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NONNEGATIVE KERNELS AND 1-RECTIFIABILITY IN THE HEISENBERG GROUP

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Let E be a 1-regular subset of the Heisenberg group $\mathbb{H}$. We prove that there exists a -1-homogeneous kernel K_1 such that if E is contained in a 1-regular curve, the corresponding singular integral is bounded in $L^2(E)$. Conversely, we prove that there exists another -1-homogeneous kernel K_2 such that the $L^2(E)$-boundedness of its corresponding singular integral implies that E is contained in a 1-regular curve. These are the first non-Euclidean examples of kernels with such properties. Both K_1 and K_2 are weighted versions of the Riesz kernel corresponding to the vertical component of $\mathbb{H}$. Unlike the Euclidean case, where all known kernels related to rectifiability are antisymmetric, the kernels K_1 and K_2 are even and nonnegative.

1. Introduction

One of the standard topics in classical harmonic analysis is the study of singular integral operators (SIOs) of the form

$$Tf(x) = \int \frac{\Omega(x-y)}{|x-y|^n} f(y)\, d\mathcal{L}^n(y),$$

where Ω is a 0-homogeneous function and $\mathcal{L}^n$ is the Lebesgue measure in $\mathbb{R}^n$; see, e.g., [Stein 1993]. A considerable amount of research has been devoted to such SIOs, and nowadays they are well understood. On the other hand if the singular integral is defined on lower-dimensional measures, the situation is much more complicated even when one considers the simplest of kernels.

As an example the reader should think of the Cauchy transform

$$C_E f(z) = \int_E \frac{f(w)}{z-w}\, d\mathcal{H}^1(w), \qquad E \subset \mathbb{C},$$

where $\mathcal{H}^1$ denotes the 1-dimensional Hausdorff measure in the complex plane. Two questions arise naturally. For which sets E is C_E bounded in $L^2(E)$? And, if C_E is bounded in $L^2(E)$, what does this imply about E? Here $L^2(E)$-boundedness means that there exists a constant $C > 0$ such that the truncated operator

$$C_E^\varepsilon f(z) = \int_{E \setminus B(z,\varepsilon)} \frac{f(w)}{z-w}\, d\mathcal{H}^1(w)$$

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satisfies $\|C_E^\varepsilon f\|_{L^2(\mathcal{H}^1|_E)} \le C\|f\|_{L^2(\mathcal{H}^1|_E)}$ for all $f \in L^2(\mathcal{H}^1|_E)$. It turns out that the $L^2(E)$-boundedness of the Cauchy transform depends crucially on the geometric structure of E.

The problem of exploring this relation has a long history and it is deeply related to rectifiability and analytic capacity; we refer to the recent book of Tolsa [2014] for an extensive treatment. One of the landmarks in the field was the characterization of the 1-regular sets E on which the Cauchy transform is bounded in $L^2(E)$. Recall that an $\mathcal{H}^1$-measurable set E is 1-*Ahlfors-regular*, if there exists a constant $1 \le C < \infty$ such that

$$C^{-1} r \le \mathcal{H}^1(B(x,r) \cap E) \le Cr$$

for all $x \in E$, and $0 < r \le \operatorname{diam} E$. It turns out that if E is 1-regular, the Cauchy transform C_E is bounded in $L^2(E)$ if and only if E is contained in a 1-regular curve. The sufficient condition is due to David [1988] and it even holds for more general smooth antisymmetric kernels. The necessary condition is due to Mattila, Melnikov and Verdera [Mattila et al. 1996]. It is a remarkable fact that their proof depends crucially on a special subtle positivity property of the Cauchy kernel related to an old notion of curvature named after Menger; see, e.g., [Melnikov and Verdera 1995; Mattila et al. 1996]. We also note that the above characterization also holds for the SIOs associated to the coordinate parts of the Cauchy kernel.

Very few things are known for the action of SIOs associated with other -1-homogeneous, 1-dimensional Calderón–Zygmund kernels (see Section 2 for the exact definition) on 1-regular sets in the complex plane. Call a kernel "good" if its associated SIO is bounded on $L^2(E)$ if and only if E is contained in a 1-regular curve. It is noteworthy that all known good or bad kernels are related to the kernels

$$k_n(z) = \frac{x^{2n-1}}{|z|^{2n}}, \quad z = (x, y) \in \mathbb{C} \setminus \{0\},\ n \in \mathbb{N}.$$

Observe that k_1 is a good kernel as it is the x-coordinate of the Cauchy kernel; see [Mattila et al. 1996]. It was shown in [Chousionis et al. 2012] that the kernels k_n, $n > 1$, are good as well, and these were the first nontrivial examples of good kernels not directly related to the Cauchy kernel. Now let

$$\kappa_t(z) = k_2(z) + t \cdot k_1(z), \quad t \in \mathbb{R}.$$

It follows by [Chousionis et al. 2012] and [Mattila et al. 1996] that κ_t is good for $t > 0$. Recently Chunaev [2016] showed that κ_t is good for $t \le -2$ and Chunaev, Mateu and Tolsa [Chunaev et al. 2016] proved that κ_t is good for $t \in (-2, -\sqrt{2})$. For $t = -1$ and $t = -\frac{3}{4}$ there exist intricate examples of sets E, due to Huovinen [2001] and Jaye and Nazarov [2013] respectively, which show that the $L^2(E)$-boundedness of the SIO associated to κ_{-1} and $\kappa_{-3/4}$ does not imply rectifiability for E. Therefore the kernels $\kappa_{-1}(z) = xy^2/|z|^4$ and $\kappa_{-3/4}(x, y) = (x^3 - 3xy^2)/|z|^4$ are bad kernels.

Notice that all the kernels mentioned so far are odd and this is very reasonable. Consider, for example, a 1-dimensional Calderón–Zygmund kernel $k : \mathbb{R} \times \mathbb{R} \setminus \{x = y\} \to \mathbb{R}^+$ which is not locally integrable along the diagonal. Take, for example, $k(x, y) = |x - y|^{-1}$. Then $\int_I k(x, y)\, dy = \infty$ for all open intervals $I \subset \mathbb{R}$. It becomes evident that defining a SIO which makes sense on lines and other "nice" 1-dimensional objects depends crucially on the cancellation properties of the kernel. Surprisingly in the Heisenberg group $\mathbb{H}$ the situation is very different.

The *Heisenberg group* $\mathbb{H}$ is $\mathbb{R}^3$ endowed with the group law

$$p \cdot q = \big(x + x',\ y + y',\ z + z' + \tfrac{1}{2}(xy' - yx')\big) \tag{1-1}$$

for $p = (x, y, t)$, $q = (x', y', t') \in \mathbb{R}^3$. We use the following metric on $\mathbb{H}$:

$$d_{\mathbb{H}} : \mathbb{H} \times \mathbb{H} \to [0, \infty), \quad d_{\mathbb{H}}(p, q) := N(q^{-1} \cdot p),$$

where $N : \mathbb{H} \to [0, \infty)$ is the *Korányi norm* in $\mathbb{H}$,

$$N(x, y, z) := ((x^2 + y^2)^2 + z^2)^{1/4}.$$

We also let

$$NH(x, y, z) = |z|^{1/2},$$

where NH stands for nonhorizontal. Note that

$$d_{\mathbb{H}}(x, y) = (|\pi(x) - \pi(y)|^4 + NH(x^{-1}y)^4)^{1/4}.$$

We also remark that the metric $d_{\mathbb{H}}$ is homogeneous with respect to the dilations

$$\delta_r : \mathbb{H} \to \mathbb{H}, \quad \delta_r((x, y, z)) = (rx, ry, r^2 z), \quad (r > 0).$$

Finally let $\Omega : \mathbb{H} \setminus \{0\} \to [0, \infty)$,

$$\Omega(p) = \frac{NH(p)}{N(p)} \tag{1-2}$$

and notice that Ω is 0-homogeneous, as $\Omega(\delta_r(p)) = \Omega(p)$ for all $r > 0$. One can also define the dilations for $r < 0$ for which the metric is still 1-homogeneous (although with absolute value).

In our first main theorem we prove that, in contrast to the Euclidean case, there exists a nonnegative, -1 homogeneous, Calderón–Zygmund kernel which is bounded in $L^2(E)$ for every 1-regular set E which is contained in a 1-regular curve. We warn the reader that from now on $\mathcal{H}^1$ will denote the 1-dimensional Hausdorff measure in $(\mathbb{H}, d_{\mathbb{H}})$.

Theorem 1.1. *Let $K_1 : \mathbb{H} \setminus \{0\} \to [0, \infty)$ be defined by*

$$K_1(p) = \frac{\Omega(p)^8}{N(p)},$$

and let E be a 1*-regular set which is contained in a* 1*-regular curve. Then the corresponding truncated singular integrals*

$$T_1^{\varepsilon} f(p) = \int_{E \setminus B_{\mathbb{H}}(p,\varepsilon)} K_1(q^{-1} \cdot p) f(q)\, d\mathcal{H}^1(q)$$

are uniformly bounded in $L^2(E)$.

