, we have

$$\operatorname{Per}(E \cup B_r, \bar{B}_r) = \mathcal{H}^{n-1}((\partial B_r) \setminus E). \tag{7-10}$$

Since it is not easy to find a complete reference for such formula in the literature, we try to give here an exhaustive proof. To this end, given a set F and $t \in [0, 1]$, we denote by $F^{(t)}$ the set of points of density t of F (see, e.g., Example 5.17 in [Maggi 2012]), that is,

$$F^{(t)} := \left\{ x \in \mathbb{R}^n : \lim_{r \to 0} \frac{|F \cap B_r(x)|}{|B_r|} = t \right\}.$$

With this notation, we observe that $B_r^{(0)} = \mathbb{R}^n \setminus \bar{B}_r$, and thus

$$B_r^{(0)} \cap \bar{B}_r = \varnothing. \tag{7-11}$$

We denote by ∂^* the reduced boundary of a set of locally finite perimeter (see, e.g., formula (15.1) in [Maggi 2012]); we recall that for any $x \in \partial^* E$ one can define the measure-theoretic outer unit normal to E, which we denote by ν_E. We also recall that, by De Giorgi's structure theorem (see, e.g., formula (15.10) in [Maggi 2012]),

$$|\mu_E| = \mathcal{H}^{n-1}\big|_{\partial^* E}. \tag{7-12}$$

We also set

$$N_r := \{x \in (\partial^* E) \cap (\partial B_r) : \nu_E = \nu_{B_r}\}.$$

We claim that, for a.e. $r \in (0, R)$,

$$\mathcal{H}^{n-1}(N_r) = 0. \tag{7-13}$$

To check this, for any $k \in \mathbb{N}$ we define

$$\beta_k := \left\{ r \in (0, R) : \mathcal{H}^{n-1}(N_r) \geqslant \frac{1}{k} \right\}.$$

Then, if $r \in \beta_k$, by (7-12) we have

$$|\mu_E|(\partial B_r) = \mathcal{H}^{n-1}\big|_{\partial^* E}(\partial B_r) = \mathcal{H}^{n-1}((\partial^* E) \cap (\partial B_r)) \geqslant \mathcal{H}^{n-1}(N_r) \geqslant \frac{1}{k}.$$

As a consequence, if $r_1, \ldots, r_j \in \beta_k$ and $r \in (0, R)$, we obtain

$$\operatorname{Per}(E, B_R) = |\mu_E|(B_R) \geqslant |\mu_E|\left(\bigcup_{i=1}^{j} (\partial B_{r_i}) \right) = \sum_{i=1}^{j} |\mu_E|(\partial B_{r_i}) \geqslant \frac{j}{k},$$

that is, $j \leqslant k \operatorname{Per}(E, B_R)$.

This says that β_k has a finite (indeed less than $k \operatorname{Per}(E, B_R)$) number of elements. Thus the following set is countable (and so is of measure zero):

$$\bigcup_{k=1}^{+\infty} \beta_k = \{r \in (0, R) : \mathcal{H}^{n-1}(N_r) > 0\} = \{r \in (0, R) : \text{(7-13) does not hold}\}.$$

This proves (7-13).

Now we use the known formula about the perimeter of the union. For instance, exploiting formula (16.12) of [Maggi 2012] (used here with $F = B_r$ and $G := \bar{B}_r$) we have

$$\operatorname{Per}(E \cup B_r, \bar{B}_r) = \operatorname{Per}(E, B_r^{(0)} \cap \bar{B}_r) + \operatorname{Per}(B_r, E^{(0)} \cap \bar{B}_r) + \mathcal{H}^{n-1}(N_r \cap \bar{B}_r).$$

In particular, using (7-11) and (7-13), we obtain

$$\operatorname{Per}(E \cup B_r, \bar{B}_r) = \operatorname{Per}(B_r, E^{(0)} \cap \bar{B}_r) \tag{7-14}$$

for a.e. $r \in (0, R)$. On the other hand, B_r is a smooth set and so (see, e.g., Example 12.6 in [Maggi 2012]) we have

$$\mathrm{Per}(B_r, E^{(0)} \cap \bar{B}_r) = \mathcal{H}^{n-1}(E^{(0)} \cap \bar{B}_r \cap (\partial B_r)) = \mathcal{H}^{n-1}(E^{(0)} \cap (\partial B_r)),$$

and so (7-14) becomes

$$\mathrm{Per}(E \cup B_r, \bar{B}_r) = \mathcal{H}^{n-1}(E^{(0)} \cap (\partial B_r)). \tag{7-15}$$

Now we set

$$S := (E^{(0)} \setminus E^c) \cup (E^c \setminus E^{(0)})$$

and we remark that $|S| = 0$ (see, e.g., formula (5.19) in [Maggi 2012]). Then, also $|S \cap B_r| = 0$. Therefore (see, e.g., Remark 12.4 in [Maggi 2012]) we get that $\mathrm{Per}(S, \mathbb{R}^n) = 0 = \mathrm{Per}(S \cap B_r, \mathbb{R}^n)$ and then (see, e.g., formula (15.15) in [Maggi 2012]) for a.e. $r \in (0, R)$ we obtain

$$\mathcal{H}^{n-1}(S \cap (\partial B_r)) = \mathrm{Per}(S \cap B_r, \mathbb{R}^n) - \mathrm{Per}(S, B_r) = 0,$$

and so, as a consequence,

$$\mathcal{H}^{n-1}(E^{(0)} \cap (\partial B_r)) = \mathcal{H}^{n-1}(E^c \cap (\partial B_r)).$$

Now we combine this and (7-15) and we finally complete the proof of (7-10).

Now we show that, for a.e. $r \in (0, R)$,

$$\mathrm{Per}(E \cup B_r, \Omega) - \mathrm{Per}(E, \Omega \setminus \bar{B}_r) = \mathrm{Per}(B_r \setminus E, \Omega) - \mathrm{Per}(E, B_r). \tag{7-16}$$

To prove this, we notice that $(E \cup B_r) \setminus \bar{B}_r = E \setminus \bar{B}_r$, and so we use Lemma 2.2 to see

$$\mathrm{Per}(E \cup B_r, \Omega) - \mathrm{Per}(E, \Omega) = \mathrm{Per}(E \cup B_r, \bar{B}_r) - \mathrm{Per}(E, \bar{B}_r).$$

As a consequence,

$$\begin{aligned}
\mathrm{Per}(E \cup B_r, \Omega) - \mathrm{Per}(E, \Omega \setminus \bar{B}_r) &= \mathrm{Per}(E \cup B_r, \bar{B}_r) - \mathrm{Per}(E, \bar{B}_r) + \mathrm{Per}(E, \Omega) - \mathrm{Per}(E, \Omega \setminus \bar{B}_r) \\
&= \mathrm{Per}(E \cup B_r, \bar{B}_r) - |\mu_E|(\bar{B}_r) + |\mu_E|(\Omega) - |\mu_E|(\Omega \setminus \bar{B}_r) \\
&= \mathrm{Per}(E \cup B_r, \bar{B}_r),
\end{aligned}$$

thanks to the additivity of the Gauss–Green measure μ_E. Then, we use (7-10) and we obtain

$$\mathrm{Per}(E \cup B_r, \Omega) - \mathrm{Per}(E, \Omega \setminus \bar{B}_r) = \mathcal{H}^{n-1}((\partial B_r) \setminus E).$$

Then, we exploit (7-9) and we complete the proof of (7-16).

Now we observe that, using (7-9) and (7-16), we obtain, for a.e. $r \in (0, R)$,

$$\mathrm{Per}(E \cup B_r, \Omega) = \mathrm{Per}(E, \Omega \setminus \bar{B}_r) + \mathcal{H}^{n-1}((\partial B_r) \setminus E). \tag{7-17}$$

Now, putting together (7-8) and (7-17), and noticing that $E \cup B_r = E \cup A$, we have

$$\begin{aligned}
\mathrm{Per}(E, B_r) &\leqslant \mathrm{Per}(E, \Omega) - \mathrm{Per}(E, \Omega \setminus \bar{B}_r) \\
&= \mathrm{Per}(E, \Omega) - \mathrm{Per}(E \cup B_r, \Omega) + \mathcal{H}^{n-1}((\partial B_r) \setminus E) \\
&= \mathrm{Per}(E, \Omega) - \mathrm{Per}(E \cup A, \Omega) + \mathcal{H}^{n-1}((\partial B_r) \setminus E).
\end{aligned}$$

Therefore, recalling (7-5) (used here with $\sigma = 1$), we conclude that

$$\text{Per}(E, B_r) \leqslant \mathcal{H}^{n-1}((\partial B_r) \setminus E). \tag{7-18}$$