There are abundant examples of 1-regular sets in $\mathbb{H}$ which are not contained in 1-regular curves. For example, one can consider suitable 1-dimensional Cantor sets in the vertical axis, $T = \{(0, 0, z) : z \in \mathbb{R}\}$, which is 2-dimensional.

We define the principal value of f at p to be

$$\text{p.v.}\, T_1 f(p) = \lim_{\varepsilon \to 0} T_1^{\varepsilon}(f)(p),$$

when the limit exists. Because the kernel is positive, we will be able to use Theorem 1.1 to easily show that the principal value operator is bounded in L^2.

Corollary 1.2. *If $f \in L^2(E)$, then* p.v. $T_1 f(x)$ *exists almost everywhere and is in $L^2(E)$. Moreover, we have that there exists a constant $C > 0$ such that*

$$\|\text{p.v.}\, T_1 f\|_{L^2(E)} \leq C\|f\|_{L^2(E)} \quad \forall f \in L^2(E).$$

Let us quickly give an intuition behind why one would expect a positive kernel like $NH(x)^m/N(x)^{m+1}$ to be bounded on Lipschitz curves. Rademacher's theorem says that Lipschitz curves in $\mathbb{R}^n$ infinitesimally resemble affine lines, and antisymmetric kernels cancel on affine lines. This is essentially what controls the singularity. In the Heisenberg setting, a Rademacher-type theorem by Pansu [1989] says that Lipschitz curves infinitesimally resemble *horizontal* lines and NH is 0 on horizontal lines. Thus, we again have control over the singularity.

Some heuristic motivation comes from the fact that the positive Riesz kernel $|z|/(x^2+y^2+z^2)^{3/2}$ defines a SIO which is trivially bounded in $\mathbb{R}^3$ for curves in the xy-plane. In this case, however, the boundedness of this SIO tells us nothing about the regularity of the xy-curve. An analogous concern in the Heisenberg group would be whether the boundedness of kernels of the form $NH(z)^p/N(z)^{p+1}$ implies anything about the regularity of the sets if the vertical direction is "orthogonal" to Lipschitz curves. While we do not know if the boundedness of the kernel of Theorem 1.1 says anything about regularity, our next result shows that there exists some p for which these vertical Riesz kernels do:

Theorem 1.3. *Let $K_2 : \mathbb{H} \setminus \{0\} \to [0, \infty)$ be defined by*

$$K_2(p) = \frac{\Omega(p)^2}{N(p)},$$

and let E be a 1-regular set. If the corresponding truncated singular integrals

$$T_2^{\varepsilon} f\,(p) = \int_{E \setminus B_{\mathbb{H}}(p,\varepsilon)} K_2(q^{-1} \cdot p) f(q)\, d\mathcal{H}^1(q)$$

are uniformly bounded in $L^2(E)$ then E is contained in a 1-regular curve.

One can interpret this statement as saying that the vertical fluctuations of a 1-regular set $E \subset \mathbb{H}$ (that is, $K_i(p^{-1} \cdot q)$ when $p, q \in E$) contain enough information to determine that it lies on a 1-regular curve.

The following question arises naturally from Theorems 1.1 and 1.3. *Does there exist some $m \in \mathbb{N}$ such that any 1-regular set E is contained in some 1-regular curve if and only if the operators*

$$T^{\varepsilon} f\,(p) = \int_E \frac{\Omega(q^{-1} \cdot p)^m}{N(q^{-1} \cdot p)} f(q)\, d\mathcal{H}^1(q)$$

are uniformly bounded in $L^2(E)$? The methods developed in this paper do not allow us to answer this question, partly because our proof for Theorem 1.1 seems to require a large power for $\Omega(p)$. This is essential because we are using a positive kernel and so are not able to use antisymmetry to gain additional control from the bilinearity, as is commonly used in these types of arguments; for example, see Section 6.2 of [Tolsa 2009]. The proof of Theorem 1.3 uses delicate estimates regarding the Korányi norm and is also not likely to be improved without a major change in the proof structure.

A motivation for the geometric study of SIOs in $\mathbb{R}^n$ is their significance in PDE and potential theory. In particular the d-dimensional Riesz transforms (the SIOs associated to the kernels $x/|x|^{d+1}$) for $d = 1$ and $d = n - 1$ play a crucial role in the geometric characterization of *removable sets* for bounded analytic functions and Lipschitz harmonic functions. Landmark contributions by David [1998], David and Mattila [2000], and Nazarov, Tolsa and Volberg [Nazarov et al. 2014a; 2014b] established that these removable sets coincide with the purely $(n-1)$-unrectifiable sets in $\mathbb{R}^n$, i.e., the sets which intersect every $(n-1)$-dimensional Lipschitz graph in a set of vanishing $(n-1)$-dimensional Hausdorff measure. For an excellent review of the topic and its connections to nonhomogeneous harmonic analysis, we refer the reader to the survey [Volberg and Èiderman 2013].

The same motivation exists in several noncommutative Lie groups as well. For example, the problem of characterizing removable sets for Lipschitz harmonic functions has a natural analogue in Carnot groups. In that case the harmonic functions are, by definition, the solutions to the sub-Laplacian equation. It was shown in [Chousionis and Mattila 2014] that in the case of the Heisenberg group, the dimension threshold for such removable sets is $\dim \mathbb{H} - 1 = 3$, where $\dim \mathbb{H}$ denotes the Hausdorff dimension of $\mathbb{H}$. See also [Chousionis et al. 2015] for an extension of the previous result to all Carnot groups. As in the Euclidean case, one has to handle a SIO whose kernel is the horizontal gradient of the fundamental solution of the sub-Laplacian. For example, in $\mathbb{H}$, such a kernel can be explicitly written as

$$K(p) := \left(\frac{x(x^2+y^2)+yz}{((x^2+y^2)^2+z^2)^{3/2}}, \frac{y(x^2+y^2)-xz}{((x^2+y^2)^2+z^2)^{3/2}} \right)$$

for $p = (x, y, z) \in \mathbb{H}$. Currently we know very little about the action of this kernel on 3-dimensional subsets of $\mathbb{H}$. Nevertheless it has motivated research on SIOs on lower-dimensional subsets of $\mathbb{H}$, e.g., [Chousionis and Mattila 2011] and the present paper, as well as the very recent study of quantitative rectifiability in $\mathbb{H}$; see [Chousionis et al. 2016].

2. Preliminaries

Although we have already defined a metric on $\mathbb{H}$, we will also need the fact that there exists a natural path metric on $\mathbb{H}$. Notice that the Heisenberg group is a Lie group with respect to the group operation defined in (1-1), and the Lie algebra of the left invariant vector fields in $\mathbb{H}$ is generated by the vector fields

$$X := \partial_x + y\partial_z, \quad Y := \partial_y - x\partial_z, \quad T := \partial_z.$$

The vector fields X and Y define the *horizontal subbundle* $H\mathbb{H}$ of the tangent vector bundle of $\mathbb{R}^3$. For every point $p \in \mathbb{H}$ we will denote the horizontal fiber by $H_p\mathbb{H}$. Every such horizontal fiber is endowed with the left invariant scalar product $\langle \cdot, \cdot \rangle_p$ and the corresponding norm $|\cdot|_p$ that make the vector fields X, Y, T orthonormal.

Definition 2.1. An absolutely continuous curve $\gamma : [a, b] \to \mathbb{H}$ will be called *horizontal* with respect to the vector fields X, Y if

$$\dot{\gamma}(t) \in H_{\gamma(t)}\mathbb{H} \quad \text{for a.e. } t \in [a, b].$$

Definition 2.2. The *Carnot–Carathéodory distance* of $p, q \in \mathbb{H}$ is

$$d_{cc}(p, q) = \inf \int_a^b |\dot{\gamma}(t)|_{\gamma(t)}\, dt,$$

where the infimum is taken over all horizontal curves $\gamma : [a, b] \to \mathbb{H}$ such that $\gamma(a) = p$ and $\gamma(b) = q$.

By Chow's theorem, the above set of curves joining p and q is not empty and hence d_{cc} defines a metric on $\mathbb{H}$. Furthermore the infimum in the definition can be replaced by a minimum. See [Bonfiglioli et al. 2007] for more details.

Remark 2.3. It follows by results of Pansu [1982a; 1982b] that any 1-regular curve is horizontal; hence the reader should keep in mind that our two main theorems (Theorems 1.1 and 1.3) essentially involve subsets of horizontal curves.

A point $p \in \mathbb{H}$ is called *horizontal* if p lies on the xy-plane. We can now define an important family of curves in the Heisenberg group.

Definition 2.4. Let $p, q \in \mathbb{H}$ such that q is horizontal. The subsets of the form

$$\{p \cdot \delta_r(q) : r \in \mathbb{R}\}$$

are called *horizontal lines*.

Observe that horizontal lines are horizontal curves with constant tangent vector. Thus, in the horizontal line above, the element q can be thought of as defining a "horizontal direction" for the line.

Note also that the horizontal lines going through a specified point in $\mathbb{H}$ span only two dimensions instead of three as in $\mathbb{R}^3$. This is a significant difference between Heisenberg and Euclidean geometry.