Now we take $r' \in (r, R)$ and we observe that $B_r \Subset B_{r'} \Subset \Omega$. Also, we see that $A \setminus \bar{B}_{r'} = \varnothing$; thus, by Lemma 2.2 (applied here with $F := \varnothing$),

$$\text{Per}(A, \mathbb{R}^n) = \text{Per}(A, \bar{B}_{r'}) \leqslant \text{Per}(A, \Omega) = \text{Per}(B_r \setminus E, \Omega).$$

As a consequence of this and of (7-16), we obtain

$$\text{Per}(A, \mathbb{R}^n) \leqslant \text{Per}(E \cup B_r, \Omega) - \text{Per}(E, \Omega \setminus \bar{B}_r) + \text{Per}(E, B_r).$$

Hence, in light of (7-17) and (7-18),

$$\text{Per}(A, \mathbb{R}^n) \leqslant 2\mathcal{H}^{n-1}((\partial B_r) \setminus E) = 2a(r).$$

This completes the proof of (7-7) when $\sigma = 1$.

When $\sigma \in (0, 1)$, to prove (7-7) we use a modification of the argument contained in formulas (5.8)–(5.12) in [Dipierro and Valdinoci 2016]. We first observe that

$$\text{Per}_\sigma(E, \Omega) - \text{Per}_\sigma(E \cup A, \Omega) = L(A, E) - L(A, (E \cup A)^c).$$

As a consequence,

$$\begin{aligned}
\text{Per}_\sigma(A, \mathbb{R}^n) &= L(A, A^c) = L(A, E) + L(A, (E \cup A)^c) \\
&= 2L(A, (E \cup A)^c) + \text{Per}_\sigma(E, \Omega) - \text{Per}_\sigma(E \cup A, \Omega).
\end{aligned}$$

This and (7-5) give

$$\text{Per}_\sigma(A, \mathbb{R}^n) \leqslant 2L(A, (E \cup A)^c) \leqslant 2L(A, B_r^c). \tag{7-19}$$

Now we recall that $A \subseteq B_r$ and so, using the change of coordinates $\zeta := x - y$, we obtain

$$\begin{aligned}
L(A, B_r^c) = \int_{A \times B_r^c} \frac{dx\,dy}{|x - y|^{n+\sigma}} &\leqslant \int_{\{(x,\zeta) \in A \times \mathbb{R}^n : |\zeta| \geqslant r - |x|\}} \frac{dx\,d\zeta}{|\zeta|^{n+\sigma}} \\
&\leqslant C \int_A \left[\int_{r-|x|}^{+\infty} \frac{\rho^{n-1}\,d\rho}{\rho^{n+\sigma}} \right] dx \leqslant C \int_A \frac{dx}{(r - |x|)^\sigma}. \tag{7-20}
\end{aligned}$$

Now we use the coarea formula (see, e.g., Theorem 2 on page 117 of [Evans and Gariepy 1992], applied here in codimension 1 to the functions $f(x) = |x|$ and $g(x) := \chi_A(x)/(r - |x|)^\sigma)$, and we deduce that

$$\begin{aligned}
\int_A \frac{dx}{(r - |x|)^\sigma} &= \int_{\mathbb{R}} \left[\int_{\partial B_t} \frac{\chi_A(x)}{(r - |x|)^\sigma}\, d\mathcal{H}^{n-1}(x) \right] dt \\
&= \int_0^r \left[\int_{\partial B_t} \frac{\chi_{E^c}(x)}{(r - t)^\sigma}\, d\mathcal{H}^{n-1}(x) \right] dt = \int_0^r \frac{\mathcal{H}^{n-1}(E^c \cap (\partial B_t))}{(r - t)^\sigma}\, dt = \int_0^r \frac{a(t)}{(r - t)^\sigma}\, dt.
\end{aligned}$$

This and (7-20) imply

$$L(A, B_r^c) \leqslant C \int_0^r \frac{a(t)}{(r - t)^\sigma}\, dt.$$

Inserting this into (7-19) we get

$$\text{Per}_\sigma(A, \mathbb{R}^n) \leqslant C \int_0^r \frac{a(t)}{(r-t)^\sigma}\, dt,$$

which gives the desired claim in (7-7) when $\sigma \in (0, 1)$.

Using (7-6) and (7-7), and possibly renaming constants, we conclude that, for a.e. $r \in (0, R)$,

$$(V(r))^{(n-\sigma)/n} \leqslant \begin{cases} C\, a(r) & \text{if } \sigma = 1, \\ C \int_0^r a(\rho)(r - \rho)^{-\sigma}\, d\rho & \text{if } \sigma \in (0, 1). \end{cases} \tag{7-21}$$

Our next goal is to show that, for any $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$, we have

$$\int_{r_o/4}^{tr_o} (V(r))^{(n-\sigma)/n}\, dr \leqslant C t^{1-\sigma} r_o^{1-\sigma}\, V(tr_o) \tag{7-22}$$

for some $C > 0$. To prove this, we integrate (7-21) in $r \in \left[\frac{1}{4}r_o, tr_o\right]$. Then, when $\sigma = 1$, we obtain (7-22) directly from (7-1). If instead $\sigma \in (0, 1)$, we obtain

$$\int_{r_o/4}^{tr_o} (V(r))^{(n-\sigma)/n}\, dr \leqslant C \int_{r_o/4}^{tr_o} \left[\int_0^r a(\rho)(r - \rho)^{-\sigma}\, d\rho \right] dr$$

$$\leqslant C \int_0^{tr_o} \left[\int_\rho^{tr_o} a(\rho)(r - \rho)^{-\sigma}\, dr \right] d\rho = \frac{C}{1-\sigma} \int_0^{tr_o} a(\rho)(tr_o - \rho)^{1-\sigma}\, d\rho$$

$$\leqslant \frac{C}{1-\sigma} \int_0^{tr_o} a(\rho)(tr_o)^{1-\sigma}\, d\rho = \frac{C\,(tr_o)^{1-\sigma}}{1-\sigma}\, V(tr_o),$$

where we used (7-1) in the last identity. This completes the proof of (7-22), up to renaming the constants.

Now we define $t_k := \frac{1}{4} + \frac{1}{2^k}$ for any $k \geqslant 2$. Let also $w_k := r_o^{-n} V(t_k r_o)$. Notice that $t_{k+1} \geqslant \frac{1}{4}$. Then we use (7-22) with $t := t_k$ and we obtain

$$C t_k^{1-\sigma} r_o^{1-\sigma}\, V(t_k r_o) \geqslant \int_{r_o/4}^{t_k r_o} (V(r))^{(n-\sigma)/n}\, dr \geqslant \int_{t_{k+1} r_o}^{t_k r_o} (V(r))^{(n-\sigma)/n}\, dr.$$

Thus, since $V(\,\cdot\,)$ is monotone,

$$C t_k^{1-\sigma} r_o^{1-\sigma}\, V(t_k r_o) \geqslant (t_k r_o - t_{k+1} r_o)(V(t_{k+1} r_o))^{(n-\sigma)/n} = \frac{r_o}{2^{k+1}}\, (V(t_{k+1} r_o))^{(n-\sigma)/n}.$$

This can be written as

$$w_{k+1}^{(n-\sigma)/n} = r_o^{\sigma-n} (V(t_{k+1} r_o))^{(n-\sigma)/n} \leqslant 2^{k+1} C t_k^{1-\sigma} r_o^{-n} V(t_k r_o) = 2^{k+1} C t_k^{1-\sigma} w_k.$$

Consequently, using that $t_k \leqslant 1$ and possibly renaming $C > 0$, we obtain

$$w_{k+1}^{(n-\sigma)/n} \leqslant C^k w_k. \tag{7-23}$$

Also, we have $t_2 = \frac{1}{2}$ and thus

$$w_2 = r_o^{-n} V\left(\tfrac{1}{2} r_o\right) \leqslant r_o^{-n} V(r_o) \leqslant \delta,$$

in view of (7-2). Then, if $\delta > 0$ is sufficiently small, we have $w_k \to 0$ as $k \to +\infty$ (see, e.g., formula (8.18) in [Dipierro et al. 2014] for explicit bounds). This and the fact that $t_k \geqslant \frac{1}{4}$ say that

$$0 = \lim_{k \to +\infty} r_o^{-n} V(t_k r_o) = \lim_{k \to +\infty} r_o^{-n} |B_{t_k r_o} \setminus E| \geqslant r_o^{-n} |B_{r_o/4} \setminus E|.$$

Hence, we have $|B_{r_o/4} \setminus E| = 0$, in contradiction with the assumption that $0 \in \partial E$ (in the measure-theoretic sense). The proof of Theorem 1.5 is thus complete.

8. Proof of Theorem 1.6

By Lemma 2.7, we have

$$u \geqslant 0 \quad \text{a.e. in } \mathbb{R}^n. \tag{8-1}$$

For any $r \in (0, R)$ we define

$$V(r) := |B_r \cap E| \quad \text{and} \quad a(r) := \mathcal{H}^{n-1}((\partial B_r) \cap E),$$

and we observe that

$$V(r) = \int_0^r a(t) \, dt; \tag{8-2}$$

see, e.g., formula (13.3) in [Maggi 2012].