It is easy to see that the homomorphic projection $\pi : \mathbb{H} \to \mathbb{R}^2$ defined by

$$\pi(x, y, z) = (x, y)$$

is 1-Lipshitz. We will also use the map $\tilde{\pi} : \mathbb{H} \to \mathbb{H}$ defined by

$$\tilde{\pi}(x, y, z) = (x, y, 0).$$

We stress that $\tilde{\pi}$ is *not* a homomorphism.

Definition 2.5 (horizontal interpolation). For $p, q \in \mathbb{H}$,

$$\overline{pq} = \{p \cdot \delta_r \tilde{\pi}(p^{-1} \cdot q) : r \in [0, 1]\}.$$

Note that $\overline{pq}$ is a horizontal segment starting from p traveling in the horizontal direction of $p^{-1} \cdot q$.

Definition 2.6. Let (X, d) be a metric space. We say that

$$k(\cdot, \cdot) : X \times X \setminus \{x = y\} \to \mathbb{R}$$

is an n-dimensional *Calderón–Zygmund* (CZ)-kernel if there exist constants $c > 0$ and η, with $0 < \eta \le 1$, such that for all $x, y \in X$, with $x \neq y$,

(1) $|k(x, y)| \le c/d(x, y)^n$,

(2) $|k(x, y) - k(x', y)| + |k(y, x) - k(y, x')| \le c d(x, x')^\eta / d(x, y)^{n+\eta}$ if $d(x, x') \le d(x, y)/2$.

For the next lemma, recall the definition (1-2) of the functions Ω.

Lemma 2.7. *Fix $m \in \mathbb{N}$, and let $k : \mathbb{H} \times \mathbb{H} \setminus \{x = y\} \to \mathbb{R}$ be defined as*

$$k(p, q) = \frac{\Omega(q^{-1} \cdot p)^m}{N(q^{-1} \cdot p)}.$$

Then k is a 1*-dimensional CZ-kernel.*

Proof. We need to verify (1) and (2) from Definition 2.6. Notice that (1) is immediate because by the definition of the Korányi norm, $NH(p) \le N(p)$ for all $p \in \mathbb{H}$. For (2) we will use the fact that the function

$$f(p) = \frac{\Omega(p)^m}{N(p)}, \quad p \in \mathbb{H} \setminus \{0\},$$

is C^1 away from the origin and it is also -1-homogeneous, that is,

$$f(\delta_r(p)) = \frac{1}{r} f(p)$$

for all $r > 0$ and $p \in \mathbb{H} \setminus \{0\}$. Hence by [Folland and Stein 1982, Proposition 1.7] there exists some constant $C > 0$ such that for all $P, Q \in \mathbb{H}$ with $N(Q) \le N(P)/2$,

$$|f(P \cdot Q) - f(P)| \le C \frac{N(Q)}{N(P)^2}.$$

Hence if $p, p', q \in \mathbb{H}$ such that $d_{\mathbb{H}}(p, p') \le d_{\mathbb{H}}(p, q)/2$,

$$\begin{aligned} |k(p, q) - k(p', q)| &= |f(q^{-1} \cdot p) - f(q^{-1} \cdot p')| \\ &= |f(q^{-1} \cdot p) - f(q^{-1} \cdot p \cdot p^{-1} \cdot p')| \le C \frac{N(p'^{-1} \cdot p)}{N(q^{-1} \cdot p)^2} = C \frac{d_{\mathbb{H}}(p', p)}{d_{\mathbb{H}}(p, q)^2}. \end{aligned} \tag{2-1}$$

Since k is symmetric, from (2-1) we deduce that k also satisfies (2) of Definition 2.6. ☐

In the sequel, we will use the notation $a \lesssim b$ or $a \gtrsim b$ to mean that there exists a universal constant C so that $a \le Cb$ or $a \ge Cb$. This universal constant can change from instance to instance. We let $a \asymp b$ mean both $a \lesssim b$ and $b \lesssim a$. Given another fixed quantity α, we let $a \lesssim_\alpha b$ and $b \lesssim_\alpha a$ mean that the quantity C can depend only on α.

3. Necessary conditions

In order to simplify notation, in the two following sections we will denote $d := d_{\mathbb{H}}$, $B(p,r) := B_{\mathbb{H}}(p,r)$ and $ab := a \cdot b$ for $a, b \in \mathbb{H}$.

Let $E \subset \mathbb{H}$ such that $\mu = \mathcal{H}^1|_E$ satisfies the 1-regularity condition

$$\xi r \le \mu(B(x,r)) \le \xi^{-1} r \quad \forall x \in E,\ r > 0,$$

for some $\xi < 1$. We now recall the construction of *David cubes* [1991]. David cubes can be constructed on any regular set of a geometrically doubling metric space. In particular, for the set E, we obtain a constant $c > 0$ and a family of partitions Δ_j of E, $j \in \mathbb{Z}$, with the following properties:

(D1) If $k \le j$, $Q \in \Delta_j$ and $Q' \in \Delta_k$, then either $Q \cap Q' = \varnothing$ or $Q \subset Q'$.

(D2) If $Q \in \Delta_j$, then $\operatorname{diam} Q \le 2^{-j}$.

(D3) Every set $Q \in \Delta_j$ contains a set of the form $B(p_Q, c2^{-j}) \cap E$ for some $p_Q \in Q$.

The sets in $\Delta := \bigcup \Delta_j$ are called David cubes, or *dyadic cubes*, of E. Notice that $\operatorname{diam} Q \asymp 2^{-j}$ if $Q \in \Delta_j$. For a cube $S \in \Delta$, we define

$$\Delta(S) := \{Q \in \Delta : Q \subseteq S\}.$$

Given a cube $Q \in \Delta$ and $\lambda \ge 1$, we define

$$\lambda Q := \{x \in E : d(x, Q) \le (\lambda - 1) \operatorname{diam} Q\}.$$

It follows from (D1), (D2), and the 1-regularity of E that $\mu(Q) \sim 2^{-j}$ for $Q \in \Delta_j$.

Define the positive symmetric -1-homogeneous kernel K by

$$K(p) = \frac{\Omega^8(p)}{N(p)} = \frac{NH(p)^8}{N(p)^9}.$$

For any $\varepsilon > 0$, we can define the truncated operator as before:

$$T_1^\varepsilon f(x) = \int_{d(y,x)>\varepsilon} K(y^{-1}x) f(x)\, d\mu(y).$$

Proof of Theorem 1.1. Our goal is to show that when E lies on a rectifiable curve, there exists a uniform bound $C < \infty$ that can depend on ξ such that

$$\|T_1^\varepsilon \chi_S\|_{L^2(S)}^2 \le C\mu(S) \quad \forall S \in \Delta,\ \forall \varepsilon > 0. \tag{3-1}$$

We then apply the $T(1)$ theorem for homogeneous spaces — see, e.g., [Deng and Han 2009; David 1991] — to deduce the uniform L^2-boundedness of T_1^ε for all $\varepsilon > 0$. We may suppose E is a 1-regular rectifiable curve, as taking a subset can only decrease the L^2-bound of $T_1^\varepsilon \chi_S$.

From now on we assume the 1-regular set E actually lies on a rectifiable curve. For $x \in E$ and $r > 0$, we define

$$\beta_E(x,r) = \inf_L \sup_{z \in E \cap B(x,r)} \frac{d(z,L)}{r},$$

where the infimum is taken over all horizontal lines.

Proposition 3.1. *There exists a constant $C \geq 1$ depending only ξ so that for any $S \in \Delta$, we have*

$$\sum_{Q \in \Delta(S)} \beta(10Q)^4 \mu(Q) \leq C\mu(S). \tag{3-2}$$

Proof. This essentially follows from Theorem I of [Li and Schul 2016b], which says that there exists some universal constant $C > 0$ such that

$$\int_{\mathbb{H}} \int_0^\infty \beta_E(B(x,t))^4 \frac{dt}{t^4} \, d\mathcal{H}^4(x) \leq C\mathcal{H}^1(E)$$

when E is simply a horizontal curve. When E is in addition 1-regular, it is a standard argument to use the Ahlfors regularity to bound this integral from below by a constant multiple — which can depend on ξ — of the left-hand side of (3-2) (after intersecting E with S). In fact, one can easily show that the integral and sum are comparable up to multiplicative constants.

One then gets

$$\sum_{Q \in \Delta(S)} \beta(10Q)^4 \mu(Q) \leq C\mathcal{H}^1(E \cap S) \lesssim_\xi \mu(S),$$

where we again used 1-regularity of E in the final inequality. □

We now fix $S \in \Delta$ a cube.