The proof of Theorem 1.6 is obtained by a contradiction argument. Namely, we suppose that, for some $r_o \in \left(0, \frac{1}{2}R\right)$ we have

$$V(r_o) = |B_{r_o} \cap E| \leqslant \delta_* r_o^n, \tag{8-3}$$

and we derive a contradiction if $\delta_* > 0$ is sufficiently small.

We let $A := B_r \cap E$. Let also $\tilde{v}$ be the minimizer of the Dirichlet energy in B_{r_o} among all the possible candidates $v : \mathbb{R}^n \to \mathbb{R}$ such that $v = u$ outside B_{r_o}, $v - u \in H_0^1(B_{r_o})$ and $v = 0$ a.e. in $E^c \cup A$ (for the existence and the uniqueness of such harmonic replacement see, e.g., page 481 in [Athanasopoulos et al. 2001]). By (8-1) and Lemma 2.3 in [Athanasopoulos et al. 2001] we have

$$\tilde{v} \geqslant 0 \quad \text{a.e. in } \mathbb{R}^n. \tag{8-4}$$

Now we set $F := E \setminus A$. We observe that $\tilde{v} = 0$ a.e. in $F^c = E^c \cup A$ by construction. This and (8-4) give that $(\tilde{v}, F)$ is an admissible pair, and recall also that $\tilde{v} - u \in H_0^1(B_{r_o}) \subseteq H_0^1(\Omega)$. Hence, the minimality of (u, E) gives

$$0 \leqslant \mathcal{E}_\Omega(\tilde{v}, F) - \mathcal{E}_\Omega(u, E) = \int_\Omega |\nabla \tilde{v}(x)|^2 \, dx - \int_\Omega |\nabla u(x)|^2 \, dx + \Phi(\mathrm{Per}_\sigma^\star(F, \Omega)) - \Phi(\mathrm{Per}_\sigma^\star(E, \Omega)).$$

Using this and the fact that $\tilde{v}$ and u coincide outside B_{r_o}, we obtain

$$\Phi(\mathrm{Per}_\sigma^\star(E, \Omega)) - \Phi(\mathrm{Per}_\sigma^\star(F, \Omega)) \leqslant \int_{B_{r_o}} |\nabla \tilde{v}(x)|^2 \, dx - \int_{B_{r_o}} |\nabla u(x)|^2 \, dx. \tag{8-5}$$

Now we take $\tilde{w}$ to be the minimizer of the Dirichlet energy in B_{r_o} among all the functions $w : \mathbb{R}^n \to \mathbb{R}$ such that $w = u$ outside B_{r_o}, $w - u \in H_0^1(B_{r_o})$ and $w = 0$ a.e. in E^c. We remark that u is a competitor with such $\tilde{w}$ and therefore

$$\int_{B_{r_o}} |\nabla \tilde{w}(x)|^2 \, dx \leqslant \int_{B_{r_o}} |\nabla u(x)|^2 \, dx.$$

Plugging this into (8-5), we deduce that

$$\Phi(\mathrm{Per}_\sigma^\star(E, \Omega)) - \Phi(\mathrm{Per}_\sigma^\star(F, \Omega)) \leqslant \int_{B_{r_o}} |\nabla \tilde{v}(x)|^2 \, dx - \int_{B_{r_o}} |\nabla \tilde{w}(x)|^2 \, dx.$$

This and Lemma 2.3 in [Caffarelli et al. 2015] imply

$$\Phi(\mathrm{Per}_\sigma^\star(E, \Omega)) - \Phi(\mathrm{Per}_\sigma^\star(F, \Omega)) \leqslant C r_o^{-2} |A| \, \|\tilde{w}\|_{L^\infty(B_{r_o})}^2. \tag{8-6}$$

Since, by Lemma 2.3 in [Athanasopoulos et al. 2001], we know that $\tilde{w} \geqslant 0$ a.e. in $\mathbb{R}^n$ and is subharmonic, we have that w in B_{r_o} takes its maximum along ∂B_{r_o}, where it coincides with u. Hence

$$\|\tilde{w}\|_{L^\infty(B_{r_o})} \leqslant \sup_{\partial B_{r_o}} u. \tag{8-7}$$

Now we observe that condition (1-20) allows us to use Theorem 1.3, which gives

$$\sup_{\partial B_{r_o}} u \leqslant C \sqrt{L_Q} \, r_o^{1 - \sigma/2}$$

for some $C > 0$. Hence (8-7) gives

$$\|\tilde{w}\|_{L^\infty(B_{r_o})} \leqslant C \sqrt{L_Q} \, r_o^{1 - \sigma/2}.$$

Thus, recalling (8-6), and possibly renaming constants, we conclude that

$$\Phi(\mathrm{Per}_\sigma^\star(E, \Omega)) - \Phi(\mathrm{Per}_\sigma^\star(F, \Omega)) \leqslant C r_o^{-\sigma} |A| L_Q. \tag{8-8}$$

Now we claim that

$$\mathrm{Per}_\sigma(E, \Omega) - \mathrm{Per}_\sigma(F, \Omega) \leqslant C c_o^{-1} r_o^{-\sigma} |A| L_Q, \tag{8-9}$$

where $c_o > 0$ is the one introduced in (1-21). To check this, we may suppose that $\lambda_1 := \mathrm{Per}_\sigma(E, \Omega) > \mathrm{Per}_\sigma(F, \Omega) =: \lambda_2$, otherwise we are done. Then, by (1-22), both λ_1 and λ_2 belong to $[0, Q]$; therefore we can make use of (1-21) and obtain

$$\Phi(\mathrm{Per}_\sigma^\star(E, \Omega)) - \Phi(\mathrm{Per}_\sigma^\star(F, \Omega)) = \Phi(\lambda_1) - \Phi(\lambda_2)$$
$$= \int_{\lambda_2}^{\lambda_1} \Phi'(t) \, dt \geqslant c_o(\lambda_1 - \lambda_2) = c_o\big(\mathrm{Per}_\sigma^\star(E, \Omega) - \mathrm{Per}_\sigma^\star(F, \Omega)\big),$$

and then it follows from (8-8) that

$$\mathrm{Per}_\sigma^\star(E, \Omega) - \mathrm{Per}_\sigma^\star(F, \Omega) \leqslant C c_o^{-1} r_o^{-\sigma} |A| L_Q. \tag{8-10}$$

Now we observe that $E \setminus \bar{B}_r = F \setminus \bar{B}_r$; therefore, using (2-2) and (2-3), we see that

$$\text{Per}_\sigma^\star(E, \Omega) - \text{Per}_\sigma^\star(F, \Omega) = \text{Per}_\sigma(E, \bar{B}_r) - \text{Per}_\sigma(F, \bar{B}_r) = \text{Per}_\sigma(E, \Omega) - \text{Per}_\sigma(F, \Omega).$$

Putting together this and (8-10) we obtain (8-9).

Now we show that, for a.e. $r \in (0, r_o)$,

$$\text{Per}_\sigma(A, \mathbb{R}^n) \leqslant \begin{cases} C\big(a(r) + c_o^{-1} r_o^{-\sigma} |A| L_Q\big) & \text{if } \sigma = 1, \\ C\big(\int_0^r a(\rho)(r-\rho)^{-\sigma} \, d\rho + c_o^{-1} r_o^{-\sigma} |A| L_Q\big) & \text{if } \sigma \in (0, 1). \end{cases} \tag{8-11}$$

To prove (8-11) we distinguish the cases $\sigma = 1$ and $\sigma \in (0, 1)$. If $\sigma = 1$, we notice that $A \setminus \bar{B}_r = (B_r \cap E) \setminus \bar{B}_r = \varnothing$; hence, by Lemma 2.2, we have

$$\text{Per}(A, \mathbb{R}^n) = \text{Per}(A, \bar{B}_r) = \text{Per}(E \cap B_r, \bar{B}_r).$$

Hence we use the formula for the perimeter associated with the intersection with balls (see, e.g., (15.14) in Lemma 15.12 of [Maggi 2012]) and we obtain

$$\begin{aligned} \text{Per}(A, \mathbb{R}^n) = |\mu_{E \cap B_r}|(\bar{B}_r) &= \mathcal{H}^{n-1}\big|_{E \cap (\partial B_r)}(\bar{B}_r) + |\mu_E|\big|_{B_r}(\bar{B}_r) \\ &= \mathcal{H}^{n-1}(E \cap (\partial B_r) \cap \bar{B}_r) + \text{Per}(E, B_r \cap \bar{B}_r) \\ &= \mathcal{H}^{n-1}(E \cap (\partial B_r)) + \text{Per}(E, B_r). \end{aligned} \tag{8-12}$$