Now define a positive, even Lipschitz function $\psi : \mathbb{R} \to \mathbb{R}$ such that $\chi_{B(0,1/2)} \leq \psi \leq \chi_{B(0,2)}$. We define

$$\psi_j : \mathbb{H} \to \mathbb{R}, \quad z \mapsto \psi(2^j N(z)),$$

and $\phi_j := \psi_j - \psi_{j+1}$. Thus, ϕ_j is supported on the annulus $B(0, 2^{1-j}) \backslash B(0, 2^{-2-j})$ in $\mathbb{H}$ and we have

$$\chi_{\mathbb{H} \backslash B(0,2^{-n+1})} \leq \sum_{n \leq N} \phi_n \leq \chi_{\mathbb{H} \backslash B(0,2^{-n-2})}. \tag{3-3}$$

For each $j \in \mathbb{Z}$, we can define $K_{(j)} = \phi_j \cdot K$ and also

$$T_{(j)} \chi_S(x) = \int_S K_{(j)}(y^{-1}x) \, d\mu(y).$$

Define $S_N = \sum_{n \leq N} T_{(n)}$. As the kernel K is positive, we can easily get the following *pointwise* estimates for any positive function f from (3-3):

$$0 \leq T_1^\varepsilon f \leq S_{n+1} f \quad \forall \varepsilon \geq 2^{-n}.$$

Thus, to show uniform bound (3-1), it suffices to show that there exists $C < \infty$ depending possibly on ξ such that

$$\|S_n \chi_S\|_{L^2(S)}^2 \leq C\mu(S) \quad \forall S \in \Delta, \ \forall n \in \mathbb{Z}.$$

We now fix $S \in \Delta_\ell$.

We will need the following lemma.

Lemma 3.2 [Li and Schul 2016a, Lemma 3.3]. *For every $a, b \in \mathbb{H}$ and horizontal line $L \subset \mathbb{H}$, we have*

$$\max\{d(a,L), d(b,L)\} \geq \frac{1}{16} \frac{NH(a^{-1}b)^2}{d(a,b)}. \tag{3-4}$$

Lemma 3.3. *For any $j \in \mathbb{Z}$ and $x \in E$, we have*

$$T_{(j)} 1(x) \lesssim_\xi \beta_E(x, 2^{1-j})^4. \tag{3-5}$$

Proof. Define the annulus $A = E \cap A(x, 2^{-2-j}, 2^{1-j})$. Then

$$T_{(j)}1(x) \le \int_E \phi_j(y^{-1}x)K(y^{-1}x)\,d\mu(y) \le 2^{j+2}\int_A \frac{NH(y^{-1}x)^8}{N(y^{-1}x)^8}\,d\mu(y) \lesssim_\xi \sup_{y\in A} \frac{NH(y^{-1}x)^8}{d(x,y)^8}.$$

It suffices to show

$$\frac{NH(y^{-1}x)^8}{d(x,y)^8} \le 8^4 \beta_E(B(x, 2^{1-j}))^4$$

when $y \in A$. This follows easily from (3-4). Indeed, as $y \in A$, we have $d(x, y) \ge 2^{-j-2}$. We can then find a horizontal line so that

$$\beta_{\{x,y\}}(B(x, 2^{1-j})) = \frac{\max\{d(x, L), d(y, L)\}}{2^{1-j}} \ge \frac{\max\{d(x, L), d(y, L)\}}{8d(x, y)} \overset{(3\text{-}4)}{\ge} \frac{NH(x^{-1}y)^2}{128d(x, y)^2}.$$

The statement now follows as $\beta_E(B(x, 2^{1-j})) \ge \beta_{\{x,y\}}(B(x, 2^{1-j}))$. □

We now have the following easy corollary.

Corollary 3.4. *Let $R \in \Delta_j$. Then for any $\alpha > 0$, we have*

$$\int_R T_{(j)}1(x)^\alpha\,d\mu(x) \lesssim_\xi \beta_E(10R)^{4\alpha}\mu(R). \tag{3-6}$$

Remark 3.5. We may replace the constant 1 function in (3-5) and (3-6) with any positive function $f \le 1$ (such as $f = \chi_S$ for some $S \in \Delta$). This is again because the kernel of T_j is positive and so respects the partial ordering of positive functions.

For any $Q \in \Delta$, we can also define

$$T_Q\chi_S := \chi_Q T_{(j(Q))}\chi_S.$$

Thus, we have

$$S_n\chi_S = \sum_{j\le n} T_{(j)}\chi_S = \sum_{j\le n}\sum_{Q\in\Delta_j} T_Q\chi_S.$$

and so

$$\|S_n\chi_S\|^2_{L^2(S)} = \sum_{j\le n}\|T_{(j)}\chi_S\|^2_{L^2(S)} + 2\sum_{j<k\le n}\langle T_{(j)}\chi_S, T_{(k)}\chi_S\rangle, \tag{3-7}$$

where the inner product $\langle\,\cdot\,,\,\cdot\,\rangle$ is integration on S. We will bound the two terms on the right-hand side separately.

Let $S^* \in \Delta_{\ell-2}$ be such that $S \subset S^*$. It follows from (D1) that S^* is unique for S. It follows from the ϕ_j factor and the fact that cubes of Δ_ℓ have diameter at most $2^{-\ell}$ that $T_{(j)}\chi_S(x) = 0$ for $x \in S \in \Delta_\ell$ whenever $j < \ell - 2$. Thus, as $S \in \Delta_\ell$, we have

$$\sum_{j\le n}\|T_{(j)}\chi_S\|^2_{L^2(S)} \le \sum_{\ell-2\le j\le n}\ \sum_{Q\in\Delta_j, Q\subseteq S}\int_Q T_{(j)}\chi_S(x)^2\,d\mu(x) \overset{(3\text{-}6)}{\lesssim_\xi} \sum_{Q\in\Delta(S^*)}\beta(10Q)^8\mu(Q). \tag{3-8}$$

We now have to bound the off-diagonal terms of (3-7). We have

$$\begin{aligned}\sum_{j\geq \ell-2}\sum_{j<k\leq n}\int_S T_{(j)}\chi_S(x)\cdot T_{(k)}(x)\chi_S\,d\mu(x) &\overset{(3\text{-}5)}{\lesssim_\xi} \sum_{j\geq\ell-2}\sum_{Q\in\Delta_j(S)}\beta(10Q)^4\sum_{k>j}\int_Q T_{(k)}\chi_S\,d\mu(x)\\ &\overset{(3\text{-}6)}{\lesssim_\xi} \sum_{Q\in\Delta(S^*)}\beta(10Q)^4\sum_{R\in\Delta(Q)}\beta(10R)^4\mu(R)\\ &\overset{(3\text{-}2)}{\lesssim_\xi} C\sum_{Q\in\Delta(S^*)}\beta(10Q)^4\mu(Q).\end{aligned}\tag{3-9}$$

Note that the constants hidden in the $\lesssim$ of (3-8) and (3-9) do not depend on S or n.

Altogether, we have

$$\|S_n\chi_S\|^2_{L^2(S)} \overset{(3\text{-}7)\text{–}(3\text{-}9)}{\lesssim_\xi} \sum_{Q\in\Delta(S^*)}\beta(10Q)^4\mu(Q) \overset{(3\text{-}2)}{\lesssim_\xi} \mu(S^*) \lesssim_{\xi,c} \mu(S),$$

where we used properties (D2), (D3), and 1-regularity of E in the last inequality. □

We now demonstrate how using a positive kernel leads to an easy proof of Corollary 1.2.

Proof of Corollary 1.2. First suppose that $f\in L^2(E)$ is a nonnegative function. Then as the kernel K_1 is positive, we have for fixed $p\in E$ that $T_1^\varepsilon f(p)$ is a monotonically increasing sequence as $\varepsilon\to 0$ and so

$$\text{p.v. } T_1 f(p) := \lim_{\varepsilon\to 0} T_1^\varepsilon f(p)$$

is a well-defined function, although it be infinity. By Theorem 1.1, we get that there exists some $C>0$ such that

$$\sup_{\varepsilon>0}\int (T_1^\varepsilon f)^2\,d\mu \leq C\int f^2\,d\mu.$$

Thus, by Fatou's lemma, we get

$$\int (\text{p.v. } T_1 f)^2\,d\mu \leq \liminf_{\varepsilon\to 0}\int (T_1^\varepsilon f)^2 \leq C\int f^2\,d\mu.$$

This then proves the corollary for nonnegative functions.

Now let $f\in L^2(E)$ be a real-valued function. We have the decomposition $f=f^+-f^-$, where $f^+=\max\{f,0\}$ and $f^-=\max\{-f,0\}$. Then

$$\max(\|f^+\|_{L^2(E)},\|f^-\|_{L^2(E)})\leq \|f\|_{L^2(E)}$$

and so we get that the principal values of f^+ and f^- under T_1 are controlled by $C\|f\|_{L^2(E)}$. Thus, the principal values have to be finite almost everywhere and so we get p.v. $T_1 f = $ p.v. $T_1 f^+ -$ p.v. $T_1 f^-$ as $L^2(E)$ functions. Additionally, we get

$$\|\text{p.v. } T_1 f\|_{L^2(E)} \leq \|\text{p.v. } T_1 f^+\|_{L^2(E)} + \|\text{p.v. } T_1 f^-\|_{L^2(E)} \leq 2C\|f\|_{L^2(E)}.$$

This proves the entire corollary. □

4. Sufficient conditions

We will need the following "triangle inequality" for this section.

Lemma 4.1 (NH^2 triangle inequality). *Let $a, b, c \in \mathbb{H}$ and let A be the (unsigned) area of the triangle in $\mathbb{R}^2$ with vertices $\pi(a), \pi(b), \pi(c)$. For the four quantities*

$$A, \quad NH(a^{-1}b)^2, \quad NH(b^{-1}c)^2, \quad NH(c^{-1}a)^2,$$

any one of these numbers is less than the sum of the other three.