On the other hand, we have $(E \setminus B_r)^c = E^c \cup B_r$; hence (see, e.g., formula (16.11) in [Maggi 2012]) we obtain that $\text{Per}(E \setminus B_r, \bar{B}_r) = \text{Per}(E^c \cup B_r, \bar{B}_r)$ for a.e. $r \in (0, r_o)$. Hence, by Lemma 2.2,

$$\begin{aligned} \text{Per}(E, \Omega) - \text{Per}(F, \Omega) &= \text{Per}(E, \bar{B}_r) - \text{Per}(F, \bar{B}_r) \\ &= \text{Per}(E, \bar{B}_r) - \text{Per}(E \setminus B_r, \bar{B}_r) = \text{Per}(E, \bar{B}_r) - \text{Per}(E^c \cup B_r, \bar{B}_r) \end{aligned} \tag{8-13}$$

for a.e. $r \in (0, r_o)$. Moreover (see, e.g., formula (7-10), applied here to the complementary set), we have

$$\text{Per}(E^c \cup B_r, \bar{B}_r) = \mathcal{H}^{n-1}((\partial B_r) \cap E),$$

so we can write (8-13) as

$$\text{Per}(E, \Omega) - \text{Per}(F, \Omega) = \text{Per}(E, \bar{B}_r) - \mathcal{H}^{n-1}((\partial B_r) \cap E).$$

In particular

$$\text{Per}(E, B_r) \leqslant \text{Per}(E, \bar{B}_r) = \text{Per}(E, \Omega) - \text{Per}(F, \Omega) + \mathcal{H}^{n-1}((\partial B_r) \cap E).$$

Then we insert this information into (8-12) and we obtain

$$\text{Per}(A, \mathbb{R}^n) \leqslant 2\mathcal{H}^{n-1}(E \cap (\partial B_r)) + \text{Per}(E, \Omega) - \text{Per}(F, \Omega).$$

Now we recall (8-9), which completes the proof of (8-11) when $\sigma = 1$, and focus on the case $\sigma \in (0, 1)$. For this, we use (1-1) and we see that

$$\text{Per}_\sigma(E, \Omega) - \text{Per}_\sigma(F, \Omega) = \text{Per}_\sigma(E, \Omega) - \text{Per}_\sigma(E \setminus A, \Omega) = L(A, E^c) - L(A, E \setminus A).$$

Therefore

$$\operatorname{Per}_\sigma(A, \mathbb{R}^n) = L(A, A^c) = L(A, E^c) + L(A, E \setminus A) = \operatorname{Per}_\sigma(E, \Omega) - \operatorname{Per}_\sigma(F, \Omega) + 2L(A, E \setminus A). \quad (8\text{-}14)$$

Now we use the fact that $A \subseteq B_r$ and the change of coordinates $\zeta := x - y$ to write

$$L(A, E \setminus A) \leqslant L(A, B_r^c) = \int_{A \times B_r^c} \frac{dx\,dy}{|x-y|^{n+\sigma}} \leqslant \int_{\{(x,\zeta) \in A \times \mathbb{R}^n : |\zeta| \geqslant r - |x|\}} \frac{dx\,d\zeta}{|\zeta|^{n+\sigma}}$$

$$\leqslant C \int_A \left[\int_{r-|x|}^{+\infty} \frac{\rho^{n-1}\,d\rho}{\rho^{n+\sigma}} \right] dx \leqslant C \int_A \frac{dx}{(r-|x|)^\sigma}. \quad (8\text{-}15)$$

Now we observe that, by the coarea formula (see, e.g., Theorem 2 on page 117 of [Evans and Gariepy 1992], applied here in codimension 1 to the functions $f(x) = |x|$ and $g(x) := \chi_A(x)/(r-|x|)^\sigma)$,

$$\int_A \frac{dx}{(r-|x|)^\sigma} = \int_{\mathbb{R}} \left[\int_{\partial B_t} \frac{\chi_A(x)}{(r-|x|)^\sigma}\,d\mathcal{H}^{n-1}(x) \right] dt$$

$$= \int_0^r \left[\int_{\partial B_t} \frac{\chi_E(x)}{(r-t)^\sigma}\,d\mathcal{H}^{n-1}(x) \right] dt = \int_0^r \frac{\mathcal{H}^{n-1}(E \cap (\partial B_t))}{(r-t)^\sigma}\,dt = \int_0^r \frac{a(t)}{(r-t)^\sigma}\,dt.$$

This and (8-15) give

$$L(A, E \setminus A) \leqslant C \int_0^r \frac{a(t)}{(r-t)^\sigma}\,dt.$$

So we substitute this and (8-9) into (8-14) and we complete the proof of (8-11) when $\sigma \in (0, 1)$.