Proof. Let us first show A is less than the sum of the other three quantities. Since everything is invariant under left translation, we may suppose $c = (0, 0, 0)$, $a = (x, y, t)$, and $b = (x', y', t')$. Then $NH(c^{-1}a)^2 = |t|$ and $NH(b^{-1}c)^2 = |t'|$ and we have

$$A = \tfrac{1}{2}|x'y - xy'| \le \left|\tfrac{1}{2}x'y - xy' - t + t'\right| + |t'| + |t| \le NH(a^{-1}b)^2 + NH(b^{-1}c)^2 + NH(c^{-1}a)^2.$$

We now show that $NH(a^{-1}b)^2$ is less than the sum of the other three quantities. We will keep the same normalization as the last case:

$$NH(a^{-1}b)^2 = \left|\tfrac{1}{2}x'y - xy' - t + t'\right| \le \tfrac{1}{2}|x'y - xy'| + |t'| + |t| \le A + NH(b^{-1}c)^2 + NH(c^{-1}a)^2. \quad \square$$

For $r < R$ and $x \in \mathbb{H}$, we can define the annulus

$$A(x, r, R) := \{y \in \mathbb{H} : d(x, y) \in (r, R)\}.$$

For three points p_1, p_2, p_3 in $\mathbb{H}$, we define

$$\partial(p_1, p_2, p_3) = \min_{\sigma \in S_3}\big\{d(p_{\sigma(1)}, p_{\sigma(2)}) + d(p_{\sigma(2)}, p_{\sigma(3)}) - d(p_{\sigma(1)}, p_{\sigma(3)})\big\}.$$

For $\alpha \in (0, 1)$, $r > 0$, and a metric space X, we let $\Sigma_X(\alpha, r)$ denote the triples of points $(p_1, p_2, p_3) \in X$ such that

$$\alpha r \le d(p_i, p_j) \le r \quad \forall i \neq j.$$

We also let $\Sigma_X(\alpha) = \bigcup_{r>0} \Sigma_X(\alpha, r)$. For notational convenience, we will drop the X subscript when we want $X = E$, where E is the 1-regular set of the hypothesis of Theorem 1.3.

Lemma 4.2. *Let $(p_1, p_2, p_3) \in \Sigma(\alpha, r)$. If for some $\varepsilon \in (0, 1/2)$ we have*

$$NH(p_i^{-1} p_j) \le \varepsilon d(p_i, p_j), \tag{4-1}$$

then the point $\pi(p_i) \in \mathbb{R}^2$ is contained in the strip around the line $\overline{\pi(p_{i+1}), \pi(p_{i+2})}$ of width $16\alpha^{-1}\varepsilon^2 r$.

Proof. We will view $\overline{\pi(p_2), \pi(p_3)}$ as the base of a triangle with top vertex $\pi(p_1)$. It suffices to bound the height. We let A denote the area of the triangle.

Suppose $A \ge 4\varepsilon^2 r^2$. We have by the NH^2 triangle inequality that

$$NH(p_2^{-1} p_3)^2 \ge A - NH(p_1^{-1} p_2)^2 - NH(p_1^{-1} p_3)^2 \overset{(4\text{-}1)}{\ge} 2\varepsilon^2 r^2.$$

This is a contradiction of (4-1).

Thus, we may assume $A \le 4\varepsilon^2 r^2$. But if $NH(p_2^{-1}p_3) \le d(p_2, p_3)/2$, then $|\pi(p_2) - \pi(p_3)| \ge d(p_2, p_3)/2 \ge \alpha r/2$. Thus, the height of the triangle is less than

$$\frac{2A}{|\pi(p_2) - \pi(p_3)|} \le \frac{16}{\alpha}\varepsilon^2 r. \qquad \square$$

Given $u, v, w \in \mathbb{H}$, we denote the largest and second largest quantities of

$$\left\{ \frac{NH(u^{-1}v)}{d(u, v)}, \frac{NH(v^{-1}w)}{d(v, w)}, \frac{NH(u^{-1}w)}{d(u, w)} \right\}$$

by $\gamma_1(u, v, w)$ and $\gamma_2(u, v, w)$, respectively.

Lemma 4.3. *For all $\alpha > 0$, there exists a constant $c_1 > 0$ such that if $(p_1, p_2, p_3) \in \Sigma(\alpha, r)$, then*

$$\partial(p_1, p_2, p_3) \le c_1 \gamma_1(p_1, p_2, p_3)^4 r.$$

Proof. Let $\gamma = \gamma_1(p_1, p_2, p_3)$, and we may suppose without loss of generality that

$$\partial(p_1, p_2, p_3) = d(p_1, p_2) + d(p_2, p_3) - d(p_1, p_3).$$

Suppose first that $\gamma < c$ for some $c > 0$ to be determined soon. Then

$$NH(p_i^{-1}p_j) \le \gamma d(p_i, p_j) < c d(p_i, p_j) \quad \forall i \ne j, \tag{4-2}$$

and so

$$|\pi(p_i) - \pi(p_j)| = \big(d(p_i, p_j)^4 - NH(p_i^{-1}p_j)^4\big)^{1/4} \ge (1 - c^4)^{1/4} d(p_i, p_j).$$

By taking c small enough, we get that $(\pi(p_1), \pi(p_2), \pi(p_3)) \in \Sigma_{\mathbb{R}^2}(\alpha/2)$ and, by Taylor expansion of the Korányi norm, that

$$d(p_i, p_j) \le |\pi(p_i) - \pi(p_j)| + \frac{NH(p_i^{-1}p_j)^4}{|\pi(p_i) - \pi(p_j)|^3} \le |\pi(p_i) - \pi(p_j)| + (1 - c^4)^{-3/4}\gamma^4 r,$$

and so

$$\partial(p_1, p_2, p_3) \le |\pi(p_1) - \pi(p_2)| + |\pi(p_2) - \pi(p_3)| - |\pi(p_1) - \pi(p_3)| + 2(1 - c^4)^{-3/4}\gamma^4 r. \tag{4-3}$$

As $(\pi(p_1), \pi(p_2), \pi(p_3)) \in \Sigma_{\mathbb{R}^2}(\alpha/2)$, we get by a Taylor approximation of the Euclidean metric that

$$|\pi(p_1) - \pi(p_2)| + |\pi(p_2) - \pi(p_3)| - |\pi(p_1) - \pi(p_3)| \lesssim_\alpha \frac{h^2}{r}, \tag{4-4}$$

where h is the height of the triangle in $\mathbb{R}^2$ defined by $\pi(p_i)$ with base $\overline{\pi(p_1), \pi(p_3)}$. From (4-1) and (4-2), we have

$$h \le 16\alpha^{-1}\gamma^2 r. \tag{4-5}$$

The result now follows from (4-3)–(4-5).

Now suppose $\gamma \ge c$. As $\partial(p_1, p_2, p_3) \le 3r$, the lemma trivially follows. $\square$

We let $E \subset \mathbb{H}$ be a set with $\mu = \mathcal{H}^1|_E$ satisfying the estimate

$$\xi r \le \mu(B(x,r)) \le \xi^{-1} r \quad \forall x \in E,\ r > 0,$$

where $\xi \le 1$.

Lemma 4.4. *Let $E \subset \mathbb{H}$ be a 1-regular set and $\alpha \in (0,1)$. There exists $c_2 \ge 1$ depending on α and ξ such that if $(p_1, p_2, p_3) \in \Sigma(\alpha, r)$, then one of the following is true*:

(1) $\gamma_1(p_1,p_2,p_3) \le c_2 \gamma_2(p_1,p_2,p_3)$.

(2) *After a possible reindexing of p_i, there exists a set $V \subseteq E \cap B(p_1, \alpha r/10)$ with $\mu(V) \ge r/c_2$ such that for every $x \in V$ we have*

$$\gamma_1(p_1,p_2,p_3) \le c_2 \gamma_2(x,p_2,p_3)$$

and $(x,p_2,p_3) \in \Sigma(c_2^{-1})$.

(3) *After a possible reindexing of p_i, there exist sets $W_1, W_2 \subseteq E \cap B(p_1, \alpha r/5)$ with $\mu(W_1), \mu(W_2) \ge r/c_2$ such that for all $(x,y) \in W_1 \times W_2$ we have*

$$\gamma_1(p_1,p_2,p_3) \le c_2 \gamma_2(p_1,x,y)$$

and $(p_1,x,y) \in \Sigma(c_2^{-1}, r)$.