Now we recall that $|A| = V(r)$ and we use the (either classical or fractional) isoperimetric inequality in the whole of $\mathbb{R}^n$ (see, e.g., Theorem 3.46 in [Ambrosio et al. 2000] when $\sigma = 1$, and [Frank et al. 2008], or Corollary 25 in [Caffarelli and Valdinoci 2011] when $\sigma \in (0, 1)$) and we deduce from (8-11) that, for a.e. $r \in (0, r_o)$,

$$(V(r))^{(n-\sigma)/n} = |A|^{(n-\sigma)/n} \leqslant \begin{cases} C\left(a(r) + c_o^{-1} r_o^{-\sigma} V(r) L_Q\right) & \text{if } \sigma = 1, \\ C\left(\int_0^r a(\rho)(r-\rho)^{-\sigma}\,d\rho + c_o^{-1} r_o^{-\sigma} V(r) L_Q\right) & \text{if } \sigma \in (0, 1), \end{cases} \quad (8\text{-}16)$$

up to renaming $C > 0$. Now we recall (8-3) and we notice that, if $r \in (0, r_o)$,

$$c_o^{-1} r_o^{-\sigma} V(r) L_Q \leqslant c_o^{-1} r_o^{-\sigma} (V(r))^{(n-\sigma)/n} (V(r_o))^{\sigma/n} L_Q \leqslant \delta_*^{\sigma/n} c_o^{-1} (V(r))^{(n-\sigma)/n} L_Q.$$

This means that, if $\delta_* > 0$ is small enough, or more precisely if

$$\delta_*^{\sigma/n} c_o^{-1} L_Q \leqslant \frac{1}{2C}, \quad (8\text{-}17)$$

we can reabsorb[4] one term in the left-hand side of (8-16): in this way, possibly renaming constants, we obtain that, for a.e. $r \in (0, r_o)$,

$$(V(r))^{(n-\sigma)/n} \leqslant \begin{cases} C\,a(r) & \text{if } \sigma = 1, \\ C \int_0^r a(\rho)(r-\rho)^{-\sigma}\,d\rho & \text{if } \sigma \in (0, 1). \end{cases}$$

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[4]It is interesting to point out that the possibility of absorbing the term $C c_o^{-1} r_o^{-\sigma} V(r) L_Q$ into the left-hand side of (8-16) crucially depends on the fact that the power produced by the (either classical or fractional) isoperimetric inequality and the one given by the growth result in Theorem 1.3 match together in the appropriate way.

This implies that, for any $t \in \left[\frac{1}{4}, \frac{1}{2}\right]$, we have

$$\int_{r_o/4}^{tr_o} (V(r))^{(n-\sigma)/n} \, dr \leqslant C t^{1-\sigma} r_o^{1-\sigma} V(tr_o) \tag{8-18}$$

for some $C > 0$. Indeed, the proof of (8-18) is obtained in the same way as that of (7-22) (the only difference is that here one has to use (8-2) in lieu of (7-1)). Then, one defines $t_k := \frac{1}{4} + \frac{1}{2^k}$ and $w_k := r_o^{-n} V(t_k r_o)$ and observes that

$$w_{k+1}^{(n-\sigma)/n} \leqslant C^k w_k. \tag{8-19}$$

Indeed, (8-19) can be obtained as in the proof of (7-23) (but using here (8-18) instead of (7-22)). Furthermore

$$w_2 = r_o^{-n} V\left(\tfrac{1}{2} r_o\right) \leqslant \delta_*,$$

thanks to (8-3). This says that

$$\text{if } \delta_* > 0 \text{ is sufficiently small (with respect to a universal constant),} \tag{8-20}$$

then $w_k \to 0$ as $k \to +\infty$ (see formula (8.18) in [Dipierro et al. 2014] for explicit bounds). Thus

$$0 = \lim_{k \to +\infty} r_o^{-n} V(t_k r_o) = \lim_{k \to +\infty} r_o^{-n} |B_{t_k r_o} \cap E| \geqslant r_o^{-n} |B_{r_o/4} \cap E|.$$

This is in contradiction with the assumption that $0 \in \partial E$ (in the measure-theoretic sense) and so the proof of Theorem 1.6 is finished. We stress that the explicit condition in (1-23) comes from (8-17) and (8-20).

Acknowledgements

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