Proof. Throughout this proof, we will give a finite series of lower bounds for c_2. The final c_2 will then just be the maximum of these lower bounds. For simplicity of notation, let $\gamma_i = \gamma_i(p_1,p_2,p_3)$. We may of course suppose that $\gamma_2 \le c\gamma_1$ for some small $c > 0$ depending on α and ξ to be determined, as otherwise condition (1) would be satisfied. Without loss of generality, we can assume that $\gamma_1 = NH(p_2^{-1}p_3)/d(p_2,p_3)$. Let A denote the area of the triangle in $\mathbb{R}^2$ with vertices $\pi(p_i)$. Then we have from the NH^2 triangle inequality that

$$NH(p_2^{-1}p_3)^2 \le NH(p_1^{-1}p_2)^2 + NH(p_1^{-1}p_3)^2 + A,$$

and so if we set $c < \alpha/2$ (while still allowing ourselves to take c smaller) then

$$A \ge \tfrac{1}{2}\alpha^2\gamma_1^2 r^2. \tag{4-6}$$

Fix $\lambda \in (0,1)$ depending only ξ so that

$$\mu(A(x,\lambda\ell,\ell)) \ge \tfrac{1}{2}\xi\ell \quad \forall x \in E,\ \ell > 0.$$

Suppose now $A(p_1, \lambda\alpha r/10, \alpha r/10)$ contains a subset S of μ-measure at least $\xi\alpha r/40$ so that

$$\frac{NH(x^{-1}p_1)}{d(x,p_1)} < c\gamma_1 \quad \forall x \in S. \tag{4-7}$$

If there is a further subset $V \subseteq S$ with $\mu(V) \ge \xi\alpha r/80$ such that $NH(x^{-1}p_2) \ge c\gamma_1 d(x,p_2)$ for each $x \in V$, then we are done as we've satisfied condition (2) for large enough c_2 if we keep p_2, p_3 and draw x from V.

Thus, suppose there is a subset $V \subseteq S$ with $\mu(V) \ge \xi\alpha r/80$ and

$$\frac{NH(x^{-1}p_2)}{d(x,p_2)} < c\gamma_1 \quad \forall x \in V. \tag{4-8}$$

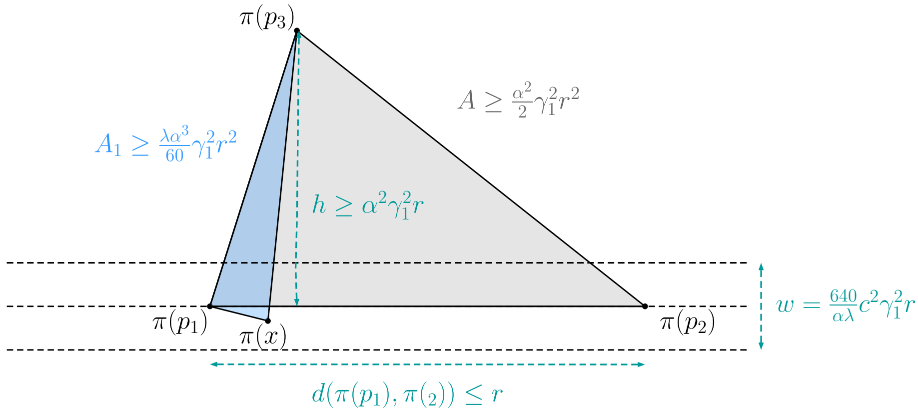

Figure 1. A denotes the area of the triangle determined by $\pi(p_i)$, $i = 1, 2, 3$, and A_1 denotes the area of the triangle determined by $\pi(p_1)$, $\pi(p_3)$ and $\pi(x)$.

Recalling

$$d(x, p_1) \in \left[\tfrac{1}{10}\lambda\alpha r, \tfrac{1}{10}\alpha r\right], \quad d(x, p_2) \in \left[\tfrac{1}{2}r, 2r\right], \quad \forall x \in V \subseteq A\left(p_1, \tfrac{1}{10}\lambda\alpha r, \tfrac{1}{10}\alpha r\right), \tag{4-9}$$

from (4-7), (4-8), and Lemma 4.2, for every $x \in V$ we get that $\pi(x)$ lies in the strip around $\overline{\pi(p_1), \pi(p_2)}$ of width

$$w = \frac{640}{\lambda\alpha}c^2\gamma_1^2 r. \tag{4-10}$$

As $NH(x^{-1}p_1) < c\gamma_1 d(x, p_1)$, we easily get (supposing c is small enough) that

$$|\pi(x) - \pi(p_1)| \geq \tfrac{1}{2}d(x, p_1) \overset{(4\text{-}9)}{\geq} \tfrac{1}{20}\lambda\alpha r. \tag{4-11}$$

As $d(p_1, p_2) \leq r$, we get that the height of the triangle given by $\pi(p_i)$ with base $\overline{\pi(p_1), \pi(p_2)}$ is then at least

$$h \geq \frac{2A}{d(p_1, p_2)} \overset{(4\text{-}6)}{\geq} \alpha^2\gamma_1^2 r.$$

Let A_1 denote the area of the triangle determined by $\pi(p_1), \pi(x), \pi(p_3)$. By (4-10), we have that w is at most some constant multiple (depending on α and λ) of $c^2 h$. Thus, if we choose c small enough to get $\pi(x)$ sufficiently close to the line $\overline{\pi(p_1), \pi(p_2)}$ compared to h, we get

$$A_1 \geq \tfrac{1}{3}h|\pi(p_1) - \pi(x)| \overset{(4\text{-}11)}{\geq} \tfrac{1}{60}\lambda\alpha^3\gamma_1^2 r^2.$$

See Figure 1 for an illustration of these triangles.

Now using the NH^2 triangle inequality, we get

$$\tfrac{1}{60}\alpha^3\lambda\gamma_1^2 r^2 \leq A_1 \leq NH(x^{-1}p_1)^2 + NH(p_1^{-1}p_3)^2 + NH(x^{-1}p_3)^2 \overset{(4\text{-}7),(4\text{-}9)}{\leq} 2c^2\gamma_1^2 r^2 + NH(x^{-1}p_3)^2.$$

Thus, if we choose c small enough compared to α and λ once and for all, we get

$$NH(x^{-1}p_3) \geq \tfrac{1}{10}\sqrt{\alpha^3\lambda}\gamma_1 r \geq \tfrac{1}{20}\sqrt{\alpha^3\lambda}\gamma_1 d(x, p_3).$$

Now we can satisfy condition (2) for sufficiently large c_2 by keeping p_2, p_3 and drawing x from V.

Thus, we may suppose that $E \cap A(p_1, \lambda\alpha r/10, \alpha r/10)$ contains a subset S so that $\mu(S) \geq \xi\alpha r/40$ and

$$NH(z^{-1}p_1) \geq c\gamma_1 d(z, p_1) \quad \forall z \in S.$$

Using the 1-regularity of E, an elementary, although tedious, packing argument shows that there exist $\eta, \tau < \lambda\alpha/100$ depending only on α and ξ and points $x', y' \in E \cap A(p_1, \lambda\alpha r/10, \alpha r/10)$ such that $d(x', y') \geq 10\tau r$ and

$$\min\bigl\{\mu(S \cap B(x', \tau r)),\ \mu(S \cap B(y', \tau r))\bigr\} \geq \eta r.$$

Note by the triangle inequality that we get

$$B(x', \tau r), B(y', \tau r) \subseteq A\bigl(p_1, \tfrac{1}{20}\lambda\alpha r, \tfrac{1}{5}\alpha r\bigr).$$

Thus, after setting c_2 large enough, we've satisfied condition (3) with $W_1 = S \cap B(x', \tau r)$ and $W_2 = S \cap B(y', \tau r)$, which would completely finish the proof of the lemma. We will present a quick sketch of the packing argument and leave the details to the reader.

Find a maximal τr-separated net $\mathcal{N}$ of $E \cap B(p_1, \alpha r)$ for $\tau > 0$ to be determined. By 1-regularity, we have $\#\mathcal{N} \gtrsim \alpha/\tau$. First use the 1-regularity of E to find $M \geq 1$ such that any subset $S \subseteq \mathcal{N}$ for which $\#S \geq M$ must contain $x', y' \in S$ so that $d(x', y') \geq 10\tau r$. Now $\{B(x, \tau r) : x \in \mathcal{N}\}$ is a covering of $B(p_1, \alpha r/10)$. Define $\mathcal{B} = \{B(x, \tau r) : x \in \mathcal{N}, \mu(S \cap B(x, r)) \geq \eta r\}$. By choosing η small enough relative to $\alpha\tau$, we can use the 1-regularity of E and the fact that $\mu(S) \gtrsim \alpha r$ to get that $\#\mathcal{B} \gtrsim \alpha\mathcal{N} \gtrsim \alpha^2/\tau$ (with no dependence on η). Now simply choose τ small enough so that $\#\mathcal{B} \geq M$. One then finds two balls $B(x', \tau r), B(y', \tau r) \in \mathcal{B}$ such that $d(x', y') \geq 10\tau r$, which finishes the sketch. □

For $x, y \in E$, we let

$$\Sigma(\alpha, r; x) := \{(y, z) \in E^2 : (x, y, z) \in \Sigma(\alpha, r)\}, \qquad \Sigma(\alpha; x, y) := \{z \in E : (x, y, z) \in \Sigma(\alpha)\}.$$

One easily has that there exists some constant $c_3 \geq 1$ depending on ξ such that

$$\frac{1}{c_3} r^2 \leq \mu \times \mu(\Sigma(\alpha, r; x)) \leq c_3 r^2, \qquad \frac{1}{c_3} d(x, y) \leq \mu(\Sigma(\alpha; x, y)) \leq c_3 d(x, y).$$

For simplicity, we will adopt the convention that the integral $\int_A f(x)\,dx$ means $\int_A f(x)\,d\mu(x)$ when $A \subseteq E$. Recall that for three points p_1, p_2, p_3 in a metric space X, the Menger curvature $c(p_1, p_2, p_3) \in \mathbb{R}$ is defined as

$$c(p_1, p_2, p_3) = \frac{1}{R},$$

where R is the radius of the circle in $\mathbb{R}^2$ passing through a triangle defined by the vertices $p_1', p_2', p_3' \in \mathbb{R}^2$, where $d(p_i, p_j) = |p_i' - p_j'|$.

Proposition 4.5. *For any $\alpha > 0$, there exists $c_4 \geq 1$ such that*

$$\iiint_{\Sigma(\alpha)} c(x, y, z)^2\,dx\,dy\,dz \leq c_4 \iiint_{\Sigma(c_4^{-1})} \frac{\gamma_1(x, y, z)^2\,\gamma_2(x, y, z)^2}{\operatorname{diam}(\{x, y, z\})^2}\,dx\,dy\,dz. \tag{4-12}$$

Proof. We have by [Hahlomaa 2005] that there exists some $\tau > 0$ depending on α such that if $(x, y, z) \in \Sigma(\alpha)$, then

$$c(x, y, z)^2 \leq \tau \operatorname{diam}(\{x, y, z\})^{-3}\partial(x, y, z). \tag{4-13}$$

By Lemma 4.3, we have that there exists $c_1 > 0$ such that

$$\iiint_{\Sigma(\alpha)} \operatorname{diam}(\{x, y, z\})^{-3} \partial(x, y, z)\, dx\, dy\, dz \le c_1 \iiint_{\Sigma(\alpha)} \frac{\gamma_1(x, y, z)^4}{\operatorname{diam}(\{x, y, z\})^2}\, dx\, dy\, dz. \tag{4-14}$$

We now decompose $\Sigma(\alpha)$ into three pieces. For $i = 1, 2, 3$, let $S_i \subseteq \Sigma(\alpha)$ denote the triples of points for which condition (i) of Lemma 4.4 holds for some $r > 0$ (that can depend on the triple of points). Note $\Sigma(\alpha) \subseteq S_1 \cup S_2 \cup S_3$, but this decomposition need not be disjoint.

It will be convenient to define the functions

$$f(x, y, z) := \frac{\gamma_1(x, y, z)^4}{\operatorname{diam}(\{x, y, z\})^2}, \quad g(x, y, z) := \frac{\gamma_1(x, y, z)^2\, \gamma_2(x, y, z)^2}{\operatorname{diam}(\{x, y, z\})^2}.$$

We trivially have that

$$\iiint_{S_1} f(x, y, z)\, dx\, dy\, dz \le c_2^2 \iiint_{S_1} g(x, y, z)\, dx\, dy\, dz. \tag{4-15}$$

When we write a triple of points $(x, y, z) \in S_2$, we will always assume y, z play the role of p_2, p_3 in condition (2). Now let $(x, y, z) \in S_2 \cap \Sigma(\alpha)$. We then have that there exists a subset with $\mu(V) \ge r/c_2$,

$$f(x, y, z) \le c_2 g(u, y, z) \quad \forall u \in V.$$

We then have

$$f(x, y, z) \le c_2 \frac{1}{\mu(V)} \int_V g(u, y, z)\, du.$$

We also have $(u, y, z) \in \Sigma(c_2^{-1})$ for all $u \in V$ and so

$$\int_{\Sigma(\alpha; y, z)} f(x, y, z)\, dx \le c_2 \frac{\mu(\Sigma(\alpha; y, z))}{\mu(V)} \int_V g(u, y, z)\, du \le c_2^2 c_3 \int_{\Sigma(c_2^{-1}; y, z)} g(u, y, z)\, du.$$

Now we have

$$\begin{aligned} \iiint_{S_2} f(x, y, z)\, dx\, dy\, dz &= \iiint_{\Sigma(\alpha)} \mathbf{1}_{S_2} f(x, y, z)\, dx\, dy\, dz \\ &\le \int_E \int_E \int_{\Sigma(\alpha; y, z)} \mathbf{1}_{S_2} f(x, y, z)\, dx\, dy\, dz \\ &\le c_2^2 c_3 \int_E \int_E \int_{\Sigma(c_2^{-1}; y, z)} g(x, y, z)\, dx\, dy\, dz \\ &\le 6 c_2^2 c_3 \iiint_{\Sigma(c_2^{-1})} g(x, y, z)\, dx\, dy\, dz. \end{aligned} \tag{4-16}$$

For S_3, we will write the points (x, y, z) with the understanding that z plays the role of p_1 in condition (3). Now let $(x, y, z) \in S_3 \cap \Sigma(\alpha/2, r)$. In a way similar to that above, we can use the properties of the conclusion of property (3) to get that

$$f(x, y, z) \le c_2^2 c_3 \iint_{\Sigma(c_2^{-1}, r; z)} g(u, v, z)\, du\, dv.$$

It is elementary to see that if $(x, y, z) \in \Sigma(\alpha)$, then

$$\int_0^\infty \mathbf{1}_{\{r:(x,y)\in\Sigma(\alpha/2,r;z)\}} \frac{dr}{r} \asymp_\alpha 1.$$

Here, we need the extra factor of $\frac{1}{2}$ in case (x, y, z) achieves tightness in the $\Sigma(\alpha)$ condition. We can now decompose the integral:

$$\begin{aligned}
\iiint_{S_3} f(x,y,z)\,dx\,dy\,dz &\lesssim_\alpha \iiint_{S_3} f(x,y,z) \int_0^\infty \mathbf{1}_{\{r:(x,y)\in\Sigma(\alpha/2,r;z)\}} \frac{dr}{r}\,dx\,dy\,dz \\
&\le \int_E \int_0^\infty \iint_{\{(x,y)\in\Sigma(\alpha/2,r;z):(x,y,z)\in S_3\}} f(x,y,z)\,dx\,dy \frac{dr}{r}\,dz \\
&\le c_2^2 c_3 \int_E \int_0^\infty \iint_{\Sigma(c_2^{-1},r;z)} g(u,v,z)\,du\,dv \frac{dr}{r}\,dz \\
&\lesssim_\alpha \int_{\Sigma(c_2^{-1})} g(x,y,z) \int_0^\infty \mathbf{1}_{\{r:(u,v)\in\Sigma(c_2^{-1},r;z)\}} \frac{dr}{r}\,du\,dv\,dz \\
&\lesssim \iiint_{\Sigma(c_2^{-1})} g(x,y,z)\,dx\,dy\,dz. \qquad (4\text{-}17)
\end{aligned}$$

In the second and penultimate inequalities, we used Fubini. We then get the conclusion from (4-13)–(4-16) and (4-17). □

Proof of Theorem 1.3. By a result of Hahlomaa [2007, p. 123], it suffices to show that for some $\alpha > 0$,

$$\iiint_{\Sigma(\alpha)\cap B(p,R)^3} c^2(y_1,y_2,y_3)\,dy_1\,dy_2\,dy_3 \lesssim R \quad \forall p\in E,\ R>0. \qquad (4\text{-}18)$$

Hence by (4-12), it is enough to prove that for some $\alpha > 0$,

$$\iiint_{\Sigma(\alpha)\cap B(p,R)^3} \frac{\gamma_1(y_1,y_2,y_3)^2\gamma_2(y_1,y_2,y_3)^2}{\operatorname{diam}(\{y_1,y_2,y_3\})^2}\,dy_1\,dy_2\,dy_3 \lesssim R \quad \forall p\in E,\ R>0. \qquad (4\text{-}19)$$

By our assumption, for all $\varepsilon > 0$ and every $f \in L^2(E)$,

$$\|T_2^\varepsilon f\|_{L^2(E)} \lesssim \|f\|_{L^2(E)}. \qquad (4\text{-}20)$$

Let $p \in E$ and $R > 0$. Applying (4-20) to $f = \chi_{B(p,R)}$, we get that there exists some $C \ge 0$ such that for every $\varepsilon > 0$,

$$\int_{E\cap B(p,R)} \int_{E\cap B(p,r)\cap B(y_1,\varepsilon)^c} \frac{NH(y_1^{-1}y_2)^2}{d(y_1,y_2)^3}\,dy_2 \int_{E\cap B(p,r)\cap B(y_1,\varepsilon)^c} \frac{NH(y_1^{-1}y_3)^2}{d(y_1,y_3)^3}\,dy_3\,dy_1 \le CR,$$

$$U_\varepsilon = \big\{(y_1,y_2,y_3)\in\Sigma(\alpha)\cap B(p,R)^3 : d(y_1,y_2)>\varepsilon,\ d(y_1,y_3)>\varepsilon\big\},$$

$$V_\varepsilon = \big\{(y_1,y_2,y_3)\in\Sigma(\alpha)\cap B(p,R)^3 : d(y_1,y_2)>\varepsilon,\ d(y_1,y_3)>\varepsilon,\ d(y_2,y_3)>\varepsilon\big\}.$$

It then easily follows from Fubini (remember that all the terms in the integrand are positive) that

$$\iiint_{U_\varepsilon} \frac{NH(y_1^{-1}y_2)^2\, NH(y_1^{-1}y_3)^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^6}\, dy_1\, dy_2\, dy_3 \leq CR. \tag{4-21}$$

Therefore,

$$CR \geq \iiint_{V_\varepsilon} \frac{NH(y_1^{-1}y_2)^2\, NH(y_1^{-1}y_3)^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^6}\, dy_1\, dy_2\, dy_3 + \iiint_{U_\varepsilon\setminus V_\varepsilon} \frac{NH(y_1^{-1}y_2)^2\, NH(y_1^{-1}y_3)^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^6}\, dy_1\, dy_2\, dy_3. \tag{4-22}$$

Using the upper regularity of μ, it is not difficult to show that

$$\iiint_{U_\varepsilon\setminus V_\varepsilon} \frac{NH(y_1^{-1}y_2)^2\, NH(y_1^{-1}y_3)^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^6}\, dy_1\, dy_2\, dy_3 \lesssim_\xi R. \tag{4-23}$$

Using (4-21)–(4-23) and letting $\varepsilon \to 0$ we deduce that

$$\iiint_{\Sigma(\alpha)\cap B(p,R)^3} \frac{NH(y_1^{-1}y_2)^2\, NH(y_1^{-1}y_3)^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^6}\, dy_1\, dy_2\, dy_3 \leq CR.$$

By permuting variables, we get

$$\iiint_{\Sigma(\alpha)\cap B(p,R)^3} \sum_{\sigma\in S_3} \frac{NH(y_{\sigma(1)}^{-1}y_{\sigma(2)})^2\, NH(y_{\sigma(1)}^{-1}y_{\sigma(3)})^2}{\operatorname{diam}(\{y_1, y, y_3\})^6}\, dy_1\, dy_2\, dy_3 \leq 6CR. \tag{4-24}$$

If $(y_1, y_2, y_3) \in \Sigma(\alpha)$, then it follows easily that

$$\begin{aligned}\frac{\gamma_1(y_1, y_2, y_3)^2\, \gamma_2(y_1, y_2, y_3)^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^2} &\lesssim \max_{\sigma\in S_3} \frac{NH(y_{\sigma(1)}^{-1}y_{\sigma(2)})^2\, NH(y_{\sigma(1)}^{-1}y_{\sigma(3)})^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^6}\\ &\leq \sum_{\sigma\in S_3} \frac{NH(y_{\sigma(1)}^{-1}y_{\sigma(2)})^2\, NH(y_{\sigma(1)}^{-1}y_{\sigma(3)})^2}{\operatorname{diam}(\{y_1, y_2, y_3\})^6},\end{aligned} \tag{4-25}$$

where the constant multiple implicit in the first inequality depends on α. We then get (4-19) from (4-24) and (4-25). □

5. Norm independence

In this short section we will show that Theorems 1.1 and 1.3 do not depend on the Korányi metric.

Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two homogeneous norms on $\mathbb{H}$ and denote by

$$d_i(p, q) = \|q^{-1}\cdot p\|_i$$

the induced metrics for $i = 1, 2$. We will also denote by $B_i(p, r)$ the balls with respect to the metric d_i for $i = 1, 2$. It is well known — see, e.g., [Bonfiglioli et al. 2007, Proposition 5.1.4] — that all homogeneous norms in a Carnot group are globally equivalent. In particular there exists some $L \geq 0$ such that

$$L^{-1}\|p\|_2 \leq \|p\|_1 \leq L\|p\|_2 \quad \text{for } p \in \mathbb{H}.$$

Let $s > 0$ and define $k_1, k_2 : \mathbb{H} \setminus \{0\} \to (0, +\infty)$ by

$$k_1(p) = \frac{|z|^s}{\|p\|_1^{2s+1}} \quad \text{and} \quad k_2(p) = \frac{|z|^s}{\|p\|_2^{2s+1}},$$

where $p = (x, y, z) \in \mathbb{H} \setminus \{0\}$. As in the proof of Lemma 2.7 one can show that the kernels $k_i,\ i = 1, 2$, are CZ kernels. Note also that

$$L^{-s-1}\, k_2(p) \le k_1(p) \le L^{s+1}\, k_2(p).$$

Let μ be a 1-regular measure on $\mathbb{H}$ and define the truncated singular integrals

$$S_1^\varepsilon f(p) = \int_{B_1(p,\varepsilon)^c} k_1(q^{-1} \cdot p) f(q)\, d\mu(q) \quad \text{and} \quad S_2^\varepsilon f(p) = \int_{B_2(p,\varepsilon)^c} k_2(q^{-1} \cdot p) f(q)\, d\mu(q)$$

for $f \in L^2(\mu)$ and $\varepsilon > 0$.

Proposition 5.1. *The operator S_1 is bounded in $L^2(\mu)$ if and only if the operator S_2 is bounded in $L^2(\mu)$.*

Proof. It suffices to show that if S_2 is bounded in $L^2(\mu)$ then S_1 is bounded in $L^2(\mu)$. We define the following auxiliary truncated singular integral for $\varepsilon > 0$ and $f \in L^2(\mu)$:

$$\widetilde{S}_2^\varepsilon f(p) = \int_{B_2(p,\varepsilon)^c} k_1(q^{-1} \cdot p) f(q)\, d\mu(q).$$

Let Q be any David cube associated to μ, as in the beginning of Section 3. Then

$$\begin{aligned}\|\widetilde{S}_2^\varepsilon \chi_Q\|_{L^2(\mu)}^2 &= \int \left(\int_{Q \cap B_2(p,\varepsilon)^c} k_1(q^{-1} \cdot p)\, d\mu(q) \right)^2 d\mu(p) \\ &\le L^{2(s+1)} \int \left(\int_{Q \cap B_2(p,\varepsilon)^c} k_2(q^{-1} \cdot p)\, d\mu(q) \right)^2 d\mu(p) \le L^{2(s+1)} \|S_2 \chi_Q\|_{L^2(\mu)}^2 \lesssim \mu(Q)\end{aligned}$$

because S_2 is bounded in $L^2(\mu)$. Hence by the $T(1)$ theorem for homogeneous spaces — see, e.g., [Deng and Han 2009; David 1991] — we deduce that $\widetilde{S}_2$ is bounded in $L^2(\mu)$.

For $f \in L^2(\mu),\ \varepsilon > 0$, and $p \in \mathbb{H}$, we have

$$\begin{aligned}|S_1^\varepsilon f(p) - \widetilde{S}_2^\varepsilon f(p)| &= \left| \int_{B_1(p,\varepsilon)^c} k_1(q^{-1} \cdot p) f(q)\, d\mu(q) - \int_{B_2(p,\varepsilon)^c} k_1(q^{-1} \cdot p) f(q)\, d\mu(q) \right| \\ &\lesssim \int_{B_1(p,\varepsilon) \setminus B_2(p,\varepsilon)} \frac{|f(q)|}{d_1(p,q)}\, d\mu(q) + \int_{B_2(p,\varepsilon) \setminus B_1(p,\varepsilon)} \frac{|f(q)|}{d_1(p,q)}\, d\mu(q).\end{aligned}$$

Note that

$$\begin{aligned}\int_{B_1(p,\varepsilon) \setminus B_2(p,\varepsilon)} \frac{|f(q)|}{d_1(p,q)}\, d\mu(q) &\le \int_{\{q : \varepsilon/L \le d_1(p,q) < \varepsilon\}} \frac{|f(q)|}{d_1(p,q)}\, d\mu(q) \\ &\le \frac{L}{\varepsilon} \int_{B_1(p,\varepsilon)} |f(y)|\, d\mu(q) \approx \frac{1}{\mu(B_1(p,\varepsilon))} \int_{B_1(p,\varepsilon)} |f(y)|\, d\mu(q) \le M_\mu^1 f(p),\end{aligned}$$

where M^1_μ denotes the Hardy–Littlewood maximal function with respect to d_1 and μ. Similarly,

$$\int_{B_2(p,\varepsilon)\setminus B_1(p,\varepsilon)} \frac{|f(q)|}{d_1(p,q)}\, d\mu(q) \lesssim M^1_\mu f(p),$$

and we have shown that

$$|S^\varepsilon_1 f(p) - \widetilde{S}^\varepsilon_2 f(p)| \lesssim M^1_\mu f(p).$$

Hence the proposition follows because we already showed that $\widetilde{S}_2$ is bounded in $L^2(\mu)$ and it is also well known that the maximal operator M^1_μ is bounded in $L^2(\mu)$. □

In particular, as a corollary to Theorems 1.1 and 1.3 and Proposition 5.1, we obtain that Theorems 1.1 and 1.3 hold respectively for the kernels

$$K_1'(p) = \frac{|z|^4}{d_{cc}(p,0)^9} \quad \text{and} \quad K_2'(p) = \frac{|z|}{d_{cc}(p,0)^3},$$

where, recalling Definition 2.2, d_{cc} stands for the Carnot–Carathéodory distance.

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